Lattice gauge theory in terms of independent Wilson loops

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We discuss the construction of a complete set of independent Wilson loops $Tr\, \mathrm{P}\exp\oint_\gamma A$ that allows one to formulate the physics of pure lattice gauge theory directly on the subspace of physical configurations, and report on progress in recasting the theory in terms of these variables.

1. THE NEW LOOP APPROACH

The mathematical ingredients necessary in the description of our general approach consist of a d-dimensional manifold Σ^d, $d = 2, 3, 4$, a Yang-Mills gauge group $G \subset Gl(n, C)$, and a representation R of G on a linear space V_R. A *path* is a continuous mapping $\gamma : [0, 1] \to \Sigma^d$. If $\gamma(0) = \gamma(1)$, γ is called a *loop*. For each loop γ, we define on the configuration space $\mathcal{A}$ of pure Yang-Mills theory the path-ordered exponential

$$U_A(\gamma) := \mathrm{P}\exp ig \oint_\gamma A_\mu(\gamma(t))\dot{\gamma}^\mu(t)dt. \qquad (1)$$

This is a mathematically well-defined object, called the *holonomy* of γ, which can be interpreted geometrically as the transformation undergone by an element of V_R after parallel-transporting it along the horizontal lift of the curve γ to the corresponding fibre bundle over Σ^d.

What we will call the "new loop approach" is the attempt to reformulate gauge theory solely in terms of the gauge-invariant traced holonomies $T_A(\gamma) := Tr\, U_A(\gamma)$, the so-called *Wilson loops*. The physical motivation for this procedure is the expectation that all physics in the confined phase of Yang-Mills theory is expressible in terms of gauge-invariant quantities only. Mathematically, this is justified by the classical equivalence theorem (which holds locally in Σ^d) [4]

$$\mathcal{A}/\mathcal{G} \simeq \{T(\gamma)\}/\{\text{Mandelstam constraints}\}. \qquad (2)$$

The left-hand side of (2) gives the usual description of the physical configuration space as the set of gauge orbits in $\mathcal{A}$ ($\mathcal{G}$ is the group of local gauge transformations), the right-hand side denotes the set of complex-valued functions on loop space, subject to a set of so-called *Mandelstam constraints*. These can be seen as deriving from identities satisfied by the traces of $n \times n$-matrices. They are algebraic equations non-linear in T, and their form depends on G and R, in particular the dimensionality n of V_R.

Our aim is to recast the theory, given in terms of the local and gauge-*covariant* gauge potentials $A(x)$, in terms of the non-local and gauge-*invariant* variables $T(\gamma)$. One hopes that such an approach may be better suited to deal with the non-linearities of non-abelian gauge theory and allow for a more direct physical interpretation, since gauge-invariance is automatically taken care of. Also it may be more amenable to a non-perturbative quantization, for example, by making use of certain algebraic structures on loop space. It is well-known that equivalence of the classical formulations does not necessarily imply the equivalence of the corresponding quantum theories.

Throughout the last 30 years, starting with the work of Mandelstam [8], these heuristically appealing ideas have led many physicists to incorporating variables based on the holonomy (1) in their descriptions of gauge theory (c.f. the review [7]). For a number of reasons, these attempts have been largely unsuccessful:

- the mathematics of the infinite-dimensional loop space is not sufficiently well developped, and we do not have a powerful differential calculus as we have in the case of finite-dimensional analysis. Only recently a

few rigorous mathematical results have become available, which however often take a somewhat different viewpoint from what is required in physical applications. In any case, we do not yet have a precise idea which part of the loop space is physically the most relevant.

- Nobody has formulated an action principle in terms of loop variables; this leaves considerable freedom for one's preferred derivation of classical or quantum equations of motion.

- Since the loop variables are non-local, standard quantization methods cannot be applied straightforwardly. There are many unresolved issues concerning a meaningful regularization and renormalization procedure in the loop formulation.

- Although the variables $T(\gamma)$ do not carry any gauge-dependence, they still do not form a set of *physical* variables because of their overcompleteness and the restrictions on them implied by the Mandelstam constraints. Their existence gives rise to many non-trivial features in the loop approach.

For example, for the case of $SU(2)$ gauge theory in the fundamental, two-dimensional representation, the non-linear Mandelstam constraints are given by

$$T(\gamma_1)T(\gamma_2) - T(\gamma_1 \circ_x \gamma_2) - T(\gamma_1 \circ_x \gamma_2^{-1}) = 0, (3)$$

for any pair of loops γ_1 and γ_2 intersecting at a point x (where $\circ_x$ denotes the loop composition in x, and γ^{-1} the inverse of γ), together with a set of inequalities which will be given in the next section. In the continuum, we do not know of a systematic way of solving the constraints (3) in order to reduce the number of redundant loop variables.

2. LATTICE RESULTS

Some of the above-mentioned problems can be tackled successfully in a regularized lattice version of the loop formulation. This amounts to approximating the set of all loops in Σ^d by the set of all closed contours that can be formed from the links of a hypercubic lattice N^d with periodic boundary conditions. For a set $\{l_i, i = 1, \ldots, n\}$ of contiguous lattice links, set $T(\gamma) = Tr\, U_{l_1} \ldots U_{l_n}$, where the U_{l_i} are the link holonomies of the standard Kogut-Susskind-Wilson approach. Note that even on a finite lattice, both the number of loop variables $T(\gamma)$ one can construct in this way, and the number of Mandelstam constraints (3) are infinite.

