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QUANTIZATION OF DISCRETE DETERMINISTIC THEORIES BY HILBERT SPACE EXTENSION

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Quantization of a theory usually implies that it is being replaced by a physically different system. In this paper it is pointed out that if a deterministic theory is completely discrete, such as a classical gauge theory on a lattice, with discrete gauge group, then there is an essentially trivial procedure to quantize it. The equations for the evolution of the physical variables are kept unchanged, but are reformulated in terms of the evolution of vectors in a Hilbert space. This transition turns a system into a conventional quantum theory, which may have more symmetries than can be seen in the original classical theory. This is illustrated in a cellular automaton, of which only the quantum version is time-reversal symmetric. Another automaton shows self-duality only after Hilbert space extension.

We discuss the importance of such observations for physics. The procedure can also be used to construct a completely finite and soluble quantum gravity model in 2 + 1 dimensions.

1. Introduction

Investigation of the problem of quantizing black holes has led to the suspicion that space-time at the Planck scale, and all relevant physical variables there, are discrete. The finiteness of the entropy of a black hole implies that the number of bits of information that can be stored there is finite and determined by the area of its horizon [1]. This gave us the idea that Nature at the Planck scale is an information processing machine like a computer, or more precisely, a cellular automaton [2].

Since Quantum Mechanics dominates our present view of the known laws of Nature it is natural to think of such a discrete theory as a *discrete quantum theory*. In some sense all quantum theories are discrete (as the name implies), but what we mean here is that if we want to describe a finite volume of space then the Hilbert space needed is finite dimensional. The evolution of its states is determined by a unitary evolution matrix $U(t_1, t_2)$ (a Schrödinger equation might not exist if also time is discrete). Besides unitarity there is no obvious restriction for the possible forms U can take, so the class of such discrete quantum theories is still a continuous set.

There is a subclass of discrete quantum theories in which we might be particularly interested: the *deterministic discrete theories*. These are theories that have a basis such that in this basis the matrices U are pure permutation matrices: at certain instants $t_1, t_2, t_3, \ldots$ the basis elements are just being permuted: $|e_1\rangle \to |e_2\rangle \to |e_3\rangle \to \ldots$. We will call this special basis the *primitive basis* [3].

In contrast with the generic quantum systems these theories are deterministic, and the most fundamental aspect of quantum mechanics, namely the spreading wave function, has disappeared*. Yet we do have a Hilbert space. Superposition of states is allowed, and the rules for computing Hilbert space matrix elements, *and their physical interpretation in terms of transition probabilities*, are as in conventional Quantum Mechanics.

Given just any discrete quantum theory, it will not be an easy matter to find out whether or not a "primitive basis" exists. In particular, given our incomplete knowledge of the quantum theory of the real world, one might ask the question whether indeed it can be cast into a deterministic frame. Usually, such questions are answered in the negative: in a deterministic theory one has the Bell inequalities [4], which are violated in a quantum world. This author is tempted to put some question marks here. We doubt whether Bell inequalities for the primitive basis elements $|e_1\rangle, |e_2\rangle, \ldots$ would really have to imply Bell inequalities for those objects that we decided to call particles with their presently employed quantum numbers. What we call particles will have to be superpositions of many different primitive basis elements, as will be explained.

In this article we will leave the nasty (and important!) EPR paradox aside and simply study discrete deterministic theories for their own sake. Let us start with a deterministic (or "classical") discrete theory, in which the law of evolution is a simple prescription of how the allowed states of the system are permuted as a function of (discrete) time. We can still talk about a "quantization procedure", although it is really nothing but the formal introduction of a Hilbert space. Each state i of the original system is now associated with a ("primitive") basis element $|e_i\rangle$ of this Hilbert space. If in some time interval δt (out of a discrete set of time intervals) the state i evolves into the state j we define the evolution operator U to give

$$U|e_i\rangle = |e_j\rangle . \tag{1.1}$$

Note that unitarity of U requires** us to limit ourselves to time reversible discrete

* That is in this basis the wave function does not spread. But if any other basis is used the wave function may spread like in any other quantum theory.

