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On the relation between Euclidean and Lorentzian 2D quantum gravity

J. Ambjørn, J. Correia and C. Kristjansen[1]

The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

R. Loll[2]

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut),
Am Mühlenberg 1, 14476 Golm, Germany

Abstract

Starting from 2D Euclidean quantum gravity, we show that one recovers 2D Lorentzian quantum gravity by removing all baby universes. Using a peeling procedure to decompose the discrete, triangulated geometries along a one-dimensional path, we explicitly associate with each Euclidean space-time a (generalized) Lorentzian space-time. This motivates a map between the parameter spaces of the two theories, under which their propagators get identified. In two dimensions, Lorentzian quantum gravity can therefore be viewed as a "renormalized" version of Euclidean quantum gravity.

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[1]E-mail: ambjorn, correia, kristjan@nbi.dk
[2]E-mail: loll@aei-potsdam.mpg.de

1 Introduction

The role of baby universes in quantum gravity has been discussed extensively and many conjectures have been made concerning their role in providing effective coupling constants for field theories as well as for gravity itself [1]. Due to the lack of a theory of (four-dimensional) quantum gravity, the "integration over baby universes" has never been performed in any explicit manner. However, there is a chance that this idea can be realized in two dimensions, where quantum gravity is a renormalizable quantum field theory. In this article we will show that starting from 2D Euclidean quantum gravity, there is indeed a sense in which one can integrate out all baby universes. The resulting "renormalized" quantum theory coincides with the theory of so-called Lorentzian 2D quantum gravity.

The Lorentzian quantum gravity model was introduced in order to have a path-integral formulation where the Lorentzian character of the metric is built in at a fundamental level. It is also closer to canonical quantization methods, where from the outset one works with globally hyperbolic space-times. The two-dimensional model can be solved explicitly and is different from Euclidean quantum gravity, by which we will mean the theory defined by a path integral over Euclidean geometries. In the continuum limit, a typical geometry coming from the path integral of Lorentzian quantum gravity is two-dimensional [2, 3]. This is in contrast with the situation in Euclidean quantum gravity, where a typical geometry has an anomalous fractal Hausdorff dimension of four [4].

We will be using a representation of Euclidean quantum gravity in terms of so-called dynamical triangulations. This method provides a reparameterization-invariant regularization of the field theory. The two-dimensional geometries will be represented by equilateral triangulations (with edges of cut-off length a). The path integral is performed by summing over all triangulations of a given topology. The continuum limit is obtained as the lattice spacing $a \to 0$. We will then give an explicit prescription for removing baby universes from a given Euclidean triangulation. One may view this procedure as analogous to the removal of tadpole graphs in a quantum field theory. The latter can usually be viewed as a redefinition of the coupling constants and a shift of the fields in the theory. In our case the effect is more drastic. We end up with a new, physically inequivalent theory.

2 The two-loop correlator and the peeling procedure

2.1 The Euclidean case

We will investigate the relation between two-dimensional Euclidean and Lorentzian quantum gravity by studying the distance-dependent two-loop correlator of the two models. The distance between two links l and l' in a given triangulation T is defined as the length of the shortest path connecting l and l', taken along the links of the dual lattice. Furthermore, the distance between a link l and a loop $\mathcal{L}$ is defined as the minimum distance of the link l to a link belonging to $\mathcal{L}$. With these definitions a loop $\mathcal{L}_2$ is said to have distance t to a loop $\mathcal{L}_1$ if all links in $\mathcal{L}_2$ have distance t to the loop $\mathcal{L}_1$.

The distance-dependent two-loop correlation function $G(l_1, l_2, t)$ is defined as

$$G(l_1, l_2, t) = \sum_{T \in \mathcal{T}(l_1, l_2, t)} g^{N(T)}, \tag{1}$$

where the sum goes over all triangulations of cylindrical topology with an exit loop $\mathcal{L}_2$ of length l_2 and an entrance loop $\mathcal{L}_1$ of length l_1 containing a marked link, the distance of $\mathcal{L}_2$ from $\mathcal{L}_1$ being equal to t. The quantity $N(T)$ is the number of triangles in the triangulation T and g is related to the cosmological constant μ by $g = e^{-\mu}$.

