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The Universe from Scratch

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Abstract

A fascinating and deep question about nature is what one would see if one could probe space and time at smaller and smaller distances. Already the 19th-century founders of modern geometry contemplated the possibility that a piece of empty space that looks completely smooth and structureless to the naked eye might have an intricate microstructure at a much smaller scale. Our vastly increased understanding of the physical world acquired during the 20th century has made this a certainty. The laws of quantum theory tell us that looking at spacetime at ever smaller scales requires ever larger energies, and, according to Einstein's theory of general relativity, this will alter spacetime itself: it will acquire structure in the form of curvature. What we still lack is a definitive *theory of quantum gravity* to give us a detailed and quantitative description of the highly curved and quantum-fluctuating geometry of spacetime at this so-called Planck scale. – This article outlines a particular approach to constructing such a theory, that of *Causal Dynamical Triangulations*, and its achievements so far in deriving from first principles why spacetime is what it is, from the tiniest realms of the quantum to the large-scale structure of the universe.

Searching for the quanta of spacetime

Armed with last century's insights into the nature of both quantum theory and general relativity, physicists believe that probing the structure of space and time at distances far below those currently accessible by our most powerful accelerators would reveal a rich geometric fabric, where spacetime itself never stands still but instead quantum-fluctuates wildly. One of the biggest challenges of theoretical physics today is to identify these fundamental "atoms" or excitations of spacetime geometry and understand how their interaction gives rise to the macroscopic spacetime we see around us and which serves as a backdrop for all known physical phenomena.

Two pillars of contemporary physics support the expectation that as we resolve the fabric of spacetime with an imaginary microscope at ever smaller scales, spacetime will turn from an immutable stage into the actor itself. First, due to Heisenberg's uncertainty relations, probing spacetime at very short distances is necessarily accompanied by large quantum fluctuations in energy and momentum - the shorter the distance, the larger the energy-momentum uncertainty. Second, according to Einstein's theory of general relativity, the presence of these energy fluctuations, like that of any form of energy, will deform the geometry of the spacetime in which it resides, imparting curvature which is detectable through the bending of light rays and particle trajectories. Taking these two things together leads to the prediction that the quantum structure of space and time at the so-called Planck scale must be highly curved and dynamical.

A long held ambition of theoretical physicists is to find a consistent description of this dynamical microstructure within a *theory of quantum gravity*, which unifies quantum theory and general relativity, and to determine its ramifications for high-energy physics and cosmology. Given the extraordinary smallness of the Planck length, how can we achieve progress in describing a physical situation that cannot be directly probed by experiment in the foreseeable future? The way this is usually done is by first postulating additional dynamical principles or fundamental symmetries at small distances, which are not accessible to direct experimental verification, second, verifying that these do not conflict with standard quantum physics or general relativity as one goes to larger scales, and third, predicting new physical phenomena that can (at least in principle) be tested, or confirmed indirectly by astrophysical observations. Examples of fundamental building principles are that the universe is made up of tiny vibrating strings, or that spacetime at the Planck scale is not a continuum, but consists of tiny discrete grains.

Research into quantum gravity falls broadly into two categories [1, 2]: non-perturbative approaches to quantum gravity, whose primary aim is to quantize the gravitational degrees of freedom per se, introducing little or no additional structure such as supersymmetry or extra dimensions, and string-theoretic approaches, where the quantization of gravity appears almost as a by-product of a unified higher-dimensional and supersymmetric "theory of everything", whose fundamental objects are strings and (mem)branes [3].

The research program that will be described in this article deals with the investigation of causal nonperturbative quantum gravity and belongs in the first category. The approach takes its name from the main technical tool it employs to try and construct a theory of quantum gravity, namely, *Causal Dynamical Triangulations*, or CDT for short.[1] What makes this approach particularly interesting is the fact that it has recently produced a number of tangible results which mark it as a serious contender for the still elusive theory of quantum gravity [5, 6, 7, 8]. Firstly, there is evidence that the theory has a good classical limit. This means that it reproduces Einstein's classical theory at sufficiently large scales: when one "zooms out" the imaginary microscope from the scale at which the quantum fluctuations take place, one eventually rediscovers the smooth four-dimensional spacetime of general relativity. Secondly, we have first indications of what the quantum structure of spacetime may be at the Planck scale.

We will explain in the following why these results are indeed remarkable and how they were obtained, in a manner hopefully accessible to those outside the field. The emphasis will be on describing the (very few) fundamental building principles that go into the construction of the theory, on explaining the main results and their physical significance, and on giving an idea of where we are headed. Before doing this, we will in the next section set out by sketching some of the problems facing research in quantum gravity, in order to provide the reader with a better idea of how the results described later appear in a larger context.

Why quantum gravity is special

Quantum gravity is quite unlike any other fundamental quantum interaction in that it describes the dynamics of an entity that in most physical situations is considered as fixed and given, namely, that of spacetime itself. Recall that the degrees of freedom of a spacetime in classical general relativity can be described by the spacetime metric $g_{\mu\nu}(x)$, a local field variable which determines the values of distance and angle measurements in spacetime, or, equivalently, how spacetime is bent and curved locally.[2] What spacetime *is* classically is determined by solving the Einstein equations for $g_{\mu\nu}(x)$, subject to boundary conditions and a particular matter content of the universe or a piece thereof. In the same manner, in order

[1]Previous reviews of CDT, describing the general theory and covering earlier results in lower dimensions can be found in [4].

[2]When using the term (quantum) spacetime, we will in the following always mean the abstract spacetime (a differential manifold in the classical case) *together* with its metric properties.

to determine what spacetime *is* from a quantum-theoretical point of view, one would like to formulate a quantum analogue of Einstein's equations, from which "quantum spacetime" should then emerge as a *solution*.

This should be contrasted with usual quantum field theory, which describes the dynamics of elementary particles and their interactions on a *fixed* spacetime background, usually that of the flat, four-dimensional Minkowski space of special relativity. Since at short distances[3] the gravitational forces are so much weaker than the electromagnetic ones, say, it is usually an excellent approximation to treat the gravitational degrees of freedom as "frozen in" and non-dynamical. The trivial geometric structure of the Minkowski metric forms merely part of the immutable background structure of how quantum field theories are formulated. On the other hand, the physical situations that quantum gravity aims to explain are not in general describable in terms of linear fluctuations of the metric field around Minkowski space or some other fixed background metric. These include the quantitative description of "empty" spacetime at very short distances of the order of the Planck scale, 10^{-35} m, and of the extreme and ultradense state our universe presumably was in when it was very young. From a technical point of view this implies that in quantum gravity one has to modify standard quantization techniques which rely (sometimes implicitly) on the presence of a fixed metric background structure. This is often phrased by saying that gravity must ultimately be quantized in a way that is both *background-independent* (i.e. does not distinguish any particular background metric at the outset) and *nonperturbative* (i.e. does not simply describe the dynamics of linear perturbations around some fixed background spacetime).

Decades of quantum gravity research, including numerous trials and errors, have convinced many of the necessity of background-independence and a nonperturbative quantization approach [9], and the last twenty years or so have seen intense efforts to develop alternative ways of quantizing applicable to the case of a dynamical spacetime geometry. Because of the complicated mathematical structures involved, this turns out to be very difficult. Singularly unhelpful in this endeavour has been the absence of experimental or observational data to guide the search for the correct theory of quantum gravity. This happens because the extreme scales that are necessarily involved in physical situations where quantum-gravitational effects are important are not accessible directly with current technology. A possible strategy to address this state of affairs is to take a rather conservative approach to theory-building, for example, to avoid going out on a limb by postulating the existence of new physical quantities and symmetries for which there is as yet no evidence. As we will see, the approach of CDT is fairly minimalist in that it takes a set of well-known physical principles and

[3]Short, but still far away from the Planck scale defined below.

tools (quantum-mechanical superposition, causality, triangulation of geometry, elements of the theory of critical phenomena) and merely adapts them to the situation of a dynamical geometry.