** However, with a little bit more work, also non time-reversible discrete theories can be shown to have a quantum mechanical extension. What one gets are interesting "quantum theories" with unphysical (zero norm) ghost states [5].

theories,

$$U^{\dagger}|e_j\rangle = |e_i\rangle\,. \tag{1.2}$$

The only thing new in this description is that we can now also ask how "quantum mechanical superpositions" such as

$$|\psi\rangle = \alpha|e_1\rangle + \beta|e_2\rangle \tag{1.3}$$

evolve as a function of time. Also, we can define a matrix H such that

$$U(\delta t) = \mathrm{e}^{-iH\delta t}\,. \tag{1.4}$$

This will turn our system into a Schrödinger system. H will necessarily be an operator that mixes and superimposes states. All eigenstates of H (and of U) will be superpositions of primitive basis elements and as such show purely quantum mechanical behaviour. Notice that in the real world atoms and molecules have uncertainties δE in their energies that are small at the Planck scale. That alone assures that they would all be superpositions of many primitive basis elements if the real world would have such a basis.

Unfortunately, eq. (1.4) does not define our hamiltonian H uniquely. To all eigenvalues we can freely add or subtract integer multiples of $2\pi/\delta t$. This is a deep and fundamental difficulty if one wants to construct a theory of the real world along these lines, because it depends on this arbitrary choice of the integers in the eigenvalues, which of the eigenstates of U should be identified to be the vacuum state.

Rather than trying to construct theories of the real world we will here study some models. One simple cellular automaton in which particles move left and right with constant velocities on a real line was considered in ref. [3]. Since there are no interactions in that model it is trivially soluble. Its quantum formulation turns out to be a quantum field theory of free fermions on the line. Thus, massless fermions in $1+1$ dimensions really are a deterministic system. We can attribute this property of $(1+1)$-dimensional fermions to the fact that one can write down wave functions for the corresponding chiral Dirac particles that do not spread (there is no zitterbewegung). Since in more than one spacial dimension Dirac wave functions always spread we were unable to extend that model to more dimensions. Note however that charged fermions in an infinite magnetic field do behave as

(1 + 1)-dimensional fermions. Therefore, in an infinite magnetic background field massless fermions do become deterministic★.

In sect. [2] we consider a different cellular automaton that, though also exactly soluble, shows much more structure. This model, that we will refer to as the "triangular $\mathbb{Z}(2)$ multiplicative automaton"★★, is one of the simplest cellular automata, and has been studied by several authors [7]. It can be formulated in an arbitrary number of dimensions, but for simplicity we take it to be in $1+1$ dimensions. It is exactly soluble (in all dimensions). The model, as described in sect. 2, can be time reversed, but the time-reversed model is very different from the original one. Amazingly, the quantum version of this model has time-reversal symmetry! The symmetry only becomes manifest if we perform a non-trivial transformation in its Hilbert space, where pure primitive basis elements are mapped onto superimposed states (sect. 3). We wish to emphasize the importance of symmetries of this sort: they only manifest themselves in the quantum mechanical description of the theory even though the quantization procedure has the looks of being trivial. Could for instance Lorentz invariance in the real world be such a symmetry?

In sect. 4 the same model is looked at from a different angle. Now the classical model lacks parity invariance, which is only restored in its quantum extension. In sect. 6 we show that a similar phenomenon occurs in the "quadrangular $\mathbb{Z}(2)$ multiplicative automaton", which becomes self-dual only in its quantum extension.

We then turn to (2 + 1)-dimensional gravity. Since the quantization of discrete theories is so easy, we suggest to first construct a deterministic discrete version of a system with general coordinate invariance and then construct its Hilbert space. This is done in sect. 7. In this theory one sees that one can also generalize it into a truly quantum mechanical system (i.e. a system with no primitive basis). The class of quantum gravities obtained this way is unfortunately rather limited. None of these models has more than 12 particles in its entire "universe". Nevertheless, they are exactly soluble quantum gravity theories.

