BPS Extensions and Gaugings of Supersymmetric Field Theories

BPS Extensions and Gaugings of Supersymmetric Field Theories

BPS Uitbreidingen en IJkingen
van Supersymmetrische Veldentheorieën

(met een samenvatting in het Nederlands)

Proefschrift

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Ivan-Sergej Herger

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Promotor: Prof. dr. B. de Wit

Spinoza Instituut en Instituut voor Theoretische Fysica
Faculteit Natuur- en Sterrenkunde
Universiteit Utrecht

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Preface

Supersymmetry has played an important role in theoretical physics for the last thirty years. In the quest for unification of the fundamental interactions, supersymmetry provides the framework that links two seemingly unrelated kinds of particles, namely bosons and fermions. Supersymmetry is a vital ingredient for the construction of string theories, which in turn are the building blocks of a conceived fundamental theory of nature, so-called M-theory.

The fundamental degrees of freedom of M-theory are not yet known. M-theory is only defined as a theory that, in certain limits, coincides with the known string theories or with supergravity theories. In the limit of low-energy interactions, i.e. at energies much smaller than the Planck mass $M_\mathrm{P} \sim 10^{19}$ $\mathrm{GeV/c^2}$, M-theory is described in terms of supersymmetric gauge theories and supergravity theories. In order to find the correct description of M-theory, it is vital to further explore new features of the effective low-energy theories. In this thesis, we study two different extensions of supergravity theories in an attempt to gain new insights into the nature of M-theory. One possibility for extending a supersymmetric field theory is to couple the original, massless theory to massive BPS multiplets. The second possibility is to construct a gauged supergravity theory by extending the abelian gauge symmetry of a supergravity theory to a non-abelian symmetry.

Outline of this thesis

This thesis is organized as follows. After introducing important concepts of supersymmetry and supergravity in chapter 1, we present a pedagogical treatment of supersymmetry in anti-de Sitter space in chapter 2. Because anti-de Sitter space is a curved space, supersymmetry exhibits some features that do not arise in supersymmetric extensions of the ordinary Poincaré algebra. For example, theories that are invariant under the supersymmetric extension of the anti-de Sitter symmetry contain so-called singletons, which are fields that contain fewer degrees of freedom than a generic local field. Anti-de Sitter space is e.g. found as the ground-state of three-dimensional supergravity theories that we are studying in chapter 6.

In chapter 3 and chapter 4, we discuss nine-dimensional supergravity theory coupled to different kinds of BPS multiplets, the so-called KKA and KKB multiplets, which both arise in the compactification of a higher-dimensional supergravity theory. From the point of view of eleven-dimensional supergravity, the KKA multiplet contains Kaluza-Klein states, and the KKB multiplet contains winding states of the supermembrane. From the point of view of ten-dimensional IIB supergravity, the role of the BPS multiplets is different. The states in the KKB multiplet correspond to Kaluza-Klein states, whereas the states in the KKA multiplet correspond to winding modes of the fundamental string and the D-string.

The explicit construction of a supergravity action coupled to the two BPS multiplets proves to be rather involved, and therefore we investigate the example of a supersymmetric field theory coupled to BPS multiplets in detail in chapter 5, namely, supersymmetric Yang-Mills theory in four dimensions. This example is considerably simpler, because only rigid supersymmetry is involved, as opposed to the nine-dimensional supergravity case, where supersymmetry transformations are local. We explicitly construct the $\mathcal{N} = 4$ super Yang-Mills theory coupled to BPS multiplets, which arise e.g. upon compactification or spontaneous symmetry breaking. The BPS fields are subsequently integrated out, and we obtain an effective action for the massless fields.

Finally, in chapter 6 we give a general classification of gauged supergravity theories in three dimensions. We derive a criterion for admissible gauge groups, and we discuss the examples of $N = 1$, $N = 2$ and $N = 9$ supersymmetry in detail.

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Ivan Herger, May 2003

1

Introduction

Supersymmetry is a vital ingredient for this thesis and we therefore start our presentation by outlining some key concepts of supersymmetry in section 1. We will not go into the details of supersymmetry here and refer the reader to chapter 2 for a more detailed introduction to supersymmetry in the special setting of anti-de Sitter space. In sections 2 and 3 of this chapter, we give a brief overview of superstring theory and superstring dualities. Finally, we discuss supergravity theories in some detail in section 4. In particular we introduce BPS-extended supergravity and gauged supergravity.

1. Supersymmetry

Physical nature consists of two kinds of fundamental particles: fermions and bosons. Fermions carry half-integer spin, whereas the spin of bosons is integer-valued [1]. "Conventional" (or "bosonic") symmetries act separately on the subspaces of the bosons and fermions, i.e. they do not mix the fields describing the two types of particles. The possible bosonic symmetries of physical theories in flat space have been classified to be a direct product of the Poincaré group with an internal symmetry group [2].

An intriguing extension of the conventional bosonic symmetries is a "fermionic" symmetry that transforms bosonic fields into fermionic fields, and vice versa. This symmetry is known as supersymmetry [3–6]. A renormalizable supersymmetric scalar field theory was first constructed in 1974 [7]. Subsequently, supersymmetry was applied to gauge theories [8], and to gravity [9, 10]. A general introduction to the principles of supersymmetry can be found for example in [11].

Let us now turn to the formulation of supersymmetry. The bosonic symmetry algebra of a field theory can be extended to a supersymmetry algebra by introducing $\mathcal{N}$ anti-commuting spinors Q^i ($i = 1 \ldots \mathcal{N}$), which are the generators of supersymmetry transformations. In flat space, the spinors Q^i obey the following anti-commutation relation,

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = P^\mu \left(\gamma_\mu C\right)_{\alpha\beta} \delta^{ij} + Z^{ij} C_{\alpha\beta}\,, \tag{1.1}$$

where P_μ is the generator of rigid space-time translations, and where C is the charge conjugation matrix. Here, Z^{ij} denotes a scalar charge that commutes with all other symmetries, a so-called central charge. Depending on the dimension of space-time, the charge conjugation matrix C can be symmetric and/or antisymmetric, and therefore Z^{ij} can also be symmetric and/or antisymmetric. Depending on the theory we are studying, there can be more than one central charge, and also central charges which are not scalars.

The number of components of a spinor depends on the dimension of space-time, which means that the number of supercharge components Q depends on the number $\mathcal{N}$ of supersymmetries and the dimension of space-time. For example, in four space-time dimensions, a Majorana spinor has four components, and therefore $\mathcal{N} = 4$ supersymmetry corresponds to $Q = 16$ supersymmetry charges. The supersymmetry charges can be rotated by the so-called R-symmetry group. The R-symmetry group, which is often labelled H_R, is defined as the largest subgroup of the automorphism group of the supersymmetry algebra that commutes with the Lorentz group. Depending on the sort of spinor of the supersymmetry charge (which in turn depends on the dimension of space-time), the group H_R is of the form $\mathrm{SO}(\mathcal{N})$, $\mathrm{U}(\mathcal{N})$, or $\mathrm{USp}(2\mathcal{N})$.

A theory that is invariant under supersymmetry contains an equal number of bosonic and fermionic states. A set of states that transform into each other under supersymmetry transformations is known as a supermultiplet, or multiplet for short. All states in the multiplet have the same mass, but they have different spin, and they may have different quantum numbers under R-symmetry or other, additional symmetries. In particular, a state with spin s transforms into a state with spin $s \pm 1/2$ under a supersymmetry transformation. The number of states that belong to a supermultiplet depends on the number of space-time dimensions, the number of supersymmetry generators, the central charges that are present in the theory and the mass[1] of the states. For example, the vector multiplet of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory contains a gauge-field, four Majorana spinors and six scalar fields. This field content comprises eight bosonic and eight fermionic degrees of freedom.

There are two different ways of formulating a supersymmetric field theory. Either one introduces a so-called superfield, which describes all the bosonic as well as fermionic states of a multiplet and one writes down an action for this superfield; this method is known as the superfield formalism. Or one retains the components of the multiplet as separate fields and writes down the action for the various fields in the so-called component formalism. While both methods

[1] To be precise, it is not the mass that determines the number of states in a multiplet, but the quadratic Casimir operator of the isometry group. In flat space, the quadratic Casimir operator is precisely the mass-squared, but e.g. in anti-de Sitter space, this is not the case, as we will see in chapter 2.

have their advantages and disadvantages, we confine ourselves to the component formalism in this thesis.

Let us point out an important feature of the supersymmetry algebra (1.1). The anti-commutator on the left-hand side of relation (1.1) is a positive definite operator, and therefore there exists a relation between the mass M (which is the zero-th component of P_μ in the rest-frame) of a state and its central charge Z^{ij}. This relation is known as the Bogomol'ny-Prasad-Sommerfeld relation, or BPS relation. For the commutator (1.1), the BPS relation is given by

$$|M| \geq |Z| \,. \tag{1.2}$$

where $|Z|^2 = Z_{ij}Z^{ij}$. States which saturate the inequality (1.2) are known as BPS-states. A multiplet of BPS states contains considerably fewer degrees of freedom than a generic massive multiplet, because a number of supersymmetry generators vanish on those states, which can easily be seen by diagonalizing (1.1).

In chapter 2, we give a more detailed introduction to supersymmetry in anti-de Sitter space. BPS-states are discussed extensively in chapter 3 in the context of nine-dimensional supergravity, and also in chapter 5 in the context of supersymmetric Yang-Mills theory in four dimensions.

2. Perturbative string theory

Supersymmetry is a tool that can be used to extend not only field theories, but also string theories. In this section, we briefly introduce and review important aspects of string theory. However, many aspects will remain undiscussed, because they are not of direct importance for this thesis. For a broader and more comprehensive introduction to perturbative string theory, the interested reader should consult e.g. [12–14].

2.1. The bosonic string

Consider a bosonic string moving in a D-dimensional flat background with Minkowski metric $\eta_{\mu\nu}$. A string can either be closed or open, and in the following we consider a closed string. The string sweeps out a two-dimensional world-sheet Σ, and we denote the coordinates on the world-sheet by σ and τ. The embedding of the world-sheet Σ in the target space is described by the D embedding coordinates $X^\mu(\tau, \sigma)$. The so-called Polyakov action of the closed bosonic string is given by

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \sqrt{\det h} \, h^{ab} \, \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \,, \tag{1.3}$$

where $h^{ab}(\tau, \sigma)$ is the world-sheet metric. The equations of motion for the coordinate fields X^μ are

$$\frac{1}{\sqrt{\det h}} \, \partial_a \left(\sqrt{\det h} \, h^{ab} \partial_b X^\mu \right) = 0 \,.$$

Solutions of the equations of motion for the closed string are periodic in σ with period π, and they decompose into a left-moving part and a right-moving part. In the conformal gauge, $h_{ab} = \eta_{ab}$, the general solutions are—up to constraints that follow from fixing the gauge—given by

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu \exp\bigl(-2in(\tau+\sigma)\bigr)\,, \tag{1.4a}$$

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \exp\bigl(-2in(\tau-\sigma)\bigr)\,. \tag{1.4b}$$

Let us now study the quantization of the string. Upon quantization, the oscillators modes α_n^μ and $\bar{\alpha}_n^\mu$, as well as x^μ and p^μ are promoted to operators. In covariant quantization one has to take into account the constraints, whereas in light-cone quantization, the constraints are solved explicitly, and the theory is described in terms of the physical degrees of freedom. We will not go into details of the quantization procedure here, we just note that for $n < 0$, the operators α_n^μ and $\bar{\alpha}_n^\mu$ act as creation operators of an infinite set of harmonic oscillators. These creation operators acting on a suitably defined vacuum state generate the spectrum of states of the string theory. The level matching condition, which is one of the constraints and which states that the number of left-moving and right-moving excitations, N_L and N_R, must be equal for the closed string, restricts the physical states to have the following masses,

$$\alpha' M^2 = 2\,(N_R + N_L - 2)\;, \tag{1.5}$$

where $N_{L,R}$ are the levels of excitation of the left-moving modes and right-moving modes, respectively. The spectrum of the closed bosonic string consists of a spinless tachyonic state with negative mass squared (corresponding to $N_L = N_R = 0$), a finite number of massless states ($N_L = N_R = 1$), and an infinite number of massive states (with $N_L + N_R > 1$). The massless states of closed string theories comprise a graviton, an antisymmetric two-tensor and a scalar. Open string theories on the other hand contain a massless vector and also a spinless tachyonic state. When one tries to quantize the bosonic string in an arbitrary dimension D, one encounters the so-called conformal anomaly, i.e. conformal invariance that is present at the classical level is not preserved at the quantum level. The only space-time dimension where the anomaly cancels and the theory becomes consistent as a quantum theory is $D = 26$.

The Polyakov action as it stands in (1.3) is incomplete because it does not contain a kinetic term for the world-sheet metric h_{ab}. The Einstein-Hilbert action provides such a kinetic term,

$$S_g = -\frac{1}{4\pi} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \sqrt{h} \; R \, . \tag{1.6}$$

In two dimensions, the action (1.6) is is a topological invariant, i.e. it is given by the Euler number of the world-sheet. In path integral quantization, the action (1.6) can therefore be used to define a perturbation theory such that the action for different world-sheets is weighted by the Euler number of the world-sheet.

We would like to point out that the actions (1.3) and (1.6) have been written down in a flat background. In a general background, the action takes the form

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, \sqrt{\det h} \Big(\big(h^{ab} G_{\mu\nu}(X) + i\varepsilon^{ab} B_{\mu\nu}(X) \big) \partial_a X^\mu \partial_b X^\nu + \alpha' \Phi(X) R \Big) \, . \tag{1.7}$$

Here, $G_{\mu\nu}(X)$ is a general metric, $B_{\mu\nu}(X)$ is an antisymmetric tensor, and $\Phi(X)$ is the dilaton field. The action (1.7) in a general background can be used to write down an effective action for the massless fields, as we will see section 4.1.

2.2. The superstring

The supersymmetric extension of the action (1.3) is the action of the superstring. Cancellation of the conformal anomaly requires that the dimension of space-time for a superstring is $D = 10$, i.e. consistent superstring theories can only be formulated in ten space-time dimensions. In order to obtain space-time supersymmetry (i.e. supersymmetry in ten space-time dimensions, as opposed to supersymmetry on the world-sheet in two dimensions, which is present by construction), some states have to be projected out of the spectrum.[2] This is consistently done by the GSO projection, which also eliminates the tachyonic mode. There are five different consistent theories of open and/or closed superstrings in ten dimensions; they are classified by the number of supersymmetries they possess and by their internal gauge symmetries. The IIA theory and the IIB theory are closed string theories and they both posses the maximal number of supersymmetries, $\mathcal{N} = 2$, and the IIB theory is chiral, i.e. both supersymmetry generators are of the same chirality.[3] The type I theory is a theory with $\mathcal{N} = 1$ supersymmetry, with gauge group SO(32) and with open as well as closed strings. Further, there are two heterotic theories with $\mathcal{N} = 1$ supersymmetry, which have gauge groups $\mathrm{E}_8 \times \mathrm{E}_8$ and SO(32), respectively.

[2]There exists a formulation of string theory with explicit space-time supersymmetry, the so-called Green-Schwarz superstring, which we will not consider here.

[3]In ten dimensions, a Majorana-Weyl spinor consists of sixteen components. Therefore, $\mathcal{N} = 1$ supersymmetry corresponds to sixteen supercharges, and $\mathcal{N} = 2$ to twice as many.

Heterotic string theories are constructed by taking a bosonic string in the left-moving sector (compactified on a sixteen-torus), and an $\mathcal{N} = 1$ superstring in the right-moving sector.

In $\mathcal{N} = 2$ string theory, the two fermions on the world-sheet can have either periodic or anti-periodic boundary conditions for both left-movers and right-movers. Periodic boundary conditions are called Ramond (R) boundary conditions, and anti-periodic boundary conditions are called Neveu-Schwarz (NS). String states in the R-R and NS-NS sectors lead to space-time bosons, and states in the R-NS and NS-R sector lead to space-time fermions.

When studying string interactions, it is often sufficient to restrict oneself to the massless states of the string theory. The interactions can then best be described by a low-energy effective field theory. The multiplet of massless states and the symmetries of the theory (general coordinate invariance and supersymmetry) determine the effective action completely up to second order in the derivatives. Namely, it is precisely the action of a supergravity theory, which we study in section 4 of this chapter. Terms that are of higher order in the derivatives are also of higher order in the parameter α', which is proportional to the square of the string length. An expansion in α' is therefore an expansion in the string length; for $\alpha' \to 0$, we obtain the supergravity action, as mentioned above.

Since we are interested in the massless string states and their interactions, let us list the massless states of the IIA and the IIB string theory as an example. The IIA superstring theory has the following massless states: a graviton $g_{\mu\nu}$ with spin 2; two spin-3/2 states of opposite chirality $\psi^{\pm}_{\mu}$, the gravitinos; a three-index tensor $C_{\mu\nu\rho}$; a two-index tensor $B_{\mu\nu}$; a vector C_{μ}; two fermions of opposite chirality, $\lambda^{\pm}$; and one scalar ϕ. The IIB superstring theory contains: a graviton $g_{\mu\nu}$; two gravitinos ψ_{μ} with equal chirality; a four-index tensor $A_{\mu\nu\rho\sigma}$ with self-dual field-strength; two two-index tensors $A^{\alpha}_{\mu\nu}$; two chiral fermions λ; and two scalars ϕ^{α}. The $\mathcal{N} = 1$ string theory with gauge group SO(32) and the two heterotic string theories with gauge groups SO(32) and $E_8 \times E_8$, all contain a Yang-Mills multiplet and a supergravity multiplet in their massless sector. They have therefore (up to the gauge group) identical low-energy effective actions, which is $\mathcal{N} = 1$ supergravity theory coupled to a Yang-Mills theory. These low-energy effective actions will be studied in chapter 5, where we consider the compactification of supersymmetric Yang-Mills theory from ten dimensions to four dimensions.

3. String dualities and M-theory

String theories exhibit interesting phenomena that are not known from field theories. One example of such a phenomenon is T-duality, which we discuss below.

Consider a closed bosonic string and compactify one of the dimensions (labelled by the index i) of the target space on a circle of radius R. Periodicity of the string and periodicity of the compactification manifold imply that the coordinate function $X^i(\tau, \sigma)$ obeys the following condition,

$$X^i(\tau, \sigma + \pi) = X^i(\tau, \sigma) + 2\pi R w \,,$$

where $w \in \mathbb{Z}$. The momentum in the i-th direction takes the discrete values $p^i = m/R$, $m \in \mathbb{Z}$, so that the general solution (1.4) is modified for the coordinate in the compactified dimension,

$$\begin{aligned} X^i_L(\tau + \sigma) &= \frac{1}{2}x^i + \alpha' \left(\frac{m}{R} + \frac{wR}{\alpha'}\right)(\tau + \sigma) + \text{oscillators}\,, \\ X^i_R(\tau - \sigma) &= \frac{1}{2}x^i + \alpha' \left(\frac{m}{R} - \frac{wR}{\alpha'}\right)(\tau - \sigma) + \text{oscillators}\,. \end{aligned}$$

The masses of the physical states are given by

$$\alpha' M^2 = \frac{\alpha' m^2}{R^2} + \frac{w^2 R^2}{\alpha'} + 2(N_L + N_R - 2)\,.$$

We identify two terms that were absent in the spectrum (1.5) of the uncompactified string. The first term is the contribution from the momentum of the string in the compact dimension, m/R, and the second term corresponds to the energy that is required to wrap the string w times around the circle. Note that the level matching condition in the compactified theory is given by $N_R - N_L = mw$, i.e. N_R and N_L do not have to be equal, but they may differ by mw. Apart from the massless states that are already present in the uncompactified case, i.e. the ones with $N_L = N_R = 1$, there are a number of new massless states: two vectors and one scalar, which arise from the reduction of the graviton and the antisymmetric tensor. They also correspond to $N_L = N_R = 1$. The vector states gauge the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry associated with the left and right isometries of the circle.

The spectrum of the compactified string theory is invariant under the duality transformation

$$R \to \frac{\alpha'}{R}\,, \qquad m \leftrightarrow w\,.$$

This duality is known as T-duality and it maps momentum modes on the compactified dimension onto winding modes of the string around the compact dimension, and vice-versa. The bosonic string theory is self-dual, i.e. the theory is mapped onto itself by the duality transformation. At certain points in the moduli space of the compactified theory, namely when the radius is self-dual ($R^2 = \alpha'$), the spectrum of the Kaluza-Klein states ($m \neq 0$) and the winding states ($w \neq 0$) becomes equal. There are four additional massless vector states with non-zero internal momentum and non-zero winding number, corresponding to $N_L = 0$, $N_R = 1$ and $m = w = \pm 1$ or $N_L = 1$, $N_R = 0$

and $m = -w = \pm 1$. The theory then exhibits an enhancement of the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ gauge symmetry to a $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ gauge symmetry. Note also that in the limit of $R \to 0$ the theory is completely equivalent to the uncompactified theory, i.e. to $R \to \infty$.

In the supersymmetric case, the spectrum of type IIA superstring theory compactified on a circle of radius R is identical to the spectrum of type IIB superstring theory compactified on a circle of radius $1/R$, i.e. the two theories are T-dual to each other. This type of T-duality relates two different theories to each other, which is conceptually different from the self-duality of the bosonic string. In particular, at the self-dual radius there is no symmetry enhancement.

There are many more duality symmetries that relate different string theories to each other in various manners. The picture that has emerged in the past few years is that of a web of dualities. It is conjectured that all string theories are different limits of one unified theory, the so-called M-theory.

Some of the string theory dualities relate a theory in a weakly coupled regime to a different theory in a strongly coupled regime. An important ingredient in verifying (certain aspects of) such a strong-weak coupling duality are D-branes. A Dp-brane is a non-perturbative, p-dimensional object in a string theory. D-branes arise in closed string theories as hyperplanes in space-time on which strings terminate, and they are dynamical objects. From a supergravity point of view, D-branes correspond to solitonic solutions of the field equations.

The tension of a D-brane is proportional to the inverse of the string coupling constant, which means that D-branes are heavy at small coupling, but light at strong coupling. In the limit where the string coupling goes to infinity, they become massless. Because the spectrum of BPS states does not change when going from the weakly coupled regime to the strongly coupled regime, D-branes can be used to check certain aspects of strong-weak coupling dualities.

An interesting result from the study of non-perturbative string theory is the emergence of an eleventh dimension in type IIA superstring theory. Among the non-perturbative objects of type IIA theory there is the D0-brane, a point particle. The mass of the D0-brane is

$$m_0 = \frac{1}{g\alpha'^{1/2}}\,, \tag{1.8}$$

where g is the string coupling constant, which we have previously set to unity. There is a bound state for every number n of D0-branes, and the non-perturbative spectrum of the IIA theory contains states whose masses are integer multiples of (1.8). The spectrum looks exactly like the Kaluza-Klein spectrum for a theory compactified on a radius $R = g\alpha'^{1/2}$. When the coupling constant goes to infinity, the radius blows up and the theory effectively decompactifies. It was therefore conjectured that the strong coupling limit of IIA superstring theory is an eleven-dimensional theory, M-theory.

From the supergravity point of view the emergence of the eleventh dimension is not really a surprise. It has long been known that the maximum number of dimensions for a supergravity theory is eleven. The eleven-dimensional supergravity theory is unique, and we describe it in more detail in the following section.

Let us note that the (up to now) elusive M-theory is only defined through its low-energy effective action, which is eleven-dimensional supergravity theory, and through its compactification to ten dimensions, which is type IIA string theory. The interpretation of the letter “M” in M-theory is left to the imaginative reader.[4]

4. Supergravity

Historically, supergravity theories were discovered in an attempt to construct a renormalizable field theory of gravity [9, 10]. Even though supergravity theories do not contain any non-renormalizable diagrams up to two loops, it was soon realized that there appear non-renormalizable infinities at three-loop order and beyond [15] and that therefore supergravity theories cannot really serve as fundamental quantum theories of gravity. Nowadays, supergravity theories are mostly studied as the low-energy effective actions of superstring theories. For a modern treatment of many aspects of supergravity theories, the reader is invited to consult [16].

The highest dimension where one can construct a supergravity theory with local Lorentz invariance is eleven. Theories with more dimensions would have states with spins greater than 2, because the smallest number of supercharges that form a spinor would be more than 32. Many other supergravity theories in lower dimensions can be directly obtained from the eleven-dimensional theory upon compactification and truncation. Let us therefore study the eleven-dimensional supergravity theory in some detail.

In eleven flat dimensions, the commutator of two supersymmetry generators Q is given by

$$\left\{Q, \bar{Q}\right\} = -iP_M \Gamma^M + \frac{1}{2} i Z_{MN} \Gamma^{MN} + \frac{1}{5!} i Z_{MNPQR} \Gamma^{MNPQR} \,. \qquad (1.9)$$

The first term on the right-hand side corresponds to a general coordinate transformation. The remaining two terms are central charge terms, they describe abelian gauge transformations related to antisymmetric tensor fields. Perturbative states in the supergravity theory are neutral with respect to the tensor field charges, but there are solitonic solutions (the so-called M2-brane and the M5-brane) that do carry charges.

The supergravity multiplet in eleven dimensions consists of the elfbein ${e_M}^A$, the Majorana gravitino Ψ_M and the three-form A_{MNP}. The Lagrangian

[4] Some suggestions can be found in [14].

contains the Einstein-Hilbert term for the elfbein, a kinetic term for the gravitinos (the Rarita-Schwinger term), a kinetic term for the three-form field and various interaction terms. It can be written as follows [17]

$$\begin{aligned}\kappa_{11}^2 \mathcal{L}_{11} = & -\frac{1}{2} E\, R - \frac{1}{2} E\, \bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P - \frac{1}{48} E\, F_{MNPQ}^2 \\ & - \frac{\sqrt{2}}{3456} \varepsilon^{MNPQRSTUVWX} F_{MNPQ} F_{RSTU} A_{VWX} \\ & - \frac{\sqrt{2}}{192} E \left(\bar{\Psi}_R \Gamma^{MNPQRS} \Psi_S + 12 \bar{\Psi}^M \Gamma^{NP} \Psi^Q \right) F_{MNPQ} ,\end{aligned} \tag{1.10}$$

where E is the determinant of the elfbein $E_M{}^A$. The field-strength F_{MNPQ} is given by $F_{MNPQ} = 24 \partial_{[M} A_{NPQ]}$ and the covariant derivative D_M is covariant with respect to local Lorentz transformations and general coordinate transformations. Note that we have omitted quartic fermion terms. The equations of motion for the bosonic fields are

$$\begin{aligned} R_{MN} &= \frac{1}{72} g_{MN} F_{PQRS} F^{PQRS} - \frac{1}{6} F_{MPQR} F_N{}^{PQR} , \\ \partial_M \left(E F^{MNPQ} \right) &= \frac{\sqrt{2}}{1152} \varepsilon^{NPQRSTUVWXY} F_{RSTU} F_{VWXY} . \end{aligned}$$

The field-strength F_{MNPQ} also satisfies the Bianchi-identity $\partial_{[M} F_{NPQR]} = 0$. The Lagrangian (1.10) is invariant under the following supersymmetry transformations,

$$\begin{aligned} \delta E_M^A &= \frac{1}{2} \bar{\epsilon} \Gamma^A \Psi_M , \\ \delta A_{MNP} &= -\frac{\sqrt{2}}{8} \bar{\epsilon} \Gamma_{[MN} \Psi_{P]} , \\ \delta \Psi_M &= D_M \epsilon + \frac{\sqrt{2}}{288} \left(\Gamma_M{}^{NPQR} - 8 \delta_M^N \Gamma_{PQR} \right) \epsilon F_{NPQR} . \end{aligned}$$

where we have omitted cubic fermion terms. Since a Majorana spinor in eleven dimensions has 32 components, $\mathcal{N} = 1$ supersymmetry corresponds to $Q = 32$ supercharge components.

4.1. Compactification

Many supergravity theories in lower dimensions can be obtained by compactifying eleven-dimensional supergravity theory on some internal manifold. In the following, we compactify the eleven-dimensional theory on a circle of radius R and study the resulting theory from a ten-dimensional point of view.

Let us begin the discussion with a simple example, namely the theory of a single, D-dimensional, abelian vector field A_M compactified on a circle. This example already contains many of the concepts we will be needing later. The Maxwell action of the vector field in D dimensions is given by

$$S = -\frac{1}{4g_D^2} \int \mathrm{d}z \, F_{MN} F^{MN} \, . \tag{1.11}$$

We decompose the vector field A_M into a vector field $\hat{A}_\mu$ and a scalar field $\hat{\phi}$, i.e. $A_M = (\hat{A}_\mu, \hat{\phi})$ and the coordinate z is split up as $z^D = (x^{D-1}, y)$. A hat on a $(D-1)$-dimensional field is used to indicate that the field depends on all D coordinates. The Fourier expansion of $\hat{A}_\mu$ and $\hat{\phi}$ on the circle with periodic coordinate y is given by

$$\hat{A}_\mu(x, y) = A_\mu(x) + \sum_{p \neq 0} A_\mu^{(p)}(x) \, e^{ipy} \, , \tag{1.12a}$$

$$\hat{\phi}(x, y) = \phi(x) + \sum_{p \neq 0} \phi^{(p)}(x) \, e^{ipy} \, . \tag{1.12b}$$

This expansion gives rise to a finite number of massless fields A_μ and ϕ, and an infinite tower of massive fields $A_\mu^{(p)}$ and $\phi^{(p)}$. The action in $D-1$ dimensions is obtained by integrating over the coordinate y of the internal manifold

$$\begin{aligned} S = {} & \frac{2\pi R}{g_D^2} \int \mathrm{d}x \, \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) \\ & - \frac{2\pi R}{g_D^2} \sum_{p \neq 0} \int \mathrm{d}x \, \left(\frac{1}{2} \left| \partial_\mu A_\nu^{(p)} - \partial_\nu A_\mu^{(p)} \right|^2 + \left| \partial_\mu \phi^{(p)} \right|^2 + p^2 \left| A_\mu^{(p)} \right|^2 \right) \, . \end{aligned} \tag{1.13}$$

We see that the zero-modes of the Fourier expansion (1.12), i.e. modes that carry no momentum in the internal direction, correspond to massless modes in the $(D-1)$-dimensional theory. In dimensional reduction, one simply neglects the higher Fourier-modes. The resulting theory contains a $(D-1)$-dimensional massless vector field and a massless scalar. One can of course also take into account the higher Fourier modes, which are massive fields from a $(D-1)$-dimensional point of view, as we can see from the second line of the action (1.13). It is important to note that in order to ensure $(D-1)$-dimensional gauge invariance one has to partially gauge-fix the theory. In the example presented here, the $\phi^{(p)}$ are Goldstone bosons for $p > 0$, and their degrees of freedom are absorbed by the $A_\mu^{(p)}$, which in turn become massive. In general, when compactifying a supergravity theory there are a number of gauge choices that one has to make for various fields in order to restrict the symmetries to the lower-dimensional space, and to identify the physical fields. The gauge-fixing procedure is discussed in great detail in chapters 3 and 4. The coupling constant in the $(D-1)$-dimensional theory is related to the D-dimensional coupling constant as $g_{D-1}^2 = g_D^2/(2\pi R)$.

Let us now turn to the compactification of the eleven-dimensional supergravity theory. For the time being, we are not interested in the massive modes,[5] and we simply consider the dimensionally reduced theory. The field content of the dimensionally reduced theory can be easily read off from the eleven-dimensional fields: the elfbein ${E_M}^A$ decomposes into a zehnbein ${e_\mu}^\alpha$, a vector field A_μ and a scalar field ϕ; the Majorana gravitino Ψ_M splits into two Majorana-Weyl gravitinos $\psi_\mu^\pm$ and two Majorana-Weyl fermions $\lambda^\pm$; and the three-form A_{MNP} produces a three-form $A_{\mu\nu\rho}$ and a two-form $A_{\mu\nu}$, which are the fields of type IIA supergravity theory in ten dimensions. The action of the IIA supergravity theory is obtained by substituting these fields into the eleven-dimensional action (1.10), and integrating over the eleventh coordinate. The bosonic part of the Lagrangian of the type IIA supergravity theory is given by

$$\begin{aligned} \kappa_{10}^2 \mathcal{L}_{10} = & -\frac{1}{2} e\, R - \frac{1}{4} e\, \partial_\mu \phi \partial^\mu \phi - \frac{3}{4} e\, \mathrm{e}^{-\phi} (H_{\mu\nu\rho})^2 \\ & - \frac{1}{8} e\, \mathrm{e}^{3\phi/2} (F_{\mu\nu})^2 - \frac{1}{48} e\, \mathrm{e}^{\phi/2} (F_{\mu\nu\rho\sigma})^2 \\ & + \frac{\sqrt{2}}{1152} \varepsilon^{\mu_1 \cdots \mu_{10}} A_{11\mu_1\mu_2} F_{\mu_3\cdots\mu_6} F_{\mu_7\cdots\mu_{10}} . \end{aligned} \tag{1.14}$$

In order to obtain the conventional normalization of the Einstein term, we have performed a field-dependent scale transformation, and the Lagrangian (1.14) is then given in the Einstein frame. We could also choose a different scale transformation and we would obtain the Lagrangian in the string frame, which would better exhibit the string theory origin of the various fields.[6] Let us just remark that from a string theory point of view, the terms in the first line are the NS-NS terms and the terms in the second line are the R-R fields. The third line consists of a Chern-Simons term, where NS-NS and R-R fields couple to each other. In the string frame, all the R-R fields couple uniformly to the dilaton ϕ, and all the NS-NS fields also couple uniformly to the dilaton.

Let us now discuss the relationship between the symmetries of the original, uncompactified, theory in D dimensions and the residual symmetries of the compactified, d-dimensional theory. The uncompactified theory possesses a number of symmetries—space-time symmetries, internal symmetries and supersymmetries. Upon compactification, some of the symmetries are (partially) broken, but there might also be new symmetries appearing that are related to the compactification manifold Σ or the gauge transformation along the compactified directions.

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[5] The massive modes appearing in the compactification are discussed in chapter 3 and in chapter 4.

[6] The supergravity Lagrangian in the string frame is obtained by constructing an effective Lagrangian for the massless fields from the action (1.7).

Supersymmetry breaking depends on the geometry of the compactification manifold Σ and on the existence of certain tensor-field charges: the number of d-dimensional spinor components that are covariantly constant on the compactification manifold Σ with the tensor-field charged turned on is equal to the number of supersymmetry charges that are conserved. For example in the compactification of the eleven-dimensional supergravity theory on a seven-sphere S^7, there are 32 supersymmetry components that are covariantly constant, provided that the field-strength F_{MNPQ} is non-vanishing in the four space-time directions.[7] The ground-state geometry of the four-dimensional space-time is consequently the anti-de Sitter space AdS_4.

In order to avoid a mixing of the massless and massive modes in the compactified theory, one has to decompose the fields in a way that is covariant with respect to the lower-dimensional gauge-symmetries and space-time symmetries. This ensures that one can make sure that the solutions for the d-dimensional field equations are also solutions of the D-dimensional field equations, i.e. one can make sure that the compactified theory can be consistently truncated to the massless modes.

4.2. BPS-extended supergravity

Upon compactification, the diffeomorphisms and gauge transformations related to the coordinates of the internal manifold give rise to additional internal symmetries in the lower-dimensional theory. The action of the massless fields is a priori invariant under SO(n), where $n = D - d$, which corresponds to a rotation of the internal dimensions, but in fact the theory is invariant under a bigger group, which is the R-symmetry group of the supersymmetry algebra in d dimensions.

It turns out that the lower-dimensional theory possesses additional global symmetries. These symmetries are known as "nonlinearly realized symmetries" or "hidden symmetries," and we denote the hidden symmetry group by G. It is not generally possible to realize the hidden symmetry G at the level of the action, but only at the level of the equations of motion.

When the maximal supergravity theory in eleven dimensions is compactified, the resulting lower-dimensional theory has in general less supersymmetry than the $Q = 32$ supercharge components of the original theory. For certain special compactification manifolds, however, all the supersymmetry is preserved, and the lower-dimensional theory is maximally supersymmetric, i.e. it is invariant under $Q = 32$ supercharge components. For example if one compactifies the theory to four dimensions, there are only two manifolds that do not break any supersymmetry: the seven-torus T^7, and the seven-sphere S^7.

[7] This is the so-called Freund-Rubin parameter.

In the latter case the action of the four-dimensional theory also contains a cosmological constant term, and the ground-state geometry is anti de-Sitter space. Both compactifications lead to $\mathcal{N} = 8$ supergravity in four dimensions.

The compactified theory contains symmetries that are induced by the symmetries of the compactification manifold, but there also are additional symmetries, whose origin cannot be directly explained from a Kaluza-Klein point of view. For example, eleven-dimensional supergravity theory compactified on a seven-torus T^7 to four dimensions, is locally invariant under the group SU(8). The equations of motion possess the hidden symmetry $G = \mathrm{E}_{7(7)}$.

In the toroidally compactified theory, the Kaluza-Klein modes are arranged into BPS multiplets. They are charged under seven "Kaluza-Klein central-charges." There are also 49 other central charges, which stem from the reduction of the two-form central charge and the five-form central charge in the eleven-dimensional supersymmetry algebra (1.9) and which are invisible from a Kaluza-Klein point of view. The 56 central charges transform in irreducible representation of the group $H_{\mathrm{R}} = \mathrm{SU}(8)$, but they also combine into a representation of $G = \mathrm{E}_{7(7)}$. Because the BPS multiplets that arise upon toroidal compactification are only a small subset of all the BPS multiplets in the lower-dimensional theory, such a compactification is inherently incomplete. In order to construct a theory that is invariant under G one should therefore add BPS multiplets that arise from wrapped two-branes and five-branes. A supergravity theory coupled to a full set of BPS-multiplets is known as a BPS-extended supergravity theory. Note that in the massless case, the theory is invariant under the continuous symmetry $\mathrm{E}_{7(7)}(\mathbb{R})$, but the theory coupled to the BPS-multiplets is only invariant under the arithmetic subgroup $\mathrm{E}_{7(7)}(\mathbb{Z})$, which is exactly the U-duality group conjectured in string theory.

4.3. Nonlinear sigma-models and gauged supergravity

Gauged supergravity theories are an important extension of conventional, ungauged supergravity theories. They arise in special compactifications of maximal supergravity theories on compact manifolds which are not flat. For example, eleven-dimensional supergravity theory compactified on a seven-torus gives rise to a gauged supergravity theory in four dimensions. Below, we first describe in some detail a vital ingredient for gauged supergravity theories, namely nonlinear sigma models, and then we state some facts about the gauging procedure.

Nonlinear sigma models and gauged supergravity theories are the ingredients of chapter 6, where we give a classification of gauged supergravity theories in three space-time dimensions.

4.3.1. *Nonlinear sigma models*

Supergravity theories in dimensions less than ten generally contain a number of scalar fields whose target space is a coset space G/H. The local group H acts linearly from the right on the matrix of scalar fields $\mathcal{V}$, and the global group G acts linearly from the left. The group G is also called the isometry group of the coset space, and in the models arising in supergravity, is usually a non-compact group. The group H is called the isotropy group of the coset space, it is the maximal compact subgroup of G and it coincides with the R-symmetry group in supergravity theories. The dynamics of the scalar fields $\mathcal{V}$ is then described by a nonlinear sigma model with target space G/H. Throughout this section we assume that the coset space G/H is symmetric.

The global transformations $g \in G$ and the local transformations $h(x) \in H$ act on $\mathcal{V}(x)$ as

$$\begin{aligned} \mathcal{V}(x) &\to g\,\mathcal{V}(x)\,, \\ \mathcal{V}(x) &\to \mathcal{V}(x)\,h^{-1}(x)\,. \end{aligned}$$

We decompose the Lie algebra $\mathfrak{g}$ of G into the Lie algebra $\mathfrak{h}$ of H and its orthogonal complement $\mathfrak{h}_\perp$. In order to obtain connection coefficients of the coset manifold, we calculate

$$\mathcal{V}^{-1}\partial_\mu \mathcal{V} = Q_\mu + P_\mu\,, \tag{1.15}$$

where $Q_\mu \in \mathfrak{h}$ and $P_\mu \in \mathfrak{h}_\perp$. One can easily show that P_μ and Q_μ transform under local transformation $h(x)$ as

$$\begin{aligned} Q_\mu &\to h Q_\mu h^{-1} + h\partial_\mu h^{-1}\,, \\ P_\mu &\to h P_\mu h^{-1}\,. \end{aligned}$$

Therefore, Q_μ is a gauge field for the group H, and P_μ is covariant under local H transformations. Both fields are invariant under global G transformations. We define the H-covariant derivative D_μ by

$$\begin{aligned} P_\mu &= \mathcal{V}^{-1}\partial_\mu \mathcal{V} - Q_\mu \\ &= \mathcal{V}^{-1}\left(\partial_\mu - Q_\mu\right)\mathcal{V} \\ &\equiv \mathcal{V}^{-1} D_\mu \mathcal{V}\,. \end{aligned}$$

The gauge field Q_μ can be used to define the covariant derivatives of the matter fields. For example, the covariant derivative of a spinor ψ is given by

$$D_\mu \psi = \left(\partial_\mu + \frac{1}{4} Q_\mu\right)\psi\,.$$

The Cartan-Maurer equations are obtained from the identity

$$\partial_{[\mu}(\mathcal{V}^{-1}\partial_{\nu]}\mathcal{V}) = -(\mathcal{V}^{-1}\partial_{[\mu}\mathcal{V})(\mathcal{V}^{-1}\partial_{\nu]}\mathcal{V})\,,$$

and they are given by

$$F_{\mu\nu}(Q) = \partial_\mu Q_\nu - \partial_\nu Q_\mu + \left[Q_\mu, Q_\nu\right] = -\left[P_\mu, P_\nu\right] \tag{1.16a}$$

$$\mathcal{D}_{[\mu} P_{\nu]} = \partial_{[\mu} P_{\nu]} + \left[Q_{[\mu}, P_{\nu]}\right] = 0\,. \tag{1.16b}$$

where $F_{\mu\nu}(Q)$ is the field-strength of the unphysical gauge field Q_μ. The physical degrees of freedom can be extracted by fixing a gauge for the local symmetry H, e.g.

$$\mathcal{V}(x) = \mathrm{e}^{\phi(x)} \,, \tag{1.17}$$

where ϕ is an element of $\mathfrak{h}_\perp$. If we act with an element $g \in G$ on $\mathcal{V}(x)$ then the resulting element is in general not in the gauge (1.17). We can reinstate the gauge by a position-dependent compensating H transformation $h(x)$. The transformation is then given by

$$e^{\phi'(x)} = g\, e^{\phi(x)}\, h^{-1}(x, g) \,.$$

The transformation $\phi(x) \to \phi'(x)$ is therefore nonlinearly realized. Often it is easier to keep the unphysical fields Q_μ in the theory and to postpone the nonlinear realization by fixing the gauge for H at a later stage.

The kinetic term for the scalar fields can be written as

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left(P_\mu P^\mu\right) \,,$$

which is clearly invariant under global G transformations and local H transformations. If one fixes the H gauge as in (1.17), the kinetic term takes the form

$$\mathcal{L} = -\frac{1}{2} g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j \,,$$

where g_{ij} is the ϕ-dependent metric of the coset-space.

4.3.2. *Gauging the sigma-model*

If the model we are looking at also describes elementary gauge fields A_μ, we can couple these gauge fields to the scalars by exploiting a subgroup G' of the global group G. That means that we introduce a local gauge symmetry G' for the scalars and treat the fields A_μ as the gauge fields of this symmetry. Note that we do not gauge the local group H, because that would convert the unphysical fields Q_μ into physical fields. The expression (1.15) for the connection coefficients is changed according to

$$\mathcal{V}^{-1}(\partial_\mu + gA_\mu)\mathcal{V} = \mathcal{V}^{-1}D_\mu\mathcal{V} = Q_\mu + P_\mu \,, \tag{1.18}$$

where $Q_\mu \in \mathfrak{h}$ and $P_\mu \in \mathfrak{h}_\perp$. Under local transformations $h(x)$ the elements of the Lie algebra transform as follows,

$$\begin{aligned} Q_\mu &\to hQ_\mu h^{-1} + h\partial_\mu h^{-1} \,, \\ P_\mu &\to hP_\mu h^{-1} \,. \end{aligned}$$

Here again, Q_μ is an H-gauge field and P_μ is a covariant quantity. We define the H and G' covariant derivative $\mathcal{D}_\mu$ as follows

$$\begin{aligned} P_\mu &= \mathcal{V}^{-1}(\partial_\mu - gA_\mu)\mathcal{V} - Q_\mu \\ &\equiv \mathcal{V}^{-1}\mathcal{D}_\mu \mathcal{V} \,. \end{aligned}$$

The Cartan-Maurer structure equations now take the form

$$\begin{aligned} F_{\mu\nu}(Q) &= -\big[P_\mu, P_\nu\big] + g\big(\mathcal{V}^{-1}F_{\mu\nu}(A)\mathcal{V}\big)\Big|_H \,, \\ \mathcal{D}_{[\mu}P_{\nu]} &= \frac{1}{2}g\big(\mathcal{V}^{-1}F_{\mu\nu}(A)\mathcal{V}\big)\Big|_{G/H} \,. \end{aligned}$$

Again, the kinetic term of the scalar fields can be written as $\mathrm{tr}\,(P_\mu P^\mu)$ if one does not impose a gauge, and in the gauge-fixed case the Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2}g_{ij}(\phi)D_\mu\phi^i D^\mu\phi^j$$

which is very similar to the ungauged version, but the partial derivatives have been replaced by covariant derivatives $D_\mu = \partial_\mu - gA_\mu$.

4.3.3. *Gauged supergravity*

The scalar sector of supergravity theories in dimensions lower than ten is described by a nonlinear sigma model with target space a homogeneous, symmetric coset space G/H. In table 1 on the following page we list the groups G and H for dimensions three up to eleven.

The theories also contain abelian vector fields that transform under the global symmetry group G. It is now possible to gauge a subgroup G' of G, and thus promote the abelian vector fields to non-abelian vector fields. Such a theory is called a gauged supergravity theory. In order to keep track of the non-abelianness of the theory, one introduces the non-abelian coupling constant g, as we have seen in section 4.3.2 above.

Gauged supergravity theories differ in a few aspects from normal supergravity theories. First of all, the field strength of the gauge fields is modified in the usual way in order to account for the non-abelianness of the gauge fields. Second, the kinetic term of the scalar fields changes because it contains a G'-covariant derivative, as we have seen in section 4.3.2. Third, covariant derivatives of additional bosonic fields have to be made G'-covariant. Furthermore, in order to preserve supersymmetry, one has to add new terms to the Lagrangian. There are masslike terms for the fermions, which are linear in the non-abelian coupling g, and there is a potential term for the scalar fields, which is quadratic in g. Because of the potential term for the scalar fields, it is in general possible to find non-trivial ground-state solutions, i.e. ground-states with non-vanishing cosmological constant.

dimension	G	H	# scalars
11	$\mathbb{1}$	$\mathbb{1}$	0
10 (IIA)	SO(1,1)/$\mathbb{Z}_2$	SO(2)	1
10 (IIB)	SL(2)	SO(2)	2
9	GL(2)	SO(2)	3
8	SL(3) × SL(2)	U(2)	7
7	SL(5)	USp(4)	14
6	SO(5,5)	USp(4) × USp(4)	25
5	$E_{6(6)}$	USp(8)	42
4	$E_{7(7)}$	SU(8)	70
3	$E_{8(8)}$	SO(16)	128

Table 1. Global symmetry groups G and local symmetry groups H for maximally symmetric supergravity theories in various dimensions.

In four dimensions, for example, the isometry group G of ungauged $\mathcal{N} = 8$ supergravity theory is the exceptional group $E_{7(7)}$, and the isotropy group H is SU(8). A gauged supergravity theory can now be constructed by gauging SO(8) $\subset E_{7(7)}$. It is easy to understand this gauged supergravity theory from an eleven-dimensional point of view; namely, it emerges in the compactification on a seven-sphere S^7, and the gauge group SO(8) is simply the isometry group of the seven-sphere.

2

Supersymmetry in anti-de Sitter space

Instead of discussing supersymmetry in flat Minkowski space, we now present an introduction to supersymmetry in anti-de Sitter space, which is an Einstein space of constant negative curvature. In the framework of this thesis, this chapter serves mainly two purposes: first to deepen our general understanding of supersymmetry applied to field theories and gravity theories, and to point out some features of supersymmetry that are generally ignored when considering supersymmetry in flat space-time. Second, this chapter prepares the reader for the study of supergravity in three dimensions in chapter 6, where we will see that anti-de Sitter space is often found as a ground-state geometry.

Field theory in anti-de Sitter space is not a new subject. Already in the thirties of the twentieth century Dirac considered wave equations that are invariant under the anti-de Sitter group [18]. Later, in 1963, he discovered the 'remarkable representation' which is now known as the singleton [19]. Shortly afterwards there was a series of papers by Fronsdal and collaborators discussing the representations of the anti-de Sitter group [20–23]. Quantum field theory in anti-de Sitter space was studied for instance in [24, 25]. Many new developments were inspired by the discovery that gauged supergravity theories have ground states corresponding to anti-de Sitter space-times [26–36]. This led to a study of the stability of these ground states with respect to fluctuations of the scalar fields [37] as well as to an extended discussion of supermultiplets in anti-de Sitter space [37–42]. In recent years, field theory in anti-de Sitter space has attracted a lot of interest because of the so-called AdS/CFT correspondence [43–46]. This conjecture states that certain supergravity theories on anti-de Sitter space-times are in some sense dual to field theories on the boundary of the anti-de Sitter space. For example, ten-dimensional IIB supergravity theory on a space-time of the form $AdS_5 \times S^5$ is conjectured to be dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions with gauge group U(n).

In the following, we are able to cover only a few of these topics. We restrict ourselves to an introduction to supersymmetry in anti-de Sitter space and discuss the presence of the so-called masslike terms in wave equations for various fields in anti-de Sitter space. Then we will analyze the various irreducible representations of the anti-de Sitter isometry group, and at the end we will consider the consequences for supermultiplets. We emphasize the issue of multiplet shortening for both multiplets of given spin and for supermultiplets. Throughout the whole chapter we make contact with supersymmetry in flat space, i.e. we show that Poincaré supersymmetry can be obtained by taking a certain limit in anti-de Sitter supersymmetry.

This chapter is based on [47].

1. Supersymmetry and anti-de Sitter space

In this section, we discuss some properties of anti-de Sitter space and consider the supersymmetry algebra in anti-de Sitter space. We will also give an example of a simple supergravity theory in anti-de Sitter space.

1.1. Properties of anti-de Sitter space

Anti-de Sitter space is a maximally symmetric space with constant negative curvature. It has $d(d+1)/2$ isometries which constitute the group $\mathrm{SO}(d-1,2)$. Anti-de Sitter space can be described as a hypersurface embedded into a $(d+1)$-dimensional embedding space. Denoting the extra coordinate of the embedding space by Y^-, so that we have coordinates Y^A with $A=-,0,1,2,\ldots,d-1$, this hypersurface is defined by

$$-(Y^-)^2-(Y^0)^2+\boldsymbol{Y}^2=\eta_{AB}\,Y^AY^B=-g^{-2}\,. \tag{2.1}$$

The parameter g is the inverse radius of the anti-de Sitter space. In the limit of $g\to 0$ one recovers flat d-dimensional Minkowski space. The hypersurface defined by (2.1) is invariant under linear transformations that leave the metric $\eta_{AB}=\mathrm{diag}\,(-,-,+,+,\ldots,+)$ invariant. These transformations constitute the isometry group $\mathrm{SO}(d-1,2)$. The $d(d+1)/2$ generators of the group $\mathrm{SO}(d-1,2)$, denoted by M_{AB}, act on the embedding coordinates by

$$M_{AB}=Y_A\frac{\partial}{\partial Y^B}-Y_B\frac{\partial}{\partial Y^A}\,,$$

where we lower and raise indices by contracting with η_{AB} and its inverse η^{AB}.