How far then have we got in our quest for a theory of quantum gravity? Maybe the best answer to this question is that it is hard to tell, as long as we do not know the final answer with some certainty. This assessment must in all honesty also include the possibility that we are still very far from the correct theory. There has been a lot of research since the late 1980s, both in nonperturbative quantum gravity as such and under the heading of string theory, at the time when the incorporation of gravitation into our understanding of the quantum world emerged as a final theoretical frontier of fundamental physics. Progress has undoubtedly been made, at the very least in terms of developing technical tools to describe quantum geometry nonperturbatively (see, for example, [10, 11, 12]). However, in order to cut a long story short, there is still *not a single theory* of quantum gravity that is both reasonably complete and internally consistent mathematically. By reasonably complete we mean that it should provide answers to some of quantum gravity's central questions, for instance, "Why is spacetime the way it is?", "What are the fundamental excitations of quantum geometry?", "What are the quantum properties of black holes?" etc., even if they are not immediately verifiable experimentally. Our entire discussion therefore must be understood against a background where we do not have a plethora of "possible" theories available (and just look for clues for how to pick the right one), but where we are still looking for the first instance of a quantum gravity theory that is sufficiently complete to make at least some predictions about the quantum behaviour of spacetime.

The dynamical principle underlying CDT

The most important theoretical tool of the CDT method to construct a quantum theory of gravity is Feynman's principle of superposing quantum amplitudes [13], the famous *path integral*, applied to spacetime geometries. Its basic idea, familiar from quantum mechanics, is to obtain a solution to the quantum dynamics of a physical system by taking a superposition of "all possible" configurations of the system, where each configuration contributes a complex weight $\exp(iS)$ to the path integral, which depends on the classical action $S = \int dt L(t)$ of the configuration, where L denotes the system's Lagrangian. For the case of a non-relativistic particle moving in a potential, the configurations are literally paths in space, i.e. continuous trajectories $\mathbf{x}(\tau)$ describing the particle's position as a function of time τ, which runs through an interval $\tau \in [0, t]$. Superposing (that is, adding or integrating up) the associated quantum amplitudes $\exp iS^{\rm part}[\mathbf{x}(\tau)]$, one obtains a solution to the Schrödinger equation of the particle. It is important

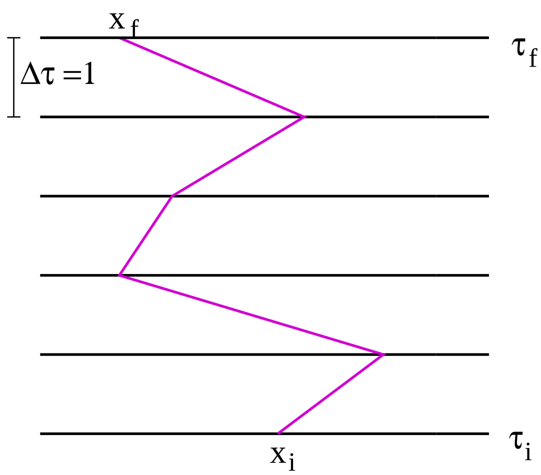

Figure 1: A typical, piecewise straight path $\mathbf{x}(\tau)$ contributing to the regularized Feynman path integral for a non-relativistic particle moving between two points $\mathbf{x}_\mathrm{i}$ and $\mathbf{x}_\mathrm{f}$.

to realize that the *individual* paths $\mathbf{x}(\tau)$ appearing in the path integral are *not* themselves physical trajectories the particle could move on, and even less solutions to the particle's classical equations of motion. Instead, they are so-called "virtual" paths, that is, a bunch of curves one can draw between fixed initial and final points $\mathbf{x}_\mathrm{i}$ and $\mathbf{x}_\mathrm{f}$ (Fig. 1). The magic of the path integral

$$G(\mathbf{x}_\mathrm{i}, \mathbf{x}_\mathrm{f}, t) := \sum\!\!\!\!\!\!\int_{\text{paths: } \mathbf{x}_\mathrm{i} \to \mathbf{x}_\mathrm{f}} \mathrm{e}^{iS^{\mathrm{part}}[\mathbf{x}(\tau)]} \tag{1}$$

is that the true quantum physics of the particle is encoded precisely in the *superposition* of all these virtual paths[4]. In order to extract these physical properties, one has to evaluate suitable quantum operators $\hat{O}$ on the ensemble of paths contributing to (1). For example, one may be interested in computing expectation values for the position or the energy of the particle, together with their quantum fluctuations. Of course, the path integral or "propagator" (1) also allows us to retrieve the classical behaviour of the particle in a particular limit (in this case, when its mass becomes big), but it contains more information, describing the full quantum dynamics of the system.

Analogously, a path integral for gravity is a superposition of all virtual "paths" our universe (or a part thereof) can follow as time unfolds. These paths are simply

[4] In using the notation $\sum\!\!\!\!\!\!\int$ in (1), we want to indicate that the path integral may be a sum or a genuine integral (or possibly a combination of both), depending on whether the configurations contributing to it are labelled by discrete or continuous parameters. In CDT we will meet an example of the former.

the different configurations for the metric field variables $g_{\mu\nu}(x)$ mentioned earlier.[5] It is important to realize that a single path is now no longer an assignment of just three numbers (the coordinates x_i of the particle) to every moment τ in time, but rather the assignment to every τ of a whole array of numbers (the space-space components $\mathbf{g}_{ij}(x) \equiv \mathbf{g}_{ij}(\mathbf{x}, \tau)$ of the metric tensor $g_{\mu\nu}(x)$) *for each spatial point* $\mathbf{x}$. This is simply a consequence of gravity being a field theory with infinitely many degrees of freedom. The path integral for gravity can thus be written as

$$G(\mathbf{g}_{\rm i}, \mathbf{g}_{\rm f}, t) := \sum\!\!\!\!\!\!\int_{\text{spacetimes: } \mathbf{g}_{\rm i} \rightarrow \mathbf{g}_{\rm f}} \mathrm{e}^{iS^{\rm grav}[g_{\mu\nu}(\mathbf{x},\tau)]}, \tag{2}$$

where $S^{\rm grav}$ now denotes the classical gravitational action associated with a spacetime metric $g_{\mu\nu}$ with initial and final boundary condition $\mathbf{g}_{\rm i}$ and $\mathbf{g}_{\rm f}$, separated by a time distance t. Like in the particle case, the individual spacetime configurations interpolating between the initial and final spatial geometries have nothing a priori to do with classical spacetimes, and are much more general objects. Again, one would expect to be able to retrieve the full quantum dynamics of spacetime from the path integral (2), which is a superposition of all possible ways in which an empty spacetime can be curved[6]. In other words, the collective behaviour of the virtual spacetimes contributing to the gravitational propagator (2) should tell us what quantum spacetime *is*. To extract this geometric information, we will again have to evaluate suitable quantum operators $\hat{O}$ on the ensemble of geometries contributing to (2). Suffice it to say that making the gravitational path integral well-defined and extracting the desired physical information is very much more difficult than in the case of the quantum particle.

The way in which CDT proceeds is by giving a precise prescription of how the path integral (2) should be computed, and in particular how the class of virtual paths should be chosen. In addition, it provides a set of technical tools to extract concrete physical information about the quantum geometry thus created by the principle of quantum superposition.

