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A NEW QUANTUM REPRESENTATION FOR CANONICAL GRAVITY AND SU(2) YANG–MILLS THEORY

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Starting from Rovelli–Smolin's infinite-dimensional graded Poisson-bracket algebra of loop variables, we propose a new way of constructing a corresponding quantum representation. After eliminating certain quadratic constraints, we "integrate" an infinite-dimensional subalgebra of loop variables, using a formal group law expansion. With the help of techniques from the representation theory of semidirect-product groups, we find an exact quantum representation of the full classical Poisson-bracket algebra of loop variables, without any higher-order correction terms. This opens new ways of tackling the quantum dynamics for both canonical gravity and Yang–Mills theory.

1. Introduction

In a recent paper [19], Rovelli and Smolin, extending earlier work by Jacobson and Smolin [13], propose a conceptually new way of consistently quantizing canonical gravity. Three ingredients are crucial in their construction: (i) Ashtekar's reformulation of the canonical theory in terms of new variables, such that its phase space can be regarded as subspace of a Yang–Mills phase space. (ii) The use of nonlocal variables on this phase space, based on closed curves in the three-manifold. (iii) The quantization of a set of basic variables that does not consist of pairs of canonically conjugate variables.

This approach is nonperturbative, in the sense that it never makes use of a perturbative expansion about a fixed background metric, the metric not even being one of the basic variables. Rather, full diffeomorphism invariance is imposed. In this formulation, they manage for the first time to find an explicit sector of physical states, i.e. simultaneous solutions to all the quantum constraint equations. However, these solutions remain formal insofar as there is neither a well-defined Hilbert space structure, nor are there any physical observables. Thus their physical interpretation is unclear. Also there remain some doubts as to the validity of their

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regularization procedure for the hamiltonian constraint, see ref. [9]. Nevertheless, their results show that the failure of the perturbation theory does not necessarily imply the nonexistence of a consistent theory of quantum gravity, and at the same time that the quantization of gravity may have to be radically different from that of an ordinary Poincaré-covariant field theory.

The purpose of this paper is to make some first steps towards the construction of a complete and rigorous quantum theory, based on the infinite-dimensional graded algebra of the nonlocal loop variables introduced in ref. [19], which is applicable to both canonical gravity and gauge theory. Although the use of loop variables in Yang–Mills theory is not a new idea (see refs. [15, 17] and references therein for examples within a path-integral context), the novelty is to have a set of such objects which form a closing Poisson-bracket algebra on phase space. This abstract algebra structure will serve as a starting point for the quantization. We will be mainly concerned with the kinematical aspects of the theory which are common to both gravity and SU(2) Yang–Mills theory.

The presentation of our results will be as follows. Sect. 2 summarizes some basic facts about the "gauge-theory formulation" of canonical gravity in terms of Ashtekar's new variables and the subsequent reformulation of this classical theory in terms of Rovelli–Smolin's loop variables. These so-called T-variables have to obey a series of quadratic constraints, which follow from their definition in terms of local phase space variables.

In sect. 3, we explain how to solve these constraints classically by taking suitable linear combinations of T-variables. This leads to a smaller set of loop variables, which still form a closing Poisson-bracket algebra. Elements L^n of this algebra inherit the grading of the T-variables ($n = 0, 1, 2, \ldots$), with algebra structure $\{L^0, L^0\} = 0$, $\{L^m, L^n\} \sim L^{m+n-1}$. Important in the construction of the L-variables is the concept of the number of self-intersections of a loop.

In sect. 4, we "integrate" the infinite-dimensional subalgebra of the L-algebra spanned by the L^0- and L^1-variables, using a formal group law expansion. The method of the formal group law is illustrated by a finite-dimensional example, and the first orders of the L-group law are worked out explicitly.

The L-group is of the form of a semi-direct product, $\mathscr{L}^0 \circledS \mathscr{L}^1$, which enables us in sect. 5 to construct a natural unitary quantum representation. It is defined in terms of state vector $\Psi[\bar{L}^0]$ that are functionals on the dual $\bar{\mathscr{L}}^0$ of $\mathscr{L}^0$. We show that also all higher-order L^n-variables can be quantized on this representation space. Due to the special structure of the quantum operators, no factor-ordering problems occur and, in contrast with the representation found previously, we reproduce exactly the classical Poisson-bracket algebra, without any higher-order correction terms. This gives us confidence in the correctness of our method. However, in the absence of a topology on the underlying loop space, it is impossible to investigate the orbit structure in $\bar{\mathscr{L}}^0$ under the $\mathscr{L}^1$-action with the usual rigour. Finally we indicate how the dynamics may be incorporated into the

emerging picture, both for Yang–Mills theory and gravity, and what are important problems that remain to be solved.

2. The Ashtekar and loop variables

Let us begin with a short description of Ashtekar's new variables for canonical gravity [3], whose discovery, inspired by earlier work of Sen [20], has given a fresh impetus to the research work in this subject over the past few years. For a more detailed motivation, the reader is referred to refs. [4,5,13]. This new hamiltonian formulation substitutes the old one, given in terms of the first and second fundamental forms, q_{ab} and K_{ab}, of the three-manifold Σ, and it depends crucially on the introduction of a spinor structure on Σ.

The new canonical pairs of variables are given by $(A_{aA}{}^{B}, \tilde{\sigma}^{a}{}_{A}{}^{B})$, where a is a spatial 3-index, and A, B denote internal spinor indices. Here $\tilde{\sigma}^{a}$ is a (density weighted) soldering form which defines an isomorphism between tangent vectors (or rather vector densities) λ^{a} on Σ and tracefree "Higgs scalars" $\lambda_{A}{}^{B}$, and A^{a} is a Lie algebra-valued spin connection one-form, the potential for the self-dual (or antiself-dual) part of the Weyl curvature on Σ [3,5]. The formalism works for any internal gauge algebra of rank one [7]. Choosing $\mathrm{su}(2) \simeq \mathrm{so}(3)$ leads to general relativity with euclidean signature. In order to recover a lorentzian signature for the four-metric, one has to use complexified su(2), $\mathrm{su}(2)_{\mathbb{C}} \simeq \mathrm{sl}(2,\mathbb{C})$. We will be mainly interested in this case, for which Ashtekar's phase space becomes complexified. The relation to the old canonical variables is complicated, but the transition can be achieved by a (complex) canonical transformation [4].

In the following we will assume that Σ is compact, without boundary, with fixed topological and differentiable structure, and will consider the trivial $\mathrm{SL}(2,\mathbb{C})$-bundle over Σ. Internal indices A on associated spinors and tensors are raised and lowered with a fixed ϵ-tensor, satisfying

$$\epsilon^{AB} = -\epsilon^{BA}, \qquad \epsilon^{CA}\epsilon_{CB} = \delta^{A}_{B}, \tag{2.1}$$

according to the convention

$$\lambda^{A} = \epsilon^{AB}\lambda_{B}, \qquad \lambda_{B} = \lambda^{A}\epsilon_{AB}. \tag{2.2}$$

Both $A_{a}{}^{AB}$ and $\tilde{\sigma}^{aAB}$ are symmetric in their spinor indices, and satisfy the canonical Poisson-bracket relation

$$\{A_{a}{}^{AB}(x), \tilde{\sigma}^{b}{}_{CD}(y)\} = \delta^{b}_{a}\delta^{(A}_{C}\delta^{B)}_{D}\delta^{3}(x,y), \tag{2.3}$$

with the parentheses indicating symmetrization. We use this form of the canonical relation, without a factor i appearing on the right-hand side. The constraints of the

theory in terms of the new variables are

$$\mathscr{D}_a\tilde{\sigma}^a \equiv \partial_a\tilde{\sigma}^a + [A_a, \tilde{\sigma}^a] = 0, \tag{2.4}$$

$$\mathrm{Tr}\,\tilde{\sigma}^a F_{ab} = 0, \qquad \mathrm{Tr}\,\tilde{\sigma}^a\tilde{\sigma}^b F_{ab} = 0, \tag{2.5), (2.6}$$

where F_{ab} denotes the curvature two-form of A_a: $F_{abA}{}^B\lambda_B = 2\mathscr{D}_{[a}\mathscr{D}_{b]}\lambda_A$. The three constraints (2.4) are exactly the Gauss-law constraints, which eliminate the internal degrees of freedom we introduced previously. On the submanifold of the phase space defined by eq. (2.4), eqs. (2.5) become the three diffeomorphism constraints and eq. (2.6) becomes the hamiltonian constraint of general relativity. This way the phase space of canonical gravity is imbedded in the phase space of an SL(2, $\mathbb{C}$) Yang–Mills theory. The polynomial form of the constraints signifies a drastic simplification compared with the old hamiltonian formulation, where nonpolynomial expressions had posed insurmountable problems in any canonical quantization program.

In the context of Yang–Mills theory, we can identify $\tilde{\sigma}^a$ with the "electric" field E^a; from the gravitational point of view $\tilde{\sigma}^a$ is essentially the "square root of the three-metric":

$$-(\det q)q^{ab} = \mathrm{Tr}\,\tilde{\sigma}^a\tilde{\sigma}^b\,. \tag{2.7}$$

We have worked with complexified quantities so far, and we still need some reality conditions on the basic variables to recover the usual formulation. How these can best be implemented in both the classical and the quantum theory, is a nontrivial issue (see refs. [4, 8] and references therein), and seems to be one of the major difficulties of this new formulation.

Defined in terms of the variables $(A, \tilde{\sigma})$, the loop variables T^n, introduced in ref. [19], form an infinite set of variables for Ashtekar's phase space. They carry a label $n = 0, 1, 2, \ldots$, counting the number of marked points on a loop, and are generalizations of the well-known Wilson loops, given by the traces of holonomies of closed paths in Σ. The T^n are complex functions depending on piecewise smooth, oriented, parametrized, nondegenerate loops in Σ, i.e. maps $S^1 \mapsto \Sigma$, with n marked points. By construction they are invariant under Yang–Mills gauge transformations, and hence constitute coordinates for the reduced (with respect to the Gauss-law constraints (2.4)) phase space. This is remarkable insofar as it gives us the possibility of quantizing the reduced theory directly, without having to deal with gauge-fixing problems. Crucial is the fact that the T^n are nonlocal variables, depending on closed loops in Σ. Even more remarkable is that they form a closing, graded Poisson-bracket algebra of the form $\{T^0, T^0\} \sim 0$, $\{T^m, T^n\} \sim T^{m+n-1}$, for $m + n \geqslant 1$, with respect to the canonical symplectic structure on Ashtekar's phase space.