Anti-de Sitter space is a homogeneous space, which means that any two points on it can be related via an isometry. It has the topology of S^1 [time] $\times$ $\mathbb{R}^{d-1}$. When unwrapping S^1 one finds the universal covering space denoted by CAdS, which has the topology of $\mathbb{R}^d$. There are many ways to coordinatize anti-de Sitter space; however, we will try to avoid using specific coordinates.

1.2. The supersymmetry algebra

The generators M_{AB} of the isometries $\mathrm{SO}(d-1,2)$ form an algebra. The commutation relations for two generators M_{AB} are given by

$$[M_{AB},M_{CD}]=\eta_{BC}\,M_{AD}-\eta_{AC}\,M_{BD}-\eta_{BD}\,M_{AC}+\eta_{AD}\,M_{BC}\,. \tag{2.2}$$

On spinors, the anti-de Sitter algebra can be realized by the following combination of gamma matrices,

$$M_{AB} = \frac{1}{2}\Gamma_{AB} = \begin{cases} \frac{1}{2}\Gamma_{ab} & \text{for} \quad A, B = a, b\,, \\ \frac{1}{2}\Gamma_{a} & \text{for} \quad A = -\,, B = a\,, \end{cases}$$

with $a, b = 0, 1, \ldots, d-1$. Our gamma matrices satisfy the Clifford property $\{\Gamma^a\,,\ \Gamma^b\} = 2\,\eta^{ab}\,\mathbb{1}$, where $\eta^{ab} = \mathrm{diag}\,(-, +, \ldots, +)$.

The supersymmetric extension of the algebra (2.2) is called the anti-de Sitter superalgebra. It contains the following additional (anti-)commutation relations,

$$\{Q_\alpha, \bar{Q}_\beta\} = -\frac{1}{2}\big(\Gamma_{AB}\big)_{\alpha\beta}\, M^{AB}\,, \tag{2.3a}$$

$$\big[M_{AB}, \bar{Q}_\alpha\big] = \frac{1}{2}\big(\bar{Q}\,\Gamma_{AB}\big)_\alpha\,. \tag{2.3b}$$

In order to recognize the relation between the anti-de Sitter superalgebra and the Poincaré superalgebra one rescales the fields as follows,

$$Q \longrightarrow \frac{1}{\sqrt{g}}Q\,,$$
$$M_{-a} \longrightarrow \frac{1}{g}M_{-a}\,.$$

One then takes the limit $g \to 0$, which corresponds to taking the radius of anti-de Sitter space to infinity and reproduces flat Minkowski space. This singular transformation is a so-called Wigner-Inönü contraction [48], and the resulting algebra is the Poincaré superalgebra. Important relations in the Poincaré superalgebra are

$$\{Q, \bar{Q}\} = -\frac{1}{2}\Gamma^a P_a\,, \tag{2.4a}$$

$$\big[M_{ab}, \bar{Q}\big] = \frac{1}{2}\left(\bar{Q}\Gamma_{ab}\right)\,, \tag{2.4b}$$

where we have defined the momentum operator $P_a \equiv M_{-a}$. One important difference between the Poincaré superalgebra and the anti-de Sitter superalgebra is that in the former, P^2 commutes with all elements of the algebra, and P^2 is therefore a Casimir operator. In the anti-de Sitter superalgebra $(M_{-a})^2$, which corresponds to P^2 in the Poincaré algebra, is not a Casimir operator. Instead, the quadratic Casimir operator for the anti-de Sitter algebra is given by

$$\mathcal{C}_2 = -\frac{1}{2}M^{AB}M_{AB}\,.$$

In general, i.e. for $D > 3$, there are also higher order Casimir operators for the anti-de Sitter algebra, but we will not be concerned with those. The quadratic

Casimir operator is related to the covariantized d'Alambertian operator $\Box_{\rm AdS}$ in anti-de Sitter space. In can be shown [16] that

$$\mathcal{C}_2 = \Box_{\rm AdS} + \mathcal{C}_2^L \,, \tag{2.5}$$

where $\mathcal{C}_2^L$ is the quadratic Casimir operator for the spin-s representation of the Lorentz group [49]. As we will see at the end of section 2, $\mathcal{C}_2^L$ vanishes for scalar fields, and the d'Alambertian equals the quadratic Casimir operator.

The structure of the anti-de Sitter algebra changes drastically for dimensions $d > 7$, see [50] and references cited therein. For $d \leq 7$ the bosonic subalgebra coincides with the anti-de Sitter algebra. As we have seen in section 1 of chapter 1, there are $\mathcal{N}$-extended versions of supersymmetry, where we introduce $\mathcal{N}$ supersymmetry generators, each transforming as a spinor under the anti-de Sitter group. These $\mathcal{N}$ generators transform under a compact group, whose generators appear as central charges in the $\{Q, \bar{Q}\}$ anticommutator. For $d > 7$ the bosonic subalgebra can no longer be restricted to the anti-de Sitter algebra and the algebra corresponding to a compact group, but one needs extra bosonic generators that transform as high-rank antisymmetric tensors under the Lorentz group. In contrast to this, there exists an ($\mathcal{N}$-extended) super-Poincaré algebra associated with flat Minkowski space of any dimension, whose bosonic generators correspond to the Poincaré group, possibly augmented with the generators of a compact group associated with rotations of the supercharges.

In this chapter, we are mainly dealing with the case $\mathcal{N} = 1$ and we always assume that $d \leq 7$.

1.3. Supergravity

Let us now study a simple supergravity theory in an unspecified number of space-time dimensions. In section 4 of chapter 1, we have already encountered the standard Einstein-Hilbert Lagrangian of general relativity and the Rarita-Schwinger Lagrangian for the gravitino field(s), cf. (1.10). We now add a cosmological term to the Lagrangian as well as a suitably chosen masslike term for the gravitino field,

$$\begin{aligned}\mathcal{L} = & -\frac{1}{2}e\,R(\omega) - \frac{1}{2}e\bar{\psi}_\mu\,\Gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho \\ & + \frac{1}{4}g(d-2)e\,\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu + \frac{1}{2}g^2(d-1)(d-2)\,e + \cdots \,,\end{aligned} \tag{2.6}$$

where the covariant derivative on a spinor ψ reads

$$D_\mu(\omega)\psi = \left(\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\right)\psi \,.$$

Here, $\omega_\mu{}^{ab}$ is the spin-connection field defined such that the torsion tensor (or a supercovariant version thereof) vanishes. As we will see shortly, the cosmological constant term in (2.6) can give rise to an anti-de Sitter space-time. The masslike term for the gravitino is required if one demands that the ground-state preserves supersymmetry.

It turns out that the action corresponding to (2.6) is locally supersymmetric, up to terms that are cubic in the gravitino field, and the supersymmetry transformation rules are given by,

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\bar\epsilon\,\Gamma^a\psi_\mu\,,\\ \delta\psi_\mu &= \left(D_\mu(\omega) + \frac{1}{2}g\Gamma_\mu\right)\epsilon\,. \end{aligned}$$

Note that the second term in the transformation rule for the gravitino is induced by the cosmological constant term in the Lagrangian and the masslike term for the gravitino.

The above demonstrates that, a priori, supersymmetry does not forbid a cosmological term, but it must be of definite sign—at least, if the ground state is to preserve supersymmetry.[1] To construct a fully supersymmetric field theory is difficult and there are strong restrictions on the number of space-time dimensions. The Lagrangian (2.6) was first written down in [53] in four space-time dimensions and the correct interpretation of the masslike term was given in [54].

The Einstein-Hilbert equation corresponding to (2.6) reads (suppressing the gravitino field),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R + \frac{1}{2}g^2(d-1)(d-2)\,g_{\mu\nu} = 0\,,$$

which implies that

$$R_{\mu\nu} = g^2(d-1)\,g_{\mu\nu}\,,\qquad R = g^2 d(d-1)\,.$$

Hence we are dealing with a d-dimensional Einstein space. The maximally symmetric solution of this equation is an anti-de Sitter space, whose Riemann curvature equals

$$R_{\mu\nu}{}^{ab} = 2g^2\,e_\mu{}^{[a}\,e_\nu{}^{b]}\,.$$

This solution leaves all the supersymmetries intact just as flat Minkowski space does. One can verify this directly by considering the supersymmetry variation of the gravitino field and by requiring that it vanishes in the bosonic background. This happens for spinors $\epsilon(x)$ satisfying

$$\left(D_\mu(\omega) + \frac{1}{2}g\Gamma_\mu\right)\epsilon = 0\,. \tag{2.7}$$

[1] For a discussion see [51, 52] and references therein.

Spinors satisfying this equation are called Killing spinors. As a direct consequence of (2.7) also $(D_\mu(\omega) + g\Gamma_\mu/2)(D_\nu(\omega) + g\Gamma_\nu/2)\epsilon$ must vanish. Antisymmetrizing this expression in μ and ν then yields the integrability condition

$$\left(-\frac{1}{4}R_{\mu\nu}{}^{ab}\,\Gamma_{ab} + \frac{1}{2}g^2\,\Gamma_{\mu\nu}\right)\epsilon = 0\,,$$

which is precisely satisfied in anti-de Sitter space.

As we have seen, anti-de Sitter space is consistent with supersymmetry. This is just as for flat Minkowski space, which has the same number of isometries but now corresponding to the Poincaré group, and which is also consistent with supersymmetry. We have already mentioned that the two cases are related; namely, the flat space results are obtained in the limit $g \to 0$.

The commutator of two supersymmetry transformations yields an infinitesimal general-coordinate transformation and a tangent-space Lorentz transformation. For example, we obtain for the vielbein,

$$\begin{aligned}\left[\delta_1, \delta_2\right] e_\mu{}^a &= \frac{1}{2}\bar\epsilon_2\,\Gamma^a\,\delta_1\psi_\mu - \frac{1}{2}\bar\epsilon_1\,\Gamma^a\,\delta_2\psi_\mu \\ &= \frac{1}{2}D_\mu\big(\bar\epsilon_2\,\Gamma^a\epsilon_1\big) + \frac{1}{2}g\,\big(\bar\epsilon_2\,\Gamma^{ab}\epsilon_1\big)\,e_{\mu b}\,.\end{aligned} \tag{2.8}$$

We remind the reader of the fact that we are dealing with an incomplete theory. For a complete theory the above result should hold uniformly on all the fields (possibly modulo field equations). As before we have ignored terms proportional to the gravitino field. In the anti-de Sitter background the vielbein is left invariant by the combination of symmetries on the right-hand side. Consequently the metric is invariant under these coordinate transformations and we have the so-called Killing equation,

$$\delta g_{\mu\nu} = D_\mu\xi_\nu + D_\nu\xi_\mu = 0\,, \tag{2.9}$$

where $\xi_\mu = (\bar\epsilon_2\,\Gamma_\mu\epsilon_1)/2$ is a Killing vector and where $\epsilon_{1,2}$ are Killing spinors. Since $D_\mu\xi_\nu = (g\bar\epsilon_2\Gamma_{\mu\nu}\epsilon_1)/2$, the right-hand side of (2.8) vanishes for this choice of supersymmetry parameters, and ξ^μ satisfies the Killing equation (2.9). As for all Killing vectors, higher derivatives can be decomposed into the Killing vector and its first derivative, e.g. $D_\mu(g\,\bar\epsilon_2\Gamma_{\nu\rho}\epsilon_1) = -g^2\,g_{\mu[\rho}\xi_{\nu]}$. The Killing vector can be decomposed into the $d(d+1)/2$ Killing vectors of the anti-de Sitter space.

2. Anti-de Sitter supersymmetry and masslike terms

Readers familiar with supersymmetry in flat space will remember that in Minkowski space all fields belonging to a supermultiplet are subject to field equations with the same mass. This follows from the fact that the momentum operator P_μ commutes with the supersymmetry charges, so that P^2 is a Casimir operator. As we have seen in the previous section, P^2 is not a Casimir operator in anti-de Sitter space, but M^2_{AB} plays that role instead. Therefore masslike terms are not necessarily the same for different fields belonging to the same anti-de Sitter supermultiplet. In the following we illustrate this phenomenon for the example of a chiral supermultiplet in four space-time dimensions. Further facts about the anti-de Sitter superalgebra will be given in section 4.

A chiral supermultiplet in four space-time dimensions consists of a scalar field A, a pseudoscalar field B and a Majorana spinor field ψ. In anti-de Sitter space the supersymmetry transformations of the fields are proportional to a spinor parameter $\epsilon(x)$, which is a Killing spinor in the anti-de Sitter space, i.e. $\epsilon(x)$ must satisfy the Killing spinor equation (2.7). We allow for two constants a and b in the supersymmetry transformations, which we parameterize as follows,

$$\begin{aligned}
\delta A &= \frac{1}{4}\bar{\epsilon}\psi \,, \\
\delta B &= \frac{1}{4} i\bar{\epsilon}\gamma_5\psi \,, \\
\delta\psi &= \not{D}\big(A + i\gamma_5 B\big)\epsilon - \big(a\,A + i b\,\gamma_5 B\big)\epsilon \,.
\end{aligned}$$

The coefficient of the first term in $\delta\psi$ has been chosen such as to ensure that $[\delta_1, \delta_2]$ yields the correct coordinate transformation $\xi^\mu D_\mu$ on the fields A and B. To determine the constants a and b and the field equations of the chiral multiplet, we consider the closure of the supersymmetry algebra on the spinor field. After some Fierz reordering we find

$$\begin{aligned}
\big[\delta_1, \delta_2\big]\psi = \xi^\mu D_\mu\psi &+ \frac{1}{16}(a-b)\,\bar{\epsilon}_2\gamma^{ab}\epsilon_1\,\gamma_{ab}\psi \\
&- \frac{1}{2}\xi^\rho\gamma_\rho\left(\not{D}\psi + \frac{1}{2}(a+b)\psi\right) .
\end{aligned}$$

We point out that derivatives acting on $\epsilon(x)$ occur in this calculation at an intermediate stage and should not be suppressed in view of (2.7). However, they produce terms proportional to g which turn out to cancel in the above commutator. Now we note that the right-hand side should constitute a coordinate transformation and a Lorentz transformation, possibly up to a field equation. Obviously, the coordinate transformation coincides with (2.8) but the correct Lorentz transformation is only reproduced provided that $a - b = 2g$. If we now denote the mass of the fermion by $m = (a+b)/2$, so that the last term is just the Dirac equation with mass m, then we find

$$a = m + g \,, \qquad b = m - g \,.$$

Consequently, the supersymmetry transformation of the ψ equals

$$\delta\psi = \not{D}(A + i\gamma_5 B)\epsilon - m(A + i\gamma_5 B)\epsilon - g(A - i\gamma_5 B)\,\epsilon \,, \tag{2.10}$$

and the fermionic field equation equals $(\not{D} + m)\psi = 0$. The second term in (2.10), which is proportional to m, can be accounted for by adding an auxiliary field to the supermultiplet. The third term, which is proportional to g, can be understood as a compensating S-supersymmetry transformation associated with auxiliary fields in the supergravity sector, see e.g. [55]. In order to construct the corresponding field equations for A and B, we consider the variation of the fermionic field equation. Again we have to take into account that derivatives on the supersymmetry parameter are not equal to zero. This yields the following second-order differential equations,

$$\left(\Box_{\text{AdS}} + 2g^2 - m(m-g)\right) A = 0\,, \tag{2.11a}$$

$$\left(\Box_{\text{AdS}} + 2g^2 - m(m+g)\right) B = 0\,, \tag{2.11b}$$

$$\left(\Box_{\text{AdS}} + 3g^2 - m^2\right) \psi = 0\,. \tag{2.11c}$$

The last equation follows from the Dirac equation. Namely, one evaluates $(\not{D}-m)(\not{D}+m)\psi$, which gives rise to the wave operator $\Box_{\text{AdS}} + [\not{D},\not{D}]/2 - m^2$. The commutator yields the Riemann curvature of the anti-de Sitter space. In an anti-de Sitter space of arbitrary dimension d this equation then reads,

$$\left(\Box_{\text{AdS}} + \frac{1}{4}d(d-1)g^2 - m^2\right)\psi = 0\,,$$

which, for $d = 4$ agrees with (2.11c). A striking feature of the above result is that the field equations (2.11) all have different mass terms, in spite of the fact that they belong to the same supermultiplet. Consequently, the role of mass is quite different in anti-de Sitter space as compared to flat Minkowski space. This will be elucidated later.

The g^2 term in the field equations for the scalar fields can be understood from the observation that the scalar d'Alambertian can be extended to a conformally invariant operator, see e.g. [55],

$$\Box + \frac{1}{4}\frac{d-2}{d-1}\,R = \Box + \frac{1}{4}d(d-2)\,g^2\,, \tag{2.12}$$

which seems the obvious candidate for a massless wave operator for scalar fields. Indeed, for $d = 4$, we do reproduce the g^2 dependence in the first two equations (2.11). Observe that the Dirac operator $\not{D}$ is also conformally invariant.

The quadratic Casimir operator for the Lorentz group in four dimensions, $\mathcal{C}_2^L$ takes the values 0 and 3/2 for scalars and spinors, respectively. Comparing these values with the field equations (2.11) and with (2.5) yields the following values for the Casimir operators of the scalar field and the spinor,

$$\mathcal{C}_2^L(\text{scalar}) = -2 + m^2\,,$$

$$\mathcal{C}_2^L(\text{spinor}) = -\frac{3}{2} + m^2\,.$$

This shows very clearly that for scalar fields and for spinor fields, the coefficient m^2 is not the mass term in the equation of motion.

3. Unitary representations of the anti-de Sitter algebra

In this section we discuss unitary representations of the anti-de Sitter algebra. For definiteness we mainly look at the case of four space-time dimensions. We will be able to construct massless and massive multiplets, and we will encounter the phenomenon of multiplet shortening for certain massive multiplets. In this context, a special representation that is not known from the Poincaré algebra is discussed, the so-called singleton representation. We refer to [20–23] for some of the original work, and to [39, 40] where some of this work was reviewed.

In order to underline the general features we start in d space-time dimensions. Obviously, the group $\mathrm{SO}(d-1,2)$ is non-compact. This implies that unitary representations will be infinitely-dimensional. The generators are then all anti-hermitian,

$$M_{AB}^{\dagger} = -M_{AB}\,.$$

Note that the covering group of $\mathrm{SO}(d-1,2)$ has the generators $\Gamma^{\mu\nu}/2$ and $\Gamma^{\mu}/2$. They act on spinors, which are finite-dimensional objects. These generators, however, have different hermiticity properties from the ones above.

The compact subgroup of the anti-de Sitter group is $\mathrm{SO}(2)\times\mathrm{SO}(d-1)$ corresponding to rotations of the compact anti-de Sitter time and spatial rotations. It is convenient to decompose the $d(d+1)/2$ generators as follows. We have seen in section 1.2 that the generator M_{-0} is related to the energy operator when the radius of the anti-de Sitter space is taken to infinity. The eigenvalues of this generator, which is associated with motions along the circle, are quantized in integer units in order to have single-valued functions, unless one goes to the covering space CAdS. So we define the energy operator H by

$$H = -iM_{-0}\,.$$

Obviously the generators of the spatial rotations are the operators M_{ab} with $a, b = 1, \ldots, d-1$. Note that we have changed notation: here and henceforth the indices $a, b, \ldots$ refer only to space-like indices. The remaining $2(d-1)$ generators M_{-a} and M_{0a} are combined into pairs of mutually conjugate operators,

$$M_a^{\pm} = -iM_{0a} \pm M_{-a}\,,$$

and we have $(M_a^+)^\dagger = M_a^-$. The anti-de Sitter commutation relations now read

$$\big[H, M_a^\pm\big] = \pm M_a^\pm\,, \tag{2.13a}$$

$$\big[M_a^\pm, M_b^\pm\big] = 0\,, \tag{2.13b}$$

$$\big[M_a^+, M_b^-\big] = -2(H\,\delta_{ab} + M_{ab})\,. \tag{2.13c}$$

Obviously, the operators $M_a^\pm$ play the role of raising and lowering operators: when applied to an eigenstate of H with eigenvalue E, application of $M_a^\pm$ yields a state with eigenvalue $E \pm 1$.

In this section we restrict ourselves to the bosonic case. Nevertheless, let us briefly indicate how some of the other (anti-)commutators of the anti-de Sitter superalgebra decompose, cf. (2.3),

$$\begin{aligned}\big\{Q_\alpha\,, Q_\beta^\dagger\big\} = {}& H\,\delta_{\alpha\beta} - \frac{1}{2} i\,M_{ab}\left(\Gamma^a\Gamma^b\Gamma^0\right)_{\alpha\beta} \\ & + \frac{1}{2}\Big(M_a^+\,\Gamma^a\left(1 + i\,\Gamma^0\right) + M_a^-\,\Gamma^a\left(1 - i\,\Gamma^0\right)\Big)_{\alpha\beta}\,,\end{aligned} \tag{2.14a}$$

$$\big[H\,, Q_\alpha\big] = -\frac{1}{2} i\big(\Gamma^0\,Q\big)_\alpha\,, \tag{2.14b}$$

$$\big[M_a^\pm\,, Q_\alpha\big] = \mp\frac{1}{2}\Big(\Gamma_a\big(1 \mp i\,\Gamma^0\big)\,Q\Big)_\alpha\,. \tag{2.14c}$$

For the anti-de Sitter superalgebra, all the bosonic operators can be expressed as bilinears of the supercharges, so that in principle one could restrict oneself to fermionic operators only and employ the projections $(1 \pm i\,\Gamma^0)Q$ as the basic lowering and raising operators. However, this is not quite what we will be doing later in section 4.

Let us now assume that the spectrum of H is bounded from below,

$$H \geq E_0\,,$$

so that in mathematical terms we are considering lowest-weight irreducible unitary representations. The lowest eigenvalue E_0 is realized on states that we denote by $|E_0, s\rangle$, where E_0 is the eigenvalue of H and s indicates the value of the total angular momentum operator. Of course there are more quantum numbers, e.g. associated with the angular momentum operator directed along some axis (in $d = 4$ there are thus $2s + 1$ degenerate states), but this is not important for the construction and these quantum numbers are suppressed. Since states with $E < E_0$ should not appear, ground states are characterized by the condition,

$$M_a^-\,|E_0, s\rangle = 0\,.$$

The representation can now be constructed by acting with the raising operators on the vacuum state $|E_0, s\rangle$. To be precise, all states of energy $E = E_0 + n$

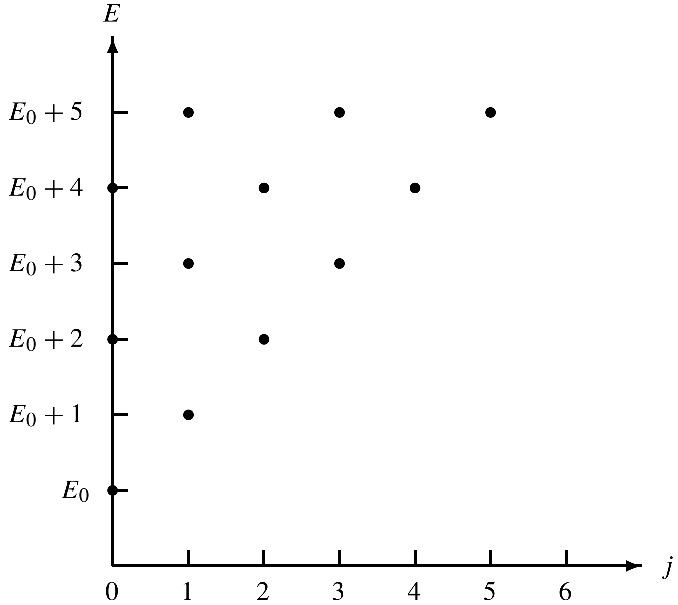

Figure 1. States of the spinless representation in terms of the energy eigenvalues E and the angular momentum j. Each point has a $(2j + 1)$-fold degeneracy.

are constructed by an n-fold product of creation operators M_a^+. In this way one obtains states of higher eigenvalues E with higher spin. The simplest case is the one where the vacuum has no spin ($s = 0$). For given eigenvalue E, the highest spin state is given by the traceless symmetric product of $E - E_0$ operators M_a^+ on the ground state. These states are shown in figure 1.

Henceforth we specialize to the case $d = 4$ in order to keep the aspects related to spin simple. To obtain spin-$1/2$ is trivial; one simply takes the direct product with a spin-$1/2$ state. That implies that every point with spin j in figure 1 generates two points with spin $j \pm 1/2$, with the exception of points associated with $j = 0$, which will simply move to $j = 1/2$. The result of this is shown in figure 2 on the next page.

Likewise one can take the direct product with a spin-1 state, but now the situation is more complicated as the resulting multiplet is not always irreducible. In principle, each point with spin j now generates three points, associated with j and $j \pm 1$, again with the exception of the $j = 0$ points, which simply move to $j = 1$. The result of this procedure is shown in figure 3 on page 31.

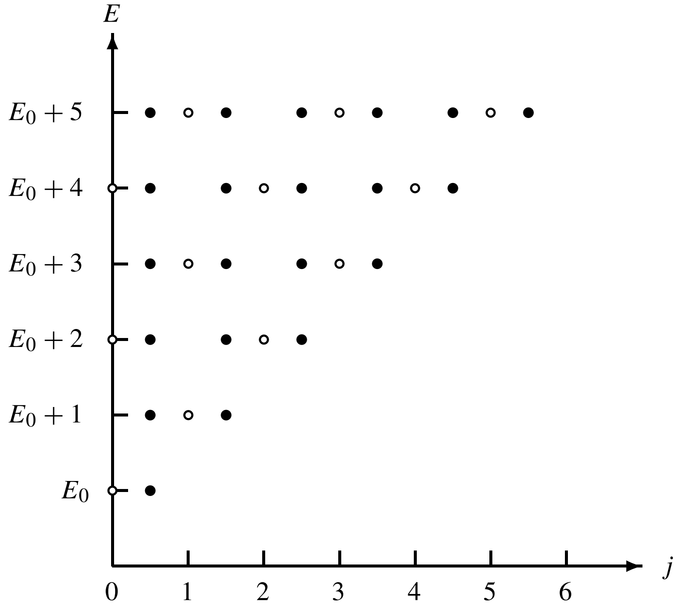

Figure 2. States of the spinor representation in terms of the energy eigenvalues E and the angular momentum; the half-integer values for $j = l + 1/2$ denote a symmetric tensor-spinor of rank l. The small circles denote the original spinless multiplet from which the spinor multiplet has been constructed by a direct product with a spinor.

Let us now turn to the quadratic Casimir operator, which for four spacetime dimensions can be written as

$$\begin{aligned}\mathcal{C}_2 &= -\frac{1}{2}M^{AB}M_{AB}\\ &= H^2 - \frac{1}{2}\{M_a^+, M_a^-\} - \frac{1}{2}(M_{ab})^2\\ &= H(H-3) - \frac{1}{2}(M_{ab})^2 - M_a^+M_a^- .\end{aligned}$$

Applying the last expression on the ground state $|E_0, s\rangle$ we derive

$$\mathcal{C}_2 = E_0(E_0 - 3) + s(s+1)\,, \tag{2.15}$$

and, since $\mathcal{C}_2$ is a Casimir operator, this result holds for any state belonging to the corresponding irreducible representation. Note, that the angular momentum operator is given by $\boldsymbol{J}^2 = -(M_{ab})^2/2$. Assuming that E_0 only takes real

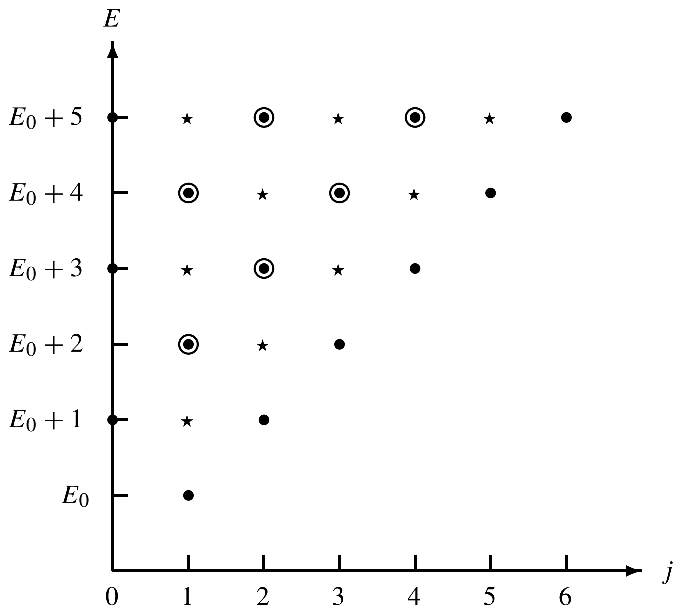

Figure 3. States of the spin-1 representation in terms of the energy eigenvalues E and the angular momentum j. Observe that there are now points with double occupancy, indicated by the circle superimposed on the dots and states transforming as mixed tensors (with rank $l = j$) denoted by a star. The double-occupancy points exhibit the structure of a spin-0 multiplet with ground state energy $E_0 + 1$. This multiplet becomes reducible and can be dropped when $E_0 = 2$, as is explained in the text. The remaining points then constitute a massless spin-1 multiplet, shown in figure 4 on the next page.

values, (2.15) imposes a lower bound on $\mathcal{C}_2$,

$$\mathcal{C}_2 \geq s(s+1) - \frac{9}{4}\,.$$

We can apply this result to an excited state (which is generically present in the spectrum) with $E = E_0 + 1$ and $j = s - 1$. Here, we assume that the ground state has $s \geq 1$. In that case we find

$$\begin{aligned}\mathcal{C}_2 &= (E_0+1)(E_0-2) + s(s-1) - \big|M_a^- |E_0+1, s-1\rangle\big|^2 \\ &= E_0(E_0-3) + s(s+1)\,,\end{aligned}$$

so that

$$E_0 - s - 1 = \frac{1}{2}\big|M_a^- |E_0+1, s-1\rangle\big|^2\,.$$

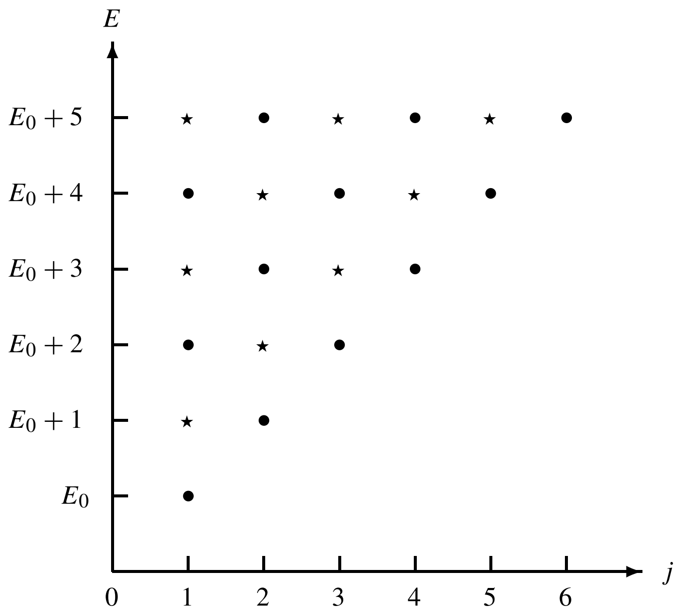

Figure 4. States of the massless $s = 1$ representation in terms of the energy eigenvalues E and the angular momentum j. Now E_0 is no longer arbitrary but it is fixed to $E_0 = 2$.

This shows that $E_0 \geq s + 1$ in order to have a unitary multiplet. When $E_0 = s+1$, however, the state $|E_0+1, s-1\rangle$ is itself a ground state, which decouples from the original multiplet, together with its corresponding excited states. This can be interpreted as the result of a gauge symmetry and therefore we call these multiplets massless. Hence massless multiplets with $s \geq 1$ are characterized by

$$E_0 = s + 1\,, \qquad \text{for} \quad s \geq 1\,.$$

For these particular values the quadratic Casimir operator is

$$\mathcal{C}_2 = 2(s^2 - 1)\,. \tag{2.16}$$

Although this result is only derived for $s \geq 1$, it also applies to massless $s = 0$ and $s = 1/2$ representations, as we shall see later. Massless $s = 0$ multiplets have either $E_0 = 1$ or $E_0 = 2$, while massless $s = 1/2$ multiplets have $E_0 = 3/2$.

One can try and use the same argument again to see if there is a possibility that even more states decouple. Consider for instance a state with the same spin as the ground state, with energy E. In that case

$$E(E-3) = E_0(E_0 - 3) + \left|M_a^- |E, s\rangle\right|^2\,. \tag{2.17}$$

For spin $s \geq 1$, this condition is always satisfied in view of the bound $E_0 \geq s+1$. But for $s=0$, one can apply (2.17) for the first excited $s=0$ state which has $E = E_0 + 2$. In that case one derives

$$2(2E_0 - 1) = \left| M_a^- |E_0 + 2, s = 0\rangle \right|^2 ,$$

so that

$$E_0 \geq \frac{1}{2} .$$

For $E_0 = 1/2$ we have the so-called singleton representation, where we have only one state for a given value of the spin. A similar result can be derived for $s = 1/2$, where one can consider the first excited state with $s = 1/2$, which has $E = E_0 + 1$. One then derives

$$2(E_0 - 1) = \left| M_a^- \left| E_0 + 1, s = 1/2 \right\rangle \right|^2 ,$$

so that

$$E_0 \geq 1 .$$

For $E_0 = 1$ we have the spin-1/2 singleton representation, where again we are left with just one state for every spin value. The existence of these singleton representations was first noted by Dirac [19]. They are shown in figure 5 on the following page. Both singletons have the same value of the Casimir operator,

$$\mathcal{C}_2 = -\frac{5}{4} .$$

In four dimensions, the spin-0 singleton and the spin-1/2 singleton are the only singleton representations. For dimensions higher than four, there are infinitely many singleton representations, which is related to the fact that the rotation group is of higher rank, so that there is a large variety of representations. Singleton representations do not have a flat space limit, and they therefore have no analogue in the Poincaré superalgebra. In order to understand this phenomenon, note that Poincaré representations correspond to plane waves, which can be decomposed into an infinite number of spherical harmonics. Therefore, for any given spin, one is dealing with an infinite tower of modes. The spectrum of the singleton, on the other hand, is different because a state has a single energy eigenvalue for any given value of the spin, as shown in figure 5 on the next page.

In the above treatment, we have come across the phenomenon of multiplet shortening for specific values of the energy and the spin of the representation. In fact, this is very similar to the multiplet shortening of massive multiplets in flat space to BPS multiplets, which occurs when the mass and the central charge obey the BPS relation (1.2).

From the above it is clear that we are dealing with the phenomenon of multiplet shortening for specific values of the energy and spin of the ground state. This can be understood more generally from the fact that the $[M_a^+, M_b^-]$

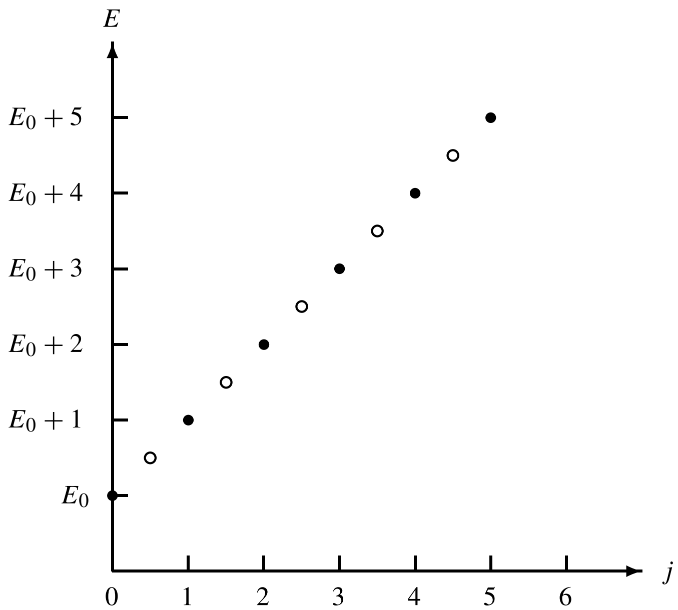

Figure 5. The spin-0 and spin-1/2 singleton representations. The solid dots indicate the states of the spin-0 singleton, the circles the states of the spin-1/2 singleton. It is obvious that singletons contain fewer degrees of freedom than a generic local field. The value of E_0, which denotes the spin-0 ground state energy, is equal to $E_0 = 1/2$. The spin-1/2 singleton ground state has an energy which is one half unit higher, as is explained in the text.

commutator acquires zero or negative eigenvalues for certain values of E_0 and s. We will return to this phenomenon in section 4 in the context of the anti-de Sitter superalgebra.

4. The $\mathcal{N} = 1$ anti-de Sitter superalgebra

In this section we return to the anti-de Sitter superalgebra. We start from the (anti-)commutation relations already established in (2.13) and (2.14). For definiteness we discuss the case of four space-time dimensions with a Majorana supercharge Q. This allows us to make contact with the material discussed in section 2. These anti-de Sitter multiplets were discussed in [37–40].

We choose conventions where the gamma matrices are given by

$$\Gamma^0 = \begin{pmatrix} -i\mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix}, \qquad \Gamma^a = \begin{pmatrix} 0 & -i\sigma^a \\ i\sigma^a & 0 \end{pmatrix},$$

where $a = 1, 2, 3$, and write the Majorana spinor Q in the form

$$Q = \begin{pmatrix} q_\alpha \\ \varepsilon_{\alpha\beta}\, q^\beta \end{pmatrix} ,$$

where $q^\alpha \equiv q_\alpha^\dagger$ and the indices $\alpha, \beta, \ldots$ are two-component spinor indices. We substitute these definitions into (2.14) and obtain

$$\left[H\,, q_\alpha\right] = -\frac{1}{2} q_\alpha\,, \tag{2.18a}$$

$$\left[H\,, q^\alpha\right] = \frac{1}{2} q^\alpha\,, \tag{2.18b}$$

$$\left\{q_\alpha\,, q^\beta\right\} = (H\,\mathbb{1} + \boldsymbol{J}\cdot\boldsymbol{\sigma})_\alpha{}^\beta\,, \tag{2.18c}$$

$$\left\{q_\alpha\,, q_\beta\right\} = M_a^-\left(\sigma^a\sigma^2\right)_{\alpha\beta}\,, \tag{2.18d}$$

$$\left\{q^\alpha\,, q^\beta\right\} = M_a^+\left(\sigma^2\sigma^a\right)^{\alpha\beta}\quad , \tag{2.18e}$$

where we have defined the angular momentum operator $J_a = -i\,\varepsilon_{abc}M^{bc}/2$. We see that the operators q_α and q^α are lowering and raising operators, respectively. They change the energy of a state by half a unit.

In analogy to the bosonic case, we study unitary irreducible representations of the OSp(1|4) superalgebra. We assume that there exists a lowest-weight state $|E_0, s\rangle$, characterized by the fact that it is annihilated by the lowering operators q_α,

$$q_\alpha|E_0, s\rangle = 0\,.$$

In principle we can now choose a ground state and build the whole representation upon it by applying products of raising operators q^α. However, we only have to study the antisymmetrized products of the q^α, because the symmetric ones just yield products of the operators M_a^+ by virtue of (2.18). Products of the M_a^+ simply lead to the higher-energy states in the anti-de Sitter representations of given spin that we considered in section 3. By restricting ourselves to the antisymmetrized products of the q^α we thus restrict ourselves to the ground states upon which the full anti-de Sitter representations are build. These ground states are $|E_0, s\rangle$, $q^\alpha|E_0, s\rangle$ and $q^{[\alpha}q^{\beta]}|E_0, s\rangle$. Let us briefly discuss these representations for different s.

The $s = 0$ case is special since it contains fewer anti-de Sitter representations than the generic case. It includes the spinless states $|E_0, 0\rangle$ and $q^{[\alpha}q^{\beta]}|E_0, 0\rangle$ with ground-state energies E_0 and $E_0 + 1$, respectively. There is one spin-1/2 pair of ground states $q^\alpha|E_0, 0\rangle$, with energy $E_0 + 1/2$. As we will see below, these states correspond exactly to the scalar field A, the pseudo-scalar field B and the spinor field ψ of the chiral supermultiplet, that we studied in section 2.

For $s \geq 1/2$ we are in the generic situation. We obtain the ground states $|E_0, s\rangle$ and $q^{[\alpha} q^{\beta]} |E_0, s\rangle$ which have both spin s and which have energies E_0 and $E_0 + 1$, respectively. There are two more (degenerate) ground states, $q^\alpha |E_0, s\rangle$, both with energy $E_0 + 1/2$, which decompose into the ground states with spin $j = s - 1/2$ and $j = s + 1/2$.

As in the purely bosonic case of section 3, there can be situations in which states decouple so that we are dealing with multiplet shortening associated with gauge invariance in the corresponding field theory. The corresponding multiplets are then again called massless. We now discuss this in a general way analogous to the way in which one discusses BPS multiplets in flat space. Namely, we consider the matrix elements of the operator $q_\alpha \, q^\beta$ between the $(2s + 1)$-degenerate ground states $|E_0, s\rangle$,

$$\begin{aligned} \big\langle E_0, s \big| \, q_\alpha q^\beta \, \big| E_0, s \big\rangle &= \big\langle E_0, s \big| \big\{ q_\alpha \, , q^\beta \big\} \big| E_0, s \big\rangle \\ &= \big\langle E_0, s \big| (E_0 \, \mathbb{1} + \boldsymbol{J} \cdot \boldsymbol{\sigma})_\alpha{}^\beta \big| E_0, s \big\rangle \, . \end{aligned} \tag{2.19}$$

This expression constitutes an hermitian matrix in both the quantum numbers of the degenerate ground-state and in the indices α and β, so that it is $(4s + 2)$-by-$(4s + 2)$. Because we assume that the representation is unitary, this matrix must be positive definite, as one can verify by inserting a complete set of intermediate states between the operators q_α and q^β in the matrix element on the left-hand side. Obviously, the right-hand side is manifestly hermitian as well, but in order to be positive definite the eigenvalue E_0 of H must be big enough to compensate for possible negative eigenvalues of $\boldsymbol{J} \cdot \boldsymbol{\sigma}$, where the latter is again regarded as a $(4s + 2)$-by-$(4s + 2)$ matrix. To determine its eigenvalues, we note that $\boldsymbol{J} \cdot \boldsymbol{\sigma}$ satisfies the following identity,

$$(\boldsymbol{J} \cdot \boldsymbol{\sigma})^2 + (\boldsymbol{J} \cdot \boldsymbol{\sigma}) = s(s + 1) \mathbb{1} \, ,$$

as follows by straightforward calculation. This shows that $\boldsymbol{J} \cdot \boldsymbol{\sigma}$ has only two (degenerate) eigenvalues (assuming $s \neq 0$, so that the above equation is not trivially satisfied), namely s and $-(s + 1)$. Hence in order for (2.19) to be positive definite, E_0 must satisfy the inequality

$$E_0 \geq s + 1 \, , \quad \text{for } s \geq \frac{1}{2} \, ,$$

If the bound is saturated, i.e. if $E_0 = s + 1$, the expression on the right-hand side of (2.19) has zero eigenvalues so that there are zero-norm states in the multiplet which decouple. In that case we must be dealing with a massless multiplet. As an example we mention the case $s = 1/2$, $E_0 = 3/2$, which corresponds to the massless vector supermultiplet in four space-time dimensions. Observe that we have multiplet shortening here without the presence of central charges.

Armed with these results we return to the masslike terms of section 2 for the chiral supermultiplet. The ground-state energy for anti-de Sitter multiplets corresponding to the scalar field A, the pseudo-scalar field B and the Majorana spinor field ψ, are equal to E_0, E_0+1 and $E_0+1/2$, respectively. The Casimir operator therefore takes the values

$$\mathcal{C}_2(A) = E_0(E_0 - 3)\,, \tag{2.20a}$$

$$\mathcal{C}_2(B) = (E_0 + 1)(E_0 - 2)\,, \tag{2.20b}$$

$$\mathcal{C}_2(\psi) = \left(E_0 + \frac{1}{2}\right)\left(E_0 - \frac{5}{2}\right) + \frac{3}{4}\,. \tag{2.20c}$$

For massless anti-de Sitter multiplets, we know that the quadratic Casimir operator is given by (2.16), so we present the value for $\mathcal{C}_2 - 2(s^2 - 1)$ for the three multiplets, i.e

$$\begin{aligned}
\mathcal{C}_2(A) + 2 &= (E_0 - 1)(E_0 - 2)\,,\\
\mathcal{C}_2(B) + 2 &= E_0(E_0 - 1)\,,\\
\mathcal{C}_2(\psi) + \frac{3}{2} &= (E_0 - 1)^2\,.
\end{aligned}$$

The terms on the right-hand side are not present for massless fields and we should therefore identify them somehow with the common mass parameter. Comparison with the field equations (2.11) shows for $g = 1$ that we obtain the correct contributions provided we make the identification $E_0 = m+1$. Observe that we could have made a slightly different identification here; the above result remains the same under the interchange of A and B combined with a change of sign in m (the latter is accompanied by a chiral redefinition of ψ).

Outside the context of supersymmetry, we could simply assign independent mass terms with a mass parameter μ for each of the fields, by equating $\mathcal{C}_2 - 2(s^2 - 1)$ to μ^2. In this way we obtain

$$E_0(E_0 - 3) - (s + 1)(s - 2) = \mu^2\,,$$

which leads to

$$E_0 = \frac{3}{2} \pm \sqrt{\left(s - \frac{1}{2}\right)^2 + \mu^2}\,. \tag{2.21}$$

For $s \geq 1/2$ we must choose the plus sign in (2.21) in order to satisfy the unitarity bound $E_0 \geq s + 1$. For $s = 0$ both signs are acceptable as long as $\mu^2 \leq 3/4$. Observe, however, that μ^2 can be negative but remains subject to the condition

$$\mu^2 \geq -\left(s - \frac{1}{2}\right)^2$$

in order that E_0 remains real. For $s = 0$, this is precisely the bound of Breitenlohner and Freedman for the stability of the anti-de Sitter background against small fluctuations of the scalar fields [37]. We can also compare $\mathcal{C}_2 - 2\left(s^2 - 1\right)$

to the conformal wave operator for the corresponding spin. This shows that (again with unit anti-de Sitter radius), $\mathcal{C}_2 = \Box_{\mathrm{AdS}} + \mathcal{C}_2^L$.

In the case of N-extended supersymmetry the supercharges transform under an SO(N) group and we are dealing with the so-called OSp($N|4$) algebras. Their representations can be constructed by elaborate methods, e.g. by the oscillator method [56], that we have not discussed in this chapter because it lies outside the scope of this thesis. However, the generators of SO(N) will now also appear on the right-hand side of the anticommutator of the two supercharges, thus leading to new possibilities for multiplet shortening. For an explicit discussion of this we refer the reader to [39].

Most of our discussion of the irreducible representations of the anti-de Sitter algebra and its superextension was restricted to four space-time dimensions, but in principle the same methods can be used for anti-de Sitter space-times of arbitrary dimension. For higher-extended supergravity, the only way to generate a cosmological constant is by elevating a subgroup of the rigid invariances that act on the gravitini to a local group, i.e. to gauge the supergravity theory. This then leads to a cosmological constant, or to a potential with possibly a variety of extrema, and corresponding masslike terms which are quadratic and linear in the gauge coupling constant, respectively. So the relative strength of the anti-de Sitter and the gauge group generators on the right-hand side of the $\{Q, \bar{Q}\}$ anticommutator is not arbitrary and because of that maximal multiplet shortening can take place so that the theory can realize a supermultiplet of massless states that contains the graviton and the gravitini. Of course, this is all under the assumption that the ground state is supersymmetric.

3

Supergravity in nine dimensions

Maximal supergravity in nine dimensions is known [57–60], but it has not been studied very extensively. The theory can be obtained by dimensional reduction of IIA or IIB supergravity in ten dimensions. In this chapter we study an extension of the nine-dimensional maximal supergravity theory by coupling various kinds of BPS multiplets to the supergravity multiplet.

In principle, the extended theory can be constructed purely in nine dimensions, but one can also use a compactification of a higher-dimensional theory as a guideline. We will follow the latter approach in this chapter. There are two kinds of 1/2-BPS multiplets that we couple to the massless theory. They arise in the compactification of eleven-dimensional supergravity theory on a two-torus, and of ten-dimensional type IIB supergravity theory on a circle. A supergravity theory coupled to BPS multiplets is called a BPS extended supergravity theory [61].

The reason for choosing the nine-dimensional theory is that it provides a reasonably simple example of such a BPS extended supergravity. Namely, the nonlinearly realized global symmetry group $G = \mathrm{GL}(2)$ is small and one is able to handle calculations much better than e.g. in four dimensions, where the hidden symmetry group is $E_{7(7)}$. Additionally the four-dimensional theory exhibits electro-magnetic duality, which would complicate the discussion. Nevertheless, it proves to be rather involved to actually write down a Lagrangian even for this simple BPS-extended supergravity theory in nine dimensions.

We will argue that the BPS-extended supergravity theory in nine dimensions goes beyond the standard eleven-dimensional supergravity theory. The new theory contains eleven-dimensional supergravity theory as a limiting case, but it also contain the ten-dimensional IIB supergravity theory in a different limit.

1. Supersymmetry algebra

We start the discussion of supersymmetry in nine dimensions by studying the supersymmetry algebra. This allows us to identify the various multiplets that can appear in the $\mathcal{N} = 2$ supergravity theory. We discuss the higher-dimensional origin and the interpretation of these BPS multiplets.

The supersymmetry algebra of maximal supergravity in nine dimensions can be obtained from the supersymmetry algebra in eleven dimensions (1.9) by straight-forward dimensional reduction on the two-torus T^2. The gamma matrices in eleven dimension, $\hat{\Gamma}^M$, are decomposed into nine-dimensional gamma matrices γ^μ and two-dimensional gamma matrices Γ^m,

$$\hat{\Gamma}^\mu = \gamma^\mu \otimes \tilde{\gamma} \, , \qquad \hat{\Gamma}^m = 1 \otimes \Gamma^m \, , \tag{3.1}$$

where $\tilde{\gamma} = -i\Gamma_9\Gamma_{10}$. A Majorana spinor in nine dimensions has 16 components, i.e. half the number as in eleven dimensions, and consequently the supersymmetry algebra consists of two Majorana spinors Q^i, i.e. $\mathcal{N} = 2$. In the first place, we are only interested in massless fields. Therefore we simply neglect the central charges of the supersymmetry algebra, and we only consider states with vanishing momenta in the directions of the internal torus. The anti-commutator of two supersymmetry transformations reads[1]

$$\left\{Q^i, \bar{Q}^j\right\} = -i\delta^{ij} P_\mu \gamma^\mu \, . \tag{3.2}$$

The $\mathcal{N} = 2$ supersymmetry algebra in nine dimensions can be realized on the supergravity multiplet consisting of 128 massless bosonic degrees of freedom and the same number of massless fermionic degrees of freedom.