There are a number of ways in which the path integral of CDT differs from that of previous approaches. In the first instance, it is genuinely nonperturbative, in that the contributing geometries can have very large curvature fluctuations at very small scales and thus be arbitrarily far away from any classical spacetime. Our summation is "democratic" in that no particular spacetime geometry is distinguished at the outset. In fact, path integral histories which have any geometric resemblance to a classical spacetime are so rare that their contribu-

[5]We are considering here only the gravitational degrees of freedom, that is, a path integral for "pure gravity", and thus a "theory for empty space". Matter fields can in principle be included without problems.

[6]These are also sometimes called "spacetime histories".

tion to the path integral is effectively negligible.[7] Secondly, as we will see in the following, the causal structure of the geometries contributing to the path sum plays an important role in the method of causal dynamical triangulations, and is a key new element in comparison with previous, so-called Euclidean path-integral approaches to quantum gravity.

Representing spacetime geometry in CDT

What we need to do next in order to make sense of the expression (2) for the nonperturbative quantum-gravitational propagator is to define the precise class of spacetime geometries (labelled above by $g_{\mu\nu}$) over which the sum or integral is to be taken. As elsewhere in quantum field theory, one is immediately confronted with the fact that unless one chooses a careful regularization for the path integral, it will be wildly divergent and simply not exist in any meaningful mathematical sense (and thus be useless for extracting physical information). "Regularizing" means making the path integral finite by introducing certain cutoff parameters for the contributing configurations, which at a later stage will be removed in a controlled manner.

Before defining a suitable class of geometries in the next section, we will first explain the nature of the regularized spacetimes used by CDT, which are called "piecewise flat geometries". Recall that the dynamical degrees of freedom of a geometry are the ways in which it is locally curved. Piecewise flat geometries are simply spaces that are flat (the same as straight or uncurved, that is, structureless from a geometric point of view) everywhere apart from small subspaces where curvature is said to be concentrated. This in a way discretizes curvature and vastly reduces the different number of ways spacetime can be curved. The type of geometry we will use is a triangulated space, also sometimes called a Regge geometry, after the physicist who first introduced it into (classical) general relativity [15]. It can be thought of as a space glued together from elementary building blocks which are (higher-dimensional generalizations of) triangles, so-called "simplices". The geometric structure of each simplex is trivial, since it is by itself flat by definition and therefore carries no curvature. Local curvature only appears along lower-dimensional interfaces when one starts gluing the simplices together.

This can be visualized most easily in the two-dimensional case. Consider a set of identical equilateral two-dimensional triangles cut out from a piece of cardboard

[7] This is completely analogous to the particle case, where it can be shown rigorously that classical paths "form a set of measure zero" with respect to the Wiener measure of the path integral [14]. Maybe surprisingly, the paths which contribute non-trivially are nowhere differentiable, and thus "consist only of corners". One expects a similarly nonclassical behaviour for the dominant configurations of the gravitational path integral.

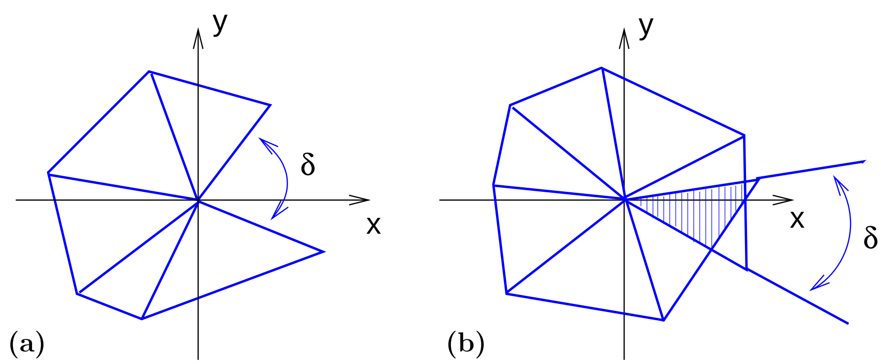

Figure 2: Example of positive and negative deficit angles δ located at a vertex of a two-dimensional triangulation. The triangulations have been cut open in order to flatten them into the two-dimensional Euclidean x-y-plane. To reconstruct the original geometry around the central vertex (which has been placed at the plane's origin), one has to identify the triangle edges as indicated by the arrows. In case (a), the angles meeting at the vertex add up to less than 2π, resulting in a positively curved space upon regluing. In case (b) the combined angle exceeds 2π, corresponding to a negatively curved space.

which is perfectly straight and unbendable (and hence flat). To obtain a larger surface, start gluing these triangles together by identifying their one-dimensional sides or edges pairwise. Points where several edges meet are also called vertices. One can obtain a piece of flat space by arranging the triangles in a regular pattern so that exactly six triangles and edges meet at each vertex. However, there are many more ways to create *curved* spaces by the same gluing procedure. Namely, whenever the number of triangles meeting at a vertex is smaller or larger than six, this vertex will carry a positive or negative curvature.[8] By "curvature" we mean the *intrinsic* curvature of the two-dimensional surface, i.e. the curvature that can be detected from within the surface – for example, by studying the trajectories of particles or light rays –, and is independent of any higher-dimensional space in which it may be imbedded. This mirrors a property of the physical theory of general relativity in four dimensions, which likewise depends only on the intrinsic geometry of spacetime. The set-up in higher dimensions is identical, with the two-dimensional triangles (or "two-simplices") substituted by the corresponding flat higher-dimensional simplices (three-simplices (or tetrahedra) in dimension 3, four-simplices in dimension 4, etc.). Generally speaking, the fundamental building blocks in dimension d are glued together pairwise along their $(d-1)$-dimensional faces, and their intrinsic curvature is concentrated on the $(d-2)$-dimensional

[8]Equivalently, one speaks of the vertex having a positive or negative deficit angle, simply because the sum of the angles of the triangles contributing at the vertex is smaller or bigger than 2π, whereas in the flat case it is exactly 2π, see Fig. 2 for illustration.

intersections of these faces.

The so-called Regge calculus [16] was originally designed to approximate smooth classical spacetimes, or, more precisely, solutions to the Einstein equations, by these piecewise flat, triangulated spaces. There are two reasons for why this is a very economical way of describing a spacetime. Firstly, only a finite amount of data is necessary to completely characterize a finite piece of spacetime, namely, the geodesic invariant lengths of all the one-dimensional edges of all the simplices involved, and the way in which the d-dimensional simplices are glued together.[9] Secondly, because no coordinate system need ever be introduced on the simplices, this formulation does not share the usual coordinate redundance of Einstein gravity described in terms of the field variables $g_{\mu\nu}(x)$. (The latter overcount the degrees of freedom, because $g_{\mu\nu}$'s which are related by coordinate transformations correspond to one and the same physical geometry.)

The use of Regge geometries in the quantum theory is not new, and CDT builds on previous attempts of both "Quantum Regge Calculus" [17] and "Dynamical Triangulations" [18] to define a theory of quantum gravity from a non-perturbative Euclidean path integral[10]. To avoid misunderstandings, it should be emphasized that the use of triangulated spacetimes differs in classical and quantum applications. The objective in the former is to approximate a single, smooth spacetime (which may or may not be known exactly by some other method) as well as possible. This can be achieved by choosing a sequence of triangulations, where in each step of the sequence the triangulation is chosen finer than in the previous step (i.e. the typical edge length is decreased) and therefore can converge to the smooth manifold in a pointwise sense. In the two-dimensional example, it is quite clear that such an approximation can be very good when the edge lengths become much smaller than the scale at which the smooth spacetime is curved.