2. The triangular $\mathbb{Z}(2)$ multiplicative cellular automaton

Let there be a space coordinate x and time coordinate t that both are limited to take integer values, and $x+t$ is even. Of these points we have variables σ that take the values ± 1. They propagate according to the rule

$$\sigma(x,t) = \sigma(x-1,t-1)\sigma(x+1,t-1)\,. \tag{2.1}$$

★ Just before completing this paper I received an interesting preprint by Zee [6] who describes a discrete *quantum* theory on a (3 + 1)-dimensional lattice which reduces to the Dirac equation in the continuum limit. Conceivably this is the beginning of a road towards a deterministic cellular automaton describing fermions in more dimensions.

★★ It is called "additive" in ref. [7], where it is generated by the polynomial $\Pi(x) = x + x^{-1}$.

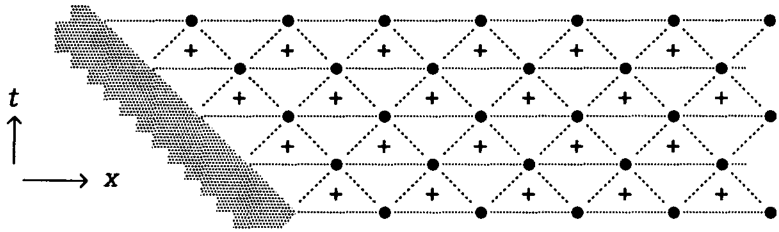

Fig. 1. The triangular $\mathbb{Z}(2)$ multiplicative model. At each triangle marked + the product of the three adjacent spins is one. In the shaded region (the boundary) all spins are +1.

It will be important that we also give the boundary condition,

$$\sigma(x,t) = +1 \quad \text{if } x < -X(t)\,, \tag{2.2}$$

for some sufficiently large $X(t)$. The boundary condition at the right will not be chosen until later.

In fig. 1 the lattice sites are indicated as dots (•), and the propagation rule is seen to correspond to the constraint that for every triangle marked with a + sign the product of the three "spins" σ at the angles be +1.

We clearly observe that, apart from the boundary condition, there is a three-fold rotational symmetry in space-time, but time reversal ($t \Leftrightarrow -t; x \Leftrightarrow \pm x$) is not an obvious symmetry.

Indeed, if the spins $\sigma(x,t)$ at one given t are all known, the spins at $t-1$ will be given by [7]

$$\sigma(x,t-1) = \prod_{x'<x} \sigma(x',t)\,, \tag{2.3}$$

which is quite different from the rule (2.1). Note that the boundary condition (2.2) is needed in deriving eq. (2.3).

In fig. 2 we show the structure of the solutions (a pattern familiar to experts in cellular automata). Fig. 2a shows how a spin at time t depends on spins at previous

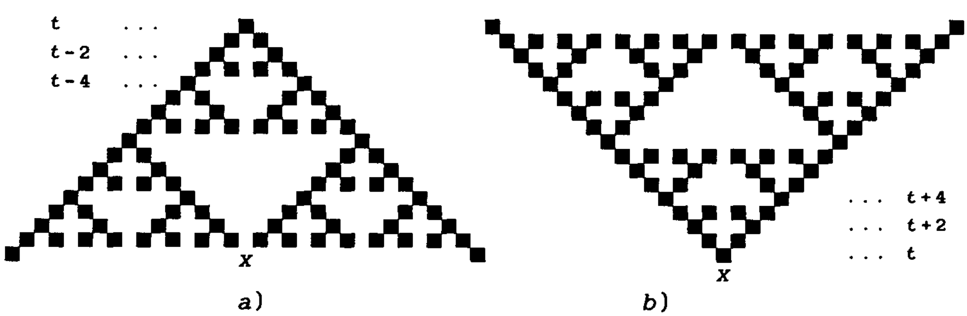

Fig. 2. (a) Dependence of $\sigma(x,t)$ on previous spins. (b) Effect of a single spin reversal at (x,t) on spins at later sites.

times $t-n$: it is the product of all spins on a black dot at time $t-n$. Fig. 2b shows which spins at future times $t+n$ flip when a single spin at time t is reversed.

