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# A PHYSICAL INTERPRETATION OF GRAVITATIONAL INSTANTONS

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A gravitational instanton solution found by Eguchi and Hanson some time ago is found to describe a special set of states in the Hilbert space of quantum gravity. We show how states of this type have to be normalized and how expectation values of operators are to be computed in such states. It then becomes clear that these states actually describe physical particles much like solitons. Being unstable they decay into large numbers of ordinary particles.

## 1. Introduction

Some time ago Eguchi and Hanson discovered a non-trivial spherically symmetric solution of Einstein's equations in 4-dimensional euclidean space [1]. It was then shown by Prasad [2] that this solution is a special case of solutions found by Gibbons and Hawking [3]. What is remarkable about the Eguchi-Hanson solution is that it is asymptotically flat and carries zero action. At first sight one would think that it therefore should contribute significantly to functional integrals like a new kind of wormholes. But this is not the case, because the asymptotical space surrounding it is only locally flat; it differs from ordinary space by the fact that points located opposite to each other with respect to the origin must be identified pairwise.

With such a peculiar boundary condition any sensible physical interpretation of this metric remained unclear until now, and most investigators suspected that its role in low-energy quantum gravity would be negligible. Extensions of the solution were found. Also electromagnetic fields with nonvanishing $F_{\mu\nu}\tilde{F}_{\mu\nu}$ could be inserted, giving rise to interesting solutions for chiral fermions, but again, their interpretation remained obscure.

We now claim that a very specific interpretation can be given. This is important because the scale of these solutions can be chosen to be much larger than the Planck length (just like the scale of black holes), so that we might arrive at predictions which are quite independent of renormalization difficulties at the Planck scale. The "particles" we find are similar to black holes in the sense that they too are surrounded by a sea of thermal conventional particles, but they differ from black holes in many respects. In particular, they allow a direct calculation of their decay or production amplitude to and from ordinary particles. Perhaps they describe the decay and formation of black holes.

The special topological nature of the Eguchi-Hanson metric is reflected in the fact that the particles surrounding the system are not truly thermal, but pairwise correlated at antigonal points, as we will explain. Thus, the physical consequence of the existence of such a metric is the existence of particle states in Hilbert space, such that these particle-like objects are surrounded by a "hot soup" with these special pairwise correlations. In more practical terms this presumably implies the existence of unstable objects decaying with strong antigonal correlations in their decay products.

An important aspect of our system is that, because of the correlations, it is *not* described by a density matrix but by a *single pure* state, in spite of its apparent thermal nature.

In the next section we summarise the properties of the Eguchi-Hanson metric, and then we explain our interpretation. How to treat a thermal heat bath which is correlated with some region far away in space and time is explained in sect. 4. An excited mode of the Eguchi-Hanson configuration corresponds with a state with leptonic chiral quantum numbers. This we show in sect. 5. There are also more exotic variants in which the fermionic boundary condition is modified (sect. 6).

## 2. The Eguchi-Hanson metric

In this section we give a resumé of some properties of certain solutions to the euclidean Einstein equations without detailed derivations, which can be found elsewhere [1–3]. We are interested in a locally euclidean metric that can be described in several ways. Gibbons and Hawking [3] found a large class of solutions of the self-duality equation

$$R_{ab\mu\nu} = -\tfrac{1}{2}\epsilon_{abcd}\sqrt{g}\,R^{cd}{}_{\mu\nu} \tag{2.1}$$

(which guarantees the vanishing of the Ricci tensor $R_{\mu\nu}$), of the following form:

$$\mathrm{d}s^2 = U^{-1}(\mathrm{d}\tau + \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{x})^2 + U(\mathrm{d}\boldsymbol{x})^2, \tag{2.2}$$

where the scalar field $U$ and the vector field $\boldsymbol{\omega}$ depend on $\boldsymbol{x}$ but not on $\tau$.

The equations (2.1) then become linear:

$$\partial_a U = \epsilon_{aij}\partial_i\omega_j\,. \tag{2.3}$$

There is a gauge redundancy. The coordinate transformation

$$\tau \to \tau + \Lambda(\boldsymbol{x}), \tag{2.4}$$

gauge transforms $\omega$:

$$\omega \to \omega - \partial \Lambda . \tag{2.5}$$

A general solution of (2.3) is

$$U = \sum_{n=1}^{s} \frac{1}{|x - x_n|} + C , \tag{2.6}$$

so that at the points $x_n$ there are magnetic monopole charges for the field $\omega$. Dirac strings emanating from these points are physically unobservable because of the invariance (2.4), (2.5).