By contrast, the objective of the quantum theory, and that of CDT in particular, is to approximate *the integral* (2) as well as possible, or, more precisely, to *define* it since there is currently no alternative, independent way of doing the computation. This is a completely different task, since the integral does not represent a single classical geometry, but a quantum superposition, where each single contributing spacetime is a highly nonclassical object, as we pointed out earlier. There is no accurate mathematical statement to guide this construction, but one would expect that the path integral should provide an "ergodic sampling" of the

[9] This can be kept track of by attaching labels to all simplices and their faces, say, and then pairing off the faces which are glued together.

[10] The essential difference between the two approaches is that in Quantum Regge Calculus one fixes a triangulation or "gluing", so that the path integral takes the form of a (multiple) integral over the lengths of the edges of that triangulation, whereas in Dynamical Triangulations one fixes *all* edge lengths to a common value a, in which case the path integral takes the form of a discrete sum over all inequivalent ways to glue the (then identical-looking) simplicial building blocks together.

space of geometries. This may seem like a very vague characterization, but one is in practice very much constrained by the requirement of making the regularized path integral mathematically well-defined and obtaining a sensible classical limit.

The short-distance cutoff a is an important part of our regularization of the spacetime geometries in the gravitational propagator. We will take the limit $a \to 0$ as part of the search for a so-called continuum limit of the path integral over the regularized geometries. This has to be done in order to obtain a final theory which does not depend on many of the arbitrary details which have gone into constructing the regularized model, which itself constitutes only an intermediate step in the construction of the theory. Using a finite "lattice spacing" a and taking $a \to 0$ (while renormalizing the coupling constants of the theory as a function of a) is a method borrowed from the theory of critical phenomena and virtually ensures that the end result does not depend on a variety of regularization details. This latter property of "universality" is only a necessary condition and does by no means guarantee that this construction leads to a viable theory of quantum gravity, as opposed to describing the dynamics of certain one-dimensional polymers, say, as we will explain further in the next section.[11]

The ensemble of virtual spacetime geometries in CDT

Now that we have introduced the regularized triangulated geometries the question still remains as to exactly what ensemble of such objects should be included in the sum over geometries in (2). Here is where CDT differs in a crucial way from previous approaches, and where the notion of "causality" comes into play. We mentioned above that precursors of CDT's nonperturbative path integral are "Euclidean" in nature. What this means is that the integration is not performed over so-called Lorentzian *spacetimes* (which have one time- and three space-directions) but over Euclidean *spaces* (which have four spatial directions, and thus no notion of time, light rays or causality). Classically, Euclidean "spacetimes" are bizarre and unphysical entities, in which moving back and forth in time is just as easy as moving back and forth in space. Their use in the (mainly perturbative) gravitational path integral was made popular in the late 1970s by the influential work of S. Hawking and collaborators on black holes and quantum cosmology in the context of Euclidean quantum gravity [19]. The reason for using them instead of

[11] One may feel tempted to postulate that the cutoff a is a fundamental shortest length scale pertaining to the existing physical world, and thus do away with any continuum limits. Apart from having to justify such an ad-hoc assumption, one would then have to face the fact that the quantum processes at this scale, which such a "theory" may describe, will by default depend on a large number of a priori arbitrary regularization parameters labelling all possible choices of fundamental building blocks and gluing rules, and thus run the danger of having no predictive value at all.

Lorentzian spacetimes of the correct physical signature[12] is mainly technical: in the Euclidean case, the weights $\exp(iS^{\text{grav}})$ are no longer complex but real numbers, which simplifies a discussion of the convergence properties of the path integral, and also makes Monte-Carlo simulations possible.[13] The potential catch is that in gravity, unlike in other quantum field theories on a flat background, there is no obvious relation between a nonperturbative path integral for Lorentzian and one for Euclidean geometries. In fact, causal dynamically triangulated gravity in dimensions two [20], three [21] and four [22, 23, 5, 6, 7, 8, 24] has for the first time provided concrete evidence that the two path integrals are genuinely inequivalent and possess completely different properties. The remainder of this section explains in more detail how the path integral for either Euclidean or Lorentzian gravity in terms of triangulated geometries is set up and evaluated, and at the same time retraces some of the history that led to the introduction of CDT.

With the ingredients that were defined in the previous section, it would seem straightforward to write down a regularized version of the gravitational propagator as

$$G^{\text{reg}}(\mathbf{T}_{\text{i}}, \mathbf{T}_{\text{f}}, t) := \sum_{\text{triangulations T}:\, \mathbf{T}_{\text{i}} \to \mathbf{T}_{\text{f}}} \mathrm{e}^{iS^{\text{reg}}[T]}, \tag{3}$$

where T denotes a triangulated spacetime, glued from four-simplices, and with two spatial triangulated boundary geometries $\mathbf{T}_{\text{i}}$ and $\mathbf{T}_{\text{f}}$ (glued from three-simplices), between which it interpolates. The gravitational action for a piecewise flat spacetime T schematically takes the form

$$S^{\text{reg}}(T) = -1/G_N\; \mathit{Curvature}(T) + \lambda\; \mathit{Volume}(T), \tag{4}$$

and there is a definite prescription for how to compute the curvature and volume of a given triangulation T in terms of the lengths of its edges and its connectivity (that is, the way the four-simplices are glued together). The two coupling constants of the theory appearing in (4) are Newton's constant G_N, governing the strength of gravitational interactions, and the cosmological constant λ, another constant of nature, which may be responsible for the "dark energy" pervading our universe [25].

As mentioned in footnote 10, all simplices used in DT are equilateral[14], and the path integral assumes the form of a discrete sum over inequivalent ways in

[12]The *signature* refers to the signs of the four eigenvalues of the symmetric matrix $g_{\mu\nu}(x)$; it is (+,+,+,+) in the Euclidean case and (-,+,+,+) in the Lorentzian case.

[13]In order to simplify notation, we will always use the notation $\exp(iS)$ to denote Boltzmann weights, with the implicit understanding that S is a real action when we talk about Lorentzian signature (and the weight thus a complex phase factor), and a purely imaginary one in a Euclidean context (and $\exp(iS)$ therefore a *real* quantity).

[14]To be precise, this is true for the Euclidean version of dynamical triangulations (DT); CDT operates with *two* different edge lengths, one for all edges that have a spatial orientation, and one for edges with a time-like orientation [23].

which the simplicial building blocks can be glued together. The only thing that remains then to be specified in (3) is whether any gluing of the building blocks is to be allowed, or whether further restrictions need to be imposed. One condition turns out to be essential for making the path integral construction well-defined. Call $\mathcal{N}(N_4)$ the number of distinct gluings of N_4 four-simplices, for a particular set of gluing rules. Clearly, this number will grow with N_4, but the important question is whether it will grow exponentially as a function of N_4 or faster, namely, "super-exponentially", for example, like $\exp(cN_4^\nu)$, with $c > 0$ and $\nu > 1$. In the latter case, and noting that $N_4(T)$ is proportional to *Volume(T)*, there is no way in which the exponential weights $\exp iS^{\rm reg}[T]$ could ever counterbalance the growth of the number of contributing geometries (the growth of the so-called "entropy" of the system). The path integral would then be too divergent to lead to a fundamental theory of gravity.