Now let us introduce Hilbert space. It is convenient to use Pauli matrices at every lattice site, so we identify

$$\sigma(x,t) = \sigma^3(x,t)\,, \tag{2.4}$$

and introduce the two others, $\sigma^1(x,t)$ and $\sigma^2(x,t)$. They satisfy the commutation rules

$$\begin{aligned} \sigma^i(x,t)\sigma^j(x,t) &= \delta^{ij} + i\varepsilon_{ijk}\sigma^j(x,t)\,, \\ \left[\sigma^i(x,t),\sigma^j(x',t)\right] &= 0 \quad \text{if } x \neq x'\,. \end{aligned} \tag{2.5}$$

(Note that we decided not to introduce Fermi anticommutation at this point.)

Now in the "quantum theory", σ^3 is simply postulated to evolve just as before,

$$\sigma^3(x,t) = \sigma^3(x-1,t-1)\sigma^3(x+1,t-1)\,. \tag{2.6}$$

Again,

$$\sigma^3(x,t-1) = \prod_{x'<x}\sigma^3(x',t)\,. \tag{2.7}$$

3. Enhanced symmetry

How do σ^1 and σ^2 evolve? Take σ^1, which is the matrix

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

Thus, the operator $\sigma^1(x,t)$ switches the sign of the spin at the site x only. All other spins at time t are left intact. As a consequence of the equation of motion (2.1), the two spins $\sigma(x-1,t+1)$ and $\sigma(x+1,t+1)$ will also switch signs. And so we derive that the two operations $\sigma^1(x,t)$ and $\sigma^1(x-1,t+1)\sigma^1(x+1,t+1)$ have the same effect on the states in our Hilbert space. Hence,

$$\sigma^1(x,t) = \sigma^1(x-1,t+1)\sigma^1(x+1,t+1)\,. \tag{3.2}$$

The effect of a sign switch at (x,t) can also be followed backwards in time. Because of our boundary condition (2.2), the spins at the far left are not allowed to switch. The only solution is that all spins at the right of the point x have switched.

Hence

$$\sigma^1(x,t) = \prod_{x'>x} \sigma^1(x',t-1)\,. \tag{3.3}$$

We see that σ^1 propagates backwards in time exactly the same way as σ^3 propagates forwards in time. Comparing eqs. (3.2) and (3.3) with eqs. (2.6) and (2.7) we observe the symmetry

$$t \Leftrightarrow -t, \qquad x \Leftrightarrow -x, \qquad \sigma^3 \Leftrightarrow \sigma^1, \qquad \sigma^2 \Leftrightarrow -\sigma^2\,. \tag{3.4}$$

The replacement of the spin variables is easily obtained in Hilbert space by rotating 180° around the axis $x_1 = x_3$; $x_2 = 0$. In fig. 1 the product of three σ^1-operators is $+1$ precisely in all unmarked triangles. So locally the $\mathbb{Z}(3)$ rotational symmetry in space-time is replaced by a $\mathbb{Z}(6)$ symmetry for the quantum system.

It is now also tempting to give a boundary condition at the far right of fig. 1,

$$\sigma^1(x,t) = +1 \quad \text{if } x > +X(t)\,. \tag{3.5}$$

This goes naturally with eq. (3.3).

Remember that in order to enhance the symmetry of this system we did not have to change anything in its "physical" properties. We merely extended the states to also include superimposed states. After all, the eigenstates for which the operators σ^1 have their eigenvalues ± 1 are superpositions of the original states $\{\sigma(x)\}$.

4. The necessity of rotating 30° in space-time

Something else happened that requires discussion. The original model was completely local in the sense that the propagation of spins only depended on the configuration of neighboring sites. Now however the propagation of σ^1 into the future, according to eq. (3.3), is not exactly local. This would make it impossible for us to construct a local or even more or less local expression for the quantum mechanical evolution operator $U(t_1, t_2)$.

But locality is not completely lost. We see this by studying the domains of the commutators of our "fields" $\sigma^i(x,t)$. From the solutions we see that

$$\left[\sigma^3(x,t), \sigma^1(0,0)\right] \tag{4.1}$$

is nonvanishing in the region marked shaded in fig. 3a. The set of commutators

$$\left[\sigma^i(x,t), \sigma^j(0,0)\right] \tag{4.2}$$

is nonvanishing within the "lightcone" sketched in fig. 3b.