Euclidean and Lorentzian quantum gravity differ by the type of triangulations allowed in (1). In the Lorentzian case one only sums over triangulations which for any $t' \leq t$ have only one (connected) loop at distance t' from the entrance loop. In the usual language of 2D quantum gravity this means that baby-universes are forbidden. Furthermore, in the Lorentzian case a loop cannot shrink to length zero. Using the distinguished variable t as a time variable, we can divide the links into space-like and (future-oriented) time-like, and equip every Lorentzian triangulation with a well-defined causal structure [2]. By contrast, in the Euclidean case there can be more than one loop at a given distance t from the entrance loop and the above construction does not work. We note, however, that in [5] a suggestion of introducing a notion of space-like and time-like edges for Euclidean triangulations was put forward. This lead to a model equivalent to 2D Euclidean quantum gravity coupled to Ising spins.

It is well-known that a combinatorial identity can be derived for the Euclidean version of (1) by a so-called peeling procedure [6]. We will show that also in the Lorentzian case a peeling procedure can be defined, which in addition enables us to set up a direct correspondence between the discrete Euclidean and Lorentzian histories.

We start by reviewing briefly the peeling formalism in the Euclidean case. In order to specify a peeling, one needs a set of moves that systematically remove building blocks at a marked link on the boundary of a triangulation and a rule for choosing a new marked link on the boundary after each move. We will work in the class of so-called unrestricted Euclidean triangulations which apart from triangles also contain double links as building blocks. (Unrestricted triangulations constitute the class of triangulations appearing in the usual matrix model formulation of 2D gravity.) This generalization does not affect any of the universal properties of the model. Consider now an unrestricted triangulation contributing to the sum in (1).

By definition it has an entrance loop with one marked link, which belongs either to a triangle or a double link. If it belongs to a triangle we remove the triangle from the triangulation. We distinguish three cases, according to whether the triangle has one, two or three links in common with the boundary. If it has more than one boundary link, additional double links are created during the move, so that the length of the original boundary loop is always increased by one (see Fig.1, where also the rule for link marking is specified). If the marked link belongs to a double

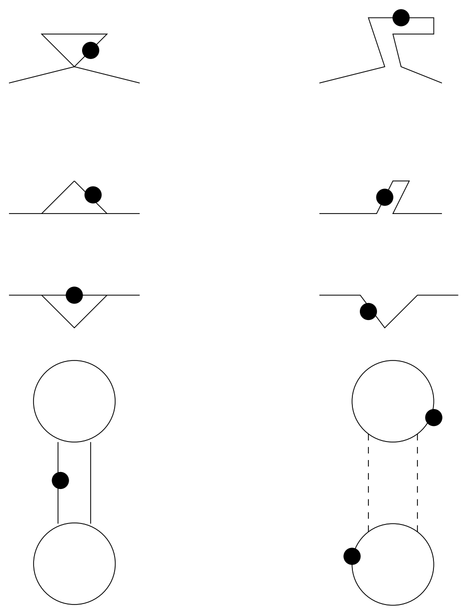

Figure 1: The minimal decomposition. In the left column the boundary before the decomposition and in the right column the boundary after the decomposition.

link we simply remove the double link. This will in general lead to a splitting of the original loop into two. On each of the resulting loops we place a mark as shown in Fig. 1. We can choose either of the two loops as our new entrance loop, associating the other one with a baby universe. This gives rise to the factor of two in eqn. (2) below. (One or both loops may of course be trivial, i.e. consist of only "a marked point".) By this procedure we associate with each Euclidean triangulation a unique sequence of peeling moves. If the entrance loop has length l_1, then on average l_1 peeling steps will move the boundary one time step ahead. Thus one identifies the deformation $\delta G(l_1, l_2, t)$ associated with a single peeling step with $\frac{1}{l_1}\frac{\partial}{\partial t}G(l_1, l_2, t)$. This deformation fulfils

$$\frac{\partial}{\partial t}G(l_1, l_2, t) = -l_1 G(l_1, l_2, t) + g l_1 G(l_1+1, l_2, t) + 2\sum_{l=0}^{l_1-2} l_1 W(l) G(l_1-l-2, l_2, t), \quad (2)$$