We define SL(2, ℂ)-matrices $U_\gamma(s_1, s_2)$ by

$$U_\gamma(s_1, s_2)^A{}_B = \left[\mathrm{P}\exp\int_{s_1}^{s_2} \mathrm{d}s A_a(\gamma(s))\dot{\gamma}^a(s)\right]^A{}_B, \tag{2.8}$$

with P indicating path ordering. The so-called holonomy $U_\gamma(s) := U_\gamma(s, s+2\pi)$ is the transformation undergone by an SL(2, ℂ)-spinor that is parallel transported along the loop γ, starting and ending at the point s. The loop variables are defined as follows

$$\begin{aligned}
T^0[\gamma] &= \mathrm{Tr}\, U_\gamma, \\
T^a[\gamma](s) &= \mathrm{Tr}\, U_\gamma(s)\tilde{\sigma}^a(\gamma(s)), \\
&\vdots \\
T^{a_1\ldots a_n}[\gamma](s_1, \ldots, s_n) & \\
&= \mathrm{Tr}\, U_\gamma(s_n, s_1)\tilde{\sigma}^{a_1}(\gamma(s_1))U_\gamma(s_1, s_2)\ldots U_\gamma(s_{n-1}, s_n)\tilde{\sigma}^{a_n}(\gamma(s_n)), \\
&\qquad\qquad 0 < s_1 < s_2 < \ldots < s_n \leqslant 2\pi, \\
&\vdots
\end{aligned} \tag{2.9}$$

where Tr denotes the trace over internal SL(2, ℂ)-indices. To form a T^n-variable we have inserted n $\tilde{\sigma}$-matrices in the trace over U_γ. By convention, in the formula for T^n we will always take the s_i to be ordered as indicated on the right-hand side, regardless of the order in which the s_i and corresponding a_i appear as arguments on the left-hand side. Note that for the case of a flat connection, $F \equiv 0$, $T^0[\gamma]$ depends only on the homotopy class of the loop γ. In Yang–Mills theory, $T^0[\gamma]$ in a certain sense measures the total magnetic flux going through γ [11].

All T-variables are invariant under loop reparametrizations which do not change the orientation. Using the fact that SL(2, ℂ) consists of (2×2)-matrices with unit determinant, contraction with the ϵ-tensor yields the important identity

$$U_\gamma(s)_{AB} = -U_{\gamma^{-1}}(s)_{BA}, \tag{2.10}$$

where the inverse γ^{-1} of the loop γ is defined by $\gamma^{-1}(s) = \gamma(2\pi - s)$. From here it follows that T^n for n even is invariant under reversal of the loop orientation

whereas T^n for n odd changes sign. The loop variables are not all independent, but obey a continuous infinity of constraints: consider two loops γ_1 and γ_2 intersecting in some point s. By inserting the identity

$$\epsilon_{AB}\epsilon^{CD} = \delta_A^C\delta_B^D - \delta_A^D\delta_B^C \tag{2.11}$$

at the point of intersection, we derive

$$T^{n_1}[\gamma_1]T^{n_2}[\gamma_2] = T^{n_1+n_2}[\gamma_1 \circ_s \gamma_2] + (-1)^{n_2}T^{n_1+n_2}\left[\gamma_1 \circ_s \gamma_2^{-1}\right], \tag{2.12}$$

which relates the T-variables depending on the separate loops γ_1 and γ_2 to those depending on the loops $\gamma_1 \circ_s \gamma_2$ and $\gamma_1 \circ_s \gamma_2^{-1}$, obtained by first going round γ_1 and then round γ_2 (respectively γ_2^{-1}).

Since the loop variables are defined in terms of the canonical pairs $(A, \tilde{\sigma})$ on Ashtekar's phase space, we can compute their Poisson-bracket algebra. The algebra closes, with the graded structure $\{T^m, T^n\} \sim T^{m+n-1}$. This is a nontrivial result, since the Poisson bracket of two functionals depending in a certain way on loops γ, connections A and soldering forms $\tilde{\sigma}$, will in general not yield an expression of the same type. The T^0- and T^1-variables form a closed subalgebra, called the "small T-algebra", with Poisson-bracket relations

$$\begin{aligned}
&\{T[\alpha], T[\beta]\} = 0,\\
&\{T^a[\gamma](s), T[\eta]\} = -\Delta^a[\gamma,\eta](s)\left(T[\gamma \circ_s \eta] - T\left[\gamma \circ_s \eta^{-1}\right]\right),\\
&\{T^a[\gamma](s), T^b[\eta](t)\} = -\Delta^a[\gamma,\eta](s)\left(T^b[\gamma \circ_s \eta] + T^b\left[\gamma \circ_s \eta^{-1}\right]\right)(u(t))\\
&\qquad + \Delta^b[\eta,\gamma](t)\left(T^a[\eta \circ_t \gamma] + T^a\left[\eta \circ_t \gamma^{-1}\right]\right)(u(s)).
\end{aligned} \tag{2.13}$$

The structure constants Δ^a are given by

$$\Delta^a[\gamma,\eta](s) = \oint_\eta \mathrm{d}t\, \delta^3(\gamma(s), \eta(t))\dot{\eta}^a(t), \tag{2.14}$$

whence it follows that $\Delta^a[\gamma, \eta^{-1}] = -\Delta^a[\gamma,\eta]$ and $\Delta^a[\gamma^{-1},\eta] = \Delta^a[\gamma,\eta]$. Note that they are really structure *constants*, since they do not any more depend on A and $\tilde{\sigma}$. The singularities appearing in Δ^a are harmless in the sense that they can be removed by a suitable smearing prescription. The general expression for the

Poisson bracket of two T-variables is

$$\{T^{a_1\dots a_m}[\gamma](s_1,\dots,s_m), T^{b_1\dots b_n}[\eta](t_1,\dots,t_n)\}$$

$$= -\sum_{i=1}^{m} \Delta^{a_i}[\gamma,\eta](s_i)\cdot\Big(T^{a_1\dots \not{a}_i\dots a_m b_1\dots b_n}[\gamma\circ_{s_i}\eta](s_1\dots \not{s}_i\dots s_m t_1\dots t_n)$$

$$+(-1)^{n+1}T^{a_1\dots \not{a}_i\dots a_m b_1\dots b_n}\big[\gamma\circ_{s_i}\eta^{-1}\big](s_1\dots \not{s}_i\dots s_m t_1\dots t_n)\Big)$$

$$+\sum_{i=1}^{n} \Delta^{b_i}[\eta,\gamma](t_i)\cdot\Big(T^{a_1\dots a_m b_1\dots \not{b}_i\dots b_n}[\eta\circ_{t_i}\gamma](s_1\dots s_m t_1\dots \not{t}_i\dots t_n)$$

$$+(-1)^{m+1}T^{a_1\dots a_m b_1\dots \not{b}_i\dots b_n}\big[\eta\circ_{t_i}\gamma^{-1}\big](s_1\dots s_m t_1\dots \not{t}_i\dots t_n)\Big) \quad (2.15)$$

with a slash denoting the omission of a parameter. This relation has a simple interpretation in terms of cutting and joining of loops, which can be encoded in a diagramatic notation as has been explained in refs. [9] and [19].

The T-variables form an almost complete set of variables on Ashtekar's phase space; they are not completely separating, since they are invariant under the discrete transformation $(A,\tilde{\sigma})\mapsto(-A,-\tilde{\sigma})$. A modified set of loop variables that is not invariant under this transformation, has been proposed in ref. [16]. Whether this affects the final conclusions of the theory is not yet known. The methods introduced in the present paper could probably still be applied. It has been conjectured [19] that it is sufficient to consider only the T^0- and T^1-variables, and not the full set of T^n-variables. However, this is not true if we consider discrete subsets of loop space (as for example is required in a lattice formulation for Yang–Mills theory, see below). In any case it will be useful to have all T-variables at our disposal, since the hamiltonian constraint for gravity is usually expressed as limit of a certain sequence of T^2-variables.

It is important to understand that our intention is to base the theory exclusively on these loop variables, without referring anymore to the canonical pairs $(A,\tilde{\sigma})$. In this sense eqs. (2.12), which are identities in terms of the old variables, now become constraints that must be imposed on T-space. Similarly, reality conditions have to be expressed as conditions on the T-variables alone. It is not at all straightforward to derive these from reality conditions on Ashtekar's variables. If we are only interested in SU(2) gauge theory, with SU(2) given by the subgroup of SL(2, $\mathbb{C}$) of unitary (2×2)-matrices with unit determinant, the reality conditions simply are $T^{n*}=T^n$. Real lorentzian gravity also corresponds to an SU(2)-subgroup of SL(2, $\mathbb{C}$), but to a different one, as there are many ways of imbedding SU(2) in SL(2, $\mathbb{C}$).

3. The L-algebra

We have seen that the T-variables form an overcomplete set of variables for the reduced phase space, because they satisfy the identities (2.12). One should not confuse this overcompleteness with the possible redundancy of the T^n-variables for $n \geqslant 2$ mentioned earlier. Note that the constraints (2.12) are quadratic in the basic variables, hence we expect difficulties if we try to impose them à la Dirac as operator constraints on the state space in the quantum theory. There is, however, the simpler possibility of removing this redundancy already at the classical level, as we will now explain.

In the following we will call "simple loop" an oriented, unparametrized loop without self-intersections, and "composite loop" a loop obtained by joining simple loops like, for example, $\gamma_1 \circ_s \gamma_2$ made out of simple loops γ_1 and γ_2. For simplicity, let us first consider the T^0-variables, in which case eq. (2.12) becomes

$$T[\gamma_1]T[\gamma_2] = T[\gamma_1 \circ_s \gamma_2] + T\left[\gamma_1 \circ_s \gamma_2{}^{-1}\right]. \tag{3.1}$$

If γ_1 and γ_2 are simple loops, this expression equates the product of their associated T-variables to a certain linear combination of T-variables associated to the composite loops $\gamma_1 \circ_s \gamma_2$ and $\gamma_1 \circ_s \gamma_2{}^{-1}$. The crucial observation now is that for this particular configuration of loops (γ_1 and γ_2, intersecting in s) only the difference, $T[\gamma_1 \circ_s \gamma_2] - T[\gamma_1 \circ_s \gamma_2{}^{-1}]$, appears on the right-hand side of the Poisson-bracket relations (2.13), whereas only the sum, $T[\gamma_1 \circ_s \gamma_2] + T[\gamma_1 \circ_s \gamma_2{}^{-1}]$, appears in constraint (3.1). This suggests a change of variables to new ones, where the constraints become explicit and can easily be solved for. It is indeed possible, by using suitable linear combinations of T-variables, to find nonredundant coordinates for the subspace of T-space defined by constraints (2.12). We will call them the L-variables, and the corresponding closing Poisson-bracket algebra the L-algebra. They inherit the grading from the T-variables, and we start by defining the L^0-variables (omitting the superscript 0 as usual). In the following $\gamma, \eta, \mu, \ldots$ denote simple loops, unless otherwise stated. We define

$$\begin{aligned}
L[\gamma] &:= T[\gamma], \\
L[\gamma \circ_s \eta] &:= \tfrac{1}{2}\left(T[\gamma \circ_s \eta] - T\left[\gamma \circ_s \eta^{-1}\right]\right), \\
L[\gamma \circ_s \eta \circ_t \mu] &:= \tfrac{1}{4}\left(T[\gamma \circ_s \eta \circ_t \mu] - T\left[\gamma \circ_s \eta \circ_t \mu^{-1}\right]\right. \\
&\qquad \left. - T\left[\gamma \circ_s \eta^{-1} \circ_t \mu^{-1}\right] + T\left[\gamma \circ_s \eta^{-1} \circ_t \mu\right]\right), \\
&\ \ \vdots
\end{aligned} \tag{3.2}$$

A general L-variable, $L[\gamma]$, depending on a loop $\gamma = \gamma_1 \circ_{s_1} \gamma_2 \circ \ldots \circ_{s_{n-1}} \gamma_n$, with $n-1$ self-intersections, i.e. a composite loop made out of n simple loops, can be expressed in terms of T's as follows. Pick a point s on one of the γ_i, on γ_1, say. Starting at s, go round the composite loop γ once, following the orientations of the individual loops γ_i. This loop γ gives a contribution $+T[\gamma_1 \circ \ldots \circ \gamma_n]$. Next consider the loop where the orientation of one of the simple loops γ_i, $i \neq 1$, is reversed, for example the loop $\gamma_1 \circ \gamma_2^{-1} \circ \gamma_3 \circ \ldots \circ \gamma_n$. The corresponding T-variable contributes with a factor $(-1)^{j_i}$, where j_i denotes the number of points on γ_i at which γ_i is joined to other simple loops γ_k via the $\circ$-operation. This way we get $(n-1)$ T-contributions from configurations where one of the γ_i has been reversed. Next we consider all configurations which can be obtained by reversing the orientation of two simple loops γ_i and γ_k, $i, k \neq 1$, (the orientation of γ_1 remains fixed throughout). They give contributions of the form $(-1)^{j_i+j_k} T[\gamma_1 \circ \ldots \circ \gamma_i^{-1} \circ \ldots \circ \gamma_k^{-1} \circ \ldots \circ \gamma_n]$. Next we consider all possible configurations with three simple loops reversed, etc. The last term comes from the configuration where all but the simple loop γ_1 have their orientation reversed. Then add up all the contributions (there are 2^{n-1} of them) and divide by 2^{n-1}. The result defines the L-variable $L[\gamma_1 \circ \ldots \circ \gamma_n]$. It is independent of the initial choice of the simple loop (which in the example we took to be γ_1) that never changes its orientation. Note that the form of L in terms of T's depends on how the various simple loops are joined together topologically.

