

LARGE N

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Keynote Address, presented at the workshop on "The Phenomenology of Large N_c QCD", Arizona State University, January, 2002.

In the first part of this lecture, the $1/N$ expansion technique is illustrated for the case of the large- N sigma model. In large- N gauge theories, the $1/N$ expansion is tantamount to sorting the Feynman diagrams according to their degree of planarity, that is, the minimal genus of the plane onto which the diagram can be mapped without any crossings. This holds both for the usual perturbative expansion with respect to powers of $\tilde{g}^2 = g^2 N$, as well as for the expansion of lattice theories in positive powers of $1/\tilde{g}^2$. If there were no renormalization effects, the \tilde{g} expansion would have a finite radius of convergence.

The zero-dimensional theory can be used for counting planar diagrams. It can be solved explicitly, so that the generating function for the number of diagrams with given 3-vertices and 4-vertices, can be derived exactly. This can be done for various kinds of Feynman diagrams. We end with some remarks about planar renormalization.

1 Introduction: The Large N Sigma Model

A simple example of a theory where new and interesting information can be obtained by studying it in the limit where the number N of fundamental field degrees of freedom becomes large, is the Large N Sigma Model¹. Let $\phi_i(x)$ be a set of N scalar fields, $i = 1, \dots, N$, and consider the Lagrangian of a renormalizable theory:

$$\mathcal{L} = -\frac{1}{2} \sum_{i=1}^N [(\partial_\mu \phi_i)^2 + m^2 \phi_i^2] - \frac{1}{8} \lambda \left(\sum_{i=1}^N \phi_i^2 \right)^2, \quad (1)$$

where λ is a coupling constant, usually taken to be positive.

The Feynman diagrams of the theory consist of propagators,

$$\begin{array}{c} i \\ \bullet \end{array} \text{---} \begin{array}{c} \bullet \\ j \end{array} \quad \frac{\delta_{ij}}{k^2 + m^2 - i\varepsilon}, \quad (2)$$

to be connected by vertices:

$$\begin{array}{c} i \\ \diagup \\ \bullet \\ \diagdown \\ l \end{array} \begin{array}{c} j \\ \diagdown \\ \bullet \\ \diagup \\ k \end{array} \quad -\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (3)$$

We write this vertex as the sum of three vertices, to be indicated the following way:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = \begin{array}{c} \diagup \\ \parallel \\ \diagdown \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \parallel \end{array} + \begin{array}{c} \diagup \\ \diagdown \\ \parallel \end{array}. \quad (4)$$

Rewriting the Lagrangian (1), using an auxiliary field $A(x)$, as

$$\mathcal{L} = -\frac{1}{2} \sum_{i=1}^N [(\partial_\mu \phi_i)^2] + \frac{1}{2\lambda} A^2 - \frac{1}{2}(A + m^2) \left(\sum_{i=1}^N \phi_i^2 \right), \quad (5)$$

we get the same Feynman rules [since integrating out A yields back Eq. (1)], but now the double line is the A propagator, $-\lambda$, and the A lines form 3-vertices with the ϕ lines. It is now relatively easy to integrate out the field ϕ in (5). We must remember that A is x dependent, so we do this perturbatively in the A field. This way, we get effective interactions in the A field, the interaction being described by ‘vertices’ formed by closed ϕ lines, see Fig. 1. Notice that the set of Feynman rules obtained this way is easily seen to be the same as the ones generated by Eqs. (1)-(4), as of course it should be.

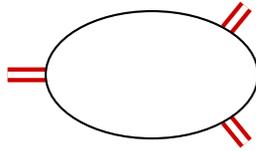


Figure 1. Effective vertex for A field.

Since (5) is only quadratic in ϕ the effective vertices only consist of single ϕ loops, and so they are easy to calculate.

Since there are N fields ϕ , each of these effective vertices go with a factor N . It is now instructive to replace the field A by \tilde{A}/\sqrt{N} , which amounts to a propagator $-\lambda N$ for \tilde{A} , whereas the ν -point effective vertices each receive an extra factor $N^{-\nu/2}$. Together with the factor N mentioned above, we find that the effective 2 point vertex goes with a factor N^0 (so it is N -independent), the 3-vertices go with $N^{-1/2}$, etc. The *tadpole diagram* (the effective vertex with $\nu = 1$) must be assumed to vanish, after application of a normal ordering procedure. It would be proportional to $N^{+1/2}$.