with the initial condition

$$G(l_1, l_2, t=0) = \delta_{l_1, l_2}, \quad (3)$$

where $W(l)$ is the disk amplitude defined by

$$W(l) = \sum_{T \in \mathcal{T}(l)} g^{N(T)}, \tag{4}$$

and the sum is over all (non-restricted) triangulations with one connected boundary component of length l. Let us introduce the generating functionals

$$G(z,y,t) = \sum_{l_1,l_2=0}^{\infty} \frac{1}{z^{l_1+1}} \frac{1}{y^{l_2+1}} G(l_1,l_2,t), \tag{5}$$

$$W(z) = \sum_{l=0}^{\infty} \frac{1}{z^{l+1}} W(l). \tag{6}$$

Here z and y can be understood as boundary cosmological constants. Then the differential equation (2) reads

$$\frac{\partial}{\partial t} G(z,y,t) = \frac{\partial}{\partial z} \left(h(z) G(z,y,t) \right), \tag{7}$$

with initial condition

$$G(z,y,t=0) = \frac{1}{zy} \frac{1}{zy-1}, \tag{8}$$

where

$$h(z) = z - gz^2 - 2W(z). \tag{9}$$

It is well known that in analogy with the procedure described above, it is possible to derive a combinatorial identity for $W(z)$ by removing a triangle or a double link from a triangulation contributing to the sum in (4), see for instance [7]. The combinatorial identity reads

$$(z - gz^2) W(z) - 1 + g(w_1(g) + z) = (W(z))^2, \tag{10}$$

where

$$w_1(g) = \sum_{T \in \mathcal{T}(1)} g^{N(T)}. \tag{11}$$

Here the terms proportional to g emerge when a triangle is removed whereas the term $(W(z))^2$ emerges when a double link is removed and the surface splits into two. We note that if we set $g = 0$ we get

$$W(z) = \frac{1}{2} \left(z - \sqrt{z^2 - 4} \right), \tag{12}$$

which is the well-known generating function for rooted branched polymers. These appear exactly because of our inclusion of double links. For $g \neq 0$, $W(z)$ is the solution of a quadratic equation and the unknown constant $w_1(g)$ is determined by noticing that the analyticity structure of $W(z)$ should not change discontinuously at $g = 0$. The solution for $W(z)$ reads

$$W(z) = \frac{1}{2} \left(z - gz^2 + g\,(z - c(g)) \sqrt{(z - c_+(g))(z - c_-(g))} \right), \tag{13}$$

where $c_-(g) \leq c_+(g) \leq c(g)$ and where all three quantities can be expressed explicitly in terms of g. At a certain value g_c of g the radius of convergence of the series (4) is reached and a continuum limit can be defined. Setting

$$g_c - g = a^2 \Lambda, \tag{14}$$

where a^2 is a scaling parameter with the dimension of area and Λ is the continuum cosmological constant we have for g close to g_c

$$\begin{aligned} c(g) &= z_c \left(1 + \frac{1}{2}\, a\sqrt{\Lambda}\right) + \mathcal{O}(a^2), & (15)\\ c_+(g) &= z_c \left(1 - a\sqrt{\Lambda}\right) + \mathcal{O}(a^2), & (16)\\ c_-(g) &= c_- + \mathcal{O}(a^2). & (17) \end{aligned}$$

It is now natural to introduce a continuum boundary cosmological constant Z by

$$z = z_c\,(1 + aZ)\,. \tag{18}$$

We thus have that $h(z)$ exactly consists of the scaling part of $-2W(z)$. Introducing a continuum time variable T by

$$T = z_c^{1/2}(z_c - c_-)^{1/2}\, g_c\, a^{1/2} t, \tag{19}$$

one obtains a differential equation for the continuum version $G(Z, Y, T)$ of the two-loop correlator, namely,

$$\frac{\partial}{\partial T} G(Z, Y, T) = -\frac{\partial}{\partial Z}\left(\left((Z - \frac{1}{2}\sqrt{\Lambda})\sqrt{Z + \sqrt{\Lambda}}\right) G(Z, Y, T)\right), \tag{20}$$

with the initial condition

$$G(Z, Y, T) = \frac{1}{Z + Y}. \tag{21}$$

Note that the continuum cosmological constants Λ and Z and the time T are only defined up to positive multiplicative constants. We have used this freedom to bring the propagator equation into the simple form (20).