The singularities at the points $x = x_n$ are also coordinate artifacts, if $\tau$ is chosen to be periodic with period $4\pi$.

As was explicitly shown by Prasad [2], the case $s = 2$, $C = 0$ can be transformed into the spherically symmetric Eguchi-Hanson metric. Put

$$n = \pm , \qquad x_{\pm} = (0, 0, \pm Z_0) , \tag{2.7}$$

$$x_1 = Z_0 \sinh\alpha \sin\theta \cos\psi , \tag{2.8}$$

$$x_2 = Z_0 \sinh\alpha \sin\theta \sin\psi , \tag{2.9}$$

$$x_3 = Z_0 \cosh\alpha \cos\theta , \tag{2.10}$$

$$\tau = 2\varphi . \tag{2.11}$$

Writing

$$8Z_0 \cosh\alpha = r^2 , \qquad 8Z_0 = a^2 , \tag{2.12}$$

one can define the new coordinates $(x, y, z, t)$ as follows:

$$x + iy = r \cos \tfrac{1}{2}\theta \, e^{\frac{1}{2}i(\psi + \varphi)} ; \tag{2.13}$$

$$z + it = r \sin \tfrac{1}{2}\theta \, e^{\frac{1}{2}i(\psi - \varphi)} . \tag{2.14}$$

In terms of these coordinates the metric becomes

$$g_{\mu\nu} = \delta_{\mu\nu} + F_1(r) x_\mu x_\nu - F_2(r) \epsilon_{\mu\alpha} x_\alpha \epsilon_{\nu\beta} x_\beta , \tag{2.15}$$

whose inverse is

$$g^{\mu\nu} = \delta_{\mu\nu} - F_2(r) x_\mu x_\nu + F_1(r) \epsilon_{\mu\alpha} x_\alpha \epsilon_{\nu\beta} x_\beta , \tag{2.16}$$

where

$$r^2 F_1 = a^4 (r^4 - a^4)^{-1} , \qquad r^2 F_2 = a^4 / r^4 , \tag{2.17}$$

and $\epsilon_{\mu\nu}$ is defined by

$$\epsilon_{\mu\nu}x_\nu = (y, -x, t, -z)\,. \tag{2.18}$$

The coordinates are limited to the range

$$x_\mu^2 = r^2 \geq a^2\,. \tag{2.19}$$

One may convince oneself that the singularity at $r = a$ is a coordinate artifact. However, this is only so if all points $x$ are identified with their antipodes $-x$. In (2.8)–(2.11) one sees that

$$0 \leq \theta < \pi\,, \qquad 0 \leq \varphi < 2\pi\,, \qquad 0 \leq \psi < 2\pi\,, \tag{2.20}$$

and in (2.13), (2.14) one sees that the mapping $\psi \to \psi + 2\pi$ maps $x$ into $-x$.

Apart from this strange disease, the metric (2.15), (2.16) is a beautiful one. It goes to flat space very rapidly as $r \to \infty$. So at first sight it is just like an instanton as these are also known in non-abelian gauge theories [4]. However, the fact that the boundary is not really an ordinary flat space turns it actually into something else. In some sense, we will argue that the name "gravitational instanton" can be kept, but the way to deal with them in a physical theory will turn out to be different from the gauge theory instantons.

## 3. Interpretation

Because of the identification $(x, -x)$ the asymptotically flat space surrounding our "instanton" is really only a half space. We could limit its description to the region $t \geq 0$ ($t$ being euclidean time). Fields such as a scalar field $\varphi(x)$ are defined on this half space, with in addition a boundary condition: at $t = 0$ one must have

$$\varphi(\boldsymbol{x}, 0) = \varphi(-\boldsymbol{x}, 0)\,. \tag{3.1}$$

Functional integrals on a half space with all sorts of boundary conditions, e.g. of the form $\varphi(\boldsymbol{x}, 0) = \varphi_0(\boldsymbol{x})$, are often considered in physics. They correspond to the computation of amplitudes such as

$$\langle 0|\varphi_0(\boldsymbol{x})\rangle\,. \tag{3.2}$$

In other words, the functional integral over a half space with some boundary condition at $t = 0$ obviously corresponds to the expression for the inner product of some chosen wave function with the vacuum. Our first observation therefore is that the Eguchi-Hanson instanton defines a very special wave function, which we could indicate as $|\boldsymbol{a}, x\rangle$, where $\boldsymbol{a}$ is a vector whose direction indicates the orientation of the solution (in (2.18) the $z$ axis differs from the $x$ and $y$ directions), with absolute value $a$, and $x$ is the location of the center (usually taken to be at the origin of course).