These considerations preclude the inclusion in the path integral (3) of a so-called "sum over topologies".[15] Therefore, the topology of the spacetimes contributing to the nonperturbative path integral has to be fixed. It is typically chosen to be a four-dimensional sphere or torus. This state of affairs is somewhat ironic, because the possibility of including a sum over topologies has often been praised as an advantage of the path integral formulation over canonical quantization methods, which employ a 3+1 split of spacetime into space and time. As we have argued, this is only true at a formal level, that is, as long as one does not perform concrete computations (and therefore has to worry about the convergence or otherwise of an expression like (3)). At least from a Euclidean point of view, there are now no further natural restrictions one may impose on the geometries, and it is from this starting point that the original approach of Dynamical Triangulations proceeded [18, 26], in order to study the properties of the theory (hopefully) defined by the continuum limit of (3).

This may be a convenient moment to make some non-technical remarks on how (C)DT evaluates the path integral and extracts physical information from it, such as the expectation values of certain geometric observables. A direct analytical evaluation of (3) – although available in lower-dimensional models – is formidably difficult. However, unlike in a variety of other approaches to quantum gravity, DT possesses a set of powerful and well-developed numerical tools, whose value can hardly be overstated. They have been adapted from statistical mechanics and the theory of critical phenomena to the case where the individual

[15]The *topology* of a spacetime describes the way in which it hangs together. For example, the topology of a two-dimensional compact and closed surface is completely characterized by the number of its "holes" or "handles". It could have the form of a surface of a ball (no holes), of a surface of a torus or bicycle inner tube (one hole), of a surface of a double-torus or two-hole doughnut (two holes), and so on. In four dimensions, the labelling of different topologies is much more involved.

configurations are curved geometries, rather than spin or field configurations on a fixed background space or lattice. The ensemble of spacetimes underlying the path integral is simulated by Monte Carlo methods [27], generating a random walk in the space of all configurations according to a probability distribution defined by (3).[16] The limitations of the computer imply that this procedure can only be implemented on a (possibly large but) finite space of geometric configurations. This is usually taken into account by performing the simulations on the ensemble of triangulations of a fixed discrete volume N_4. By repeating the numerical measurements for a variety of different N_4's, one then tries to extrapolate the results in a systematic way to the physically relevant limit $N_4 \to \infty$. This well-known technique is known as "finite-size scaling".

Now, what kind of "quantum geometry" does one expect to see with the help of these tools? If all goes well, the quantum superposition (3) of geometries should be able to reproduce a classical spacetime at large scales L, that is, in the classical limit. However, at small scales l, with $a \ll l \ll L$, one expects quantum fluctuations to dominate, with a resulting highly nonclassical behaviour of the geometry. To cut a long story short, this was unfortunately not what was found for the Euclidean dynamically triangulated path integral studied in the 1990s. This was not immediately realized, but emerged gradually as more numerical simulations were performed [28]. It turned out that the quantum geometry generated by Euclidean DT could be in either one of two "phases". In the first one the geometry was completely crumpled, and in the other totally polymerized, that is, degenerated into thin branching threads[17]. These structures persist also at large scales, and as a result the DT path integral appears to have no meaningful classical limit, and therefore does not satisfy a necessary criterion for a theory of quantum gravity. (One can only wonder how long it may have taken to realize this, had one not been in a state to perform extensive simulations of the model.)

The starting point of CDT was the hypothesis that this failure may have to do with the unphysical *Euclidean* nature of the construction, and that one may be able to rectify the situation by encoding the causal structure of *Lorentzian* spacetimes explicitly in the choices of building blocks and gluing rules. Several years passed since this initial conjecture, in which CDT's causal quantization program was implemented and its viability tested in lower dimensions [20, 21]. Namely, superpositions like (3) can be defined also by considering spacetimes glued from two- or three-dimensional building blocks. This gives rise to simplified toy models which share some, but by no means all properties of the true CDT path

[16]For the Euclidean path integral, one can directly use the real weights $\exp iS^{\rm reg}[T]$. For the Lorentzian case of CDT, in order to obtain a probability distribution from (3), one first has to apply a so-called "Wick rotation" which converts the complex to real weights [23] (see also footnote 13).

[17]See the next section for a geometric characterization of these phases.

integral. On the plus side, they can be tackled both analytically and numerically, and compared with other quantization approaches to Einstein gravity in two and three spacetime dimensions. These extensive investigations showed unequivocally that the causal, Lorentzian path integral in all cases gave different results from the corresponding Euclidean path integral. Encouraged by this and a number of further interesting and new results pertaining to lower-dimensional quantum gravity, the investigation moved on to the four-dimensional case in 2004.

Causality implies four-dimensionality!

We will now describe the first piece of evidence which showed that CDT can reproduce at least *some* aspects of classical geometry correctly. This concerns a point where previous related quantization attempts have failed, namely, to generate a geometric object that can be said to be *four-dimensional* on sufficiently large scales.

It may come as a surprise that a superposition of locally four-dimensional geometries can give anything that is *not* again four-dimensional. After all, we have obtained our geometric building blocks by cutting out small pieces from a four-dimensional flat space. However, as is illustrated by Euclidean dynamically triangulated models, it is perfectly possible that the dimension comes out not as four, and this is indeed what seems to happen *generically*. The crumpled and polymeric phases of the Euclidean model mentioned in the previous section are characterized by a so-called Hausdorff dimension which takes the values infinity and two respectively.[18]

How can one obtain spaces with such strange dimensionalities? Roughly speaking, the Hausdorff dimension is obtained by comparing the typical linear size r of a convex subspace of a given space (e.g. its diameter) with its volume $V(r)$. If the leading behaviour is $V(r) \sim r^{d_H}$, the space is said to have the Hausdorff dimension d_H. To obtain an effectively infinite-dimensional space from gluing N_4 four-dimensional simplices with edge length a, one may consider a sequence of triangulations whose volume goes to infinity, $N_4 \to \infty$, where the gluing for each fixed N_4 is chosen such that every single building block shares a given vertex. That is, no matter how large N_4 gets, all building blocks of the triangulated space crowd around a single point. This is a procedure which is compatible with the gluing rules, but gives rise to a space whose dimensionality diverges, simply because its linear size always stays at the cutoff length a. Conversely, one can get an effectively one-dimensional space by gluing the four-dimensional building blocks into a long and thin tube. That is, as $N_4 \to \infty$ and $a \to 0$, one

[18] Interestingly, one observes the same behaviour when one starts from three- instead of four-dimensional building blocks. This result is believed to hold generically in Euclidean DT models, as long as the dimensionality of the elementary building blocks is at least three.

keeps three out of the four directions at a size of the order of the cutoff a, and only grows the geometry along the fourth direction.

This argument shows that there are spaces with "exotic" dimensionality which can be obtained as limiting cases of regular simplicial manifolds. Of course, the relevant question for the gravitational path integral is whether geometries of this nature indeed *dominate the path integral in the continuum limit*. This is a genuinely dynamical question which cannot be decided a priori. It depends on the relative weight of "energy" and "entropy", that is to say, the Boltzmann weight of a given geometry (which in turn is a function of the values of the bare coupling constants) *and* the number of geometries with a given, fixed Boltzmann weight. Thus it may happen that an exotic geometry (for example, one of the highly crumpled objects above) has a very large Boltzmann weight and is therefore "energetically favoured", but that there are relatively speaking far fewer of such objects in the ensemble than there are of the more "normal" geometries, such that the contribution of the former will in the end play no role in the path integral in the continuum limit. As we have seen, this is not what happens in Euclidean dynamically triangulated models for quantum gravity whose state sums, depending on the values of the coupling constants, are dominated by exotic geometries which are either maximally crumpled ($d_H = \infty$) or of the form of so-called branched polymers (with $d_H = 2$).