2.2 The Lorentzian case

One may invert the peeling procedure introduced in the previous section to obtain a Euclidean triangulation from a sequence of peeling moves. Its geometry is given in the form of a bulk cylindrical geometry connecting the entrance and exit loops, "decorated" with baby universe out-growths. This is the case because some of the peeling steps remove just a single triangle (which we now associate with the bulk cylinder), while the removal of a double link in general amounts to the deletion of an entire baby universe.

As mentioned earlier, the difference between Euclidean and Lorentzian gravity can be traced to the presence or absence of baby universes. There is already an explicit construction of Euclidean from Lorentzian histories through the addition of baby universes [2]. Our current aim is to go the other way. The peeling procedure suggests an obvious map from Euclidean to Lorentzian discrete geometries. Given a representation of a Euclidean geometry as a sequence of peeling moves, consider a step where a double link-cum-baby universe is removed, see Fig. 2. Decompose the

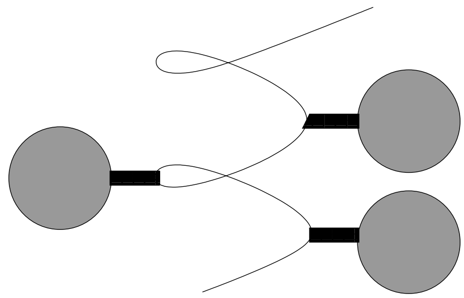

Figure 2: A representation of a Euclidean triangulation as a sequence of peeling moves along a "spiral-like" one-dimensional curve. Black rectangles are double links and grey circles baby universes.

baby universe into a branched polymer attached to the marked link and a smaller baby universe attached to the branched polymer in such a way that the branched polymer consists of the largest possible number of links. Next, replace the step where the double link-cum-baby universe is removed by a step where only the double link and the branched polymer is taken way. Do the same for all similar steps in the sequence of peeling moves. The resulting sequence of peeling moves obviously represents a geometry without baby universes. It is clear that the class of geometries one generates this way is larger than the class of Lorentzian triangulations introduced in [2]. Firstly, it contains all of the regular Lorentzian triangulations of [2], as can be checked explicitly. Secondly, the boundaries may now contain double links. We will refer to this generalized class of geometries as "Lorentzian geometries". This will be justified below by showing that they lead to a theory in the same universality class as the original Lorentzian model and the extensions discussed in [8]. The differential equation for the two-loop correlator of Lorentzian geometries is obtained by replacing $2W(z)$ in (9) by $W_{BP}(z)$, and thus reads

$$\frac{\partial}{\partial \tilde{t}} \tilde{G}(\tilde{z}, \tilde{y}, \tilde{t}) = \frac{\partial}{\partial \tilde{z}} \left(\tilde{h}(\tilde{z}) \tilde{G}(\tilde{z}, \tilde{y}, \tilde{t}) \right), \tag{22}$$

with initial condition

$$\tilde{G}(\tilde{z}, \tilde{y}, \tilde{t} = 0) = \frac{1}{\tilde{z}\tilde{y}} \frac{1}{1 - \tilde{z}\tilde{y}}, \tag{23}$$

where

$$\tilde{h}(\tilde{z}) = \tilde{z} - \tilde{g}\tilde{z}^2 - W_{BP}(\tilde{z}). \tag{24}$$

We have introduced tildes on all the Lorentzian variables in order to distinguish them from the Euclidean ones Let us note here that it is easy to generalize this model of Lorentzian quantum gravity to models with arbitrary polygons as building blocks. Using the newly developed technology of [9] it may be possible to solve all of these models exactly. Here we only consider the simplest model. We wish to locate a critical point $(\tilde{g}_c, \tilde{z}_c)$ at which we can define a non-trivial continuum limit by the scaling relations

$$\tilde{g} = \tilde{g}_c(1 - c_1 a^2 \tilde{\Lambda}), \qquad \tilde{z} = \tilde{z}_c \left(1 + c_2 a \tilde{Z}\right), \tag{25}$$

where c_1 and c_2 are numerical constants. The only way to obtain a non-trivial differential equation in the continuum limit is by demanding that $\tilde{h}(\tilde{z}) \sim \mathcal{O}(a^2)$. This determines the critical values of $\tilde{z}$ and $\tilde{g}$ to be

$$\tilde{z}_c = \frac{4}{\sqrt{3}}, \qquad \tilde{g}_c = \frac{3\sqrt{3}}{16}. \tag{26}$$