Turning to the massive states, we can augment the supersymmetry algebra (3.2) by central charge terms. Again, we could directly construct the terms in nine-dimensions, but we prefer to deduce them from the eleven-dimensional supersymmetry algebra (1.9). In order to obtain only point-like central charges in nine dimensions, we assume that the two-form central charge Z_{MN} only takes values in the ninth and tenth dimension, and we set the five-form central charge Z_{MNPQR} to zero. The supersymmetry algebra in nine dimensions then takes the form

$$\left\{Q^i, \bar{Q}^j\right\} = -i\delta^{ij} P_\mu \gamma^\mu + Z^{ij} \, , \tag{3.3}$$

where the central charge[2] is given by

$$\begin{aligned} Z^{ij} &= Z_{9\,10}\delta^{ij} - (P_9\tau_3 - P_{10}\tau_1)^{ij} \, , \\ &= M\big(a\,(\cos\theta\,\tau_3 + \sin\theta\,\tau_1)^{ij} + b\,\delta^{ij}\big) \, . \end{aligned} \tag{3.4}$$

In this way, the central charge (3.4) decomposes into a singlet of SO(2), which is proportional to δ^{ij}, and a doublet, which is a linear combination of τ_1 and τ_3. In this reduction from eleven dimensions, the doublet is formed by the momenta of the supergravity fields in the two internal directions. The singlet originates from a solitonic state of the supergravity theory, the so-called M2-brane, that is wrapped around the two-torus. The mass of a BPS state in its rest

[1] Dirac-conjugated spinors in eleven and in nine dimensions are related by $\bar{\psi}_{11} = i\psi^\dagger\Gamma_0 = i\psi^\dagger\gamma_0\tilde{\gamma} = \bar{\psi}_9\tilde{\gamma}$.

[2] We have taken $\Gamma_9 = \tau_1$, $\Gamma_{10} = \tau_3$, which implies $\tilde{\gamma} = \tau_2$. Here, τ_i are the Pauli matrices.

frame is given by

$$M = \sqrt{P_9^2 + P_{10}^2} + |Z_{9\,10}| \ . \tag{3.5}$$

From the perspective of the eleven-dimensional supermembrane [62, 63], the mass formula (3.5) can be rewritten as

$$M = \frac{1}{A\tau_2}\,|q_1 + \tau q_2| + T_m A|p|\,, \tag{3.6}$$

where p is the number of times that the membrane wraps around the torus with modular parameter τ. Here, $q_{1,2}$ are the momenta along the torus directions and T_m denotes the tension of the supermembrane.

We can also deduce the supersymmetry algebra (3.3) from the IIB supersymmetry algebra in ten dimensions [64], which is given by

$$\left\{Q^i, \bar{Q}^j\right\} = -i\delta^{ij}\left(\mathbb{P}\Gamma^M\right)P_M + \left(\mathbb{P}\Gamma^M\right)Z_M^{ij}\,,$$

where $\mathbb{P} = (1 + \Gamma^{10})/2$ projects onto states with positive chirality. Upon compactification on a circle, the supersymmetry algebra decomposes as

$$\left\{Q^i, \bar{Q}^j\right\} = -i\delta^{ij}P_\mu\gamma^\mu + Z^{ij}\ ,$$

with the central charge

$$Z^{ij} = -P_9\delta^{ij} + \left(Z_9^F\tau_3 - Z_9^D\tau_1\right)^{ij}\ .$$

The origin of the singlet and the doublet charges in the reduction of the IIB supergravity theory is different from their origin in the reduction of the eleven-dimensional theory. Here, the singlet originates from the momentum of the supergravity fields on the internal circle, and the doublet is related to the winding of the fundamental string and the D1-string around the internal direction.

When one diagonalizes the anti-commutator (3.3), the right-hand side decomposes into four eight-dimensional blocks of unit matrices with coefficients equal to M times $(1+a+b)$, $(1-a-b)$, $(1-a+b)$ and $(1+a-b)$. Whenever one or more of these coefficients vanish, the algebra can be realized on a much smaller number of states and the corresponding states are BPS-states. This is the phenomenon of multiplet shortening, that we have already discussed in chapter 1 and in chapter 2.

We can distinguish a number of cases. For $a = \pm 1$ and $b = 0$, half of the components of the supersymmetry charge in (3.3) are zero on the states, which means that we are dealing with 1/2-BPS states. The multiplet contains the momentum states of eleven-dimensional supergravity (and consequently of the IIA supergravity theory in ten dimensions), cf. (3.4). This multiplet is the so-called KKA multiplet. Setting $a = 0$ and $b = \pm 1$, we obtain a different kind of 1/2-BPS multiplet which comprises the momentum modes of the type IIB supergravity theory in ten dimensions. This multiplet is the so-called KKB multiplet. Finally, the cases $\pm a \pm b = \pm 1$ lead to 1/4-BPS states, i.e. to states that are annihilated under one quarter of the original supersymmetry charges.

These multiplets correspond to string theory states that carry both momentum and winding.

We will discuss the field content of the KKA and KKB multiplets in more detail in chapter 4. In particular we write down Lagrangians that describe the massive fields and their interaction with the massless fields and we discuss the relation between the two BPS multiplets in more depth.

2. Maximal supergravity

Let us first describe the theory of the massless fields, i.e. maximal $\mathcal{N} = 2$ supergravity in nine dimensions and its relation to IIA and IIB supergravity in ten dimensions. Maximal supergravity in nine dimensions coincides with the dimensionally reduced version of both the IIA and IIB supergravities in ten dimensions. The field content of the supergravity multiplet follows directly from the reduction of the supergravity multiplet in eleven dimensions. Alternatively, the field content can also be obtained by reducing the IIB supergravity theory. We use both approaches below, but we will mainly focus on the former.

Not only the field content but also some of the quantum numbers of the fields can be directly inferred from the supergravity multiplet in eleven dimensions by studying the decompositions of the various symmetries: the Lorentz symmetry in eleven dimensions is broken to $\mathrm{SO}(1,8) \times \mathrm{SO}(2) \subset \mathrm{SO}(1,10)$, where the SO(2) plays the role of the R-symmetry group in nine dimensions. Similarly, representations of the SO(9) helicity group in eleven dimensions decompose into representations of the SO(7) helicity group in nine dimensions. The diffeomorphism invariance of the torus gives rise to a global $\mathrm{GL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1)$ symmetry. The group $\mathrm{SL}(2,\mathbb{R})$ corresponds to transformations of the modular parameter of the internal torus, and the group $\mathrm{SO}(1,1)$ describes rescalings of the torus. When we include the BPS multiplets into the theory later in this chapter, the global $\mathrm{GL}(2,\mathbb{R})$ is broken to the arithmetic subgroup $\mathrm{SL}(2,\mathbb{Z})$.

On the IIB side, the quantum numbers arise in a somewhat different way. The $\mathrm{SO}(1,9)$ Lorentz symmetry in ten dimensions is broken to $\mathrm{SO}(1,8)$ in nine dimensions. The helicity group in ten dimensions is SO(8), and it reduces to SO(7) in nine dimensions. The $\mathrm{SL}(2,\mathbb{R})$ symmetry does not originate from symmetries of the internal manifold as in the reduction of the eleven-dimensional theory, the symmetry already exists in ten dimensions as a strong-weak coupling self-duality symmetry. The group $\mathrm{SO}(1,1)$ corresponds to rescalings of the compactification circle.

Let us now discuss the decomposition of the eleven-dimensional fields. The graviton g_{MN} in eleven dimensions transforms in the **44** representation of the SO(9) helicity group, and it splits up into the following SO(7) helicity representations and their associated massless fields in nine dimensions: a graviton

field	multiplicity	dimension of helicity representation
g_{MN}	1	$\frac{1}{2}D(D-3)$
$g_{\mu\nu}$	1	$\frac{1}{2}d(d-3)$
${A_\mu}^m$	$D-d$	$d-2$
${e_m}^a$	$\frac{1}{2}(D-d)(D-d+1)$	1

Table 2. The metric in D dimensions decomposes into massless fields in d dimensions, describing states with the given helicities.

$g_{\mu\nu}$, two vector fields ${A_\mu}^m$ and three scalars ϕ^m and σ. The SO(9) helicity representations of the corresponding states decompose into SO(7) representations as follows,

$$\begin{array}{ccccccccc} g_{MN} & \longrightarrow & g_{\mu\nu} & + & {A_\mu}^m & + & \phi^m & + & \sigma \\ \mathbf{44} & \longrightarrow & \mathbf{27} & + & \mathbf{7}+\mathbf{7} & + & \mathbf{1}+\mathbf{1} & + & \mathbf{1}\,, \end{array} \tag{3.7}$$

where $m = 1, 2$ is an SL(2, $\mathbb{R}$) index. The three scalar fields are proportional to the metric in the internal dimensions: the scalar field σ is related to the determinant of the torus metric g_{mn}. The two scalar fields ϕ^m take values in the coset space SL(2,$\mathbb{R}$)/SO(2), and they are described by a nonlinear sigma model. The gravi-photons ${A_\mu}^m$ form a doublet of SL(2,$\mathbb{R}$).

These results are easily generalized to a reduction from D dimensions to d dimensions. The helicity group in the unreduced space-time is SO($D-2$), whereas it is SO($d-2$) in the reduced space-time. The metric decomposes as in (3.7), but the representations are of course different. A list of the representations and their multiplicities can be found in table 2.

The eleven-dimensional three-form field A_{MNP} describes states in the **84** representation of the SO(9) helicity group, which decomposes in the following way into SO(7) helicity representations in nine dimensions,

$$\begin{array}{ccccccc} A_{MNP} & \longrightarrow & A_{\mu\nu\rho} & + & {A_{\mu\nu}}^m & + & B_\mu \\ \mathbf{84} & \longrightarrow & \mathbf{35} & + & \mathbf{21}+\mathbf{21} & + & \mathbf{7}\,. \end{array}$$

The nine-dimensional fields are a three-form field $A_{\mu\nu\rho}$, two two-form fields ${A_{\mu\nu}}^m$, and one vector field B_μ. Note that the vector field B_μ which is a singlet under SL(2,$\mathbb{R}$) originates from the reduction of the three-form.

The gravitino in eleven dimensions, ψ_M splits into two gravitini ψ_μ^a and four fermions χ_m^a,

$$\begin{array}{ccccc} \psi_M & \longrightarrow & \psi_\mu^a & + & \chi_m^a \\ \mathbf{128} & \longrightarrow & \mathbf{48}+\mathbf{48} & + & \mathbf{8}+\mathbf{8}+\mathbf{8}+\mathbf{8}\,, \end{array}$$

$D = 11$	$D = 9$	IIB	weight
$G_{\mu\nu}$	$g_{\mu\nu}$	$G_{\mu\nu}$	0
$A_{\mu\,9\,10}$	B_μ	$G_{\mu\,9}$	-4
$G_{\mu\,9}$, $G_{\mu\,10}$	${A_\mu}^m$	${A_{\mu\,9}}^m$	3
$A_{\mu\nu\,9}$,$A_{\mu\nu\,10}$	${A_{\mu\nu}}^m$	${A_{\mu\nu}}^m$	-1
$A_{\mu\nu\rho}$	$A_{\mu\nu\rho}$	$A_{\mu\nu\rho\,9}$	2
$G_{9\,9}$, $G_{9\,10}$, $G_{10\,10}$	ϕ^m	ϕ^m	0
	σ	$G_{9\,9}$	7

Table 3. The bosonic fields of eleven-dimensional, nine-dimensional, and type IIB supergravity. We have also included the SO(1, 1) scaling weight of the various fields in the Einstein frame.

where $a = 1, 2$ is the SO(2) R-symmetry index. The doubling of the gravitinos in nine dimensions is due to the fact that a Majorana spinor in nine dimensions contains 16 components, whereas it contains 32 components in eleven dimension.

Dimensional reduction of the IIB theory gives rise to exactly the same field content, but the origin of the fields is of course different. The IIB origin of the massless fields is listed in detail in table 3.

The three abelian gauge fields ${A_\mu}^m$ and B_μ play an important role in the interpretation of the BPS multiplets which we will be discussing in the next section. From an eleven-dimensional point of view, the SL(2) doublet ${A_\mu}^m$ is derived from the reduction of the metric. It therefore couples to all the massive Kaluza-Klein states, i.e. the massive Kaluza-Klein states carry two charges with respect to the two abelian gauge fields ${A_\mu}^m$. From a IIB point of view the role of the ${A_\mu}^m$ is rather different. The fields originate from the doublet of tensor fields in ten dimension, which means that they couple to the winding states of the fundamental string and of the D1-string in the IIB theory on the circle.

A similar analysis holds for the gauge field B_μ, which is an SL(2) singlet. In the reduction of the IIB theory it derives from the metric and therefore couples to the massive Kaluza-Klein states, i.e. the massive IIB Kaluza-Klein states are charged under B_μ. In the compactification from eleven dimension, the field B_μ originates from the three-index tensor field, and consequently it couples to the winding modes of the M2-brane on the torus.

3. BPS multiplets

As described in section 1, there are two inequivalent 1/2-BPS multiplets in nine dimensions, the KKA and the KKB multiplet, which we study in the following. We derive their field content from the eleven-dimensional supergravity theory and the ten-dimensional IIB supergravity theory, respectively.

In order to identify the massive physical fields in nine dimensions, it is necessary to impose certain gauge choices on the fields. This gauge fixing is very similar to employing the unitary gauge in spontaneously broken theories. Since we are only interested in the classical theory in this section, we do not have to worry about Faddeev-Popov ghosts. We write down the gauge choices for the various fields in chapter 4, where we also discuss the precise relation between the lower-dimensional fields and the original higher-dimensional fields.

3.1. The KKA multiplet

The massive Kaluza-Klein modes of eleven-dimensional supergravity compactified on a torus make up the KKA multiplet, which contains 128 bosonic and 128 fermionic degrees of freedom. All of the KKA modes are charged with respect to the two gravi-photons ${A_\mu}^m$, and the charges q^m of the fields are equal to their mass m, i.e. $q^2 = m^2$. The charges q^m form a two-dimensional lattice, which breaks the global symmetry group SL(2, $\mathbb{R}$) of the massless theory to the arithmetic subgroup SL(2, $\mathbb{Z}$). Under this SL(2, $\mathbb{Z}$), the charge-lattice of the KKA multiplets is mapped onto itself. To be precise, a multiplet with charges (p, q) is mapped onto a multiplet with charges (p', q') as follows,

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} ,$$

with integers a, b, c, d which are subject to $ad - bc = 1$. A theory of massless fields coupled to a "lattice" of massive fields is therefore invariant under a global SL(2, $\mathbb{Z}$) symmetry. From the perspective of IIB supergravity theory, the two vector fields ${A_\mu}^m$ originate from the SL(2)-invariant tensor fields, as we have mentioned above. They couple to the winding states of the fundamental string and the D1-string.

The massless fields of the eleven-dimensional theory transform in representations of the helicity group SO(9). In nine dimensions, the massive fields transform under the spin-group SO(8), and the SO(9) representations split into SO(8) representations as follows,

$$\begin{aligned} \mathbf{44} &\longrightarrow \mathbf{1} + \mathbf{8}_v + \mathbf{35}_v \\ \mathbf{84} &\longrightarrow \mathbf{28} + \mathbf{56}_v \\ \mathbf{128} &\longrightarrow \mathbf{8}_s + \mathbf{8}_c + \mathbf{56}_s + \mathbf{56}_c \, . \end{aligned}$$

Each of these SO(8) representations corresponds to a state in the BPS multiplet, and we will identify the corresponding fields below. The KKA multiplet

field	multiplicity	representation
g_{MN}	1	$\frac{1}{2}(D-2)(D-1)-1$
$g_{\mu\nu}$	1	$\frac{1}{2}d(d-1)-1$
${A_\mu}^m$	$D-d-1$	$d-1$
${e_m}^a$	$\frac{1}{2}(D-d)(D-d-1)$	1

Table 4. The massless metric in D dimensions splits up into massive fields in d dimensions. The states described by the D-dimensional metric transforms in a representation of the helicity group SO($D-2$), whereas the states described by the d-dimensional massive fields transform in a representation of the spin group SO($d-1$).

therefore consists of the following SO(8) representations,

$$\underbrace{\mathbf{1}+\mathbf{8}_v+\mathbf{28}+\mathbf{35}_v+\mathbf{56}_v}_{\text{bosons}}+\underbrace{\mathbf{8}_s+\mathbf{8}_c+\mathbf{56}_s+\mathbf{56}_c}_{\text{fermions}}\ .$$

Note that the KKA multiplet contains the same representations as the supergravity multiplet of the IIA theory. This can easily be understood by noting that the massless fields of the IIA theory transform under the helicity group SO(8), which is also the spin group of the massive nine-dimensional fields. The associated fields are different, though, because the states describe massive and massless fields, respectively. Let us now discuss the decomposition of the individual fields.

§ The elfbein ${E_M}^A$ in eleven dimensions, which transforms in the **44** of the helicity group SO(9) splits up into the following representations of the spin-group SO(8) in nine dimension,

$$\begin{array}{ccccccc} {E_M}^A & \longrightarrow & {e_\mu}^\alpha & + & A_\mu & + & \sigma \\ \mathbf{44} & \longrightarrow & \mathbf{35}_v & + & \mathbf{8}_v & + & \mathbf{1}\,. \end{array}$$

The SO(8) representations of the states are attributed to the various massive fields as follows: the representation $\mathbf{35}_v$ corresponds to a symmetric and traceless two-index field, which we identify as the massive neunbein ${e_\mu}^\alpha$, or equivalently the massive graviton $g_{\mu\nu}$. The representation $\mathbf{8}_v$ corresponds to a massive vector field A_μ, and the representation **1**, finally, corresponds to a scalar field in nine dimensions. For a general compactifications table 4 shows how the representation of the state associated with metric splits up into representations and the associated fields in the lower dimension.

§ The three-index tensor field A_{MNP} splits up into a massive three-form field $A_{\mu\nu\rho}$ and a massive two-form field $A_{\mu\nu}$,

$$\begin{array}{ccccc} A_{MNP} & \longrightarrow & A_{\mu\nu\rho} & + & A_{\mu\nu} \\ \mathbf{84} & \longrightarrow & \mathbf{56}_v & + & \mathbf{28}\,. \end{array}$$

The representation $\mathbf{56}_v$ is associated with a massive anti-symmetric three-index tensor field $A_{\mu\nu\rho}$, and $\mathbf{28}$ corresponds to a massive anti-symmetric two-index tensor field $A_{\mu\nu}$ in nine dimensions.

§ The gravitino ψ_M in eleven dimensions splits up into two Rarita-Schwinger fields ψ_μ and two spinor fields χ in nine dimensions. The decomposition goes as follows,

$$\begin{array}{ccccc} \psi_M & \longrightarrow & \psi_\mu^a & + & \chi^a \\ \mathbf{128} & \longrightarrow & \mathbf{56}_s + \mathbf{56}_c & + & \mathbf{8}_s + \mathbf{8}_c\,. \end{array}$$

As we have mentioned earlier, the doubling of the number of spinors in nine dimensions is due to the split of a 32-component spinor in eleven dimensions to two 16-component spinors in nine dimensions.

3.2. The KKB multiplet

The KKB multiplet consists of the massive Kaluza-Klein modes of type IIB supergravity theory. The fields are charged with respect to the gravi-photon B_μ, and the charges form a one-dimensional lattice. The multiplet consists of states in the following representations of the SO(8) spin group,

$$\underbrace{\mathbf{1}+\mathbf{1}+\mathbf{28}+\mathbf{28}+\mathbf{35}_v+\mathbf{35}_c}_{\text{bosons}}+\underbrace{\mathbf{8}_s+\mathbf{8}_s+\mathbf{56}_s+\mathbf{56}_s}_{\text{fermions}}$$

These are exactly the same representations as are contained in the massless IIB supergravity multiplet. The reason for this is obvious: in ten dimensions, the helicity group for massless fields is SO(8), and in nine dimensions the spin group for massive fields is also SO(8). In the process of the compactification, the states themselves do not change, only the fields describing the states. The degrees of freedom corresponding to the internal coordinate are absorbed by the other degrees of freedom in order to make up a massive field in nine dimensions. We will discuss this effect for the individual fields below.

The IIB supergravity theory in ten dimensions is invariant under SL(2, $\mathbb{R}$), which describes the strong-weak coupling self-duality of the theory. If we include the solitonic string modes into the theory, the symmetry is broken to the arithmetic subgroup, SL(2, $\mathbb{Z}$). Again, this global symmetry can be found back in nine dimensions. For the KKB multiplets, the SL(2, $\mathbb{Z}$) acts on the fields within one multiplet, i.e. it does not mix multiplets of different Kaluza-Klein charges. There is also an SO(1, 1) symmetry in nine dimensions, which corresponds to rescalings of the compact coordinate, and which is broken in the coupled theory. Let us now discuss the decomposition of the individual fields in the IIB theory.

§ The zehnbein $E_M{}^A$ in the IIB supergravity theory reduces to a massive neunbein $e_\mu{}^\alpha$ in nine dimensions,

$$\begin{aligned} E_M{}^A &\longrightarrow e_\mu{}^\alpha \\ \mathbf{35}_v &\longrightarrow \mathbf{35}_v \,. \end{aligned}$$

As alluded to above, we can observe a mechanism that holds for the compactification of all fields: the SO(8) helicity representation $\mathbf{35}_v$ of the ten-dimensional state is re-interpreted as an SO(8) spin representation of the nine-dimensional massive state. The massive field associated with $\mathbf{35}_v$ is a symmetric traceless two-index field, the massive graviton.

§ The two two-index tensor fields $A_{MN}{}^m$ in ten dimensions, which transform as a doublet under SL(2, $\mathbb{Z}$), reduce to two massive two form-fields $A_{\mu\nu}{}^m$ in nine dimensions,

$$\begin{aligned} A_{MN}{}^m &\longrightarrow A_{\mu\nu}{}^m \\ \mathbf{28}+\mathbf{28} &\longrightarrow \mathbf{28}+\mathbf{28}\,. \end{aligned}$$

§ The two scalars ϕ^m in ten dimensions reduce to two massive scalar fields ϕ^m in nine dimensions,

$$\begin{aligned} \phi^m &\longrightarrow \phi^m \\ \mathbf{1}+\mathbf{1} &\longrightarrow \mathbf{1}+\mathbf{1}\,. \end{aligned}$$

Note that the scalar fields both in ten dimensions and nine dimensions transform as a doublet under the SL(2, $\mathbb{Z}$) symmetry.

§ The four-form field A^+_{MNPQ} in ten dimensions with self-dual field strength reduces to a massive four-form field $A_{\mu\nu\rho\sigma}$ in nine dimensions, which is subject to a self-duality constraint,

$$\begin{aligned} A^+_{MNPQ} &\longrightarrow A_{\mu\nu\rho\sigma} \\ \mathbf{35}_c &\longrightarrow \mathbf{35}_c \,. \end{aligned}$$

While it is clear that we are dealing with a massive four-index tensor field in nine dimensions, it is a priori not obvious how the self-duality condition that holds in ten dimensions is interpreted in nine dimensions. We will comment on this in more detail in chapter 2.5.

§ The two gravitinos ψ^a_M in ten dimensions reduce to two massive Rarita-Schwinger fields ψ^a_μ in nine dimensions,

$$\begin{aligned} \psi^a_M &\longrightarrow \psi^a_\mu \\ \mathbf{56}_s+\mathbf{56}_s &\longrightarrow \mathbf{56}_s+\mathbf{56}_s \,. \end{aligned}$$

§ The fermions λ^a in ten dimensions reduce to two massive spinor fields λ^a in nine dimensions,

$$\lambda^a \longrightarrow \lambda^a$$
$$\mathbf{8}_s + \mathbf{8}_s \longrightarrow \mathbf{8} + \mathbf{8}\,.$$

Let us summarize the analysis of the BPS multiplets in nine dimensions before we proceed to the next chapter. The discussion of the multiplets in this chapter was based solely on group theoretical considerations. We studied how the helicity representations of the higher-dimensional theory decompose into representations of the lower-dimensional helicity group and spin group. This allowed us to identify the massless and massive fields in nine dimensions. In chapter 4, we are going to construct a theory describing the massless and massive fields in nine dimensions.

There is a difference in the structure of the BPS multiplets when compactifying one dimension (e.g. IIB supergravity) or two and more dimensions (e.g. eleven-dimensional supergravity). In the former case the helicity group in ten dimensions is identical to the spin group in nine dimensions, and the representations of the states do not change in the process of the compactification. In the latter case the spin group in nine dimensions is a subgroup of the helicity group in eleven dimensions, and the representations of the states are decomposed accordingly.

4

BPS-extended supergravity in nine dimensions

Now that we have discussed the massless $\mathcal{N} = 2$ supergravity multiplet and two BPS multiplets in nine dimensions, we proceed to formulate a BPS-extended field theory that describes the states comprised by these multiplets and their interactions with the states of the massless supergravity multiplet.

Although we have described what kind of fields to expect in the lower-dimensional theory, we have not yet explicitly stated how the fields in the lower-dimensional theory are defined in terms of the higher-dimensional fields. This chapter aims to fill this gap. In the first section we make a number of general observations about Kaluza-Klein theories. We then present explicit calculations needed in the compactification of both the IIB theory on a circle and the eleven-dimensional supergravity theory on a torus. In the last section we comment on the BPS-extended supergravity theory, and its significance for the construction of M-theory.

1. Kaluza-Klein theories

Supergravity theories which exhibit spontaneous compactification of space-time are known as Kaluza-Klein theories. Compactification of a D-dimensional theory means that the ground-state geometry is locally a product space, i.e. it is of the form $\mathcal{M}_D = \mathcal{M}_d \times \mathcal{M}_n$, where $\mathcal{M}_n$ is an n-dimensional compact manifold, and $\mathcal{M}_d$ is the d-dimensional non-compact space-time. A compactification is called spontaneous (as opposed to "ad-hoc"), if it occurs through a physical mechanism and is not imposed by hand.[1] Kaluza-Klein theories have been studied in great detail, for a review see e.g. [65].

Kaluza-Klein theories are of particular interest since they make it possible to study a D-dimensional field theory in a d-dimensional setting. Kaluza-Klein theories can also explain the emergence of non-abelian gauge groups in lower dimensions. Namely, if the compact manifold $\mathcal{M}_n$ has an isometry group G, then the d-dimensional theory contains massless vector fields which gauge the

[1] While we will only be dealing with spontaneous compactifications in this chapter, we present an example of an ad-hoc compactification in chapter 5 where we will study ten-dimensional supersymmetric Yang-Mills theory compactified on a six-torus. Since this theory does not contain gravity, the compactification on a six-torus is imposed by hand.

group G. Examples of internal manifolds that are frequently studied are the n-torus T^n, with corresponding gauge group $\mathrm{U}(1)^n$, and the n-sphere S^n, which gives rise to the gauge group $\mathrm{SO}(n+1)$. If the internal manifold is compact, then the spectrum of the lower-dimensional theory has a mass gap and one can consistently decouple the massive modes from the massless modes.

The compactification procedure has been discussed many times before, but the focus has mostly been put on the massless modes in the compactification. In this section we will take the conventional Kaluza-Klein procedure one step further by including the massive Kaluza-Klein modes. We will discuss the significance of the massive modes as well as the problems encountered in such an approach. We start the discussion in section 1.1 with the analysis of the vielbein (or metric), which is the most important ingredient in the compactification. We then study the reduction of a general tensor with using the example of an abelian vector field in section 1.2.

1.1. Compactification of the vielbein

The vielbein in D dimensions, $E_M{}^A$, transforms as follows under general coordinate transformations with parameter ξ^M and local Lorentz transformations (frame rotations) with parameter $\Lambda^A{}_B$,

$$\delta E_M{}^A = \partial_M \xi^N E_N{}^A + \xi^N \partial_N E_M{}^A + \Lambda^A{}_B E_M{}^B \,. \tag{4.1}$$

The coordinates z^M are decomposed as $z^M = (x^\mu, y^m)$, where x^m are coordinates on the d-dimensional space-time and y^m are coordinates on the compact internal manifold. By means of a local Lorentz transformation, the vielbein $E_M{}^A$ and its inverse $E_A{}^M$ can always be brought into block-triangular form,

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} , \quad E_A{}^M = \begin{pmatrix} e_\alpha{}^\mu & -B_\mu{}^m\, e_\alpha{}^\mu \\ 0 & e_a{}^m \end{pmatrix} . \tag{4.2}$$

This gauge choice breaks the D-dimensional Lorentz group $\mathrm{SO}(1, D-1)$ to $\mathrm{SO}(1, d-1) \times \mathrm{SO}(n)$, where $\mathrm{SO}(1, d-1)$ is the Lorentz group in d dimensions; the compact group $\mathrm{SO}(n)$ is interpreted as an internal symmetry group in the d-dimensional theory. The line element corresponding to the vielbein (4.2) is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{mn} \left(dx^m + B_\mu{}^m dx^\mu\right) \left(dx^n + B_\mu{}^n dx^\mu\right) \,.$$

Let us now study the general coordinate transformations of the fields $e_\mu{}^\alpha$, $B_\mu{}^m$ and $e_m{}^a$ that appear in the parameterization (4.2). In principle we can deduce the transformation rules in a straightforward way by writing out the relevant components in the original transformation rules (4.1). However, in order to remain in the gauge defined by (4.2), i.e. $e_m{}^\alpha = 0$, we have to uniformly modify the general coordinate transformations by a local Lorentz transformation with parameter $\Lambda^\alpha{}_b$, defined by

$$\delta e_m{}^\alpha = \partial_m \xi^\mu e_\mu{}^\alpha + \Lambda^\alpha{}_b e_m{}^b = 0 \,. \tag{4.3}$$

Taking into account the compensating Lorentz transformation, the d-dimensional fields transform as follows under D-dimensional general coordinate transformations

$$\delta e_\mu{}^\alpha = \left(\partial_\mu - B_\mu{}^n \partial_n\right) \xi^\nu e_\nu{}^\alpha + \xi^\nu \partial_\nu e_\mu{}^\alpha + \xi^m \partial_m e_\mu{}^\alpha , \tag{4.4a}$$

$$\begin{aligned} \delta B_\mu{}^m = {} & \left(\partial_\mu - B_\mu{}^n \partial_n\right) \xi^\nu B_\nu{}^m + \xi^\nu \partial_\nu B_\mu{}^m + g^{mn} g_{\mu\nu} \partial_n \xi^\nu \\ & + \left(\partial_\mu - B_\mu{}^n \partial_n\right) \xi^m + \xi^n \partial_n B_\mu{}^m , \end{aligned} \tag{4.4b}$$

$$\delta e_m{}^a = \xi^\mu \partial_\mu e_m{}^a + \xi^n \partial_n e_m{}^a + \partial_m \xi^\mu B_\mu{}^n e_n{}^a + \partial_m \xi^n e_n{}^a . \tag{4.4c}$$

Let us stress that the transformation rules for general coordinate transformations (4.4) are completely equivalent to the original D-dimensional transformation rules (4.1), they are simply written in a special parameterization.

The d-dimensional theory contains a number of important symmetries, which follow from the general coordinate transformations (4.4) in D dimension. First, there are general coordinate transformations in d dimensions which are generated by $\xi^\mu(x)$. In principle, the $\xi^\mu(x, y)$ generate a much larger symmetry group. The y-dependent modes, however, correspond to symmetries which are spontaneously broken in the ground state. These symmetries are not considered here, because they are involved in the transfer of the degrees of freedom from $B_\mu{}^m$ to a massive spin-2 field in d dimensions. Second, there are a number of internal symmetry transformations related to $\xi^m(x, y)$, which we discuss below.

The original transformation rules (4.4) of the three fields under the residual symmetries $\xi^\mu(x)$ and $\xi^m(x, y)$ reduce to

$$\delta e_\mu{}^\alpha = \partial_\mu \xi^\nu e_\nu{}^\alpha + \xi^\nu \partial_\nu e_\mu{}^\alpha + \xi^m \partial_m e_\mu{}^\alpha , \tag{4.5a}$$

$$\delta B_\mu{}^m = \partial_\mu \xi^\nu B_\nu{}^m + \xi^\nu \partial_\nu B_\mu{}^m + \left(\partial_\mu - B_\mu{}^n \partial_n\right) \xi^m + \xi^n \partial_n B_\mu{}^n , \tag{4.5b}$$

$$\delta e_m{}^a = \xi^\mu \partial_\mu e_m{}^a + \xi^n \partial_n e_m{}^a + \partial_m \xi^n e_n{}^a . \tag{4.5c}$$

Clearly, the first transformation rule in (4.5) describes the general coordinate transformations with parameter ξ^μ of a d-dimensional vielbein $e_\mu{}^\alpha$. The transformation rule for $B_\mu{}^m$ contains a term of the form $\partial_\mu \xi^m$ and we read off that $B_\mu{}^m$ transforms as a gauge field in d dimensions. The fields $e_m{}^a$, finally, transform as scalars under general coordinate transformations ξ^μ. The transformation rules (4.5) are those that we will be using henceforth. As it stands, all the fields in these transformation rules still depend on both coordinates x^μ and y^m. In order to obtain a d-dimensional theory, the fields can be expanded in terms of some basis of functions on the internal manifold, e.g. a basis of eigenfunctions of the Laplacian, $Y^{\mathcal{A}}(y)$,

$$\phi(x, y) = \sum_{\mathcal{A}} \phi^{\mathcal{A}}(x) Y^{\mathcal{A}}(y) , \tag{4.6}$$

where $\mathcal{A}$ collectively denotes all the indices of the eigenfunctions of the Laplacian. Subsequently one integrates over the internal coordinates y^m in order to eliminate the dependence on the internal coordinates. Every field $\phi(x, y)$ in the D-dimensional theory gives rise to an infinite number of fields $\phi^{\mathcal{A}}(x)$ in d dimensions.

As we have mentioned above, the d-dimensional theory can also contain internal symmetries that are related to the transformations ξ^m. Assume that the metric of the internal manifold $\mathcal{M}_n$ has isometries $K_i{}^m(y)$, $i = 1 \ldots I$, i.e. the metric admits I Killing vectors $K_i{}^m(y)$. The Killing vectors $K_i{}^m(y)$ generate a group G, and the structure constants $f_{ij}{}^k$ of the generating algebra are given by the commutation relation

$$\left[K_i, K_j\right] = f_{ij}{}^k K_k \, .$$

We restrict the transformation parameters ξ^m to those that leave the metric of the internal space invariant, which means that the $\xi^m(x, y)$ are a linear combination of Killing vectors $K_i{}^m(y)$,

$$\xi^m(x, y) = \xi^i(x) K_i{}^m(y) \, . \tag{4.7}$$

In such a background, the expansion (4.6) applied to the vector field $B_\mu{}^m(x, y)$ takes the form

$$B_\mu{}^m(x, y) = B_\mu{}^i(x) K_i{}^m(y) + \cdots ,$$

where $B_\mu{}^i(x)$ is the coefficient of the lowest component in the harmonic expansion of $B_\mu{}^m(x, y)$, and where the omitted terms correspond to higher components. The transformation rules (4.5) for the field $B_\mu{}^i(x)$ are given by

$$\delta B_\mu{}^i(x) = \partial_\mu \xi^\nu B_\nu{}^i + \xi^\nu \partial_\nu B_\mu{}^i + \partial_\mu \xi^i + f_{jk}{}^i B_\mu{}^j \xi^k \, .$$

This is precisely the transformation rule of a gauge field in d dimensions under general coordinate transformations ξ^μ and under gauge transformations ξ^i, as we have expected. Therefore, the fields $B_\mu{}^m$ transform as gauge fields in d dimensions and the isometry group G of the internal manifold is the gauge group.[2]

1.2. Compactification of tensor fields

When studying the compactified theory from the lower-dimensional point of view, one has to make sure that quantities that carry space-time indices transform as tensors in d dimensions. If $\hat{X}_M$ is a covariant vector in D dimensions, we cannot blindly assume that the components $\hat{X}_\mu$ transform as a covariant vector in d dimensions, and that the $\hat{X}_m$ transform as scalars under

[2] The possibility of obtaining non-abelian gauge theories from a Kaluza-Klein setup has first been discussed in the lecture notes [66], as a problem for the students.

d-dimensional diffeomorphisms. Let us illustrate this problem with an example. In D dimensions, a vector field $\hat{X}_M$ transforms as follows under general coordinate transformations,

$$\delta \hat{X}_M = \partial_M \xi^N \hat{X}_N + \xi^N \partial_N \hat{X}_M$$

The transformation rule for the components $\hat{X}_\mu$ contains a term $\partial_\mu \xi^m \hat{X}_m$ which transforms $\hat{X}_\mu$ into $\hat{X}_m$, i.e. the fields $\hat{X}_\mu$ and $\hat{X}_m$ are mixed under general coordinate transformations. Such a mixing is not desirable and prevents us from identifying $\hat{X}_\mu$ as a vector field in d dimensions. Note that since ξ_μ does not depend on y^m, as discussed above, the term $\partial_m \xi^\mu \hat{X}_\mu$ in the variation of $\hat{X}_m$ vanishes.

In order to identify the correct covariant quantities we use a procedure first developed by Cremmer and Julia [67], which we formulate as follows:

(1) Transform all space-time indices of the D-dimensional fields (which we marked with a hat) to tangent-space indices, using the vielbein $E_M{}^A$ and its inverse as given in (4.2), for example $\hat{X}_\alpha = e_\alpha{}^\mu \hat{X}_\mu - e_\alpha{}^\mu B_\mu{}^m \hat{X}_m$.

(2) Transform the tangent space quantities back to space-time quantities in d dimensions—if appropriate–using the d-dimensional vielbeins $e_\mu{}^\alpha$ and $e_m{}^a$ and their inverses, e.g. $X_\mu = e_\mu{}^\alpha X_\alpha$ or $X_m = e_m{}^a X_a$.

For the contravariant vectors we then obtain the following relations[3]

$$X_\mu = \hat{X}_\mu - B_\mu{}^m \hat{X}_m \quad \text{and} \quad X_m = \hat{X}_m \,.$$

Let us now discuss the compactification of an abelian gauge field $\hat{A}_M$ in some detail in order to better understand the general procedure that we have just described. In D dimensions, the abelian gauge field $\hat{A}_M$ transforms under general coordinate transformations with parameters $\xi^M(x, y)$ and under abelian gauge transformations with parameter $\lambda(x, y)$ as follows,

$$\delta \hat{A}_M = \partial_M \xi^N \hat{A}_N + \xi^N \partial_N \hat{A}_M + \partial_M \lambda \,. \tag{4.8}$$

We decompose the vector field $\hat{A}_M$ into $(\hat{A}_\mu, \hat{A}_m)$ and identify the tangent-space quantities, i.e. $\hat{A}_\alpha = A_\alpha$ and $\hat{A}_a = A_a$, as outlined above. The quantities with space-time indices are then given by

$$\hat{A}_\mu = e_\mu{}^\alpha \hat{A}_\alpha + B_\mu{}^m e_m{}^a \hat{A}_a \quad \text{and} \quad A_\mu = e_\mu{}^\alpha A_\alpha \,,$$

---

[3]The relations are different for covariant and contravariant vectors, i.e. the covariant vectors are identified as $X^\mu = \hat{X}^\mu$ and $X^m = \hat{X}^m + \hat{X}^\mu B_\mu{}^m$.

and similarly for the fields $\hat{A}_m$ and A_m. We can easily see that the two fields A_μ and A_m, defined by

$$\begin{aligned} A_\mu &= \hat{A}_\mu - B_\mu{}^m \hat{A}_m \,, \\ A_m &= \hat{A}_m \,, \end{aligned}$$

transform as a vector field and as a set of n scalar fields in d dimensions. Namely, the vector field A_μ transforms as

$$\delta A_\mu = \partial_\mu \xi^\nu A_\nu + \xi^\nu \partial_\nu A_\mu + \xi^m \partial_m A_\mu + \left(\partial_\mu - B_\mu{}^n \partial_n\right) \lambda \tag{4.9}$$

under the original general coordinate transformations and gauge transformations (4.8). The scalar fields A_m transform as

$$\delta A_m = \xi^\mu \partial_\mu A_m + \xi^n \partial_n A_m + \partial_m \xi^n A_n + \partial_m \lambda \,. \tag{4.10}$$

The reduction of the higher-dimensional field-strength $\hat{F}_{MN}$ is very similar. We define the following lower-dimensional field-strengths

$$F_{\mu\nu} = \hat{F}_{\mu\nu} + 2B_{[\mu}{}^m \hat{F}_{\nu]m} + B_\mu{}^m B_\nu{}^n \hat{F}_{mn} \,, \tag{4.11a}$$

$$F_{\mu n} = \hat{F}_{\mu n} - B_\mu{}^m F_{mn} \,, \tag{4.11b}$$

$$F_{mn} = \hat{F}_{mn} \,, \tag{4.11c}$$

In terms of the d-dimensional fields A_μ and A_m, the field-strength tensors are given by

$$\begin{aligned} F_{\mu\nu} &= 2\mathcal{D}_{[\mu} A_{\nu]} + 2G_{\mu\nu}{}^m A_m \,, \\ F_{\mu n} &= \mathcal{D}_\mu A_m - \partial_n A_\mu \,, \\ F_{mn} &= 2\partial_{[m} A_{n]} \,, \end{aligned}$$

where we have introduced the gravi-photon field-strength $G_{\mu\nu}{}^m = 2\mathcal{D}_{[\mu} B_{\nu]}{}^m$. We can easily verify that the field-strengths $F_{\mu\nu}$, $F_{\mu n}$ and F_{mn} are invariant under the abelian gauge transformations in (4.9) and (4.10) and also under the non-abelian gauge transformations generated by the restricted parameters ξ^m. The Lagrangian in the higher-dimensional theory is then reduced to the Lagrangian of the lower-dimensional theory,

$$\begin{aligned} \mathcal{L}_{\mathrm{V}} &= -\frac{1}{4}\hat{E}\left(\hat{F}_{MN}\right)^2 = -\frac{1}{4}\hat{E}\hat{g}^{MP}\hat{g}^{NQ}\hat{F}_{MN}\hat{F}_{PQ} \,, \qquad (4.12)\\ &= -\frac{1}{4}e\Delta\left(g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + 2g^{\mu\nu}g^{mn}F_{\mu m}F_{\nu n} + g^{mp}g^{nq}F_{mn}F_{pq}\right) \,, \\ &= -\frac{1}{4}e\Delta\Big(\left(2\mathcal{D}_{[\mu}A_{\nu]} + G_{\mu\nu}{}^m A_m\right)^2 + 2\left(\mathcal{D}_\mu A_m - \partial_m A_\mu\right)^2 + 2\partial_{[m}A_{n]}\Big) \,. \end{aligned}$$

where e and Δ are the determinant of the space-time vielbein $e_\mu{}^\alpha$ and of the internal vielbein $e_m{}^a$, respectively.

Let us now discuss the fate of the abelian gauge symmetry in the lower-dimensional theory when compactifying on a torus. We use part of the y-dependent gauge parameter $\lambda(x, y)$ to transform away one massive scalar field, by imposing the unitary gauge,

$$\partial^m A_m(x, y) = 0 \,. \tag{4.13}$$

In order for this gauge-fixing to be independent of gauge-transformations, the parameter λ for the gauge transformation needs to fulfill the relation

$$\partial^m \partial_m \lambda(x, y) = 0 \,.$$

The general solution to this equation that also respects the periodicity of A_μ and $F_{\mu\nu}$ is of the form

$$\lambda(x, y) = \lambda(x) + \frac{\lambda_m y^m}{R} \,, \tag{4.14}$$

where λ_m is a constant. With the gauge parameter defined in this way, the vector fields $A_\mu(x, y)$ and the scalar fields $A_m(x, y)$ transform as

$$\delta A_\mu(x, y) = \partial_\mu \lambda(x) - \frac{B_\mu{}^m \lambda_m}{R} \tag{4.15a}$$

$$\delta A_m(x, y) = \lambda_m \tag{4.15b}$$

under the original gauge transformations.

2. Lagrangians for massless and massive fields

In this section we study the Lagrangians of both the massless and the massive fields in the nine-dimensional $\mathcal{N} = 2$ supergravity theory. Where possible the results are presented in a dimension-independent way and with as few assumptions as possible about the internal manifold. However, for calculations that involve the dimensionality of space-time at crucial points, we restrict ourselves to specific dimensions, i.e. we start in eleven or ten dimensions and compactify to nine dimensions on a torus and a circle, respectively.

2.1. Einstein-Hilbert action

The action of a supergravity theory in D space-time dimensions contains the Einstein-Hilbert term,

$$S_{EH} = -\frac{1}{2} \int \mathrm{d}^D x \ \hat{E} \ \hat{R} \,, \tag{4.16}$$

where $\hat{E}$ is the determinant of the vielbein, and $\hat{R}$ is the Ricci scalar. In the following, we compactify the theory on an n-torus to d dimensions, and we present the form of the action in d dimensions when retaining not only the massless fields, but also the massive fields.

As explained above, the vielbein and its inverse can always be brought into block-triangular form by means of a local Lorentz transformation. However, the parameterization (4.2) gives rise to an Einstein-Hilbert term in the d-dimensional theory that does not have the standard normalization; it is multiplied by the determinant of the internal vielbein. In order to obtain the standard normalization of the Einstein-Hilbert term in d dimensions, we slightly modify the expressions (4.2) by extracting the determinant Δ from the internal vielbein $e_m{}^a$ and multiplying $e_\mu{}^\alpha$ with an appropriate power of Δ. The vielbein then takes the form

$$\hat{E}_M{}^A = \begin{pmatrix} \Delta^{-\zeta}\, e_\mu{}^\alpha & \Delta^{1/n}\, B_\mu{}^m\, e_m{}^a \\ 0 & \Delta^{1/n}\, e_m{}^a \end{pmatrix} , \tag{4.17a}$$

$$\hat{E}_A{}^M = \begin{pmatrix} \Delta^{\zeta}\, e_\alpha{}^\mu & -\Delta^{\zeta}\, B_\mu{}^m\, e_\alpha{}^\mu \\ 0 & \Delta^{-1/n}\, e_a{}^m \end{pmatrix} . \tag{4.17b}$$

where we have defined $\zeta = 1/(d-2)$. Note that the internal vielbein $e_m{}^a$ now has unit determinant, because we have extracted a factor $\Delta^{1/n}$. In the parameterization (4.17), the Einstein-Hilbert action takes the form

$$S_{\rm EH} = -\frac{1}{2}\int \mathrm{d}^d x\, \mathrm{d}^n y\; e\; \left(R + \mathcal{L}_{\rm K} + \mathcal{L}_{\rm M} + \mathcal{L}_{\rm I}\right) , \tag{4.18}$$

where $e = \det e_\mu{}^\alpha$ is the determinant of the d-dimensional vielbein, R is the Ricci scalar in d dimensions and where $\mathcal{L}_K$ is given by

$$\begin{aligned} \mathcal{L}_{\rm K} = &-\Delta^{2(\zeta+1/n)}\left(\mathcal{D}_{[\mu}B_{\nu]}{}^m\right)^2 - \left(e_{(a}{}^m\,\mathcal{D}_\mu e_{mb)}\right)^2 \\ &-(\zeta+1/n)\left(\mathcal{D}_\mu \ln\Delta\right)^2 . \end{aligned} \tag{4.19}$$

Here, $\mathcal{L}_{\rm K}$ contains the kinetic terms for the gravi-photons $B_\mu{}^m$ and for the scalar fields $e_m{}^a$ and Δ.

In the compactified Einstein-Hilbert action (4.18), $\mathcal{L}_{\rm M}$ and $\mathcal{L}_{\rm I}$ contain mass-terms and interaction terms of the d-dimensional theory, respectively. We will give explicit expressions for these two parts of the Lagrangian below. The covariant derivative is defined as above, namely as $\mathcal{D}_\mu = \partial_\mu - B_\mu{}^m\partial_m$. Note that all the terms in the Lagrangian still depend on the internal coordinates y at this point of the calculation. One obtains the d-dimensional theory by Fourier-expanding all the fields on the torus T^n, and consequently integrating over the internal coordinates y^m. This procedure gives rise to a finite number of massless fields, and an infinite number of massive fields.

Let us now discuss the gauge choices one has to make in order to identify the correct massive physical degrees of freedom in the lower-dimensional theory. In the D-dimensional transformation rule for $B_\mu{}^m$, equation (4.4), we can use the y-dependent modes of the term $g^{mn}g_{\mu\nu}\partial_m\xi^\nu$ to eliminate the corresponding y-dependent mode. The $d-2$ degrees of freedom associated with the eliminated vector field are absorbed by the vielbein $e_\mu{}^\alpha$. Very similarly, we can fix the y-dependency of $e_m{}^a$ such that n of the degrees of freedom associated with $e_m{}^a$ are eliminated. One of those degrees of freedom is absorbed by $e_\mu{}^\alpha$ to make up a massive spin-2 field, and the remaining $n-1$ degrees of freedom are required by the $n-1$ vector fields $B_\mu{}^m$ to become massive. For an overview of the degrees of freedom and the multiplicities of the fields in this gauge fixing, we refer to tables 2 and 4 in chapter 3.

If the internal manifold is a circle S^1, we impose the gauge

$$\partial_m B_\mu{}^m = 0 \quad \text{and} \quad \partial_m (\ln \Delta) = 0\,. \tag{4.20}$$

where m denotes the single coordinate on the internal manifold. In this way, all the massive degrees of freedom of the vector field $B_\mu{}^m$ and of the scalar field Δ are absorbed by the corresponding y-dependent components of the vielbein $e_\mu{}^\alpha$, which then constitute physical massive spin-2 fields in d dimensions. The Lagrangians $\mathcal{L}_\mathrm{M}$ and $\mathcal{L}_\mathrm{I}$ of equation (4.19) are given by

$$\begin{aligned}
\mathcal{L}_\mathrm{M} &= -\Delta^{-2(\zeta+1)} \left(e_{(\alpha}{}^\mu \partial_m e_{\mu\beta)}\right)^2 + \frac{1}{4}\Delta^{-2(\zeta+1)}(\partial_m \ln e)^2\,,\\
\mathcal{L}_\mathrm{I} &= -2(\mathcal{D}_{[\mu} B_{\nu]}{}^m)(e_\alpha{}^\mu \partial_m e^{\nu\alpha})\,.
\end{aligned}$$

Here, $\mathcal{L}_\mathrm{M}$ contains mass terms for the space-time vielbein $e_\mu{}^\alpha$ and its determinant e. Because of the gauge conditions (4.20), there are no mass terms for the gravi-photon fields $B_\mu{}^m$ and for the scalar field Δ. The interaction Lagrangian $\mathcal{L}_\mathrm{I}$ contains a single interaction term of the (massless) gravi-photon with the massive vielbein.

If the internal manifold has dimension two or more, i.e. for $n \geq 2$, the gauge choice (4.20) for the vector field can still be imposed. For the scalar fields, we choose instead

$$\partial_m e_a{}^m = 0\,.$$

In the d-dimensional theory, there will be one massive vielbein, $n-1$ massive vector fields, and $n(n-1)/2$ massive scalar fields, cf. table 4. The kinetic part of the Lagrangian, $\mathcal{L}_\mathrm{K}$, remains as in (4.19), and the Lagrangians $\mathcal{L}_\mathrm{M}$ and $\mathcal{L}_\mathrm{I}$ are now given by

$$\begin{aligned}
\mathcal{L}_\mathrm{M} = &-\left(\partial^m B_\mu{}^n\right)^2 - \Delta^{-2(\zeta+1/n)}\left(e_{(\alpha}{}^\mu \partial_m e_{\mu\beta)}\right)^2 + \Delta^{-2(\zeta+1/n)}(\partial_m \ln \Delta)^2\\
&- \Delta^{-2(\zeta+1/n)}\left(\partial_{[m} e_{n]}{}^a\right)^2 - 2\Delta^{-2(\zeta+1/n)}\left(e_a{}^m \partial_n e_m{}^b\right)\left(e_b{}^p \partial_n e_p{}^a\right)\\
&+ \frac{1}{4}\Delta^{-2(\zeta+1/n)}(\partial_m \ln e)^2\,,
\end{aligned}\tag{4.21}$$

and

$$\begin{aligned}
\mathcal{L}_\mathrm{I} = &(\partial^{(m} B_\mu{}^{n)})(e_{am}\mathcal{D}^\mu e_n{}^a) - 2(\mathcal{D}_{[\mu} B_{\nu]}{}^m)(e_\alpha{}^\mu \partial_m e^{\nu\alpha})\\
&- (\zeta + 1/n)\Delta^{-2(\zeta+1/n)}(\partial_m \ln \Delta)(\partial^m \ln e)\,.
\end{aligned}$$

We notice that there are mass terms for the vielbein, the gravi-photon and for the scalar fields. This is in accordance with group-theoretical analysis carried out in chapter 3, see for instance table 4. Additionally, there are many interaction terms between the massless and the massive fields.

