SPIN-2001/11
ITP-UU-01/18
hep-th/0105105

QUANTUM MECHANICS AND DETERMINISM*

Gerard 't Hooft

Institute for Theoretical Physics
Utrecht University, Leuvenlaan 4
3584 CC Utrecht, the Netherlands

and

Spinoza Institute
Postbox 80.195
3508 TD Utrecht, the Netherlands

e-mail: `g.thooft@phys.uu.nl`
internet: `http://www.phys.uu.nl/~thooft/`

Abstract

It is shown how to map the quantum states of a system of free scalar particles one-to-one onto the states of a completely deterministic model. It is a classical field theory with a large (global) gauge group. The mapping is now also applied to free Maxwell fields. Lorentz invariance is demonstrated.

* Presented at PASCOS 2001, Eighth International Symposium on Particles, Strings and Cosmology, University of North Carolina at Chapel Hill, April 10-15, 2001.

1. Introduction.

Numerous attempts to reconcile General Relativity with Quantum Mechanics appear to lead to descriptions of space and time at the Planck scale where notions such as locality, unitarity and causality are in jeopardy. In particular Super String theories only allow for computations of on-shell amplitudes, so that the local nature of the laws of physics becomes obscure. A further complication is the holographic principle†, which states that the total dimensionality of Hilbert space is controlled rather by the surface than by the volume of a given region in space.

As this situation is fundamentally different from what we have both in non-relativistic Quantum Mechanics and in relativistic quantum field theories, including the Standard Model, it appears to be quite reasonable to reconsider the foundations of Quantum Mechanics itself. Exactly to what extent a completely deterministic hidden variable theory is feasible is still not understood; leaving this question aside for the time being, it may nevertheless be instructive to formulate some general and useful mathematical starting points.

In a previous paper [3] it was shown how a quantum field theory of free scalar particles can be mapped onto a classical field theory. The ontological states of the classical theory form the basis of a Hilbert space, and these states evolve in accordance with a Schrödinger equation that coincides with the Schrödinger equation of the quantum theory [4]. Thus, the classical model can be used to 'interpret' the quantum mechanics of the quantum theory. The price one pays is two-fold: first, a large invariance group ('gauge group') must be introduced in the classical system, and we must restrict ourselves to those observables which are invariant under the transformations of this group. Since the gauge transformations are non-local, the observables are not obviously well-defined locally. Secondly, the procedure is known to work only in some very special cases: either massless non-interacting fermions (in ≤ 4 space-time dimensions), or free scalar particles (massless or massive, in any number of space-time dimensions).

There are reasons to suspect however that more general models might exist that allow such mappings, and it may not be entirely unreasonable to conjecture that physically reasonable models will eventually be included. In this contribution, we show how to handle free Maxwell fields, a result that is not quite trivial because of our desire to keep rotational covariance.

Secondly, we show how to handle Lorentz transformations. The examples treated here turn out to be Lorentz-invariant classical theories. For the Maxwell case, this is not a trivial derivation, requiring an interplay between our ontological gauge group and the Weyl gauge group.

In Sects. 2 and 3, we briefly resume the argument from Ref [3] that the quantum harmonic oscillator corresponds to classical circular motion, and that free scalar quantized fields can be linked to a classical theory with a special kind of gauge invariance, such that

† The general notion underlying this principle can be found in Ref [1]. The word "holography" was mentioned in print, I think, first in Ref [2].

all its Fourier modes are purely circular degrees of freedom.

Lorentz transformations are discussed in Sect. 4. The Maxwell case is handled in Sect. 5. We add a brief discussion of the situation for fermions in Sect. 6, and the interpretation of our gauge transformations in terms of a dissipation theory (Sect. 7).

2. Harmonic oscillators.

We start with a *deterministic* system, consisting of a set of N states,

$$\{(0),(1),\cdots,(N-1)\}$$

on a circle. Time is discrete, the unit time steps having length τ (the continuum limit is left for later). The evolution law is:

$$t\rightarrow t+\tau \quad : \qquad (\nu)\rightarrow(\nu+1 \mod N)\,. \tag{2.1}$$

Introducing a basis for a Hilbert space spanned by the states (ν), the evolution operator can be written as

$$U(\Delta t=\tau)\ =\ e^{-iH\tau}\ = e^{-\frac{\pi i}{N}}\cdot\begin{pmatrix}0 & & & & 1\\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0\end{pmatrix}\,. \tag{2.2}$$

The phase factor in front of the matrix is of little importance; it is there for future convenience. Its eigenstates are denoted as $|n\rangle$, $n=0,\cdots,N-1$.