For later convenience we set $c_2 = \frac{1}{3^{3/4}}$ and $c_1 = \frac{1}{\sqrt{3}}$. We also introduce a continuum version $\tilde{T}$ of $\tilde{t}$ by

$$\tilde{t} = \frac{1}{a}\tilde{T}. \tag{27}$$

Note that the relative dimensions of the time variable and the area are different for Euclidean and Lorentzian quantum gravity. Then we derive the following evolution equation for the continuum two-loop correlator $\tilde{G}(\tilde{Z}, \tilde{Y}, \tilde{T})$

$$\frac{\partial}{\partial \tilde{T}} \tilde{G}(\tilde{Z}, \tilde{Y}, \tilde{T}) = -\frac{\partial}{\partial \tilde{Z}} \left(\left(\tilde{Z}^2 - \tilde{\Lambda} \right) \tilde{G}(\tilde{Z}, \tilde{Y}, \tilde{T}) \right), \tag{28}$$

with initial condition

$$\tilde{G}(\tilde{Z}, \tilde{Y}, \tilde{T}) = \frac{1}{\tilde{Z} + \tilde{Y}}. \tag{29}$$

This differential equation coincides with the one derived in [2], using a completely different strategy.

Let us now turn to a discussion of the disk amplitude $\tilde{W}(l)$ of Lorentzian gravity. At the discrete level it may be defined as the sum $\sum_{\tilde{t}=1}^{\infty} \tilde{G}(l, l_2 = 0, \tilde{t})$. This reflects the fact that a Lorentzian geometry with a regular initial spatial geometry at $\tilde{t} = 0$ can only terminate at a finite $\tilde{t}$ if we allow for a singularity where space contracts to a point.

We can derive a combinatorial identity for the generating functional $\tilde{W}(\tilde{z})$ in the same way as for the Euclidean disk amplitude $W(z)$. The combinatorial identity reads

$$(\tilde{z} - \tilde{g}\tilde{z}^2)\tilde{W}(\tilde{z}) - 1 + \tilde{g}(\tilde{w}_1(\tilde{g}) + \tilde{z}) = W_{BP}(\tilde{z})\tilde{W}(\tilde{z}). \tag{30}$$

As in the case of the two-loop correlator, the only modification of the equation concerns the term associated with the removal of a double link. If, in the Euclidean case, the marked link is part of a double link, we remove the double link and are left with two disjoint disks. If, in the Lorentzian case, the marked link is part of a double link, we remove the double link and are left with one disk and one branched polymer. In this case, by setting $\tilde{g}=0$, we get

$$\tilde{W}(\tilde{z}) = W_{BP}(\tilde{z}). \tag{31}$$

This underlines the consistency of our construction. For $\tilde{g} \neq 0$ on the other hand we obtain

$$\tilde{W}(\tilde{z}) = \frac{2(1-\tilde{g}(\tilde{w}_1(\tilde{g})+\tilde{z}))}{\tilde{z}-2\tilde{g}\tilde{z}^2+\sqrt{\tilde{z}^2-4}}. \tag{32}$$