From this definition follows the identity

$$L\left[\gamma_1 \circ \ldots \circ \gamma_i^{-1} \circ \ldots \circ \gamma_n\right] = (-1)^{j_i} L[\gamma_1 \circ \ldots \circ \gamma_i \circ \ldots \circ \gamma_n] \tag{3.3}$$

for L under reversal of a simple loop γ_i. Note that under simultaneous reversal of all simple loops, we get

$$L[\gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_n] = L\left[\gamma_1^{-1} \circ \gamma_2^{-1} \circ \ldots \circ \gamma_n^{-1}\right] \tag{3.4}$$

since $\sum_{i=1}^{n} j_i$ is always an even number. There is a similar construction for the higher-order L^n in terms of linear combinations of the T^n. Again we start by defining $L^n[\gamma] := T^n[\gamma]$ for simple loops γ, and then work our way up for L^n depending on composite loops $\gamma_1 \circ \ldots \circ \gamma_k$, following the algorithm given above for the construction of the L^0's. The only difference is that each simple loop γ_i has now two natural numbers associated with it, the number j_i of "joining points" (with other loops) and the number h_i of marked points (insertions of $\tilde{\sigma}$-matrices) on γ_i. As a consequence the contribution from a configuration with γ_i reversed now picks up a relative factor of $(-1)^{j_i+h_i}$.

Suppose, for example, that the two simple loops γ_1 and γ_2 intersect in s, and that γ_1 has a marked point in t $(\neq s)$. Then the relevant L^1-variable is

$$L^a[\gamma_1 \circ_s \gamma_2](t) = \tfrac{1}{2}\left(T^a[\gamma_1 \circ_s \gamma_2](t) + T^a\left[\gamma_1 \circ_s \gamma_2^{-1}\right](t)\right). \tag{3.5}$$

Comparing with eq. (2.12), we see that it is the difference $T^a[\gamma_1 \circ_s \gamma_2](t) - T^a[\gamma_1 \circ_s \gamma_2{}^{-1}](t)$, which appears in the corresponding constraint equation, but the sum enters in the Poisson-bracket relations (2.13).

Given the above definition, the set of all $L^n[\gamma]$ has the following properties: (a) Its members form a closing Poisson-bracket algebra which we will call the L-algebra. (b) From knowledge of the values of all L-variables, and using the constraint equations (2.12), one can derive the values of all T-variables. (c) The reality conditions for the SU(2)-case are simply $L^{n*} = L^n$.

We will not give the general proofs of the statements (a) and (b) here, which are straightforward but somewhat tedious combinatorial exercises. The "small L-algebra" of L^0's and L^1's reads (c.f. eq. (2.13)):

$$\begin{aligned}
\{L[\alpha], L[\beta]\} &= 0, \\
\{L^a[\gamma](s), L[\eta]\} &= -\Delta^a[\gamma,\eta](s) L[\gamma \circ_s \eta], \\
\{L^a[\gamma](s), L^b[\eta](t)\} &= -\Delta^a[\gamma,\eta](s) L^b[\gamma \circ_s \eta](t) + \Delta^b[\eta,\gamma](t) L^a[\eta \circ_t \gamma](s).
\end{aligned} \tag{3.6}$$

It is easy to check that all these equations have matching symmetry properties of their left- and right-hand sides, under orientation reversal of the loops η and γ. The general expression for the Poisson bracket of two L-variables is

$$\begin{aligned}
&\{L^{a_1\cdots a_m}[\gamma](s_1,\ldots,s_m), L^{b_1\cdots b_n}[\eta](t_1,\ldots,t_n)\} \\
&\quad = -\sum_{i=1}^{m} \Delta^{a_i}[\gamma,\eta](s_i) \cdot L^{a_1\cdots \not{a}_i \cdots a_m b_1 \cdots b_n}[\gamma \circ_{s_i} \eta](s_1,\ldots,\not{s}_i,\ldots,s_m,t_1,\ldots,t_n) \\
&\qquad + \sum_{i=1}^{n} \Delta^{b_i}[\eta,\gamma](t_i) \cdot L^{a_1\cdots a_m b_1 \cdots \not{b}_i \cdots b_n}[\eta \circ_{t_i} \gamma](s_1,\ldots,s_m,t_1,\ldots,\not{t}_i,\ldots,t_n).
\end{aligned} \tag{3.7}$$

We have seen that the L-algebra still possesses some redundancy, since L-variables whose arguments differ only by the reversal of one or more simple loops are linearly dependent by virtue of eq. (3.3). If we consider only discrete subsets of the set of all loops, it is easy to eliminate the redundant L-variables: just fix an arbitrary orientation for all simple loops γ_i once and for all, and consider only L-variables that depend on the γ_i and their composites, but not on the $\gamma_i{}^{-1}$. The resulting subset still closes under Poisson brackets, and captures the full information about the theory. Note that, modulo factors of -1, the L-variables depend only on graphs, i.e. on the images of loops in Σ. If there are self-intersections,

more than one loop (even modulo reparametrizations) shares the same set of image points in Σ. However, there seems to be no way of getting rid of this "memory of orientation", as expressed in the factors $(-1)^{j_i+h_i}$ mentioned above, and hence no way of working with variables that just depend on loop graphs in Σ.

To get an idea of how the L^0-variables are connected with the actual entries of the holonomy matrices U, let us consider two simple loops α and β which intersect in some point. For simplicity we take the matrices to be SU(2)-valued, with standard parametrization

$$U_\alpha = \begin{pmatrix} \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \end{pmatrix}, \tag{3.8}$$

α_i real, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1$, and similarly for U_β. Hence we have

$$L[\alpha] \equiv T[\alpha] = \operatorname{Tr} U_\alpha = 2\alpha_1 ,$$

$$L[\beta] \equiv T[\beta] = \operatorname{Tr} U_\beta = 2\beta_1 ,$$

$$L[\alpha \circ \beta] = \tfrac{1}{2}\left(T[\alpha \circ \beta] - T\left[\alpha \circ \beta^{-1}\right]\right) = -2(\alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4) . \tag{3.9}$$

Thus $L[\alpha \circ \beta]$ measures "off-trace"-entries of holonomy matrices. Note that $\frac{1}{2}(T[\alpha \circ \beta] + T[\alpha \circ \beta^{-1}]) = 2\alpha_1\beta_1$, which we have discarded as a redundant variable, indeed does not contain any new information. For a composite loop $\alpha \circ \beta \circ \gamma$ (where β intersects both α and γ, but α and γ do not intersect each other), we find $L[\alpha \circ \beta \circ \gamma] = -2\beta_1(\alpha_2\gamma_2 + \alpha_3\gamma_3 + \alpha_4\gamma_4)$ etc.

In order to make the Poisson-bracket T- and L-algebras well defined and unambiguous, we have to think carefully about what class of loops we wish to employ, and whether we want to impose certain conditions on the structure constants Δ. One always has to check the consistency of such restrictions with the structure of the algebra and the imposition of the constraints (2.12). Note that the $\circ$-operation can be ambiguous if more than two lines meet in the point of cutting and rejoining, or if two or more marked points on a loop coincide. In such cases we can either require the corresponding Δ-functions in the Poisson-bracket algebra to vanish, or introduce some averaging procedure over all possibilities of rejoining. These issues have not been discussed carefully so far, and usually the algebra relations and constraint equations are considered only for nonsingular loop configurations, where loops are supposed to (self-)intersect at most in a discrete number of points. From this point of view also "squares" of loops, obtained by going round a given loop twice, are singular configurations. For example, the T^0-constraint in eq. (2.12) evaluated for $\gamma = \eta$ becomes

$$T[\gamma]T[\gamma] = T[\gamma \circ \gamma] + T\left[\gamma \circ \gamma^{-1}\right] . \tag{3.10}$$

The second term on the right-hand side is the T-value associated to the "unit" (= point) loop, i.e. the number 2. We can compute $T[\gamma^2]$ once $T[\gamma]$ is known, since they are not independent variables. Hence we will not include terms like $L[\gamma^n]$, $n \geqslant 2$, in the L-algebra, and for consistency set $\Delta^a[\gamma, \gamma] \equiv 0$. Also we do not include loops with partial self-overlaps, for example, so-called eyeglass loops, consisting of two separate loops, joined together by two line sections lying on top of each other [10, 19]. Their corresponding L-variables do not contain any independent information.

4. The formal group law

In the last section, we presented a complete set of variables for the reduced (with respect to the Gauss law constraints) phase space, which in addition did not contain any more quadratic constraints of type (2.12). The L-variables constitute a basic set of variables in the sense that all relevant quantities (for instance the hamiltonian) of the theory are in principle expressible in terms of them. The nonstandard feature of course is that this basic set does not consist of pairs of canonically conjugate variables. However, as explained in ref. [19], we consider this as a merit of the formulation and a necessary feature of a genuinely nonperturbative quantization.

Note that for Yang–Mills theory the L-variables are already observables in the literal sense of the word, whereas in gravity we still have to implement the diffeomorphism constraints, since the L-variables themselves are not invariant under spatial diffeomorphisms.

Given thus the primary importance of the classical L-algebra, we have to ask whether there is any systematic way of constructing a corresponding quantum theory. In such a quantum theory we expect to recover operator relations of the form $[\hat{L}^m, \hat{L}^n] \sim i\hbar \hat{L}^{m+n-1}$, and wave functionals $\Psi[L]$ depending on part but not all of the L^n (since the $\hat{L}^n$ do not all commute). Certainly life would be much easier if we had a corresponding L-group, for then we could just try to employ one of the group-theoretical quantization schemes (see, for example refs. [1, 12]), based on the ideas of Geometric Quantization, and the quantization would essentially consist in finding unitary, irreducible representations of this group.

