

Imposing $\det E > 0$ in discrete quantum gravity

R. Loll*

*Max-Planck-Institut für Gravitationsphysik
Schlaatzweg 1
D-14473 Potsdam, Germany*

Abstract

We point out that the inequality $\det E > 0$ distinguishes the kinematical phase space of canonical connection gravity from that of a gauge field theory, and characterize the eigenvectors with positive, negative and zero-eigenvalue of the corresponding quantum operator in a lattice-discretized version of the theory. The diagonalization of $\hat{\det E}$ is simplified by classifying its eigenvectors according to the irreducible representations of the octagonal group.

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* e-mail: loll@aei-potsdam.mpg.de

1 Introduction

It is sometimes stated that the unconstrained phase space of pure gravity in the Ashtekar formulation [1] is that of a Yang-Mills theory. This however is not quite true. The origin of this subtlety has nothing to do with the complexification of the connection form in the original formulation, and is indeed also present in the purely real connection formulation [2], which is the subject of this letter.

Recall that in this Hamiltonian form of Lorentzian gravity, the basic canonical variable pair (A_a^i, E_i^a) consists of an $su(2)$ -valued gauge potential A and a densitized, inverse dreibein E . Denoting the dreibein (the “square root of the three-metric”) by e_a^i , $e_a^i e_{bi} = g_{ab}$, with its inverse e_i^a satisfying $e_i^a e_a^j = \delta_i^j$, E can be expressed as $E_i^a = (\det e_b^j) e_i^a$. Taking the determinant of this equation, one obtains $\det E_i^a = (\det e_b^j)^2$, which (for non-degenerate metrics) is always positive and non-vanishing ($\det e_b^j$ alone may assume values $\pm\sqrt{\det g_{ab}}$).

However, once one chooses the E_i^a 's as the basic variables, the inequality $\det E > 0$ *has to be imposed as an extra condition* to recover the correct gravitational phase space. This is analogous to Hamiltonian metric formulations for gravity where the condition $\det g > 0$ must be imposed on the symmetric 3-tensors g_{ab} , constituting half of the canonical variables. Similar conditions also appear in other gauge-theoretic reformulations of gravity. One crucial question is how such a condition is to be translated to the quantum theory. Fortunately this is possible in the case of connection gravity, at least in a lattice-discretized version of the theory.

If one quantizes connection gravity along the lines of a non-abelian gauge field theory, as is usually done, and as is suggested by the kinematical resemblance of the two, an operator condition like $\det \hat{E} > 0$ is *not* automatically satisfied. Since $\det E$ is classically a third-order polynomial in the momenta E_i^a ,

$$\det E = \frac{1}{3!} \eta_{abc} \epsilon^{ijk} E_i^a E_j^b E_k^c, \quad (1.1)$$

and since in the Yang-Mills-like quantization the momenta are represented by i times differentiation with respect to A , $\det \hat{E} \Psi > 0$ is a differential condition for physical wave functions Ψ , and an obvious candidate for a quantization of the classical inequality $\det E > 0$. There already exists a well-defined, self-adjoint lattice operator with discrete spectrum, which is the quantized version of a discretization of the classical function $\det E$ [3]. We call this operator $\hat{D}(n)$, where n labels the vertices of a three-dimensional lattice with cubic topology, and $\hat{D}(n)$ is written in terms of the symmetrized link momenta $\hat{p}(n, \hat{a})$ as

$$\hat{D}(n) := \frac{1}{3!} \eta_{\hat{a}\hat{b}\hat{c}} \epsilon^{ijk} \hat{p}_i(n, \hat{a}) \hat{p}_j(n, \hat{b}) \hat{p}_k(n, \hat{c}), \quad (1.2)$$

where

$$\hat{p}_i(n, \hat{a}) = \frac{i}{2} (X_+^i(n, \hat{a}) + X_-^i(n - 1_{\hat{a}}, \hat{a})), \quad (1.3)$$