Note that the denominator of (32) is equal to $2\tilde{h}(\tilde{z})$. For $\tilde{g} < \frac{1}{4}$ this function has one zero $\tilde{z}_1(\tilde{g})$, which behaves as $\frac{1}{\tilde{g}} + \mathcal{O}(\tilde{g})$ when $\tilde{g}$ is close to zero. Hence a pole appears in the transition from (31) to (32) and we must choose the unknown constant $\tilde{w}_1(\tilde{g})$ such that this pole is cancelled. At $\tilde{g} = \frac{1}{4}$ the function $\tilde{h}(\tilde{z})$ acquires an additional zero, $\tilde{z}_2(\tilde{g})$, and the critical point $\tilde{g}_c = \frac{3\sqrt{3}}{16}$ corresponds to the situation where the two zeros $\tilde{z}_1(\tilde{g})$ and $\tilde{z}_2(\tilde{g})$ coincide. In the vicinity of the critical point we have the expansions

$$\tilde{z}_1(\tilde{g}) = \frac{4}{\sqrt{3}} + \frac{16}{3^{7/4}}(\tilde{g}_c-\tilde{g})^{1/2}, \tag{33}$$

$$\tilde{z}_2(\tilde{g}) = \frac{4}{\sqrt{3}} - \frac{16}{3^{7/4}}(\tilde{g}_c-\tilde{g})^{1/2}. \tag{34}$$

Using (33), (34) and (25), we derive the continuum disk amplitude

$$\tilde{W}(\tilde{Z}) = \lim_{a\to 0} a\,\tilde{W}(\tilde{z}) = \frac{4}{3^{5/4}} \frac{1}{\tilde{Z}+\sqrt{\tilde{\Lambda}}}. \tag{35}$$

This expression coincides with the one found in [2] by completely different means.

3 Renormalizing Euclidean gravity to Lorentzian gravity

We will now show that it is possible to "renormalize" the Euclidean coupling constants and the Euclidean time such that the equation governing the time evolution of the Euclidean two-loop function is mapped to the corresponding equation for the Lorentzian two-loop function.

We start from the discrete version of the differential equation for the time development of the Euclidean universes, (7), and make an ansatz

$$\tilde{z} = \tilde{z}(z,g), \qquad \tilde{t} = \frac{1}{b}t, \qquad \tilde{g} = \tilde{g}(g), \tag{36}$$

where b may depend on g and $\tilde{g}$, but not on z or $\tilde{z}$. In principle it should be possible to determine the required map between the coupling constants of the two theories directly from the peeling procedure which maps Euclidean to Lorentzian histories. However, it turns out to be simpler to use an indirect argument. *If* there exists a transformation of the form (36) which maps the Euclidean differential equation into the Lorentzian differential equation, then clearly it must include a wave-function renormalization of the two-loop correlator

$$\tilde{G}(\tilde{z},\tilde{y},\tilde{t}) \propto \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} G(z(\tilde{z}),y(\tilde{y}),b\tilde{t}), \tag{37}$$

and furthermore

$$b\,h(z)\frac{\partial \tilde{z}}{\partial z} = \tilde{h}(\tilde{z}), \tag{38}$$

or, equivalently,

$$\int \frac{dz}{h(z)} = b \int \frac{d\tilde{z}}{\tilde{h}(\tilde{z})} + k, \tag{39}$$

where k is an integration constant.

Using the explicit expressions (9) and (24) for $h(z)$ and $\tilde{h}(\tilde{z})$, we can carry out the integrations in (39) and find

$$\begin{aligned}
&-\frac{1}{\sqrt{\delta(g)}} \log\left\{\frac{2\delta(g)+\epsilon(g)(z-c(g))+2\sqrt{\delta(g)}\sqrt{(z-c_+(g))(z-c_-(g))}}{z-c(g)}\right\} \qquad (40)\\
&= b\left\{\frac{1}{2\tilde{g}^2}\left[\frac{1-2\tilde{g}\tilde{z}_1(\tilde{g})}{2\tilde{z}_1(\tilde{g})\left(\tilde{z}_1(\tilde{g})-\frac{3}{4\tilde{g}}\right)}\log(\tilde{z}-\tilde{z}_1(\tilde{g}))+\frac{1-2\tilde{g}\tilde{z}_2(\tilde{g})}{2\tilde{z}_2(\tilde{g})\left(\tilde{z}_2(\tilde{g})-\frac{3}{4\tilde{g}}\right)}\log(\tilde{z}-\tilde{z}_2(\tilde{g}))\right.\right.\\
&\quad\left.\left.+\frac{\sqrt{\tilde{z}_1(\tilde{g})^2-4}}{4\tilde{z}_1(\tilde{g})^2\left(\tilde{z}_1(\tilde{g})-\frac{3}{4\tilde{g}}\right)}\log\left[\sqrt{\tilde{z}_1(\tilde{g})^2-4}\sqrt{\tilde{z}^2-4}+\tilde{z}\tilde{z}_1(\tilde{g})-4\right]+\ldots\right]\right\}+k,
\end{aligned}$$