The finding that "dimensionality" is turned into a dynamical quantity is a consequence of the fact that the nonperturbative gravitational path integral contains highly nonclassical geometries which are curved (and even highly curved) at the cutoff scale a. It can and indeed does happen that geometries with such an unruly short-scale behaviour dominate the path integral as $a \to 0$. As already remarked earlier, this is exactly what one would expect in analogy with the path integral for the particle, which in the continuum limit is dominated by totally nonclassical paths with "infinitely many corners". It is important to emphasize that the short-scale picture of geometry that arises in CDT is completely different from that of the classical theory. If one looks at a piece of classical spacetime – no matter how curved – with an ever finer resolution, it will *always* eventually start looking like a piece of flat spacetime, namely, when the observed scale becomes much smaller than the characteristic scale at which the space is curved. By contrast, a typical "quantum spacetime" generated by our nonperturbative path integral construction will *never* resemble a flat space, no matter how fine we choose the resolution of our virtual magnifying glass.

Having understood that quantum geometry will necessarily look very nonclassical at short scales, we presumably are still left with many possibilities for the precise microstructure that is generated by various prescriptions for setting up the gravitational path integral. Can we formulate criteria for recognizing when a particular prescription stands a chance of leading to the correct theory of quantum gravity? Fortunately, the answer is yes, and the criteria in question have to do with reproducing features of classical geometry at sufficiently large scales. As alluded to above, the simplest such test is whether the quantum geometry has the correct dimension four at large distances. A path integral which does not pass this test simply does not qualify as a candidate for a theory of quantum gravity.

The art is then to come up with a path integral which allows for large short-scale fluctuations in curvature, but in such a way that the resulting large-scale geometry nevertheless does not degenerate completely, so that a sensible classical limit may exist. The method of causal dynamical triangulations has for the first time in the history of the nonperturbative gravitational path integral given us an explicit example of such a geometry. What has been found to be crucial in its derivation are certain causal rules one imposes on the triangular building blocks, which make explicit reference to the Lorentzian structure of the individual geometries contributing to the path integral. The new and intriguing physical insight that can therefore be deduced from this result is that causality at sub-Planckian scales may be responsible for the fact that our universe is four-dimensional [5, 8]. A related lesson that has been made explicit by the dynamical triangulations approach in general is the fact that once geometric excitations are "let loose" in a nonperturbative formulation of quantum gravity, just about anything can happen. Not even the dimensionality of (what we thought of as) the spacetime emerging from the quantum superposition has to come out right. At the same time one could therefore also worry that other nonperturbative quantum gravity approaches may suffer related pathologies, which have only gone undetected because one has not been able to determine expectation values like that of the Hausdorff dimension $\langle d_H \rangle$ explicitly.

The reader may by now be curious about the precise nature of the causality conditions present in the CDT approach. They are simply that each spacetime appearing in the sum over geometries should have a specific form. Namely, it should be a geometric object which can be obtained by evolving a purely spatial geometry in time, in such a way that its spatial topology (the way in which space hangs together) is unchanged as a function of time. An example of a forbidden spacetime is one where an initially connected space splits into two or several components, or the converse process where several components of a space reunite into a single one [29]. Spacetimes with so-called wormholes also fall into this category and will therefore not be included in the sum over geometries. So, what is wrong with these geometries? Why do they violate causality? Let us start by explaining why these geometries are pathological *from a classical point of view*. Imagine a three-dimensional space that undergoes a branching process as time progresses (Fig. 3). Initially the space consists of a single piece (or component), which simply means that any point in the space can be reached from any other point along a continuous path. At some moment in time, the

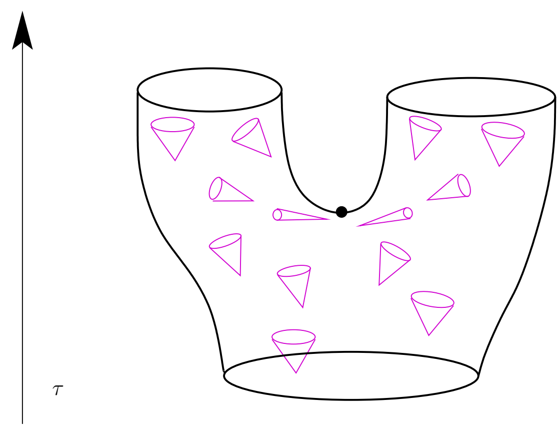

Figure 3: A "trouser" spacetime as example of a spacetime with topology change. Two-dimensional rendering of a space (here depicted as a one-dimensional circle) splitting into two components as time τ progresses. The smooth assignment of light cones must break down at the marked branching point, because the light cone "does not know where to turn".

space splits into two components which then remain cut off from each other. Classically this represents a highly singular process, with nothing to suggest such processes actually occur in nature.[19] From a spacetime point of view, something in these processes goes wrong with the light cone structure. The assignment of light cones to spacetime points cannot be smooth, since there must be at least one point in spacetime (precisely the branching point) where it cannot be decided whether a light ray arriving from the past should be continued to the future in one or the other of the two spatial components. Since the light cones define the causal structure of spacetime, this is an example of a geometry where causality is violated. The classical Einstein equations simply cannot describe such topology-changing spacetimes. Two more things should be noted: firstly, the absence of branching points (and their time inverses, the joining points) from a Lorentzian geometry is invariant under diffeomorphisms because different notions of time always share the same overall direction of their "time flow".[20] In order to introduce branching points and their associated "baby universes" (those parts of the universe that branch out from the "mother universe", never to return), one would need to reverse the time flow in entire open regions of spacetime, which cannot be done by an allowed coordinate transformation. Secondly, in the Euclidean theory, which has no distinguished (class of) time direction(s), one

[19] If this were the case, we would see whole chunks of space (together with their contents) suddenly disappear.

[20] We are not considering the possibility of a complete reversion of the time flow, which exchanges past and future globally.

simply cannot talk about the absence or presence of analogous branching points in a meaningful (i.e. coordinate-invariant) manner.

Returning to our discussion of the *quantum* theory, the premise of CDT is to use the Lorentzian structure of its contributing geometries explicitly and exclude all spacetimes with topology changes and therefore acausal behaviour.[21] Although classical considerations of causality have motivated a similar implementation of causality in CDT, it should be emphasized that such constraints on the path integral histories can never be derived conclusively from the classical theory. After all, the individual path integral geometries are never going to be smooth classical objects (let alone solutions to the equations of motion), nor even close to classical geometries. There is hence no obvious reason to forbid any particular quantum fluctuations of the geometry, including those that include topology changes. In principle, a quantum superposition of acausal spacetimes could lead to a quantum spacetime where causality by some mechanism is restored dynamically, at least macroscopically. However, although this is a theoretical possibility, it is not what one has observed in the Euclidean version of DT which does not have such causality restrictions, and which goes wrong already in trying to reconstruct a *four*-dimensional space. By the same token, the fact that individual path integral geometries in CDT are causal is also not by itself sufficient to guarantee that the quantum geometry it generates has again the same property. Whether this is indeed the case is not yet known, and requires a more detailed knowledge about the local geometric structure than is currently available. For example, one would like to ascertain that at a sufficiently coarse-grained level the quantum geometry possesses a well-defined light cone structure by defining and measuring suitable quantum observables. – However, there are already a number of important facts known about the large- and small-scale structure of the quantum spacetime emerging in CDT from first principles, which form the subject of the following section.

What *is* the quantum spacetime generated by CDT?