It is straightforward to calculate the massive sector in the compactification of the Einstein-Hilbert action by Fourier-expanding (4.19) and (4.21) and consequently integrating over the coordinates of the internal manifold. The calculation is rather lengthy, and we refrain from giving further details. Instead, we now turn to the compactification from eleven and ten dimensions to nine dimensions, and give the explicit Lagrangian for the massless modes. In a toroidal reduction from eleven dimensions, the massless sector originating from the Einstein-Hilbert action takes the following form,

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{2}\, e \left(R - \mathrm{e}^{-6\sigma/7} \big(\partial_{[\mu} B^m_{\nu]}\big)^2 - \big|V^{-1}\partial_\mu V\big|^2 - \frac{8}{7}(\partial_\mu \sigma)^2 \right) ,$$

where $\Delta = \exp(-2\sigma/3)$ and where we have defined the SL(2,$\mathbb{R}$)/SO(2) valued scalar fields $V_m{}^a = e_m{}^a$. An analogous reduction of the Einstein-Hilbert term in the IIB supergravity Lagrangian yields

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{2} e \left(R - \frac{1}{7}\left(\partial_\mu \tilde{\sigma}\right)^2 \right) .$$

It is important to note that the field $\tilde{\sigma}$ is not equal to the field σ. The former is the determinant of the einbein on the circle, and the latter is the determinant of the zweibein on the torus.

2.2. Antisymmetric tensor fields

Both the eleven-dimensional supergravity theory and the IIB supergravity theory in ten dimensions contain a number of antisymmetric tensor fields. As an example of how compactification affects antisymmetric tensor fields, we study the compactification of the three-index tensor field from eleven dimensions to nine dimensions.

The antisymmetric tensor in eleven dimensions $\hat{A}_{MNP}$ transforms under abelian tensor gauge transformations with infinitesimal parameter $\hat{\Lambda}_{MNP}$ as follows,

$$\delta \hat{A}_{MNP} = 3\partial_{[M} \hat{\Lambda}_{NP]} \,. \tag{4.22}$$

The kinetic term of the tensor field in the eleven-dimensional Lagrangian is given by

$$\mathcal{L}_{\mathrm{tensor}} = -\frac{1}{48}\, \hat{E}\, \hat{F}^2_{MNPQ} \,, \tag{4.23}$$

where the field strength $\hat{F}_{MNPQ}$ is defined as $\hat{F}_{MNPQ} = 4\partial_{[M}\hat{A}_{NPQ]}$, cf. (1.10).

The reduction to nine dimensions is done using the procedure by Cremmer and Julia [67] as outlined in section 1 above. The eleven-dimensional fields $\hat{A}_{MNP}$ and the nine-dimensional fields $A_{\mu\nu\rho}$ are equal for tangent-space indices, e.g $\hat{A}_{\alpha\beta\gamma} = A_{\alpha\beta\gamma}$. This identification allows us to define fields in nine dimensions that are true tensor fields and that do not transform under gauge transformations pointing along the directions of the internal manifold. For space-time indices, the higher-dimensional fields are defined in terms of the lower-dimensional fields as follows,

$$\hat{A}_{\mu\, mn} = A_{\mu\, mn} \,, \tag{4.24a}$$

$$\hat{A}_{\mu\nu\, m} = A_{\mu\nu\, m} + 2B_{[\mu}{}^{n} A_{\nu]\, mn} \,, \tag{4.24b}$$

$$\hat{A}_{\mu\nu\rho} = A_{\mu\nu\rho} + 3B_{[\mu}{}^{m} A_{\nu\rho]\, m} + 3B_{[\mu}{}^{m} B_{\nu}{}^{n} A_{\rho]\, mn} \,. \tag{4.24c}$$

Note that no tensor field with three internal indices arises in the reduction from eleven to nine dimensions. Using the vielbein (4.17) with $\zeta = 1/7$ and $n = 2$, we solve (4.24) for the tensor-fields that are covariant in nine dimensions,

$$\begin{aligned}
A_{\mu\nu\rho} &= \Delta^{-3/7}\big(\hat{A}_{\mu\nu\rho} - 3B_{[\mu}{}^{m} \hat{A}_{\nu\rho]\, m} + 6B_{[\mu}{}^{m} B_{\nu}{}^{n} \hat{A}_{\rho]\, mn}\big) \,, \\
A_{\mu\nu\, m} &= \Delta^{-2/7+1/2}\big(\hat{A}_{\mu\nu\, m} - 2B_{[\mu}{}^{n} \hat{A}_{\nu]\, mn}\big) \,, \\
A_{\mu\, mn} &= \Delta^{-1/7+1} \hat{A}_{\mu\, mn} \,.
\end{aligned}$$

The nine-dimensional theory therefore contains one three-index gauge field $A_{\mu\nu\rho}$, two two-index gauge fields $A_{\mu\nu\, m}$ and one vector field $A_{\mu\, mn}$. These fields transform as follows under the higher-dimensional tensor gauge transformations

$$\delta A_{\mu\nu\rho} = 3\mathcal{D}_{[\mu} \Lambda_{\nu\rho]} \,, \tag{4.25a}$$

$$\delta A_{\mu\nu\, p} = 2\mathcal{D}_{[\mu} \Lambda_{\nu]\, p} + \partial_p \Lambda_{\mu\nu} \,, \tag{4.25b}$$

$$\delta A_{\mu\, pq} = \mathcal{D}_{\mu} \Lambda_{pq} + 2\partial_{[p} \Lambda_{q]\, \mu} \,. \tag{4.25c}$$

The gravi-photon fields $B_\mu{}^m$ do not transform under these tensor gauge transformations. Similarly to the case of the abelian vector field discussed above, we can restrict the gauge transformation parameters and gauge away the unphysical degrees of freedom of the various fields.

We define the following field-strength tensors,

$$F_{\mu\nu\rho\sigma} = 4\mathcal{D}_{[\mu} A_{\nu\rho\sigma]} + 6G_{[\mu\nu}{}^{m} A_{\rho\sigma]\, m} \,,$$

$$\begin{aligned}
F_{\mu\nu\rho\, q} = {} & 3\mathcal{D}_{[\mu} A_{\nu\rho]q} - 3\partial_q B_{[\mu}{}^{m} A_{\nu\rho]m} - \partial_q A_{\mu\nu\rho} \\
& - 3G_{[\mu\nu}{}^{p} A_{\rho]pq} + 6B_{[\mu}{}^{m} B_{\nu}{}^{n} \partial_q A_{\rho]mn} \,,
\end{aligned}$$

$$F_{\mu\nu\, pq} = 2\mathcal{D}_{[\mu} A_{\nu]pq} + 2\partial_{[p} A_{q]\mu\nu} + 4\partial_{[p} B_{[\mu}{}^{n} A_{\nu]q]n} \,.$$

These field strengths are invariant under the gauge transformations (4.25) and also under the gauge transformations generated by the gravi-photon fields $B_\mu{}^i$, as can easily be checked.

Let us now turn to the massless sector in the compactification of the kinetic term (4.23) of the three-index tensor field. Using the nine-dimensional fields that we have defined, and neglecting all dependence on the internal coordinates, we derive the following terms in the nine-dimensional Lagrangian

$$\mathcal{L}_{\text{tensor}} = -\frac{1}{48}\, e\Big(\mathrm{e}^{-4\sigma/7} F^2_{\mu\nu\rho\sigma} + 4\,\mathrm{e}^{2\sigma/7} F^2_{\mu\nu\rho\, q} + 6\,\mathrm{e}^{8\sigma/7} F^2_{\mu\nu\, pq}\Big)\,.$$

The reduction of the Chern-Simons term $F \wedge F \wedge A$ in the eleven-dimensional Lagrangian (1.10) is straightforward if we neglect the dependence of the fields on the internal coordinates, but we will not present it here.

The IIB theory also contains various tensor fields. There is an SL(2) doublet of two-index tensor fields A^m_{MN}, which can be compactified to nine dimensions using the methods discussed in this section. The two-index tensor field gives rise to a two-index tensor field in nine-dimensions, and to an SL(2) doublet of vector fields, cf. section 3.2 of the previous chapter. The five-index tensor field of IIB supergravity will be discussed in section 2.5, as its self-duality constraint requires a special analysis.

2.3. Rarita-Schwinger fields

Let us now discuss the reduction of the Rarita-Schwinger field from eleven dimensions to nine dimensions. The reduction from ten dimensions to nine dimensions that is required for the IIB theory compactified on a circle is straightforward; it only requires a rescaling of the fields and we will not discuss it here. Generalizations to other dimensions can be carried out using the same methods. However, because the details of the compactifications depend crucially on the dimension of the spinors in the various space-time dimensions a unified treatment is not easily possible and we will not attempt it here.

In order to obtain the expressions for the fermionic terms in the compactified space, it is important to study the fate of the symmetries of the original theory. As discussed in section 1.1, the compactification on a circle S^1 or on a torus T^2 gives rise to one and two new U(1) gauge symmetries, respectively, stemming from the isometries of the internal manifold. Since the gravitinos in the lower-dimensional theory should transform as vector-spinors under general coordinate transformations, we transform all higher-dimensional space-time indices to tangent-space indices. The procedure is closely related to the procedure for vector fields outlined in section 1. From the discussion of the massless multiplet in nine-dimensions in section 2 of the preceding chapter, we know that the Rarita-Schwinger field ψ_A in eleven dimensions splits into two massless Rarita-Schwinger fields and two massless fermions in nine-dimensions. There is also an infinite number of massive fields, namely a two-dimensional lattice of massive spin-3/2 fields and of massive spin-1/2 fields.

We now turn to the compactification of the kinetic term for the gravitinos in eleven-dimensions (the Rarita-Schwinger term) to nine dimensions. According to (1.10), the Rarita-Schwinger term in the eleven-dimensional action is given by,

$$\mathcal{L} = \frac{1}{2}\,\hat{E}\,\hat{\bar{\psi}}_A \Gamma^{ABC}\,\hat{D}_B \hat{\psi}_C\,. \tag{4.26}$$

If we simply split the Rarita-Schwinger fields in $\hat{\psi}_\alpha$ and $\hat{\psi}_a$ and rewrite the Lagrangian (4.26) in terms these new fields, then the kinetic terms in nine dimensions is not diagonal. In order to diagonalize the kinetic terms in the Lagrangian, we define the spin-3/2 field ψ_α and the spin-1/2 field ψ_a as

$$\begin{aligned}\psi_\alpha &= \Delta^{1/14}\left(\hat{\psi}_\alpha + \frac{1}{7}\tilde{\gamma}\gamma_\alpha \Gamma^a \hat{\psi}_a\right)\,,\\ \psi_a &= \Delta^{1/14}\hat{\psi}_a\,,\end{aligned}$$

where γ_α are the nine-dimensional gamma matrices. Here, Γ_a and $\tilde{\gamma}$ are the gamma matrices in two dimensions which have been defined in section 1 of chapter 3. The Lagrangian (4.26) is then written as the sum of a kinetic part $\mathcal{L}_{\mathrm{K}}$ and an interaction part $\mathcal{L}_{\mathrm{M}}$, which are given by

$$2e^{-1}\Delta^{3/7}\mathcal{L}_{\mathrm{K}} = \bar{\psi}_\alpha \gamma^{\alpha\beta\gamma} D_\beta \psi_\gamma + \bar{\psi}_a\left(\delta^{ab} + \frac{1}{7}\Gamma^a\Gamma^b\right)\gamma^\alpha D_\alpha \psi_b\,, \tag{4.27a}$$

$$\begin{aligned}2e^{-1}\Delta^{3/7}\mathcal{L}_{\mathrm{M}} = &-\bar{\psi}_\alpha \tilde{\gamma}\Gamma^a \gamma^{\alpha\beta} D_a \psi_\beta - \frac{72}{49}\bar{\psi}_a \tilde{\gamma}\Gamma^a\Gamma^b\Gamma^c D_b \psi_c\\ &-\hat{\bar{\psi}}_\alpha\left(\delta^{ab} + \frac{1}{7}\Gamma^a\Gamma^b\right)\gamma^\alpha D_a \psi_b\\ &-\bar{\psi}_a\left(\delta^{ab} + \frac{1}{7}\Gamma^a\Gamma^b\right)\gamma^\alpha D_b \hat{\psi}_\alpha\end{aligned} \tag{4.27b}$$

Note that the covariant derivatives D_α are still the original eleven-dimensional ones. In order to bring the kinetic term of the spinor fields ψ_a into the standard form, we define two spinors χ_{ABC} and λ_A which transform irreducibly as doublets under the R-symmetry group SO(2), i.e.

$$\chi_{ABC} = -\frac{\sqrt{2}}{2}\left(\Gamma^a_{(AB}\psi_{aC)} - \frac{1}{2}\delta_{(AB}\Gamma^a_{C)D}\psi_{aD}\right)\,, \tag{4.28a}$$

$$\lambda_A = -\frac{3}{\sqrt{14}}\Gamma^a_{AD}\psi_{aD}\,, \tag{4.28b}$$

where $A = 1, 2$ are SO(2) spinor indices. Here, χ_{ABC} is completely symmetric in the SO(2) spinor indices and traceless, i.e. $\chi_{ABB} = 0$. The definitions (4.28) are invertible, and the original fields ψ_a can be expressed as

$$\psi^a_C = -\frac{\sqrt{2}}{2}\Big(\Gamma^a_{AB}\chi_{ABC} + \left(\Gamma^a\Gamma^b\right)_{CA}\Gamma^b_{BD}\chi_{ABD}\Big) + \frac{\sqrt{14}}{6}\Gamma^a_{CA}\lambda_A\,.$$

In order to identify the physical degrees of freedom, we impose the gauge

$$\left(\delta^{ab} + \frac{1}{7}\Gamma^a\Gamma^b\right) D_a\psi_b = 0\,. \tag{4.29}$$

This gauge choice eliminates the interaction terms in the $\mathcal{L}_{\mathrm{M}}$ of (4.27b). The mass terms for the gravitinos ψ_α and for the spinors λ are still present in $\mathcal{L}_{\mathrm{M}}$. Note that because of the gauge condition (4.29), there is only one mass term, either for χ_{ABC} or for λ_A. This means that the massive degrees of freedom of two spinors are transferred to the two gravitinos, which are then physical massive fields in nine dimensions.

At this point, all fields still depend on the coordinates y^m of the internal manifold. Integration over y^m yields the terms in the nine-dimensional theory. We concentrate on the massless modes, and write down the kinetic terms for the Rarita-Schwinger fields and the fermion fields in nine dimensions. They are given by

$$\mathcal{L}_{\mathrm{K}} = \frac{1}{2}\, e\, \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + \frac{1}{2}\, e\, \Delta^{-3/7}\big(\bar{\chi}_{ABC}\not{\partial}\chi_{ABC} + \bar{\lambda}_A\not{\partial}\lambda_A\big)\,,$$

where we have transformed the tangent-space indices of the gravitino to space-time indices using the rescaled vielbein $\Delta^{1/7}e_\mu{}^a$.

2.4. Compactification of a nonlinear sigma model

The scalar sector of IIB supergravity is described by a nonlinear sigma model based on the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. In order to write down the Lagrangian for the scalar fields in nine dimensions, we discuss the compactification of a general sigma model with coset space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.[4]

The degrees of freedom of the scalar fields are described by a matrix-valued field $\mathcal{V}(x,y) \in \mathrm{SL}(n,\mathbb{R})$ transforming as

$$\mathcal{V}(x,y) \to g\,\mathcal{V}(x,y)\,h^{-1}(x,y)\,,$$

where g denotes a constant element of $\mathrm{SL}(n,\mathbb{R})$ and $h(x,y)$ is a local $\mathrm{SO}(n)$ transformation. In view of the gauge invariance that depends on both x and y, and the fact that we are dealing with a group element, the split into massive and massless degrees of freedom is not entirely straightforward.

The best approach is to write $\mathcal{V}(x,y)$ as the product of two $\mathrm{SL}(n,\mathbb{R})$ elements,

$$\mathcal{V}(x,y) = \mathcal{V}_0(x)\,\mathcal{V}_1(x,y)\,, \tag{4.30}$$

and to require that $\mathcal{V}_0$ describes the massless modes in the torus compactification. To do this, one can first fix the $\mathrm{SO}(n)$ gauge freedom and define a coset representative. Subsequently one considers the logarithm of $\mathcal{V}(x,y)$ and expands it in terms of Fourier modes on the torus. Dropping the y-dependent

[4]The results of this section were obtained in collaboration with B. de Wit and H. Nicolai.

modes in this expansion yields $\mathcal{V}_0(x)$. However, $\mathcal{V}_0(x)$ is itself a coset representative so that it is defined up to multiplication by an x-dependent SO(n) transformation acting from the right. This leads to a corresponding ambiguity for $\mathcal{V}_1(x, y)$. Hence $\mathcal{V}_0(x)$ parameterizes a nonlinear sigma model in the lower-dimensional space, so that it transforms according to

$$\mathcal{V}_0(x) \to g\, \mathcal{V}_0(x)\, h_0^{-1}(x)\,,$$

where $h_0(x)$ is an x-dependent SO(n) transformation, and $\mathcal{V}_1(x, y)$ transforms under an x-dependent SO(n) transformation from the left and, provided one again relaxes the original gauge condition, under an x- and y-dependent SO(n) transformation from the right,

$$\mathcal{V}_1(x, y) \to h_0(x)\, \mathcal{V}_1(x, y)\, h_1^{-1}(x, y)\, h_0^{-1}(x)\,,$$

where we defined $h(x, y) = h_0(x)\, h_1(x, y)$. All the massive Kaluza-Klein degrees of freedom thus reside in $\mathcal{V}_1(x, y)$. The SO(n) symmetry corresponding to $h_1(x, y)$ can now be fixed by going to a "unitary gauge",

$$\mathcal{V}_1(x, y) = \exp\bigl(\phi(x, y)\bigr)\,,$$

where $\phi(x, y)$ is a symmetric traceless $(n\times n)$-matrix, such that $\mathcal{V}_1(x, y)$ transforms under the residual x-dependent SO(n) transformations according to

$$\begin{aligned}\mathcal{V}_1(x, y) &\to h_0(x)\, \mathcal{V}_1(x, y)\, h_0^{-1}(x)\,,\\ \phi(x, y) &\to h_0(x)\, \phi(x, y)\, h_0^{-1}(x)\,.\end{aligned}$$

Therefore the massive fields $\phi(x, y)$ transform covariantly under x-dependent SO(n) gauge transformations but not under SL($n, \mathbb{R}$).

The split (4.30) of $\mathcal{V}(x, y)$ exhibits clearly how the massive Kaluza-Klein degrees of freedom behave with respect to the local symmetries of the massless theory. To describe the Lagrangian we consider the SL($n, \mathbb{R}$) Lie-algebra valued expression

$$\begin{aligned}P_M + Q_M &\equiv \mathcal{V}^{-1}\partial_M \mathcal{V}\\ &= \mathcal{V}_1^{-1} P_M^0 \mathcal{V}_1 + \mathcal{V}_1^{-1} D_M^0 \mathcal{V}_1 + Q_M^0\,,\end{aligned}$$

where Q_M and P_M belong to the Lie algebra of SO(n) and its complement in the Lie algebra of SL($n, \mathbb{R}$), respectively. Decomposing the index M into μ and m as before, we have the y-independent quantities

$$P_\mu^0 + Q_\mu^0 \equiv \mathcal{V}_0^{-1}\partial_\mu \mathcal{V}_0$$

(obviously, $Q_m^0 = P_m^0 = 0$). The derivative D_M^0 is covariant with respect to x-dependent SO(n) gauge transformations,

$$\begin{aligned}D_\mu^0 \mathcal{V}_1 &\equiv \partial_\mu \mathcal{V}_1 + \bigl[Q_\mu^0, \mathcal{V}_1\bigr]\,,\\ D_m^0 \mathcal{V}_1 &\equiv \partial_m \mathcal{V}_1\,.\end{aligned}$$

To write down an action coupling the massless sector and the massive Kaluza-Klein modes in an SO(n) invariant way, we expand

$$P_M = P^0_M + D^0_M\phi + \left[P^0_M, \phi\right] + \frac{1}{2}\left[\left[P^0_M, \phi\right], \phi\right] + \frac{1}{2}\left[D^0_M\phi, \phi\right] + \cdots ,$$

projected on the complement of the Lie algebra of SO(n). Because the coset space SL($n, \mathbb{R}$)/SO(n) is symmetric some of the terms in P_M will trivially vanish. What remains is to substitute the expression for P_M into

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\left(P_\mu^2\right) - \frac{1}{2}\,\mathrm{tr}\left(P_m^2\right),$$

which will lead to an action that is non-polynomial in ϕ. Let us repeat, however, that this action is invariant under x-dependent gauge transformations, as well as under a global SL($n, \mathbb{R}$) symmetry which acts exclusively in the massless sector. Once we fix an SO(n) gauge, the SL($n, \mathbb{R}$) symmetry becomes nonlinearly realized and acts also on the massive fields.

Before fixing an SO(n) gauge, the SL($n, \mathbb{R}$) symmetry does not act on the massive modes in this simplified model. This is not so when the SL($n, \mathbb{R}$) originates from the dimensional reduction in the more complicated models based on (super)gravity in higher dimensions. Upon performing a Kaluza-Klein reduction (not a truncation!) on the torus T^n, the global symmetry will still act on the massive modes, but it will be broken to an arithmetic subgroup such as $G(\mathbb{Z})$. To see how this comes about, recall that $G = \mathrm{SL}(n, \mathbb{R})$ and $H = \mathrm{SO}(n)$ are precisely the symmetries that one obtains upon dimensional reduction of pure Einstein theory on a torus T^n. As we saw in chapter 4, the Kaluza-Klein gauge field ${B_\mu}^m$ couples via the derivative operator,

$$\mathcal{D}_\mu = \partial_\mu - {B_\mu}^m \partial_m$$

(with vanishing affine connection for the torus). When the theory is compactified on a torus T^n, the derivative operator ∂_m will only admit discrete eigenvalues $\boldsymbol{q} = (q_1, \ldots, q_m)$. These eigenvalues lie on an n-dimensional lattice, the lattice of Kaluza-Klein charges. It is the presence of this lattice that leads to the breaking SL($n, \mathbb{R}$) $\to$ SL($n, \mathbb{Z}$): the group SL($n, \mathbb{Z}$) acts on the vectors $\boldsymbol{q}$ labeling the Kaluza-Klein modes, rather than on the fields themselves. The massless modes have $\boldsymbol{q} = 0$ and transform under SL($n, \mathbb{Z}$) in the way described above for the nonlinear sigma model.

2.5. Self-dual tensor field

The compactification of the five-index field-strength F_{MNPQR} which appears in type IIB supergravity requires special attention. The field-strength is self-dual and therefore it is not possible to define an action for the fields A_{MNPQ} in ten dimensions. In nine dimensions, it is, however, possible to define a Lagrangian as we discuss below. The original self-duality condition in ten dimensions gives rise to a duality condition for the massive fields in nine dimensions.

In the IIB theory, the four-index tensor field A_{MNPQ} is a real field, and a priori it consists of 70 components. Its field strength F_{MNPQR} is subject to the Bianchi identity

$$\partial_{[M} F_{NPQRS]} = 0\,. \tag{4.31}$$

and to the self-duality condition

$$F_{M_1\ldots M_5} = \frac{1}{5!}\varepsilon_{M_1\ldots M_5 N_1\ldots N_5} F^{N_1\ldots N_5}\,. \tag{4.32}$$

This self-duality condition reduces the components of A_{MNPQ} to 35, and consequently the five-index tensor field transforms in the $\mathbf{35}_c$ representation of SO(8). The Bianchi identity (4.31) and the self-duality condition (4.32) give rise to the equation

$$\partial_M F^{MNPQR} = 0\,, \tag{4.33}$$

which is the standard equation of motion of a five-index tensor field.

In order to carry out the compactification, we Fourier-expand the field A_{MNPQ} on a circle with radius R, and concentrate on one massive mode with mode number n. For simplicity we consider the self-dual tensor field in a background with no gravi-photon field. We decompose the index M into $(\mu, 10)$ and denote the compact coordinate by y, so that the expansion of A_{MNPQ} can be written as

$$\begin{aligned} A_{\mu\nu\rho\sigma}(x,y) &\equiv A_{\mu\nu\rho\sigma}(x) + \mathrm{e}^{iny/R} B_{\mu\nu\rho\sigma}(x) + \text{h.c.}\,,\\ A_{10\,\mu\nu\rho}(x,y) &\equiv A_{\mu\nu\rho}(x) + \mathrm{e}^{iny/R} B_{\mu\nu\rho}(x) + \text{h.c.}\,. \end{aligned}$$

The analysis of the massless modes is straightforward. We define the field-strengths in nine dimensions, $F_{\mu\nu\rho\sigma\tau} = 5\partial_{[\mu} A_{\nu\rho\sigma\tau]}$ and $F_{\mu\nu\rho\sigma} = 4\partial_{[\mu} A_{\nu\rho\sigma]}$. The Bianchi identity (4.31) in ten-dimensions gives rise to Bianchi-identities for both $F_{\mu\nu\rho\sigma\tau}$ and $F_{\mu\nu\rho\sigma}$. Furthermore, the self-duality condition (4.32) yields

$$F_{\mu_1\ldots\mu_4} = \frac{1}{5!}\varepsilon_{\mu_1\ldots\mu_4\nu_1\ldots\nu_5} F^{\nu_1\ldots\nu_5}\,,$$

where $\varepsilon_{\mu_1\ldots\mu_9} = \varepsilon_{10\mu_1\ldots\mu_9}$ is the nine-dimensional Levi-Civita symbol. This is simply the Poincaré duality between a three-index tensor field and a four-index tensor field in nine dimensions. This means that we are free to formulate the massless sector in nine dimensions in terms of either the three-index field or the four-index field and both descriptions are equivalent.

Let us now turn to the massive modes. We define the Fourier-expansion of the field-strength F_{MNPQR} as follows,

$$F_{\mu\nu\rho\sigma\tau}(x,y) \equiv F_{\mu\nu\rho\sigma\tau}(x) + \mathrm{e}^{iny/R} G_{\mu\nu\rho\sigma\tau}(x) + \text{h.c.}\,, \tag{4.34a}$$

$$F_{10\,\mu\nu\rho\sigma}(x,y) \equiv F_{\mu\nu\rho\sigma}(x) + \mathrm{e}^{iny/R} G_{\mu\nu\rho\sigma}(x) + \text{h.c.}\,. \tag{4.34b}$$

This implies that the field-strengths for the massive fields are given by

$$G_{\mu\nu\rho\sigma\tau} = 5\ \partial_{[\mu} B_{\nu\rho\sigma\tau]}\,, \tag{4.35a}$$

$$G_{\mu\nu\rho\sigma} = 4\ \partial_{[\mu} B_{\nu\rho\sigma]} - i\frac{n}{R} B_{\mu\nu\rho\sigma}\,. \tag{4.35b}$$

A field-dependent gauge transformation of the massive four-index tensor field $B_{\mu\nu\rho\sigma}$ given by $\delta B_{\mu\nu\rho\sigma} = -4i(R/n)\partial_{[\mu} B_{\nu\rho\sigma]}$ does not change (4.35a) and (4.35b) simplifies to

$$G_{\mu\nu\rho\sigma} = -i\frac{n}{R} B_{\mu\nu\rho\sigma}\,.$$

If we substitute this result in the Fourier-expansion (4.34) and subsequently evaluate the self-duality condition (4.32) for the massive modes, we obtain

$$B_{\mu_1\ldots\mu_4} = i\frac{1}{5!}\frac{R}{n}\varepsilon_{\mu_1\ldots\mu_4\nu_1\ldots\nu_5} G^{\mu_1\ldots\mu_5}\,. \tag{4.36}$$

This condition reduces the number of components for a massive field in nine dimensions from 70 to 35, and the condition is known as a self-duality condition [68]. The self-duality condition (4.36) can be obtained as the equation of motion from the Lagrangian

$$\mathcal{L} = B_{\mu\nu\rho\sigma} B^{\mu\nu\rho\sigma} - i\frac{1}{5!}\frac{R}{n}\varepsilon_{\mu_1\ldots\mu_4\nu_1\ldots\nu_5} B^{\mu_1\ldots\mu_4} G^{\nu_1\ldots\nu_5}\,.$$

We would also like to point out that conventional equation of motion for a massive tensor field,

$$\partial_\mu G^{\mu\nu\rho\sigma\tau} - \frac{n^2}{R^2} B^{\nu\rho\sigma\tau} = 0\,,$$

follows directly from the self-duality condition (4.36) and from the Bianchi identity for the massive fields.

3. BPS-extended supergravity

In this chapter we have described how one can construct a nine-dimensional BPS-extended supergravity theory from the eleven-dimensional supergravity theory and from ten-dimensional IIB supergravity. We would like to point out the fact that the BPS-extended supergravity theory in nine dimensions consisting of the massless $\mathcal{N} = 2$ multiplet coupled to an infinite lattice of KKA multiplets is not simply a reduction of the eleven-dimensional theory on a torus, but it is in fact equivalent to it. Similarly, the BPS-extended supergravity theory consisting of the massless multiplet and an infinite tower of KKB multiplets is equivalent to the IIB supergravity theory.

From a purely nine-dimensional point of view, it is natural to couple not only one type of BPS multiplet to the massless multiplet, but to include both KKA and KKB multiplets. One can then construct a theory consisting of the SL(2, $\mathbb{Z}$) invariant lattice of KKA multiplets and a tower of KKB multiplets.

This new BPS-extended theory contains eleven-dimensional supergravity theory, as well as the ten-dimensional IIA and IIB supergravity theories in certain decompactification limits. The theory is in a way twelve-dimensional, with three compact coordinates; however, no field in the theory simultaneously depends on all twelve coordinates [61].

We have already made some comments on this BPS-extended supergravity theory throughout chapters 3 and 4, but we would like to repeat some of the important points here. The theory contains three abelian vector fields, ${A_\mu}^m$ and B_μ. The two fields ${A_\mu}^m$ transform as a doublet under $\mathrm{SL}(2,\mathbb{Z})$. From an eleven-dimensional point of view they are the gravi-photons that appear in the toroidal compactification, and they couple to the states in the KKA multiplets, and the SL(2) transformations correspond to modular transformations of the torus. From a IIB point of view, the fields ${A_\mu}^m$ originate from the two-index tensor fields, and they couple to the winding states of the elementary string and the D1-string. The SL(2) is in this case identified with the S-duality transformations of the IIB theory, which mix the elementary string states and the D1-string states. The gauge field B_μ is an SL(2) singlet. It is the gravi-photon field in the reduction of the IIB theory on a circle, and the states in the KKB multiplets are charged under it. In the reduction of the eleven-dimensional theory, it originates from the tensor field, and couples to the winding states of the M2-brane on the torus.

The global symmetry group of the massless theory, $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1)$ is broken by the inclusion of the KKA and KKB multiplets. The residual symmetry group that leaves the charge lattice invariant is an arithmetic subgroup, namely $\mathrm{SL}(2,\mathbb{Z})$. The BPS-extended supergravity theory contains two mass-scales m_{KKA} and m_{KKB}, associated to the KKA and KKB states, respectively. The BPS mass formula in nine dimensions is then given by

$$M_{\mathrm{BPS}}(\boldsymbol{q}, p) = m_{\mathrm{KKA}} \mathrm{e}^{3\sigma/7} \left| \boldsymbol{q} \cdot \boldsymbol{\phi} \right| + m_{\mathrm{KKB}} \mathrm{e}^{-4\sigma/7} \left| p \right|$$

where $\boldsymbol{q}$ is the KKA charge vector and p is the KKB charge. We can compare this mass formula with the mass formula (3.6) for the M2-brane on the torus, and obtain the relation

$$m_{\mathrm{KKA}}^2 m_{\mathrm{KKB}} \propto T_m \,.$$

The most important feature of this BPS-extended supergravity theory is that it can possibly help to give new insights into M-theory in eleven dimensions. The theory is not very well understood yet, but is has been used as evidence for the fact that M-theory is the non-perturbative theory of the eleven-dimensional supermembrane [61]. The theory can also shed new light on higher order invariants in eleven-dimensional supergravity theory. In this context, it has been shown that one-loop corrections to R^4 terms receive only contributions from the KKA and KKB multiplets [69].

5

BPS-extended supersymmetric Yang-Mills theory

Field theories that are invariant under rigid supersymmetry transformations do not contain gravity and are easier to handle than theories with local supersymmetry invariance. For this reason we now turn to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions and couple BPS supermultiplets to the massless theory. In this way we obtain a BPS-extended supersymmetric Yang-Mills theory. We start from a Kaluza-Klein reduction of a 10-dimensional supersymmetric gauge theory, and thus follow an approach that is very similar to the one taken in chapters 3 and 4. As opposed to a simple dimensional reduction [70], we retain massive modes, which constitute BPS supermultiplets in four dimensions. An alternative method is based on considering a spontaneously broken realization of the gauge group directly in four space-time dimensions. Both these approaches lead to massive multiplets that must be BPS, as the highest spin of the (massive) multiplet is equal to one. There are a few differences in the two approaches. The Kaluza-Klein approach leads to an infinite lattice of BPS multiplets transforming in the adjoint representation of the non-abelian gauge group and the gauge group in the lower-dimensional theory is the same as the gauge group in the uncompactified theory. For the spontaneously broken theories, the gauge group is broken to a subgroup; the precise residual gauge symmetry depends on the vacuum expectation value of the scalar fields. Furthermore, there are only a finite number of BPS multiplets, transforming in different representations of the residual gauge group.

We start from supersymmetric Yang-Mills theory in ten space-time dimensions. Upon compactification of six dimensions on a torus, we obtain $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, coupled to towers of massive BPS multiplets that contain Proca fields. To bring the theory into the conventional four-dimensional form requires a series of gauge conditions, which induce non-polynomial behavior into the supersymmetry transformation rules. Our goal here is to obtain the theory to quadratic order in the BPS fields but to all orders in the massless fields. The interactions are all governed by the non-abelian gauge coupling constant.

Unlike pure four-dimensional Yang-Mills theory, which is superconformally invariant even at the quantum level, the interactions with the BPS multiplets are not superconformally invariant. We restrict ourselves to a single BPS supermultiplet in an arbitrary representation and evaluate the one-loop corrections to the massless effective theory. We find that the BPS supermultiplets do not lead to coupling constant renormalization, as is to be expected. We will integrate out the massive fields using the proper time method [71]. For a general setup this is rather difficult but in the case where gauge fields and scalars are constant, we are able to calculate all the induced F^n terms in the effective action as well as the effective potential to all orders. In appropriate limits, our findings agree with [72], where the effective action was computed in the context of the purely massless theory. The question to what extent the BPS extended supersymmetric Yang-Mills theory is renormalizable is subtle and we will briefly comment on it.

This chapter is based on [73].

1. Supersymmetric Yang-Mills theory

As is well known, the highest space-time dimension in which a supersymmetric Yang-Mills theory can exist, is equal to ten. Here we present an outline of this ten-dimensional theory and briefly discuss its compactification on a six-dimensional torus. This discussion will enable us to establish the notation and to explain a number of features that are relevant for what follows. The Yang-Mills supermultiplet in ten dimensions consists of a gauge field A_M and a Majorana-Weyl spinor ψ, which both take values in the Lie algebra associated with a gauge group G. For a compact group G the fields are anti-hermitian matrices. The action is given by

$$S_{10} = \frac{1}{g_{10}^2} \int \mathrm{d}^{10}x \; \mathrm{tr}_{\mathrm{R}} \left(\frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} \bar{\psi} \Gamma^M D_M \psi \right) , \qquad (5.1)$$

where g_{10} is the gauge coupling constant which is of mass-dimension minus three. The trace tr_{R} is taken in some Lie algebra representation $\mathbf{R}$. In (5.1) we used the following definitions for the covariant derivative and the field-strength,

$$\begin{aligned} D_M \psi &= \partial_M \psi - [A_M, \psi] , \\ F_{MN} &= \partial_M A_N - \partial_M A_N - [A_M, A_N] . \end{aligned}$$

The action is invariant under non-abelian gauge transformations and under rigid supersymmetry transformations,

$$\delta A_M = \bar{\epsilon} \Gamma_M \psi , \qquad \delta \psi = -\frac{1}{2} F_{MN} \Gamma^{MN} \epsilon ,$$

where ϵ is a constant infinitesimal Majorana-Weyl spinor. The supersymmetry transformations close up to fermionic equations of motion. For instance, on the gauge fields, the anticommutator of two supercharges yields

$$\big[\delta_Q(\epsilon_1) , \delta_Q(\epsilon_2) \big] A_M = 2 \, \bar{\epsilon}_2 \Gamma^N \epsilon_1 \, F_{NM} , \qquad (5.2)$$

which represents precisely a (covariant) translation. Observe the absence of higher order fermionic terms in the action and transformation rules. The coupling to a six-rank tensor field of supergravity (the dual of the two-rank anti-symmetric tensor field [74, 75]) leaves supersymmetry unbroken for constant values,

$$S_\theta \propto \int \mathrm{d}^{10}x\ \varepsilon^{M_1\cdots M_{10}}\, A_{M_1\cdots M_6}\ \mathrm{tr_R}\,\big(F_{M_7M_8}F_{M_9M_{10}}\big)\,. \tag{5.3}$$

From the ten-dimensional theory one obtains a four-dimensional theory by compactifying six dimensions on a torus. The ten-dimensional coordinates x^M then decompose into four space-time coordinates x^μ and six 'internal' torus coordinates y^m, i.e. $x^M = (x^\mu, y^m)$. The fields that are constant on the torus then comprise the massless $\mathcal{N} = 4$ Yang-Mills supermultiplet in four dimensions, consisting of a gauge field, six scalar fields and four Majorana spinors. The higher Fourier modes on the torus are associated with BPS fields, whose mass and charge are inversely proportional to the size of the torus.

Let us first briefly discuss the internal six-torus defined as $T^6 = \mathbb{R}^6/\Lambda$, where Λ is a lattice in $\mathbb{R}^6$. There are two options to deal with it. Either one chooses a diagonal torus metric, in which case the lattice is not necessarily orthogonal, or one chooses an orthogonal lattice and allows the metric to be more complicated. The latter approach is convenient in order to set up the Fourier series, especially when one would also switch on a dynamic (supergravity) background, where the torus metric itself is associated with dynamic degrees of freedom. For these reasons we choose the latter option and parameterize the torus in terms of the coordinates y^m each having a periodicity interval equal to $2\pi R$, where R is some length. In the presence of spinor fields we need the sechsbein on the torus, which we denote by $e_m{}^a$. In the Kaluza-Klein context, the sechsbein depends on x and y, but in the background it is just constant. Constant metrics can still be transformed by linear coordinate transformations that constitute the group $\mathrm{GL}(6,\mathbb{R})$ but for the torus these transformations have to be restricted to the arithmetic subgroup $\mathrm{SL}(6,\mathbb{Z})$ that leaves the periodicity lattice and the torus volume invariant.[1] The sechsbein transforms also under SO(6) tangent-space rotations. As the sechsbein is itself a six-by-six matrix, it takes the form of an SL(6)/SO(6) coset representative times a volume factor. The former is parameterized by torus moduli which transform nonlinearly under the action of $\mathrm{SL}(6,\mathbb{Z})$.

We can now expand every single-valued field in a Fourier series, i.e.

$$\phi(x,y) = \phi_0(x) + \sum_{\boldsymbol{q}\neq 0} \phi^{(\boldsymbol{q})}(x)\, \mathrm{e}^{i\, q_m y^m/R}\,.$$

[1] Note that the massless sector is not affected by the lattice and remains invariant under $\mathrm{SL}(6,\mathbb{R})$.

The periodicity in the coordinates y^m then requires that the charges q_m are elements of $\mathbb{Z}$.

The decomposition of a ten-dimensional Majorana-Weyl spinor in terms of four-dimensional Majorana spinors, is standard. First one writes the 32×32 gamma matrices Γ^M as direct products of 4×4 and 8×8 matrices according to

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}\,, \qquad \Gamma^a = \gamma^5 \otimes \hat{\Gamma}^a\,, \tag{5.4}$$

where γ^μ and $\hat{\Gamma}^a$ are the gamma matrices appropriate to four-dimensional and six-dimensional spinors, which generate two Clifford algebras which commute in view of $\left[\gamma^\mu, \hat{\Gamma}^a\right] = 0$. All gamma matrices are hermitian, with the exception of Γ^0 and γ^0. We use $\gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^0$ and similarly $\gamma^7 = -i\hat{\Gamma}^1\hat{\Gamma}^2\hat{\Gamma}^3\hat{\Gamma}^4\hat{\Gamma}^5\hat{\Gamma}^6$, so that the chirality operator for the ten-dimensional spinor reads $\Gamma^{11} = \gamma^5 \otimes \gamma^7$. Under SO(6) the spinors transform according to the $\mathbf{4} \oplus \bar{\mathbf{4}}$ representation of its covering group SU(4). Because we are dealing with Majorana-Weyl spinors in ten dimensions, the four-dimensional chirality and the six-dimensional chirality are linked. As a result the positive chirality spinors ψ^i transform according to the $\mathbf{4}$ representation and the negative chirality spinors ψ_i transform according to the $\bar{\mathbf{4}}$ representation of SU(4). Here $i, j = 1, \ldots, 4$ denote SU(4) indices. Further details are presented in the appendix, but here we note that our conventions imply that

$$\hat{\Gamma}^a = \begin{pmatrix} 0 & i\left(\gamma^a\right)^{ij} \\ i\left(\gamma^a\right)_{ij} & 0 \end{pmatrix}, \tag{5.5}$$

where the $\left(\gamma^a\right)^{ij}$ are antisymmetric in $[ij]$. Here and henceforth we use the convention that complex conjugation is always effected by raising or lowering of SU(4) indices, i.e. $\left((\gamma^a)_{ij}\right)^* = (\gamma^a)^{ij}$. We also note the following important identities,

$$\left(\gamma^a\right)^{ik}\left(\gamma^b\right)_{kj} + \left(\gamma^b\right)^{ik}\left(\gamma^a\right)_{kj} = -2\,\delta^{ab}\,\delta^i_j\,, \tag{5.6a}$$

$$\left(\gamma^a\right)^{ij}\left(\gamma^a\right)_{kl} = 4\,\delta^{[i}{}_k\,\delta^{j]}{}_l\,, \tag{5.6b}$$

$$\left(\gamma^a\right)_{ij}\left(\gamma^a\right)_{kl} = -2\,\varepsilon_{ijkl}\,, \tag{5.6c}$$

$$\left(\gamma^a\right)^{ij} = -\frac{1}{2}\varepsilon^{ijkl}\left(\gamma^a\right)_{kl}\,. \tag{5.6d}$$

In the remainder of this section we concentrate on the massless modes, which are constant on T^6. The massive modes will be discussed in the next section. The conversion to SU(4) notation, indicated above for the spinors, can be made uniform by redefining the scalars (which correspond to the gauge field components in the torus directions) according to

$$\phi_{ij} = \frac{1}{2} i \big(\gamma^m\big)_{ij} \, A_m \, , \qquad A_m = \frac{1}{2} i \big(\gamma_m\big)_{ij} \, \phi^{ij} = -\frac{1}{2} i \big(\gamma_m\big)^{ij} \, \phi_{ij} \, .$$

We note that these redefinitions involve the torus vielbein. In view of (5.6) we have

$$\phi^{ij} \equiv \big(\phi_{ij}\big)^* = \frac{1}{2} \varepsilon^{ijkl} \, \phi_{kl} \, .$$

The dimensionally reduced SU(4) invariant action is then given by

$$\begin{aligned} S_4^{(0)} = \frac{1}{g_4^2} \int \mathrm{d}^4 x \; \mathrm{tr}_{\mathrm{R}} \bigg(& \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} D_\mu \phi_{ij} \, D^\mu \phi^{ij} + \frac{1}{4} \big[\phi_{ij} , \phi_{kl}\big]\big[\phi^{ij} , \phi^{kl}\big] \\ & + \frac{1}{2} \bar\psi^i \overset{\leftrightarrow}{\not{D}} \, \psi_i - \bar\psi_i \left[\phi^{ij} , \psi_j\right] 1 - \bar\psi^i \left[\phi_{ij} , \psi^j\right] \bigg) \, , \end{aligned} \tag{5.7}$$

where $F_{\mu\nu}$ is the four-dimensional field-strength and the covariant derivate D_μ contains only the four-dimensional gauge field A_μ, e.g.

$$D_\mu \phi^{ij} = \partial_\mu \phi^{ij} - \left[A_\mu \, , \phi^{ij}\right] .$$

The coupling constants are related by $g_{10}^2 = g_4^2 \, (2\pi R)^6 \sqrt{g}$, where $\sqrt{g}$ is the determinant of the vielbein on the six-torus. The action (5.3) contributes exclusively to the massless modes and leads precisely to the topological invariant $\int \mathrm{d}^4 x \; \varepsilon^{\mu\nu\rho\sigma} \, F_{\mu\nu} F_{\rho\sigma}$, with a corresponding theta parameter

$$\theta \propto \int A_{mnpqrs} \, \mathrm{d}y^m \wedge \cdots \wedge \mathrm{d}y^s \, .$$

The action (5.7) is invariant under the rigid supersymmetry transformations,

$$\begin{aligned} \delta A_\mu &= \bar\epsilon^i \gamma_\mu \psi_i + \bar\epsilon_i \gamma_\mu \psi^i \, , \\ \delta\phi^{ij} &= \bar\epsilon^i \psi^j - \bar\epsilon^j \psi^i + \varepsilon^{ijkl} \, \bar\epsilon_k \psi_l \, , \\ \delta\psi_i &= -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon_i - 2 \not{D} \phi_{ij} \, \epsilon^j - 2 \left[\phi_{ij} , \phi^{jk}\right] \epsilon_k \, , \\ \delta\psi^i &= -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon^i - 2 \not{D} \phi^{ij} \, \epsilon_j - 2 \left[\phi^{ij} , \phi_{jk}\right] \epsilon^k \, . \end{aligned}$$

Modulo the fermionic field equations, these transformations close upon anticommutation. On the bosons the supersymmetry commutator yields

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] A_\mu = \xi^\nu \, F_{\nu\mu} + D_\mu \big(\xi^{ij} \phi_{ij} + \xi_{ij} \phi^{ij}\big) \, , \tag{5.8a}$$

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] \phi^{ij} = \xi^\mu \, D_\mu \phi^{ij} + \big[\xi^{kl} \phi_{kl} + \xi_{kl} \phi^{kl} , \, \phi^{ij}\big] \, , \tag{5.8b}$$

where $\xi_\mu = 2 \left(\bar\epsilon_2^i \gamma_\mu \epsilon_{1\,i} + \bar\epsilon_{2\,i} \gamma_\mu \epsilon_1^i\right)$ and $\xi^{ij} = -4 \, \bar\epsilon_2^{[i} \epsilon_1^{j]}$. Here, we recognize a (covariant) translation and a (field-dependent) gauge transformation. Observe that the Lagrangian and the supersymmetry transformation rules are all consistent with SU(4).

2. Yang-Mills theory coupled to a BPS multiplet

When compactifying on T^6 to four space-time dimensions, one obtains a massless Yang-Mills theory coupled to an infinite tower of Kaluza-Klein states, which comprise BPS supermultiplets. They carry charges associated with the reciprocal lattice of T^6 and transform according to the adjoint representation of the gauge group. From the supersymmetry algebra (5.2) in ten space-time dimensions, one observes that the central charge is associated with translations in the torus coordinates and thus proportional to $\hat{\Gamma}^m \partial_m$. This leads to a central charge $-i[\xi^{ij}\, \mathcal{Z}_{ij}(q) + \xi_{ij}\, \mathcal{Z}^{ij}(q)]$ with

$$\mathcal{Z}_{ij}(q) = \frac{i q_m}{2R}\,(\gamma^m)_{ij}\ . \tag{5.9}$$

This charge determines the mass and part of the couplings of the BPS states. Observe that it depends on the torus moduli and is self-dual; it satisfies

$$\mathcal{Z}_{ik}(q)\,\mathcal{Z}^{kj}(q) = -\frac{1}{4}\frac{q^2}{R^2}\delta_i^{\,j}\ .$$

In principle, the central charges may consist of self-dual and anti self-dual components. We will denote the first one as 'electric' for reasons to be explained below. The 'magnetic' central charge is anti-self-dual. The presence of the purely 'electric' central charge breaks the SU(4) automorphism group of the supersymmetry algebra to USp(4) $\sim$ SO(5). This is the generic automorphism group for massive $\mathcal{N} = 4$ BPS supermultiplets in four dimensions. A massive $\mathcal{N} = 4$ multiplet of lowest spin contains the states corresponding to one spin-1, four spin-1/2 and five spin-0 particles. The central charge, here induced by a shift of the torus coordinates, acts by a phase transformation, so that all the massive states are doubly degenerate.

When following an alternative approach and generating the BPS multiplets by a spontaneous breaking of the gauge group directly in four space-time dimensions, the central charges $\mathcal{Z}_{ij}$ are generated by the vacuum-expectation values of the fields ϕ_{ij} which take their values in the Cartan subalgebra. In general the residual gauge group is non-abelian, and the massive fields transform in some representation of the residual gauge group; the precise representation depends on the original gauge group and the residual gauge group. Let us now discuss the two approaches in some detail.

2.1. Kaluza-Klein compactification

We first study the Kaluza-Klein approach and include the nontrivial Fourier modes on T^6. Because we intend to only retain terms that are quadratic in these modes, we restrict ourselves to the ones with torus momentum $\pm\boldsymbol{q}$. Hence we write for the ten-dimensional fields,

$$
\begin{aligned}
A_\mu(x,y) &= A_\mu(x) + B_\mu(x)\,\mathrm{e}^{iq_m y^m/R} + \bar{B}_\mu(x)\,\mathrm{e}^{-iq_m y^m/R}\,, \\
\frac{1}{2} i(\gamma^m)_{ij}\, A_m(x,y) &= \phi_{ij}(x) + B_{ij}(x)\,\mathrm{e}^{iq_m y^m/R} + \frac{1}{2}\varepsilon_{ijkl}\, B^{kl}(x)\,\mathrm{e}^{-iq_m y^m/R}\,, \\
\psi^i(x,y) &= \psi^i(x) + \chi^i(x)\,\mathrm{e}^{iq_m y^m/R} + C^{-1}\,\bar{\chi}^{i\mathrm{T}}(x)\,\mathrm{e}^{-iq_m y^m/R}\,, \\
\bar{\psi}^i(x,y) &= \bar{\psi}^i(x) - \chi^{i\mathrm{T}}(x)\,C\,\mathrm{e}^{iq_m y^m/R} + \bar{\chi}^i(x)\,\mathrm{e}^{-iq_m y^m/R}\,,
\end{aligned}
$$

where $\bar{\chi}^i = i(\chi_i)^\dagger\gamma^0$ and C denotes the charge conjugation matrix in four dimensions. Observe that complex conjugation does not act on the Lie-algebra generators. The counting of states presented earlier is in agreement with this field representation, as we have a complex spin-1 field B_μ, six complex spin-0 fields B_{ij} and four Dirac spinors with chiral components χ^i and χ_i. However, a proper assessment of the degrees of freedom should take into account that the model is still invariant under gauge transformations which depend on both x^μ and y^m. The y-dependence mixes the various Fourier modes. To interpret the degrees of freedom in a four-dimensional context (and to be able, as we do above, to restrict ourselves consistently to a single Fourier mode) it is best to eliminate the y-dependent gauge transformations by an appropriate gauge condition, such as

$$
\mathcal{Z}_{ij}(q)\, B^{ij} = 0\,. \tag{5.10}
$$

This condition is in fact equivalent to the condition (4.13) that we have discussed in section 1 of chapter 4. Even though this gauge condition is natural, it is not the only one possible. In section 3.4 we will be calculating induced higher order corrections to the Lagrangian in a background where ϕ_{ij} is covariantly constant and where the gauge field-strengths are abelian and constant. There, we will impose a slightly different gauge condition which is more convenient in that case.