Here $\ldots$ means terms similar to the previous one with $\tilde{z}_1(\tilde{g})$ replaced by $\tilde{z}_2(\tilde{g})$, $\tilde{z}_3(\tilde{g})$ and $\tilde{z}_4(\tilde{g})$ where $\tilde{z}_1(\tilde{g})$, $\tilde{z}_2(\tilde{g})$, $\tilde{z}_3(\tilde{g})$ and $\tilde{z}_4(\tilde{g})$ are the four roots of the polynomial $z^4-\frac{1}{\tilde{g}}\,z^3+\frac{1}{\tilde{g}^2}$ with $\tilde{z}_1(\tilde{g})$ and $\tilde{z}_2(\tilde{g})$ still playing the roles described on the previous page. Furthermore δ and ϵ are defined by

$$\begin{aligned}
\delta(g) &= (c(g)-c_+(g))(c(g)-c_-(g)),\\
\epsilon(g) &= 2c(g)-c_+(g)-c_-(g).
\end{aligned}$$

For any value of k, g, $\tilde{g}$ and b, the relation (40) defines a mapping $\tilde{z}=\tilde{z}(z)$. However, not all values of b and k correspond to situations we can encounter in the peeling procedure. Firstly, the two theories have to approach criticality simultaneously, since macroscopic surfaces and boundaries are created at the critical points. Secondly, we insist on a canonical scaling dimension of the cosmological and boundary cosmological constants. This amounts to saying that the scaling of the continuum cosmological constants is governed by the same power of the lattice cut-off a. Since we have assumed that $\tilde{g}$ is only a function of g, and not of z and $\tilde{z}$, this implies that $\tilde{\Lambda} \propto \Lambda$, with a proportionality coefficient independent of the cut-off. In the scaling limit all dependence on $\tilde{z}_3(\tilde{g})$ and $\tilde{z}_4(\tilde{g})$ disappears and using the scaling relations (14)–(18), (25), (33) and (34), we see that in order for the mapping to persist in the continuum limit, b and k must scale according to

$$b = a^{1/2}\Lambda^{1/4}\beta, \qquad k = a^{-1/2}\Lambda^{-1/4}\kappa, \tag{41}$$

for two dimensionless constants β and κ. To fix the relevant power of Λ in (41), we have used the fact that b and k cannot depend on z or $\tilde{z}$. *Note that b and k behave in a singular way when we approach the continuum limit.* This is an unavoidable consequence of interpolating between two universality classes of 2D geometries with different scaling behaviour (recall that in the previous section we showed that t scales as $t \sim a^{-1/2}$, whereas $\tilde{t} \sim a^{-1}$).

It is now straightforward to compute the continuum version of (40),

$$\frac{1}{\sqrt{\Lambda}} \log \left\{ \frac{\sqrt{\frac{2}{3}}\Lambda^{1/4}\sqrt{Z+\sqrt{\Lambda}}+\sqrt{\Lambda}}{\sqrt{\frac{2}{3}}\Lambda^{1/4}\sqrt{Z+\sqrt{\Lambda}}-\sqrt{\Lambda}} \right\} = \frac{\beta}{\sqrt{\tilde{\Lambda}}} \log \left\{ \frac{\tilde{Z}+\sqrt{\tilde{\Lambda}}}{\tilde{Z}-\sqrt{\tilde{\Lambda}}} \right\} + \frac{\kappa}{\sqrt{\Lambda}}, \tag{42}$$

where we have absorbed some additional numerical constants into β and κ. We will argue that in order for this map to be associated with the cutting procedure we must choose $\kappa = 0$. Taking the limit $Z \to \infty$ implies that the boundary length $L \to 0$. Now, a Lorentzian boundary is shorter than a Euclidean one, since it was obtained from the latter by cutting away baby universes which at a given time contributed to the boundary length. This is the physical motivation for requiring that $Z \to \infty$ should imply $\tilde{Z} \to \infty$, i.e. $\kappa = 0$. Moreover, a rescaling of both Z and $\sqrt{\Lambda}$ with a common factor allows us to choose