and $X_{\pm}^i(n, \hat{a})$ denote the left- and right-invariant vector fields on the group manifold associated with the link $l = (n, \hat{a})$, with commutators $[X_{\pm}^i, X_{\pm}^j] = \pm \epsilon^{ijk} X_{\pm}^k$. (For convenience we have rescaled $D(n)$ by a factor of $\frac{1}{6}$ with respect to the definition in [3].) The square root of $\hat{D}(n)$ (whenever it is defined) is the so-called volume operator, and some of its spectral properties have been investigated both on the lattice and in the continuum. The latter is relevant because it turns out that self-adjoint volume operators can be defined in the continuum loop representation of quantum gravity [4,5,6]. After regularization their action on fixed, imbedded spin network states is very much like that of a lattice operator. In particular, the finite volume operators of [5,6] (up to overall factors and modulus signs) coincide on suitable geometries with (1.2) (this is explained in more detail in [7]). The volume operator and its discretized version have emerged as important ingredients in the construction of the quantum Hamiltonian constraint. Note that the non-polynomial quantities appearing in canonical connection gravity can always be rewritten in polynomial form modulo arbitrary powers of $\det E$. Thus, if one can explicitly quantize $\det E$, arbitrary functions of $\det E$ can be quantized in terms of its spectral resolution. If inverse powers of $\det E$ appear, one in addition has to identify the zero-eigenstates of $\hat{\det E}$ [8].

There is therefore clearly a need for a better understanding of the spectral properties of the operator $\hat{\det E}$. There exist general formulae for its matrix elements, obtained in various preferred orthogonal bases of wave functions [4,6]. Since one does not expect to be able to establish general analytic formulae for the spectrum itself, the limits for evaluating it numerically are given by the size of the matrices that are to be diagonalized and the computing power available. We will below describe a way of reducing the matrix size, by establishing a set of superselection sectors on which $\hat{D}(n)$ can be diagonalized separately. They have their origin in discrete geometric symmetries of the operator and the Hilbert space on which it is defined.

Our discussion will take place within the lattice theory, but for the reasons mentioned above, results about the lattice spectrum translate, at least partially, into results about the continuum spectrum.

2 Characterization of eigenstates

It was already noted during earlier investigations of the volume spectrum [9,3] that non-vanishing eigenvalues of the operator $\det \hat{E}$ always appear in pairs of opposite sign. That this is also true in general can be seen as follows (the argument is similar to the one used to prove that three-valent spin network states necessarily have vanishing volume [9]). We work on the gauge-invariant sector \mathcal{H}^{inv} of the lattice gauge-theoretic Hilbert space, whose elements are linear combinations of Wilson loops, i.e. of traces of closed lattice holonomies. A convenient way of labelling a basis of states is given by $|j_l, \vec{v}_n\rangle$ (so-called spin network states), where $j_l = 0, 1, 2, \dots$ labels the $su(2)$ -representation associated with each lattice link $l = (n, \hat{a})$, and \vec{v}_n is a set of linearly independent intertwiners (contractors of Wilson lines) compatible with the j_l at each lattice vertex n . Note that these states are real functions of the $SU(2)$ -lattice holonomies.

Since $\hat{D}(n)$ only acts locally at n , we need only consider the part of Hilbert space associated with the single vertex n and the six links intersecting at n . Moreover, $\hat{D}(n)$ leaves the flux line numbers j_l alone, and therefore acts non-trivially only on the finite-dimensional spaces of the linearly independent intertwiners labelled by \vec{v}_n .

Consider an orthogonal basis of states $\{\phi_i\}$ in one of these finite-dimensional spaces, and assume that Ψ is an eigenstate of $\hat{D}(n)$, $\hat{D}(n)\Psi = d\Psi$. Since $\hat{D}(n)$ is a self-adjoint operator, d is a real number. In this basis, the decomposition for Ψ reads $\Psi = \sum_i (a_i + ib_i)\phi_i$, $a_i, b_i, \in \mathbb{R}$. Since the explicit operator expression for $\hat{D}(n)$ is purely imaginary, as can be seen from (1.2,3), it immediately follows from

$$\hat{D}(n) \sum_i (a_i + ib_i)\phi_i = d \sum_i (a_i + ib_i)\phi_i, \quad (2.1)$$

by taking the complex conjugate that

$$\hat{D}(n) \sum_i (a_i - ib_i)\phi_i = -d \sum_i (a_i - ib_i)\phi_i. \quad (2.2)$$

The consequences can be summarized as follows: if Ψ is an eigenstate of $\hat{D}(n)$ with eigenvalue d , then its complex conjugate Ψ^* is also an eigenstate, with eigenvalue $-d$. If an eigenstate Ψ is a purely real or a purely imaginary linear combination of spin network states, then its eigenvalue must necessarily be $d = 0$.