This law can be represented by a Hamiltonian using the notation of quantum physics:

$$H|n\rangle=\frac{2\pi(n+\frac{1}{2})}{N\tau}|n\rangle\,. \tag{2.3}$$

The $\frac{1}{2}$ comes from the aforementioned phase factor. Next, we apply the algebra of the $SU(2)$ generators L_x, L_y and L_z, so we write

$$N\stackrel{\rm def}{=}2\ell+1 \quad , \qquad n\stackrel{\rm def}{=}m+\ell \quad , \qquad m=-\ell,\cdots,\ell\,. \tag{2.4}$$

Using the quantum numbers m rather than n to denote the eigenstates, we have

$$H|m\rangle=\frac{2\pi(m+\ell+\frac{1}{2})}{(2\ell+1)\tau}|m\rangle \qquad \hbox{or} \qquad H=\tfrac{2\pi}{(2\ell+1)\tau}\left(L_z+\ell+\tfrac{1}{2}\right)\,. \tag{2.5}$$

This Hamiltonian resembles the harmonic oscillator Hamiltonian with angular frequencies $\omega=2\pi/(2\ell+1)\tau$, except for the fact that there is an upper bound for the energy. This upper bound disappears in the continuum limit, if $\ell\to\infty$, $\tau\downarrow 0$. Using L_x and L_y, we can make the correspondence more explicit. Write

$$\begin{aligned} L_\pm|m\rangle &\stackrel{\text{def}}{=} \sqrt{\ell(\ell+1)-m(m\pm 1)}\ |m\pm 1\rangle\ ; \\ L_\pm &\stackrel{\text{def}}{=} L_x\pm iL_y \quad ; \qquad [L_i,L_j]=i\varepsilon_{ijk}L_k\ , \end{aligned} \tag{2.6}$$

and define

$$\hat{x}\stackrel{\text{def}}{=}\alpha L_x \quad , \qquad \hat{p}\stackrel{\text{def}}{=}\beta L_y \quad ; \qquad \alpha\stackrel{\text{def}}{=}\sqrt{\frac{\tau}{\pi}} \quad , \qquad \beta\stackrel{\text{def}}{=}\frac{-2}{2\ell+1}\sqrt{\frac{\pi}{\tau}}\ . \tag{2.7}$$

The commutation rules are

$$[\hat{x},\hat{p}]=\alpha\beta iL_z=i(1-\frac{\tau}{\pi}H)\,, \tag{2.8}$$

and since

$$L_x^2+L_y^2+L_z^2=\ell(\ell+1)\,, \tag{2.9}$$

we have

$$H=\tfrac12\omega^2\hat{x}^2+\tfrac12\hat{p}^2+\frac{\tau}{2\pi}\left(\frac{\omega^2}{4}+H^2\right)\ . \tag{2.10}$$

The coefficients α and β in Eqs. (2.7) have been tuned to give (2.8) and (2.10) their most desirable form.

Now consider the continuum limit, $\tau\downarrow 0$, with $\omega=2\pi/(2\ell+1)\tau$ fixed, for those states for which the energy stays limited. We see that the commutation rule (2.8) for $\hat{x}$ and $\hat{p}$ becomes the conventional one, and the Hamiltonian becomes that of the conventional harmonic oscillator. There are no other states than the legal ones, and their energies are bounded, as can be seen not only from (2.10) but rather from the original definition (2.5). Note that, in the continuum limit, both $\hat{x}$ and $\hat{p}$ become continuous operators.

The way in which these operators act on the 'primordial' or 'ontological' states (ν) of Eq. (2.1) can be derived from (2.6) and (2.7), if we realize that the states $|m\rangle$ are just the discrete Fourier transforms of the states (ν). This way, also the relation between the eigenstates of $\hat{x}$ and $\hat{p}$ and the states (ν) can be determined. Only in a fairly crude way, $\hat{x}$ and $\hat{p}$ give information on where on the circle our ontological object is; both $\hat{x}$ and $\hat{p}$ narrow down the value of ν of our states (ν).

The most important conclusion from this section is that there is a close relationship between the quantum harmonic oscillator and the classical particle moving along a circle. The period of the oscillator is equal to the period of the trajectory along the circle. We started our considerations by having time discrete, and only a finite number of states. This is because the continuum limit is a rather delicate one. One cannot directly start with the continuum because then the Hamiltonian does not seem to be bounded from below.

The price we pay for a properly bounded Hamiltonian is the square root in Eq. (2.6); it may cause complications when we attempt to introduce interactions, but this is not the subject of this paper.

3. Free scalar particles.

Now consider the Klein-Gordon equation describing a quantized field ϕ,

$$(\Delta - \mu^2)\phi - \ddot{\phi} = 0\,, \tag{3.2}$$

where the dots refer to time differentiation. It represents coupled harmonic oscillators. We decouple them by diagonalizing the equation, that is, we consider the Fourier modes. For each Fourier mode, with given $\pm\vec{k}$, there are two quantum harmonic oscillators, because the Fourier coefficients are complex.