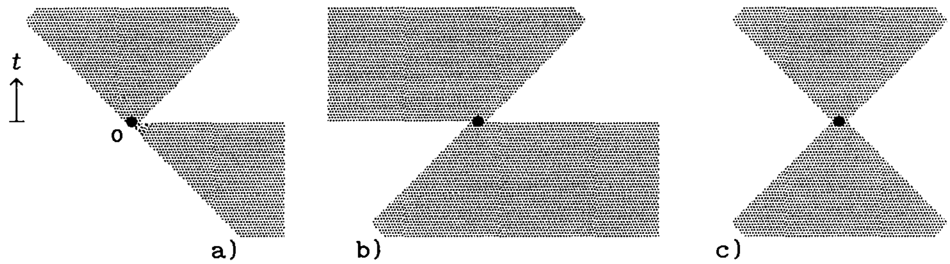

Fig. 3. (a) Domain of (4.1). (b) Domain of (4.2): the quantum light-cone. (c) The light-cone of the classical theory.

It is not known how special this "quantum enlargement" of the light-cone is to this particular model. Anyway, we now see what should be done if we want to construct the quantum evolution operator $U(t_1, t_2)$. Choose new coordinates (z, τ) in terms of which the light-cone of fig. 3b is put in the correct position,

$$x = \tfrac{1}{2}(3z - \tau), \quad t = \tfrac{1}{2}(z + \tau), \quad z = \tfrac{1}{2}(x + t), \quad \tau = \tfrac{1}{2}(3t - x), \quad z + \tau \text{ even}. \tag{4.3}$$

In terms of these coordinates we have

$$\sigma^3(z, \tau) = \sigma^3(z - 1, \tau - 1)\sigma^3(z, \tau - 2), \tag{4.4}$$

$$\sigma^1(z, \tau) = \sigma^1(z + 1, \tau - 1)\sigma^1(z, \tau - 2). \tag{4.5}$$

From their construction we should remember that (4.4) implies (4.5) and vice versa. What we see here is that *parity invariance* is restored in the quantum extended model. A "physicist" living in this world would attach parity quantum numbers to each of the particles he sees. But the particles he describes this way cannot be identified classically in terms of the original spin variables.

5. Quantum field theory

Now the spins are only at the sites with $z + \tau$ even. If we define

$$\sigma^i(z, \tau) = \sigma^i(z, \tau - 1) \quad \text{if } z + \tau \text{ is odd}, \tag{5.1}$$

then we see that eq. (4.4) updates alternatingly the spins at $z =$ even and the spins at $z = \text{odd}$. The even spins are updated by the operator

$$U_1 = \prod_{z=\text{even}} \left(\sigma^1(z)\right)^{\frac{1}{2}(1-\sigma^3(z-1))} = \exp \sum_{z=\text{even}} \tfrac{1}{4}\pi i\eta\left(1-\sigma^1(z)\right)\left(1-\sigma^3(z-1)\right), \tag{5.2}$$

and the odd spins by

$$U_2 = \exp \sum_{z=\text{odd}} \tfrac{1}{4}\pi i\eta\left(1-\sigma^1(z)\right)\left(1-\sigma^3(z-1)\right), \tag{5.3}$$

where η is an arbitrary sign: $\eta = \pm 1$.

We were able to rewrite the product as the exponent of a sum because all individual factors commute. Notice that the same operators are obtained if we start from eq. (4.5) instead of eq. (4.4).

The evolution over large even time intervals Δt is generated by the operator

$$U(\Delta\tau) = (U_1U_2)^{\Delta\tau/2}. \tag{5.4}$$

Now U_1 and U_2 do not commute. If we want to write eq. (5.4) as a single exponent we can use the Baker–Campbell–Haussdorff series as described in ref. [3],

$$\mathrm{e}^A\mathrm{e}^B = \exp\left(A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[A-B,[A,B]] + \tfrac{1}{24}[[A,[A,B]],B] + \ldots\right). \tag{5.5}$$