$$\sqrt{\Lambda} = \frac{1}{\beta}\sqrt{\tilde{\Lambda}}, \tag{43}$$

in such a way that the map (42) simplifies to

$$\frac{\tilde{Z}}{\sqrt{\tilde{\Lambda}}} = \sqrt{\frac{2}{3}}\,\frac{\sqrt{Z+\sqrt{\Lambda}}}{\Lambda^{1/4}}, \tag{44}$$

and the relation between the continuum times T and $\tilde{T}$ becomes

$$\tilde{T} = \frac{T}{\beta\Lambda^{1/4}}. \tag{45}$$

One can of course verify directly that the (non-analytic) change of variables (44) and (45) supplemented by the wave function renormalization (37) transforms eq. (20) for Euclidean quantum gravity into eq. (28) for Lorentzian gravity.[3]. To map the actual two-loop correlator of Euclidean quantum gravity $G_0(Z, Y, T)$ (i.e. the solution to (20) with initial condition (21)) onto the actual two-loop correlator of Lorentzian

[3]Note that some additional constants have been absorbed into β in going from (40) to (42)

quantum gravity $\tilde{G}_0(\tilde{X}, \tilde{Y}, \tilde{T})$ (i.e. the solution to (28) with initial condition (29)) an additional dressing factor is needed More precisely one has

$$\tilde{G}_0(\tilde{Z}, \tilde{Y}, \tilde{T}) = K(\bar{Z}(T, Z), Y)\; G_0(Z, Y, T), \tag{46}$$

where

$$K(\bar{Z}(T, Z), Y) = \frac{\bar{Z}(T, Z) + Y}{\tilde{Z}(\bar{Z}(T, Z)) + \tilde{Y}(Y)} \frac{\sqrt{Z + \sqrt{\Lambda}}}{\sqrt{\bar{Z}(T, Z) + \sqrt{\Lambda}}}, \tag{47}$$

and where $\bar{Z}(T, Z)$ is the solution to characteristic equation for (20):

$$\frac{\partial \bar{Z}(T, Z)}{\partial T} = -(Z - \frac{1}{2}\sqrt{\Lambda})\sqrt{Z + \sqrt{\Lambda}}, \quad \bar{Z}(T{=}0, Z) = Z. \tag{48}$$

Summing up, we have shown that "integrating over baby" universes can be described as a renormalization of the cosmological constants and the time variable combined with a dressing of the two-loop correlator.

4 Discussion

We have shown how one can analyze the geometric structure of Lorentzian and Euclidean triangulations by representing them as a sequence of peeling moves defining a one-dimensional 'spiral-like' path. Along this path, one meets in both cases either triangles or double links. The difference between the Euclidean and Lorentzian cases is that in the former a double link may have a branched polymer plus an entire baby universe attached to it whereas in the latter a double link can have at most a branched polymer attached to it. At the level of sequences of peeling moves, this allows us to define a many-to-one map from Euclidean to Lorentzian geometries. This is reminiscent of the situation one encounters in analyzing the fractal structure of random walks versus branched polymers. There, moving along a path connecting two vertices, in the random walk case one may have only a single link emerging at each vertex. For the branched polymer case, there can be an entire polymer attached at each vertex, see e.g. [7]. In this context it may be of interest to point out that pure Lorentzian quantum gravity has been proven to be equivalent to a model of random walks [8].

This analogy is further strengthened by the nature of the continuum renormalization of the coupling constants presented in section 3. Apart from trivial rescalings, only the *boundary* cosmological constant is subject to a renormalization. This reflects the fact that the peeling decomposition proceeds along a one-dimensional path. Moreover, the functional form of the relation between the Lorentzian and Euclidean boundary cosmological constants is very similar to that between the cosmological constants of random walks and branched polymers, since in both cases the cosmological constant of the simpler model has a square-root dependence on the cosmological constant of the more extended model. Furthermore, in both cases the fractal dimension of the decorated system is twice that of the simpler system.

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