This provides a first characterization of positive-, negative- and zero-eigenstates of the operator $\hat{D}(n)$. That there should be such a one-to-one map between states of positive and negative volume is plausible from a physical point of view, since for Yang-Mills configurations there is no preferred orientation for triples of E -fields.

A first practical consequence for the computation of spectrum and eigenstates of \hat{D} is the following. Although \hat{D} does not commute with complex conjugation, its square \hat{D}^2 (which also is a well-defined self-adjoint operator) does. Therefore \hat{D}^2 can be diagonalized already on the subspace of real states. Assume now that χ is such a real eigenstate of \hat{D}^2 , $\hat{D}^2(n)\chi = v^2\chi$, $v \neq 0$. It follows immediately that its image under \hat{D} is an (imaginary) eigenstate of \hat{D}^2 since $\hat{D}^2(\hat{D}\chi) = v^2\hat{D}\chi$. Consider the linear combination of these two states under the action of \hat{D} ,

$$\hat{D}\left(\chi \pm \frac{1}{|v|}\hat{D}\chi\right) = \pm\frac{1}{|v|}\hat{D}^2\chi + \hat{D}\chi = \pm|v|\left(\chi \pm \frac{1}{|v|}\hat{D}\chi\right). \quad (2.3)$$

Thus, we can read off a recipe for constructing positive volume eigenstates: take any eigenstate χ of \hat{D}^2 with non-zero eigenvalue v^2 , then $\chi + \frac{1}{|v|}\hat{D}\chi$ is an eigenvector of \hat{D} with eigenvalue $|v|$.

3 The role of the octagonal group

In order to simplify the task of finding eigenstates of \hat{D} , we will construct operators that commute with it and among themselves, and can therefore be diagonalized simultaneously. The finite-dimensional matrices associated with the action of \hat{D} on vertex states of given flux line numbers decompose into block-diagonal form, and the blocks can be diagonalized individually.

The key observation is that the classical lattice function $D(n) \equiv \det E(n)$ is invariant under the action of the discrete group \mathcal{O} of 24 elements, called the octagonal or cubic group [10]. They can be thought of as the permutations of the three (oriented) lattice axes meeting at the intersection n which do not change the orientation of the local coordinate system they define. By contrast, $D(n)$ changes sign under the total space reflection T (i.e. under simultaneous inversion of the three axes). It is sometimes convenient to consider the discrete group of 48 elements $\mathcal{O} \times T$.

As a result of this classical symmetry, eigenstates of $\hat{D}(n)$ can be classified according to the irreducible representations of \mathcal{O} . This set-up is familiar to lattice gauge theorists,

because it has been employed in analyzing the glueball spectrum of the Hamiltonian in four-dimensional $SU(3)$ -lattice gauge theory [11]. Adapted to the present $SU(2)$ -context, certain further simplifications occur which have to do with how the gauge-invariant sector of the lattice theory is labelled by the spin network states.

One way of labelling local spin network states at a vertex n is the following. Fix a local coordinate system at n and label the three incoming links as $(-1, -2, -3)$, and the corresponding outgoing ones as $(1, 2, 3)$, and the corresponding link fluxes by j_i , $i = \pm 1, 2, 3$. (The j_i cannot be chosen totally freely but must be such that suitable gauge-invariant routings of flux lines through the intersection exist.) To take care of the intertwiners, call $j_{m,n}$ the number of spin- $\frac{1}{2}$ -flux lines coming in at link m and going out at link n . Both m and n can take positive and negative values, but $m = n$ is excluded, since it corresponds to a trivial retracing of a link. Since the flux lines appearing in spin network states are not sensitive to orientation, there are 15 numbers $j_{m,n}$. They are subject to a number of constraints since the total number of flux lines j_i associated with a given incoming or outgoing link is assumed fixed. Our reason for choosing this label set for the contractors is their simple transformation behaviour under the cubic group.