We have emphasized in the last section the importance of the emergence of classical geometry as a test for potential quantum gravity theories. The dimensionality of spacetime is only one of many quantum observables one may try to evaluate in order to determine the properties of the ground state geometry generated by CDT at various length scales. It is the coarsest such variable, because the dimensionality of a spacetime – at least in classical differential geometry – precedes

[21] It is rather straightforward to implement the causality conditions on the triangulated geometries of CDT. Each spacetime is built from layers of fixed duration (one "length step" in proper time), and one implements gluing rules for the simplices which ensure that no change of spatial topology can occur during the step [23].

that of specifying a metric structure.

Talking about observables, one must keep in mind that an innocent-looking question like "what is the value of the metric tensor $g_{\mu\nu}$ at point x?" is among the most difficult to answer in a nonperturbative approach like ours. Firstly, although CDT histories come with a notion of proper time, they do not otherwise carry any natural coordinate system. Even if we introduced coordinate systems on the individual triangulated spacetimes, there is no way to mark "the same point" simultaneously in all of them. This is a consequence of the fact that individual points do not have any physical significance in empty space; in the absence of matter there is simply nothing we could "mark" the point x with. We are thus forced to phrase any question about local curvature properties, say, in terms of quantities that *are* meaningful in the context of a diffeomorphism-invariant theory, for example, n-point correlation functions where the location of each of the n points has been averaged over spacetime.[22]

The correlation function that has been studied up to now in CDT measures the correlation between the volumes $V_{\text{space}}(\tau)$ of spatial slices (slices of constant time τ) some fixed proper-time distance $\Delta\tau$ apart, that is, a suitably normalized version of the expectation value

$$\langle V_{\text{space}}(0) V_{\text{space}}(\Delta\tau)\rangle = \sum_{\tau=0}^{t} \langle V_{\text{space}}(\tau) V_{\text{space}}(\tau+\Delta\tau)\rangle, \qquad (5)$$

where the ensemble average is taken over simplicial spacetimes with time extension t and with fixed four-volume [5, 6, 8]. One piece of evidence for the four-dimensionality of spacetime at large distances is the fact that in order to map the functions $\langle V_{\text{space}}(0) V_{\text{space}}(\Delta\tau)\rangle$ on top of each other for different values of the spacetime volume N_4, the time distance $\Delta\tau$ has to be rescaled by the power N_4^{1/D_H}, where the "cosmological Hausdorff dimension" is $D_H = 4$ within measuring accuracy [5, 8]. This means that what we would like to call a continuum "time" really scales with the correct fraction of the total spacetime volume. Such a "canonical scaling" is what one would have expected naïvely, but is absolutely not ensured a priori in the presence of large geometric quantum fluctuations, even though the individual building blocks at the cutoff scale are four-dimensional.[23]

Before looking at another striking result on dimensionality to have come out of CDT, let us review what else we know about the large-scale geometry of the quantum spacetime dynamically generated by CDT. This concerns a result which enables us to make contact with (quantum) cosmology. Recall the remarkable fact that almost every aspect of today's standard model of cosmology, describing the

[22] Two-point functions of this type have been measured previously in Euclidean DT [30].

[23] Further, independent evidence that the volumes $V_{\text{space}}(\tau)$ of the spatial slices also scale canonically as N_4 is increased, $V_{\text{space}} \sim N_4^{3/D_H}$, with $D_H = 4$, can be found in [8].

large-scale structure of our universe, is based on a radical truncation of (the geometric sector of) Einstein's theory to a *single* global degree of freedom, the so-called *scale factor* $a(\tau)$. It describes the linear size (or "scale") of the universe as a function of time τ.[24] This truncation is justified if the universe is homogeneous and isotropic at the largest scales, which means that it looks the same everywhere and in all spatial directions, something that is usually assumed to be true. An entirely different question is whether one can extract information about the *quantum* behaviour of the universe (for example, very close to the big bang where quantum effects should come into play) by quantizing the classically truncated system of just a single geometric variable $a(\tau)$. One may wonder whether in this way one is not missing important physics contained in the quantum fluctuations of all the local gravitational degrees of freedom which the cosmological description ignores.

Having in hand an explicit construction of quantum geometry à la CDT where no such truncation is present, one can ask what predictions it makes for the dynamics of the scale factor, and compare those to standard quantum cosmology. The answer obtained is intriguing: it is indeed possible to extract an effective action for the scale factor from CDT by integrating out all other degrees of freedom in the full quantum theory. The resulting action takes the *same* functional form as the standard action of a "minisuperspace" cosmology for a closed universe, *up to an overall sign* [6]. The collective effect of the local gravitational excitations seems to result in the same kind of contribution as that coming from the scale factor itself, but with the opposite sign. One way to understand this from an analogous continuum point of view may be in terms of so-called Faddeev-Popov determinants, which contribute to the effective action as a result of gauge-fixing [31]. The potentially far-reaching consequences of this result for quantum cosmology are currently being explored. What has already been established is that the computer-generated quantum geometry can in the semiclassical approximation be understood as a so-called "bounce", a particular type of solution to the Euclidean equation of motion (see Fig. 4). On the basis of this, the infamous "wave function of the universe" $\Psi_0(a)$ [32, 33] has been computed in CDT as a function of the scale factor a [6].

However, what is also clear from the computer simulations is that the semiclassical approximation is no longer an adequate description of the observed behaviour of the scale factor when the latter becomes small. This is of course to be expected and is an indicator for new quantum-gravitational effects appearing at short distances. Having gathered some nontrivial evidence that CDT's quantum geometry reproduces well-known aspects of classical general relativity on

[24] As far as we can tell, our present universe not only expands, but expands at an ever increasing rate, that is, both $\dot{a}(\tau) > 0$ and $\ddot{a}(\tau) > 0$. A "big bang" or "big crunch" corresponds classically to a singular point where $a = 0$.

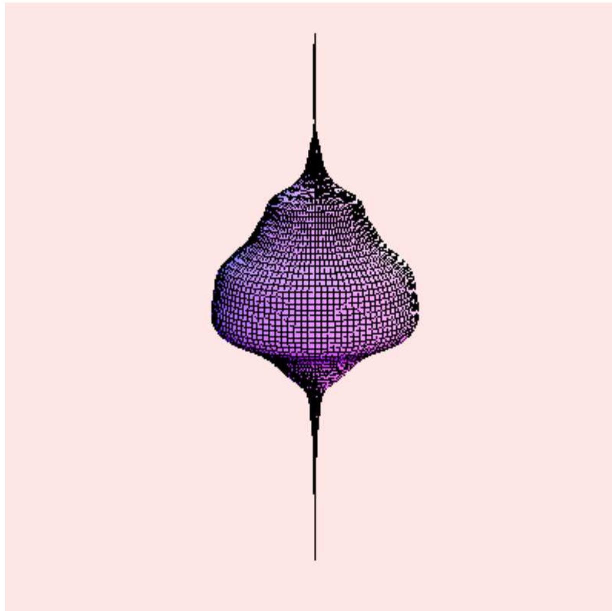

Figure 4: The overall shape of the extended quantum universe generated by CDT is determined by a bounce solution of the Euclidean effective action for the scale factor $a(\tau)$. The figure is a Monte Carlo snapshot of a typical universe of $N_4 = 91.100$ four-simplices and illustrates the behaviour of $a(\tau)$ (the circumference of the rotational body) as a function of the proper time τ (vertical axis).

sufficiently large scales, the main focus of research has to be on what the actual *quantum modifications* of this structure are. This is the place where new quantum physics will appear, and our effort will go into describing it in both a qualitative and quantitative manner.