Thus, the functional integral in the Eguchi-Hanson metric defines the amplitude

$$\langle 0|\boldsymbol{a}, x\rangle, \tag{3.3}$$

but it will not be difficult to find other matrix elements such as

$$\langle \boldsymbol{p}_1, \boldsymbol{p}_2|\boldsymbol{a}, x\rangle, \tag{3.4}$$

where $|\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\rangle$ is an $n$-particle state. It is found by allowing source insertions in the Eguchi-Hanson space, which give, if $\phi$ is a scalar field,

$$J(x_1)\dots J(x_n)\langle 0|\phi(x_1)\dots\phi(x_n)|\boldsymbol{a}, x\rangle, \tag{3.5}$$

after which we put $J(x_i)$ on mass shell as usual. In particular, one-graviton matrix elements can very easily be found via

$$\langle 0|g_{\mu\nu}(x_1)|\boldsymbol{a}, x\rangle = g_{\mu\nu}(x_1), \tag{3.6}$$

which is just the metric (2.15).

An important question however is how $|\boldsymbol{a}, x\rangle$ is normalized. How do we compute for instance

$$\langle \boldsymbol{a}, \boldsymbol{x}, t|\boldsymbol{b}, \boldsymbol{x}', t\rangle = ? \tag{3.7}$$

Here, $t$ is euclidean time. We will argue that for equal times this matrix element is infinite or meaningless. Now we can also consider

$$\langle \boldsymbol{a}, \boldsymbol{x}, t|\mathrm{e}^{-\beta H}|\boldsymbol{b}, \boldsymbol{x}', t\rangle = \langle \boldsymbol{a}, \boldsymbol{x}, t|\boldsymbol{b}, \boldsymbol{x}', t+\beta\rangle, \tag{3.8}$$

which corresponds to a two-instanton configuration: one with the Eguchi-Hanson boundary conditions having the origin at $(\boldsymbol{x}, t)$ and one mirror image of that around the point $(\boldsymbol{x}', t+\beta)$, where $t$ and $t+\beta$ are euclidean time coordinates. Space-time is limited to the region $t \le x_4 \le t+\beta$. The functional integral corresponding to this configuration seems to make a lot of sense, as long as $\beta > a+b$, even if $\boldsymbol{a} = \boldsymbol{b}$ and $\boldsymbol{x} = \boldsymbol{x}'$.

Thus, the states $|\boldsymbol{a}, x\rangle$ cannot be properly normalized, but the states

$$|\boldsymbol{a}, \boldsymbol{x}, t\rangle_\beta \equiv \mathrm{e}^{-\frac{1}{2}\beta H}|\boldsymbol{a}, \boldsymbol{x}, t\rangle \tag{3.9}$$

can be. Note that insertion of the operator $\mathrm{e}^{-\frac{1}{2}\beta H}$ implies that the origin is shifted over a distance $\frac{1}{2}\beta$ in euclidean space.

Since the hamiltonian vanishes in the vacuum (assuming that the cosmological constant vanishes) the insertion of $e^{-\frac{1}{2}\beta H}$ does not affect the vacuum matrix element (3.3). Similarly, there will be no difficulty in calculating

$$\langle \boldsymbol{p}, \ldots | \boldsymbol{a}, \boldsymbol{x}, 0 \rangle_{\beta} \tag{3.10}$$

using source insertions, instead of (3.4).

However, closer study of these expressions reveals that there still is a divergence even if $\beta$ is large. This is a simple volume effect, to be explained in the next section.

## 4. Thermally correlated spaces

Let us concentrate on regions far away from the "instanton"'s origin. There we have ordinary flat space, but the state $|\boldsymbol{a}, \boldsymbol{x}, 0\rangle_0$ still features perfect correlations from the constraint

$$\varphi(\boldsymbol{x}, 0) = \varphi(-\boldsymbol{x}, 0). \tag{4.1}$$

This implies that even far away from the origin we have a significant deviation from the vacuum. What do we have far away from the origin?