In the context of a spontaneously broken gauge theory the condition (5.10) is identical to the one that is imposed in the unitary gauge. Upon this condition the vector fields B_μ are Proca fields which describe massive spin-1 states, whereas the B_{ij} subject to (5.10) now correspond to five complex scalar fields. All fields transform in the adjoint representation of the gauge group, which is y-independent. Hence the residual gauge invariance will be associated with the massless fields of the $\mathcal{N}=4$ Yang-Mills theory.

An immediate consequence of the gauge condition (5.10) is that the supersymmetry transformations for the BPS fields have to be modified by extra field-dependent gauge transformations so as to preserve the gauge condition. To exhibit this in detail, let us list the supersymmetry transformations for the massive fields prior to gauge-fixing,

$$\delta B_\mu = \bar\epsilon^i \gamma_\mu \chi_i + \bar\epsilon_i \gamma_\mu \chi^i \,, \tag{5.11a}$$

$$\delta B_{ij} = \bar\epsilon_i \chi_j - \bar\epsilon_j \chi_i + \varepsilon_{ijkl}\, \bar\epsilon^k \chi^l \,, \tag{5.11b}$$

$$\begin{aligned} \delta \chi^i =& - D_\mu B_\nu\, \gamma^{\mu\nu} \epsilon^i + \Big(2i\mathcal{Z}^{ij} \not{B} - 2\big[\phi^{ij}, \not{B}\big] - \varepsilon^{ijkl} \not{D} B_{kl}\Big)\epsilon_j \\ &+ \Big(4i\mathcal{Z}^{ik} B_{kj} + i\,\delta^i_j \mathcal{Z}^{kl} B_{kl} - 4\big[\phi^{ik}, B_{kj}\big] - \delta^i_j\big[\phi^{kl}, B_{kl}\big]\Big)\epsilon^j \,. \end{aligned} \tag{5.11c}$$

Here we use the derivative D_μ which is covariant with respect to x-dependent gauge transformations, e.g. $D_\mu B_{ij} = \partial_\mu B_{ij} - [A_\mu, B_{ij}]$. In principle the supersymmetry transformation rules for the massive fields could also contain terms which are cubic in the massive fields. However, since we are only interested in terms in the Lagrangian that are quadratic in the massive fields, the cubic terms in the supersymmetry transformations are neglected.

The supersymmetry variation of the massless spinor field acquires terms quadratic in the massive fields,

$$\begin{aligned} \delta' \psi^i =& \big[B_\mu, \bar B_\nu\big]\gamma^{\mu\nu}\epsilon^i + \Big(4\big[B^{ij}, B_{jk}\big] + \delta^i_k\big[B^{lm}, B_{lm}\big]\Big)\epsilon^k \\ &- 2i\left(\big[B^{ij}, B_\mu\big] + \frac{1}{2}\varepsilon^{ijkl}\big[B_{kl}, \bar B_\mu\big]\right)\gamma^\mu \epsilon_j \,. \end{aligned}$$

The supersymmetry transformations of the massless bosonic fields are linear in the fermions, and as a result there can be no terms that are quadratic in the massive fields.

Under gauge transformations associated with Fourier modes of charge $\pm\boldsymbol{q}$, with complex Lie-algebra valued parameter Λ, we have

$$\begin{aligned} \delta B_\mu &= D_\mu \Lambda \,, \\ \delta B_{ij} &= i\mathcal{Z}_{ij}\Lambda - \big[\phi_{ij}, \Lambda\big] \,, \\ \delta \chi^i &= -\big[\psi^i, \Lambda\big] \,, \end{aligned}$$

for the massive fields, while the massless fields transform as,

$$\delta A_\mu = -\big[\bar B_\mu, \Lambda\big] - \big[B_\mu, \bar\Lambda\big] \,, \tag{5.12a}$$

$$\delta \phi_{ij} = -\frac{1}{2}\varepsilon_{ijkl}\big[B^{kl}, \Lambda\big] - \big[B_{ij}, \bar\Lambda\big] \,, \tag{5.12b}$$

$$\delta \psi^i = -C^{-1}\big[\bar\chi^{i\mathrm{T}}, \Lambda\big] - \big[\chi^i, \bar\Lambda\big] \,. \tag{5.12c}$$

At this point we will allow a slight generalization and assume that the massive fields (and the parameter Λ, introduced above) are no longer in the adjoint representation, but in an arbitrary representation of the gauge group, (the massless fields must remain in the adjoint representation). We then use a notation where the massive fields are no longer Lie-algebra valued but are written as row and column vectors. The Lie-algebra valued expressions associated with the massless fields are thus defined in the representation appropriate for the massive ones. This is the representation that we denoted by **R** in (5.1) and (5.7). Note that the choice for the representation affects the definition of the gauge coupling constant. The above extension to arbitrary representations for the massive fields is such that our results will also be directly applicable to the case of spontaneously broken four-dimensional supersymmetric gauge theories.

With this change of notation the commutators in (5.11) change according to, e.g. $[\phi, B] \to \phi\, B$, where on the right-hand side, ϕ refers to the massless Lie-algebra valued expression and B to the column vector of massive fields. Similar changes occur in (5.12). With these changes the variation of the gauge condition (5.10) under the combined transformation reads

$$\delta\left(\mathcal{Z}^{ij} B_{ij}\right) = 2\,\mathcal{Z}^{ij}\,\bar{\epsilon}_i \chi_j + 2\,\mathcal{Z}_{ij}\,\bar{\epsilon}^i \chi^j + \left(i\,|\mathcal{Z}|^2 - \mathcal{Z}\cdot\phi\right) \Lambda\,.$$

Here $\mathcal{Z}\cdot\phi = \mathcal{Z}^{ij}\phi_{ij}$ is a Lie-algebra valued expression with real coefficients. To preserve the gauge condition (5.10), the supersymmetry transformation are thus accompanied by a compensating field-dependent gauge transformation with complex parameter

$$\Lambda = 2i\,\frac{|\mathcal{Z}|^2 - i\,\mathcal{Z}\cdot\phi}{|\mathcal{Z}|^4 + (\mathcal{Z}\cdot\phi)^2}\,\left(\mathcal{Z}^{ij}\,\bar{\epsilon}_i \chi_j + \mathcal{Z}_{ij}\,\bar{\epsilon}^i \chi^j\right)\,.$$

From this result one can easily write down the resulting supersymmetry transformations for the various fields. The transformation rules of the massive vector multiplets remain linear in the massive fields, whereas the transformation rules of the massless fields acquire corrections quadratic in the massive fields. All the resulting expressions take a rather complicated form. As an example we give the transformation for B_{ij},

$$\begin{aligned}\delta B_{ij} = {}& \left(\delta_{ij}^{mn}\,\mathcal{Z}^{kl} - \delta_{ij}^{kl}\,\mathcal{Z}^{mn}\right)\,(\mathcal{Z} + i\phi)_{kl}\;\frac{1}{|\mathcal{Z}|^2 + i\,\mathcal{Z}\cdot\phi} \\ & \times \left(2\bar{\epsilon}_m \chi_n + \varepsilon_{mnpq}\,\bar{\epsilon}^p \chi^q\right)\,.\end{aligned} \tag{5.13}$$

In the abelian case the result for the supersymmetry variations can be compared to corresponding expressions for massive vector supermultiplets [76] which were obtained in the context of $\mathcal{N} = 1$ supergravity in four dimensions.

A few remarks about the supersymmetry algebra are in order. Clearly, the supersymmetry transformations of the massive bosonic fields are nonpolynomial in the massless fields, i.e. equation (5.13) contains terms ϕ^n of any power n. The supersymmetry transformations of the fermionic fields, on the other hand, are polynomial if we discard terms of the form $\epsilon\chi^2$, as we have done throughout. It is therefore somewhat surprising that the action, which is polynomial and which we write down below, is nevertheless supersymmetric. Moreover, the commutator of two supersymmetry transformations on a bosonic field closes, and all the non-polynomial parts cancel. The supersymmetry algebra (5.11) in the unitary gauge (5.10) closes on the bosonic fields. On the massless fields, the algebra remains unchanged, i.e. it is still given by (5.8). The commutator of two supersymmetry transformations on the massive bosons are given by

$$\begin{aligned}
\big[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\big]B_\mu &= \xi^\nu\big(\mathcal{D}_\nu B_\mu-\mathcal{D}_\mu B_\nu\big)-\big[\xi^{ij}\phi_{ij}+\xi_{ij}\phi^{ij},B_\mu\big]\\
&\qquad -i\big(\xi_{ij}\mathcal{Z}^{ij}+\xi^{ij}\mathcal{Z}_{ij}\big)B_\mu\,,\\
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)]B_{ij} &= \xi^\mu\mathcal{D}_\mu B_{ij}+\big[\xi^{kl}\phi_{kl}+\xi_{kl}\phi^{kl},B_{ij}\big]\\
&\qquad -i\big(\xi_{kl}\mathcal{Z}^{kl}+\xi^{kl}\mathcal{Z}_{kl}\big)B_{ij}\,.
\end{aligned}$$

As in the massless case, the supersymmetry transformation rules contain a covariant translation and a field-dependent gauge transformation. The additional term proportional to $(\xi_{kl}\mathcal{Z}^{kl}+\xi^{kl}\mathcal{Z}_{kl})$ is the central charge term, and this term is of course proportional to the mass of the field since we are dealing with a BPS multiplet. The closure of the supersymmetry algebra on the bosonic fields suggests that it might be possible to redefine the fields in the massive multiplet in such a way that the Lagrangian remains polynomial, and that also the supersymmetry transformation rules become polynomial.

We now turn to the action. Here the change of notation implies that $\mathrm{tr}(\bar B[\phi,B])\;\to\;-\bar B\,\phi\,B$. The minus sign is extracted because the group-invariant metric induced by the trace is negative definite for a compact group. For convenience, we restrict ourselves to compact groups, such that $\bar B_\mu$ transforms in the conjugate representation of B_μ, and similarly for the scalars and the fermions. Note also that ϕ_{ij} is hermitian in the combined gauge indices and SU(4) indices, as is $i\mathcal{Z}_{ij}$ and $F_{\mu\nu}$. With this in mind we derive the Lagrangian (suppressing terms that vanish because of the condition (5.10)),

$$\begin{aligned}
C_{\mathrm{R}}^{-1}g_4^2\,\mathcal{L} = &-\tfrac{1}{2}\big|D_\mu B_\nu-D_\nu B_\mu\big|^2-\big|\mathcal{Z}\big|^2\big|B_\mu\big|^2-\big|D_\mu B_{ij}\big|^2-\big|\mathcal{Z}\big|^2\big|B_{ij}\big|^2\\
&-\bar B^\mu F(A)_{\mu\nu}B^\nu+\Big(B^{ij}\overset{\leftrightarrow}{D}_\mu\,\phi_{ij}\Big)B^\mu+\bar B^\mu\Big(\phi^{ij}\overset{\leftrightarrow}{D}_\mu\,B_{ij}\Big)\\
&-2i\,\mathcal{Z}_{ij}\,\big(\bar B_\mu\phi^{ij}B^\mu+B^{kl}\phi^{ij}B_{kl}\big) \qquad\qquad (5.14)\\
&+B^{ij}\Big(\phi^{kl}\phi_{kl}B_{ij}+\big(\phi_{ij}\phi^{kl}-2\phi^{kl}\phi_{ij}\big)B_{kl}\Big)+\bar B^\mu\,\phi^{ij}\phi_{ij}\,B_\mu\\
&-\bar\chi^i\not{D}\chi_i-\bar\chi_i\not{D}\chi^i-2\bar\chi^i\big(i\mathcal{Z}_{ij}-\phi_{ij}\big)\chi^j-2\bar\chi_i\big(i\mathcal{Z}^{ij}-\phi^{ij}\big)\chi_j\\
&-\big(\bar\chi^i\gamma^\mu\psi_i+\bar\chi_i\gamma^\mu\psi^i\big)\,B_\mu-\bar B_\mu\big(\bar\psi^i\gamma^\mu\chi_i+\bar\psi_i\gamma^\mu\chi^i\big)
\end{aligned}$$

$$- 2\bar{\chi}^i \psi^j \, B_{ij} - 2B^{ij} \, \bar{\psi}_i \chi_j - \varepsilon_{ijkl} \, B^{kl} \, \bar{\psi}^i \chi^j - \varepsilon^{ijkl} \, \bar{\chi}_i \psi_j \, B_{kl} \,,$$

where C_R is a group theoretical factor proportional to the second-order Casimir operator of the gauge group representation. The Lagrangian is manifestly invariant under the USp(4) subgroup of SU(4) and under the U(1) associated with the central charge, under which B_μ, B_{ij}, χ^i and χ_i transform with equal strength.

The Lagrangian (5.14), which is quadratic in the massive fields and does not contain four-fermi interactions, is at this level invariant under supersymmetry transformations.

2.2. Spontaneous symmetry breaking

Let us now compare the Lagrangian (5.14) with the Lagrangian one obtains for the four-dimensional theory based on (5.7), in a spontaneously broken realization of the gauge group induced by nonzero vacuum-expectation values of the scalar fields ϕ_{ij}. The potential for the scalar fields is given by

$$V(\phi) = -\frac{1}{4} \,\mathrm{tr} \big[\phi_{ij}\,, \phi_{kl}\big]\big[\phi^{ij}\,, \phi^{kl}\big] \,,$$

and it clearly vanishes when ϕ_{ij} takes values in the Cartan subalgebra associated with an abelian subgroup of the gauge group G. The vacuum expectation value of the scalar fields is denoted by $\mathcal{Z}_{ij}$. In the vacuum, the gauge symmetry group G is broken to a subgroup of G, which we denote by H. Maximal symmetry breaking occurs when the residual symmetry group H coincides with the Cartan subgroup, but in general H need not be abelian.

In order to make the discussion more concrete, we consider the case where the original gauge group G is SU$(N+1)$, and the vacuum expectation values of the scalars are chosen such that the residual gauge symmetry group H is SU$(N) \times$ U(1). We decompose the $N^2 + 2N$ anti-hermitian generators $\hat{\tau}^A$ of SU$(N+1)$ in the fundamental representation accordingly,

$$\hat{\tau}^a = \begin{pmatrix} \tau^a & 0 \\ 0 & 0 \end{pmatrix} , \qquad \hat{\tau}^0 = \frac{i}{\sqrt{2N(N+1)}} \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -N \end{pmatrix} , \tag{5.15a}$$

$$\hat{\tau}^+ = \begin{pmatrix} 0 & \tau^+ \\ 0 & 0 \end{pmatrix} , \qquad \hat{\tau}^- = \begin{pmatrix} 0 & 0 \\ \tau^- & 0 \end{pmatrix} . \tag{5.15b}$$

Here, τ^a are the $N^2 - 1$ generators of the residual SU(N) in the fundamental representation, $\hat{\tau}^0$ is the generator of the residual U(1), and $\hat{\tau}^\pm$ are the $2N$ generators of the broken symmetry. The generators are normalized such that $\mathrm{tr}\,(\hat{\tau}^A \hat{\tau}^B) = -\delta^{AB}/2$. In this basis, the vacuum expectation value $\mathcal{Z}_{ij}$ of the scalar fields is proportional to the generator $\hat{\tau}^0$. The residual gauge symmetry is SU$(N) \times$ U(1), since the generators $\hat{\tau}^a$ commute with $\hat{\tau}^0$, and obviously $\hat{\tau}^0$ commutes with itself.

To exhibit the effect of spontaneous symmetry breaking, we decompose the fields in terms of the generators defined in (5.15),

$$\hat{A}_\mu = \sqrt{\frac{2N}{N+1}} A^0_\mu \hat{\tau}^0 + \begin{pmatrix} A_\mu & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_\mu \\ \bar{B}_\mu & 0 \end{pmatrix} , \tag{5.16a}$$

$$\hat{\phi}_{ij} = \sqrt{\frac{2N}{N+1}} \left(\phi^0_{ij} + \mathcal{Z}_{ij} \right) \hat{\tau}^0 + \begin{pmatrix} \phi_{ij} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B_{ij} \\ \frac{1}{2}\varepsilon_{ijkl} B^{kl} & 0 \end{pmatrix} , \tag{5.16b}$$

$$\hat{\psi}^i = \sqrt{\frac{2N}{N+1}} \psi^i \hat{\tau}^0 + \begin{pmatrix} \psi^i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \chi^i \\ C^{-1} \bar{\chi}^{i\mathrm{T}} & 0 \end{pmatrix} , \tag{5.16c}$$

$$\hat{\bar{\psi}}^i = \sqrt{\frac{2N}{N+1}} \bar{\psi}^i \hat{\tau}^0 + \begin{pmatrix} \bar{\psi}^i & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\chi^{i\mathrm{T}} C \\ \bar{\chi}^i(x) & 0 \end{pmatrix} . \tag{5.16d}$$

where the hatted fields are the fields in the unbroken phase. The massless fields $\left(A^0_\mu, A_\mu\right)$, $\left(\phi^0_{ij}, \phi_{ij}\right)$ and $\left(\psi^{i0}, \psi^i\right)$ form a vector supermultiplet. The scalars and the fermions transform in the adjoint representation of the residual gauge group H, and $\left(A^0_\mu, A_\mu\right)$ are the gauge fields in the broken phase. The fields B_μ, B_{ij} and χ_i are massive and they transform in the fundamental representation of H. Their mass squared is equal to $|\mathcal{Z}|^2$. The covariant derivative of a massive field is given by e.g.

$$D_\mu B_{ij} = \partial_\mu B_{ij} - A_\mu B_{ij} - i A^0_\mu B_{ij} ,$$

Note that the U(1) charge of the massive fields is unity, which, as we have explained above, is due to the normalization that we have chosen in the decomposition (5.16). The Lagrangian one obtains in the broken phase is very similar to the one obtained in the Kaluza-Klein approach, Eq. (5.14). The Lagrangian consists of three properly normalized pieces,

$$g^2 \mathcal{L} = \frac{N+1}{2N} g^2 \mathcal{L}_{\mathrm{U}(1)} + g^2 \mathcal{L}_{\mathrm{SU}(N)} + g^2 \mathcal{L}_{\mathrm{BPS}} .$$

The first two pieces describe the purely massless theory, and since the U(1) generators commute with the SU(N) generators, the massless U(1) fields decouple from the massless SU(N) fields. Both $\mathcal{L}_{\mathrm{U}(1)}$ and $\mathcal{L}_{\mathrm{SU}(N)}$ are still of the form (5.7), but coupling constant of the U(1) piece has been multiplied by an N-dependent factor due to the normalization of the U(1) fields in the decomposition (5.16). Except for the gauge group, the Lagrangian for the massive fields is exactly the same as in the Kaluza-Klein approach, and it is given by (5.14).

We again allow the massive fields to transform in an arbitrary representation **R** of SU(N), very similar to the Kaluza-Klein case. This implies that the SU(N) generators τ^a are now in the representation **R** and the massive fields are column vectors upon which the representation matrices act.

The unitary gauge $Z_{ij}B^{ij} = 0$ reduces the number of scalars to the five physical ones. The massive fields then form a (complex) BPS multiplet. Similar to the Kaluza-Klein approach, one has to add a uniform compensating gauge transformation to the supersymmetry transformations of the various fields in order to remain in the unitary gauge.

As opposed to the Kaluza-Klein approach where we obtained an infinite number of BPS multiplets whose charges are on an $\mathrm{SL}(6,\mathbb{Z})$-invariant lattice, there is only a finite number of BPS multiplets in the spontaneously broken theory and there is no $\mathrm{SL}(6,\mathbb{Z})$ symmetry that acts on the multiplet charges.

3. Integrating out BPS fields

In this section we study the effect of integrating out the massive fields in order to obtain an effective action for massless fields in four space-time dimensions. Some of the results that we obtain are by themselves not new, but they have not been obtained in a field-theoretic context. We restrict ourselves to the semi-classical approximation (i.e. one closed loop), so that we only need terms in the action that are quadratic in the massive fields. These corrections have to be added to the ones originating from the pure gauge theory, many of which have appeared in the literature. For instance, it has been established explicitly that the beta-function vanishes to three loops [77] (and it is expected to vanish to all orders), so that the theory is superconformally invariant as a quantum theory [78]. Also contributions containing terms of higher powers of the field-strengths have been determined, such as the terms proportional to $F^4/|\phi^{ij}|^4$ and their supersymmetric completion, which receive perturbative contributions at one-loop, and are completely known [72, 79, 80]. We should stress that all the higher order contributions from the pure gauge theory are scale invariant, unlike the contributions originating from the BPS supermultiplets.

The evaluation of the one-loop diagrams is rather subtle. In the above discussion we were forced to adopt a unitary gauge, so that the contributions from massive vector fields are generically more divergent than the behavior expected on the basis of a renormalizable field theory. Calculations in the unitary gauge are notoriously difficult for that reason. We have arguments why such divergencies should be softer or should even disappear. For instance, consider the situation where the BPS multiplets arise from a spontaneously broken non-abelian gauge theory. Because the original theory is renormalizable, we know that in this case one BPS multiplet coupled to a non-abelian gauge theory constitutes a renormalizable theory. We expect that the analogous theory obtained in the Kaluza-Klein approach is also renormalizable. The difference between these two cases lies only in the representation of the gauge group, but the calculations do not depend in a crucial way on the actual representations. The fact that ten-dimensional supersymmetric Yang-Mills theory is not renormalizable implies that the non-renormalizable divergencies arise in the coupling of infinitely many BPS multiplets to the vector multiplet in four dimensions. However, as has been pointed out in [81], the relation between the (one-loop) quantum properties of $\mathcal{N}$ supersymmetric Yang-Mills theory in four dimensions and the $\mathcal{N}=1$ supersymmetric Yang-Mills theory in ten dimensions is not straightforward, and needs to be studied with caution.

The divergencies that one encounters in the unitary gauge are partly gauge artefacts and this aspect forces us to restrict ourselves to the calculation of gauge-invariant (i.e. on-shell) quantities. In terms of the corresponding effective action one encounters similar subtleties because there may be terms that vanish when applying the classical field equations on the massless fields. These should be regarded as off-shell contributions. One expects that those contributions are not of physical relevance, but in any case, they cannot be calculated reliably.

Another difficulty is related to the fact that there exists no off-shell formulation for $\mathcal{N}=4$ gauge theories. This has two important implications. First of all, it is not easy to determine the restrictions implied by supersymmetry on the one-loop effective actions. Secondly, the one-loop calculations will not be manifestly supersymmetric and may require additional finite subtractions. Assuming that the underlying theory is renormalizable, those subtractions will only be of the renormalizable type. This means that the counterterms will not exhibit supersymmetry, just as one expects when performing calculations in the Wess-Zumino gauge [82]. Here it is important that our calculations are at least gauge invariant. At one loop the pattern of subtractions is not too complicated.

Assuming that the theory is renormalizable, the terms in the effective action that are of the non-renormalizable type should be free of ultraviolet divergences. But as explained above, this only holds for terms that are on shell. Obvious contributions to consider are the ones of dimension six and eight, which are expected to contain terms cubic and quartic in the field-strengths. The contributions of dimension four should be proportional to the original Lagrangian and the proportionality coefficient corresponds to the coupling constant renormalization. However, to recover the original supersymmetric Lagrangian one must include finite subtractions in order to ensure supersymmetry.

We first turn to the evaluation of the one-loop contributions of the BPS states to the classical Lagrangian (5.7). The renormalization of the coupling constant is most easily extracted from the two-point function of the gauge fields, which means that we determine the one-loop contributions proportional to $\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})$. Because of the gauge invariance, the vertices are rather restricted and the diagrams are the usual vacuum-polarization diagrams caused by massive scalar, spinor and vector loops. The scalar loops and spinor loops are standard diagrams of the renormalizable type and involve only a logarithmic divergence. The vector loops are not of the renormalizable type. This is reflected in the fact that these vacuum polarization graphs have infinities to any order in the external momentum. One can verify that this has to be a gauge artefact, for instance, by comparing to similar calculations in a spontaneously broken gauge theory in a continuous variety of renormalizable gauges. In the limiting case where these gauges tend to the unitary gauge, one encounters these divergences. Naturally, physical quantities are protected against such spurious infinities, and therefore we restrict most of our calculations to on-shell amplitudes.

To ensure gauge invariance we use dimensional regularization, i.e. we use symmetric integration, drop terms in the momentum integrals that are polynomials in the momenta or that are total divergences, and allow shifts of the integration variables. It is not necessary to evaluate the resulting momentum integrals. Because straightforward dimensional regularization is not consistent with supersymmetry, we proceed with caution and evaluate the diagrams in n space-time dimensions, keeping the number of spinor components equal to n_s, without assigning specific values to n and n_s until the end.

In order to calculate the Feynman diagrams, we need the various propagators of the massive bosonic fields,

$$\begin{aligned}\langle B_{ij}\, B^{kl}\rangle &= \left(\delta_{ij}^{kl} - \frac{\mathcal{Z}_{ij}\,\mathcal{Z}^{kl}}{|\mathcal{Z}|^2}\right)\frac{1}{p^2+|\mathcal{Z}|^2}\,,\\ \langle B_\mu\, \bar{B}_\nu\rangle &= \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{|\mathcal{Z}|^2}\right)\frac{1}{p^2+|\mathcal{Z}|^2}\,.\end{aligned}$$

Note that the propagator for the scalar fields contains a projection on the physical states, and the propagator for the massive vector fields is given in the unitary gauge.

As for the massive fermions, the propagators are given by

$$\begin{aligned}\langle \chi^i\, \bar{\chi}_j\rangle &= \frac{-i\,\not{p}\delta^i_j}{p^2+|\mathcal{Z}|^2}\,\frac{1-\gamma_5}{2}\,,\\ \langle \chi_j\, \bar{\chi}^i\rangle &= \frac{-i\,\not{p}\delta^i_j}{p^2+|\mathcal{Z}|^2}\,\frac{1+\gamma_5}{2}\,,\\ \langle \chi^i\, \bar{\chi}^j\rangle &= \frac{2i\,\mathcal{Z}^{ij}}{p^2+|\mathcal{Z}|^2}\,\frac{1+\gamma_5}{2}\,,\\ \langle \chi_i\, \bar{\chi}_j\rangle &= \frac{2i\,\mathcal{Z}_{ij}}{p^2+|\mathcal{Z}|^2}\,\frac{1-\gamma_5}{2}\,.\end{aligned}$$

We have suppressed the overall factor $C_{\mathrm{R}}^{-1}\, g_4^2$, because it does not contribute to the one-loop graphs.

3.1. Calculation of the β-function

We now turn to the explicit results contributing to $\mathrm{tr}\,(F_{\mu\nu}^2)$. The relevant diagrams are listed in figure 6 on the next page. First there are the self-energy

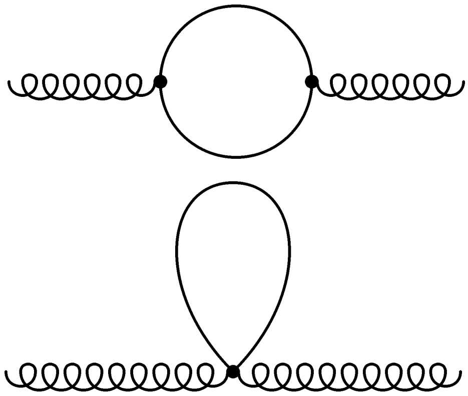

Figure 6. The self-energy diagrams of the vector fields. The upper diagram appears for massive vectors, fermions and scalars in the loop, whereas the lower diagram occurs only for massive vectors and scalars.

diagrams induced by the minimal coupling to the charged vector bosons. They yield

$$-\operatorname{tr}\left(F_{\mu\nu}^2\right)\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(\frac{1}{4}\frac{1}{(p^2+|\mathcal{Z}|^2)^2}+\frac{2}{n}\frac{|\mathcal{Z}|^2\,p^2}{(p^2+|\mathcal{Z}|^2)^4}\right).$$

Note that this expression does not depend on the coupling constant since it is a one-loop contribution. To the above contributions one has to add the contributions from the magnetic moment coupling $\bar{B}^\mu F_{\mu\nu}(A)B^\nu$, that appears explicitly in the Lagrangian,

$$-\frac{3}{2}\operatorname{tr}\left(F_{\mu\nu}^2\right)\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\,\frac{1}{(p^2+|\mathcal{Z}|^2)^2}.$$

Combining the two previous expressions shows that the massive spin-1 fields give rise to

$$\operatorname{tr}\left(F_{\mu\nu}^2\right)\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(-\frac{7}{4}\frac{1}{(p^2+|\mathcal{Z}|^2)^2}-\frac{2}{n}\frac{|\mathcal{Z}|^2\,p^2}{(p^2+|\mathcal{Z}|^2)^4}\right).$$

A similar calculation for the five complex scalars and the four Dirac fermions yields, respectively,

$$5\operatorname{tr}\left(F_{\mu\nu}^2\right)\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(\frac{1}{4(n-1)}\frac{1}{(p^2+|\mathcal{Z}|^2)^2}-\frac{2}{n(n-1)}\frac{|\mathcal{Z}|^2\,p^2}{(p^2+|\mathcal{Z}|^2)^4}\right),$$

$$4\operatorname{tr}\left(F_{\mu\nu}^2\right)\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(\frac{n_s(n-2)}{8(n-1)}\frac{1}{(p^2+|\mathcal{Z}|^2)^2}+\frac{n_s}{n(n-1)}\frac{|\mathcal{Z}|^2\,p^2}{(p^2+|\mathcal{Z}|^2)^4}\right).$$

The sum of these contributions is equal to zero, provided one sets $n = n_s = 4$. This implies that the BPS states do not induce any (i.e. neither finite nor infinite) perturbative renormalization of the coupling constant g_4. To the best of our knowledge, this is not implied by any known non-renormalization theorem. One can invoke $\mathcal{N} = 2$ arguments here and observe that the perturbative contributions to the coupling from ghost fields in harmonic superspace are precisely opposite to the contribution from a hypermultiplet, while the vector multiplet itself does not contribute. Since the $\mathcal{N} = 4$ vector multiplet comprises precisely one $\mathcal{N} = 2$ vector and one hypermultiplet, it follows that there should be an exact cancellation.[2]

In order to calculate the β-function, we regularize the divergent integrals encountered above using dimensional regularization. For $n = 4 + \varepsilon$ we obtain in the limit $\varepsilon \to 0$, cf. [83],

$$\int \frac{\mathrm{d}^n p}{i(2\pi)^n} \frac{1}{(p^2 + |\mathcal{Z}|^2)^2} \sim -\frac{1}{8\pi^2} \frac{\mu^\varepsilon}{\varepsilon} .$$

The sum of the divergent one-loop contributions is then given by

$$-\mathrm{tr}\left(F_{\mu\nu}^2\right) \frac{1}{16\pi^2} \frac{\mu^\varepsilon}{\varepsilon} \left(-\frac{7}{2} n_1 + \frac{2}{3} n_{1/2} + \frac{1}{6} n_0\right) \tag{5.17}$$

where we have kept the number of massive complex spin-1, spin-1/2 and spin-0 fields arbitrary and equal to n_1, $n_{1/2}$ and n_0, respectively. The bare coupling g_B is defined in terms of the running coupling g_4 and a counterterm to absorb the one-loop infinities. We obtain the relation

$$\frac{1}{g_\mathrm{B}^2} = \frac{1}{g_4^2} + \frac{1}{4\pi^2} \frac{\mu^\varepsilon}{\varepsilon} \left(-\frac{7}{2} n_1 + \frac{2}{3} n_{1/2} + \frac{1}{6} n_0\right) .$$

The β-function is then given by the expression

$$\mu \frac{\partial}{\partial \mu} g_4^{-2}(\mu) = -2 \frac{\beta(g_4)}{g_4^3} = -\frac{1}{4\pi^2} \left(-\frac{7}{2} n_1 + \frac{2}{3} n_{1/2} + \frac{1}{6} n_0\right) . \tag{5.18}$$

As we have already mentioned, the contributions of the one-loop β-function vanish identically for the BPS multiplets of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, where the field content is one massive vector field, four massive fermions and five massive scalars.

This result can be compared with the one-loop beta function originating from a non-abelian gauge theory with n_1 gauge fields, $n_{1/2}$ Majorana fermions and n_0 real scalars, all transforming in the adjoint representation of the gauge group. The β-function for the purely massless theory can be computed in a number of ways, most easily using the background field method [84, 85]. The

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[2] We thank I. Buchbinder for explaining this to us.

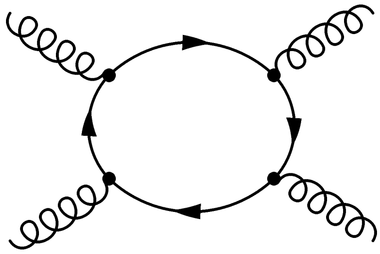


Figure 7. The diagram with four external vector fields A_μ and a massive fermion χ^i in the loop.

value of the β-function is given by

$$\mu\frac{\partial}{\partial\mu}g_4^{-2}(\mu) = -2\,\frac{\beta(g_4)}{g_4^3} = -\frac{1}{8\pi^2}\left(-\frac{11}{3}n_1 + \frac{2}{3}n_{1/2} + \frac{1}{6}n_0\right) . \qquad (5.19)$$

For every gauge group generator that is spontaneously broken a scalar disappears, and its contribution will be contained in the contribution of the massive vector field. Therefore, for a massive vector field the factor $-11/3$ changes into $-11/3 + 1/6 = -7/2$, which is indeed the factor found in (5.18). Observe that the overall factor 2 difference between (5.19) and (5.18) is due to the fact that the massive fields are complex.

Strictly speaking, the self-energy diagrams that we have calculated above only give rise to the abelian part of $\mathrm{tr}\,(F_{\mu\nu}^2)$ in the Lagrangian. Because of gauge invariance, this is sufficient. Nevertheless, let us now verify the calculations by considering diagrams with four external gauge fields A_μ which carry no external momentum. Diagrams of this sort give rise to $\mathrm{tr}\,[A_\mu, A_\nu]^2$ terms in the Lagrangian, which are the ones that differentiate the abelian theory from the non-abelian theory. Again, as in the abelian case above, the gauge fields do not receive a multiplicative renormalization (because of gauge invariance), and we can directly access the gauge coupling renormalization through this four-point function. We calculate all diagrams with no external moments, and use dimensional regularization to extract the infinities. Let us first have a look at the diagrams with massive fermions in the loop. There is only one type, schematically shown in figure 7. There are six diagrams of this type, corresponding to different attachments of the external lines, and sum of them yields

$$n_{1/2}\left[A_\mu, A_\nu\right]^2 \int \frac{\mathrm{d}^n p}{i(2\pi)^n}\,\frac{n_s(n-2)}{8(n-1)}\,\frac{1}{(p^2+\mathcal{Z}^2)^2}\,, \qquad (5.20)$$

There are three types of diagrams with massive scalars in the loop, shown in figure 8 on the next page. Adding them up and making sure that all the

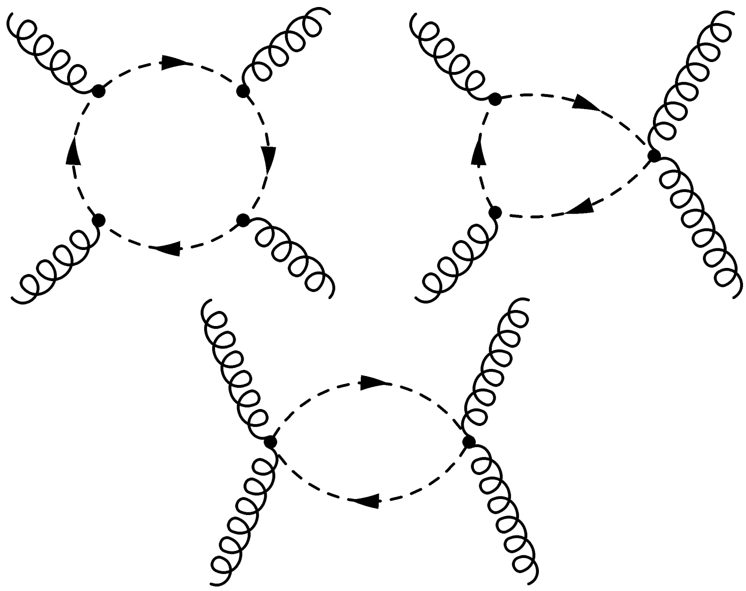


Figure 8. The three diagrams with four external vector fields A_μ and with massive scalar fields B_{ij} in the loop. Similar diagrams also exist with a vector field in the loop.

different attachments of the external lines are included, we obtain

$$n_0 \left[A_\mu, A_\nu\right] \int \frac{\mathrm{d}^n p}{i(2\pi)^n} \frac{1}{4(n-1)} \frac{1}{(p^2 + \mathcal{Z}^2)^2} \,.$$

Finally, the massive vectors in the loop contribute through four kinds of diagrams. Summing these diagrams with all inequivalent attachments of the external lines yields

$$-n_1 \left[A_\mu, A_\nu\right] \int \frac{\mathrm{d}^n p}{i(2\pi)^n} \frac{7}{4} \frac{1}{(p^2 + \mathcal{Z}^2)^2} \,.$$

In four dimensions, i.e. for $n = n_s = 4$, the divergent part of the sum of the vector, fermion and scalar contributions is given by

$$-\mathrm{tr}\left[A_\mu, A_\nu\right]^2 \frac{1}{16\pi^2} \frac{\mu^\varepsilon}{\varepsilon} \left(-\frac{7}{2} n_1 + \frac{2}{3} n_{1/2} + \frac{1}{6} n_0 \right) \tag{5.21}$$

This result needs to be compared with the result (5.17) that we obtained by calculating the two-point function. As expected, the results coincide. This confirms that for the BPS multiplets of the $\mathcal{N} = 4$ theory, i.e. for the $n_1 = 1$, $n_{1/2} = 4$ and $n_0 = 5$, the β-function is zero, and the gauge coupling constant is not renormalized. Let us note that throughout the calculations, we have neglected finite integrals. Since the finite contributions vanished in the calculation of the two-point function above, we believe that they do not contribute to the four-point function either.

The fact that there is no renormalization of the coupling constant does not imply that there are no other quantum corrections to the dimension-four Lagrangian. This is a consequence of the fact that we are not dealing with an off-shell formulation, so that fields belonging to the same supermultiplet can acquire different renormalizations. The discussion below will demonstrate these features.

3.2. Renormalization of the fermions and scalars

While the masslessness of the gauge fields is ensured by gauge invariance, the fact that the other fields remain massless at the one-loop order is less obvious. It turns out that the fermions remain massless, simply because the only possible mass term must be proportional to $\mathcal{Z}_{ij}\,\bar\psi^i\psi^j$ and its hermitian conjugate. However, this term vanishes because the fermions ψ^i are Majorana fields. Of course, they should also vanish by supersymmetry and chiral invariance, but the preservation of these symmetries is less obvious. In any case, there is no induced fermion mass term, but there is a multiplicative field renormalization, as reflected in the following one-loop correction,

$$\mathcal{L} = \mathrm{tr}\left(\bar\psi^i \overleftrightarrow{\not\partial}\, \psi_i\right) \int \frac{\mathrm{d}^n p}{i(2\pi)^n}\, \frac{n^2-2n-4}{2(n-2)}\, \frac{1}{(p^2+|\mathcal{Z}|^2)^2}\,. \tag{5.22}$$

The masslessness of the scalars is less obvious. An explicit calculation for the induced terms quadratic in the scalars with at most two derivatives, leads to

$$\begin{aligned}\mathcal{L} = {}& \mathrm{tr}\left|\partial_\mu\phi^{ij}\right|^2 \int \frac{\mathrm{d}^n p}{i(2\pi)^n} \left(n_s - 2\,\frac{2n-3}{n-2}\right) \frac{1}{(p^2+|\mathcal{Z}|^2)^2}\,, \\ & + \mathrm{tr}\left|\phi^{ij}\right|^2 \int \frac{\mathrm{d}^n p}{i(2\pi)^n}\, \frac{4+n-2n_s}{p^2+|\mathcal{Z}|^2}\,, \\ & + \mathrm{tr}\left|\mathcal{Z}_{ij}\,\partial_\mu\phi^{ij}\right|^2 \\ & \times\int \frac{\mathrm{d}^n p}{i(2\pi)^n} \left(\frac{4(4+n-2n_s)}{n}\, \frac{p^2}{(p^2+|\mathcal{Z}|^2)^4} + \frac{2}{|\mathcal{Z}|^2}\, \frac{1}{(p^2+|\mathcal{Z}|^2)^2}\right), \\ & - \mathrm{tr}\left|\mathcal{Z}_{ij}\,\phi^{ij}\right|^2 \int \frac{\mathrm{d}^n p}{i(2\pi)^n}\, \frac{2(4+n-2n_s)}{(p^2+|\mathcal{Z}|^2)^2}\,.\end{aligned} \tag{5.23}$$

For $n = n_s = 4$ we find that all scalar fields remain massless. There are (infinite) renormalizations of the kinetic terms (which break supersymmetry and SU(4)) which can be absorbed by multiplicative renormalizations of the various fields. These renormalizations are different for different fields belonging to the same supermultiplet. This phenomenon is to be expected as we are not dealing with an off-shell formulation [82]. Observe that all the corrections exhibited so far are of the renormalizable type.

3.3. Miscellaneous terms

On the basis of the multiplicative renormalizations of the fields noted in (5.22) and (5.23), it follows that supersymmetry requires that the one-loop corrections to the Yukawa term should take the form,

$$\mathcal{L} = -\left(\bar{\psi}^i\big[\phi_{ij},\, \psi^j\big] + 2\,\frac{\mathcal{Z}_{ij}\,\mathcal{Z}^{kl}}{|\mathcal{Z}|^2}\,\bar{\psi}^i\big[\phi_{kl},\, \psi^j\big] + \text{h.c.}\right) \times \int \frac{\mathrm{d}^n p}{i\,(2\pi)^n}\,\frac{1}{(p^2+|\mathcal{Z}|^2)^2}\,. \quad (5.24)$$

This result has indeed been reproduced by explicit calculation, which is quite a nontrivial check on our methods and results so far. Similar results can be obtained for the couplings to the gauge fields, but unlike in the case above, their consistency is not linked to supersymmetry.

3.4. Induced F^n-terms and the proper-time method

Up to now, we have made use of Feynman diagrams to calculate the one-loop corrections to terms in the classical action (5.7). There are, however, terms that appear at one-loop level which have mass-dimension higher than four and which are therefore not contained in the classical action. When working in a renormalizable gauge, the new terms appear with a finite coefficient at one loop, as opposed to the corrections to the dimension four terms calculated previously. In order to calculate these higher order terms, it is no longer convenient to evaluate Feynman diagrams. For example, an F^4 term in the effective action originates from a diagram with four external gauge fields with different momenta; calculating such a one-loop diagram is not trivial.

We therefore need an effective method to integrate out the massive fields from the Lagrangian (5.14). Since the Lagrangian is quadratic in the massive fields, the corresponding path integral is Gaussian. However, the operator whose determinant we need to calculate is rather complicated and we cannot solve the path integral in all generality. Nevertheless, for certain sectors of the theory the determinant can be explicitly calculated and we are able to obtain all corresponding higher order terms in the effective action for the massless fields, as we will see below. The calculation makes use of the proper time method [71, 84].

Consider the Lagrangian (5.14) that we obtained from a Kaluza-Klein reduction from ten dimensions. We restrict the field-strength $F_{\mu\nu}$ to an abelian subgroup of the gauge group and we take $F_{\mu\nu}$ (covariantly) constant. In a special gauge, this allows us to write the gauge field in terms of the field-strength, namely $A_\mu = -F_{\mu\nu}x^\nu/2$. Further, we assume that the scalar fields ϕ_{ij} belong to the same abelian subalgebra as $F_{\mu\nu}$, i.e. $[F_{\mu\nu}, \phi_{ij}] = 0$, and that they are (covariantly) constant. For convenience, we restrict ourselves to compact gauge groups. The Lagrangian of this theory is given by

$$\begin{aligned}\mathcal{L}_{\text{BPS}} = & -\frac{1}{2}\left|D_\mu B_\nu - D_\nu B_\mu\right|^2 - \bar{B}_\mu|\mathcal{Z}+i\phi|^2 B_\mu - \bar{B}^\mu F_{\mu\nu}B^\nu \\ & - i B^{ij}(\mathcal{Z}+i\phi)_{ij}D^\mu B_\mu - i\bar{B}^\mu(\mathcal{Z}+i\phi)^{ij}D_\mu B_{ij} \\ & - \left|D_\mu B_{ij}\right|^2 - B^{ij}\left|\mathcal{Z}+i\phi\right|^2 B_{ij} + B^{ij}(\mathcal{Z}+i\phi)_{ij}(\mathcal{Z}+i\phi)^{kl}B_{kl} \\ & - \bar{\chi}^i \not{D}\chi_i - \bar{\chi}_i\not{D}\chi^i - 2i\bar{\chi}^i(\mathcal{Z}+i\phi)_{ij}\chi^j - 2i\bar{\chi}_i(\mathcal{Z}+i\phi)^{ij}\chi_j\,.\end{aligned} \tag{5.25}$$

where $|\mathcal{Z}+i\phi|^2 = (\mathcal{Z}+i\phi)^{ij}(\mathcal{Z}+i\phi)_{ij} = |\mathcal{Z}|^2 - |\phi|^2 + 2i\mathcal{Z}^{ij}\phi_{ij}$. As opposed to the Lagrangian (5.14), which is written in the unitary gauge (5.10), we have not imposed any gauge condition. The massive fields transform under abelian gauge transformations,

$$\begin{aligned}\delta B_\mu &= \left(\partial_\mu - A_\mu\right)\Lambda\,, \\ \delta B_{ij} &= i\left(\mathcal{Z}+i\phi\right)_{ij}\Lambda\,.\end{aligned}$$

We add a gauge fixing Lagrangian $\mathcal{L}_{\text{GF}}$ to the initial Lagrangian (5.25),

$$\mathcal{L}_{\text{GF}} = -\left(D^\mu\bar{B}_\mu - iB^{ij}(\mathcal{Z}+i\phi)_{ij}\right)\left(D^\nu B_\nu + i(\mathcal{Z}+i\phi)^{kl}B_{kl}\right).$$

This gauge fixing Lagrangian corresponds to the 't Hooft-Feynman gauge in spontaneously broken gauge theories [86] and it has the virtue that the coupling between B_{ij} and B_μ vanishes. Because the gauge transformation parameter Λ is complex, the gauge fixing requires that we include two Faddeev-Popov ghost terms in the Lagrangian,

$$\mathcal{L}_{\text{FP}} = -\left|D_\mu\eta\right|^2 - \bar{\eta}\left|\mathcal{Z}+i\phi\right|^2\eta - \left|D_\mu\zeta\right|^2 - \bar{\zeta}\left|\mathcal{Z}+i\phi\right|^2\zeta\,.$$

The combined Lagrangian $\mathcal{L} = \mathcal{L}_{\text{BPS}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}$ is given by

$$\begin{aligned}\mathcal{L} = & -\left|D_\mu B_\nu\right|^2 - \bar{B}^\mu\left|\mathcal{Z}+i\phi\right|^2 B_\mu - 2\bar{B}^\mu F_{\mu\nu}B^\nu \\ & - \left|D_\mu\eta\right|^2 - \bar{\eta}\left|\mathcal{Z}+i\phi\right|^2\eta - \left|D_\mu\zeta\right|^2 - \bar{\zeta}\left|\mathcal{Z}+i\phi\right|^2\zeta \\ & - \left|D_\mu B_{ij}\right|^2 - B^{ij}\left|\mathcal{Z}+i\phi\right|^2 B_{ij} \\ & - \bar{\chi}^i\not{D}\chi_i - \bar{\chi}_i\not{D}\chi^i - 2i\bar{\chi}^i(\mathcal{Z}+i\phi)_{ij}\chi^j - 2i\bar{\chi}_i(\mathcal{Z}+i\phi)^{ij}\chi_j\,.\end{aligned} \tag{5.26}$$

Note that the coefficient of the term $\bar{B}^\mu F_{\mu\nu}B^\nu$ changed because we integrated by parts when combining the kinetic term of the vector fields with the gauge fixing Lagrangian $\mathcal{L}_{\text{GF}}$.

The field equations for the vector fields B_μ and the scalars B_{ij} are given by

$$\Big(\big(D^2 - |\mathcal{Z} + i\phi|^2\big)\eta_{\mu\nu} - 2F_{\mu\nu}\Big)\, B_\nu = 0\,,$$
$$\Big(D^2 - |\mathcal{Z} + i\phi|^2\Big)\, B_{ij} = 0\,.$$

The field equations for the fermions are written in matrix notation,

$$\begin{pmatrix} 2i(\mathcal{Z}+i\phi)_{ij} & \not{D}\,\delta_i^j \\ \not{D}\,\delta_j^i & 2i(\mathcal{Z}+i\phi)^{ij} \end{pmatrix} \begin{pmatrix} \chi^j \\ \chi_j \end{pmatrix} = 0\,.$$

Let us now introduce the proper time method. Historically, the proper-time method goes back to Schwinger [71]. Subsequent works using the proper-time method [87–90] treated the effect of massive fields on the effective Lagrangian for the massless fields in a variety of both supersymmetric and non-supersymmetric contexts. We denote a generic complex bosonic field by B and write the gauge-covariant derivative as

$$D_\mu B = \big(\partial_\mu - A_\mu\big) B\,,$$

As above, the generators of the gauge transformations are anti-hermitian matrices. Here, the massive field B can carry additional space-time indices, but also indices of internal symmetries. The action for the massive bosonic field B in n space-time dimensions can be written as

$$S_{\mathrm{B}} = \int \mathrm{d}^n x\ \bar{B}\big(D^2 - M^2\big) B\,.$$

The effective mass M is a function of the constant scalar fields ϕ and the constant field strength $F_{\mu\nu}$. Integrating out the massive fields B induces one-loop contributions to the classical action of the massless fields. The part of the action containing the one-loop contributions is denoted by S_1 and it is given by

$$\begin{aligned} \exp\left(\frac{i}{\hbar} S_1[B]\right) &= \int \mathcal{D}B\mathcal{D}\bar{B}\ \exp\left(\frac{i}{\hbar} S_B\right) \\ &= \frac{1}{\det\big(D^2 - M^2\big)} \\ &= \exp\left(-\int \mathrm{tr}\ \ln\big(D^2 - M^2\big)\right)\,, \end{aligned}$$

where the right-hand side is properly normalized and the trace is taken over space-time indices and possible internal symmetry indices. One should remember that the operator $(D^2 - M^2)$ depends on two space-time points x and y, and therefore the integral runs over the space-time coordinates x and y. The one-loop effective action for the massless fields is then given by

$$\begin{aligned} S_{\text{eff}} &= S_0 + S_1 \\ &= S_0 + i\hbar \int \operatorname{tr} \ln\bigl(D^2 - M^2\bigr) . \end{aligned}$$

The variation of S_1 with respect to M^2 yields

$$\begin{aligned} \delta S_1 &= -i\hbar \int \operatorname{tr} \frac{\delta M^2}{D^2 - M^2} \\ &= -i\hbar \int \operatorname{tr}\ \delta M^2\ G(x,x) , \end{aligned} \tag{5.27}$$

where $G(x, y)$ is the Green's function of the massive field B, and the integral is now taken only over the space-time coordinate x. The trace runs over possible space-time indices and internal symmetry indices. Therefore, from $G(x, y)$ we can determine S_1 by integrating (5.27). For future convenience we note that for fermions the corresponding expression is given by

$$\delta S_1 = i\hbar \int \operatorname{tr}\ \delta M\ G(x,x) ,$$

where the trace is now taken over spinor indices. Comparing with the bosonic case, there is a relative minus sign, which is due to the fermionic determinant. The details of the fermionic case are treated in more detail below.