If now we choose the limit $N \rightarrow \infty$, while $\tilde{\lambda} = \lambda N$ is kept fixed, then we see that a perturbative theory emerges, where all effective vertices with ν external lines are suppressed by factors $N^{1-\nu/2}$, so that $1/\sqrt{N}$ acts as a coupling constant. Only the 2-point vertex is non-perturbative, so it has to be computed exactly, and added to the 'bare' term $-\tilde{\lambda}$ to obtain the \tilde{A} propagator. Fortunately, the 2-point vertex correction $F(m, q)$, where q is the momentum of the \tilde{A} field and m is the ϕ mass, is easy to compute. In n spacetime dimensions,

$$\begin{aligned} F(m, q) &= \frac{1}{2} (2\pi)^{-4} \int d^n k \frac{1}{(k^2 + m^2 - i\varepsilon)[(k+q)^2 + m^2 - i\varepsilon]} \\ &= C(n) \int_0^1 dx [m^2 + q^2 x(1-x)]^{\frac{n}{2}-2}, \end{aligned} \quad (6)$$

where $C(n) = \frac{1}{2}(4\pi)^{-n/2} \Gamma(2 - \frac{n}{2})$.

This has to be added to the bare \tilde{A} propagator term, so the series of diagrams contributing to the dressed propagator amounts to:

$$= + \text{[diagram: circle with double lines]} + \text{[diagram: two circles with double lines]} + \dots = \frac{-\tilde{\lambda}}{1 + \tilde{\lambda}F(m, q)}. \quad (7)$$

This is the exact propagator to lowest order in $1/N$. The effective Feynman diagrams describing the $1/N$ corrections are sketched in Fig. 2. In many respects, the $1/N$ expansion is like the coupling constant expansion in a field theory. Thus, the effective propagators in diagrams such as Fig. 2 can form closed loops.

At $n = 4$, the function $C(n)$ has a pole, which has to be absorbed into a redefinition of $1/\tilde{\lambda}$, as usual. At $n = 6$ or higher, there are divergences which require more troublesome subtractions, since there the theory is non-renormalizable. It is instructive to look at the q dependence of this effective propagator. At $n < 4$, this propagator is regular at high $\pm q^2$, since F tends to zero. F is complex if $q^2 < -4m^2$. If m is low enough, and/or $\tilde{\lambda}$ large enough, then a new pole may emerge: an (unstable) bound state of two ϕ particles.

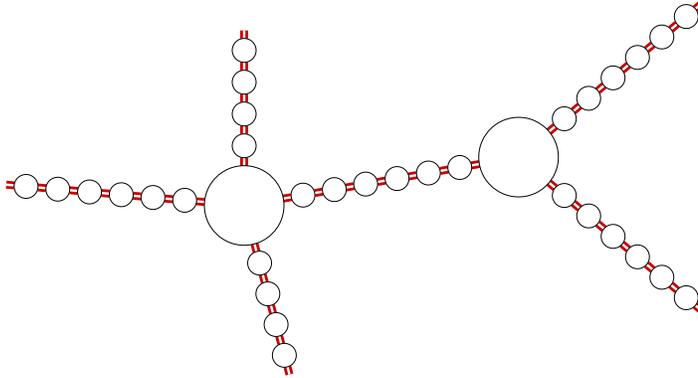


Figure 2. Effective Feynman graphs for the $N \rightarrow \infty$ Sigma Model.

If, however, $n \geq 4$, the function F diverges at *large* $\pm q^2$. Note that the *sign* of F after the required subtraction flips, so that for large q^2 F behaves as

$$F(m, q^2) \rightarrow -\frac{1}{32\pi^2} \log q^2 . \quad (8)$$

Consequently, in Eq. (7) a pole at some large *positive* value of q^2 is inevitable. Thus, we have a tachyonic pole in the propagator: the theory has a Landau ghost. This ghost is directly related to the fact that the theory is not asymptotically free. At very large momenta, the effective quartic coupling is negative, which implies a fundamental instability of the theory. This is an important lesson from the study of the $1/N$ expansion of the Sigma Model: lack of asymptotic freedom in the ultraviolet makes the theory inconsistent. There is no tendency to go to an ultraviolet fixed point. The theory does not show any inclination to “cure itself.” In the literature, we often see the belief that models with a decent-looking Lagrangian are well-defined, “if only we knew how to do the mathematics.” Here, we happen to be able to do the mathematics, and find the model to be inconsistent at $n \geq 4$ due to its bad ultraviolet behaviour, just like any other non-renormalizable system.