Let us isolate some region $V_1$ far away from the origin, diagonally opposite to region $V_2$ which we give the same shape as $V_1$. In a basis where the fields $\varphi$ are diagonal, the wave function $|\boldsymbol{a}, 0\rangle_0$ that we are interested in is

$$|\boldsymbol{a}, 0\rangle_0 = C \prod_{x} \int \mathrm{d}\varphi |\varphi(\boldsymbol{x})\rangle_1 |\varphi(-\boldsymbol{x})\rangle_2, \tag{4.2}$$

where $C$ is some normalization factor. Let $|\Omega\rangle_{1,2}$ be the vacuum states in $V_{1,2}$, and $|\Omega\rangle \equiv |\Omega\rangle_1 |\Omega\rangle_2$. Then, since the Eguchi-Hanson action vanishes, we expect $C$ to be determined by

$$\langle \Omega | \boldsymbol{a}, 0 \rangle_0 = 1. \tag{4.3}$$

Obviously, (4.2) is very different from the vacuum state, and it is no big surprise that its norm comes out to be infinite.

Now consider the state

$$|\boldsymbol{a}, 0\rangle_{\beta} = e^{-\frac{1}{2}\beta H} |\boldsymbol{a}, 0\rangle_0. \tag{4.4}$$

Let $A_1$ be an operator describing an observable in space $V_1$. Write

$$\langle A_1 \rangle \equiv \langle\langle A_1 \rangle\rangle / \langle\langle 1 \rangle\rangle, \qquad \langle\langle A \rangle\rangle \equiv {}_{\beta}\langle \boldsymbol{a}, 0 | A | \boldsymbol{a}, 0 \rangle_{\beta}. \tag{4.5}$$

Then

$$\langle\langle A_1 \rangle\rangle = |C|^2 \int \mathrm{d}\varphi_1 \int \mathrm{d}\varphi_2 \langle \varphi_1 | e^{-\frac{1}{2}\beta H} A_1 e^{-\frac{1}{2}\beta H} | \varphi_2 \rangle \langle \varphi_1 | e^{-\beta H} | \varphi_2 \rangle. \tag{4.6}$$

By PCT invariance we must have

$$\langle \varphi_1(\boldsymbol{x},0)|\mathrm{e}^{-\beta H}|\varphi_2(\boldsymbol{x},0)\rangle = \langle \varphi_2^*(-\boldsymbol{x},0)|\mathrm{e}^{-\beta H}|\varphi_1^*(-\boldsymbol{x},0)\rangle^*, \tag{4.7}$$

and since we are in euclidean space and $\varphi$ was assumed to be a real field, we may ignore the asterisks in (4.7) (for complex fields the asterisks would have to be added in (4.2)).

We now see that completeness can be used to derive

$$\langle\langle A_1\rangle\rangle = |C|^2\,\mathrm{Tr}\,A_1\,\mathrm{e}^{-2\beta H}. \tag{4.8}$$

From (4.5) we see that the expectation values for all operators which describe observables in $V_1$ only, derive from a density matrix

$$\rho = \mathrm{e}^{-2\beta H}/\mathrm{Tr}\,\mathrm{e}^{-2\beta H}. \tag{4.9}$$

Clearly an observer in $V_1$ experiences a temperature $T = 1/2\beta$.

The same holds for an observer in $V_2$. But $V_1$ and $V_2$ are correlated. The expectation value for the product operator $A = A_1A_2$, where $A_2$ acts in $V_2$, follows from

$$\langle\langle A_1A_2\rangle\rangle = |C|^2\,\mathrm{Tr}\,A_1\,\mathrm{e}^{-\beta H}A_2\,\mathrm{e}^{-\beta H}. \tag{4.10}$$

Indeed, the combined space $V_1 + V_2$ is not in a quantum-mechanically mixed state but in a pure state. We can write:

$$|\boldsymbol{a},0\rangle_\beta = \sum_i \mathrm{e}^{-\beta E_i}|E_i\rangle_1|E_i\rangle_2, \tag{4.11}$$

where $|E_i\rangle$ are the energy eigenstates.