The result is that we can write (5.4) as

$$U(\Delta\tau) = \mathrm{e}^{-iH\Delta\tau}, \tag{5.6}$$

where the hamiltonian H obeys

$$H = \sum_z \mathscr{H}(z), \tag{5.7}$$

$$\mathscr{H}(z) = -\eta\tfrac{1}{8}\pi\left(1-\sigma^1(z)\right)\left(1-\sigma^3(z-1)\right) + (-)^{z+1}\tfrac{1}{32}\pi^2\left(1-\sigma^1(z+1)\right)\sigma^2(z)\left(1-\sigma^3(z-1)\right) + \ldots, \tag{5.8}$$

where $(-)^{z+1}$ is -1 for z even and $+1$ for z odd.

This does not give a closed form for this hamiltonian or its eigenvalues and eigenstates. *If* one would be allowed to terminate the series to obtain a sensible approximation to the true system then we see an interesting, highly nontrivial hamiltonian density emerging. All terms depend only on the variables σ^i at the point z and its immediate neighbors, so in the continuum (or rather thermodynamic) limit this hamiltonian density would be a local one. Because the σ-operators are bounded there would surely be a lowest-energy eigenstate, which we could call the "vacuum".

Actually we suspect that (5.8) has a rather complicated vacuum, at least in the most interesting case $\eta = -1$ (in that case the vacuum expectation values for the operators A and B in eq. (5.5) will all tend to be small so that we may have some reason to hope for a reasonable convergence of the series, see later). We expect the vacuum to have an N-fold degeneracy (in a 2^N-dimensional Hilbert space; N is the number of points z in space).

Our argument goes as follows. It is not difficult to see that *all* terms in eq. (5.8) have the form

$$\left(1-\sigma^1(z_2)\right)F\{\boldsymbol{\sigma}(z_1+1),\dots,\boldsymbol{\sigma}(z_2-1)\}\left(1-\sigma^3(z_1)\right), \tag{5.9}$$

where $z_2 > z_1$, and F is some function of the spins at the sites in between. This annihilates all states that have, given some arbitrary point z_0,

$$\begin{aligned} &\sigma^3(z)=1 \qquad \text{for } z<z_0, \qquad \sigma^1(z)=1 \quad \text{for } z>z_0, \\ &\boldsymbol{\sigma}(z_0) \qquad \text{arbitrary}. \end{aligned} \tag{5.10}$$

We can define the state $|z_0\rangle$ by requiring in addition

$$\sigma^3(z_0) = -1, \tag{5.11}$$

so that all states $|z_0\rangle$ with different z_0 are mutually orthogonal.

We have

$$H|z_0\rangle = 0. \tag{5.12}$$

Take now the case $\eta = -1$. Then for all other states the expectation value of the first term in eq. (5.8) is positive. If we assume that the other terms will be unable to beat that, then the states $|z_0\rangle$ form an N-fold degenerate lowest-energy state, i.e. vacuum.

There is a caveat however, which makes the above derivation of the vacuum structure uncertain. It is not difficult to argue that the series (5.8) will not converge. Decent behavior of the series can be expected only if we sandwich it between energy eigenstates whose energy eigenvalues are less than 2π apart. We should *not* expect anything better than that because the exact hamiltonian has the ambiguity of integer multiples of 2π in its eigenvalues.

In a thermodynamic limit, where only low-lying energy states are considered, this model may produce an interesting quantum field theory. By performing the Jordan–Wigner transformation [8] one can turn the σ-fields into fermionic fields $\psi(z,\tau)$. We were unable to do this in such a way that interactions among these fermions would be avoided. Most likely therefore we have interacting fermions here (which can be described in many ways).

Let us observe that the patterns of fig. 2 which correspond to the complete "solution" of this system have an important scaling invariance: they map into themselves when scaled by a factor of 2. This suggests that the fermionic system described above is covariant under a renormalization group transformation, so that it may belong to one of the more interesting classes of conformally-invariant fermion models in two dimensions (see e.g. ref. [9]).