In principle, the procedure to be followed may appear to be straightforward: introduce a dynamical degree of freedom moving on a circle, with angular frequencies $\omega(\vec{k}) = (\vec{k}^2 + \mu^2)^{1/2}$, for each of these modes. The question is, how to introduce a model for which the Fourier modes contain just such degrees of freedom; an ordinary classical field, to be denoted as $\{\varphi(\vec{x},t)\}$, would contain Fourier modes, $\hat{\varphi}(\vec{k},t)$, which do not only have a circular degree of freedom, but also an amplitude:

$$\varphi(\vec{x},t) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int \mathrm{d}^3\vec{k}\,\hat{\varphi}(\vec{k},t)\,e^{i\vec{k}\cdot\vec{x}}\ . \tag{3.1}$$

$\hat{\varphi}(\vec{k},t)$ are all classical oscillators. They are not confined to the circle, but the real parts, $\Re(\hat{\varphi}(\vec{k},t))$, and the imaginary parts, $\Im(\hat{\varphi}(\vec{k},t))$ of every Fourier mode each move in a two-dimensional phase space. If we want to reproduce the quantum system, we have to replace these two-dimensional phase spaces by one-dimensional circles.

The trick that will be employed is to introduce a new kind of gauge invariance. In a given classical oscillator

$$\dot{x} = y \quad , \qquad \dot{y} = -\omega^2 x\ , \tag{3.2}$$

we want to declare that the amplitude is unobservable; only the *phase* is physical. So, we introduce transformations of the form

$$x \to \Lambda x \quad , \qquad y \to \Lambda y\ , \tag{3.3}$$

where the transformation parameter Λ is a real, positive number.

Now, however, a certain amount of care is needed. We do not want to destroy translation invariance. A space translation would mix up the real and imaginary parts of $\hat{\varphi}(\vec{k},t)$ and $\dot{\hat{\varphi}}(\vec{k},t)$. This is why it is not advised to start from the real part and the imaginary part, and subject these to the transformations (3.3) separately. Rather, we note that, at each $\vec{k}$, there are two oscillatory modes, a positive and a negative frequency. Thus, in general,

$$\begin{aligned}\hat{\varphi}(\vec{k},t) &= A(\vec{k})\,e^{i\omega t} + B(\vec{k})\,e^{-i\omega t}\,;\\ \dot{\hat{\varphi}}(\vec{k},t) &= i\omega A(\vec{k})\,e^{i\omega t} - i\omega B(\vec{k})\,e^{-i\omega t}\,,\end{aligned} \tag{3.4}$$

where $\omega = (\vec{k}^2 + \mu^2)^{1/2}$. It is these amplitudes that we may subject to the transformations (3.3), so that we only keep the circular motions $e^{\pm i\omega t}$.

Thus, we introduce the 'gauge transformation'

$$\begin{aligned} A(\vec{k}) &\to R_1(\vec{k})A(\vec{k}) \ , \\ B(\vec{k}) &\to R_2(\vec{k})B(\vec{k}) \ , \end{aligned} \tag{3.5}$$

where $R_1(\vec{k})$ and $R_2(\vec{k})$ are *real, positive* functions of $\vec{k}$. The *only* quantities invariant under these two transformations are the phases of A and B, which is what we want. In terms of φ and $\dot{\varphi}$, the transformation reads:

$$\begin{aligned} \varphi &\to \tfrac{R_1+R_2}{2}\,\varphi + \tfrac{R_1-R_2}{2i\omega}\,\dot{\varphi} \ , \\ \dot{\varphi} &\to \tfrac{R_1+R_2}{2}\,\dot{\varphi} + \tfrac{i\omega(R_1-R_2)}{2}\,\varphi \ . \end{aligned} \tag{3.6}$$

Writing

$$\begin{aligned} \hat{K}_1(\vec{k}) &= \tfrac{R_1+R_2}{2} \ , \\ \hat{K}_2(\vec{k}) &= \tfrac{R_2-R_1}{2\omega} \ , \end{aligned} \tag{3.7}$$

and Fourier transforming, we see that, in coordinate space,

$$\begin{aligned} \varphi(\vec{x},t) &\to \int \mathrm{d}^3\vec{y}\,\big(K_1(\vec{y})\,\varphi(\vec{x}+\vec{y},t) + K_2(\vec{y})\,\dot{\varphi}(\vec{x}+\vec{y},t)\big)\,, \\ \dot{\varphi}(\vec{x},t) &\to \int \mathrm{d}^3\vec{y}\,\big(K_1(\vec{y})\,\dot{\varphi}(\vec{x}+\vec{y},t) + K_2(\vec{y})\,(\Delta-\mu^2)\varphi(\vec{x}+\vec{y},t)\big)\,. \end{aligned} \tag{3.8}$$