Unfortunately, what we have said about the T- and L-algebras so far, does not even suffice to make them into well-defined, infinite-dimensional Lie algebras. The crucial missing ingredient is a topology on the space of loops, telling us what it means for a $T[\alpha]$ or $L[\alpha]$ to be "smooth in α", and when we regard two loops α and β as being "close to each other". This is a nontrivial task which we do not attempt to solve in the present work. We however have to keep in mind then that all expressions involving loop variables we write down remain in some sense formal. As long as we have not specified a topology, we are in fact treating the loop argument γ in $L^n[\gamma]$ as a discrete, and not continuous label or index. (For the

case of gravity, the discussion may simplify if we decide to work with diffeomorphism-invariant quantities at the classical level, see the comments in the next section.) Nevertheless we can use our knowledge of the structure constants of the L-algebra to construct an L-group law in terms of a formal power series, on whose "formality" we will comment in due course.

A great deal of information about an analytic group is already contained in its corresponding Lie algebra or, equivalently, its set of structure constants. A given set of structure constants can be used to construct a so-called formal group law [21], which at least locally describes a corresponding analytic group (if it exists). This method has been applied and proved very useful in the context of a group quantization scheme to find explicit local group laws for the affine Kac–Moody groups and the Virasoro group [2]. We will just give a short summary of the method.

For a commutative ring R with unit, consider the formal power series ring $\mathrm{R}[X]$, in n variables $X=(X_1,\dots,X_n)$. Let X' be another such set of n variables. A formal group law in n variables is an n-tuple $F=(F^1,\dots,F^n)$ of formal power series $F^\alpha \in \mathrm{R}[X,X']$, with $F(X,0)=X$, $F(0,X')=X'$:

$$\begin{aligned} F^\alpha(X',X) = {} & X'^\alpha + X^\alpha + \sum_{i,j} B^\alpha_{i,j} X'^i X^j + \sum_{i,j,l} \left(\frac{1}{2!} B^\alpha_{ij,l} X'^i X'^j X^l + \frac{1}{2!} B^\alpha_{i,jl} X'^i X^j X^l \right) \\ & + \sum_{i,j,l,m} \left(\frac{1}{3!} B^\alpha_{ijl,m} X'^i X'^j X'^l X^m + \frac{1}{2!}\frac{1}{2!} B^\alpha_{ij,lm} X'^i X'^j X^l X^m \right. \\ & \left. + \frac{1}{3!} B^\alpha_{i,jlm} X'^i X^j X^l X^m \right) + \dots, \qquad \alpha,i,j,\dots = 1,\dots,n, \end{aligned} \tag{4.1}$$

such that $F(X'',F(X',X)) = F(F(X'',X'),X)$. The X^i denote the n generators of an n-dimensional Lie algebra, which we can identify with n local group parameters. The constants B in the formal group law are determined by the structure constants C^α_{ij} of the Lie algebra, up to an isomorphism. From the associative law there follows a series of equations for the B's:

$$\begin{aligned} &\text{2nd order:} && B^\alpha_{i,j} - B^\alpha_{j,i} = C^\alpha_{ij}, \\ &\text{3rd order:} && B^\alpha_{i,jl} - B^\alpha_{ij,l} = \sum_p \left(B^\alpha_{p,l} B^p_{i,j} - B^\alpha_{i,p} B^p_{j,l} \right), \\ &\text{4th order:} && B^\alpha_{i,jlm} - B^\alpha_{ij,lm} = \sum_p \left(B^\alpha_{p,lm} B^p_{i,j} - B^\alpha_{i,p} B^p_{j,lm} - B^\alpha_{i,mp} B^p_{j,l} - B^\alpha_{i,lp} B^p_{j,m} \right), \\ & && B^\alpha_{ij,lm} - B^\alpha_{ijl,m} = \sum_p \left(B^\alpha_{p,m} B^p_{ij,l} + B^\alpha_{pj,m} B^p_{i,l} + B^\alpha_{pi,m} B^p_{j,l} - B^\alpha_{ij,p} B^p_{l,m} \right) \\ & && \vdots \end{aligned} \tag{4.2}$$

At the nth order ($n \geqslant 3$) there are $n-2$ independent equations. Given the C^{α}_{ij}, one can solve these equations order by order. Obviously the B's appearing on the left-hand sides of these equations are not uniquely determined, and a consistent "gauge" choice has to be made at each order.

In order to illustrate the method, we first discuss a finite-dimensional example, which also bears some resemblance to the loop case analyzed later. Consider the commutator algebra of two variables x and y,

$$[x,x]=0, \qquad [x,y]=x, \qquad [y,y]=0. \tag{4.3}$$

This may be realized as a Poisson bracket algebra on $\mathbb{R}^2$ with the standard symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}p$ and the identification $x \equiv x$ and $y \equiv xp$. The only nonvanishing structure constants are $C^x_{xy} = -C^x_{yx} = 1$. The first equation that has to be solved is

$$B^x_{x,y} - B^x_{y,x} = C^x_{xy}. \tag{4.4}$$

We present two different "gauge choices":

(a) Set $B^x_{x,y} = -B^x_{y,x} = \frac{1}{2}C^x_{xy} = \frac{1}{2}$, all other $B^{\alpha}_{i,j} \equiv 0$. The nonvanishing third-order equations are

$$B^x_{x,yy} - B^x_{xy,y} = \left(B^x_{x,y}\right)^2 = \tfrac{1}{4}, \qquad B^x_{y,yx} - B^x_{yy,x} = -\left(B^x_{y,x}\right)^2 = -\tfrac{1}{4}. \tag{4.5}$$

We set $B^x_{yy,x} = B^x_{x,yy} = \frac{1}{4}$, all other $B^{\alpha}_{ij,k}$ and $B^{\alpha}_{i,jk} \equiv 0$. The nonvanishing fourth-order equations are

$$B^x_{x,yyy} - B^x_{xy,yy} = B^x_{x,yy}B^x_{x,y} = \tfrac{1}{8}, \qquad B^x_{yy,yx} - B^x_{yyy,x} = -B^x_{yy,x}B^x_{y,x} = \tfrac{1}{8}. \tag{4.6}$$

Now set $B^x_{x,yyy} = \frac{1}{8}$, $B^x_{yyy,x} = -\frac{1}{8}$ and all fourth-order B's $\equiv 0$, and proceed similarly for the higher orders. Then we find for the formal group law

$$\begin{aligned} x'' &= x' + x + \tfrac{1}{2}x'y - \tfrac{1}{2}y'x + \tfrac{1}{8}(y')^2x + \tfrac{1}{8}x'y^2 - \tfrac{1}{48}(y')^3x + \tfrac{1}{48}x'y^3 + \text{h.o.} \\ &= \mathrm{e}^{y/2}x' + \mathrm{e}^{-y'/2}x, \\ y'' &= y' + y. \end{aligned} \tag{4.7}$$

(b) Now set $B^x_{x,y} = 1$, $B^x_{y,x} = 0$, and all other $B^{\alpha}_{i,j} \equiv 0$. As in (a), we can make the following choices for nonvanishing B's at higher orders: $B^x_{x,yy} = 1$, $B^x_{x,yyy} = 1$, etc. Then we get the following version for the group law:

$$x'' = \mathrm{e}^y x' + x, \qquad y'' = y' + y \tag{4.8}$$

(c) Similarly, for an initial choice $B^x_{x,y}=0$, $B^x_{y,x}=-1$, etc., we arrive at the group law

$$x''=x'+\mathrm{e}^{-y'}x\,,\qquad y''=y'+y\,. \tag{4.9}$$

All these local group laws are equivalent, at the algebra level they correspond to trivial redefinitions of the generators. In the group laws we can add up the contributions of all orders, since they form convergent series for all values of x and y. This way we get a genuine group law with global parameters x and y, and the group is the semi-direct product $\mathbb{R}\circledS\mathbb{R}$.

In the following we will apply the same algorithm to the small L-algebra, i.e. the subalgebra of the L-algebra formed by the loop variables L^0 and L^1. Like in the finite-dimensional example, we have a semidirect-product structure, with the nonabelian subgroup $\mathscr{L}^1$ of the $L^a[\gamma](s)$ acting on the abelian subgroup $\mathscr{L}^0$ of the $L[\alpha]$. We neither have a topology, nor an integration measure defined on loop space. The higher-order terms in the formal power series for the group law we are going to write down contain integrals over all of loop space and are therefore ill-defined. Even if we had a well-defined integration, the formal power series would be unlikely to converge, because of the infinite dimensionality of the spaces involved. Nevertheless, from the formal expansion we can still read off the structure of each order in terms of L^0- and L^1-variables.

In what follows we will treat the loop variable γ as a discrete index, use discrete sums over loops, delta-functions with discrete loop arguments, and similarly treat marked points on a given loop as being a discrete set of objects. This is certainly justified if we consider the L-algebra restricted to a finite or countable number of loops, as may happen either in a lattice formulation, or (in gravity) if we work with diffeomorphism equivalence classes of loops rather than individual loops. If we use some truly infinite-dimensional version of the full L-algebra, it is understood that all these discrete objects will have to be replaced with continuous analogues.

The structure constants of the small L-algebra are

$$\begin{aligned}
C^{(\lambda)}_{(\gamma,s,a)(\eta)} &= -\Delta_a[\gamma,\eta](s)\delta(\gamma\circ_s\eta,\lambda)\,,\\
C^{(\mu,x,k)}_{(\gamma,s,a)(\eta,t,b)} &= \Delta_b[\eta,\gamma](t)\delta(\gamma\circ_t\eta,\mu)\delta(u(s),x)\delta_a{}^k\\
&\quad -\Delta_a[\gamma,\eta](s)\delta(\eta\circ_s\gamma,\mu)\delta(u(t),x)\delta_b{}^k\,.
\end{aligned} \tag{4.10}$$

An index (η) corresponds to a variable $L[\eta]$, an index (γ,s,a) to a variable $L^a[\gamma](s)$, and we have introduced delta-functions depending on loops and points on loops, with obvious meaning. Spatial indices are raised and lowered with the three-metric q.

We have seen that an "asymmetrical" choice for the second-order B's led to a simple group law in the finite-dimensional example, and we will now try the ansatz

$$B^{(\lambda)}_{(\gamma,s,a),(\eta)} = C^{(\lambda)}_{(\gamma,s,a)(\eta)}, \qquad B^{(\lambda)}_{(\eta),(\gamma,s,a)} = 0,$$

$$B^{(\mu,x,k)}_{(\gamma,s,a),(\eta,t,b)} = -B^{(\mu,x,k)}_{(\eta,t,b),(\gamma,s,a)} = \tfrac{1}{2}C^{(\mu,x,k)}_{(\gamma,s,a)(\eta,t,b)}, \qquad \text{all other } B^{\alpha}_{i,j} \equiv 0. \quad (4.11)$$

With this choice the only nonvanishing third-order equations are

$$B^{(\alpha)}_{(\gamma,s,a),(\eta,t,b)(\beta)} - B^{(\alpha)}_{(\gamma,s,a)(\eta,t,b),(\beta)}$$
$$= \sum_{(\mu,x,k)} B^{(\alpha)}_{(\mu,x,k),(\beta)} B^{(\mu,x,k)}_{(\gamma,s,a),(\eta,t,b)} - \sum_{(\lambda)} B^{(\alpha)}_{(\gamma,s,a),(\lambda)} B^{(\lambda)}_{(\eta,t,b),(\beta)}$$

$$B^{(\lambda,v,c)}_{(\mu,x,k),(\gamma,s,a)(\eta,t,b)} - B^{(\lambda,v,c)}_{(\mu,x,k)(\gamma,s,a),(\eta,t,b)}$$
$$= \sum_{(\rho,y,d)} \left(B^{(\lambda,v,c)}_{(\rho,y,d),(\eta,t,b)} B^{(\rho,y,d)}_{(\mu,x,k),(\gamma,s,a)} - B^{(\lambda,v,c)}_{(\mu,x,k),(\rho,y,d)} B^{(\rho,y,d)}_{(\gamma,s,a),(\eta,t,b)} \right). \quad (4.12)$$