Herewith we unravelled the nature of the states introduced in the previous chapter when seen at large distances from the origin. Because of (4.3) the constant of proportionality in (4.11) must be one. But then it will be clear that the norm of this state will diverge with the volume. In flat space one will have

$$_\beta\langle\boldsymbol{a},0|\boldsymbol{a},0\rangle_\beta = \sum_i \mathrm{e}^{-2\beta E_i} = \mathrm{e}^{-2\beta F(2\beta)}, \tag{4.12}$$

where $F(2\beta)$ is the free energy of the vacuum at inverse temperature $2\beta$. This is an extensive quantity, whose value grows linearly with the volume $V_1$:

$$\langle\boldsymbol{a},0|\boldsymbol{a},0\rangle = \mathrm{e}^{f(2\beta)V_1}. \tag{4.13}$$

$V_1$ is half the total volume of three-space ($V = V_1 + V_2$).

## 5. Chirality

The Eguchi-Hanson configuration is topologically non-trivial in such a way that a Maxwell field $F_{\mu\nu}$ can be inserted [5] that is self-dual:

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu} . \tag{5.1}$$

The explicit form is [5]:

$$A_\mu = 2Na^2\epsilon_{\mu\nu}x_\nu r^{-4} . \tag{5.2}$$

The existence of a spin structure and the boundary condition at $\infty$ require $N$ to be integer (more about this in the next section). We have

$$\int \mathrm{d}^4x\, F_{\mu\nu}F_{\mu\nu} = 16\pi^2N^2/e^2 , \tag{5.3}$$

so that $n = N^2$ is the number of chiral zero modes of the euclidean Dirac equation.

We also have

$$\langle\Omega|\boldsymbol{a}, N, \boldsymbol{x}\rangle = \mathcal{O}(\mathrm{e}^{-\pi N^2/\alpha}) , \qquad \alpha = e^2/4\pi , \tag{5.4}$$

so that the inner products are now very different. Eq. (5.4) holds for the pure Einstein-Maxwell system. If light charged fermions are present, such as the electron, we get an additional factor $m_\mathrm{e}$ from each Dirac zero mode. On the other hand, if a chiral symmetry breaking source is inserted, e.g.

$$J(x_1)\bar{\psi}(x_1)\psi(x_1) , \tag{5.5}$$

then $m_\mathrm{e}$ is replaced by $J(x_1)$. Hence, If $m_\mathrm{e}$ is small,

$$|\langle\Omega|\bar{\psi}\psi|\boldsymbol{a},1,\boldsymbol{x}\rangle| \gg |\langle\Omega|\boldsymbol{a},1,\boldsymbol{x}\rangle| , \tag{5.6}$$

also at finite $\beta$, and this is easy to interpret: apparently the state $|\boldsymbol{a},1,\boldsymbol{x}\rangle_\beta$ has two units of chiral fermion number.

## 6. The fermionic boundary condition

The antipodal constraint corresponds to eq. (4.1) for a scalar field. In particular it should hold also far away from the origin. For vector fields one expects

$$A_\mu(\boldsymbol{x},0) = -A_\mu(-\boldsymbol{x},0) , \tag{6.1}$$

and for spinors

$$\psi(\boldsymbol{x},0) = P\psi(-\boldsymbol{x}.\ \ ) , \tag{6.2}$$

where $P$ is an element of the group $SU(2) \otimes SU(2)$ corresponding to the reflection $x_\mu \to -x_\mu$. At first sight one might think that $P$ is only defined up to a sign, but this is not so. $\psi(x)$ must be continuous along a path from $x$ to $-x$ and this path defines an element of the covering group of SO(4) unambiguously. Indeed, the only allowed sign in the self-dual Eguchi-Hanson instanton is:

$$P = -\gamma_5 . \tag{6.3}$$

The bra state $\langle \boldsymbol{a}, \boldsymbol{x}, 0|$ is a parity reflection of the ket state $|\boldsymbol{a}, \boldsymbol{x}, 0\rangle$ in space-time, so if the ket state is self-dual then the bra state is *anti*-self-dual, and there the boundary condition will be

$$\psi(\boldsymbol{x}, \beta) = +\gamma_5 \psi(-\boldsymbol{x}, \beta) . \tag{6.4}$$

Consequently, the matrix element

$$_\beta\langle \boldsymbol{a}, \boldsymbol{x}, 0|\psi(x_1) \dots |\boldsymbol{a}, \boldsymbol{x}, 0\rangle_\beta \tag{6.5}$$

will change its sign when $x_1$ runs from $(\boldsymbol{x}_1, 0)$ to $(\boldsymbol{x}_1, 2\beta)$. This implies that fermionic fields $\psi$ have antiperiodic boundary conditions as these occur naturally [6] in functional integrals for thermodynamic systems at temperature $1/2\beta$.