Since R_1 and R_2 are real in $\vec{k}$-space, the kernels K_1 and K_2 obey the constraints:

$$K_1(\vec{y}) = K_1(-\vec{y}) \quad ; \qquad K_2(\vec{y}) = -K_2(-\vec{y})\,, \tag{3.9}$$

and they are time-independent. The classical Klein-Gordon equation is obviously invariant under these transformations. The fact that R_1 and R_2 are constrained to be positive, amounts to limiting oneself to the homogeneous part of the gauge group.[‡]

Note that the requirement that physical observables are invariant under these transformations, essentially reduces the set of physically observable dynamical degrees of freedom by half. This is essentially what Quantum Mechanics does: in Quantum Mechanics, a complete specification of only coordinates *or* only momenta suffices to fix the degrees of freedom at a given time; in a classical theory, one would normally have to specify coordinates as well as momenta.

Although the gauge group is a very large one, it should still be characterized as a *global* gauge group, since the kernels K_i depend on $\vec{y}$ and not on $\vec{x}$, and they are time independent.

[‡] Alternatively, one could consider dropping such limitations, which would require dividing the angular frequencies ω by a factor 2, because then the phase angles are defined *modulo* 180° only. We believe however that the resulting theory will be physically unacceptable.

4. Lorentz invariance.

The gauge transformation (3.8) is non-local in space, but local in time. Hence, one could question whether the selection of gauge invariant observables is Lorentz invariant. Of course, rotational invariance is evident from the notation, even though the kernels $K_i(\vec{y})$ are in general *not* rotationally invariant. The fact that Lorentz invariance still holds, is guaranteed by the equations of motion of the field $\varphi(\vec{x},t)$. This is derived as follows.

Consider an infinitesimal Lorentz rotation in the x-direction, given the fields $\varphi(\vec{x},0)$ and $\dot\varphi(\vec{x},0)$ at $t=0$:

$$\varphi' = \varphi + \varepsilon\delta\varphi \quad , \qquad \dot\varphi' = \dot\varphi + \varepsilon\delta\dot\varphi\,, \tag{4.1}$$

where ε is infinitesimal. Of course, φ transforms as a Lorentz scalar and $\dot\varphi$ as a Lorentz vector. We have

$$\begin{aligned} \delta\varphi &= x\partial_t\varphi + t\partial_x\varphi = x\dot\varphi + t\partial_x\varphi\ ; \\ \delta\dot\varphi &= x\partial_t\dot\varphi + t\partial_x\dot\varphi + \partial_x\varphi = x(\Delta-\mu^2)\varphi + t\partial_x\dot\varphi + \partial_x\varphi\ . \end{aligned} \tag{4.2}$$

This transforms the gauge transformation (3.8) into

$$\begin{aligned} &\phi' \to K_1'\varphi' + K_2'\dot\varphi'\ ; \\ &\dot\varphi' \to K_1'\dot\varphi' + K_2'\varphi'\ . \end{aligned} \tag{4.3}$$

with $K_i' = K_i + \delta K_i$, and

$$\begin{aligned} &\delta K_1 = (\Delta-\mu^2)(x\,K_2 - K_2\,x) - \partial_x K_2\ ; \\ &\delta K_2 = x\,K_1 - K_1\,x\ . \end{aligned} \tag{4.4}$$

In these expressions, $K_i\varphi$ stands short for $\int \mathrm{d}^3\vec{y}\,K_i(\vec{y})\varphi(x{+}\vec{y},t)$, and $x\,K_2{-}K_2\,x$ replaces $K_2(\vec{y})$ by $-y_x\,K_2(\vec{y})$. We easily read off from these expressions that the transformed kernels $K_i'(\vec{y})$ have the correct symmetry properties (3.9) under reflection in $\vec{y}$.

This explicit calculation confirms a more abstract argument, which simply observes that multiplying the amplitudes A and B in Eq. (3.4) by real numbers should be a Lorentz invariant procedure.

We conclude from this section that, if in one Lorentz frame observables are required to be invariant under all gauge rotations (3.8) that obey the symmetry constraints (3.9), then this continues to hold in any other Lorentz frame. The theory is Lorentz invariant.

5. Maxwell fields.

Handling theories with multiples of bosonic fields may seem to be straightforward. In general, however, simple symmetries such as isospin or rotational invariance are not guaranteed. Take the simple example of a vector field $\vec{A}$ in 3-space. We now have three quantum harmonic oscillators for each $\vec{k}$, which would have to be linked to three circular degrees of freedom. This would lead us to a gauge transformation of the form (3.8) for each of the three vector components A_i separately. Clearly this is not rotationally invariant. The kernels $K_i(\vec{y})$ would have to be replaced by symmetric matrices instead, but this would give us a gauge group that is far too large.