The scalar Green's function $G(x, y)$ is defined by

$$\bigl(D_x^2 - M^2\bigr)G(x,y) = \delta(x,y) . \tag{5.28}$$

For constant abelian field-strengths and constant scalar fields, this equation can be solved. The fact that $F_{\mu\nu}$ is constant implies that the Green's function is translationally invariant up to a gauge transformation. This can be seen by applying the transformations $x \to x + a$ and $y \to y + a$ to equation (5.28). The fields $A_\mu(x)$ are not invariant under this translation, but the variation takes the form of a gauge transformation with parameter $\Lambda(x, a) = -F_{\mu\nu}x^\mu a^\nu/2$. Therefore, the Green's function also has to be gauge transformed, so that it changes according to[3]

$$G(x+a, y+a) = e^{\Lambda(x,a)} G(x,y) e^{-\Lambda(y,a)} .$$

Therefore, the Green's function is given by

$$G(x,y) = \exp\left(-\frac{1}{2} F_{\mu\nu} x^\mu y^\nu\right) \tilde{G}(x-y) .$$

[3]Remember that for a compact gauge group the field-strength $F_{\mu\nu}$ and the gauge transformation parameter Λ are anti-hermitian matrices, so that there are no explicit factors i.

and equation (5.28) translates into

$$\left(\partial^2 - F_{\mu\nu}(x-y)^\mu\partial^\nu - \frac{1}{4}\big(F^2\big)_{\mu\nu}(x-y)^\mu(x-y)^\nu - M^2\right)\tilde{G}(x-y) \\ = \delta(x,y)\,,$$

where multiple matrix products of $F_{\mu\nu}$ are written as

$$\big(F^m\big)_{\mu\nu} = F_\mu{}^{\rho_1}F_{\rho_1}{}^{\rho_2}\cdots F_{\rho_{m-1}\nu}\,.$$

We introduce the Fourier-transformed Green's function,

$$\tilde{G}(x-y) = \frac{1}{(2\pi)^n}\int \mathrm{d}^n p\ \mathrm{e}^{ip\cdot(x-y)}G(p)\,,$$

which obeys the equation

$$\left(-p^2 - F_{\mu\nu}p^\mu\frac{\partial}{\partial p_\nu} + \frac{1}{4}\big(F^2\big)_{\mu\nu}\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu} - M^2\right)G(p) = 1\,. \tag{5.29}$$

The Green's function $G(p)$ is a function of the momenta p^μ, the mass M, and the field-strength $F_{\mu\nu}$. Lorentz-invariance and the fact that $F_{\mu\nu}$ is antisymmetric restricts the form of $G(p)$ to be a function of $p^\mu(F^{2m})_{\mu\nu}p^\nu$. The second term in (5.29) does not contribute, because $(F^{2m})_{\mu\nu}$ is symmetric in μ and ν, whereas $F_{\mu\nu}$ is antisymmetric. We make the following ansatz for the Green's function in momentum-space [71],

$$G(p) = \int_0^\infty \mathrm{d}s\ \exp\big(-M^2 s - p_\mu A^{\mu\nu}(s)p_\nu - C(s)\big)\,. \tag{5.30}$$

The functions $A^{\mu\nu}(s)$ and $C(s)$ can be determined in terms of the constant scalar field ϕ and the gauge field-strength $F_{\mu\nu}$. We substitute the ansatz (5.30) into (5.29), and obtain

$$\int_0^\infty \mathrm{d}s\ \left(-M^2 - \frac{1}{2}\mathrm{tr}\,F^2A - p\big(1-AF^2A\big)p\right)\exp\big(-M^2s - pAp - C\big) = 1\,,$$

where we have suppressed the space-time indices for $A_{\mu\nu}$ and $(F^2)_{\mu\nu}$. This equation can now be solved by partial integration. We set

$$\frac{\partial A}{\partial s} = 1 - AF^2A \quad \text{and} \quad \frac{\partial C}{\partial s} = \frac{1}{2}\mathrm{tr}\,F^2A\,,$$

and readily obtain the solutions

$$A(s) = F^{-1}\tanh(Fs)\,,$$
$$C(s) = \frac{1}{2}\,\mathrm{tr}\,\ln\,\cosh(Fs)\,,$$

with the boundary conditions $A(s=0)=C(s=0)=0$. The solutions are defined in terms of power series of $F_{\mu\nu}$. The Green's function $G(x,x)$, which is the integral of $G(p)$, is then given by

$$\begin{aligned} G(x,x) &= \frac{1}{(2\pi)^n}\int \mathrm{d}^n p\ G(p) \qquad (5.31)\\ &= \frac{i}{(4\pi)^{n/2}}\int_0^\infty \frac{\mathrm{d}s}{s^{n/2}}\exp\left(-M^2 s-\frac{1}{2}\mathrm{tr}\ \ln(A/s)-C\right)\\ &= \frac{i}{(4\pi)^{n/2}}\int_0^\infty \frac{\mathrm{d}s}{s^{n/2}}\exp\left(-M^2 s-\frac{1}{2}\mathrm{tr}\ \ln\frac{\sinh(Fs)}{Fs}\right)\,, \end{aligned}$$

The induced one-loop action can now be obtained by integrating $G(x,x)$ with respect to M^2 and is given by

$$S_1=\frac{\hbar}{(4\pi)^{n/2}}\int \mathrm{d}^n x\int_0^\infty \frac{\mathrm{d}s}{s^{1+n/2}}\,\mathrm{e}^{-M^2 s}\sqrt{\det\frac{Fs}{\sinh(Fs)}}\,. \qquad (5.32)$$

For multicomponent bosonic fields the discussion is very similar. Namely, the Green's function is then matrix-valued and it obeys the equation

$$\Big(\big(D_x^2\big)_{ab}-\big(M^2\big)_{ab}\Big)G_{bc}(x,y)=\delta(x,y)\delta_{ac}$$

where a,b,c denote space-time indices, or indices of an internal symmetry. Assuming that $F_{\mu\nu}$ commutes with M, the solution (5.31) for the Green's function is still valid, however all the terms are now matrix-valued quantities. Note that the trace in (5.31) is only taken over space-time indices of $(F^{2m})_{\mu\nu}$. Similarly, the induced one-loop action (5.32) contains now a trace over additional space-time indices and internal symmetry indices.

The proper time method for fermions is a bit more subtle. The Green's function for the massive fermions is defined by

$$\begin{pmatrix} M & \not{D}\\ \not{D} & M^\dagger\end{pmatrix}G(x,y)=-\delta(x,y)\,\mathbb{1}\,,$$

where in the case at hand $M=2i(\mathcal{Z}+i\phi)_{ij}$ and $M^\dagger=2i(\mathcal{Z}+i\phi)^{ij}$. The Green's function is now a block matrix in the chiral subspaces, i.e.

$$G(x,y)=\begin{pmatrix} G^{++}(x,y) & G^{+-}(x,y)\\ G^{-+}(x,y) & G^{--}(x,y)\end{pmatrix}\,.$$

Extracting the phase factor from the Green's function as above, the Fourier transform $G(p)$ of $\tilde{G}(x-y)$ obeys the equation

$$\begin{pmatrix} M & i\not{p}\\ i\not{p} & M^\dagger\end{pmatrix}G(p)=-\mathbb{1}\,,$$

where $\not{P}$ is defined as

$$\not{P} = \gamma^{\mu}\left(p_{\mu} - \frac{1}{2}F_{\mu\nu}\frac{\partial}{\partial p_{\nu}}\right) .$$

In order to obtain $G(x, x)$, it suffices to determine $G_{\mathrm{sym}}(p) = G(p)/2 + G(-p)/2$, which is block-diagonal. We can easily verify that $G_{\mathrm{sym}}(p)$ obeys the equation

$$\begin{pmatrix} -\not{P}^2 - M^{\dagger}M & 0 \\ 0 & -\not{P}^2 - M^{\dagger}M \end{pmatrix} G_{\mathrm{sym}}(p) = \begin{pmatrix} M^{\dagger} & 0 \\ 0 & M \end{pmatrix} .$$

The explicit form of the operator $\not{P}^2$ is given by

$$-\not{P}^2 = -p^2 + F_{\mu\nu}p^{\mu}\frac{\partial}{\partial p_{\nu}} + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + \left(F^2\right)_{\mu\nu}\frac{\partial}{\partial p_{\mu}}\frac{\partial}{\partial p_{\nu}} .$$

Note that there is an additional mass term of the form $\gamma^{\mu\nu}F_{\mu\nu}/2$. Using the previous results for the bosonic case we are able to write down the Green's function

$$G(x, x) = \frac{i}{(4\pi)^{n/2}}\int_0^{\infty}\frac{\mathrm{d}s}{s^{1+n/2}}\sqrt{\det\frac{Fs}{\sinh(Fs)}}\begin{pmatrix} M^{\dagger} & 0 \\ 0 & M \end{pmatrix}$$
$$\times \exp\begin{pmatrix} -sMM^{\dagger} + s\gamma^{\mu\nu}F_{\mu\nu}/2 & 0 \\ 0 & -sM^{\dagger}M + s\gamma^{\mu\nu}F_{\mu\nu}/2 \end{pmatrix}$$

Therefore, the variation of the action S_1 is given by

$$\delta S_1 = i\hbar\int \mathrm{tr}\left(\delta M\; G^{++} + \delta M^{\dagger}\; G^{--}\right) .$$

In order to carry out the integration with respect to M, we note that M can be written as $M = U\tilde{M}V^{-1}$, where $\tilde{M}$ is a diagonal matrix (containing the eigenvalues of M), and U and V are unitary matrices. The integration over U and V could in principle give rise to so-called Wess-Zumino-Witten terms [91, 92], but in the anomaly free $\mathcal{N} = 4$ supersymmetric Yang-Mills theory they are absent. We are left with the integration over the diagonal matrix $\tilde{M}$. Finally, the effective one-loop contribution of a massive fermion to the effective action is expressed as

$$S_1 = -\frac{\hbar}{2(4\pi)^{n/2}}\int \mathrm{d}^n x \int_0^{\infty}\frac{\mathrm{d}s}{s^{1+n/2}}\,\mathrm{tr}\,\mathrm{e}^{-M^2 s}\sqrt{\det\frac{Fs}{\sinh(Fs)}}\exp\left(\frac{s}{2}\gamma^{\mu\nu}F_{\mu\nu}\right) .$$

The overall difference of a minus sign with respect to the bosonic case is due to the fermionic determinant, as explained above. The factor 2 in the denominator of the action relative to the bosonic case is due to the fact that $\delta\tilde{M}\tilde{M} = \delta\tilde{M}^2/2$. The trace in the effective action is over the full spinor space and all internal symmetry indices.

In total, there are four contributions to the one-loop effective action stemming from massive vector fields, massive fermions and massive scalars and Faddeev-Popov ghosts, respectively. The contributions of the four types of massive fields add up, and the total induced action is given by

$$S_1 = S_1^{\mathrm{V}} + S_1^{\mathrm{F}} + S_1^{\mathrm{S}} + S_1^{\mathrm{ghosts}} .$$

We express the one-loop contributions of massive fields to the effective action as

$$S_1^{\mathrm{I}} = \frac{C_{\mathrm{I}}}{(4\pi)^{n/2}} \int \mathrm{d}^n x \int_0^\infty \frac{\mathrm{d}s}{s^{1+n/2}} \,\mathrm{tr}\; \mathrm{e}^{-M_{\mathrm{I}}^2 s} \sqrt{\det \frac{Fs}{\sinh(Fs)}} \,, \tag{5.33}$$

where M_{I} denotes the effective mass term for the different field types. The trace is taken over space-time indices, internal symmetry indices and spinor indices. When dealing with complex fields, the coefficient C_{I} takes the values $C_{\mathrm{V}} = C_{\mathrm{S}} = 1$ for the vector fields and the scalars, $C_{\mathrm{F}} = -1/2$ for the fermions, and C_{gh} for the Faddeev-Popov ghosts.

In order to evaluate the one-loop contributions to the effective action (5.33) we first calculate the determinant factor, then we evaluate the trace over $\exp(-M^2 s)$ and expand the result in terms of the field-strength F before we finally integrate over s. To this end, we decompose the field-strength $F_{\mu\nu}$ as

$$F_{\mu\nu} = F_{\mu\nu}^{+} + F_{\mu\nu}^{-} \,, \tag{5.34}$$

where $F_{\mu\nu}^{\pm}$ are the self-dual and anti-selfdual part of $F_{\mu\nu}$, respectively. Further, we denote the complex eigenvalues of $F_{\mu\nu}^{+}$ by $\pm f_+/2$, and similarly for $F_{\mu\nu}^{-}$. We also note the identity $F_{\mu\nu}^2 = F_{\mu\nu}^{+2} + F_{\mu\nu}^{-2} = -f_+^2 - f_-^2$.

We first calculate the factor containing the determinant, which is common for all fields,

$$\begin{aligned} \sqrt{\det \frac{Fs}{\sinh(Fs)}} &= \frac{s(f_+ + f_-)/2}{\sinh\big(s(f_+ + f_-)/2\big)} \frac{s(f_+ - f_-)/2}{\sinh\big(s(f_+ - f_-)/2\big)} \\ &= \frac{s^2\big(f_+^2 - f_-^2\big)}{2\cosh(sf_+) - 2\cosh(sf_-)} \\ &= 1 - \frac{s^2}{12}\big(f_+^2 + f_-^2\big) + \frac{s^4}{576}\big(f_+^4 + f_-^4 - 2f_+^2 f_-^2\big) + \mathcal{O}(s^6) \,, \end{aligned} \tag{5.35}$$

where, in the last line, we have given the first few terms in the expansion of s. In order to evaluate the trace over $\exp(-M_I^2 s)$ we write down the mass terms for the four different types of fields,

$$M^2 = \begin{cases} |\mathcal{Z} + i\phi|^2 \eta_{\mu\nu} - 2F_{\mu\nu} & \text{for vector fields,} \\ |\mathcal{Z} + i\phi|^2 - \gamma^{\mu\nu} F_{\mu\nu}/2 & \text{for fermions,} \\ |\mathcal{Z} + i\phi|^2 & \text{for scalars and Faddeev-Popov ghosts}\,. \end{cases}$$

For the vector fields, we have

$$\begin{aligned}\mathrm{tr}\,\exp(2sF_{\mu\nu}) &= \mathrm{tr}\,\exp(2sF^{+}_{\mu\nu})\exp(2sF^{-}_{\mu\nu})\\ &= 4\cosh(sf_{+})\cosh(sf_{-})\,,\end{aligned}$$

and for the fermions, the trace is given by

$$\begin{aligned}\mathrm{tr}\,\exp(\gamma^{\mu\nu}F_{\mu\nu}/2) &= \mathrm{tr}\,\frac{1-\gamma_5}{2}\exp(\gamma^{\mu\nu}F^{+}_{\mu\nu}/2)\\ &\quad+\mathrm{tr}\,\frac{1+\gamma_5}{2}\exp(\gamma^{\mu\nu}F^{-}_{\mu\nu}/2)\\ &= 2\cosh(sf_{+})+2\cosh(sf_{-})\,.\end{aligned}$$

For arbitrary numbers of massive fields (n_1 vector fields, $n_{1/2}$ fermions, n_0 scalar fields and n_{gh} Faddeev-Popov ghosts), the action takes the form

$$\begin{aligned}S_1 = \frac{\hbar}{(4\pi)^{n/2}}\int \mathrm{d}^n x \int_0^\infty \frac{\mathrm{d}s}{s^{1+n/2}}\,\frac{s^2\big(f_+^2-f_-^2\big)}{2\cosh(sf_+)-2\cosh(sf_-)}\\ \times\,\mathrm{e}^{-|\mathcal{Z}+i\phi|^2 s}\begin{cases}4n_1\cosh(sf_+)\cosh(sf_-) & \text{vector field,}\\ -n_{1/2}\big(\cosh(sf_+)+\cosh(sf_-)\big) & \text{fermion,}\\ n_0 & \text{scalar field,}\\ -n_{\mathrm{gh}} & \text{Faddeev-Popov ghost.}\end{cases}\end{aligned} \tag{5.36}$$

The total action induced by the one-loop contributions of the various fields is the sum of all four terms in (5.36).

Let us now discuss the action (5.36) in some detail for the field content of the 1/2-BPS multiplet of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory that we have integrated out. These fields transform according to some irreducible representation $\mathbf{R}$ of the gauge group. The underlying supermultiplet contains one vector field B_μ that combines with one ghost and therefore has three physical degrees of freedom. The second ghost field cancels against one of the six scalar fields B_{ij}, such that five physical scalar fields remain. Furthermore, there are four fermions. Each of these fields is degenerate such as to constitute the same representation $\mathbf{R}$ of the gauge group. The sum of all contributions to the action

(5.36) is therefore proportional to

$$
\begin{aligned}
4\cosh(sf^+)\cosh(sf^-) &- 4\cosh(sf^+) - 4\cosh(sf^-) + 4 \\
&= 16\sinh^2(sf^+/2)\sinh^2(sf^-/2) \\
&= s^4 f_+^2 f_-^2 + \frac{s^6}{12}\big(f_+^2 f_-^4 + f_+^4 f_-^2\big) \\
&\quad + \frac{s^8}{720}\big(2f_+^2 f_-^6 + 2f_+^6 f_-^2 + 5f_+^4 f_-^4\big) + \mathcal{O}(s^{10})\,.
\end{aligned} \tag{5.37}
$$

We immediately see that there is no constant term, which reflects the fact that there is an equal number of fermions and bosons in the theory. Further, the terms of order s^2, which are proportional to F^2 vanish identically. This is in agreement with our previous calculation of these corrections terms in section 3.1. We note further that there are no terms of the form $F_+^{2m} + F_-^{2m}$ so that all the terms in the expansion are mixed $F_+^{2m} F_-^{2m'}$ terms.

Let us finally carry out the integration over the parameter s in order to obtain the complete action induced by the one-loop corrections of the massive BPS fields. We use the expansion (5.35) and make use of the identity

$$
\int_0^\infty \frac{\mathrm{d}s}{s^{1+n/2}} \mathrm{e}^{-M^2 s} s^{2m} = \Gamma(2m - n/2)\frac{1}{(M^2)^{2m-n/2}}\,, \tag{5.38}
$$

where $m \geq 2$.[4] The induced action for the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is then given by

$$
\begin{aligned}
S_1 = \frac{\hbar}{16\pi^2}\,\mathrm{tr}_{\mathbf{R}} \int \mathrm{d}^4x \left(\frac{F_+^2 F_-^2}{|\mathcal{Z} + i\phi|^4} + \frac{5F_+^4 F_-^4 + 2F_+^2 F_-^6 + 2F_+^6 F_-^2}{6|\mathcal{Z} + i\phi|^{12}} \right) \\
+ \mathcal{O}\big(F^{10}/|\mathcal{Z} + i\phi|^{16}\big)\,.
\end{aligned} \tag{5.39}
$$

where the trace is over the representation $\mathbf{R}$ of the gauge group.

We observe that the first term that receives contributions from the BPS fields is quartic in the field-strengths. This is in accordance with calculations in a string theoretical context [93, 94] where it was stated that the F^4 terms are the first terms to receive contributions from BPS states, and furthermore the F^4 terms only receive contribution from BPS states, i.e. not from other arbitrary massive states. In our calculation, the F^4 terms are of the form

$$
\frac{F_+^2 F_-^2}{|\mathcal{Z} + i\phi|^4}\,. \tag{5.40}
$$

This is similar to the expression obtained in [72] where the higher order terms were studied in the purely massless $\mathcal{N} = 4$ case. There, the analogous term

[4] We do not need to evaluate the (divergent) terms with $m = 0$ and $m = 1$, because, as discussed above, they vanish in the case at hand. In the following, we therefore assume $n = 4$.

is conformally invariant and given by F^4/ϕ^4. The explicit appearance of the mass term $\mathcal{Z}$ in Eq. (5.40) breaks conformal invariance. In the limit of $\mathcal{Z} \to 0$ our results coincide with those of [72] and conformal invariance is restored.[5] Note also that in the action (5.39) all terms proportional to F^6 vanish, which is again in agreement with the results found in [72].

The effective Lagrangian (5.39) can also be used to calculate the effective potential for the scalar fields. The scalar fields in (5.39) only appear in terms which are proportional to powers of the field-strength F, i.e. there are no terms consisting solely of scalar fields. Therefore, the one-loop corrections to the effective potential for scalar fields in the Cartan subalgebra of the gauge group are zero. This means that the minima of the potential are unchanged, and the valley structure of the potential is preserved.

We would like to repeat that the calculations in this section and in the previous section were based on the coupling of one 1/2-BPS multiplet to the massless $\mathcal{N} = 4$ vector multiplet. In order to calculate the effective action in the context of the Kaluza-Klein theory we would, however, need to couple the whole six-dimensional lattice of BPS multiplets to the massless vector multiplet. All the terms that we have obtained for the effective action, e.g. in (5.39), are derived for a single BPS multiplet, and we therefore need to sum over the whole six-dimensional charge lattice to obtain the full terms. The F^4 terms in (5.39) are encoded by an SL(6, $\mathbb{Z}$) invariant function $\mathcal{F}^4$, given by

$$\mathcal{F}^4(\Lambda^6, \phi) = \left(\sum_{\Lambda^6} \frac{1}{|\mathcal{Z} + i\phi|^4} \right) F_+^2 F_-^2 \, .$$

The sum over the lattice is an Eisenstein series [95], and the whole Lagrangian is clearly invariant under the 'electric' duality group SL(6, $\mathbb{Z}$). As we have not included magnetic charges, the theory is not invariant under electro-magnetic duality at this stage.

Let us conclude this section by recalling that the proper-time method that we have presented above is limited to fields which belong to the Cartan subalgebra of the gauge group. For fields with non-commuting components, we have to return to the approach taken previously, namely to the evaluation of individual Feynman diagrams. As an example we calculate dimension six terms in the effective action in the following section.

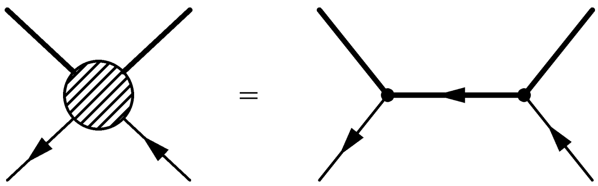
3.5. Higher order terms in the effective action

To evaluate non-abelian terms in the effective action that are of dimension six and higher is far from trivial. For a non-abelian gauge group, there are gauge invariant terms of dimension six proportional to $\mathrm{tr}(F_\mu{}^\nu \, F_\nu{}^\rho \, F_\rho{}^\mu)$, which vanish

[5] In the context of spontaneously broken gauge theories $\mathcal{Z} \to 0$ corresponds to vanishing vacuum expectation value of the scalar fields. For Kaluza-Klein theories, the same limit corresponds to a decompactification.

for abelian groups. Since our methods restrict meaningful calculations to those of on-shell quantities, it is relevant to note that, up to equations of motion and Bianchi identities, this term can be written as $\mathrm{tr}\,(D_\rho F_{\mu\nu}\, D^\rho F^{\mu\nu})$. Since a classification of the supersymmetric extensions of these terms has not been given as of to date, we calculate the one-loop terms quartic in the spinor fields, which are of the same dimension. All these terms arise from box diagrams with four external ψ-lines with zero momentum.

Let us thus turn to the explicit calculation of the one-loop ψ^4 terms in the effective action. To this end we first construct an effective four-point vertex of two massless fermions $\bar\psi$ and ψ and two massive bosons, which can be either vector fields or scalars, exchanging a virtual massive fermion χ. We make use of the Lagrangian (5.14). Schematically, the effective vertex is given by



where the solid line with an arrow denotes the propagator of a massive fermion, and the solid lines without arrows denote massive bosonic fields. The expression for this effective vertex is

$$
\begin{aligned}
& i\bar B_\mu\big(-\bar\psi^i\gamma^\mu \not{p}\gamma^\nu\psi_i - \bar\psi_i\gamma^\mu\not{p}\gamma^\nu\psi^i \\
&\qquad\qquad + 2\mathcal{Z}_{ij}\,\bar\psi^i\gamma^\mu\gamma^\nu\psi^j + 2\mathcal{Z}^{ij}\,\bar\psi_i\gamma^\mu\gamma^\nu\psi_j\big)B_\nu \\
& + 2i\,B^{ij}\big(2\delta_i^k(\bar\psi_j\not{p}\psi^l - \bar\psi^l\not{p}\psi_j) + \delta_{ij}^{kl}\,\bar\psi^m\not{p}\psi_m \\
&\qquad\qquad - 4(\bar\psi_i\psi_m\,\mathcal{Z}^{mk} + \mathcal{Z}_{im}\,\bar\psi^m\psi^k)\delta_j^l\big)B_{kl} \\
& - 2i\,\bar B_\mu\bigg(\bar\psi^i\gamma^\mu\not{p}\psi^j + \frac{1}{2}\varepsilon^{ijkl}\,\bar\psi_k\gamma^\mu\not{p}\psi_l \\
&\qquad\qquad + 2\mathcal{Z}^{ik}(\bar\psi_k\gamma^\mu\psi^j - \bar\psi^j\gamma^\mu\psi_k)\bigg)B_{ij} \\
& - 2i\,B^{ij}\bigg(\bar\psi_i\not{p}\gamma^\mu\psi_j + \frac{1}{2}\varepsilon_{ijkl}\,\bar\psi^k\not{p}\gamma^\mu\psi^l \\
&\qquad\qquad + 2(\bar\psi_i\gamma^\mu\psi^k - \bar\psi^k\gamma^\mu\psi_i)\,\mathcal{Z}_{kj}\bigg)B_\mu\,.
\end{aligned}
$$

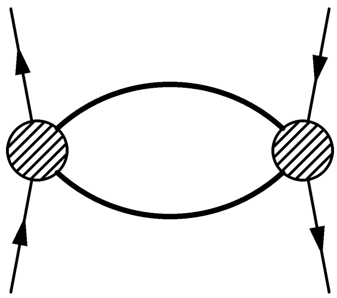


Figure 9. One-loop diagrams with four external massless fermions ψ.

For the calculation of the ψ^4 terms, one simply connects the massive bosonic fields of two effective vertices and evaluates the corresponding diagrams of the form given in figure 9.

There are three different diagrams, depending on the type of the virtual propagators in the loop. First, the vector-vector diagram generates four terms, which are given by

$$
\begin{aligned}
&\operatorname{tr}\left(\bar\psi^i\gamma_\mu\psi_i - \bar\psi_i\gamma_\mu\psi^i\right)^2 \\
&\qquad\qquad \times \frac{1}{n}\left(\frac{3}{(p^2+\mathcal{Z}^2)^3} - \frac{3|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \operatorname{tr}\left(\bar\psi^i\gamma^\mu\psi_i + \bar\psi_i\gamma^\mu\psi^i\right)^2 \\
&\times \frac{1}{2n}\left(-\frac{1}{|\mathcal{Z}|^4(p^2+\mathcal{Z}^2)} - \frac{2n-3}{|\mathcal{Z}|^2(p^2+\mathcal{Z}^2)^2} + \frac{n-1}{(p^2+\mathcal{Z}^2)^3} + \frac{(n+1)|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \operatorname{tr}\left(\mathcal{Z}_{ij}\bar\psi^i\gamma^{\mu\nu}\psi^j + \mathcal{Z}^{ij}\bar\psi_i\gamma^{\mu\nu}\psi_j\right)^2 \\
&\qquad\qquad \times \frac{1}{n}\left(\frac{1}{|\mathcal{Z}|^2(p^2+\mathcal{Z}^2)^3} + \frac{n/2-1}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \operatorname{tr}\left(\mathcal{Z}_{ij}\bar\psi^i\psi^j + \mathcal{Z}^{ij}\bar\psi_i\psi_j\right)^2 \\
&\qquad\qquad \times \frac{1}{2}\left(-\frac{1}{|\mathcal{Z}|^4(p^2+\mathcal{Z}^2)^2} - \frac{n-1}{(p^2+\mathcal{Z}^2)^4}\right) .
\end{aligned}
\tag{5.41}
$$

The trace is taken over the adjoint representation of the gauge group. Additionally, all terms are integrated over the momentum p in n space-time dimensions. The combinatorial factor for the vector-vector diagram equals $1/2$, and the expression (5.41) already accounts for this factor.

Second, there is the scalar-scalar diagram, which yields the following terms

$$
\begin{aligned}
&\mathrm{tr}\left(\bar\psi_i\gamma_\mu\psi^j - \bar\psi^j\gamma_\mu\psi_i\right)\left(\bar\psi_j\gamma_\mu\psi^i - \bar\psi^i\gamma_\mu\psi_j\right) \\
&\qquad\qquad \times \frac{1}{n}\left(-\frac{2}{(p^2+\mathcal{Z}^2)^3} + \frac{2|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \mathrm{tr}\left(\bar\psi^i\gamma_\mu\psi_i + \bar\psi_i\gamma_\mu\psi^i\right)^2 \\
&\qquad\qquad \times \frac{1}{2n}\left(-\frac{5}{(p^2+\mathcal{Z}^2)^3} + \frac{5|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \frac{\mathcal{Z}^{ij}\mathcal{Z}_{kl}}{|\mathcal{Z}|^2}\,\mathrm{tr}\left(\bar\psi_i\gamma_\mu\psi^k - \bar\psi^k\gamma_\mu\psi_i\right)\left(\bar\psi_j\gamma_\mu\psi^l - \bar\psi^l\gamma_\mu\psi_j\right) \\
&\qquad\qquad \times \frac{1}{n}\left(\frac{8}{(p^2+\mathcal{Z}^2)^3} - \frac{8|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \left(\frac{\mathcal{Z}^{ij}\mathcal{Z}^{kl}}{|\mathcal{Z}|^2}\mathrm{tr}\left(\bar\psi_i\psi_k\bar\psi_l\psi_j + \bar\psi_i\psi_k\bar\psi_j\psi_l\right) + \text{h.c.}\right) \\
&\qquad\qquad \times \frac{8|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^2} \\
&+ \left(\bar\psi_i\psi_j\bar\psi^j\psi^i + \bar\psi^j\psi^i\bar\psi_i\psi_j + \bar\psi_j\psi_i\bar\psi^j\psi^i + \bar\psi^i\psi^j\bar\psi_i\psi_j\right) \\
&\qquad\qquad \times \frac{2|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^2} \\
&+ \frac{1}{|\mathcal{Z}|^2}\left(\mathcal{Z}^{ij}\bar\psi_i\psi_j + \mathcal{Z}_{ij}\bar\psi^i\psi^j\right)^2 \times \frac{-10|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^2}\,.
\end{aligned}
\tag{5.42}
$$

Also for the scalar-scalar diagram the combinatorial factor, which has already been included in the expression (5.42), equals $1/2$,

Finally, the vector-scalar diagram gives rise to the following terms,

$$
\begin{aligned}
&\left(\bar\psi_i\gamma_\mu\psi^j - \bar\psi^j\gamma_\mu\psi_i\right)\left(\bar\psi_j\gamma_\mu\psi^i - \bar\psi^i\gamma_\mu\psi_j\right) \\
&\qquad\qquad \times \frac{1}{n}\left(-\frac{2}{(p^2+\mathcal{Z}^2)^3} + \frac{2(n-1)|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right) \\
&+ \left(\bar\psi^i\gamma_\mu\psi_i - \bar\psi_i\gamma_\mu\psi^i\right)^2 \\
&\qquad\qquad \times \frac{1}{n}\left(-\frac{1}{(p^2+\mathcal{Z}^2)^3} + \frac{(n-1)|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right)
\end{aligned}
\tag{5.43}
$$

$$
\begin{aligned}
&+\frac{\mathcal{Z}^{ij}\mathcal{Z}_{kl}}{|\mathcal{Z}|^2}\big(\bar\psi_i\gamma_\mu\psi^k-\bar\psi^k\gamma_\mu\psi_i\big)\big(\bar\psi_j\gamma_\mu\psi^l-\bar\psi^l\gamma_\mu\psi_j\big)\\
&\qquad\qquad\qquad\times\frac{1}{n}\left(-\frac{8}{(p^2+\mathcal{Z}^2)^3}+\frac{8(n-1)|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right)\\
&+\frac{1}{|\mathcal{Z}|^2}\big(\mathcal{Z}^{ij}\bar\psi_i\gamma^{\mu\nu}\psi_j+\mathcal{Z}_{ij}\bar\psi^i\gamma^{\mu\nu}\psi^j\big)^2\\
&\qquad\qquad\qquad\times\left(-\frac{4}{(p^2+\mathcal{Z}^2)^3}+\frac{4|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^4}\right)\\
&+\big(\bar\psi^i\psi^j\bar\psi_i\psi j-\bar\psi^i\psi^j\bar\psi_j\psi i\big)\\
&\qquad\qquad\qquad\times\left(-\frac{4}{(p^2+\mathcal{Z}^2)^2}+\frac{4|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^3}\right)\\
&+\frac{1}{|\mathcal{Z}|^2}\big(\mathcal{Z}^{ij}\bar\psi_i\psi_j+\mathcal{Z}_{ij}\bar\psi^i\psi^j\big)^2\\
&\qquad\qquad\qquad\times\left(\frac{4}{(p^2+\mathcal{Z}^2)^2}-\frac{4|\mathcal{Z}|^2}{(p^2+\mathcal{Z}^2)^3}\right).
\end{aligned}
$$

The combinatorial factor of the vector-scalar diagram equals 1.

Combining the results for the three diagrams is rather tedious and requires the repeated use of Fierz identities. The Fierz identities are not trivial since also the generators of the gauge group are involved. Some examples of Fierz identities that we used are

$$\mathrm{tr}\big(\bar\psi^j\gamma^\mu\psi_i\ \bar\psi_j\gamma^\mu\psi^i\big)=-2\,\mathrm{tr}\big(\bar\psi^i\psi^j\ \bar\psi_i\psi_j\big)\,, \tag{5.44a}$$

$$\mathrm{tr}\big(\bar\psi^j\gamma^\mu\psi_i\ \bar\psi^i\gamma^\mu\psi_j\big)=\mathrm{tr}\big(\bar\psi_j\gamma^\mu\psi^j\ \bar\psi_i\gamma^\mu\psi^i\big)\,, \tag{5.44b}$$

$$\varepsilon^{ijkl}\mathrm{tr}\big(\bar\psi_i\psi_j\ \bar\psi_k\psi_l\big)=0\,. \tag{5.44c}$$

Note that we are allowed to carry out the Fierz transformations in four dimensions, even though we are using dimensional regularization throughout, because the fermions ψ are outside of the n-dimensional integral and can be regarded as purely four-dimensional objects.

After applying the Fierz identities and summing the terms of all the diagrams (5.41), (5.42) and (5.43), we are left with the following terms

$$
\begin{aligned}
&\Big(\big[\bar\psi^i,\gamma^\mu\psi_i\big]\Big)^2\\
&\quad\times\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(\frac{C_n}{(p^2+|\mathcal{Z}|^2)^2}-\frac{(n-6)/8n}{(p^2+|\mathcal{Z}|^2)^3}+\frac{(n-4)/8n}{(p^2+|\mathcal{Z}|^2)^4}\right)
\end{aligned}
\tag{5.45}
$$

$$+\left([\bar{\psi}_i,\psi_j]+\frac{1}{2}\epsilon_{ijkl}[\bar{\psi}^k,\psi^l]\right)^2$$
$$\times\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(\frac{1}{(p^2+|\mathcal{Z}|^2)^2}-\frac{(2n-4)/n}{(p^2+|\mathcal{Z}|^2)^3}+\frac{(2n-4)/n}{(p^2+|\mathcal{Z}|^2)^4}\right)$$
$$+\left(\mathcal{Z}_{ij}[\bar{\psi}^i,\psi^j]+\mathcal{Z}^{ij}[\bar{\psi}_i,\psi_j]\right)^2$$
$$\times\int\frac{\mathrm{d}^n p}{i(2\pi)^n}\left(-\frac{2}{(p^2+|\mathcal{Z}|^2)^2}+\frac{4}{(p^2+|\mathcal{Z}|^2)^3}+\frac{2(n-6)}{(p^2+|\mathcal{Z}|^2)^4}\right)$$

where, using dimensional regularization, the coefficient C_n is given by $8nC_n = 2n-3-2/(n-2)$.

Let us discuss this result. All the terms in (5.45) have the same commutator structure, so that they are proportional to the square of the structure constants of the gauge group. The individual diagrams contain anti-commutator terms, but they cancel at the end. Further, each of the three terms is logarithmically divergent. The ultra-violet divergence of the terms (5.45) might at first sight be surprising in view of the fact that the $\mathcal{N}=4$ supersymmetric Yang-Mills theory is a renormalizable theory. Remember, however, that the calculations in this section were done in the unitary gauge, and that in principle they are only meaningful on-shell. The question therefore arises whether our result can indeed be regarded as an on-shell result.

All the non-vanishing terms (5.45) that follow from the one-loop calculation above have precisely the structure of Born diagrams. Namely, the structure of the first term coincides with the Born diagram where a gauge field A_μ is exchanged, as shown in figure 10 on the next page. The second and the third term can be combined into the following form,

$$\left([\bar{\psi}_i,\psi_j]+\frac{1}{2}\epsilon_{ijpq}[\bar{\psi}^p,\psi^q]\right)$$
$$\times\left(\alpha\delta^{ij}_{kl}-\beta\frac{\mathcal{Z}^{ij}\mathcal{Z}_{kl}}{|\mathcal{Z}|^2}\right)\left([\bar{\psi}^k,\psi^l]+\frac{1}{2}\epsilon^{klrs}[\bar{\psi}_r,\psi_s]\right)$$

where α and β are related to the (divergent) integrals in (5.45). A term of this form corresponds to a Born diagram where a massless scalar field ϕ_{ij} is exchanged.

This observation is significant in view of the fact that virtual corrections to the Born graphs are also ultraviolet divergent. These corrections also diverge for kinematical reasons when the fermion momenta are taken to zero and the boson propagator diverges. Therefore, on one hand our results are incomplete because we have not included the Born diagrams, and on the other hand the results from the calculation of the Born diagrams are ill-defined. This result

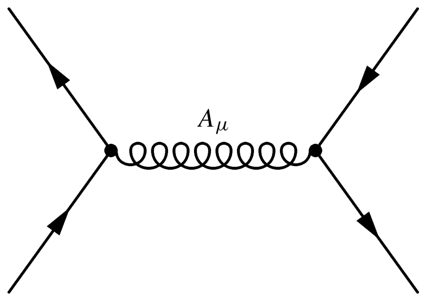


Figure 10. The Born diagram with the exchange of a virtual massless vector A_μ.

is in accordance with what we would expect from a background field calculation when the terms (5.45) belong to an expression quadratic in the field equations. The above shows that the contributions (5.45) should not be regarded as on-shell effects so that there is no dimension six contribution to the effective action.

Appendix. Gamma matrices

In this appendix we collect some formulae on gamma matrices that are needed in the Kaluza-Klein approach of chapter 5. The decomposition of the ten-dimensional gamma matrices in terms of four-dimensional and six-dimensional matrices was given in (5.4). We use tangent-space indices $a, b, \ldots = 1, \ldots, 6$ in six dimensions (for the four-dimensional ones there is no distinction between space-time indices and tangent-space indices). Spinor indices of SO(1, 3) are usually suppressed, whereas spinor indices $\alpha, \beta, \ldots = 1, \ldots, 8$ of SO(6) will be written explicitly in what follows. In section 2 we have already defined $\Gamma^{11} = \gamma^5 \otimes \gamma^7$. The charge conjugation matrix equals $C_{(10)} = C_{(4)} \otimes C_{(6)}$, where $C_{(4)}$ is the antisymmetric and $C_{(6)}$ the symmetric charge conjugation matrix in four Minkowskian and six Euclidean dimensions, respectively. We list a number of Fierz identities for the six gamma matrices $\hat{\Gamma}^a_{\alpha\beta}$,

$$
\begin{aligned}
\hat{\Gamma}^a_{\alpha\beta}\hat{\Gamma}^a_{\gamma\delta} = \frac{6}{8}\Big(\delta_{\alpha\delta}\delta_{\gamma\beta} - \gamma^7_{\alpha\delta}\gamma^7_{\gamma\beta}\Big)& \\
- \frac{1}{2}\Big(\hat{\Gamma}^a_{\alpha\delta}\hat{\Gamma}^a_{\gamma\beta} + \big(\hat{\Gamma}^a\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^a\gamma^7\big)_{\gamma\beta}\Big)& \\
- \frac{1}{8}\Big(\hat{\Gamma}^{ab}_{\alpha\delta}\hat{\Gamma}^{ab}_{\gamma\beta} - \big(\hat{\Gamma}^{ab}\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^{ab}\gamma^7\big)_{\gamma\beta}\Big)&\,,
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Gamma}^{abc}_{\alpha\beta}\hat{\Gamma}^{abc}_{\gamma\delta} = {} & -15\Big(\delta_{\alpha\delta}\delta_{\gamma\beta} - \gamma^7_{\alpha\delta}\gamma^7_{\gamma\beta}\Big) \\
& - \frac{3}{2}\Big(\hat{\Gamma}^{ab}_{\alpha\delta}\hat{\Gamma}^{ab}_{\gamma\beta} - \big(\hat{\Gamma}^{ab}\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^{ab}\gamma^7\big)_{\gamma\beta}\Big)\,, \\
6\,\delta_{\alpha\beta}\delta_{\gamma\delta} - \hat{\Gamma}^{ab}_{\alpha\beta}\hat{\Gamma}^{ab}_{\gamma\delta} = {} & \frac{9}{2}\Big(\delta_{\alpha\delta}\delta_{\gamma\beta} + \gamma^7_{\alpha\delta}\gamma^7_{\gamma\beta}\Big) \\
& + 2\Big(\hat{\Gamma}^{a}_{\alpha\delta}\hat{\Gamma}^{a}_{\gamma\beta} - \big(\hat{\Gamma}^{a}\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^{a}\gamma^7\big)_{\gamma\beta}\Big) \\
& - \frac{1}{4}\Big(\hat{\Gamma}^{ab}_{\alpha\delta}\hat{\Gamma}^{ab}_{\gamma\beta} + \big(\hat{\Gamma}^{ab}\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^{ab}\gamma^7\big)_{\gamma\beta}\Big)\,, \\
2\delta_{\alpha\beta}\delta_{\gamma\delta} - \hat{\Gamma}^{ab}_{\alpha\beta}\hat{\Gamma}^{ab}_{\gamma\delta} = {} & 4\Big(\delta_{\alpha\delta}\delta_{\gamma\beta} + \gamma^7_{\alpha\delta}\gamma^7_{\gamma\beta}\Big) \\
& + \frac{3}{2}\Big(\hat{\Gamma}^{a}_{\alpha\delta}\hat{\Gamma}^{a}_{\gamma\beta} - \big(\hat{\Gamma}^{a}\gamma^7\big)_{\alpha\delta}\big(\hat{\Gamma}^{a}\gamma^7\big)_{\gamma\beta}\Big) \\
& + \frac{1}{12}\hat{\Gamma}^{abc}_{\alpha\delta}\hat{\Gamma}^{abc}_{\gamma\beta}\,.
\end{aligned}
$$

Note also that $6\hat{\Gamma}^{abc} = i\varepsilon^{abcdef}\,\hat{\Gamma}^{def}\gamma^7$. We now choose a basis where

$$
C_{(6)} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^7 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.
$$

In this basis the gamma matrices $\hat{\Gamma}^a$ decompose according to (5.5). The spinors now decompose into four-dimensional positive-chirality spinors ψ^i and negative-chirality spinors ψ_i, respectively, where the SU(4) indices i, j were introduced in section 2. Because the spinors in $D = 10$ are Majorana-Weyl, the chiral spinors in four dimensions transform in conjugate SU(4) representations.

6

Gauged supergravities in three dimensions

Supergravity theories with vector gauge fields can usually be modified by the introduction of charges that couple to these fields. This can be done in a variety of ways corresponding to different gauge groups. Especially for theories with a high degree of supersymmetry, these gaugings constitute the only known supersymmetric deformations.

In three space-time dimensions the situation is special in two respects. First of all, pure extended supergravity is topological. Non-topological theories are constructed by coupling supergravity to matter. In three dimensions the obvious matter supermultiplets are scalar multiplets, so that the resulting Lagrangians take the form of a nonlinear sigma model coupled to supergravity. These theories have been constructed and classified in [96]. It turns out that the number of supersymmetries[1] is restricted to $N \leq 16$, implying that there are at most 32 supercharges.[2] For increasing N the target space becomes highly constrained. For $N = 9$, $N = 10$, $N = 12$ and $N = 16$, the target spaces are symmetric spaces and they are unique. For example the target space of the $N = 9$ theory is the symmetric coset manifold $\mathrm{F}_{4(-20)}/\mathrm{SO}(9)$. Second, the gauging of these theories seems impossible at first sight, because of the lack of vector gauge fields. However, one can introduce a Chern-Simons term in three dimensions, which is topological just as pure supergravity itself, and the corresponding gauge fields can be coupled to the nonlinear sigma model by gauging a subgroup of the target space isometries. Such gaugings have for example been constructed in [97–99] for $N = 8$ and $N = 16$ and in [100, 101] for some abelian gauge groups in $N = 2$. The gauging is characterized by the embedding of the gauge group into the isometry group. The embedding is defined by a symmetric tensor, which defines the so-called T-tensors. The viability of the gauging depends in a subtle manner on the properties of the T-tensors.

Gauged supergravity theories in three dimensions exhibit ground-states with an anti-de Sitter geometry. This makes gauged supergravities interesting

[1] We denote the number of supersymmetries by N in this chapter, as opposed to $\mathcal{N}$ in the rest of this thesis.

[2] There is no helicity in three dimensions and therefore, unlike in higher dimensions, one cannot obtain an upper bound on N from the condition that the maximal helicity is equal to two. The bound $N \leq 16$ follows from restrictions on the target space of the nonlinear sigma model.

from the point of view of the AdS/CFT correspondence, which relates supergravity theories on anti-de Sitter geometries to conformal field theories on the boundary of the anti-de Sitter space. Three-dimensional anti-de Sitter spaces are of particular interest because the corresponding two-dimensional conformal field theories have many applications and are well understood [102, 103].

This chapter is organized as follows. In section 1 we summarize and reformulate the results of [96] for the ungauged theories. Subsequently we analyze the presence of invariances of the Lagrangian related to the isometries of the target space metric. Then we discuss the gauging of possible subgroups of the isometry group in section 2. We derive the potential and the masslike terms in the general case and derive the extra conditions that must be satisfied in order to preserve supersymmetry. In section 3 we analyze these restrictions in detail for selected values of N. We conclude with an analysis of supersymmetric minima of the scalar potential that are present in the gauged theories.

This chapter is based on [104].

1. Nonlinear sigma models coupled to supergravity

In this section we summarize and elaborate on the construction of three-dimensional nonlinear sigma models coupled to supergravity. For the derivation and conventions we refer to [96]. The fields of the nonlinear sigma model are scalar fields ϕ^i and spinor fields χ^i, with $i = 1, \ldots, d$; the supergravity fields are the dreibein $e_\mu{}^a$, the spin-connection field ω_μ^{ab} and N gravitino fields ψ_μ^I with $I = 1, \ldots, N$. The gravitinos transform under the R-symmetry group $\mathrm{SO}(N)$, which is not necessarily a symmetry group of the Lagrangian. The scalar fields parameterize a target space endowed with a Riemannian metric $g_{ij}(\phi)$.

1.1. Target-space geometry

Pure supergravity is topological in three dimensions and exists for an arbitrary number N of supercharges and corresponding gravitinos [105]. Its coupling to a nonlinear sigma model requires the existence of $N - 1$ almost complex structures $f^{Pi}{}_j(\phi)$, labelled by $P = 2, \ldots, N$, which are hermitian,

$$g_{ij}\, f^{Pj}{}_k + g_{kj}\, f^{Pj}{}_i = 0\,,$$

and generate the $\mathrm{SO}(N-1)$ Clifford algebra,

$$f^{Pi}{}_k\, f^{Qk}{}_j + f^{Qi}{}_k\, f^{Pk}{}_j = -2\,\delta^{PQ}\,\delta^i_j\,.$$

The $\mathrm{SO}(N-1)$ Clifford algebra can be lifted to an $\mathrm{SO}(N)$ Clifford algebra by introducing a matrix notation and defining an additional element $\hat{f}^1$, i.e.

$$\hat{f}^P = \begin{pmatrix} 0 & f^P \\ f^P & 0 \end{pmatrix}\,, \qquad \hat{f}^1 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\,.$$

We now define the N generators of the SO(N) Clifford algebra as $\hat{f}^I = (\hat{f}^1, \hat{f}^P)$, where here and henceforth $I, J = 1, \ldots, N$. Clearly, they fulfill the Clifford algebra property

$$\hat{f}^I \hat{f}^J + \hat{f}^J \hat{f}^I = -2\,\delta^{IJ}\,.$$

Note that the generators $\hat{f}^I$ are $(2d \times 2d)$-matrices, as opposed to the f^P which are $(d \times d)$-matrices. From the $\hat{f}^I$ one constructs $N(N-1)/2$ tensors f^{IJ}, which are defined by

$$\frac{1}{2}\left(\hat{f}^I \hat{f}^J - \hat{f}^J \hat{f}^I\right) \equiv \begin{pmatrix} f^{IJ} & 0 \\ 0 & f^{IJ} \end{pmatrix}.$$

The f^{IJ} are $(d \times d)$-matrices and they act as the generators of the group SO(N). They are anti-symmetric in both types of indices, i.e.

$$f^{IJ}_{ij} = -f^{JI}_{ij} = -f^{IJ}_{ji}\,.$$

and they satisfy the relations

$$f^{IJ}\,f^{KL} = f^{[IJ}\,f^{KL]} - 4\,\delta^{[I[K}\,f^{L]J]} - 2\,\delta^{I[K}\,\delta^{L]J}\,\mathbb{1}\,, \tag{6.1a}$$

$$f^{IJ}\,f^{KL} - f^{KL}\,f^{IJ} = 4\,\delta^{K[I}\,f^{J]L} - 4\,\delta^{L[I}\,f^{J]K}\,, \tag{6.1b}$$

$$\left(f^{IJ}\right)^2 = -\,\mathbb{1}\,, \quad (I, J \text{ fixed}) \tag{6.1c}$$

$$f^{IK}\,f^{KJ} = (N-1)\,\delta^{IJ}\,\mathbb{1} - (N-2)\,f^{IJ}\,, \tag{6.1d}$$

$$f^{IJ\,ij}\,f^{KL}{}_{ij} = 2d\,\delta^{I[K}\,\delta^{L]J} - \delta_{N,4}\,\varepsilon^{IJKL}\,\mathrm{Tr}(J)\,. \tag{6.1e}$$

The tensor $J^i{}_j$ is only relevant for $N = 4$. It is defined by

$$J = \frac{1}{6}\,\varepsilon_{PQR}\,f^P\,f^Q\,f^R\,,$$

so that

$$f^P\,f^Q = -\delta^{PQ}\,\mathbb{1} - \varepsilon^{PQR}\,J\,f^R\,.$$

Furthermore, the tensor J satisfies,

$$\begin{aligned} Jf^P &= f^P J\,, \\ J^2 &= \mathbb{1}\,, \\ J_{ij} &= J_{ji}\,, \\ J &= \frac{1}{24}\varepsilon^{IJKL} f^{IJ} f^{KL}\,, \end{aligned}$$

and it has eigenvalues equal to ± 1. It turns out that J is covariantly constant, which implies that the target space is locally the product of two separate Riemannian spaces of dimension $d_\pm$, where $d_+ + d_- = d$ and $d_\pm$ are both multiples of 4. These two spaces correspond to inequivalent $N = 4$ supermultiplets. Hence the case $N = 4$ is rather special, and the identity (6.1e) can be written as

$$f^{IJ\,ij}\, f^{KL}{}_{ij} = 4\left(d_+\, \mathbb{P}_+^{IJ,KL} + d_-\, \mathbb{P}_-^{IJ,KL}\right) ,$$

with the projectors,

$$\mathbb{P}_\pm^{IJ,KL} = \frac{1}{2}\delta^{I[K}\,\delta^{L]J} \mp \frac{1}{4}\epsilon^{IJKL} . \tag{6.2}$$

For rigidly supersymmetric nonlinear sigma models, the number of supersymmetries is equal to $N = 1$, $N = 2$ or $N = 4$, and the Lagrangians are manifestly invariant under $\mathrm{SO}(N)$ R-symmetry transformations acting exclusively on the fermion fields through multiplication with the complex structures. The case $N = 3$ is not distinct from $N = 4$, because the existence of two complex structures necessarily implies the existence of a third one. In case of $N = 4$, the Lagrangian is a sum of two separate Lagrangians corresponding to the $d_\pm$-dimensional target spaces. For $N = 3$ and $N = 4$ the target spaces are hyperkähler.