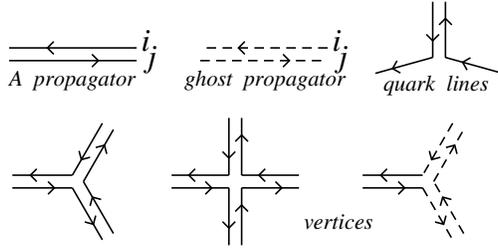


Figure 3. Feynman rules for $U(N)$ gauge theories.

2 $1/N$ for Gauge Theories

Many gauge theories are asymptotically free², so here a large N expansion could be much more promising. The large N expansion for gauge theories can be formulated for all large Lie groups, and the final results are all very similar. Let us first consider the gauge group $U(N)$. The vector fields are Hermitean matrices $A_{\mu j}^i$, where both indices i and j run from 1 to N . Writing

$$F_{\mu\nu j}^i = \partial_{\mu} A_{\nu j}^i - \partial_{\nu} A_{\mu j}^i + ig [A_{\mu}, A_{\nu}]_j^i, \quad (9)$$

we have

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \text{Tr}(F_{\mu\nu} F_{\mu\nu}) - \bar{\psi}(i\gamma D + m_{\psi})\psi \\ &= -\text{Tr}\left(\frac{1}{2}(\partial_{\mu} A_{\nu})^2 - \frac{1}{2}\partial_{\mu} A_{\nu}\partial_{\nu} A_{\mu} + ig\partial_{\mu} A_{\nu}[A_{\mu}, A_{\nu}] - \frac{1}{4}g^2[A_{\mu}, A_{\nu}]^2\right) \\ &\quad - \bar{\psi}(i\gamma_{\mu}\partial_{\mu} + ig\gamma_{\mu}A_{\mu} + m_{\psi})\psi. \end{aligned} \quad (10)$$

A transparent notation for the Feynman diagrams³ is one where the propagators are represented by double lines, each line indicating how the index is transported, see Fig. 3. The mathematical details of these rules depend on the gauge fixing procedure, which we shall not further expand upon.

Everywhere in a Feynman diagram where an index loop closes, we get a factor N from counting the values that the index can have. This is the only way the dimension N of the gauge group enters. Now consider a general diagram with V_3 3-point vertices, V_4 4-point vertices, P propagators and F index loops. Assume, temporarily, no external lines (a gauge-invariant coupling to an external source can for instance be introduced by means of a 2 quark vertex insertion, $J\bar{\psi}\psi$; this then is a 2 point vertex, counted by V_2). Write $V = V_2 + V_3 + V_4$.

First, by drawing dots at both ends of every propagator, we see immediately by counting dots that

$$2P = \sum_n nV_n. \quad (11)$$

We now turn the diagram into a polyhedron by defining every index loop to be a facet. The facets form a polyhedron if there are no quark loops. Then, we have Euler's theorem for polyhedra:

$$F - P + V = 2 - 2H. \quad (12)$$

Here, H is the *genus*, or number of 'holes' in the polyhedron. This theorem is easily proven by induction. Since all 3-point vertices in Fig. 3 come with a factor g , and all 4-point vertices with a factor g^2 , the g and N dependence of the amplitude is seen to be³

$$g^{V_3+2V_4} N^F = g^{2P-2V} N^F = (g^2 N)^{F+2H-2} N^{2-2H}. \quad (13)$$

In the limit where $N \rightarrow \infty$, while $\tilde{g}^2 = g^2 N$ is kept fixed, we see that the N dependence of the diagrams is N^{2-2H} , so that diagrams without holes ($H = 0$) dominate.

If quarks are present, we simply declare the quark loops to be facets as well, but they lack a factor N . Therefore, we also get a factor N^{-Q} , where Q is the number of quark loops. Thus, the $1/N$ expansion is found to be an expansion with respect to the number of quark loops and the number of holes in the polyhedra. In many respects, this expansion resembles an expansion of a string theory in powers of the string coupling constant.