In some special cases, this difficulty can be removed, and an example at hand is the Maxwell theory. We consider pure Maxwell fields without any charged sources. Again, we first diagonalize the Hamiltonian by going to Fourier space. As is well-known, the photon has two polarizations, so at every $\vec{k}$, there are now two quantum oscillators with positive ω and two with negative ω.

Let us choose $\vec{k}$ to lie along the z-axis. Then $\hat{A}_x(\vec{k},t)$ and $\hat{A}_y(\vec{k},t)$ should rotate into one another under rotations about the z-axis. Can we choose two rotators in such a way that this rotation symmetry is kept?

Consider again the discretized case, where we have a finite time interval τ. The two rotators have the same angular velocity $\omega = \pm|\vec{k}|$. The Hamiltonian (normalized to $2(\frac{1}{2}\omega)$ for the ground state) is therefore

$$H = |\omega|(L_z^{(1)} + L_z^{(2)} + 2\ell + 1\,. \tag{5.1}$$

These states are illustrated in Fig. 1, for the case $\ell = 3$.

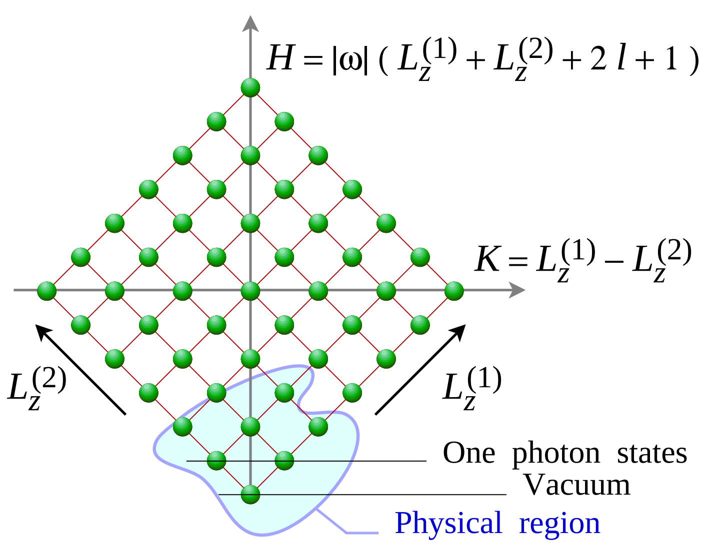

Figure 1. Two-boson states with a $U(1)$ exchange symmetry. H generates time translations: $\theta_1 \to \theta_1 + \omega\tau$, $\theta_2 \to \theta_2 + \omega\tau$, whereas K generates rotations about the $\vec{k}$ axis: $\theta_1 \to \theta_1 + \tau$, $\theta_2 \to \theta_2 - \tau$.

States with $H=(n+1)\omega$ contain n-photons, whose total spin ranges from $-n$ to n. This is the range of the operator $K=L_z^{(1)}-L_z^{(2)}$. Thus, we have to arrange the states in accordance with their spin values along the axis of the momentum variable $\vec{k}$ (which we take to be the z-axis). This way, the energy eigenstates of our circular degrees of freedom will automatically also be eigenstates of transverse rotations. We are lead to consider the field modes $A_\pm$, each of which has positive and negative frequencies:

$$A_\pm \stackrel{\text{def}}{=} A_x \pm iA_y \quad ; \qquad \hat{A}_\pm(\vec{k},t)\rightarrow A_\pm e^{i\omega t}+B_\pm e^{-i\omega t}\,. \tag{5.2}$$

We can be assured that this decomposition is covariant under rotations about the z-axis.

As before, we restrict ourselves to the phase components of these degrees of freedom, while the amplitudes themselves are subject to gauge transformations. These gauge transformations are

$$A_\pm\rightarrow K_\pm A_\pm \quad , \qquad B_\pm\rightarrow L_\pm B_\pm\ , \tag{5.3}$$

with $K_\pm$ and $L_\pm$ both real and positive.