For the third-order B's we choose

$$B^{(\alpha)}_{(\gamma,s,a)(\eta,t,b),(\beta)} = -\tfrac{1}{2} \sum_{(\mu,x,k)} C^{(\alpha)}_{(\mu,x,k)(\beta)} C^{(\mu,x,k)}_{(\gamma,s,a)(\eta,t,b)} + \sum_{(\lambda)} C^{(\alpha)}_{(\gamma,s,a)(\lambda)} C^{(\lambda)}_{(\eta,t,b)(\beta)}$$
$$= \tfrac{1}{2} \sum_{(\lambda)} \left(C^{(\alpha)}_{(\lambda)(\eta,t,b)} C^{(\lambda)}_{(\beta)(\gamma,s,a)} + C^{(\alpha)}_{(\gamma,s,a)(\lambda)} C^{(\lambda)}_{(\eta,t,b)(\beta)} \right),$$

$$B^{(\alpha)}_{(\gamma,s,a),(\eta,t,b)(\beta)} = 0,$$

$$B^{(\lambda,v,c)}_{(\mu,x,k),(\gamma,s,a)(\eta,t,b)} = \tfrac{1}{4} \sum_{(\rho,y,d)} \left(C^{(\lambda,v,c)}_{(\rho,y,d)(\eta,t,b)} C^{(\rho,y,d)}_{(\mu,x,k)(\gamma,s,a)} + C^{(\lambda,v,c)}_{(\rho,y,d)(\gamma,s,a)} C^{(\rho,y,d)}_{(\mu,x,k)(\eta,t,b)} \right),$$

$$B^{(\lambda,v,c)}_{(\mu,x,k)(\gamma,s,a),(\eta,t,b)} = \tfrac{1}{4} \sum_{(\rho,y,d)} \left(C^{(\lambda,v,c)}_{(\mu,x,k)(\rho,y,d)} C^{(\rho,y,d)}_{(\gamma,s,a)(\eta,t,b)} + C^{(\lambda,v,c)}_{(\gamma,s,a)(\rho,y,d)} C^{(\rho,y,d)}_{(\mu,x,k)(\eta,t,b)} \right),$$

$$\text{all other } B^{\alpha}_{i,jk} \text{ and } B^{\alpha}_{ij,k} = 0. \quad (4.13)$$

(The second equality of the first equation in (4.13) follows from the Jacobi identity, see below.) Before writing down the group law up to this order, we will have a look at the Jacobi identities. For the given set of structure constants, we get just two

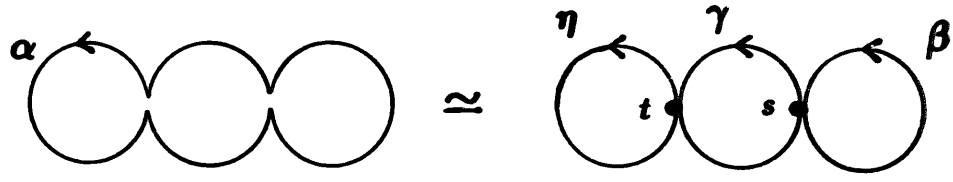


Fig. 1. Loops $\alpha, \eta, \gamma, \beta$ for which the Jacobi identity does not vanish identically. Further explanation is given in the text.

different nontrivial identities:

$$\sum_{(\lambda)} C^{(\alpha)}_{(\lambda)(\eta,t,b)} C^{(\lambda)}_{(\beta)(\gamma,s,a)} + \sum_{(\mu,x,k)} C^{(\alpha)}_{(\mu,x,k)(\beta)} C^{(\mu,x,k)}_{(\gamma,s,a)(\eta,t,b)}$$

$$+ \sum_{(\lambda)} C^{(\alpha)}_{(\lambda)(\gamma,s,a)} C^{(\lambda)}_{(\eta,t,b)(\beta)} = 0,$$

$$\sum_{(\rho,y,d)} \Big(C^{(\lambda,v,c)}_{(\rho,y,d)(\eta,t,b)} C^{(\rho,y,d)}_{(\mu,x,k)(\gamma,s,a)} + C^{(\lambda,v,c)}_{(\rho,y,d)(\mu,x,k)} C^{(\rho,y,d)}_{(\gamma,s,a)(\eta,t,b)}$$

$$+ C^{(\lambda,v,c)}_{(\rho,y,d)(\gamma,s,a)} C^{(\rho,y,d)}_{(\eta,t,b)(\mu,x,k)} \Big) = 0. \quad (4.14)$$

In order to interpret these identities, one has to evaluate them for fixed indices, for example, for the first identity for fixed (α), (β), (γ, s, a) and (η, t, b). Of course, only for special choices of these loops the Jacobi identity does not vanish identically. We will just give one example, all other cases are in principle the same. Consider the loops given in fig. 1, where η, γ and β intersect each other in the marked points s and t as indicated, and it is understood that α has been constructed by joining the three loops η, γ and β in the usual fashion. Only two terms in the Jacobi identity are nonzero, and the first equation in (4.14) becomes

$$-\Delta_b[\eta,\lambda](t)\Delta_a[\gamma,\beta](s)\delta(\eta \circ_t \lambda, \alpha)\delta(\gamma \circ_s \beta, \lambda)$$

$$+ \Delta_a[\mu,\beta](x)\Delta_b[\eta,\gamma](t)\delta(\mu \circ_x \beta, \alpha)\delta(\gamma \circ_t \eta, \mu)\delta(u(s), x) = 0. \quad (4.15)$$

This just expresses the associativity of joining loops, i.e. $\eta \circ_t (\gamma \circ_s \beta) = (\eta \circ_t \gamma) \circ_s \beta$. A similar analysis applies to the other Jacobi identity. For fixed (λ, v, c), (η, t, b), (μ, x, k) and (γ, s, a), for example, as in fig. 2, again only two terms survive in the

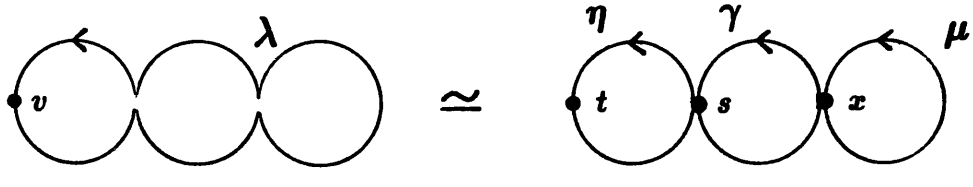


Fig. 2. Same as fig. 1 for loops λ, η, γ and μ.

Jacobi identity, which now tells us that we get the same contribution, no matter in what order we joined the loops together to construct λ: $\eta \circ_s (\gamma \circ_x \mu) = (\eta \circ_s \gamma) \circ_x \mu$. It is now straightforward to write down the first few orders of the formal group law,

$$
\begin{aligned}
L[\alpha]'' &= L[\alpha]' + L[\alpha] + \sum_{\gamma,\eta} C^{(\alpha)}_{(\gamma,s,a)(\eta)} L^a[\gamma]'(s) L[\eta] \\
&\quad + \tfrac{1}{2} \sum_{\gamma,\eta,\beta} \tfrac{1}{2} \sum_{\lambda} \left(C^{(\alpha)}_{(\lambda)(\eta,t,b)} C^{(\lambda)}_{(\beta)(\gamma,s,a)} + C^{(\alpha)}_{(\gamma,s,a)(\lambda)} C^{(\lambda)}_{(\eta,t,b)(\beta)} \right) \\
&\quad \times L^a[\gamma]'(s) L^b[\eta]'(t) L[\beta] + \text{h.o.}, \\
L^c[\lambda]''(v) &= L^c[\lambda]'(v) + L^c[\lambda](v) + \tfrac{1}{2} \sum_{\gamma,\eta} C^{(\lambda,v,c)}_{(\gamma,s,a)(\eta,t,b)} L^a[\gamma]'(s) L^b[\eta](t) \\
&\quad + \tfrac{1}{2} \sum_{\mu,\gamma,\eta} B^{(\lambda,v,c)}_{(\mu,x,k),(\gamma,s,a)(\eta,t,b)} L^k[\mu]'(x) L^a[\gamma](s) L^b[\eta](t) \\
&\quad + \tfrac{1}{2} \sum_{\mu,\gamma,\eta} B^{(\lambda,v,c)}_{(\mu,x,k)(\gamma,s,a),(\eta,t,b)} L^k[\mu]'(x) L^a[\gamma]'(s) L^b[\eta](t) + \text{h.o.}
\end{aligned}
\tag{4.16}
$$

After some algebra we obtain the final form of the group law

$$
\begin{aligned}
L[\alpha]'' &= L[\alpha]' + L[\alpha] - \sum_{\gamma,\eta} \Delta_a[\gamma,\eta](s) \delta(\gamma \circ_s \eta, \alpha) L^a[\gamma]'(s) L[\eta] \\
&\quad + \tfrac{1}{2} \sum_{\gamma,\eta,\beta} \Delta_b[\eta, \gamma \circ_s \beta](t) \Delta_a[\gamma,\beta](s) \delta(\eta \circ_t \lambda, \alpha) \\
&\quad \times L^a[\gamma]'(s) L^b[\eta]'(t) L[\beta] + \text{h.o.}, \\
L^c[\lambda]''(v) &= L^c[\lambda]'(v) + L^c[\lambda](v) + \tfrac{1}{2} \sum_{\gamma,\eta} \Delta_b[\eta,\gamma](t) \delta(\gamma \circ_t \eta, \lambda) \delta(u(s), v) \\
&\quad \times \left(L^c[\gamma]'(s) L^b[\eta](t) - L^b[\eta]'(t) L^c[\gamma](s) \right) \\
&\quad - \tfrac{1}{4} \sum_{\mu,\gamma,\eta} \Delta_a[\gamma,\mu](s) \Delta_b[\eta, \gamma \circ_s \mu](t) \delta(\eta \circ_t (\gamma \circ_s \mu), \lambda) \\
&\quad \times \delta(u(x), v) \big(L^c[\mu]'(x) L^a[\gamma](s) L^b[\eta](t) \\
&\quad + L^b[\eta]'(t) L^a[\gamma](s) L^c[\mu](x) - 2 L^c[\mu]'(x) L^b[\eta]'(t) L^a[\gamma](s) \\
&\quad - 2 L^a[\gamma]'(s) L^c[\mu](x) L^b[\eta](t) + L^a[\gamma]'(s) L^b[\eta]'(t) L^c[\mu](x) \\
&\quad + L^a[\gamma]'(s) L^c[\mu]'(x) L^b[\eta](t) \big) + \text{h.o.}
\end{aligned}
\tag{4.17}
$$

It is understood that a group element is given by the assignment of a complex number to each loop α and a triple of complex numbers to each pair $(\lambda, \lambda(v))$ of a loop λ and a marked point v on λ. Thinking of the set of loops as being discrete, it makes sense to evaluate the group law of the functional $L[\alpha]$ at some fixed loop α_0, which we will denote by $L[\alpha]|_{\alpha_0}$. The group law for an L-variable, evaluated at a loop α_0 or a loop with a marked point, $(\gamma_0, \gamma_0(s))$, depends on the number of self-intersections of the loop. Let us first look at the group law of an $L[\alpha]|_{\alpha_0}$.
(i) If α_0 is a simple loop, the group law reduces to

$$L[\alpha]''|_{\alpha_0} = L[\alpha]'|_{\alpha_0} + L[\alpha]|_{\alpha_0}. \tag{4.18}$$

(ii) If α_0 is a composite loop with one self-intersection, we have

$$L[\alpha]''|_{\alpha_0} = L[\alpha]'|_{\alpha_0} + L[\alpha]|_{\alpha_0} - \sum_{\gamma,\eta} \Delta_a[\gamma,\eta](s)\delta(\gamma \circ_s \eta, \alpha) L^a[\gamma]'(s) L[\eta]|_{\alpha=\alpha_0}. \tag{4.19}$$

The sum contains two terms, because the loop γ with marked point may have been either on the "right"- or the "left-hand" side. Note that even if the δ-function is nonvanishing for a term in the sum, there is still no contribution if $\tilde{\sigma}^a(s)$ is perpendicular to the tangent vector $\dot{\eta}^a(s)$, i.e. if $q_{ab}(s)\tilde{\sigma}^a(s)\dot{\eta}^b(s) = 0$.
(iii) If α_0 has two self-intersections, also the next, third-order term in the group law is nonzero.