Now one may also ask whether a boundary condition of the form

$$\psi(\boldsymbol{x}, 0) = -P\psi(-\boldsymbol{x}, 0) = +\gamma_5 \psi(-\boldsymbol{x}, 0) \tag{6.6}$$

is possible for the ket state, instead of condition (6.2). In the Eguchi-Hanson metric as described above this turns out *not* to be allowed; such a field $\psi$ would be singular in regular coordinates at $x^2 = a^2$. A gauge transformation with Dirac string singularity at $x^2 = a^2$ would restore (6.2). Suppose we require (6.6) at large $x^2$ in the absence of a vector potential. Then one can convince oneself that close to $x^2 = a^2$ a vector field of the type (5.1)–(5.2) has to be inserted into this space, but now with $N$ half-odd-integer: $N = k + \frac{1}{2}$, $k$ integer. The Atiyah-Singer index turns out to be

$$n = N^2 - \tfrac{1}{4} = k^2 + k . \tag{6.7}$$

Only then a spin structure with boundary condition (6.6) is allowed [7]. The $-\frac{1}{4}$ in (6.7) is a topological effect from our non-trivial boundary condition. We checked the index theorems by explicitly computing the solutions of the Dirac equation in these metrics.

One reason to be interested in the aspects of this particular instanton is the fact that for $N = \pm\frac{1}{2}$ the corresponding states $|\boldsymbol{a}, \pm\frac{1}{2}\rangle$ have inner products

$$\langle 0|\boldsymbol{a}, \pm\tfrac{1}{2}\rangle = \mathcal{O}(\mathrm{e}^{-\pi/4\alpha}) , \tag{6.8}$$

a number that might relate some widely separated constants of nature. For instance, by comparing the known numbers we remark that

$$Gm_e^2 \cong \left(\alpha\sqrt{2}\right)^{-1} e^{-\pi/4\alpha}\,(?)\,, \tag{6.9}$$

in natural units ($G$ is Newton's constant and $m_e$ is the electron mass). Since we can take $a$ large with respect to $m_e^{-1}$ it makes sense to compare the physical, on mass shell values of $\alpha$ and $m_e$ in (6.8) and (6.9).

## 7. Discussion

The Eguchi-Hanson configurations may seem to be a rather exotic consequence of general relativity in euclidean space, but as we have seen, they represent states in Hilbert space that do not seem to have very exotic properties. Replacing euclidean time by $\frac{1}{2}\beta + it$ we see that in Minkowski space we are dealing with an "event" (so that the name "instanton" makes sense). The fact that the metric in Minkowski space is complex (like all self-dual metrics) and has zero energy is not a problem since we are dealing with off-diagonal matrix elements (see eq. (3.6)). Diagonal expectation values for the $\beta$-states can easily be seen to be real.

As far as we can see we are dealing with just very unstable objects, decaying rapidly into the vacuum, surrounded by radiation with a strong antipodal correlation (our states are symmetric under $CPT$). It should be possible to identify them with existing objects, perhaps "micro-black holes" in a rapidly decaying state. We think it is important that they may be large. There is no limit to the value of $a$. Because of this, one may use known field equations to compute their properties, as soon as $a \gtrsim 0.01\ \mathrm{GeV}^{-1}$.

The value of $a$ is indeed free. One may or may not decide to integrate over $\boldsymbol{a}$, depending on what state one is interested in:

$$|\psi\rangle = \int \mathrm{d}\boldsymbol{a} f(\boldsymbol{a})|\boldsymbol{a}, x\rangle\,, \tag{7.1}$$

with arbitrary $f$. This is different from the situation with the instantons in gauge theories, where integration over the sizes $a$ and the form of $f$ are prescribed.

The amplitude

$$\langle \boldsymbol{a}, \boldsymbol{x}, t|e^{-\beta H}|\boldsymbol{b}, \boldsymbol{x}', t\rangle\,, \tag{7.2}$$

corresponds to a functional integral in a euclidean space-time with two Eguchi-Hanson objects (one self-dual, one anti-self-dual, so the entire configuration is neither). Asymptotically this space-time is a torus ($R^3 \otimes S^1$), with periodicity $2\beta$. A closed form for this metric is not known to the author.

Other configurations mentioned in ref. [5] are surrounded by an asymptotically flat space with more than two points identified, dividing $R^4$ by a discrete subgroup of the rotation group, with the dodecahedral group as an extreme example. It would be interesting to see what these metrics imply for our Hilbert space.

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