At $t=0$, we get, plugging (5.2) into (5.3):

$$\begin{aligned}
\hat{A}_i &\rightarrow \hat{K}_1\hat{A}_i + i\varepsilon_{ijk}k_j\,\hat{K}_2\hat{A}_k + i\hat{L}_1\dot{\hat{A}}_i - \varepsilon_{ijk}k_j\,\hat{L}_2\dot{\hat{A}}_k\ ;\\
\dot{\hat{A}}_i &\rightarrow -i\omega^2\hat{L}_1\hat{A}_i + \varepsilon_{ijk}k_j\,\omega^2\hat{L}_2\hat{A}_k + \hat{K}_1\dot{\hat{A}}_i + i\varepsilon_{ijk}k_j\,\hat{K}_2\dot{\hat{A}}_k\ ,
\end{aligned} \tag{5.4}$$

where $\omega=|k|$ and

$$\begin{aligned}
\hat{K}_1 &= \tfrac{1}{4}(K_+ + K_+ + L_+ + L_-) \quad , \qquad \hat{K}_2 = \tfrac{1}{4\,\omega}(-K_+ + K_- - L_+ + L_-)\ ,\\
\hat{L}_1 &= \tfrac{1}{4\,\omega}(-K_+ - K_- + L_+ + L_-) \quad , \qquad \hat{L}_2 = \tfrac{1}{4\,\omega^2}(K_+ - K_- - L_+ + L_-)\ .
\end{aligned} \tag{5.5}$$

In coordinate space this transformation reads

$$\begin{aligned}
A_i(\vec{x}) \rightarrow \int \mathrm{d}^3\vec{y}\Big(&K_1(\vec{y})A_i(\vec{x}+\vec{y}) + K_2(\vec{y})\varepsilon_{ijk}\partial_j A_k(\vec{x}+\vec{y})\\
&+ L_1(\vec{y})\dot{A}_i(\vec{x}+\vec{y}) + L_2(\vec{y})\varepsilon_{ijk}\partial_j\dot{A}_k(\vec{x}+\vec{y})\Big)\ ;\\
\dot{A}_i(\vec{x}) \rightarrow \int \mathrm{d}^3\vec{y}\Big(&L_1(\vec{y})\Delta A_i(\vec{x}+\vec{y}) + L_2(\vec{y})\varepsilon_{ijk}\partial_j\Delta A_k(\vec{x}+\vec{y})\\
&+ K_1(\vec{y})\dot{A}_i(\vec{x}+\vec{y}) + K_2(\vec{y})\varepsilon_{ijk}\partial_j\dot{A}_k(\vec{x}+\vec{y})\Big)\ ,
\end{aligned} \tag{5.6}$$

where $K_i(\vec{y})$ and $L_i(\vec{y})$ now obey the following symmetry conditions:

$$K_i(-\vec{y}) = K_i(\vec{y}) \quad , \qquad L_i(-\vec{y}) = -L_i(\vec{y})\ . \tag{5.7}$$

Thus, we indeed obtained a transformation rule that leaves the theory manifestly rotationally covariant.

Lorentz invariance is far less trivial to establish for this system. In general, transversality of the fields is guaranteed only if a Lorentz transformation is associated with a Weyl gauge transformation. Remember that the transformation (5.6) is defined in a given Cauchy surface $t = 0$. So, let us start with a given field $A_i(\vec{x}, t)$ with $\partial_i A_i = 0$ and $A_0 = 0$. After an infinitesimal Lorentz transformation in the z-direction, we have $A_\mu \to A'_\mu = A_\mu + \varepsilon\delta A_\mu$, with ε infinitesimal, and

$$\begin{aligned} \delta A_{1,2} &= z\dot{A}_{1,2} + t\,\partial_3 A_{1,2} + \partial_{1,2}\Lambda \;, \\ \delta A_3 &= z\dot{A}_3 + t\,\partial_3 A_3 + \partial_3\Lambda \;, \\ \delta A_0 &= A_3 + \partial_0\Lambda \;. \end{aligned} \tag{5.8}$$

We find

$$\partial_i \delta A_i = \dot{A}_3 + \Delta\Lambda \,, \tag{5.9}$$

so, since we want the transformed fields to remain transverse, we must choose the gauge transformation Λ to be

$$\Lambda = -\Delta^{-1}\dot{A}_3 \,, \tag{5.10}$$

which also implies

$$\partial_0\Lambda = -\Delta^{-1}\partial_0^2 A_3 = -A_3 \,, \tag{5.11}$$

so that A_0 also vanishes after the transformation. Thus we find, at $t = 0$,

$$\delta A_i = z\dot{A}_i - \Delta^{-1}\partial_i\dot{A}_3 \quad , \qquad \delta\dot{A}_i = z\Delta A_i + \partial_3 A_i - \partial_i A_3 \;. \tag{5.12}$$

Let us write our gauge transformation (5.6) in the following shortcut notation,

$$\begin{aligned} \tilde{A}_i &= K_1 A_i + K_2\,\varepsilon_{ijk}\,\partial_j A_k + L_1\dot{A}_i + L_2\,\varepsilon_{ijk}\,\partial_j\dot{A}_k \;, \\ \dot{\tilde{A}}_i &= L_1\Delta A_i + L_2\,\varepsilon_{ijk}\,\partial_j\Delta A_k + K_1\dot{A}_i + K_2\,\varepsilon_{ijk}\,\partial_j\dot{A}_k \;, \end{aligned} \tag{5.13}$$