6. The quadrangular $\mathbb{Z}(2)$ multiplicative automaton

The quadrangular interaction is defined if instead of (2.1) we take

$$\sigma^3(x,t)=\sigma^3(x-1,t-1)\sigma^3(x+1,t-1)\sigma^3(x,t-2)\,. \tag{6.1}$$

Again the spins are only defined on the even lattice sites ($x+t=$ even). Since now two time layers are involved, states in Hilbert space are characterized by identifying the values of σ at two consecutive times $t, t+1$. If of these spins one is switched by the operation of a σ^1-operator then other spins at earlier or later times will switch also. The pattern of switching spins is now given by fig. 4.

Notice that symmetry suggests that we assign to these operators σ^1 a location at *odd* lattice sites ($x+t=$ odd). The operator that produced fig. 4 is defined to be $\sigma^1(0,t+1)$. Indeed in that case it is not difficult to establish that

$$\sigma^1(x,t)=\sigma^1(x-1,t-1)\sigma^1(x+1,t-1)\sigma^1(x,t-2)\quad \text{if } x+t \text{ odd}\,. \tag{6.2}$$

Thus we have extended the model to the odd (dual) lattice sites. The translation

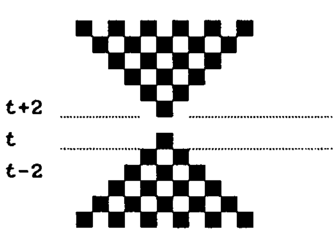

Fig. 4. Effect of $\sigma^1(0,t+1)$ on the spins at other times. Spins that flip are colored black. Hilbert space is defined by the spins at times t and $t+1$ so that these are kept fixed. This is the region within the dotted lines.

operators (in space and in time) for this model may now include odd distances, provided they are accompanied by the rotation $1 \Leftrightarrow 3$ in spin space. The model is then invariant also under these translations. Alternatively, we may say that our model is self-dual.

It would be interesting to speculate that the quantization procedure proposed in this paper might become applicable one day for constructing lattice approximations to physically relevant quantum field theories such as QCD. One would obtain a cellular automaton in three spacial dimensions that would be ideal for computer simulations. As yet we will resist such temptations and consider another application, which is to obtain more conceptual understanding of Quantum Gravity.

7. Gravitation in 2 + 1 dimensions

As stated in sect. 1, we expect quantum theories including the gravitational force to be essentially discrete. It may therefore be a good idea to search for such theories by first taking a deterministic discrete theory that shows some of the symmetries we would like to have, and then quantizing it by Hilbert space extension. It should hardly be necessary to point out the advantages such a procedure would have. There would be no infinity problems in the set-up of such a theory; no mysticism (such as a "many world hypothesis") would be needed to understand "Quantum Cosmology", etc.

The problem remains then how to define the vacuum state, or, how to fix the 2π ambiguity in the energy eigenvalues.

In this section we show an example of such a model. It is well known that the gravitational force in $2+1$ dimensions is nearly trivial. Space-time surrounding a massive object is locally flat, but globally it has the metric structure of a cone. Newton's equations are therefore exactly soluble in $2+1$ dimensions.

Quantum field theories for scalar particles are super-renormalizable and therefore also easy to handle in $2+1$ dimensions. Can we not combine these "easy" theories to construct a model for gravitating quantized scalar particles?

Partial successes [10] and claims that there is a renormalizable theory of this sort [11] were reported, but the complete mathematical structure of such theories remains obscure. We will now propose a simple finite model. The reasoning behind it goes as follows.

In classical $(2+1)$-dimensional gravity a spinless particle is characterized by giving its mass, position and velocity. In center-of-mass coordinates, where the particle stays at the origin of the two space coordinates, the conical structure of the metric of the surrounding space-time is described by first giving a local rectangular coordinate frame, and then specifying how this frame has to be rotated if we follow a closed contour around the particle. This rotation can be given as an

SO(2, 1) matrix L,

$$L = \begin{pmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.1}$$

where μ is the conical deficiency angle, which is proportional to the mass m of our particle.

If the particle is not situated at the origin but at a point $\boldsymbol{x} = \boldsymbol{a}$, then a closed loop around the particle brings about an element P of the Poincaré group,

$$x' = Px = L(x - a) + a = Lx + (1 - L)a, \tag{7.2}$$

where $x = (\boldsymbol{x}, 0)$ and $a = (\boldsymbol{a}, 0)$.