We just illustrate the higher-order contributions graphically in figs. 3 and 4. Due to the configurations in fig. 3, there are four contributions in the second order term. The configurations in fig. 4 give four contributions in the third order term. For composite loops α_0 with more, say, $(n-1)$ self-intersections, the formal group law terminates with the nth-order contribution. The corresponding coefficients

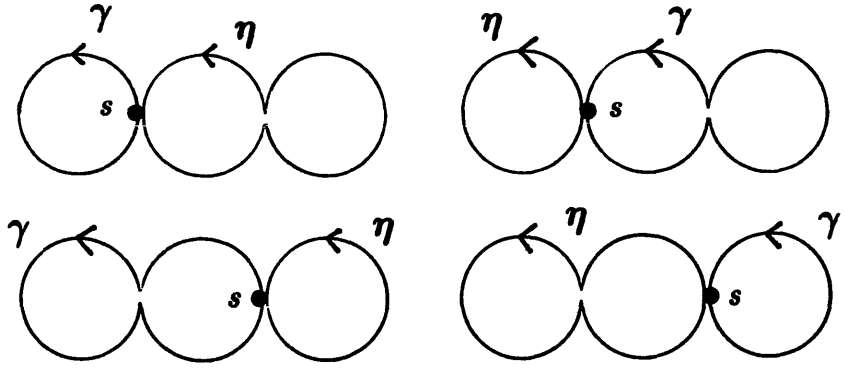


Fig. 3. Configurations which contribute to the group law in the second order. α has two self-intersections.

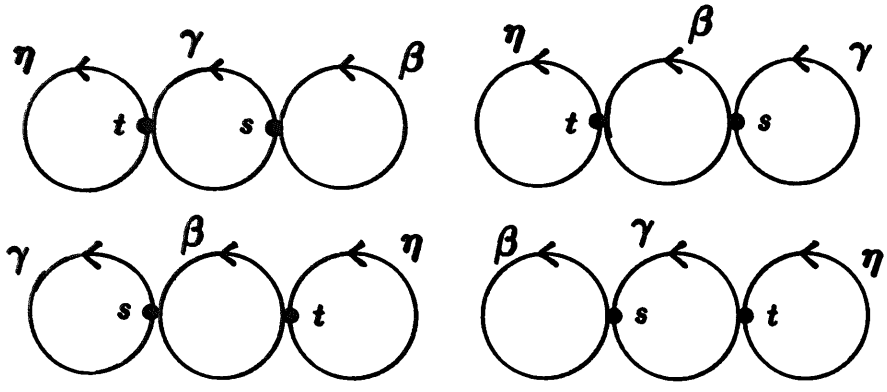


Fig. 4. Same as fig. 3 in the third order.

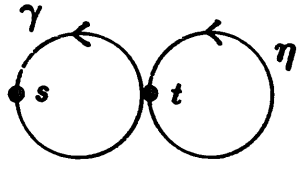


Fig. 5. Configuration which gives a second-order contribution for a loop λ_0 with one self-intersection.

and the combinatorics can be calculated in a straightforward manner. At the ith order, the number of configurations that contribute is given by the number of different ways of writing α_0 as the loop product of i simple or composite loops.

For the L^1-variables we proceed in a similar fashion, and the same remarks concerning the number of self-intersections do apply. For a loop λ_0 with one self-intersection, we get just first- and second-order contributions, the latter of which contains just one term from the configuration in fig. 5. If λ_0 has two self-intersections, the second-order contributions come from configurations given in fig. 6 and the third-order contribution comes from the configuration in fig. 7. The combinatorics we get is different, due to the additional marked point. Note that the structure of the $\mathscr{L}^0$-part of the group law is completely analogous to the corresponding term in eq. (4.9) in the finite-dimensional example. Schematically we can write

$$L^{0''} = L^{0'} + \exp(-\Delta \cdot \delta(\ ) \cdot L^{1'}) L^0 . \tag{4.20}$$

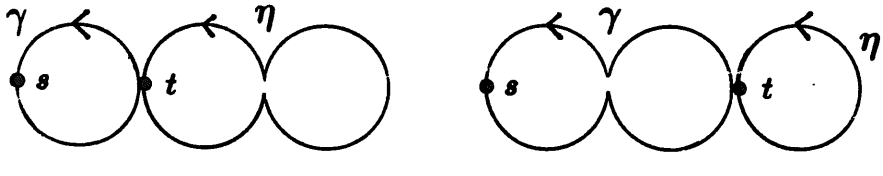


Fig. 6. Configurations which give second-order contributions for λ_0 with two self-intersections.

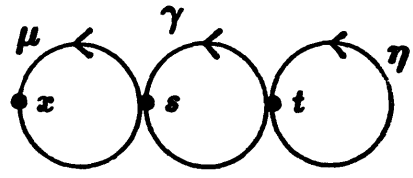


Fig. 7. Same as fig. 6 for the third-order contribution.

The structure of the higher-order contributions in the group law follows from straightforward diagrammatic rules. Also there are no problems in principle to extend the group law to L^n's depending on loops with two and more marked points, but this is not directly relevant to the present work.

5. The quantum theory

Taking the group law (4.17) as our starting point, we will now derive a quantum theory, in the spirit of the remarks made at the beginning of sect. 4. Since the small L-group is a semidirect-product group, $\mathscr{L}^0 \circledS \mathscr{L}^1$, we will attempt to construct natural unitary representations of this group, using well-known techniques from the Mackey theory of induced representations [12, 14]. The representation we find will allow as well for a natural quantization of the variables L^n, $n \geqslant 2$. All provisos we have made about the formality of the L-variables apply of course similarly to the quantum theory, and we will continue treating loop labels as being discrete.

An ultimate quantum theory will have to involve a more careful treatment of the underlying loop spaces, and therefore at this intermediate stage we will not concern ourselves with the search of well-defined Hilbert space structures. Anyway their precise form will depend both on the theory and on the gauge group. For Yang–Mills theory, in the context of a lattice formulation, we will indicate later how they may be made rigorous. For gravity, the definition of a scalar product is usually disposed till after imposing all the constraints. One also has to analyze to what extent concepts like "the orbit structure under the $\mathscr{L}^1$-action" are still meaningful when we are dealing with a formal group.

Nevertheless, we must emphasize that our results are no more formal than those obtained in ref. [19]. In contrast to the representation given there, we find a quantum representation of the full L-algebra (the difference between L- and T-algebra does not matter here) which closes without any additional higher-order terms in the commutation relations. We consider it of utmost importance to make maximal use of the group and algebra structure in the construction of the quantum theory, as we do. This, together with appropriately defined limits as the number of loops increases and their size decreases, offers a new way to finding a rigorous quantization in this nonperturbative approach.

Guided by our knowledge of the representation theory of semidirect-product groups, we expect that the quantum theory will involve wave functionals defined on some appropriate dual $\bar{\mathscr{L}}^0$ of the space $\mathscr{L}^0$, and the study of the induced action of $\mathscr{L}^1$ on $\bar{\mathscr{L}}^0$. The scalar product of an element $\bar{L} \in \bar{\mathscr{L}}^0$ and some $L' \in \mathscr{L}^0$ we define to be

$$\langle \bar{L}, L' \rangle := \sum_{\alpha} \bar{L}^*[\alpha] L'[\alpha], \tag{5.1}$$

the star denoting complex conjugation. This gives us a natural isomorphism between elements of $\mathscr{L}^0$ and $\bar{\mathscr{L}}^0$, as long as they are chosen in such a way as to make the sum in eq. (5.1) convergent.

Let us first look at the action of a "one-parameter subgroup" $l_{L^a[\gamma_0](s)}$ on $\mathscr{L}^0$, for some fixed loop γ_0. From the group law (4.17) we deduce

$$l_{L^a[\gamma_0](s)}(L[\alpha]) = L[\alpha] - \sum_{\eta} \Delta_a[\gamma_0, \eta](s) \delta(\gamma_0 \circ_s \eta, \alpha) L^a[\gamma_0](s) L[\eta]. \tag{5.2}$$

The sum breaks off here since we do not consider configurations containing squares γ_0^2 of loops. This action can only be nontrivial if L depends on loops α which contains γ_0 in the sense that $\alpha = \gamma_0 \circ \eta$. Note also that the action is necessarily trivial if $L[\alpha]$ depends only on simple loops α.

The induced contragredient action on $\bar{\mathscr{L}}^0$ is

$$l'_{L^a[\gamma_0](s)}(\bar{L}[\alpha]) = \bar{L}[\alpha] - \sum_{\lambda} \Delta_a[\gamma_0, \alpha](s) \delta(\alpha, \lambda \circ'_s \gamma_0) L^{a*}[\gamma_0](s) \bar{L}[\lambda], \tag{5.3}$$

where $\lambda \circ'_s \gamma_0$ denotes the "γ_0-shrink" of λ, i.e. the loop β that results from shrinking γ_0 in $\lambda = \beta \circ \gamma_0$ to a point. Action (5.3) can only be nontrivial if $\bar{L}[\alpha]$ depends on loops α which go through point s, but α need not have any self-intersections. The first terms of the full contragredient action of $\mathscr{L}^1$ on $\bar{\mathscr{L}}^0$ are

$$\begin{aligned} \bar{L}[\alpha] \to \bar{L}[\alpha] &- \sum_{\eta, \gamma} \Delta_a[\gamma, \alpha](s) \delta(\alpha, \eta \circ' \gamma) L^{a*}[\gamma](s) \bar{L}[\eta] \\ &+ \tfrac{1}{2} \sum_{\beta, \gamma, \eta} \Delta_b[\eta, \gamma \circ \alpha](t) \Delta_a[\gamma, \alpha](s) \delta((\beta \circ' \eta) \circ' \gamma, \alpha) \\ &\times L^{a*}[\gamma](s) L^{b*}[\eta](t) \bar{L}[\beta] + \text{h.o.}, \end{aligned} \tag{5.4}$$

which follows from the corresponding action of $\mathscr{L}^1$ on $\mathscr{L}^0$:

$$L[\alpha] \to L[\alpha] - \sum_{\eta,\gamma} \Delta_a[\gamma,\eta](s)\delta(\gamma\circ\eta,\alpha)L^a[\gamma](s)L[\eta]$$

$$+\tfrac{1}{2}\sum_{\gamma,\eta,\beta} \Delta_b[\eta,\gamma\circ\beta](t)\Delta_a[\gamma,\beta](s)\delta(\eta\circ(\gamma\circ\beta),\alpha)$$

$$\times L^a[\gamma](s)L^b[\eta](t)L[\beta] + \text{h.o.} \tag{5.5}$$

Note that the actual numerical values of the terms arising from the $\mathscr{L}^1$-action are irrelevant, the important thing is whether they vanish or not. We also see that the contragredient $\mathscr{L}^1$-action on $\bar{\mathscr{L}}^0$ is actually different from the action on $\mathscr{L}^0$, and in what sense the two actions are dual to each other.