After the Lorentz transformation (5.12), we have

$$\begin{aligned} \tilde{A}'_i &= \tilde{A}_i + \varepsilon\delta\tilde{A}_i \quad , \qquad K'_i = K_i + \varepsilon\delta K_i \;, \\ \dot{\tilde{A}}'_i &= \dot{\tilde{A}}_i + \varepsilon\delta\dot{\tilde{A}}_i \quad , \qquad L'_i = L_i + \varepsilon\delta L_i \;, \end{aligned} \tag{5.14}$$

where the $\delta\tilde{A}_i$ and $\delta\dot{\tilde{A}}_i$ obey equations similar to (5.12). After a little algebra, where we have to use the fact that the fields are transverse,

$$\partial_i A_i = \partial_i\dot{A}_i = 0 \,, \tag{5.15}$$

we find that both of the equations (5.13) are obeyed by the Lorentz transformed fields iff

$$\begin{aligned} \delta K_1 &= (z\,L_1 - L_1\,z)\Delta - L_1\,\partial_3 \quad , \qquad \delta L_1 = z\,K_1 - K_1\,z \;, \\ \delta K_2 &= (z\,L_2 - L_2\,z)\Delta - 2L_2\,\partial_3 \quad , \qquad \delta L_2 = z\,K_2 - K_2\,z - K_2\,\Delta^{-1}\partial_3 \;. \end{aligned} \tag{5.16}$$

Eq. (5.16) is written in operator notation. The kernel functions $K_i(\vec{y})$ and $L_i(\vec{y})$ transform by

$$\begin{aligned} \delta K_1(\vec{y}) &= -y_3\Delta L_1(\vec{y}) - \partial_3 L_1(\vec{y}) \quad , \qquad \delta L_1(\vec{y}) = -y_3 K_1(\vec{y}) \ , \\ \delta K_2(\vec{y}) &= -y_3\Delta L_2(\vec{y}) - 2\partial_3 L_2(\vec{y}) \quad , \qquad \delta L_2(\vec{y}) = -y_3 K_2(\vec{y}) - \partial_3\Delta^{-1}K_2(\vec{y}) \ . \end{aligned} \tag{5.17}$$

It is important to note that the transformed kernels still obey the (anti-)symmetry requirements (5.7). This proves that the theory is indeed Lorentz covariant.

6. Massless spinors.

Spinor fields cannot be treated in exactly the same manner. At each Fourier mode of a Dirac field there are only a few quantum degrees of freedom that can take only two values. At time steps separated at a fixed distance $1/\pi\omega$, where $\omega = (\vec{k}^2+\mu^2)^{1/2}$, we may have elements of an ontological basis, but not in between, so, unlike the bosonic case, we cannot take the limit of continuous time.

However, an other approach was described in Ref [5]. There, we note that the first-quantized chiral wave equation can be given an ontological basis. The massless, two-component case is described by the Hamiltonian

$$H = \vec{\sigma}\cdot\vec{p}\ . \tag{6.1}$$

Take the following set of operators,

$$\{\hat{p}, \quad \hat{p}\cdot\vec{\sigma}, \quad \hat{p}\cdot\vec{x}\}\ , \tag{6.2}$$

where $\hat{p}$ stands for the momentum *modulo* its length and *modulo* its sign:

$$\hat{p} = \pm\vec{p}/|p| \quad ; \qquad \hat{p}_x > 0\ . \tag{6.3}$$

These operators all commute with one another. Only the commutator $[\hat{p}\cdot\vec{x},\ \hat{p}]$ requires some scrutiny. In momentum space we see

$$[\vec{p}\cdot\vec{x},\ \hat{p}] = i\left(\vec{p}\cdot\frac{\partial}{\partial\vec{p}}\right)\hat{p} = 0\,, \tag{6.4}$$

because $\hat{p}$ has unit length; its length does not change under dilatations. Not only does the set (6.2) commute with one another at fixed time t, the operators commute at all times. This is because $\hat{p}$ and $\hat{p}\cdot\vec{\sigma}$ commute with the Hamiltonian, whereas

$$\hat{p}\cdot\vec{x}(t) = \hat{p}\cdot\vec{x}(0) + \hat{p}\cdot\sigma\, t\ . \tag{6.5}$$

For this reason, the set of basis elements in which the set of operators (6.2) are diagonal, evolve by permutation, and we can say that the evolution of these elements is deterministic.