For other gauge groups, the result is very similar. If we take $SU(N)$ instead of $U(N)$, we are essentially subtracting a $U(1)$ component, whose coupling is $g = \tilde{g}/\sqrt{N}$, so even at large \tilde{g} , the $U(1)$ subtractions are still perturbative. For $SO(N)$, we have an additional reality constraint on the fields A :

$$A_{\mu j}^i = -A_{\mu i}^j = A_{\mu j}^{i*}. \quad (14)$$

This means that the arrows disappear from the double line propagators in Fig. 2. The surfaces of the planar diagrams are now non-oriented, so that non-orientable diagrams are allowed: Klein bottles and Möbius strips. They, however, have $H > 0$ or $Q > 0$, and hence they are also suppressed at large N , so that the dominating diagrams are essentially the same as the ones for $U(N)$ and $SU(N)$.

3 $1/N$ for Theories on a Lattice.

It is instructive to consider the $1/N$ expansion for lattice gauge theories⁴. Consider a lattice with lattice sites, connected by lattice links. The gauge fields live on the links, the quark fields live on the sites. On a link with length a , connecting the sites x and $x + e_\mu$ (where e_μ is a unit vector of length a on the lattice), we write

$$U_\mu(x) \stackrel{\text{def}}{=} \exp(ia g A_\mu). \quad (15)$$

The gauge-invariant Wilson action (in n dimensions) is

$$S = \frac{a^{n-4}}{2g^2} \sum_{x,\mu,\nu} \text{Tr} [U_\mu(x)U_\nu(x + e_\mu)U_\mu^\dagger(x + e_\nu)U_\nu^\dagger(x)]. \quad (16)$$

The two indices of the field variable $U_{\underline{\mu}i}^j(x)$ refer to gauge transformations in x and $x + e_\mu$, respectively, so these are two different indices, to be distinguished by underlining one of them. Since $U(x)$ is unitary, we have

$$U_{\underline{i}j}^k U_{\underline{j}k}^i = \delta_{\underline{i}}^k; \quad U_{\underline{i}j}^k U_{\underline{j}k}^i = \delta_{\underline{i}}^k, \quad (17)$$

from which we derive that the ‘functional’ integral over the U values will yield⁵:

$$\langle U_{\underline{i}j}^k U_{\underline{k}l}^j \rangle = \frac{1}{N} \delta_{\underline{i}}^l \delta_{\underline{k}}^j, \quad (18)$$

[simply by observing that (18) is the only term with the required gauge invariance, so all that has to be determined is the prefactor]. Similarly, we have only four possible terms for the quartic averaging expression. Two of these turn out to be leading:

$$\langle U_{\underline{i}j}^k U_{\underline{k}l}^j U_{\underline{m}n}^l U_{\underline{p}q}^n \rangle = \frac{1}{N^2} \left(\delta_{\underline{i}}^l \delta_{\underline{k}}^j \delta_m^q \delta_{\underline{p}}^n + \delta_{\underline{k} \leftrightarrow \underline{p}}^l \delta_{\underline{m} \leftrightarrow \underline{q}}^j \right) + \mathcal{O}(N^{-3}). \quad (19)$$

The effect of these expressions is that when we do the functional integral, simply by inserting the complete expansion of the exponential of the action S , at every link every factor U has to be paired with a factor U^\dagger . Every such pair produces a factor $1/N$. As in the continuum case, we construct a polyhedron from the elementary plaquettes. Let F be the number of plaquettes (the number of terms of S in our ‘diagram’, coming from expanding e^{iS}), let P be the number of pairings, and V be the number of indices that end up being summed. We find a factor $1/N$ for each pairing, a factor N for each index, which amounts to a factor N for each point on our polyhedron, and a factor $1/g^2$ for every plaquette (apart from a harmless numerical coefficient).

All in all, the N and g dependence arises as a factor

$$N^V (1/N)^P (g^2)^{-F} = (g^2 N)^{-F} N^{V-P+F} = (\tilde{g}^2)^{-F} N^{2-2H}, \quad (20)$$

where again Euler was applied, and $\tilde{g}^2 = g^2 N$. So again we find that the diagrams only depend on \tilde{g} , and the N dependence of the diagrams is as in the continuum theory. This result is not quite trivial, since we are looking at a different corner of the theory, and the transition from the small \tilde{g} to the large \tilde{g} limit need not commute with the large N limit, but apparently it does.