This way of labelling still contains a large redundancy in the form of so-called Mandelstam constraints. This is partially eliminated by choosing a smaller label set: again fix an orientation of the three axes, and consider only intertwiners with non-vanishing $\{j_{-1,1}, j_{-1,2}, j_{-1,3}, j_{-2,1}, j_{-2,2}, j_{-2,3}, j_{-3,1}, j_{-3,2}, j_{-3,3}\}$. It can easily be shown that all other intertwiners can be written as linear combinations of this set, by virtue of the Mandelstam identities. Moreover, $\hat{D}(n)$ maps the set into itself. However, the symmetry group \mathcal{O} does not leave it invariant; only a six-dimensional subgroup (which we will call $\mathcal{O}^{(6)}$) maps the set into itself. Dropping the minus signs in front of the negative subscripts of the j_{mn} in the reduced 9-element set, let us rearrange the data in a 3×3 -matrix J ,

$$J := \begin{pmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{pmatrix}. \quad (3.1)$$

The non-trivial elements of $\mathcal{O}^{(6)}$ in this notation are represented by

$$\begin{aligned}
R_1(J) &:= \begin{pmatrix} \dot{j}_{11} & \dot{j}_{31} & \dot{j}_{21} \\ \dot{j}_{13} & \dot{j}_{33} & \dot{j}_{23} \\ \dot{j}_{12} & \dot{j}_{32} & \dot{j}_{22} \end{pmatrix}, \quad R_2(J) := \begin{pmatrix} \dot{j}_{33} & \dot{j}_{23} & \dot{j}_{13} \\ \dot{j}_{32} & \dot{j}_{22} & \dot{j}_{12} \\ \dot{j}_{31} & \dot{j}_{21} & \dot{j}_{11} \end{pmatrix}, \quad R_3(J) := \begin{pmatrix} \dot{j}_{22} & \dot{j}_{12} & \dot{j}_{32} \\ \dot{j}_{21} & \dot{j}_{11} & \dot{j}_{31} \\ \dot{j}_{23} & \dot{j}_{13} & \dot{j}_{33} \end{pmatrix}, \\
S_1(J) &:= \begin{pmatrix} \dot{j}_{22} & \dot{j}_{23} & \dot{j}_{21} \\ \dot{j}_{32} & \dot{j}_{33} & \dot{j}_{31} \\ \dot{j}_{12} & \dot{j}_{13} & \dot{j}_{11} \end{pmatrix}, \quad S_2(J) := \begin{pmatrix} \dot{j}_{33} & \dot{j}_{31} & \dot{j}_{32} \\ \dot{j}_{13} & \dot{j}_{11} & \dot{j}_{12} \\ \dot{j}_{23} & \dot{j}_{21} & \dot{j}_{22} \end{pmatrix}.
\end{aligned} \tag{3.2}$$

We will also use the total space reflection T ,

$$T(J) := \begin{pmatrix} \dot{j}_{11} & \dot{j}_{21} & \dot{j}_{31} \\ \dot{j}_{12} & \dot{j}_{22} & \dot{j}_{32} \\ \dot{j}_{13} & \dot{j}_{23} & \dot{j}_{33} \end{pmatrix}. \tag{3.3}$$

Since T commutes with all elements of $\mathcal{O}^{(6)}$, adjoining it we obtain a 12-element group $\mathcal{O}^{(6)} \times T \equiv \mathcal{O}^{(6)} \times \mathbf{Z}_2$. The multiplication table for the group $\mathcal{O}^{(6)}$ is given in Table 1.

	$\mathbb{1}$	R_1	R_2	R_3	S_1	S_2
$\mathbb{1}$	$\mathbb{1}$	R_1	R_2	R_3	S_1	S_2
R_1	R_1	$\mathbb{1}$	S_1	S_2	R_2	R_3
R_2	R_2	S_2	$\mathbb{1}$	S_1	R_3	R_1
R_3	R_3	S_1	S_2	$\mathbb{1}$	R_1	R_2
S_1	S_1	R_3	R_1	R_2	S_2	$\mathbb{1}$
S_2	S_2	R_2	R_3	R_1	$\mathbb{1}$	S_1

Table 1 Multiplication table for the subgroup $\mathcal{O}^{(6)}$ of the octagonal group.