When coupling to supergravity, the Lagrangian and the supersymmetry transformations depend on $\mathrm{SO}(N)$ target-space connections denoted by $Q_i^{IJ}(\phi)$. These connections are nontrivial in view of

$$R_{ij}^{IJ}(Q) \equiv \partial_i\, Q_j^{IJ} - \partial_j\, Q_i^{IJ} + 2Q_i^{K[I}\, Q_j^{J]K} = \frac{1}{2} f_{ij}^{IJ} . \tag{6.3}$$

For local supersymmetry N can take the values $N = 1, \ldots, 6$ and $8, 9, 10, 12$ or 16. The situation regarding $\mathrm{SO}(N)$ symmetry is more subtle in this case, as we shall discuss in due course. The $N = 3$ theory is no longer equivalent to an $N = 4$ theory, since it has only 3 gravitinos. In view of the three almost complex structures, the target space is a quaternionic space. For $N = 4$ the target space decomposes locally into a product of two quaternionic spaces of dimension $d_\pm$. The f_{ij}^{IJ} are covariantly constant, both with respect to the Christoffel connection $\Gamma_{ij}{}^k$ and the $\mathrm{SO}(N)$ connection Q_i^{IJ},

$$D_i(\Gamma, Q)\, f_{jk}^{IJ} \equiv \partial_i f_{jk}^{IJ} - 2\,\Gamma_{i[k}{}^l\, f_{j]l}^{IJ} + 2\, Q_i^{K[I}\, f_{jk}^{J]K} = 0\,. \tag{6.4}$$

For local supersymmetry we are therefore dealing with almost complex structures, except for the cases $N = 1$ and $N = 2$. Equation (6.4) implies an integrability condition for the target-space Riemann tensor R_{ijkl},

$$R_{ijmk}\, f^{IJ\,m}{}_l - R_{ijml}\, f^{IJ\,m}{}_k = -f_{ij}^{K[I}\, f_{kl}^{J]K} , \tag{6.5}$$

where we made use of (6.3). Contracting (6.5) with f^{MNkl} gives, for general $N > 2$,

$$R_{ijkl}\, f^{IJ\,kl} = \frac{1}{4} d\, f^{IJ}_{ij}\,, \tag{6.6}$$

so that the target space has nontrivial SO(N) holonomy, while contracting (6.5) with g^{jl}, using the cyclicity of the Riemann tensor and the above result, yields (for $N > 2$)

$$R_{ij} \equiv R_{ikjl}\, g^{kl} = c\, g_{ij}\,, \tag{6.7}$$

with $c = N - 2 + d/8 > 0$. Hence the target space must be an Einstein space.[3]

Following [96] we introduce a complete set of linearly independent antisymmetric tensors h^{α}_{ij}, labeled by indices α, that commute with the almost complex structures, i.e.

$$h^{\alpha}_{ik}\, f^{IJ\,k}{}_j - h^{\alpha}_{ik}\, f^{IJ\,k}{}_j = 0\,. \tag{6.8}$$

For $N = 2$, there is only one tensor f^{IJ} which commutes with itself, so that this decomposition is not meaningful. For $N > 2$ we must have $h^{\alpha}_{ij}\, f^{IJ\,ij} = 0$. The tensors h^{α}_{ij} generate a subgroup $\mathrm{H}' \subset \mathrm{SO}(d)$ that commutes with the group SO(N) generated by the tensors f^{IJ}_{ij}. They can be normalized according to $h^{\alpha}_{ij} h^{\beta\,ij} \propto \delta^{\alpha\beta}$ and are covariantly constant with respect to the Christoffel connection and a new connection $\Omega_i^{\alpha\beta}$,

$$D_i(\Gamma) h^{\alpha}_{jk} - \Omega_i^{\alpha\beta}\, h^{\beta}_{jk} = 0\,.$$

The Riemann tensor can be written as (for $N > 2$)

$$R_{ijkl} = \frac{1}{8}\Big(f^{IJ}_{ij}\, f^{IJ}_{kl} + C_{\alpha\beta}\, h^{\alpha}_{ij}\, h^{\beta}_{kl} \Big)\,,$$

where $C_{\alpha\beta}(\phi)$ is a symmetric tensor. This result implies that the holonomy group is contained in $\mathrm{SO}(N) \times \mathrm{H}' \subset \mathrm{SO}(d)$ which must act irreducibly on the target space. For $N = 4$ this result is modified because of the product structure [96]. In table 5 on the following page we list the group H' for $N > 2$, together with the number of supermultiplets k, and the number of bosonic states in such a multiplet, d_N.

[3] For $N = 3$ this is in accordance with the fact that quaternionic spaces of $d > 4$ are always Einstein [106]. In the case at hand, the result also holds true for a $d = 4$ target space. For $N = 4$ the equations (6.6) and (6.7) read

$$R_{ijkl}\, f^{IJ\,kl} = \frac{1}{2}\left(d_+\, \mathbb{P}_+{}^{IJ,KL} + d_-\, \mathbb{P}_-{}^{IJ,KL} \right) f^{KL}_{ij}\,,$$
$$R_{ij} = \left(2 + \frac{1}{8} d \right) g_{ij} + \frac{1}{8}(d_+ - d_-) J_{ij}\,,$$

and we have a product space of two quaternionic manifolds, which are both Einstein. For $N = 2$ the target space is Kähler and f^{IJ} is a complex structure. The SO(2) holonomy is undetermined.

N	d_N	k	H'
16	128	1	$\mathbb{1}$
12	64	1	Sp(1)
10	32	1	U(1)
9	16	1	$\mathbb{1}$
8	8	k	SO(k)
6	8	k	U(k)
5	8	k	Sp(k)
4	4	$k_\pm$	Sp($k_\pm$)
3	4	k	Sp(k)

Table 5. The groups H' for all $N > 2$. Here, d_N denotes the number of bosonic states in a supermultiplet, and k stands for the number of supermultiplets. For the case of $N = 4$ supersymmetry, there are two independent quaternionic subspaces corresponding to k_+ and k_- inequivalent supermultiplets [96].

1.2. Lagrangian and invariances

Let us now turn to the Lagrangian and supersymmetry transformations. In the following it is convenient to adopt an SO(N) covariant notation which allows to select the $N-1$ almost almost complex structures from the f^{IJ} tensors by specifying some arbitrary unit N-vector α_I and identifying the almost complex structures with $\alpha_J f^{JI}$. By extending the fermion fields χ^i to an overcomplete set χ^{iI}, defined by

$$\chi^{iI} = \left(\chi^i, f^{Pi}{}_j\, \chi^j\right) ,$$

we can write the Lagrangian and transformation rules in a way that does no longer depend explicitly on the almost complex structures. The fact that we have only d fermion fields, rather than dN, can be expressed by the covariant SO(N) constraint,

$$\chi^{iI} = \mathbb{P}^{I\,i}_{Jj}\, \chi^{jJ} \equiv \frac{1}{N}\left(\delta^{IJ}\delta^i_j - f^{IJ\,i}{}_j\right)\chi^{jJ} . \tag{6.9}$$

The trace of this projector equals $\mathbb{P}^{I\,i}_{I\,i} = d$, which confirms that the total number of fermion fields is not altered. We should stress here, that the introduction of χ^{iI} is a purely notational exercise and we do not aim at implementing the constraint (6.9) at the Lagrangian level. At every step in the computation one may change back to the original notation by choosing $\chi^i = \alpha_I \chi^{iI}$. The covariant notation does not imply that the theory is SO(N) invariant, but the covariant setting allows us to treat the N supersymmetries and the corresponding gravitinos on equal footing and it facilitates the various derivations in later sections.

The supersymmetry transformations read

$$\delta e_\mu{}^a = \frac{1}{2}\,\bar\epsilon^I \gamma^a\,\psi^I_\mu\;, \tag{6.10a}$$

$$\delta\psi^I_\mu = D_\mu\epsilon^I - \frac{1}{8} g_{ij}\;\bar\chi^{iI}\gamma^\nu\chi^{jJ}\;\gamma_{\mu\nu}\,\epsilon^J - \delta\phi^i\;Q_i^{IJ}\,\psi^J_\mu\;, \tag{6.10b}$$

$$\delta\phi^i = \frac{1}{2}\,\bar\epsilon^I\,\chi^{iI}\;, \tag{6.10c}$$

$$\delta\chi^{iI} = \frac{1}{2}\big(\delta^{IJ} - f^{IJ}\big)^i{}_j\;\hat{\partial\!\!\!/}\phi^j\,\epsilon^J - \delta\phi^j\big(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\,\chi^{iJ}\big)\;, \tag{6.10d}$$

where the supercovariant derivative $\hat\partial_\mu\phi^i$ and the covariant derivative $D_\mu\epsilon^I$ are defined by

$$\begin{aligned}
\hat\partial_\mu\phi^i &= \partial_\mu\phi^i - \frac{1}{2}\bar\psi^I_\mu\chi^{iI}\;,\\
D_\mu\epsilon^I &= \left(\partial_\mu + \frac{1}{2}\omega^a_\mu\,\gamma_a\right)\epsilon^I + \partial_\mu\phi^i\;Q_i^{IJ}\,\epsilon^J\;.
\end{aligned}$$

Observe that the terms proportional to $\delta\phi$ in $\delta\chi^{iI}$ do not satisfy the same constraint (6.9) as χ^{iI} itself, because the projection operator $\mathbb{P}^{I\,i}_{J\,j}$ itself transforms under supersymmetry. As in [96], we use the Pauli-Källén metric with hermitian gamma matrices γ^a, satisfying $\gamma_a\gamma_b = \delta_{ab} + i\varepsilon_{abc}\gamma^c$.

Let us now turn to the Lagrangian, which reads

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\,i\,\varepsilon^{\mu\nu\rho}\left(e_\mu{}^a\,R_{\nu\rho a} + \bar\psi^I_\mu D_\nu\psi^I_\rho\right)\\
&-\frac{1}{2}e\,g_{ij}\left(g^{\mu\nu}\,\partial_\mu\phi^i\,\partial_\nu\phi^j + \frac{1}{N}\bar\chi^{iI}D\!\!\!\!/\,\chi^{jI}\right)\\
&+\frac{1}{4}e\,g_{ij}\;\bar\chi^{iI}\gamma^\mu\gamma^\nu\psi^I_\mu\,(\partial_\nu\phi^j + \hat\partial_\nu\phi^j)\\
&+\frac{1}{48N^2}\,e\,\left(3\left(g_{ij}\;\bar\chi^{iI}\chi^{jI}\right)^2 - 2(N-2)\left(g_{ij}\;\bar\chi^{iI}\gamma^a\chi^{jJ}\right)^2\right)\\
&-\frac{1}{24N^2}\,e\,R_{ijkl}\;\bar\chi^{iI}\gamma_a\chi^{jI}\;\bar\chi^{kJ}\gamma^a\chi^{lJ}\;.
\end{aligned} \tag{6.11}$$

Here we used the covariant derivatives

$$\begin{aligned}
D_\mu\psi^I_\nu &= \left(\partial_\mu + \frac{1}{2}\omega^a_\mu\,\gamma_a\right)\psi^I_\nu + \partial_\mu\phi^i\;Q_i^{IJ}\,\psi^J_\nu\;,\\
D_\mu\chi^{iI} &= \left(\partial_\mu + \frac{1}{2}\omega^a_\mu\,\gamma_a\right)\chi^{iI} + \partial_\mu\phi^j\left(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\,\chi^{iJ}\right)\;.
\end{aligned}$$

We emphasize that the above results coincide with the results of [96], written in a different form. The conversion makes use of (6.5). The Lagrangian and transformation rules are consistent with target-space diffeomorphisms and field-dependent SO(N) R-symmetry rotations, acting on ψ^I_μ, χ^{iI} and Q_i^{IJ} according to

$$\delta\psi^I = \Lambda^{IJ}(\phi)\,\psi^J_\mu\,, \tag{6.12a}$$

$$\delta\chi^{iI} = \Lambda^{IJ}(\phi)\,\chi^{iJ}\,, \tag{6.12b}$$

$$\delta Q_i^{IJ} = -D_i\Lambda^{IJ}(\phi)\,. \tag{6.12c}$$

Combining (6.12c) with (6.3), one concludes that the f^{IJ} should also be rotated,

$$\delta f^{IJ} = 2\,\Lambda^{K[I}(\phi)\,f^{J]K}\,. \tag{6.13}$$

The target-space diffeomorphisms and field-dependent SO(N) R-symmetry rotations correspond to reparametrizations within certain equivalence classes, but do not, in general, constitute an invariance.

In the remainder of this section we discuss the invariances of these models, other than supersymmetry, space-time diffeomorphisms and local Lorentz transformations. The target space of the nonlinear sigma model may have isometries, generated by Killing vector fields $X^i(\phi)$. Some of these isometries can be extended to invariances of the full Lagrangian, possibly after including a field-dependent SO(N) transformation according to (6.12) and (6.13). Hence, we combine an isometry characterized by a Killing vector field X^i with a special SO(N) transformation whose parameters depend on $X^i(\phi)$ and on the scalar fields. Denoting the infinitesimal SO(N) transformations by $\mathcal{S}^{IJ}(X,\phi)$, we require invariance of the target-space metric, the SO(N) connections and the almost complex structures, up to a uniform SO(N) rotation, i.e.

$$\mathcal{L}_X\,g_{ij} = 0\,, \tag{6.14a}$$

$$\mathcal{L}_X Q_i^{IJ} + D_i\mathcal{S}^{IJ}(\phi,X) = 0\,, \tag{6.14b}$$

$$\mathcal{L}_X f_{ij}^{IJ} - 2\,\mathcal{S}^{K[I}(\phi,X)\,f_{ij}^{J]K} = 0\,, \tag{6.14c}$$

where $\mathcal{L}_X$ denotes the isometry transformation. The Lagrangian (6.11) is then invariant under the combined transformations

$$\delta\phi^i = X^i(\phi)\,, \tag{6.15a}$$

$$\delta\psi^I_\mu = \mathcal{S}^{IJ}(\phi,X)\,\psi^J_\mu\,, \tag{6.15b}$$

$$\delta\chi^{iI} = \chi^{jI}\partial_j X^i + \mathcal{S}^{IJ}(\phi,X)\,\chi^{iJ}\,. \tag{6.15c}$$

The fermion transformations can be rewritten covariantly,

$$\delta\psi^I_\mu = \mathcal{V}^{IJ}(\phi,X)\,\psi^J_\mu - \delta\phi^i\,Q_i^{IJ}\,\psi^J_\mu\,, \tag{6.16a}$$

$$\begin{aligned}\delta\chi^{iI} = {}& D_j X^i\,\chi^{jI} + \mathcal{V}^{IJ}(\phi,X)\,\chi^{iJ}\\ &- \delta\phi^j\big(\Gamma^i_{jk}\,\chi^{kI} + Q_j^{IJ}\,\chi^{iJ}\big)\,,\end{aligned} \tag{6.16b}$$

where $\mathcal{V}^{IJ}(\phi, X) \equiv X^j Q_j^{IJ}(\phi) + \mathcal{S}^{IJ}(\phi, X)$. Using (6.3) and (6.4), one verifies that (6.14b) and (6.14c) can be written as,

$$D_i \mathcal{V}^{IJ}(\phi, X) = \frac{1}{2} f_{ij}^{IJ}(\phi)\, X^j(\phi) \,, \tag{6.17a}$$

$$f^{IJk}{}_{[i}(\phi)\; D_{j]} X_k(\phi) = f_{ij}^{K[I}(\phi)\; \mathcal{V}^{J]K}(\phi, X) \,. \tag{6.17b}$$

Equation (6.17a) shows that $\mathcal{V}^{IJ}(\phi, X)$ can be regarded as the moment map associated with the isometry X^i. Equation (6.17b) is just the integrability condition associated with (6.17a), so it is not independent. After contraction with $f^{MN\,ij}$, it leads to

$$f^{IJ\,ij}\; D_i X_j = \begin{cases} \frac{1}{2} d\, \mathcal{V}^{IJ} \,, & \text{for } N \neq 1, 2, 4, \\ \Big(d_+\, \mathbb{P}_+^{IJ,KL} + d_-\, \mathbb{P}_-^{IJ,KL} \Big) \mathcal{V}^{KL} \,, & \text{for } N = 4. \end{cases} \tag{6.18}$$

From combining the above equations it follows that $\Delta \mathcal{V}^{IJ} = \frac{1}{4} d\, \mathcal{V}^{IJ}$, where Δ equals the SO(N) covariant Laplacian. This result applies to $N > 2$, with obvious modifications for $N = 4$. The above analysis shows that there are no restrictions for $N > 2$ to extend an isometry to a symmetry of the Lagrangian. For $N = 2$ this is different, as the isometry should be holomorphic, i.e. it should leave the complex structure invariant. In this case $\mathcal{V}^{IJ}$ is determined up to an integration constant. This constant is related to an invariance under constant SO(2) transformations of the fermions, as we will see in section 3.2.

For $N = 4$ we note that the almost complex structures $\mathbb{P}_\pm^{IJ,KL}\, f^{KL}$ live in the two separate quaternionic subspaces. The same is true for $\mathbb{P}_\pm^{IJ,KL}\, \mathcal{V}^{KL}$, which according to (6.17) depends only on the corresponding subspace coordinates. Note, however, that when one of the subspaces is trivial, e.g. when $d_- = 0$, then $\mathbb{P}_-^{IJ,KL}\, \mathcal{V}^{KL}$ corresponds to a triplet of arbitrary constants. This is a consequence of the fact that the model has an SO(3) rigid invariance which acts exclusively on the fermions.

Let us point out that the supersymmetry transformations do not commute with the isometries, as one can verify most easily on the fields ϕ^i, where one derives

$$\big[\delta_{\rm Q}(\epsilon), \delta_{\rm G}(X)\big] = \delta_{\rm Q}(\epsilon') \,, \tag{6.19}$$

with $(\epsilon^I)' = \mathcal{S}^{IJ}(\phi, X)\, \epsilon^J$.

The isometries that can be extended to an invariance of the Lagrangian, generate an algebra $\mathfrak{g}$. Denoting $\{X^{\mathcal{M}}\}$ as a basis of generators, we have

$$X^{\mathcal{M}i}\, \partial_i X^{\mathcal{N}} - X^{\mathcal{N}i}\, \partial_i X^{\mathcal{M}} = f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, X^{\mathcal{K}} \,, \tag{6.20}$$

with structure constants $f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}$. Closure of the algebra implies that the corresponding induced SO(N) rotations, $\mathcal{S}^{\mathcal{M}\,IJ} \equiv \mathcal{S}^{IJ}(\phi, X^{\mathcal{M}})$, satisfy,

$$\big[\mathcal{S}^{\mathcal{M}}, \mathcal{S}^{\mathcal{N}}\big]^{IJ} = -f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{S}^{\mathcal{K}\,IJ} + \big(X^{\mathcal{M}i}\, \partial_i \mathcal{S}^{\mathcal{N}\,IJ} - X^{\mathcal{N}i}\, \partial_i \mathcal{S}^{\mathcal{M}\,IJ}\big) \,.$$

For $N = 2$, the left-hand side of this equation vanishes, and an additional constant term appears, as we will see in section 3.2. Using (6.20) and the second equation (6.14), the above equation can be rewritten as

$$\left[\mathcal{V}^{\mathcal{M}},\, \mathcal{V}^{\mathcal{N}}\right]^{IJ} = -f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{V}^{\mathcal{K}\,IJ} + \frac{1}{2}\, f_{ij}^{IJ}\, X^{\mathcal{M}i} X^{\mathcal{N}j}\ , \tag{6.21}$$

with $\mathcal{V}^{\mathcal{M}IJ} \equiv \mathcal{V}^{IJ}(\phi, X^{\mathcal{M}})$.

We now note that the second equation of (6.17) implies that $D_i X_j - \frac{1}{4}\, f_{ij}^{MN}\, \mathcal{V}^{MN}$ commutes with the almost complex structures, so that it can be decomposed in terms of the antisymmetric tensors h_{ij}^{α} that were introduced in (6.8),

$$D_i X_j^{\mathcal{M}} - \frac{1}{4} f_{ij}^{IJ}\, \mathcal{V}^{\mathcal{M}\,IJ} \equiv h_{ij}^{\alpha}\, \mathcal{V}^{\mathcal{M}}{}_{\alpha}\ . \tag{6.22}$$

Using the general result for Killing vectors, $D_i D_j X_k = R_{jkil}\, X^l$, we can evaluate the derivative of $\mathcal{V}^{\mathcal{M}}{}_{\alpha}$. Introducing furthermore the notation $\mathcal{V}^{\mathcal{M}\,i} \equiv X^{\mathcal{M}i}$, we establish the following system of linear differential equations,

$$D_i \mathcal{V}^{\mathcal{M}\,IJ} = \frac{1}{2}\, f_{ij}^{IJ}\, \mathcal{V}^{\mathcal{M}\,j}\ , \tag{6.23a}$$

$$D_i \mathcal{V}^{\mathcal{M}}{}_j = \frac{1}{4}\, f_{ij}^{IJ}\, \mathcal{V}^{\mathcal{M}\,IJ} + h_{ij}^{\alpha}\, \mathcal{V}^{\mathcal{M}}{}_{\alpha}\ , \tag{6.23b}$$

$$D_i \mathcal{V}^{\mathcal{M}}{}_{\alpha} = \frac{1}{8}\, C_{\alpha\beta}\, h_{ij}^{\beta}\, \mathcal{V}^{\mathcal{M}\,j}\ . \tag{6.23c}$$

We will return to these equations in section 3, where we discuss the admissible gauge groups for various N.

2. Gauged isometries

In this section we elevate a subgroup of the isometries to a local symmetry by making the parameters space-time dependent. The particular subgroup is encoded in an embedding matrix $\Theta_{\mathcal{M}\mathcal{N}}$ which defines the Killing vectors that generate the gauge group by

$$X^i = g\, \Theta_{\mathcal{M}\mathcal{N}}\, \Lambda^{\mathcal{M}}(x)\, X^{\mathcal{N}\,i}\ , \tag{6.24}$$

with space-time dependent parameters $\Lambda^{\mathcal{N}}(x)$ and a gauge coupling constant g. Unless the gauge group coincides with the full group of isometries, the embedding matrix acts as a projector which reduces the number of independent parameters. In order for this subset of Killing vectors to generate a group, the embedding matrix $\Theta_{\mathcal{M}\mathcal{N}}$ must satisfy the following condition,

$$\Theta_{\mathcal{M}\mathcal{P}}\, \Theta_{\mathcal{N}\mathcal{Q}}\, f^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{R}} = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\, \Theta_{\mathcal{P}\mathcal{R}}\ ,$$

for certain constants $\hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$, which are subsequently identified as the structure constants of the gauge group [107]. When the gauge group is smaller than the isometry group, the embedding matrix reduces the number of independent components of $\hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ as compared to the structure constants of the full isometry group. One can verify that the validity of the Jacobi identity for the gauge group structure constants follows directly from the Jacobi identity associated with the full group of isometries.

2.1. Gauge fields

The next step is to introduce gauge fields $A_\mu^{\mathcal{M}}$ corresponding to the gauge group parameters $\Lambda^{\mathcal{M}}(x)$ and include them into the definition of the covariant derivatives. For example, we have

$$\mathcal{D}_\mu \phi^i = \partial_\mu \phi^i + g\, \Theta_{\mathcal{M}\mathcal{N}}\, A_\mu^{\mathcal{M}}\, X^{\mathcal{N}\,i}\,, \tag{6.25}$$

which transforms under local isometries according to

$$\mathcal{D}_\mu \phi^i \to \mathcal{D}_\mu \phi^i + g\, \Theta_{\mathcal{M}\mathcal{N}}\, \Lambda^{\mathcal{M}}\, \partial_j X^{\mathcal{N}\,i}\; D_\mu \phi^j\,,$$

provided that the gauge fields transformation as

$$\Theta_{\mathcal{M}\mathcal{N}}\, \delta A^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}}\big(-\partial_\mu \Lambda^{\mathcal{M}} + g\, \hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}}\, A_\mu^{\mathcal{P}}\, \Lambda^{\mathcal{Q}}\big)\,.$$

The covariant field strengths follow from the commutator of two covariant derivatives, e.g.

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\, \phi^i = g\, \Theta_{\mathcal{M}\mathcal{N}}\, F_{\mu\nu}^{\mathcal{M}}\, X^{\mathcal{N}i}\,, \tag{6.26}$$

and take the form

$$\Theta_{\mathcal{M}\mathcal{N}}\, F_{\mu\nu}^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}}\big(2\, \partial_{[\mu} A_{\nu]}^{\mathcal{M}} - g\, \hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}}\, A_\mu^{\mathcal{P}} A_\nu^{\mathcal{Q}}\big)\,. \tag{6.27}$$

The gauge transformations on the fermion fields follow from (6.16) upon substitution of (6.24). From this we derive the covariant derivatives for the spinor fields,

$$\begin{aligned}\mathcal{D}_\mu \psi_\nu^I &= \left(\partial_\mu + \frac{1}{2}\omega_\mu^a \gamma_a\right) \psi_\nu^I + \partial_\mu \phi^i\, Q_i^{IJ}\, \psi_\nu^J \\ &\quad + g\, \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}}\, \mathcal{V}^{\mathcal{N}\,IJ}\, \psi_\nu^J\,,\end{aligned} \tag{6.28a}$$

$$\begin{aligned}\mathcal{D}_\mu \chi^{iI} &= \left(\partial_\mu + \frac{1}{2}\omega_\mu^a \gamma_a\right) \chi^{iI} + \partial_\mu \phi^j \left(\Gamma_{jk}^i\, \chi^{kI} + Q_j^{IJ}\, \chi^{iJ}\right) \\ &\quad + g\, \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}} \left(\delta_j^i\, \mathcal{V}^{\mathcal{N}\,IJ} - \delta^{IJ} g^{ik}\, D_k \mathcal{V}^{\mathcal{N}}{}_j\right) \chi^{jJ}\,.\end{aligned} \tag{6.28b}$$

In view of the commutator (6.19), the covariant derivative on the supersymmetry parameter acquires also an additional covariantization,

$$\mathcal{D}_\mu \epsilon^I = \left(\partial_\mu + \frac{1}{2}\omega_\mu^a\, \gamma_a\right) \epsilon^I + \partial_\mu \phi^i\, Q_i^{IJ}\, \epsilon^J + g\, \Theta_{\mathcal{M}\mathcal{N}}\, A_\mu^{\mathcal{M}}\, \mathcal{V}^{\mathcal{N}\,IJ}\, \epsilon^J\,.$$

We would like to stress that in this section we only make use of the previous results (6.17) that apply to arbitrary N.

The extra minimal couplings (6.25) induce modifications of the supersymmetry variations and of the Lagrangian. As long as we are dealing with first derivatives, these new couplings do not lead to complications as they are controled by gauge covariance. However, commutators of the new covariant derivatives lead to new covariant terms proportional to the field strength (6.27). These terms, which cause a violation of supersymmetry, take the form,

$$\delta\mathcal{L} = -\frac{1}{2} i\, g\, \Theta_{\mathcal{M}\mathcal{N}} F^{\mathcal{N}}_{\nu\rho}\, \epsilon^{\mu\nu\rho} \left(\mathcal{V}^{\mathcal{N}\,IJ}\, \bar\psi^I_\mu \epsilon^J + \frac{1}{2}\, \mathcal{V}^{\mathcal{N}}{}_i\, \bar\chi^{iI} \gamma_\mu \epsilon^I \right) .$$

They are canceled by introducing a Chern-Simons term for the vector fields,

$$\mathcal{L}_{\rm CS} = \frac{1}{4}\, i g\, \varepsilon^{\mu\nu\rho}\, A^{\mathcal{M}}_\mu\, \Theta_{\mathcal{M}\mathcal{N}} \left(\partial_\nu A^{\mathcal{N}}_\rho - \frac{1}{3} g\, \hat f_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{N}}\, A^{\mathcal{P}}_\nu A^{\mathcal{Q}}_\rho \right) , \tag{6.29}$$

provided the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ is symmetric and provided we assume the following supersymmetry transformations for $A^{\mathcal{M}}_\mu$,

$$\Theta_{\mathcal{M}\mathcal{N}}\, \delta A^{\mathcal{M}}_\mu = \Theta_{\mathcal{M}\mathcal{N}} \left(2\, \mathcal{V}^{\mathcal{M}\,IJ}\, \bar\psi^I_\mu \epsilon^J + \mathcal{V}^{\mathcal{M}}{}_i\, \bar\chi^{iI} \gamma_\mu \epsilon^I \right) . \tag{6.30}$$

The embedding tensor is gauge invariant and therefore satisfies $\hat f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{Q}}\, \Theta_{\mathcal{Q}\mathcal{P}} + \hat f_{\mathcal{M}\mathcal{P}}{}^{Q}\, \Theta_{\mathcal{Q}\mathcal{N}} = 0$, which implies

$$\Theta_{\mathcal{P}\mathcal{L}} \left(f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{M}} \Theta_{\mathcal{N}\mathcal{K}} + f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N}} \Theta_{\mathcal{M}\mathcal{K}} \right) = 0\, . \tag{6.31}$$

Multiplying this equation with $\mathcal{V}^{\mathcal{M}IJ}$ or $\mathcal{V}^{\mathcal{M}}{}_i$, and $\mathcal{V}^{\mathcal{N}KL}$ or $\mathcal{V}^{\mathcal{N}}{}_j$, one derives a number of identities which imply that the following tensors

$$T^{IJ,KL} = \mathcal{V}^{\mathcal{M}IJ} \Theta_{\mathcal{M}\mathcal{N}} \mathcal{V}^{\mathcal{N}KL}\, , \tag{6.32a}$$

$$T^{IJi} = \mathcal{V}^{\mathcal{M}IJ} \Theta_{\mathcal{M}\mathcal{N}} \mathcal{V}^{\mathcal{N}i}\, , \tag{6.32b}$$

$$T^{ij} = \mathcal{V}^{\mathcal{M}i} \Theta_{\mathcal{M}\mathcal{N}} \mathcal{V}^{\mathcal{N}j}\, , \tag{6.32c}$$

transform covariantly under the gauged isometries. These are the so-called T-tensors. We note the following identities for the T-tensors,

$$\begin{aligned} D_{(i} T_{jk)} &= 0\, , \\ D_{(i} T^{IJ}{}_{j)} &= \frac{1}{2} T_{k(i}\, f^{IJk}{}_{j)}\, , \\ D_i T^{IJ,KL} &= \frac{1}{2} f^{IJ}_{ij}\, T^{KLj} + \frac{1}{2} f^{KL}_{ij}\, T^{IJj}\, . \end{aligned}$$

The covariance under the gauged isometries also allows the derivation of identities quadratic in the T-tensors. Two examples of such identities are,

$$\begin{aligned} & T^{MN\,i}\, T^{KL\,j}\, f^{IJ}_{ij} + T^{MN\,i}\, T^{IJ\,j}\, f^{KL}_{ij} \\ & \qquad = 4\, T^{MN,P[I}\, T^{J]P,KL} + 4\, T^{MN,P[K}\, T^{L]P,IJ}\, , \end{aligned}$$

$$T^{ki}\, T^{KL\,j}\, f^{IJ}_{ij} + T^{ki}\, T^{IJ\,j}\, f^{KL}_{ij} = 4\, T^{P[I\,k}\, T^{J]P,KL} + 4\, T^{P[K\,k}\, T^{L]P,IJ}\, .$$

At this point one can also derive the field equation for the vector fields, which reads,

$$\Theta_{\mathcal{M}\mathcal{N}}\Big(\hat{F}^{\mathcal{M}}_{\mu\nu} + 2\,ie\varepsilon_{\mu\nu\rho}\,\hat{D}^{\rho}\phi^{i}\,\mathcal{V}^{\mathcal{M}}{}_{i} \\ + \frac{1}{12}\bar{\chi}^{iI}\gamma_{\mu\nu}\chi^{jJ}\left(g_{ij}\,\mathcal{V}^{\mathcal{M}IJ} - \delta^{IJ}\,D_i\mathcal{V}^{\mathcal{M}}{}_{j}\right)\Big) = 0\,, \quad (6.33)$$

where $\hat{F}^{\mathcal{M}}_{\mu\nu}$ denotes the supercovariant curvature.

2.2. Constraints from supersymmetry

The supersymmetry variations of the vector fields in (6.25) and (6.28) give rise to additional supersymmetry variations of order g. The variations linear in the spinor fields are

$$\delta\mathcal{L} = -eg\,\Theta_{\mathcal{M}\mathcal{N}}\Big(2\,\mathcal{V}^{\mathcal{M}\,IJ}\,\bar{\psi}^{I}_{\mu}\epsilon^{J} + \mathcal{V}^{\mathcal{M}}{}_{i}\,\bar{\chi}^{iI}\gamma_{\mu}\epsilon^{I}\Big)\mathcal{V}^{\mathcal{N}}{}_{j}\,\mathcal{D}^{\mu}\phi^{j}\;. \quad (6.34)$$

They are cancelled by introducing mass-like terms for the gravitinos and the fermions,

$$(eg)^{-1}\mathcal{L}_g = \frac{1}{2}A_1^{IJ}\,\bar{\psi}^{I}_{\mu}\,\gamma^{\mu\nu}\,\psi^{J}_{\nu} + A_{2\,j}^{IJ}\,\bar{\psi}^{I}_{\mu}\,\gamma^{\mu}\chi^{jJ} + \frac{1}{2}A_3{}^{IJ}_{ij}\,\bar{\chi}^{iI}\chi^{jJ}\;, \quad (6.35)$$

accompanied by additional modifications of the supersymmetry transformation rules

$$\delta_g\psi^{I}_{\mu} = g\,A_1^{IJ}\gamma_{\mu}\,\epsilon^{J}\;, \quad (6.36a)$$

$$\delta_g\chi^{iI} = -gN\,A_2^{JIi}\,\epsilon^{J}\;. \quad (6.36b)$$

Obviously the tensors A_1 and A_3 are symmetric, i.e. $A_1^{IJ} = A_1^{JI}$ and $A_3{}^{IJ}_{ij} = A_3{}^{JI}_{ji}$. Furthermore, in view of (6.9), the tensors A_2 and A_3 are subject to the constraints,

$$\mathbb{P}^{Jj}_{I\,i}\,A_{2\,j}^{KJ} = A_{2\,i}^{KI}\;, \quad (6.37a)$$

$$\mathbb{P}^{Jj}_{I\,i}\,A_3{}^{JK}_{jk} = A_3{}^{IK}_{ik}\;. \quad (6.37b)$$

The variations of (6.35) and the additional variations (6.36) of the original Lagrangian together cancel the terms (6.34), provided that A_2 and A_3 take the following form,

$$A_{2\,i}^{IJ} = \frac{1}{N}\Big(D_iA_1^{IJ} + 2\,T^{IJ}{}_{i}\Big)\,, \quad (6.38a)$$

$$A_3{}^{IJ}_{ij} = \frac{1}{N^2}\Big(-2\,D_{(i}D_{j)}A_1^{IJ} + g_{ij}\,A_1^{IJ} + A_1^{K[I}\,f_{ij}^{\;J]K} \\ + 2\,T_{ij}\,\delta^{IJ} - 4\,D_{[i}T^{IJ}{}_{j]} - 2\,T_{k[i}\,f^{IJk}{}_{j]}\Big)\;. \quad (6.38b)$$

Here, the tensor A_3 has the required symmetry structure. For the convenience of the reader we also give the dependent result,

$$D_i A_{2j}^{IJ} = \frac{1}{2} g_{ik} A_1^{IK} \mathbb{P}_{Jj}^{Kk} - \frac{1}{2} N A_3{}_{ij}^{IJ} + T_{ik} \mathbb{P}_{Jj}^{Ik} . \tag{6.39}$$

In addition, we need to ensure that both A_2 and A_3 as defined in (6.38) satisfy the projection constraints (6.37). In view of (6.39), it is sufficient to impose this constraint on A_2, which implies the following two equations,

$$f^{K(I\,j}{}_i\, D_j A_1^{J)K} + (N-1)\, D_i A_1^{IJ} + 2\, D_i T^{IK,JK} = 0\,, \tag{6.40a}$$

$$f^{K[I\,j}{}_i \left(D_j A_1^{J]K} + 2\, T^{J]K}{}_j \right) - 2(N-1)\, T^{IJ}{}_i = 0\,. \tag{6.40b}$$

The identity (6.40a) leads to a number of results. By iterating this equation (i.e. by resubstituting the result for $D_i A_1$), we derive

$$\begin{aligned} f^{KI\,j}{}_i\, D_j \Big(4\, T^{JL,KL} + (N-2) A_1^{JK} \Big) - D_i \Big(4\, T^{IL,JL} + (N-2) A_1^{IJ} \Big) & \\ = f^{IJ\,j}{}_i\, D_j A_1^{KK} + \delta^{IJ}\, D_i A_1^{KK}\,. & \quad (6.41) \end{aligned}$$

This result can be combined with equation (6.40b) to eliminate A_1^{IJ} and to find a linear constraint for the components $T^{IJ}{}_i$ of the T-tensor

$$(N-4)\, T^{IJ}{}_i + 2\, f_{ij}^{K[I}\, T^{J]K\,j} - \frac{1}{N-1} \left(f^{IJ} f^{KL} \right)_{ij} T^{KL\,j} = 0\,. \tag{6.42}$$

Applying this constraint to the combination $f_{ij}^{IJ}\, T^{KL\,j} + f_{ij}^{KL}\, T^{IJ\,j}$ the resulting equation may be integrated to

$$\begin{aligned} (N-2)\Big(T^{IJ,KL} - T^{[IJ,KL]} \Big) - 4\, \delta^{[I[K}\, T^{L]M,J]M} & \\ + \frac{2}{N-1}\, \delta^{I[K} \delta^{L]J}\, T^{MN,MN} = 0\,. & \quad (6.43) \end{aligned}$$

In principle, this equation holds up to a covariantly constant term. Because of (6.3), covariantly constant terms cannot exist, unless they are SO(N) invariant and therefore constant. However, the above equation does not contain a singlet contribution so that it is in fact exact for any $N > 1$.

Vice versa one may show that the covariant derivative of (6.43) implies (6.42), such that these two equations are in fact equivalent. It is not straightforward to disentangle various independent equations, due to the nontrivial properties (6.1) of the f^{IJ}. By employing a set of SO(N) projection operators, one can systematize this analysis [104]. For example, the following equation is not independent,

$$\begin{aligned} (N-8)\, D_i T^{[IJ,KL]} - \frac{1}{N-1} \big(f^{[IJ} f^{KL]} \big)_{ij}\, D^j T^{MN,MN} & \\ + 5 \big(f^{M[I} f^{JK} \big)_{ij} T^{L]M\,j} = 0\,. & \end{aligned}$$

Now we use (6.42) to rewrite the $f^{KIj}{}_i\, D_j T^{JL,KL}$ term in (6.41). Combining the result with (6.40) to remove the $f^{KIj}{}_i\, D_j A^{JK}$ terms, we may integrate the resulting equation to obtain

$$4\,T^{IL,JL} + (N-2)A_1^{IJ} - \frac{2}{N-1}\,T^{MN,MN}\,\delta^{IJ} = (N-2)\mu\,\delta^{IJ}\ , \quad (6.44)$$

with a yet undetermined constant μ.

To sum up, we have shown that supersymmetry at linear order in the gauge coupling constant g determines the tensors A_1, A_2, A_3 according to (6.38) and (6.44) in terms of the T-tensor (6.32) while the latter satisfies the equivalent constraints (6.42) and (6.43).

Before proceeding to the remaining terms in the action and transformation rules, we take a brief look at the supersymmetry algebra. The supersymmetry commutator leads to a covariantized translation, and a supersymmetry and Lorentz transformation with parameters proportional to χ^2. When switching on the gauge coupling, there is an extra Lorentz transformation, but more importantly, also a local isometry. The commutators on the bosonic fields are given by

$$\begin{aligned}
\big[\delta(\epsilon_1),\delta(\epsilon_2)\big]\, e_\mu{}^a &= \mathcal{D}_\mu \xi^\nu e_\nu{}^a + \xi^\nu \mathcal{D}_\nu e_\mu{}^a\ ,\\
\big[\delta(\epsilon_1),\delta(\epsilon_2)\big]\, \phi^i &= \xi^\mu \mathcal{D}_\mu \phi^i + g\Theta_{\mathcal{M}\mathcal{N}}\Lambda^{\mathcal{M}} X^{\mathcal{N}\,i}\ ,\\
\big[\delta(\epsilon_1),\delta(\epsilon_2)\big]\, A_\mu^{\mathcal{M}} &= \xi^\nu F_{\mu\nu}^{\mathcal{M}} + \mathcal{D}_\mu \Lambda^{\mathcal{M}}\ ,
\end{aligned}$$

where the parameter of the gauge transformations is $\Lambda^{\mathcal{M}} = 2\mathcal{V}^{\mathcal{M}\,IJ}\,\bar\epsilon_2^I\epsilon_1^J$ and where $\xi^\mu = (\bar\epsilon_2\gamma^\mu\epsilon_1)/2$. Strictly speaking, the commutator on the gauge field is subject to the projection with the embedding matrix $\Theta_{\mathcal{M}\mathcal{N}}$. The supersymmetry established so far guarantees the closure of the algebra to that order, except for the gauge fields which appear multiplied by a coupling constant. Their closure, up to the field equations (6.33), implies that

$$\Theta_{\mathcal{M}\mathcal{N}}\big(2\,\mathcal{V}^{\mathcal{N}\,K(I}\,A_1^{J)K} + \mathcal{V}^{\mathcal{N}i}D_i A_1^{IJ}\big) = 0\ . \quad (6.45)$$

This result implies that the function A_1 is gauge covariant; in particular, its trace is invariant, i.e. $\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{M}i}\,D_i A^{II} = 0$. This is in agreement with equation (6.44), since we have already proven that the T-tensors are gauge covariant. Moreover, equations (6.38) show that the tensors A_2 and A_3 are covariant as well, as they depend on the T-tensors and A_1 and covariant derivatives thereof. Again we can derive certain identities from (6.45) that involve some of the T-tensors and A_1, such as

$$T^{IJ\,i}\,D_i A_1^{KL} + 2\,T^{IJ,M(K}A_1^{L)M} = 0\ , \quad (6.46a)$$

$$T^{ij}\,D_j A_1^{KL} + 2\,T^{M(K\,i}A_1^{L)M} = 0\ . \quad (6.46b)$$

In order to preserve supersymmetry to order g^2 one determines the corresponding variations linear in ψ_μ^I and χ^{iI}. They reveal the need for a gauge invariant scalar potential in the Lagrangian,

$$\begin{aligned}\mathcal{L}_{g^2} &= \frac{4\,eg^2}{N}\left(A_1^{IJ}A_1^{IJ} - \frac{1}{2}N\,g^{ij}\,A_{2i}^{I\,J}A_{2j}^{I\,J}\right)\,, \qquad (6.47)\\ &= \frac{4\,eg^2}{N^2}\left(N\,A_1^{IJ}A_1^{IJ} - \frac{1}{2}g^{ij}\,D_iA_1^{IJ}\,D_jA_1^{IJ} - 2\,g^{ij}\,T_i^{IJ}\,T_j^{IJ}\right)\,.\end{aligned}$$

We note that the variation of the scalar potential is given by

$$\partial_i\mathcal{L}_{g^2} = e\,g^2\Big(3\,A_1^{IJ}\,A_{2i}^{I\,J} + N\,A_{3}{}_{ij}^{IJ}\,A_2^{IjJ}\Big)\,, \qquad (6.48)$$

by virtue of (6.38), (6.39) and (6.45). In order for all supersymmetry variations of the potential to cancel, the following two quadratic equations must be satisfied,

$$\begin{aligned}&2\,A_1^{IK}A_1^{KJ} - N\,A_2^{IiK}\,A_{2i}^{J\,K}\\ &\qquad = \frac{1}{N}\,\delta^{IJ}\left(2\,A_1^{KL}\,A_1^{KL} - N\,A_2^{KiL}A_{2i}^{KL}\right)\,, \qquad (6.49a)\end{aligned}$$

$$\begin{aligned}&3\,A_1^{IK}\,A_{2j}^{K\,J} + N\,g^{kl}\,A_{2k}^{IK}\,A_{3}{}_{lj}^{KJ}\\ &\qquad = \mathbb{P}_{Jj}^{I\,i}\left(3\,A_1^{KL}\,A_{2i}^{KL} + N\,g^{kl}\,A_{2k}^{KL}\,A_{3}{}_{li}^{KL}\right)\,. \qquad (6.49b)\end{aligned}$$

It may be shown after some computation that these relations are a direct consequence of (6.46) upon using (6.38) and (6.44) and determine the free constant in the latter equation to be $\mu = 0$.

What remains is to analyze the supersymmetry variations cubic in the fermion fields. There are four types of terms, schematically written as ψ^3, $\chi\psi^2$, $\chi^2\psi$, and χ^3. Supersymmetry variations proportional to ψ^3 cancel provided that

$$\delta^{[I[K}A_1^{L]J]} = -T^{IJ,KL} + T^{[IJ,KL]}\,, \qquad (6.50)$$

which is in agreement with equation (6.44). The variations that are proportional to $\chi\psi^2$ cancel by virtue of (6.38a). We have not verified the cancellation of the supersymmetry variations proportional to $\chi^2\psi$ and χ^3. However, we expect that all these terms vanish by means of the constraints derived so far; in the case of the maximal $N = 16$ theory this has been verified explicitly [98].

2.3. Summary

Let us summarize our findings of this section. A gauge group G_0 with embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ that describes the minimal couplings according to (6.25) is consistent with supersymmetry if the associated T-tensor (6.32) satisfies the constraint

$$
\begin{aligned}
T^{IJ,KL} = T^{[IJ,KL]} - \frac{4}{N-2}\,\delta^{[I[K}\,T^{L]M,MJ]} \\
- \frac{2}{(N-1)(N-2)}\,\delta^{I[K}\delta^{L]J}\,T^{MN,MN}\;,
\end{aligned}
\tag{6.51}
$$

from which all further consistency conditions can be derived. The Lagrangian of the ungauged theory is modified by a Chern-Simons term (6.29), mass-like fermionic terms (6.34) and a scalar potential (6.47). For $N > 2$, the scalar tensors A_1, A_2, and A_3 describing these new terms are uniquely given in terms of the T-tensor by means of (6.38) and (6.44),

$$
\begin{aligned}
A_1^{IJ} &= -\frac{4}{N-2}\,T^{IM,JM} + \frac{2}{(N-1)(N-2)}\,\delta^{IJ}\,T^{MN,MN}\;, \\
A_{2}{}^{IJ}_{j} &= \frac{N}{N-2}\,\mathbb{P}^{Mm}_{Jj}\left(2\,T_{IMm} + \frac{\delta_{IM}}{N-1}\,f^{KL}{}_{m}{}^{n}\,T_{KLn}\right)\;, \\
A_{3}{}^{IJ}_{ij} &= \frac{1}{N}\,g_{ik}\,A_1^{IK}\,\mathbb{P}^{Kk}_{Jj} - \frac{2}{N}\,D_i A_{2}{}^{IJ}_{j} + \frac{2}{N}\,T_{ik}\,\mathbb{P}^{Ik}_{Jj}\;.
\end{aligned}
$$

The consistency constraint (6.51) has a simple group theoretical meaning in SO(N). We denote the irreducible parts of $T_{IJ,KL}$ under SO(N) by

$$
\ydiagram{1,1} \times_{\mathrm{sym}} \ydiagram{1,1} = 1 + \ydiagram{2} + \ydiagram{2,2} + \ydiagram{1,1,1,1}\,, \tag{6.52}
$$

with each box representing a vector representation[4] of SO(N). Then, equation (6.51) expresses that the projection equation

$$
\mathbb{P}_{\ydiagram{2,2}}\,T_{IJ,KL} = 0\;, \tag{6.53}
$$

holds on the entire scalar manifold. We will refer to this representation as the obstruction to a consistent gauging.

Note that the constraint (6.51) is well-defined even for $N = 1$ and $N = 2$, but degenerates into an identity. Any subgroup of the isometry group can be gauged, as we will see in the next section. Furthermore, these theories also allow supersymmetric deformations that are not induced by gaugings, which will also be discussed in the next section.

[4]We use the standard Young tableaux for the orthogonal groups, i.e. the four representations on the right-hand side of the decomposition (6.52) have dimensions 1, $N(N+1)/2-1$, $N(N-3)(N+1)(N+2)/12$, and $\binom{N}{4}$, respectively. For $N = 8$, the last representation is in fact reducible. However, this does not affect the argument here.

3. Admissible gauge groups

Let us now discuss the admissible gauge groups for different values of N. Rather than giving an exhaustive classification for all possible values of N, we concentrate on the cases $N = 1$, $N = 2$ and $N = 9$. The cases $N = 1$ and $N = 2$ are special, since the constraint for the T-tensor (6.51) degenerates into an identity. For $N > 4$ the target-space manifolds are homogeneous spaces G/H, which allows us to lift the constraint (6.51) to a projection equation for the embedding tensor $\Theta_{\mathcal{MN}}$. As an example of a theory with $N > 4$, we discuss the case $N = 9$ in some detail.

3.1. $N = 1$ supersymmetry

For the simple case of $N = 1$, the target space of the ungauged theory is a Riemannian manifold of arbitrary dimension d, as has been found in [96]. The consistency conditions for the gauged theory simplify considerably, e.g. the quadratic constraints (6.49) become identities.