If the particles do not interact, the numbers of particles and antiparticles are fixed in time, and therefore also the second quantized theory is deterministic. The only caveat here, is the fact that the negative energy states must (nearly) all be occupied; they form the Dirac sea. Since energy is not an ontological observable here, the process of filling the Dirac sea is a delicate one, and therefore it is advised to introduce a cut-off: one may assume that the values of $\hat{p} \cdot \vec{x}$ form a dense but discrete lattice. The time required to hop from one lattice point to the next is then a (tiny) time quantum. If furthermore, at every value of the unit vector $\hat{p}$, we introduce an infrared cut-off as well then the number of states in the Dirac sea is finite, and filling the Dirac sea is straightforward.

As in the bosonic case, discreteness in time leads to an ambiguity in the Hamiltonian due to periodicity. In this case the most obvious choice of the Hamiltonian is the symmetric one, half of the first-quantized states having positive energy and half negative.

We see that there is some similarity with the bosonic case. For fermions, we may consider $\hat{p} \cdot \vec{x}$ to be one of the three coordinates of a 'classical particle', that carries with it the value of $\hat{p}$ as a flag. Replacing the particle by substituting its coordinates $\vec{x}$ by $\vec{x} + \vec{y}$ with $\hat{p} \cdot \vec{y} = 0$ (*i.e.*, a transverse displacement), may here also be seen as a gauge transformation. All observables must be invariant under this gauge transformation.

7. Dissipation.

The classical counterparts of the quantized models discussed here are time-reversible, just as the quantum theories themselves. If we restrict ourselves to gauge-invariant observables, the classical theories described here are mathematically equivalent to the quantum systems, the latter being familiar but very simple quantum field models.

However, one might be inclined to go one step further. The large gauge groups may in practice be difficult to incorporate in more complicated — hence more interesting — settings, ones where interactions and symmetry patterns are more realistic. This is why it may be of importance to find an *interpretation* of our gauge invariance. Consider the set of infinitesimal gauge transformations out of which the finite ones can be generated. It could be that these represent motions that cannot be predicted or tracked back into the past: information concerning these motions is lost [5, 6]. Our fermionic particles, for instance, perform random, uncontrollable steps sideways, while the projection of their motion along the direction of $\hat{p}$ varies uniformly in time.

Viewed from this perspective, it seems that we end up with perfectly reasonable hidden variable theories. Those properties of the system that can be followed during macroscopic time intervals form *equivalence classes* of states.

In the models presented here, the equivalence classes are large, but they typically multiply the number of degrees of freedom with a factor two or so. In quantum gravity, we expect equivalence classes much larger than that. In the vicinity of black holes, only the degrees of freedom at the horizon represent observables; all other degrees of freedom appear not to be independent of those. For the interpretation and understanding of black holes, this has been a formidable obstacle. If, however, the basis of our physical Hilbert space can be seen to be spanned by equivalence classes, then we can simply infer that black holes may also be represented as classical objects, where equivalence classes are large because information is divulged by the horizon.

References.

1. G. 't Hooft, "On the quantization of space and time", in Proc. of the 4th seminar on Quantum Gravity, May 25-29, 1987, Moscow, USSR. Eds. M.A. Markov, World Scientific, Singapore, New Jersey, Hong Kong, 1988, pp. 551-567, and in: *Themes in Contemporary Physics II. Essays in honor of Julian Schwinger's 70th birthday.* Eds. S. Deser and R.J. Finkenstein. World Scientific, Singapore (1989) 77-89.
   See also: G. 't Hooft, "Dimensional reduction in quantum gravity". In *Salam-festschrift: a collection of talks*, World Scientific Series in 20th Century Physics, vol. 4, ed. A. Ali, J. Ellis and S. Randjbar-Daemi (World Scientific, 1993), gr-qc/9310026.
2. L. Susskind, *J. Math. Phys.* **36** (1995) 6377, hep-th/9409089.
3. G. 't Hooft, "Determinism in Free Bosons", SPIN-2001/07, hep-th/0104080.
4. G. 't Hooft, "How Does God Play Dice? (Pre-)Determinism at the Planck Scale", an Essay in honour of John S. Bell, SPIN-2001/09 / ITP-UU-01/15, hep-th/0104219.
5. G. 't Hooft, "Quantum Gravity as a Dissipative Deterministic System", SPIN-1999/07, gr-qc/9903084; *Class. Quant. Grav.* **16** (1999) 3263; "Determinism and Dissipation in Quantum Gravity", presented at *Basics and Highlights in Fundamental Physics*, Erice, August 1999, SPIN-2000/07, hep-th/0003005.
6. M. Blasone, P. Jizba and G. Vitiello, "Dissipation and Quantization", hep-th/0007138.