If the particle has a certain velocity $\boldsymbol{v}$ then the matrix L is replaced by its Lorentz transform QLQ^{-1}, where Q is the Lorentz transformation that gives the particle its velocity $\boldsymbol{v}$.

We see that all essential features, namely the mass, position and velocity of the particle are uniquely determined* by P. The simplest classical (2 + 1)-dimensional gravity model consists of just specifying the trajectories of all particles i (all straight lines) and the corresponding elements P_i of the Poincaré group. The model becomes nontrivial if we realize that the parameters in P_1 for a particle 1 change when we follow a contour around particle 2, into $P_2 P_1 P_2^{-1}$. Which branch one is supposed to take depends on how a particle passes the observer $\mathscr{O}$.

Now our program is to turn this into a discrete theory, after which quantization by Hilbert space extension is easy. A discrete theory is obtained by putting space-time on a lattice. There are various choices, the most obvious one being a rectangular lattice. Thus all points in space-time are given by 3-vectors whose components x_1, x_2 and t are integers. This forces us to limit ourselves to the subgroup $\mathscr{P}(2, 1, \mathbb{Z})$ of the Poincaré group, which has only integers in the matrix L and the displacement vector a. This still is an infinite subgroup. An example of a nontrivial Lorentz transformation is

$$L = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}. \tag{7.3}$$

The particles in our model must all generate elements of this subgroup of P. Thus, in eq. (7.1) μ is a multiple of 90°. Clearly, the particles are rather heavy, and

* This is *not* always true for a spinning particle. Angular momentum is connected to a time translation in P, but in that case the particle's position becomes somewhat ambiguous. The position of a spinless particle is seen to be given by the one-dimensional set of solutions to the equation $Px = x$. For a spinning particle this has no solution. This is the reason why we limit ourselves to spinless particles.

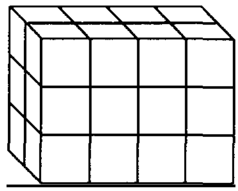

Fig. 5. Example of a stationary (2 + 1)-dimensional universe with 8 particles.

since the total energy in an open universe cannot exceed $2\pi = 360°$, and in a closed universe equals $4\pi = 720°$, the total number of these particles cannot be more than 4 and 8 in the two cases. A static universe with 8 particles is pictured in fig. 5. The particles are at the corners of this block of cubicles, where we see conical deficiency angles of 90°.

In this example the particles do not move as a function of time (the newtonian force is zero in 2 + 1 dimensions). Moving particles are obtained by performing Lorentz transformations such as (7.3). The lowest possible non-zero velocity is $v/c = \sqrt{8/9}$.

Our model is not yet completely described by merely giving the allowed elements of the Poincaré group. We must give a unique prescription of what happens when two particles collide head-on. (On a lattice this is of course a concrete possibility.) One could devise some deterministic algorithm $(P_1, P_2 \Rightarrow P_3, P_4)$ or we could at this point decide to deviate from this paper's philosophy and reintroduce truly quantum mechanical superpositions,

$$|P_1, P_2\rangle \Rightarrow \alpha|P_3, P_4\rangle + \beta|P_5, P_6\rangle + \dots, \tag{7.4}$$

where $P_3, P_4, \dots$ are certain functions of P_1 and P_2, and $\alpha, \beta, \dots$ are complex numbers. Either way, the Hilbert space formulation of this model is a truly finite quantum gravity theory in 2 + 1 dimensions.

By considering other than rectangular lattices one finds generalizations of this model. A triangular lattice allows us to have deficiency angles of 60° so that the total number of particles in a closed universe can be 12. The stationary configuration then has the shape of an icosaedre.

We see that Hilbert space is spanned by all possible particle configurations on such a lattice. It is a Heisenberg picture that one gets: the operators evolve as a function of time but not the states. Time is simply defined by some prescription for obtaining Cauchy surfaces as is done conventionally in unquantized General Relativity.

Our conclusion is that although the models of this section may be too simple to be useful as real theories of the world, they may give important conceptual insights in what a Quantum Theory of Gravity could be like.

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