4 Divergence of Perturbation Expansion

Gauge theories can now be expanded in terms of several parameters: we have the \tilde{g} expansion, the $1/\tilde{g}$ expansion and the $1/N$ expansion. Of these, only $1/N$ acts as a genuine interaction parameter, describing the strength of the interactions among mesons. Physical arguments would consequently suggest that the $1/N$ expansion produces terms of the form $a_k N^{-k}$ where a_k will diverge faster than exponentially for large k values, typically containing $k!$ terms. In contrast, the \tilde{g} expansion generates only planar diagrams. In principle, one could expect better convergence for this expansion. Indeed, the number R_k of planar diagrams with k loops will be bounded by an expression of the form C^k for some finite constant C .

Unfortunately, even the \tilde{g} expansion will generate factors $k!$, not due to the number of diagrams being large, but because of renormalization effects. These effects already occur in the large N sigma model of Sect. 1. Consider the $\tilde{\lambda}$ expansion of the propagators of Figure 2. The expansion term going as $\tilde{\lambda}^{k+1}$ contains exactly k loops. Each of these loops generates an expression $F(m, q)$, which, for large q , diverges roughly like $\log(q^2 + m^2)$. Imagine this propagator itself being part of a closed loop, and imagine that a sufficient number of subtractions has been carried out in order to make the integral convergent, so that some renormalized amplitude results. We see that the k^{th} term requires the calculation of an integral of the form

$$\int d^4 q \frac{[\log(q^2 + m^2)]^k}{(q^2 + m^2)^3}; \quad (21)$$

Writing $q^2 + m^2 \approx m^2 e^x$, we see that integrals of the form

$$\int_0^\infty dx x^k e^{-x} = k! \quad (22)$$

arise.

Such $k!$ terms being very similar to the ones generated by instanton effects in quantum field theory, this effect was dubbed ‘renormalon’. The renormalon is closely associated with the Landau ghost. Indeed, replacing the coupling constant $\tilde{\lambda}$ by a running coupling constant, we notice that the Landau ghost mass increases exponentially with the inverse coupling constant:

$$m_{\text{gh}} \propto e^{16\pi^2/\tilde{\lambda}}. \quad (23)$$

In some very special cases, the Landau ghost can be avoided altogether. This is if the theory is both asymptotically free in the ultraviolet and infrared convergent due to mass terms⁶. Such theories can be constructed, making judicious use of gauge vector, scalar *and* spinor fields. Integrals of the type (22) do still occur in that case, but the perturbation expansion can then be Borel resummed⁶.

QCD is far more divergent than that. Here, we have no mass term to curb the infrared divergences. Instanton effects are to be expected at finite N , and possibly even at infinite N , and infrared renormalons invalidate simple attempts to prove convergence of Borel resummation⁷. Simple physical considerations, however, could be used to argue that unique procedures should exist to resum the planar diagrams.

5 Counting Planar Diagrams

Doing such resummations analytically is way beyond our present capabilities. What we can do is carry out such summations in a simple, zero-dimensional theory. This result will give us an analytic method to *count* planar diagrams. The ‘field theory’ is described by the action^{8,9}

$$S(M) = \text{Tr} \left(-\frac{1}{2}M^2 + \frac{1}{3}gM^3 + \frac{1}{4}\lambda M^4 \right), \quad (24)$$

where M is an $N \times N$ -dimensional matrix, in the limit

$$N \rightarrow \infty, \quad g, \lambda \rightarrow 0, \quad N g^2 = \tilde{g}^2 \quad \text{and} \quad N \lambda = \tilde{\lambda} \quad \text{fixed.} \quad (25)$$

Henceforth, the tilde ($\tilde{}$) will be omitted. Precise calculations of the generating functions for the number of planar diagrams, with various types of restrictions on them, were carried out in Ref.¹⁰. Different cases were studied, all related to one another by exact equations. Part of that paper is based on pioneering work by Koplik, Neveu and Nussinov⁸— which in turn makes use of earlier work by Tutte⁹— and on the work by Brézin, Itzykson, Parisi and Zuber¹¹. The latter apply matrix theory to do the integration with the action (1). It is instructive instead to work directly from the functional equations, as these will be easier to handle in the QCD case, and they are also more

transparent in diagrammatic approaches. The relations are read off directly from the diagrams.