It is easy to generate all allowed intertwiner configurations J , given flux line assignments j_i , $i = -1, -2, -3, 1, 2, 3$, for the in- and outgoing links. The elements of the rows and columns of J simply have to add up to the appropriate j_i , for example, $\sum_{i=1}^3 j_{1,i} = j_{-1}$, $\sum_{i=1}^3 j_{i,1} = j_1$. Another advantage of this form is that the still remaining Mandelstam constraints can be expressed as simple linear combinations of J -matrices, and are all of the form

$$\begin{aligned}
& \begin{pmatrix} \dot{j}_{11} + 1 & \dot{j}_{12} & \dot{j}_{13} \\ \dot{j}_{21} & \dot{j}_{22} + 1 & \dot{j}_{23} \\ \dot{j}_{31} & \dot{j}_{32} & \dot{j}_{33} + 1 \end{pmatrix} - \begin{pmatrix} \dot{j}_{11} + 1 & \dot{j}_{12} & \dot{j}_{13} \\ \dot{j}_{21} & \dot{j}_{22} & \dot{j}_{23} + 1 \\ \dot{j}_{31} & \dot{j}_{32} + 1 & \dot{j}_{33} \end{pmatrix} \\
& - \begin{pmatrix} \dot{j}_{11} & \dot{j}_{12} + 1 & \dot{j}_{13} \\ \dot{j}_{21} + 1 & \dot{j}_{22} & \dot{j}_{23} \\ \dot{j}_{31} & \dot{j}_{32} & \dot{j}_{33} + 1 \end{pmatrix} + \begin{pmatrix} \dot{j}_{11} & \dot{j}_{12} + 1 & \dot{j}_{13} \\ \dot{j}_{21} & \dot{j}_{22} & \dot{j}_{23} + 1 \\ \dot{j}_{31} + 1 & \dot{j}_{32} & \dot{j}_{33} \end{pmatrix} \\
& + \begin{pmatrix} \dot{j}_{11} & \dot{j}_{12} & \dot{j}_{13} + 1 \\ \dot{j}_{21} + 1 & \dot{j}_{22} & \dot{j}_{23} \\ \dot{j}_{31} & \dot{j}_{32} + 1 & \dot{j}_{33} \end{pmatrix} - \begin{pmatrix} \dot{j}_{11} & \dot{j}_{12} & \dot{j}_{13} + 1 \\ \dot{j}_{21} & \dot{j}_{22} + 1 & \dot{j}_{23} \\ \dot{j}_{31} + 1 & \dot{j}_{32} & \dot{j}_{33} \end{pmatrix} = 0
\end{aligned} \tag{3.4}$$

Obviously, (3.4) is not to be understood as a matrix equation; the matrices are only labels for Hilbert space elements. The operator \hat{D} is cubic in derivatives, and can therefore be written as a sum of terms, each of which acts on some triplet of spin- $\frac{1}{2}$ flux lines routed through the intersection n . Its explicit form can be derived in a straightforward way, but is too long to be reproduced here. It can be found in our forthcoming publication [12]. Its form is a linear combination (with j_{mn} -dependent coefficients) of matrices J whose entries differ at most by $\Delta j_{mn} = \pm 1$ from the input matrix. This gives us a general formula for matrix elements, albeit in a non-orthogonal basis.

One finds the following relations under conjugation with elements of $\mathcal{O}^{(6)} \times T$:

$$R_i \hat{D} R_i = \hat{D}, \quad i = 1, 2, 3, \quad S_i \hat{D} S_i = \hat{D}, \quad i = 1, 2, \quad T \hat{D} T = -\hat{D}. \tag{3.5}$$

Next, we are interested in the representation theory of these discrete groups. $\mathcal{O}^{(6)}$ contains three conjugacy classes of elements namely, $\{\mathbb{1}\}$, $\{R_1, R_2, R_3\}$ and $\{S_1, S_2\}$. Following [10], one establishes the existence of three irreducible representations: two one-dimensional ones (called A_1 and A_2) and one two-dimensional one (called E). They can be identified by the values of their characters, i.e. the traces of the matrices representing the group elements (which only depend on the conjugacy class). The enlarged group $\mathcal{O}^{(6)} \times T$ has six conjugacy classes and six irreducible representations, since each of the previous representations gives rise to one of positive and one of negative parity, denoted by a subscript $+$ or $-$. The possible orbits sizes through single elements J under the action of $\mathcal{O}^{(6)} \times T$ are 1, 2, 3, 6 and 12, and they have a well-defined irreducible representation content [12].