The tensor A_1 has just one component, which takes the value $A_1 = F$, where F is a function on the target space. According to (6.45) the tensor A_1, and therefore also the function F, is gauge invariant,

$$\Theta_{\mathcal{MN}} X^{\mathcal{N}i} \partial_i F = 0 \; . \tag{6.54}$$

Reading off the values for A_2 and A_3 from equation (6.38), we have

$$A_1 = F \; , \tag{6.55a}$$

$$A_{2\,i} = \partial_i F \; , \tag{6.55b}$$

$$A_{3\,ij} = g_{ij} F - 2 D_i \partial_j F + 2 T_{ij} \; , \tag{6.55c}$$

with $T_{ij} = X_i^{\mathcal{M}} \Theta_{\mathcal{MN}} X_j^{\mathcal{N}}$ from (6.32).

As a consequence of (6.54), any subgroup of isometries may be gauged by choosing e.g. the trivial solution of constant F. Note that in the $N = 1$ case, the gravitino ψ_μ is never charged under the gauge group, as can be seen directly from (6.28a), and the gauging is restricted to the matter sector.

It is worth mentioning that there exist deformations of the original theory that are not induced by gaugings. These are the cases where $\Theta_{\mathcal{MN}} = 0$ and $F \neq 0$ and they are described by the Lagrangian (6.11) together with the mass-like terms (6.35) and the scalar potential (6.47) where the tensors A_1, A_2, A_3 are given by (6.55).

The scalar potential V (6.47) is given by

$$V = -e g^2 \left(4F^2 - 2 g^{ij}\, \partial_i F \partial_j F \right) \; .$$

The condition for a supersymmetric ground state is that $A_{2i} = \partial_i F = 0$ at the minimum of the potential. This is trivially satisfied for a constant function F. In that case the potential gives rise to a negative cosmological constant in the Einstein equations, and the corresponding supersymmetric ground-state geometry is an anti-de Sitter space or Minkowski space if $F = 0$. It is also possible to construct so-called "domain-wall" solutions, which are (partially supersymmetric) solutions that interpolate between two different vacua [108]. In the context of the AdS/CFT correspondence, domain-walls are relevant for RG flows in the dual field theories.

3.2. $N = 2$ supersymmetry

The target space of the nonlinear sigma model coupled to $N = 2$ supergravity theory is a Kähler manifold. Some (partial) results for abelian gaugings have been obtained in [100, 101]. Using the results of the previous section, we present the gauging of an arbitrary subgroup of Kähler isometries. The $N = 2$ gaugings we construct in three dimensions share some similarities with the $N = 1$ gaugings of four-dimensional supergravity [109–111].

For $N = 2$, many of the quantities introduced in sections 1 and 2 simplify considerably. It is therefore convenient to introduce the notation

$$f = f^{12}\,, \qquad Q_i = Q_i^{12}\,, \qquad \mathcal{V} = \mathcal{V}^{12}\,,$$
$$T_i = T_i^{12}\,, \qquad T = T^{IJ,IJ} = 2T^{12,12}\,.$$

To avoid confusion, we keep using the notation Λ^{12} and $\mathcal{S}^{12}$ for the parameters of the SO(2) transformations. We also note the relation between the T-tensors, $\partial_i T = 2 f_i{}^j T_j$. Further, we have

$$\partial_i Q_j - \partial_j Q_i = \frac{1}{2} f_{ij}\,, \tag{6.56a}$$
$$D_i(\Gamma) f_j{}^k = 0\,, \tag{6.56b}$$

where Γ_{ij}^k is the Christoffel connection. For a Kähler manifold as target space it is convenient to decompose the d real fields into $d/2$ complex ones and their complex conjugates, $\phi^i \to (\phi^i, \bar\phi^{\bar\imath})$ in a basis where $f_i{}^j = i\,\delta_i^j$, $f_{\bar\imath}{}^{\bar\jmath} = -i\,\delta_{\bar\imath}^{\bar\jmath}$. From the fact that f is hermitian, it follows that only the components $g_{i\bar\jmath} = g_{\bar\imath j}$ are non-zero, and therefore $f_{i\bar\jmath} = i g_{i\bar\jmath} = -f_{\bar\jmath i}$. The fact that f is covariantly constant then leads to

$$\partial_i g_{j\bar k} = \partial_j g_{i\bar k}\,,$$

which implies that the metric can locally be written as

$$g_{i\bar\jmath} = \partial_i \partial_{\bar\jmath} K\,,$$

where $K(\phi, \bar\phi)$ is the so-called Kähler potential. The projectors defined in (6.9), have a holomorphic and an anti-holomorphic component,

$$\mathbb{P}^{Ii}_{Jj} = \frac{1}{2}\delta^i_j\big(\delta^{IJ} + i\epsilon^{IJ}\big)\,, \quad \mathbb{P}^{I\bar\imath}_{J\bar\jmath} = \frac{1}{2}\delta^{\bar\imath}_{\bar\jmath}\big(\delta^{IJ} - i\epsilon^{IJ}\big)\,,$$

and the T-tensors obey the relation

$$\partial_i T = 2i T_i\,. \tag{6.57}$$

The Kähler potential K is defined up to a Kähler transformation,

$$K(\phi,\bar{\phi}) \to K(\phi,\bar{\phi}) + \Lambda(\phi) + \bar{\Lambda}(\bar{\phi})\,. \tag{6.58}$$

A solution of (6.56a) is provided by

$$Q_i = -\frac{1}{4} i\,\partial_i K \quad \text{and} \quad Q_{\bar{\imath}} = \frac{1}{4} i\,\partial_{\bar{\imath}} K\,. \tag{6.59}$$

This solution is not unique as it is subject to field-dependent gauge transformations. By adopting (6.59) we have removed this gauge freedom, but the Kähler transformations now act on Q in the form of a field-dependent SO(2) gauge transformation. This induces an SO(2) gauge transformation with parameter

$$\Lambda^{12}(\phi,\bar{\phi}) = \frac{i}{4}\big(\Lambda(\phi) - \bar{\Lambda}(\bar{\phi})\big)\,. \tag{6.60}$$

Consequently, all quantities transforming nontrivially under SO(2) become now subject to Kähler transformations induced by (6.60). Note that transformations where Λ equals an imaginary constant, correspond to SO(2) transformations acting exclusively on the fermions and not on the Kähler potential. These transformations constitute an invariance group of the ungauged Lagrangian and they are in the center of the full group of combined isometries and SO(2) transformations of the fermions.

According to (6.14a) and (6.14c) only holomorphic isometries of the target space can be extended to symmetries of the Lagrangian. Such isometries, parameterized by Killing vector fields $(X^i, X^{\bar{\imath}})$, preserve both the metric and the complex structure, i.e.

$$\mathcal{L}_X g = \mathcal{L}_X f = 0\,.$$

The invariance of the complex structure implies that X^i and $X^{\bar{\imath}}$ must be holomorphic and anti-holomorphic, respectively. The invariance of the metric gives rise to the Killing equations

$$\begin{aligned} D_i X_{\bar{\jmath}} + D_{\bar{\jmath}} X_i &= 0 \\ D_i X_j + D_j X_i &= 0\,. \end{aligned}$$

Because of the holomorphicity of X^i, the second condition is automatically satisfied, whereas the first condition implies that the Kähler potential remains invariant under the isometry up to a Kähler transformation. We write this special Kähler transformation in terms of a holomorphic function $\mathcal{S}(\phi)$, i.e

$$\delta K(\phi,\bar{\phi}) = -X^i\partial_i K - X^{\bar{\imath}}\partial_{\bar{\imath}} K = 4i\big(\mathcal{S} - \bar{\mathcal{S}}\big)\,. \tag{6.61}$$

According to (6.17a), the function $\mathcal{V}$, which is defined as

$$\mathcal{V} = X^i Q_i + X^{\bar{\imath}} Q_{\bar{\imath}} + \mathcal{S}^{12} = -\frac{i}{4} X^i \partial_i K + \frac{i}{4} X^{\bar{\imath}} \partial_{\bar{\imath}} K + \mathcal{S}^{12}$$

must satisfy the equation,

$$\partial_i \mathcal{V} = \frac{i}{2} g_{i\bar{j}} X^{\bar{j}} .$$

As the right-hand side can be written as a derivative, this equation can now be solved and we obtain

$$\mathcal{S}^{12}(\phi, \bar{\phi}) = \mathcal{S}(\phi) + \bar{\mathcal{S}}(\bar{\phi}) . \tag{6.62}$$

Consequently we have,

$$\begin{aligned} \mathcal{V} &= -\frac{i}{4}\big(X^i \partial_i K - X^{\bar{\imath}} \partial_{\bar{\imath}} K\big) + \mathcal{S} + \bar{\mathcal{S}} \\ &= -\frac{i}{2} X^i \partial_i K + 2\mathcal{S} . \end{aligned}$$

When $\mathcal{S}$ is a real constant and $X^i = 0$ we are again dealing with the subgroup that acts exclusively on the fermions and which commutes with all other symmetry transformations.

When there are noncommuting isometries we must insist that the field-dependent SO(2) transformations close on the fermions. For every generator $X^{\mathcal{M}}$ we distinguish a function $\mathcal{S}^{\mathcal{M}}$, which is determined up to a real constant. The resulting condition decomposes into a holomorphic and an anti-holomorphic part and is defined by

$$f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\, \mathcal{S}^{\mathcal{K}} = X^{\mathcal{M}i} \partial_i \mathcal{S}^{\mathcal{N}} - X^{\mathcal{N}i} \partial_i \mathcal{S}^{\mathcal{M}} + f_0^{\mathcal{M}\mathcal{N}} . \tag{6.63}$$

The (complex) constants $f_0^{\mathcal{M}\mathcal{N}}$ satisfy the cocycle condition $f_0^{\mathcal{M}[\mathcal{N}}\, f^{\mathcal{K}\mathcal{L}]}{}_{\mathcal{M}} = 0$. For semi-simple Lie algebras, to which we restrict the discussion here, these constants can be absorbed by suitable shifts into the functions $\mathcal{S}^{\mathcal{M}}$. In general, one needs to introduce an additional vector field corresponding to the constant SO(2) transformations labelled by an index zero [104].

Now we turn to the gaugings, assuming an embedding matrix Θ, and determine the various quantities involved. It is convenient to decompose the tensor A_1^{IJ} in terms of a singlet part $A_1^{11} + A_1^{22}$ and a complex quantity

$$\mathrm{e}^{K/2} W \equiv \frac{1}{2}\big(A_1^{22} - A_1^{11}\big) + i A_1^{12} .$$

Kähler transformations are induced by the SO(2) transformations (6.60),

$$\delta\big(e^{K/2} W\big) = 2i \Lambda^{12}\big(e^{K/2} W\big) .$$

This implies that W transforms under Kähler transformations according to

$$\delta W = -\Lambda(\phi) W . \tag{6.64}$$

Imposing the equations (6.40) then leads directly to the following result,

$$\begin{aligned} \partial_i\big(A_1^{11} + A_1^{22} + 2T\big) &= 0 , \\ \partial_i \bar{W} = \partial_{\bar{\imath}} W &= 0 . \end{aligned}$$

Hence, W can be identified as the holomorphic superpotential. Here, T is defined by $T = 2\mathcal{V}^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}$. Gauge covariance of A_1 yields the relations

$$\Theta_{\mathcal{M}\mathcal{N}}\big(X^{\mathcal{N}i}\partial_i W + 4i\,\mathcal{S}^{\mathcal{M}}W\big) = 0\,, \tag{6.65a}$$

$$\Theta_{\mathcal{M}\mathcal{N}}\big(X^{\mathcal{N}i}\partial_i T + X^{\mathcal{N}\bar{\imath}}\partial_{\bar{\imath}} T\big) = 0\,, \tag{6.65b}$$

The tensor A_2 can be derived from (6.38a). Its components are given by

$$A_{2\,i}^{1\,1} = -\frac{1}{2}\partial_i T - \frac{1}{2}e^{K/2}D_i W\,, \tag{6.66a}$$

$$A_{2\,i}^{1\,2} = -\frac{i}{2}\partial_i T - \frac{i}{2}e^{K/2}D_i W\,, \tag{6.66b}$$

$$A_{2\,i}^{21} = \frac{i}{2}\partial_i T - \frac{i}{2}e^{K/2}D_i W\,, \tag{6.66c}$$

$$A_{2\,i}^{22} = -\frac{1}{2}\partial_i T + \frac{1}{2}e^{K/2}D_i W\,. \tag{6.66d}$$

The projection equation for $N = 2$, (6.37a), reads

$$A_{2\,i}^{I\,1} = -i\,A_{2\,i}^{I\,2}\,. \tag{6.67}$$

It can easily be verified that the components of A_2 as given in (6.66) fulfill this projection constraint.

We can check the consistency of our results so far by inserting A_1 and A_2 into the quadratic constraints (6.49). It turns out that the constraints cancel provided that W and T satisfy the condition

$$g^{\bar{\imath}j}\partial_{\bar{\imath}}T D_j W = 2TW\,.$$

with the Kähler covariant derivative $D_i W \equiv \partial_i W + \partial_i K\, W$, which is already implied by (6.64). Indeed, this condition follows directly from (6.65a). Therefore the tensors A_1 and A_2 that we have constructed above satisfy the constraints that we have found in section 2. We refrain from giving the explicit form of the tensor A_3, which can be constructed from (6.38b).

The above analysis shows that there is no restriction on the T-tensor T, and therefore all subgroups of the isometry group of the Kähler manifold are allowed gauge groups, as long as the superpotential W obeys (6.65a).

The scalar potential (6.47) of the gauged theory is given by

$$V = -4g^2\Big(T^2 - g^{i\bar{\imath}}\partial_i T\partial_{\bar{\imath}}T + e^K\,|W|^2 - \frac{1}{4}\,g^{i\bar{\imath}}\,e^K\,D_i W D_{\bar{\imath}}\overline{W}\Big)\,.$$

Note that in three dimensions, the scalar potential is quartic in the moment map $\mathcal{V}$, since the T-tensor T is quadratic in $\mathcal{V}$. This is in contrast with e.g. four dimensions, where the scalar potential is quadratic in $\mathcal{V}$.

We conclude by pointing out that there are, similar to the $N = 1$ case, two kinds of deformations of the original theory. On one hand, there are the gaugings, which are characterized by the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ and on the other hand there are the deformations described by the holomorphic superpotential

W, which are not induced by a gauging. Whereas gaugings are possible for all N, the deformations by a superpotential do not appear for higher N, due to the increasing number of restrictions that supersymmetry imposes on the deformations.

3.3. $N = 9$ supersymmetry

For $N > 4$, it has been shown [96] that the target space manifolds of the ungauged theories are symmetric homogeneous spaces G/H. In this section, we show that the consistency condition for admissible gauge groups can be translated into a projection equation for the embedding tensor Θ. This provides a very efficient way to classify and construct possible gaugings and has first been applied to the maximal gaugings in [97, 98]. Note that the case of $N = 9$ supersymmetry discussed in this section is somewhat special because the group H' is trivial, just as for $N = 16$. This means that the holonomy group is SO(9), as was indicated in table 5 on page 114. However, the discussion presented here can be carried out for general $N > 4$ with only minor modifications [104].

The scalar fields of the $N = 9$ theory can be described by a nonlinear sigma model with target space $\mathrm{F}_{4(-20)}/\mathrm{SO}(9)$. The scalars are combined into a $F_{4(-20)}$-valued matrix $V = V(\phi^i)$, on which the rigid action of $F_{4(-20)}$ is realized by left multiplication, while SO(9) acts as local symmetry from the right. The 52 generators $t^{\mathcal{M}}$ of the $\mathrm{F}_{4(-20)}$ Lie algebra decompose into 36 generators X^{IJ} of SO(9), which is the maximal compact subgroup of $\mathrm{F}_{4(-20)}$, and into 16 noncompact generators Y^A. Under SO(9), the 16 generators Y^A transform in the spinor representation and the various commutators are given by [96]

$$\begin{aligned}
\left[X^{IJ}, X^{KL}\right] &= \delta^{IK} X^{IL} - \delta^{IK} X^{JL} - \delta^{JL} X^{IK} + \delta^{IL} X^{JK} , \\
\left[X^{IJ}, Y_A\right] &= -\frac{1}{2}\Gamma^{IJ}_{AB} Y^A , \\
\left[Y^A, Y^B\right] &= \frac{1}{4}\Gamma^{IJ}_{AB} X^{IJ} ,
\end{aligned}$$

where Γ^{IJ}_{AB} are antisymmetrized products of the SO(9) gamma matrices. The Lie-algebra valued function $V^{-1}\partial_i V$ is decomposed as

$$V^{-1}\partial_i V = \frac{1}{2}\, Q_i^{IJ}\, X^{IJ} + e_i^A\, Y^A ,$$

where Q_i is the composite connection and e_i^A is the vielbein which may be used to convert curved target-space indices into flat SO(9) indices. The target-space metric g_{ij} is defined as

$$g_{ij} = e_i^A\, e_j^B\, \delta_{AB} .$$

The Cartan-Maurer equations (1.16) for Q_i^{IJ} yield

$$\partial_i Q_j^{IJ} - \partial_j Q_i^{IJ} + 2Q_i^{K[I} Q_j^{J]K} = -\frac{1}{2}\Gamma^{IJ}_{AB} e_i^A e_j^B ,$$

and consequently, using (6.3), we identify

$$f_{ij}^{IJ} = -\Gamma^{IJ}_{AB} e_i^A e_j^B .$$

The matter fermion fields are redefined by converting their target-space indices to the spinor indices of SO(9)

$$\chi^A \equiv \frac{1}{N} e_i^B \, \Gamma^I_{BA} \, \chi^{iI} .$$

All the general formulae obtained above may be conveniently translated noting that the projector $\mathbb{P}$ from (6.9) factorizes according to

$$\mathbb{P}^{I\,i}_{J\,j} = \frac{1}{N}\left(g^{ik} e_k^A \Gamma^I_{AC}\right)\left(\Gamma^J_{BC} e_j^B\right) .$$

The variation of the scalar fields V under isometry transformations generated by the $t^{\mathcal{M}}$ define $\mathcal{V}$ as

$$\begin{aligned} V^{-1}\delta_{\mathcal{M}} V \equiv V^{-1} t^{\mathcal{M}} V &\equiv \mathcal{V}^{\mathcal{M}}{}_{\mathcal{N}}\, t^{\mathcal{N}} \qquad (6.68)\\ &= \frac{1}{2}\, \mathcal{V}^{\mathcal{M}}{}_{IJ}\, X^{IJ} + \mathcal{V}^{\mathcal{M}}{}_{A}\, Y^A . \end{aligned}$$

and thus constitute the adjoint representation of G. The variation (6.68) describes the isometry generated by left invariant vector fields

$$X^{\mathcal{M}i} = g^{ij} e_j^A \mathcal{V}^{\mathcal{M}}{}_A ,$$

with $D_i X^{\mathcal{M}j} = 0$ and $\mathcal{S}^{\mathcal{M}IJ} = 0$. For any coset space, the $\mathcal{V}^{IJ}$ take the form [16]

$$\mathcal{V}^{\mathcal{M}IJ} = X^{\mathcal{M}j} Q_j^{IJ} + \mathcal{S}^{\mathcal{M}IJ} = g^{ij} e_i^A \mathcal{V}^{\mathcal{M}}{}_A\, Q_j^{IJ} + \mathcal{S}^{\mathcal{M}IJ} ,$$

where $\mathcal{S}^{IJ}$ is a compensating SO(9) transformation. Therefore, $\mathcal{V}$ is identified with the moment map defined in section 1. Correspondingly, the components of the T-tensor (6.32) are given by

$$T_{IJ,KL} = \mathcal{V}^{\mathcal{M}}{}_{IJ}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{KL} , \qquad (6.69a)$$

$$T_{IJj} = e_j^A\, \mathcal{V}^{\mathcal{M}}{}_{IJ}\Theta_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_A \equiv e_j^A\, T_{IJ,A} , \qquad (6.69b)$$

$$T_{ij} = e_i^A e_j^B \mathcal{V}^{\mathcal{M}}{}_A \Theta \mathcal{V}^{\mathcal{M}}{}_B \equiv e_i^A e_j^B T_{AB} , \qquad (6.69c)$$

i.e. they may be combined into a tensor $T_{\mathcal{A}\mathcal{B}}$, with tangent space indices $\mathcal{A} = (IJ, A)$ by dressing the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ by the matrix $\mathcal{V}$ in the adjoint representation of $F_{4(-20)}$,

$$T_{\mathcal{A}\mathcal{B}} = \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}\, \Theta_{\mathcal{M}\mathcal{N}}\, \mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}} . \qquad (6.70)$$

Let us now discuss the conditions that a group has to satisfy in order to be an admissible gauge group. Recall that all consistency conditions for a supersymmetric gauging can be derived from the single equation (6.53) for the T-tensor, i.e. from the vanishing of the SO(9) representation $\mathbf{495} = \boxplus$ in the tensor $T_{IJ,KL}$. In order to satisfy this equation on the entire scalar manifold $\mathrm{F}_{4(-20)}/\mathrm{SO}(9)$, the structure (6.70) of the T-tensor shows that in fact the entire $\mathrm{F}_{4(-20)}$-orbit of the SO(9) representation **495** must vanish. The tensor $T_{\mathcal{A}\mathcal{B}}$ transforms as the symmetrized tensor product of the 52-dimensional adjoint representations, i.e. explicit form

$$\mathbf{52} \times_{\mathrm{sym}} \mathbf{52} = \mathbf{1} + \mathbf{324} + \mathbf{1053}\,. \tag{6.71}$$

Under SO(9), the above $\mathrm{F}_{4(-20)}$-representations decompose as follows,

$$\begin{aligned} \mathbf{1053} &\to \mathbf{126} + \mathbf{432} + \mathbf{495}\,, && (6.72a)\\ \mathbf{324} &\to \mathbf{1} + \mathbf{9} + \mathbf{16} + \mathbf{44} + \mathbf{126} + \mathbf{128}\,, && (6.72b)\\ \mathbf{1} &\to \mathbf{1}\,. && (6.72c) \end{aligned}$$

A priori, the T-tensor therefore contains the following SO(9) representations,

$$\begin{aligned} T_{IJ,KL} &= \mathbf{1} + \mathbf{44} + \mathbf{126} + \mathbf{495}\,,\\ T_{IJ,A} &= \mathbf{16} + \mathbf{128} + \mathbf{432}\,,\\ T_{A,B} &= \mathbf{1} + \mathbf{9} + \mathbf{126}\,. \end{aligned}$$

The unique irreducible representation of $\mathrm{F}_{4(-20)}$ appearing in the sum in (6.71) which, upon breaking $\mathrm{F}_{4(-20)}$ to SO(9), contains the representation **495** is the **1053** representation. The condition (6.53) is then equivalent to

$$\mathbb{P}_{\mathbf{1053}}\, T_{\mathcal{A}\mathcal{B}} = 0\,. \tag{6.73}$$

Consequently, the T-tensors of the $N = 9$ theory are restricted to the following SO(9) representations

$$\begin{aligned} T_{IJ,KL} &= \mathbf{1} + \mathbf{44} + \mathbf{126}\,, && (6.74a)\\ T_{IJ,A} &= \mathbf{16} + \mathbf{128}\,, && (6.74b)\\ T_{A,B} &= \mathbf{1}' + \mathbf{9} + \mathbf{126}\,. && (6.74c) \end{aligned}$$

The representations $\mathbf{1}$ and $\mathbf{1}'$ which appear in $T_{IJ,KL}$ and $T_{A,B}$, respectively, are not related, since they originate from the different representations of $\mathrm{F}_{4(-20)}$. On the other hand, the two **126** representations both originate from the **324** representation and they are therefore identical.

Moreover, since (6.73) is a $\mathrm{F}_{4(-20)}$-covariant condition, the structure of the T-tensor (6.70) shows that this constraint is further equivalent to

$$\mathbb{P}_{\mathbf{1053}}\, \Theta_{\mathcal{M}\mathcal{N}} = 0\,. \tag{6.75}$$

The underlying group structure thus allows to translate the field dependent form of the consistency condition (6.51) into a single condition for the constant embedding tensor of the gauge group G_0. This projection equation comprises all the consistency conditions of the gauged theory. For a given subgroup $G_0 \subset \mathrm{F}_{4(-20)}$, characterized by its embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$, equation (6.75) provides a simple and efficient criterion to check whether this subgroup may be consistently gauged retaining all supersymmetries. We will refer to the solutions of (6.75) as the admissible gauge groups G_0.

As a direct consequence of the projection equation (6.73), we observe that the Cartan-Killing form of $\mathrm{F}_{4(-20)}$ is a solution to equation (6.75) as it corresponds to the singlet in the decomposition of (6.72). This means that the full global symmetry group $\mathrm{F}_{4(-20)}$ is an admissible gauge group. The potential of the corresponding gauged theory reduces to a constant since all scalars fields can be gauged away.

The components $\Theta_{IJ,KL}$ of the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ describe the possible compact gaugings. They have to satisfy (6.51), and it is therefore of the form

$$\Theta_{IJ,KL} = \theta\, \delta_{IJ}^{KL} + \delta_{[I[K}\, \Xi_{L]J]} + \Xi_{IJKL}\ , \tag{6.76}$$

with a traceless symmetric tensor Ξ_{IJ} and a completely antisymmetric tensor Ξ_{IJKL}. For $N = 9$, the tensor Ξ_{IJKL} corresponds to the **126** representation of SO(9). We conclude that Ξ_{IJKL} has to vanish for compact gaugings, since otherwise according to the decomposition (6.74) the component $\Theta_{A,B}$ would also contain the **126** representation, and the gauging would be noncompact.

Let us now consider in some more detail compact gauge groups with embedding tensor of the form

$$\Theta_{IJ,KL} = \theta\, \delta_{IJ}^{KL} + \delta_{[I[K}\, \Xi_{L]J]}\ . \tag{6.77}$$

It is straightforward to verify, that the choice

$$\Xi_{IJ} = \begin{cases} (2 - 2p/9)\, \delta_{IJ} & \text{for } \ I \le p\,, \\ -2p/9\, \delta_{IJ} & \text{for } \ I > p\,, \end{cases}$$

with $\theta = 2p/9 - 1$, describes the embedding of $\mathrm{SO}(p) \times \mathrm{SO}(9-p) \subset \mathrm{SO}(9)$ as $\Theta = \Theta_{\mathrm{SO}(p)} - \Theta_{\mathrm{SO}(9-p)}$, i.e. the relative coupling constant between the two factors is -1. This ratio is fixed by the requirement that the embedding tensor must take the form (6.77). Likewise, one may check that no product $\mathrm{SO}(p_1) \times \ldots \times \mathrm{SO}(p_n)$ with more than two factors can be embedded into SO(9) with an embedding tensor of the form (6.77). This severely restricts the possible choices of compact gauge groups.

We do not discuss the noncompact gauge groups here, and simply state that the groups $\mathrm{G}_{2(-14)} \times \mathrm{SL}(2)$ and $\mathrm{Sp}(2, 1) \times \mathrm{SU}(2)$, which are maximal subgroups of $\mathrm{F}_{4(-20)}$, are among the admissible gauge groups.

3.4. Stationary solutions

Let us finally discuss possible stationary solutions of gauged supergravity theories in three dimensions. The scalar potential of the gauged Lagrangian is given by (6.47), The condition for stationarity of the scalar potential follows from the variation of the potential, (6.48) and is given by

$$3A_1^{IK}A_{2\,j}^{KJ} + Ng^{mn}A_{3\,jm}^{\,JK}A_{2\,n}^{IK} = 0\,. \tag{6.78}$$

Unbroken supersymmetry implies that the supersymmetry variations of the spinor fields (6.36) vanish in the ground state. For the fermion fields χ^{iI}, this implies

$$A_{2\,i}^{JI}\epsilon^J = 0\,.$$

At the stationary point an eigenvector ϵ of A_2 is necessarily a linear combination of eigenvectors of A_1, as can be seen by contracting (6.78) with ϵ^I. Contracting the quadratic constraint (6.49a) with ϵ^J and substituting the scalar potential (6.47) yields

$$A_1^{IJ}A_1^{JK}\epsilon^K = -\frac{1}{4g^2}V_0\epsilon^I\,, \tag{6.79}$$

where V_0 is the value of the potential at the stationary point. Therefore the eigenvalues of A_1 are given by

$$4\,\alpha^2 = -\frac{V_0}{g^2}\,.$$

Consequently, the number of unbroken supersymmetries at a stationary point is given by the number of zero eigenvalues of A_2 and the corresponding A_1 eigenvalues α, given by the above condition. In particular, preserving all N supersymmetries is equivalent to imposing $A_{2\,j}^{IJ} = 0$. In that case, the potential V simplifies to

$$V = -\frac{4g^2}{N}A_1^{IJ}A_1^{IJ}\,,$$

which is a non-positive function, since it is quadratic in A_1^{IJ}. The Einstein equations derived from the gauged Lagrangian are given by

$$R_{\mu\nu} = 2\;V_0\;g_{\mu\nu}\,,$$

and therefore the maximally supersymmetric stationary solutions with $A_{2i}^{IJ} = 0$ correspond to maximally symmetric spaces with non-positive cosmological constant, which are anti-de Sitter spaces, or Minkowski-space in the limit of $V_0 = 0$. As we have explained in chapter 2, a supersymmetric ground-state solution is only possible if the cosmological constant is negative or zero.

Let us conclude this chapter with a summary of the main results. We have constructed the possible gaugings for extended supergravity theories in three dimensions. We found that the viability of a gauging is encoded in a single constraint on the T-tensor, cf. (6.43). For $N > 4$ the target-space of the nonlinear sigma model is a homogeneous symmetric space G/H, and the constraint on the T-tensor can be lifted to a projection equation for the embedding tensor Θ, cf. (6.73). For the special cases of $N = 1$ and $N = 2$ we found that all subgroups of the target-space isometries can be gauged. Additionally, there are deformations of the $N = 1$ and $N = 2$ theories that are not induced by gaugings.

References

[1] W. Pauli, "The connection between spin and statistics," *Phys. Rev.* 55 (1940) 716–722.

[2] S. R. Coleman and J. Mandula, "All possible symmetries of the S matrix," *Phys. Rev.* 159 (1967) 1251–1256.

[3] Y. A. Gol'fand and E. P. Likhtman, "Extension of the algebra of Poincaré group generators and violation of P invariance," *JETP Lett.* 13 (1971) 323–326.

[4] D. V. Volkov and V. P. Akulov, "Possible universal neutrino interaction," *JETP Lett.* 16 (1972) 438–440.

[5] A. Neveu and J. H. Schwarz, "Factorizable dual model of pions," *Nucl. Phys.* B31 (1971) 86–112.

[6] P. Ramond, "Dual theory for free fermions," *Phys. Rev.* D3 (1971) 2415–2418.

[7] J. Wess and B. Zumino, "A Lagrangian model invariant under supergauge transformations," *Phys. Lett.* B49 (1974) 52.

[8] J. Wess and B. Zumino, "Supergauge invariant extension of quantum electrodynamics," *Nucl. Phys.* B78 (1974) 1.

[9] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara, "Progress toward a theory of supergravity," *Phys. Rev.* D13 (1976) 3214–3218.

[10] S. Deser and B. Zumino, "Consistent supergravity," *Phys. Lett.* B62 (1976) 335.

[11] M. F. Sohnius, "Introducing supersymmetry," *Phys. Rept.* 128 (1985) 39–204.

[12] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*. Cambridge University Press, 1987.

[13] D. Lüst and S. Theisen, "Lectures on string theory," *Lect. Notes Phys.* 346 (1989) 1–346.

[14] J. Polchinski, *String Theory*. Cambridge University Press, 1998.

[15] S. Deser, J. H. Kay, and K. S. Stelle, "Renormalizability properties of supergravity," *Phys. Rev. Lett.* 38 (1977) 527.

[16] B. de Wit, "Supergravity," hep-th/0212245.

[17] E. Cremmer, B. Julia, and J. Scherk, "Supergravity theory in 11 dimensions," *Phys. Lett.* B76 (1978) 409–412.

[18] P. A. M. Dirac, "The electron wave equation in de Sitter space," *Ann. Math.* 36 (1935) 657.

[19] P. A. M. Dirac, "A remarkable respresentation of the 3+2 de Sitter group," *J. Math. Phys.* 4 (1963) 901–909.

[20] C. Fronsdal, "Elementary particles in a curved space," *Rev. Mod. Phys.* 37 (1965) 221.

[21] C. Fronsdal, "Elementary particles in a curved space. II," *Phys. Rev.* D10 (1974) 589–598.

[22] C. Fronsdal and R. B. Haugen, "Elementary particles in a curved space. III," *Phys. Rev.* D12 (1975) 3810–3818.

[23] C. Fronsdal, "Elementary particles in a curved space. IV. Massless particles," *Phys. Rev.* D12 (1975) 3819.

[24] S. J. Avis, C. J. Isham, and D. Storey, "Quantum field theory in anti-de Sitter space-time," *Phys. Rev.* D18 (1978) 3565.

[25] D. W. Düsedau and D. Z. Freedman, "Lehmann spectral representation for anti-de Sitter quantum field theory," *Phys. Rev.* D33 (1986) 389.

[26] D. Z. Freedman, "Supergravity with axial-gauge invariance," *Phys. Rev.* D15 (1977) 1173.
[27] D. Z. Freedman and A. Das, "Gauge internal symmetry in extended supergravity," *Nucl. Phys.* B120 (1977) 221.
[28] D. Z. Freedman and J. H. Schwarz, "$N = 4$ supergravity theory with local SU(2) $\times$ SU(2) invariance," *Nucl. Phys.* B137 (1978) 333.
[29] B. de Wit and H. Nicolai, "Extended supergravity with local SO(5) invariance," *Nucl. Phys.* B188 (1981) 98.
[30] B. de Wit and H. Nicolai, "$N = 8$ supergravity with local SO(8) $\times$ SU(8) invariance," *Phys. Lett.* B108 (1982) 285.
[31] B. de Wit and H. Nicolai, "$N = 8$ supergravity," *Nucl. Phys.* B208 (1982) 323.
[32] J. Gates, S. James and B. Zwiebach, "Gauged $N = 4$ supergravity theory with a new scalar potential," *Phys. Lett.* B123 (1983) 200.
[33] M. Pernici, K. Pilch, and P. van Nieuwenhuizen, "Gauged maximally extended supergravity in seven dimensions," *Phys. Lett.* B143 (1984) 103.
[34] C. M. Hull, "More gaugings of $N = 8$ supergravity," *Phys. Lett.* B148 (1984) 297–300.
[35] F. Giani, M. Pernici, and P. van Nieuwenhuizen, "Gauged $N = 4$ $d = 6$ supergravity," *Phys. Rev.* D30 (1984) 1680.
[36] M. Günaydin, L. J. Romans, and N. P. Warner, "Compact and noncompact gauged supergravity theories in five dimensions," *Nucl. Phys.* B272 (1986) 598.
[37] P. Breitenlohner and D. Z. Freedman, "Stability in gauged extended supergravity," *Ann. Phys.* 144 (1982) 249.
[38] W. Heidenreich, "All linear unitary irreducible representations of de Sitter supersymmetry with positive energy," *Phys. Lett.* B110 (1982) 461–464.
[39] D. Z. Freedman and H. Nicolai, "Multiplet shortening in Osp(N,4)," *Nucl. Phys.* B237 (1984) 342.
[40] H. Nicolai, "Representations of supersymmetry in anti-de Sitter space," in *Supersymmetry and Supergravity '84. Proceedings of the Trieste Spring School*, B. de Wit, P. Fayet, and P. van Nieuwenhuizen, eds. World Scientific, 1984.
[41] M. Günaydin and N. P. Warner, "Unitary supermultiplet of Osp(8|4, $\mathbb{R}$) and the spectrum of the S^7 compactification of eleven-dimensional supergravity," *Nucl. Phys.* B272 (1986) 99.
[42] M. Günaydin, P. van Nieuwenhuizen, and N. P. Warner, "General construction of the unitary representations of anti-de Sitter superalgebras and the spectrum of the S^4 compactification of eleven-dimensional supergravity," *Nucl. Phys.* B255 (1985) 63.
[43] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* 2 (1998) 231–252, hep-th/9711200.
[44] E. Witten, "Anti-de Sitter space and holography," *Adv. Theor. Math. Phys.* 2 (1998) 253–291, hep-th/9802150.
[45] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," *Phys. Lett.* B428 (1998) 105–114, hep-th/9802109.
[46] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* 323 (2000) 183–386, hep-th/9905111.
[47] B. de Wit and I. Herger, "Anti-de Sitter supersymmetry," *Lect. Notes Phys.* 541 (2000) 79–100, hep-th/9908005.
[48] E. Inönü and E. P. Wigner, "On the contraction of groups and their represenations," *Proc. Nat. Acad. Sci.* 39 (1953) 510–524.
[49] K. Pilch and A. N. Schellekens, "Formulae for the eigenvalues of the Laplacian on tensor harmonics on symmetric coset spaces," *J. Math. Phys.* 25 (1984) 3455.
[50] W. Nahm, "Supersymmetries and their representations," *Nucl. Phys.* B135 (1978) 149.
[51] S. Ferrara, "Algebraic properties of extended supergravity in de-Sitter space," *Phys. Lett.* B69 (1977) 481.

[52] B. de Wit and A. Zwartkruis, "SU(2,2/1,1) supergravity and $N = 2$ supersymmetry with arbitrary cosmological constant," *Class. Quant. Grav.* 4 (1987) L59.
[53] P. K. Townsend, "Cosmological constant in supergravity," *Phys. Rev.* D15 (1977) 2802–2804.
[54] S. Deser and B. Zumino, "Broken supersymmetry and supergravity," *Phys. Rev. Lett.* 38 (1977) 1433.
[55] B. de Wit, "Multiplet calculus," in *Supersymmetry and Supergravity '82. Proceedings of the Trieste Spring School*, S. Ferrara, J. Taylor, and P. van Nieuwenhuizen, eds. World Scientific, 1983.
[56] M. Günaydin and C. Saclioglu, "Oscillator like unitary representations of noncompact groups with a Jordan structure and the noncompact groups of supergravity," *Commun. Math. Phys.* 87 (1982) 159.
[57] S. J. Gates, H. Nishino, and E. Sezgin, "Supergravity in $d = 9$ and its coupling to the non-compact σ model," *Class. Quant. Grav.* 3 (1986) 21.
[58] N. Khviengia and Z. Khviengia, "$D = 9$ supergravity and p-brane solitons," hep-th/9703063.
[59] E. Bergshoeff, C. M. Hull, and T. Ortin, "Duality in the type II superstring effective action," *Nucl. Phys.* B451 (1995) 547–578, hep-th/9504081.
[60] P. Meessen and T. Ortin, "An $\mathrm{SL}(2,\mathbb{Z})$ multiplet of nine-dimensional type II supergravity theories," *Nucl. Phys.* B541 (1999) 195–245, hep-th/9806120.
[61] M. Abou-Zeid, B. de Wit, D. Lüst, and H. Nicolai, "Space-time supersymmetry, IIA/B duality and M-theory," *Phys. Lett.* B466 (1999) 144–152, hep-th/9908169.
[62] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Supermembranes and eleven-dimensional supergravity," *Phys. Lett.* B189 (1987) 75–78.
[63] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Properties of the eleven-dimensional super membrane theory," *Ann. Phys.* 185 (1988) 330.
[64] P. K. Townsend, "P-brane democracy," hep-th/9507048.
[65] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, "Kaluza-Klein supergravity," *Phys. Rept.* 130 (1986) 1–142.
[66] B. S. DeWitt, "Dynamical theory of groups and fields," in *Relativity, Groups and Topology*, C. DeWitt and B. S. DeWitt, eds., pp. 587–822. 1964.
[67] E. Cremmer and B. Julia, "The SO(8) supergravity," *Nucl. Phys.* B159 (1979) 141.
[68] P. K. Townsend, K. Pilch, and P. van Nieuwenhuizen, "Selfduality in odd dimensions," *Phys. Lett.* 136B (1984) 38.
[69] B. de Wit and D. Lüst, "BPS amplitudes, helicity supertraces and membranes in M- theory," *Phys. Lett.* B477 (2000) 299–308, hep-th/9912225.
[70] L. Brink, J. H. Schwarz, and J. Scherk, "Supersymmetric Yang-Mills theories," *Nucl. Phys.* B121 (1977) 77.
[71] J. S. Schwinger, "On gauge invariance and vacuum polarization," *Phys. Rev.* 82 (1951) 664–679.
[72] I. L. Buchbinder and E. A. Ivanov, "Exact $N = 4$ supersymmetric low-energy effective action in $N = 4$ super Yang-Mills theory," hep-th/0211067.
[73] B. de Wit and I. Herger, "Field theory of $N = 4$ BPS supermultiplets." In preparation.
[74] E. Bergshoeff, M. de Roo, and B. de Wit, "Extended conformal supergravity," *Nucl. Phys.* B182 (1981) 173.
[75] A. H. Chamseddine, "Interacting supergravity in ten-dimensions: The role of the six-index gauge field," *Phys. Rev.* D24 (1981) 3065.
[76] A. Van Proeyen, "Massive vector multiplets in supergravity," *Nucl. Phys.* B162 (1980) 376.
[77] M. T. Grisaru, M. Roček, and W. Siegel, "Zero three loop beta function in $N = 4$ super Yang-Mills theory," *Phys. Rev. Lett.* 45 (1980) 1063–1066.

[78] D. R. T. Jones, “Charge renormalization in a supersymmetric Yang-Mills theory,” *Phys. Lett.* B72 (1977) 199.
[79] M. Dine and N. Seiberg, “Comments on higher derivative operators in some SUSY field theories,” *Phys. Lett.* B409 (1997) 239–244, hep-th/9705057.
[80] B. de Wit, M. T. Grisaru, and M. Roček, “Nonholomorphic corrections to the one-loop $N = 2$ super Yang-Mills action,” *Phys. Lett.* B374 (1996) 297–303, hep-th/9601115.
[81] E. S. Fradkin and A. A. Tseytlin, “Quantum properties of higher dimensional and dimensionally reduced supersymmetric theories,” *Nucl. Phys.* B227 (1983) 252.
[82] S. Kovacs, “A perturbative re-analysis of $N = 4$ supersymmetric Yang-Mills theory,” hep-th/9902047.
[83] B. de Wit and J. Smith, *Field theory in particle physics. Vol. 1*. North-Holland, 1986.
[84] B. S. DeWitt, “Quantum theory of gravity. II. the manifestly covariant theory,” *Phys. Rev.* 162 (1967) 1195–1239.
[85] L. F. Abbott, “The background field method beyond one loop,” *Nucl. Phys.* B185 (1981) 189.
[86] G. ’t Hooft, “Renormalizable Lagrangians for massive Yang-Mills fields,” *Nucl. Phys.* B35 (1971) 167–188.
[87] M. R. Brown and M. J. Duff, “Exact results for effective Lagrangians,” *Phys. Rev.* D11 (1975) 2124–2135.
[88] M. J. Duff and M. Ramon-Medrano, “On the effective Lagrangian for the Yang-Mills field,” *Phys. Rev.* D12 (1975) 3357.
[89] T. Hagiwara, “An effective Lagrangian for multi-photon processes and a nonlinear Born-Infeld Lagrangian,” *Nucl. Phys.* B189 (1981) 135.
[90] P. Di Vecchia, R. Musto, F. Nicodemi, and R. Pettorino, “The anomaly term in the $N = 2$ supersymmetric gauge theory,” *Nucl. Phys.* B252 (1985) 635.
[91] J. Wess and B. Zumino, “Consequences of anomalous Ward identities,” *Phys. Lett.* B37 (1971) 95.
[92] E. Witten, “Global aspects of current algebra,” *Nucl. Phys.* B223 (1983) 422–432.
[93] C. Bachas and E. Kiritsis, “F^4 terms in $N = 4$ string vacua,” *Nucl. Phys. Proc. Suppl.* 55B (1997) 194–199, hep-th/9611205.
[94] M. Bianchi and J. F. Morales, “RG flows and open/closed string duality,” *JHEP* 08 (2000) 035, hep-th/0006176.
[95] N. A. Obers and B. Pioline, “Eisenstein series and string thresholds,” *Commun. Math. Phys.* 209 (2000) 275–324, hep-th/9903113.
[96] B. de Wit, A. K. Tollstén, and H. Nicolai, “Locally supersymmetric $d = 3$ nonlinear sigma models,” *Nucl. Phys.* B392 (1993) 3–38, hep-th/9208074.
[97] H. Nicolai and H. Samtleben, “Maximal gauged supergravity in three dimensions,” *Phys. Rev. Lett.* 86 (2001) 1686–1689, hep-th/0010076.
[98] H. Nicolai and H. Samtleben, “Compact and noncompact gauged maximal supergravities in three dimensions,” *JHEP* 04 (2001) 022, hep-th/0103032.
[99] H. Nicolai and H. Samtleben, “$N = 8$ matter coupled AdS_3 supergravities,” *Phys. Lett.* B514 (2001) 165–172, hep-th/0106153.
[100] N. S. Deger, A. Kaya, E. Sezgin, and P. Sundell, “Matter coupled AdS_3 supergravities and their black strings,” *Nucl. Phys.* B573 (2000) 275–290, hep-th/9908089.
[101] M. Abou-Zeid and H. Samtleben, “Chern-Simons vortices in supergravity,” *Phys. Rev.* D65 (2002) 085016, hep-th/0112035.
[102] P. Ginsparg, “Applied conformal field theory,” hep-th/9108028.
[103] A. N. Schellekens, “Introduction to conformal field theory,” *Fortsch. Phys.* 44 (1996) 605–705.
[104] B. de Wit, I. Herger, and H. Samtleben, “Gauged locally supersymmetric $D = 3$ nonlinear sigma models.” In preparation.

[105] A. Achucarro and P. K. Townsend, "A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories," *Phys. Lett.* B180 (1986) 89.

[106] D. V. Alekseevskii, "Riemannian spaces with unusual holonomy groups," *Funct. Anal. Appl.* 2 (1968) 97–105.

[107] B. de Wit, H. Samtleben, and M. Trigiante, "On Lagrangians and gaugings of maximal supergravities," *Nucl. Phys.* B655 (2003) 93–126, hep-th/0212239.

[108] K. Skenderis and P. K. Townsend, "Gravitational stability and renormalization-group flow," *Phys. Lett.* B468 (1999) 46–51, hep-th/9909070.

[109] J. Bagger and E. Witten, "The gauge invariant supersymmetric nonlinear sigma model," *Phys. Lett.* B118 (1982) 103–106.

[110] J. A. Bagger, "Coupling the gauge invariant supersymmetric nonlinear sigma model to supergravity," *Nucl. Phys.* B211 (1983) 302.

[111] E. Cremmer, S. Ferrara, L. Girardello, and A. Van Proeyen, "Yang-Mills theories with local supersymmetry: Lagrangian, transformation laws and superhiggs effect," *Nucl. Phys.* B212 (1983) 413.

Samenvatting

Supersymmetrie speelt een belangrijke rol in de hedendaagse theoretische fysica. De quantummechanica kent twee fundamenteel verschillende soorten deeltjes, namelijk bosonen en fermionen, welke door supersymmetrie met elkaar in verbinding gebracht worden. Supersymmetrie is een belangrijk onderdeel van supergravitatie en van de verschillende stringtheorieën. De theorie die alle superstring theorieën verenigt is de zogenaamde M-theorie.

Op dit moment is niet bekend wat de fundamentele vrijheidsgraden van M-theorie zijn. De theorie is alleen in bepaalde limieten gedefinieerd. In die limieten komt M-theorie overeen met de bekende string– of supergravitatie-theorieën. Als we alleen maar naar kleine energieën kijken, i.e. als de energie veel kleiner is dan de Planck massa $M_P \sim 10^{19}$ GeV/c^2, dan kunnen we M-theorie beschrijven als een supersymmetrische ijktheorie en als supergravitatie theorie. Om de juiste beschrijving van M-theorie te vinden is het belangrijk om nieuwe eigenschappen van de effectieve lage-energie theorieën te bestuderen. Dit proefschrift gaat over twee verschillende uitbreidingen van supergravitatie-theorieën, die we bestuderen om mogelijk nieuwe feiten over M-theorie te leren kennen. Een eerste uitbreiding van supergravitatie-theorieën is het koppelen van de massaloze toestanden van de oorsponkelijke theorie aan massieve toestanden. Een tweede soort uitbreiding komt tot stand door de abelse ijkgroep van een supergravitatie theorie niet-abels te maken.

Dit proefschrift bevat de volgende hoofdstukken. In hoofdstuk 1 introduceren we supersymmetrie en supergravitatie. Belangrijke concepten van supersymmetrie en supergravitatie worden besproken die in de rest van het proefschrift weer terug komen. Hoofdstuk 2 bevat een pedagogische inleiding in supersymmetrie in anti-de Sitter ruimtes. Omdat anti-de Sitter ruimtes een kromming hebben vertoont supersymmetrie eigenschappen die niet voorkomen bij gewone supersymmetrie in een vlakke ruimte. De theorie bevat bijvoorbeeld zogenaamde singletons, die veel minder vrijheidsgraden bevatten dan een gewoon lokaal veld. Anti-de Sitter ruimtes komen ook voor als supersymmetrische oplossingen van supergravitatie theorieën in drie dimensies, welke in hoofdstuk 6 beschreven worden.

In hoofdstuk 3 bediscussiëren we supergravitatie in negen-dimensionale ruimte-tijd gekoppeld aan verschillende soorten BPS multipletten, de zogenaamde KKA en KKB multipletten. Deze BPS multipletten ontstaan in de compactificatie van supergravitatie in tien of elf ruimte-tijd dimensies. Vanuit het oogpunt van elf-dimensionale supergravitatie bevat het KKA multiplet Kaluza-Klein toestanden en het KKB multiplet bevat windingstoestanden van de supermembranen. De interpretatie van de BPS multipletten is verschillend vanuit het oogpunt van IIB supergravitatie in tien dimensies. De toestanden in het KKB multiplet komen overeen met Kaluza-Klein toestanden en de toestanden in het KKA multiplet komen overeen met windingstoestanden van de fundamentele string en de D-string.

De expliciete constructie van een supergravitatie actie met de twee BPS multipletten is zeer ingewikkeld. Daarom onderzoeken we in hoofdstuk 5 een eenvoudiger voorbeeld van een BPS uitbreiding, namelijk de vier-dimensionale supersymmetrische Yang-Mills theorie. Dit voorbeeld is veel eenvoudiger omdat de theorie alleen globale supersymmetrie transformaties bevat en geen lokale supersymmetrie transformaties zoals een supergravitatie theorie. Wij construeren expliciet de $\mathcal{N} = 4$ theorie gekoppeld aan BPS multipletten die bijvoorbeeld van de compactificatie of van een spontane breking van een ijksymmetrie afkomen. De BPS velden worden vervolgens uit geïntegreerd, en op die manier krijgen we een effectieve actie voor de massaloze velden.

In hoofdstuk 6 geven we tenslotte een algemene classificatie van geijkte supergravitatie theorieën in drie dimensies. We stellen een criterium op voor toelaatbare ijkgroepen, en we bediscussiëren de consequenties daarvan aan de hand van $N = 1$, $N = 2$ en $N = 9$ supersymmetrie.

Curriculum Vitae

I was born on 2 November 1971 in Fribourg, Switzerland. After my secondary school years at the Kantonsschule Zug, I moved to Basel in 1992 to study physics. During my studies, I spent one year as an exchange student at the State University of Saint-Petersburg, Russia. My diploma thesis about quantum mechanics in topologically non-trivial spaces was carried out under the supervision of Dr. Stefan Weigert in the group of Prof. Dr. Harry Thomas at the Institute for Theoretical Physics at Basel University and lead to the diploma for theoretical physics in 1997. With the aid of a TMR scholarship by the Swiss National Science Foundation I started working as an "Assistent in Opleiding" (AIO) at Utrecht University in 1998, under the supervision of Prof. Dr. Bernard de Wit. The results of the research carried out during the last five year are presented in this thesis. During my thesis I was able to attend many schools and conferences, most notably in Corfu, Trieste, Paris, Berlin, and Mumbai.