A delicate problem then is the choice of boundary conditions for these equations. They can be derived by carefully considering the holomorphic structure that the generating functions are required to have. Once this is understood, a fundamental solution is obtained for the generating function describing *the numbers of all planar diagrams for all multiparticle connected Green functions, with given numbers V_3 of three-point vertices, V_4 four-point vertices, and E external lines.*

We denote the number of all connected planar diagrams by $N_{(E,V_3,V_4)}$. The generating function $F(z, g, \lambda)$ is defined by

$$F(z, g, \lambda) \equiv \sum_{\substack{E=1, \\ V_3=0, V_4=0}}^{\infty, \infty, \infty} z^{E-1} g^{V_3} \lambda^{V_4} N_{(E-1, V_3, V_4)} . \quad (26)$$

Here, it is for technical reasons that we start counting at $E = 1$.

The recursion equation for the generating function $F \stackrel{\text{def}}{=} F(z, g, \lambda)$, is

$$F = z + g \left(F^2 + \frac{F - F_{(0)}}{z} \right) + \lambda \left(F^3 + 2F \frac{F - F_{(0)}}{z} + \frac{F - F_{(0)} - F_{(1)}z}{z^2} + F_{(0)} \frac{F - F_{(0)}}{z} \right) . \quad (27)$$

Here, $F_{(0)} = F(0, g, \lambda)$ is the tadpole diagram ($E = 1$), and $F_{(1)} = \frac{\partial}{\partial z} F(z, g, \lambda)|_{z=0}$ is the propagator ($E = 2$). Written diagrammatically, we have

$$\begin{aligned}
 & \text{---} \boxed{F} \text{---} = \text{---} + \text{---} \begin{array}{c} \boxed{F} \\ \boxed{F} \end{array} \text{---} + \text{---} \boxed{F} \text{---} + \text{---} \begin{array}{c} \boxed{F} \\ \boxed{F} \\ \boxed{F} \end{array} \text{---} \\
 & \quad \quad \quad \text{(a)} \quad \quad \quad \text{(b)} \quad \quad \quad \text{(c)} \quad \quad \quad \text{(d)} \\
 & + \text{---} \begin{array}{c} \boxed{F} \\ \boxed{F} \end{array} \text{---} + \text{---} \begin{array}{c} \boxed{F} \\ \boxed{F} \end{array} \text{---} + \text{---} \boxed{F} \text{---} + \text{---} \boxed{F} \boxed{F} \text{---} . \quad (28) \\
 & \quad \quad \quad \text{(e)} \quad \quad \quad \text{(f)} \quad \quad \quad \text{(g)} \quad \quad \quad \text{(h)}
 \end{aligned}$$

Note that inside the 1-loop terms the tadpole diagrams had to be removed by hand, and in the 2-loop term the propagator had to be removed. Such equations can be solved! There are three methods:

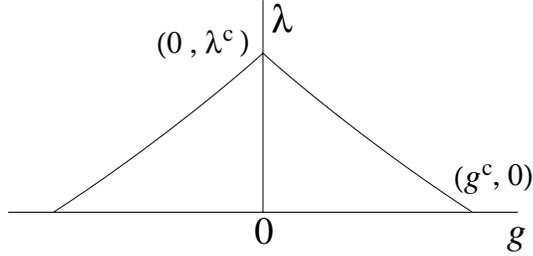


Figure 4. Line of critical λ and g values to a high precision. The lines are not exactly straight, and in fact form a single curve with a cusp singularity. The corners are given by the values $\lambda^c = \frac{1}{12}$ and $g^c = 1/\sqrt{12\sqrt{3}}$.

$$a = 4 : \quad \text{also no dressed 3-vertices:} \quad \text{---} \circ \text{---} \quad (34)$$

$$a = 5 : \quad \text{also no dressed 4-vertices:} \quad \text{---} \circ \text{---} \quad (35)$$

In each of these cases, there is a domain in the g - λ plane where the zero-dimensional theory converges. The location of the boundaries of these domains then tells us how the number of diagrams of the specific types increase at high orders. For example, the last curve, of the case $a = 5$, is given by the parametric equations

$$g_5^c = \frac{2t^2(6-t^2)^2(6-3t^2+t^3)}{(24-72t-12t^2+56t^3-10t^4-10t^5+3t^6)^{3/2}},$$

$$\lambda_5^c = (3456 - 31104t + 25920t^2 + 55296t^3 - 180576t^4 - 44064t^5 + 247824t^6 - 27936t^7 - 147672t^8 + 52760t^9 + 38076t^{10} - 24432t^{11} - 1762t^{12} + 4506t^{13} - 921t^{14} - 202t^{15} + 114t^{16} - 18t^{17} + t^{18}) \times (24 - 72t - 12t^2 + 56t^3 - 10t^4 - 10t^5 + 3t^6)^{-3}. \quad (36)$$