After these preliminaries we are now ready to define a "unitary" representation of the group $\mathscr{L}^0 \circledS \mathscr{L}^1$. State vectors will be given by functionals $\Psi[\bar{L}]$ on the dual of $\mathscr{L}^0$, i.e. $\|\Psi[\bar{L}]\|^2$ represents a probability density of configurations $\bar{L}[\alpha]$. A one-parameter group U of "rotations" $\in \mathscr{L}^1$ is defined by

$$(U(L^a[\gamma_0](s))\Psi)[\bar{L}[\alpha]] := \Psi\left[l'_{L^a[\gamma_0](s)}\bar{L}[\alpha]\right], \tag{5.6}$$

and a one-parameter group V of translations $\in \mathscr{L}^0$ by

$$(V(L[\beta_0])\Psi)[\bar{L}[\alpha]] := \mathrm{e}^{-i\langle \bar{L}[\alpha], L[\beta_0]\rangle}\,\Psi[\bar{L}[\alpha]]\,. \tag{5.7}$$

We have assumed the existence of some invariant Hilbert space measure. The corresponding self-adjoint operators are found to be

$$\hat{L}^a[\gamma_0](s)\Psi[L[\alpha]] = -i\sum_{\beta}\Delta^a[\gamma_0,\beta](s)L[\gamma_0\circ_s\beta]\frac{\delta}{\delta L[\beta]}\Psi[L[\alpha]]\,, \tag{5.8}$$

$$\hat{L}[\beta_0]\Psi[L[\alpha]] = L[\beta_0]\Psi[L[\alpha]]\,, \tag{5.9}$$

where we have identified elements $\bar{L}[\alpha]$ with their duals in $\mathscr{L}^0$. Hence in this representation the generators of $\mathscr{L}^0$ are diagonal and represented by multiplication with L^0. It is straightforward to compute the commutator algebra of these quantities:

$$[\hat{L}[\alpha_1],\hat{L}[\alpha_2]] = 0\,,$$

$$[\hat{L}^a[\gamma](s),\hat{L}[\alpha]] = -i\hbar\Delta^a[\gamma,\alpha](s)\hat{L}[\gamma\circ_s\alpha]\,,$$

$$[\hat{L}^a[\gamma](s),\hat{L}^b[\eta](t)] = -i\hbar\Delta^a[\gamma,\eta](s)\hat{L}^b[\gamma\circ_s\eta](t)$$

$$+i\hbar\Delta^b[\eta,\gamma](t)\hat{L}^a[\eta\circ_t\gamma](s)\,, \tag{5.10}$$

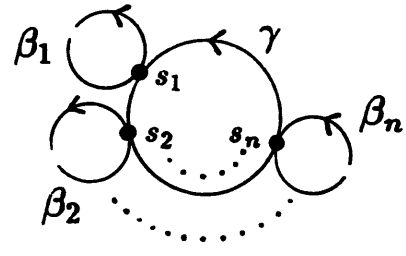
which is just the quantum version of the classical Poisson-bracket algebra, with factors of $i\hbar$. Note that no factor ordering problems occur in the definition of $\hat{L}^a[\gamma_0](s)$, since the loop $\gamma_0 \circ_s \beta$ appearing in the argument of L is always bigger than, i.e. different from β, which means that $L[\gamma_0 \circ_s \beta]$ and $\delta/\delta L[\beta]$ always commute. Comparison with the canonical commutator $[\hat{A}, \hat{\tilde{\sigma}}] \sim i\hbar$ of the local quantum operators of the Ashtekar approach fixes the scale in our L-representation in such a way that factors of $\hbar$ enter the commutation relations (5.10) as indicated. This corresponds to the insertion of a factor $\hbar$ on the right-hand side of eq. (5.8).

We will now show that the higher L^n-observables ($n \geqslant 2$) possess natural quantum analogues on the representation space of wave functionals $\Psi[L]$. It turns out that a straightforward generalization of the definitions (5.8) and (5.9) leads to the correct results. We define

$$\hat{L}^{a_1 \dots a_n}[\gamma](s_1, \dots, s_n) = (-i\hbar)^n \sum_{\beta_1 \dots \beta_n} \Delta^{a_1}[\gamma, \beta_1](s_1) \dots \Delta^{a_n}[\gamma, \beta_n](s_n)$$
$$\times L\big[\big(\dots\big(\big(\gamma \circ_{s_n} \beta_n\big) \circ_{s_{n-1}} \beta_{n-1}\big) \dots\big) \circ_{s_1} \beta_1\big] \frac{\delta}{\delta L[\beta_n]} \cdots \frac{\delta}{\delta L[\beta_1]} \,. \tag{5.11}$$

In order to compute the commutator algebra of these objects, it is useful to introduce a diagrammatic notation where $\hat{L}^{a_1 \dots a_n}[\gamma](s_1, \dots, s_n)$ is represented by

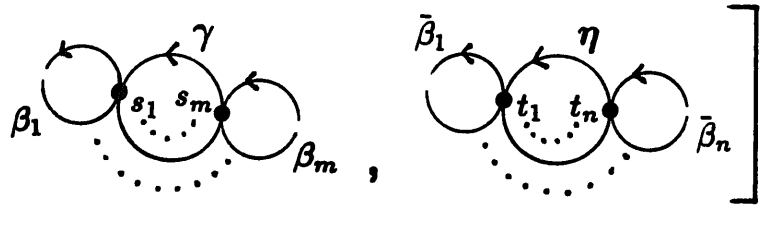
$$(-i\hbar)^n \Delta^{a_1}[\gamma, \beta_1](s_1) \dots \Delta^{a_n}[\gamma, \beta_n](s_n) \cdot$$



with the "open" loops β_i representing the differentials $\delta/\delta L[\beta_i]$. The commutator of two $\hat{L}$-operators is given by

$$\big[\hat{L}^{a_1 \dots a_m}[\gamma](s_1, \dots, s_m), \hat{L}^{b_1 \dots b_n}[\eta](t_1, \dots, t_n)\big]$$
$$\cong (-i\hbar)^{m+n} \Delta^{a_1}[\gamma, \beta_1](s_1) \dots \Delta^{a_m}[\gamma, \beta_m](s_m) \Delta^{b_1}[\eta, \bar{\beta}_1](t_1) \dots \Delta^{b_n}[\eta, \bar{\beta}_n](t_n)$$

$$\times \Bigg[\quad , \quad \Bigg] \tag{5.12}$$



The commutator of the two diagrams is evaluated in the following way:

(i) Start with the diagram on the left-hand side in the commutator bracket. In the point s_1, substitute the open loop β_1 by the closed loop η from the right-hand side, which still has all the open loops $\bar{\beta}_1$ to $\bar{\beta}_n$ glued to it. Substitute β_1 by η in the structure constants multiplying this term. The result is a diagram consisting of a loop $\gamma \circ_{s_1} \eta$ with $n+m-1$ open loops, $\beta_2, \ldots, \beta_m, \bar{\beta}_1, \ldots, \bar{\beta}_n$, joined to it, multiplied with a factor of $(-i\hbar)^{m+n}\Delta^{a_1}[\gamma, \eta](s_1) \ldots \Delta^{b_n}[\eta, \bar{\beta}_n](t_n)$.

(ii) Repeat the process for all open loops β_i and sum over all the contributions.

(iii) Consider then the diagram on the right-hand side in the commutator bracket. In the point t_1, substitute the open loop $\bar{\beta}_1$ by the closed loop γ, with all open loops $\beta_1, \ldots, \beta_m$ joined to γ as before. Substitute $\bar{\beta}_1$ by γ in the structure constants.

(iv) Repeat this for all open loops $\bar{\beta}_i$, sum over all the contributions and subtract them from the expression obtained in (ii).

In principle there are not only contributions from open loops acting on closed loops, as we have described, but also from open loops acting on open loops. However, these contributions always come in pairs with opposite sign and cancel each other. Using these diagrammatic rules it is easy to show that the commutator algebra of the quantum $\hat{L}$-operators is precisely the same as the corresponding classical Poisson-bracket algebra (3.7), apart from a factor of $i\hbar$ appearing everywhere on the right-hand side. It is remarkable that, at least at this formal level, one has found a consistent, anomaly-free quantization of an infinite-dimensional algebra. In our derivation it was not important that we had eliminated the constraints in eq. (2.12), a similar result is obtained if one quantizes the original Poisson-algebra of the T-variables.

The absence of factor-ordering problems is traced back to the special form of the quantum operators. A generic $\hat{L}^n$-operator consists of terms of the form

$$L\left[\left(\ldots\left(\gamma \circ_{s_n} \beta_n\right) \circ \ldots\right) \circ_{s_1} \beta_1\right] \delta/\delta L[\beta_n] \ldots \delta/\delta L[\beta_1],$$

and clearly the loop $(\ldots(\gamma \circ_{s_n} \beta_n) \circ \ldots) \circ_{s_1} \beta_1$ appearing in the argument of L is always bigger than all the loops $\beta_1, \ldots, \beta_n$ joined together. It is possible to construct finite-dimensional algebras with similar properties. On the phase space $\mathbb{R}^{2n}$, with coordinates (x_i, p_i), $i = 1, \ldots, n$, consider the following set of polynomial functions:

$$\begin{aligned}
&x_i, \\
&x_i p_j, && i > j, \\
&x_i p_j p_k, && i > j+k, \\
&x_i p_j p_k p_l, && i > j+k+l, \\
&\vdots \quad && i, j, \ldots \in \{1, \ldots, n\}.
\end{aligned} \tag{5.13}$$

Objects of this type form a finite-dimensional closing (under Poisson brackets) subalgebra of all functions on $\mathbb{R}^{2n}$, for any n. If one quantizes this algebra on the usual representation space $L^2(\mathbb{R}^n)$, i.e. according to the rules $\hat{x}_i \equiv x_i$, $\hat{p}_i \equiv -i\hbar\partial/\partial x_i$, one also obtains an anomaly-free quantum algebra without any factor-ordering ambiguities. It is straightforward to check that all these algebras are solvable and nilpotent, with the rank of solvability (nilpotency) increasing as n increases. It would be interesting to know whether these algebras have appeared elsewhere in mathematics or physics, and to study the limit as $n \to \infty$.

Let us now turn again to the quantization of the small L-algebra. Up to now we have not commented on the irreducibility or otherwise of the quantum representation constructed above. From comparison with the finite-dimensional case we expect that this representation is reducible to smaller sectors of the Hilbert space $\mathscr{H}_L$ of the $\Psi[L]$, according to the action of $\mathscr{L}^1$ on $\bar{\mathscr{L}}^0$, as is implicit in the integrated relation (5.6). Such sectors of $\mathscr{H}_L$ would essentially be given by wave functions with support on one of the orbits of $\mathscr{L}^1$ in $\bar{\mathscr{L}}^0$.