An important result in this context was obtained in [5], where it was shown that for $G = SU(2)$ and $d = 2, 3, 4$ one can solve the highly coupled set of lattice Mandelstam constraints (3), and give an explicit local description of the (finite-dimensional) physical configuration space

$$\mathcal{C}_{\text{phys}} = \mathcal{A}/\mathcal{G}|_{\text{lattice}} =$$
$$= \{T(\gamma),\ \gamma \text{ a lattice loop}\}/\{\text{M.c.}\} \qquad (4)$$

in terms of a set of independent loop variables. To describe the result, it is convenient to perform a linear transformation to another set of loop variables, defined by

$$L_1(\gamma_i) = \tfrac{1}{2}T(\gamma_i),$$
$$L_2(\gamma_i, \gamma_j) = \tfrac{1}{4}(T(\gamma_i \circ \gamma_j^{-1}) - T(\gamma_i \circ \gamma_j)). \qquad (5)$$

It turns out that for a complete description of $\mathcal{C}_{\text{phys}}$ it suffices to consider variables $L_1(\gamma_i)$, where γ_i is a plaquette loop, and $L_2(\gamma_i, \gamma_j)$, where (γ_i, γ_j) is a (not necessarily co-planar) pair of neighbouring plaquette loops.

The first step in the proof consists in deriving an algorithm for expressing traced holonomies of composite loops, $T(\gamma_1 \circ \ldots \circ \gamma_n)$ as functions of T-variables depending on fewer than n elementary loops. This still does not lead to an algebraically independent set of lattice loop variables, but to an educated guess as to what such an independent set may be.

The second step involves the computation of determinants of certain large matrices, in order to show that the chosen subset of Wilson loops does indeed span the space $\mathcal{C}_{\text{phys}}$ locally. For this a simple FORTRAN program with a MATH/LIBRARY subroutine was used (see [5] for more details). This second step depends on

the dimension of the lattice because of the different lattice geometries.

In a sense the solution for the independent degrees of freedom is surprisingly simple: it is sufficient to look at small lattice loops (hence the formulation is "not very non-local"). For example, in $d = 3$, associated with each unit cube, there are six independent Wilson loops, three of type L_1 and three of type L_2, which is exactly the number of physical degrees of freedom per unit cube in three dimensions (recall the T-variables for $SU(2)$ are real-valued).

Another remarkable fact is that the set of independent variables $\{L_i\}$ in all dimensions is invariant under lattice translations, in contrast with the usual gauge-covariant and maximally gauge-fixed description in terms of link holonomies. Thus, the loop approach seems qualitatively different from the standard covariant formalism, in which no complete gauge-fixing exists. Moreover, the weak-field limit in our formulation assumes the simple form $L_2 \to 0$.

The next step must be to rewrite the theory in terms of the independent L-variables. This is not straightforward, since the explicit dependence $U_{l_i} = U_{l_i}(L_j)$ is not known for general lattices. Note that in our formulation, the usual Wilson action S_W is just a linear (!) function of the L_1-variables. In reexpressing the partition function,

$$\int [dU_l] e^{-S_W[U_l]} \rightsquigarrow \int [dL_1 dL_2] e^{-S_W[L_1]}, \qquad (6)$$

non-triviality arises through the measure $[dL_i]$ and a set of inequalities the L-variables must satisfy, namely,

$$-1 \leq L_i \leq 1 \qquad (7)$$

and

$$L_2(\alpha, \alpha) L_2(\beta, \beta) L_2(\gamma, \gamma) - L_2(\alpha, \alpha) L_2(\beta, \gamma)^2$$
$$-L_2(\beta, \beta) L_2(\alpha, \gamma)^2 - L_2(\gamma, \gamma) L_2(\alpha, \beta)^2$$
$$+2 L_2(\alpha, \beta) L_2(\alpha, \gamma) L_2(\beta, \gamma) \geq 0. \qquad (8)$$

When deriving equations of motion from an action like $S_W(L_i)$, the infinitesimal variations must obey these inequalities.

As a first step, explicit relations of the form $U_\gamma = U_\gamma(L_i)$, with γ a lattice loop, where derived

in [6], but the Jacobian $|\frac{\partial U}{\partial L}|$ so far is known only for the one-point model in $d = 2$.

3. Outlook

Only a few numerical results have been obtained in a pure loop approach, and all of them in a Hamiltonian framework. The only work we are aware of is that of Furmanski and Kolawa [2] in $d = 3 + 1$, and that of Gambini, Leal and Trias [3] and Brügmann [1], both in $d = 2 + 1$ dimensions. These are all small-scale calculations for $G = SU(2)$. They have in common that quantum states are labelled by lattice loops, the Hamiltonian operator $\hat{H}$ acts by fusion and fission of loop arguments, and the spectrum is calculated in a truncated loop basis.

The main problem in these approaches is how to deal efficiently with the overcompleteness of the loop basis, due to the Mandelstam constraints. All of them corroborate the validity of the loop approach as such, but the data obtained are insufficient to claim superiority to other methods, and so far no scaling has been observed.

The discovery of the independent loop variables opens new possibilities both in the Lagrangian and Hamiltonian loop formulation of lattice gauge theory. In the former, the foremost problem is that of determining the exact form of the measure $[dL_i]$, and hence the induced coupling between neighbouring lattice loops. Related questions in the canonical formulation concern the form of the Hamiltonian $H(L_i)$ and the choice of appropriate canonical momenta.

The answers will of course be non-trivial, but will have the advantage of being formulated in terms of *physical* variables. In particular, the point of view we have presented here may lead to new ideas how to construct a meaningful, intrinsically gauge-invariant loop perturbation theory, and thereby throw new light on a corresponding continuum loop formulation.

It will be interesting to see how non-trivial topological properties of gauge fields on a torus T^d are expressed in terms of the independent Wilson loops. Note also that fermions can be incorporated naturally into the loop picture, forming gauge-invariant quantities by "gluing" fermions

to the ends of an open lattice path, again using the holonomy associated with that path.

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