These, as well as the other expressions that were derived in Ref. ¹⁰, are all exact. The domains of convergence for the various cases are displayed in Fig. 5. The last boundary, that of region # 5, is the one given by Eqs. (36). Figures 4 and 5 show the actual domains with great precision. All curves have

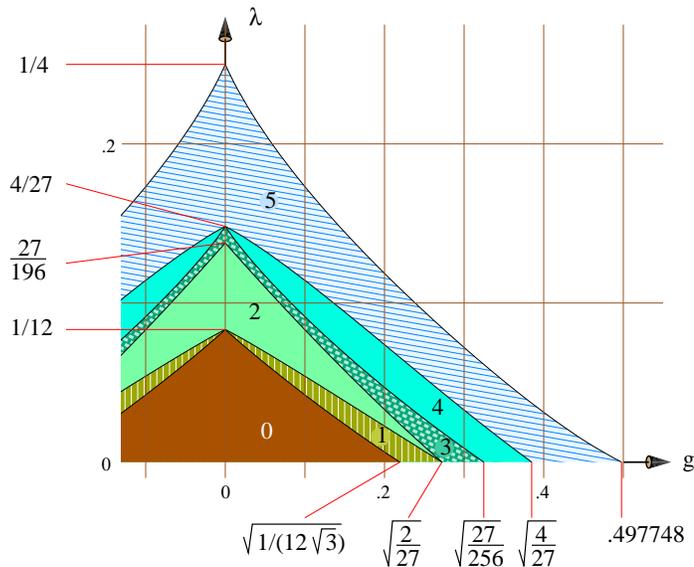


Figure 5. Critical regions after imposing different constraints.

cusps at $g = 0$. Many of our results were obtained and/or checked using the computer program *Mathematica*.

6 The $g^2 N$ Expansion

We conclude with a short discussion on constructing perturbative planar QCD amplitudes at all orders without divergences.

A renormalization scheme that appears to be not so well-known⁶ can be set up in the following way. It works particularly well for planar field theories in 4 space-time dimensions, but it also applies to several other quantum field theories, notably QED. Presumably, QCD with $N_c = 3$ can also be covered along these lines, but some technical details have not been worked out to my knowledge. Our scheme consists of first collecting all one-particle irreducible 2-, 3- and 4-point diagrams, and formally considering the non-local quartic effective action generated by these diagrams.

Next, consider all diagrams using the Feynman rules derived from this

action, but *with the limitation that only those diagrams that are absolutely ultraviolet convergent are included*. No superficially ultraviolet divergent subgraph is accepted. To be precise, we omit all diagrams with 4 or less external lines, as well as all diagrams containing any non-trivial irreducible subdiagram with 4 or less external lines. They are the diagrams of type 5 of the previous section.

Clearly, one expects that the 1PI subgraphs with 4 or less external lines have already been taken care of by our use of the quartic effective action instead of the bare Lagrangian. The important issue to be addressed is, whether the counting of all diagrams was done correctly so as to obtain the required physical amplitudes. But this is not difficult to prove:

Theorem: The above procedure correctly reproduces the complete amplitudes for the original theory. No diagrams are over-counted or under-counted.

The proof of the theorem is to be found in Refs^{6,10}. One deduces that *if all one-particle irreducible 2-, 3-, and 4-point vertices are known, all amplitudes can be derived without encountering any divergent (sub-)graph(s)*.

Subsequently, what has to be done to complete a perturbative computational scheme, is to establish an algorithm to compute the irreducible 2-, 3-, and 4-point vertices (and, if they occur, the tadpole diagrams as well). Actually, this is simple. Consider a 4-point 1PI diagram $\Gamma(p_1, p_2, p_3, p_4)$. Here p_μ are the external momenta, and $p_1 + p_2 + p_3 + p_4 = 0$. Now consider the difference

$$\Gamma(p_1 + k, p_2 - k, p_3, p_4) - \Gamma(p_1, p_2, p_3, p_4) \equiv k_\mu \Delta^\mu(p_1, \underline{k}, p_2 - k, p_3, p_4). \quad (37)$$

The underlining here refers to the fact that, in the function Δ^μ the external line with momentum k follows distinct Feynman rules.