CDT has already given us first insights into what the microstructure of quantum spacetime may be. The evidence comes from yet another way of probing the effective dimensionality of spacetime. The idea is to define a diffusion process (equivalently, a random walk) on the triangulated geometries in the path integral over spacetimes, and to deduce geometric information of the underlying quantum spacetime from the behaviour of the diffusion as a function of the diffusion time σ inherent to the process. The beauty of this procedure is its wide applicability, since diffusion processes cannot just be defined on smooth manifolds, but on much more general spaces, such as our triangulations and even on fractal structures [34]. The quantity we are interested in is the so-called "spectral dimension", which is really the effective dimension of the carrier space "seen" by the diffusion process. It can be extracted from the return probability $P(\sigma)$ which measures the probability of a random walk to have returned to its origin after diffusion time σ (or σ evolution steps if the diffusion is implemented discretely). For diffusion

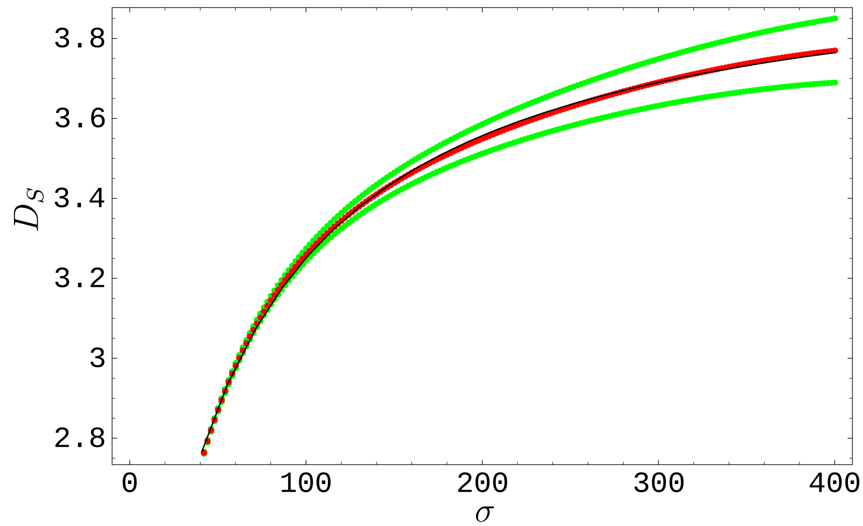

Figure 5: The spectral dimension D_S of the universe as function of the diffusion time σ, as measured in CDT for a quantum universe of $N_4 = 181.000$ building blocks. Extrapolating from this data, one obtains the estimates $D_S(\sigma = \infty) = 4.02 \pm 0.1$ in the limit of large distances and $D_S(\sigma = 0) = 1.80 \pm 0.25$ for the short-distance spectral dimension [7, 8]. The two outer curves represent error bars.

on a flat d-dimensional manifold, we have the exact relation $P(\sigma) = 1/(4\pi\sigma)^{d/2}$. For general spaces we *define* the spectral dimension $D_S(\sigma)$ as the logarithmic derivative[25]

$$D_S(\sigma) := -2\, \frac{d \log P(\sigma)}{d \log \sigma}. \tag{6}$$

Note that in general this dimension will depend on σ: small values of σ probe the small-distance properties of the underlying space, and large values its large-distance geometry.[26] The spectral dimension extracted for the quantum geometry of CDT is a twofold average over the starting point of the diffusion process (which is initially peaked at a given four-simplex) and over all geometries contributing to the path integral [7, 8]. The result of the measurement is quite striking and plotted in Fig. 5. What one observes is indeed a scale-dependence of the space-time dimension! At large distances it approaches the value four asymptotically, in agreement with the dimension obtained previously from scaling arguments, and in agreement with our classical expectation. However, as we probe the geometry at ever shorter distances (and *before* we enter the region where the simulations

[25] The complete expression for the return probability has correction terms because of the finite size of the computer-generated geometries which we are suppressing for simplicity. A more detailed discussion can be found in [8].

[26] As usual in a random walk, the linear distance probed will be of the order of $\sqrt{\sigma}$.

become unreliable due to discretization effects), this dimension decreases continuously to an extrapolated value of two within measuring accuracy. Such a scale-dependence has never before been observed in statistical models of quantum gravity and is a clear indication that spacetime behaves highly nonclassically at short distances close to the Planck length. Further investigations of a number of critical exponents and dimensions associated with the geometric structure of spatial slices and "sandwiches" (of time extension $\Delta\tau = 1$) [8] suggest the presence of a *fractal microstructure* of quantum spacetime, whose details are the subject of ongoing research.

In an independent development, a similar smooth running from four to two of the spectral dimension has been derived within a renormalization group approach to quantum gravity in the continuum [35], which posits (and provides some evidence for) the existence of a nontrivial fixed point in the ultraviolet (i.e. short-distance) regime of quantum gravity [36]. Although this coincidence by no means proves that either formulation is correct, it is nevertheless remarkable that the same unexpected result has been obtained in two very different approaches to quantum gravity. If the result can indeed be shown to be part and parcel of a viable quantum gravity theory, its implications for how we view spacetime and how we compute quantum processes of the other fundamental interactions on spacetime may be profound. For example, it could provide a natural ultraviolet cutoff for scattering amplitudes in high-energy physics.

Conclusions and outlook

This article has offered an overview of some of the fundamental issues addressed by quantum gravity, and has described a particular attempt to arrive at a consistent quantum theory of gravity, through the use of causal dynamical triangulations (CDT). As we hope to have illustrated, this approach has yielded a number of concrete results concerning the emergence of classical geometry from a Feynman-type superposition of spacetimes, provided appropriate care was taken to eliminate spacetimes with acausal features from the nonperturbative gravitational path integral. Although the derivation of the four-dimensionality of spacetime "from scratch" is an unprecedented result, more features of the classical theory still need to be established, for example, the presence of attractive gravitational forces as expressed by Newton's law. Assuming that this can be accomplished, the really interesting and new physics lies of course beyond the classical approximation. Here the challenge will be to extract more detailed information about the short-scale structure of quantum spacetime and, if possible, to uncover concrete physical consequences that may in principle be detectable. As we have seen, CDT offers already some tantalizing glimpses of what spacetime may look like at or near the Planck scale.

It is unlikely that the construction we have presented here will satisfy everyone's prejudices of how a quantum theory of gravity should be constructed, be it through invoking this or that kind of fundamental discrete structure at the Planck scale or according to this or that favourite symmetry principle. This need not necessarily be a reason for concern: if we can find *one* way to Rome, we will be able to find many others. That is to say, we believe (in the spirit of "universality") that there is at most one theory which describes the nontrivial quantum dynamics of intrinsic spacetime geometry, and that in order to construct it, we should just get a few "basic things" right, among them presumably some genuinely nonperturbative features (like the inclusion of locally highly curved geometries which are "very far away" from any classical spacetimes), and possibly a principle of "microcausality", like that implemented in CDT to ensure the emergence of an extended, four-dimensional spacetime.

What remains to be shown is that a single such theory with the correct properties exists. Because of the minimalist input we have used in our construction (no new symmetries, no new dynamical fields or other extended objects, no additional spacetime dimensions, and thus no associated new free parameters) we are unlikely to run into the converse, "M-theoretic" problem of having vast numbers of possible vacua [37] and therefore possible theories of quantum gravity, with a continuum of different physical predictions. The paradigm of spacetime beginning to emerge from CDT is that of a scale-invariant, fractal and effectively lower-dimensional structure at the Planck scale, which only at a larger scale acquires well-known features of geometry which accord with our classical intuition. The deeper reasons for how and why this comes about remain to be understood.

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