However, as long as we do not have a well-defined Lie group structure, also the notion of an orbit in $\bar{\mathscr{L}}^0$ under the $\mathscr{L}^1$-action must to some extent remain formal. There is no way of summing up an infinite series as, for example, in eq. (5.4). The most we can do is to look for special points $\bar{L}[\alpha]$ in the space $\bar{\mathscr{L}}^0$ for which all higher-order terms in eq. (5.4) vanish identically, i.e. for which $\bar{L}[\alpha]$ is invariant. The orbit through $\bar{L}[\alpha]$ is then zero-dimensional. A sufficient condition for this to happen is $\bar{L}[\alpha]|_{\alpha_0} = 0$, for all α_0 that are not simple, i.e. possess self-intersections.

Let us first analyze the condition $\bar{L}[\alpha \circ \beta] = 0$, for α, β simple. Recalling definition (3.2) and the constraint

$$\bar{T}[\alpha \circ \beta] + \bar{T}[\alpha \circ \beta^{-1}] = \bar{T}[\alpha]\bar{T}[\beta], \tag{5.14}$$

the condition becomes

$$2\bar{T}[\alpha \circ \beta] = \bar{T}[\alpha]\bar{T}[\beta] \Rightarrow 2\,\mathrm{Tr}\, U_\alpha U_\beta = \mathrm{Tr}\, U_\alpha\, \mathrm{Tr}\, U_\beta . \tag{5.15}$$

Setting $\alpha = \beta$, we derive $\mathrm{Tr}\, U_\alpha = \pm 2$. Let us first consider the case with $\mathrm{Tr}\, U_\alpha = 2$. A general SL(2, $\mathbb{C}$)-matrix of this form we write as $U_\alpha = \begin{pmatrix} a+1 & a_2 \\ a_3 & -a+1 \end{pmatrix}$, with $a_2 a_3 = -a^2$. Evaluating condition (5.15) on two different matrices U_α and U_β, we find for the general form of such a matrix

$$U_\alpha = \begin{pmatrix} a+1 & ca \\ -a/c & -a+1 \end{pmatrix}, \tag{5.16}$$

for some complex number $c \neq 0$. In other words: if, for all loops α, U_α is of the form (5.16) for fixed, but arbitrary c, then eq. (5.15) is fulfilled and $\bar{L}[\alpha \circ \beta] = 0$. Moreover, whenever the holonomy matrices U_α take their values in the $\mathbb{C}$-subgroup

of $SU(2)_{\mathbb{C}}$ defined by eq. (5.16), all variables $\bar{L}[\gamma]$, for nonsimple γ with any number of self-intersections, vanish identically. This follows from the definition of the L^0-variables as sums of an even number of T^0-variables, half of which contribute with a plus-, the other half with a minus-sign.

Note that this yields degenerate points of the mapping $(A, \tilde{\sigma}) \mapsto (T^0, T^1)$ from Ashtekar's phase space to the space of the small T-algebra: it is now easy to find configurations $\tilde{\sigma}_0(x)$ such that all phase space points $(A, \tilde{\sigma}_0)$, where A varies in the subalgebra corresponding to eq. (5.16), are mapped into the point $(T^0[\alpha], T^1[\gamma](s)) \equiv (2, 0)$ (a similar result holds for the L-variables). This degeneracy disappears when we restrict ourselves to holomorphic connections A (c.f. the comments in ref. [13]). Also for the Yang–Mills case, with SU(2)-matrices defined by eq. (3.9), these considerations are irrelevant, since the only group element of the form (5.16) is the unit matrix. The case $\mathrm{Tr}\, U_\alpha = -2$ can be treated in an analogous manner, but holonomy matrices U_α of this type do not form a subgroup, because they do not contain the unit matrix.

We will call the quantum representation defined by the operators (5.8), (5.9) and (5.11), together with some space of wave functionals $\Psi[L]$, the L-representation, to distinguish it both from Ashtekar's self-dual and Rovelli–Smolin's loop representation. It is the most natural quantum representation if one starts from the T- or L-algebra of nonlocal loop variables, and it leads to an anomaly-free quantization of the full algebra (formally, since we have to feed in more information about the loop space and introduce a regularization for quantum operators). Although up to this point our kinematical considerations were valid for both gravity and SU(2)-gauge theory, this will not be true for the further development of the theories.

In our opinion, it is even misleading to overemphasize the gauge theory aspects of canonical gravity in the new formulation. One has made a first step in the right direction by basing the theory on loops, and not on points. This is important since the notion of a point simply has no meaning in a diffeomorphism-invariant theory, whereas the set of diffeomorphism equivalence classes of loops has a very rich structure in three dimensions.

Having thus "delocalized" gravity theory, it seems paradoxical to try and get well-defined expressions for quantities that again depend on points in three-space, by taking appropriate limits of nonlocal quantities. Not surprisingly, such attempts are riddled with problems: for example, in order to define the (local) hamiltonian constraint operator $\hat{\mathscr{H}}_\perp$, one needs a regularization prescription. This has to be non-standard, since in Ashtekar's formulation the three-metric q_{ab}, which is necessary in most of the standard regularization schemes, is not a basic variable, and hence not readily available. The point-splitting regularizations used in refs. [9, 13, 19] rely on the introduction of an auxiliary metric defined in a coordinate patch. Unfortunately, in the limit as the regulator is removed, this metric dependence cannot be eliminated completely.

Similar remarks apply to the idea of putting the theory (in the loop formulation) on the lattice. Any lattice formulation of gravity explicitly breaks diffeomorphism invariance, and how to recover this invariance in the continuum limit seems to be an unsolved problem (see ref. [18] for related comments).

The idea that is somewhat implicit in the work by Rovelli and Smolin, and that one would like to pursue further, is that of solving the diffeomorphism constraints already at the classical level. Like the T-variables, the L-variables carry a natural representation of the group of diffeomorphisms on Σ, by virtue of their loop dependence. Natural coordinates for the reduced theory are then those L-variables that are constant along the orbits of the diffeomorphism group. Unfortunately, only the L^0-variables project down to the reduced space in an easy way, whereas for the higher-order L-variables this is problematic due to their dependence on marked points. (Similarly, since wave functions in the quantum theory depend only on L^0's, it is straightforward to single out those constant under diffeomorphisms, but not easy to find quantum observables constructed from operators $\hat{L}^n$.) If one could find an easy way of integrating out this dependence without disturbing the Poisson-algebra structure too much, this would lead to a complete set of coordinates on the reduced space (from which diffeomorphism-invariant quantities could be constructed).

Assume we had a set of such variables. They will depend on generalized knot and link classes, following the usual reasoning. The space they span we will call the reduced L-space for the moment. Therefore, all that remains to be done is to define a natural reduced, non-local hamiltonian in terms of these variables. Recall that the hamiltonian constraint $\mathscr{H}_\perp(x)$ is also not strictly invariant under diffeomorphisms, but transforms like a scalar density. Its Poisson brackets with the diffeomorphism constraints $\mathscr{H}_a$ are given by

$$\{\mathscr{H}_\perp(x), \mathscr{H}_a(y)\} = i\partial_a \delta^3(x,y)\mathscr{H}_\perp(x). \tag{5.17}$$

Only on the submanifold defined by the constraint $\mathscr{H}_\perp \equiv 0$ the right-hand side vanishes and $\mathscr{H}_\perp$ is actually diffeomorphism invariant. This means that $\mathscr{H}_\perp$ does not project down to the reduced L-space, unless we are on the submanifold $\mathscr{H}_\perp \equiv 0$. Thus we have to construct a hamiltonian (using suitable integration or smearing) on the reduced L-space that coincides with some appropriate nonlocal analogue of the usual hamiltonian $\mathscr{H}_\perp$ on the subspace defined by $\mathscr{H}_\perp \equiv 0$.

Regardless of whether it is possible to solve the diffeomorphism constraints already classically, one has to carry out the quantization, along the lines of the L-representation introduced in this paper. One expects that the canonical group structures underlying the construction of the L-representation will simplify the search for a rigorous scalar product for the quantum theory, as well as physical states and observables. This is momentarily under investigation.

Let us go back now to the case of Yang–Mills theory, where one can push the kinematical structure still further, even before introducing a hamiltonian in terms of L-variables. We will sketch one way of how quantities may become physically meaningful, i.e. how we can encode the fact that we are dealing with an SU(2)-gauge theory. To this end one has to make a choice of Hilbert space and scalar product compatible with the L-algebra structure and the physical interpretation. The $\hat{L}^n$ must be self-adjoint operators (with real spectrum), because their classical counterparts take real values. However, this is a necessary but not sufficient condition, since the reality of the trace is a property of both SU(2)-matrices in the defining representation and of SL(2,$\mathbb{R}$)-matrices, which form another subgroup of the SL(2,$\mathbb{C}$)-matrices (those with real entries). A further restriction on the trace of an SU(2)-matrix is that it can only take values between -2 and $+2$ on the real axis (c.f. eq. (3.9)). By construction, the same is true for all L^0-variables.

In agreement with this physical interpretation, wave functionals $\Psi[L]$ will only have support for values of L in the interval $[-2,2]$. (This differs from the quantization of the algebra (5.13), where we do not have any physical reasons to assume that the spectrum of the $\hat{x}_i$ is not the whole real axis.) In contrast with gravity theory, here one can and should make use of a lattice formulation to get approximate results. For example, we can start with a finite number of loops covering Σ, construct an explicit basis of wave functions, choose boundary conditions such that the $\hat{L}$-operators become truly self-adjoint, find a discretized version of the hamiltonian and a suitable quantization. Since $\|\Psi[L]\|^2$ can be interpreted as a probability density of configurations $L[\alpha]$, in the limit of strong fields and big loops, $\|\Psi[L[\alpha_0]]\|$ as a function of $L[\alpha_0]$ will be roughly symmetric about $L[\alpha_0]=0$. However, as the loops become smaller or the field weaker, the holonomy matrix for a given loop α_0 will be close to the unit matrix, hence $L[\alpha_0]$ close to 2, and the probability density strongly peaked at values $L[\alpha_0]$ close to, but smaller than 2.

Of course, in order to recover the full theory, we will have to take some limit as the number of loops increases and also as loops shrink down to points. Luckily, for Yang–Mills we do have a background metric, and can employ the usual field-theoretic methods to regularize and renormalize the theory in this limit. On the other hand, the system should exhibit some of its nonperturbative features already before these limits are performed.

One of the virtues of the new loop formulation, initiated in ref. [19], is that it approaches the theory "from the other end" by taking an intrinsically nonlocal and nonperturbative starting point. In contrast with previous loop formulations of gauge field theories, one has an additional Poisson-algebra structure of preferred loop observables. For the case of gravity, this at the moment seems to be the only viable way of constructing a consistent quantum theory, but also for Yang–Mills theory we expect to gain new insights. In particular, it gives us the possibility of working entirely on the reduced phase space, thus avoiding the usual gauge fixing problems. One way of exploiting the algebra structure in constructing a quantiza-

tion has been described in the present work. It seems worthwhile to study the classical L-group from a purely group-theoretical point of view, for example, its subgroups, possible central extensions, and the structure of the $\mathscr{L}^1$-orbits. This would also enhance our understanding of the quantum theory. Another challenge is the generalization of the present formulation to other gauge groups, most importantly to SU(3). The further study of old theories in terms of loop variables may still yield many surprises.

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