If we follow a path inside the diagram, we can consider the entire expression for Δ^μ as being built from expressions containing differences. For instance, in the propagators:

$$\frac{1}{(p+k)^2 + m^2} - \frac{1}{p^2 + m^2} = \frac{k_\mu (-2p - k)^\mu}{[(p+k)^2 + m^2](p^2 + m^2)}, \quad (38)$$

or else, in the 3-vertices:

$$(p+k)_\nu - p_\nu = k_\mu \delta_\nu^\mu. \quad (39)$$

We notice that the expressions for Δ^μ are all (superficially) ultraviolet convergent! Actually, one may set up unambiguous Feynman rules for $\Delta^\mu(p_1, \underline{k}, p_2, -kp_3, p_4)$ and observe that this amplitude exactly behaves as a 5-point diagram, hence it is (superficially) convergent.

For the 3-point and the 2-point diagrams, one can do exactly the same thing by differentiating more than once. In practice, what one finds is that there is a set of rules containing fundamental irreducible 2-, 3- and 4-point vertices, and in addition rules to determine their differences at different values of their momenta. The complete procedure thus leads to the following situation.

We start by postulating the so-called ‘primary vertex functions’. These are not only the irreducible 2-point functions $\Gamma_{[2]}(p, -p)$, the irreducible 3-point functions $\Gamma_{[3]}(p_1, p_2, -p_1 - p_2)$ and the irreducible 4-point functions $\Gamma_{[4]}(p_1, \dots, p_4)$, but also, in addition, the difference functions $\Delta_{[2]}^\mu(p, \underline{k}, -p - k)$ and $\Delta_{[3]}^\mu(p_1, \underline{k}, p_2 - k, -p_1 - p_2)$, and finally the functions $U_{[2]}$, obtained by differentiating $\Delta_{[2]}^\mu$ once more:

$$\Delta_{[2]}^\mu(p_1 + q, p_2 - q, -p_1 - p_2) - \Delta_{[2]}^\mu(p_1, p_2, -p_1 - p_2) \equiv q_\nu U_{[2]}^{\mu\nu}(p_1 - q, \underline{q}, p_2, -p_1 - p_2), \quad (40)$$

where one of the other external lines, p_1 or p_2 , is underlined. The double underlining is here to denote that the two entries are to be treated distinctly (because of the factor k_μ , the functions $U_{[2]}^{\mu\nu}$ are not symmetric under interchange of k and q).

These primary vertex functions are derived by first considering the differences for $\Gamma_{[4]}$, $\Delta_{[3]}^\mu$ and $U_{[2]}^{\mu\nu}$ at two different sets of external momenta. These expressions are handled as if they were irreducible 5-point diagrams. These are expanded in terms of planar diagrams, where all irreducible subgraphs of 4 or less external lines are bundled to form the primary vertex functions. At one of the the edges of such a diagram, we then encounter one of the functions Δ^μ or $U^{\mu\nu}$.

This way, we arrive at difference equations for the primary vertices, with on the r.h.s. again the primary vertices. The primary vertices $\Gamma_{[3]}$, $\Delta_{[2]}^\mu$ and $\Gamma_{[2]}$ are then obtained by integrating equations such as (37) with respect to the external momentum k . The *integration constants* will be associated with the values of the vertices and propagators in the far ultraviolet region, where the theory is asymptotically classical. This completes the procedure to obtain all amplitudes by iteration.

Technical implementation of our scheme requires that in all diagrams, an unambiguous path can be defined from one external line to another. In QED, one may use the paths defined by the electron lines. In a planar theory, one may define the paths to run along the edges of a diagram.

Three remarks are in order:

- i.* The procedure is effectively a renormalization group procedure. The functions Δ^μ and $U^{\mu\nu}$ play the role of beta functions.
- ii.* The procedure is essentially still perturbative, since the planar diagrams must still be summed. Our beta functions are free of ultraviolet divergences, but the summation over planar diagrams may well diverge.
- iii.* The procedure only works if the integrations do not lead to clashes. This implies that it is not to be viewed as a substitute for regularization procedures such as dimensional regularization. We still need dimensional regularization if we want to *prove* that the method is unambiguous, which, of course, it is in the case of planar QCD.

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