It follows from (3.5) that \hat{D} obeys the (anti-)commutation relations

$$[\hat{D}, R_i] = 0, \quad i = 1, 2, 3, \quad [\hat{D}, S_1 + S_2] = 0, \quad [\hat{D}, T]_+ = 0. \tag{3.6}$$

We conclude that \hat{D} does not alter the $\mathcal{O}^{(6)}$ -quantum numbers, but maps positive- into negative-parity states and vice versa. In practice it is convenient to work with the operator \hat{D}^2 . A maximal subset of operators commuting both among themselves and with \hat{D}^2 is, for example, $\{R_1 + R_2 + R_3, S_1 + S_2, T\}$. This of course implies that \hat{D}^2 may be diagonalized separately on the eigenspaces of these operators, reducing the problem to a smaller one.

One further observation turns out to be useful. Since parity-odd wave functions are constructed by weighted sums (with factors ± 1) of spin network states, which may sometimes vanish, there are always fewer states transforming according to the representations A_i^-, E^- , than those transforming according to A_i^+, E^+ . The most efficient way of diagonalizing \hat{D} is therefore to start from the set of wave functions transforming according to one of the negative-parity irreducible representations, diagonalize \hat{D}^2 , construct the images under \hat{D} of the resulting set of states (which all have positive parity), and then form complex linear combinations to obtain eigenstates of \hat{D} , as explained in the previous section. The number of zero-volume states is then given by the difference of positive- and negative-parity states.

As an application of this scheme, we have analyzed the irreducible representation content of some of the Hilbert spaces corresponding to flux line numbers $(j_{-1}, j_{-2}, j_{-3}, j_1, j_2, j_3) = (j, j, j, j, j, j)$, i.e. for genuine six-valent intersections [12]. In this case, $\mathcal{O}^{(6)} \times T$ maps the Hilbert space into itself. Matrix sizes are reduced considerably when the various superselection sectors are considered separately, and the eigenvalues of \hat{D} could be found easily up to flux line numbers of order $j = 10$. For example, considering only the $\mathcal{O}^{(6)}$ -invariant sector, solution of the eigenvalue problem for $j = 1, 2, 3, \dots$ requires the diagonalization of square matrices of size $1, 2, 5, 8, 14, 20, 30, 40, \dots$, to be compared with a total number of states $5, 15, 34, 65, 111, 175, 260, 369, \dots$, if the $\mathcal{O}^{(6)} \times T$ -action is not taken into account. We also found that on these subsectors of Hilbert space, all eigenvalues already occur in the invariant A_1^- -sector, and are non-degenerate, that is, their corresponding eigenvectors are automatically orthogonal. Whether the \mathcal{O} -invariant sector is also distinguished on physical grounds depends on how the continuum limit of the lattice theory is taken, and on how the diffeomorphism symmetry is realized, both of which are still unresolved issues.

4 Summary

We have explained the need for the condition $\det E > 0$ on physical states in connection gravity, both classically and quantum-mechanically. Since the spectrum of the local lattice operator $\det \hat{E}(n)$ is discrete, there is no problem in principle in eliminating states with negative or vanishing eigenvalue of $\det \hat{E}$. Eigenvalues come in pairs of opposite sign, and

the corresponding eigenstates are related by complex conjugation. Eigenstates of a definite sign can be constructed once the eigenstates of $(\det \hat{E})^2$ are known.

These considerations make the evaluation of the spectrum of $\det \hat{E}$ even more urgent, apart from its central importance as an ingredient in kinematical and dynamical operators in canonical quantum gravity. We were able to make progress in this task by taking into account superselection sectors related to the symmetry properties of $\det \hat{E}(n)$ under the action of the cubic group and space reflection.

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