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A categorical approach to continuous logic using MV-algebras

Master's Thesis

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1 Introduction

Ever since the 1920s, logicians have studied formal logical systems which, unlike classical first order logic, admit more than two truth values. As early as 1930 even, Łukasiewicz and Tarski presented in [15] a logic with an infinite truth set, thus introducing the concept of an infinite valued logic. This logic eventually formed the basis of a much larger class of mathematical logics, called fuzzy logics. A very complete overview of fuzzy logics and their properties can be found in [12].

Another interesting example of an infinite valued logic is presented in [9]. Here, the authors present a logic that was designed to have models which can be interpreted as metric structures. Using a very general framework that allowed for many different truth spaces, Chang and Keisler show that many concepts of first-order logic can be generalized to the new infinite valued logic. However, this generalization came with a price. The amount of extra structure that was necessary to accommodate for the different truth spaces made the logic somewhat unfeasible to work with, and no further research in this direction was done for quite a while. Since then, other mathematicians have introduced new logics to study metric spaces, such as Henson's logic for Banach structures, introduced by Henson in [13], and the concept of compact abstract theories, introduced by Ben Yaacov in [1]. Both of these logics only allow the two classical truth values.

Expanding on the latter two concepts, Ben Yaacov and Usvyatsov introduced continuous logic. An introduction to continuous logic is given in [6]. Continuous logic is a generalization of classical first-order logic, in the sense that continuous logic is based on the notions of a language which contains terms and formulas and the interpretation of a language in structures. The main differences are that the truth space of continuous logic is the continuous interval $[0, 1]$, and that the structures in continuous logic are all necessarily endowed with a complete metric. Because of these differences, the logical connectives and the quantifiers are also different. Generally speaking, connectives are continuous functions $u : [0, 1]^n \to [0, 1]$ and the quantifiers are the supremum and infemum functions.

In [6] the authors show that continuous logic is equivalent to special cases of both Henson's logic and the notion of compact abstract theories. To further strengthen the former equivalence, Ben Yaacov presented in [3] a variant of continuous logic which is equipped with the truth space $[0, \infty)$. This variant is called unbounded continuous logic, and was specifically designed to resemble Henson's logic, whilst also clearly being a generalisation of bounded continuous logic.

In this thesis, we will study continuous logic using categorical logic. Categorical logic is the study of logic using category theory. In particular, category theory stresses the functional behaviour of logical connectives and quantifiers in certain categories. One way to formalise this concept is with the notion of a hyperdoctrine, which is in general a pseudo-functor $\mathbf{P} : \mathcal{C}^{op} \to Preorder$. In the classical setting $\mathcal{C}$ is the category of sets, and every set X is sent to its poset of subobjects, but for different kinds of logics different hyperdoctrines are possible.

Recently, Figueroa attempted in his master's thesis to analyse continuous logic using the constructions known from categorical first-order logic. He defines a functor CMT with which he wants to interpret continuous logic. He does this by constructing a category of partial equivalence relations over a certain hyperdoctrine, which has certain nice properties, and defining another hyperdoctrine CL, with the category of partial equivalence relations as a base, and shows that CMT can be embedded into CL in such a way that CMT inherits all the relevant properties to make it into a hyperdoctrine. However, he does not manage to fully interpret continuous logic in this hyperdoctrine. Instead, the hyperdoctrine only manages to interpret a relatively weak fragment of continuous logic.

In this thesis, we will look at a different categorical approach for interpreting continuous logic in a hyperdoctrine. To do so, we look back at the precursors of continuous logic, and try to incorporate concepts found there into a hyperdoctrine. Specifically, we look at the category of MV-algebras, and try to use this as the codomain of this hyperdoctrine. We show that there is a natural way to do so, and furthermore, that continuous logic can indeed be fully interpreted in such a hyperdoctrine.

In section 2, we look at some precursors of continuous logic. In doing so, we also define some concepts that we will use later on when looking at the categorical side of continuous logic. In section 3 we define continuous logic. After that, we show the relation between continuous logic and the precursors we defined in the previous section, and we also give an example of a theory. In section 4, we look at Figueroa's attempt to interpret continuous logic in a hyperdoctrine. To do so, we first state some prerequisites, and look closely at the concept of hyperdoctrines. After that, we look at Figueroa's approach, and shortly discuss the results. In section 5, we try to adapt Figueroa's work to suit continuous logic more properly by making a few small changes. These changes turn out to be interesting, but insufficient to get to where we want. In section 6 we revisit MV-algebras, which were also noted in section 2. We give a proper definition, and show that we can use them to define a new class of hyperdoctrines, one of which can be used to fully interpret continuous logic in a hyperdoctrine. In section 7, we look at the unbounded variant of continuous logic, and illustrate the relation with Hensons's logic for Banach structures. In section 8, we conclude with a small discussion and some suggestions for further results.

2 Precursors of continuous logic

In this section, we look at some precursors of continuous logic. The logics that we will look at can be divided into two categories. First, we look at some logics with an infinite truth set that extend classical first order logic. After that, we look at two examples of logics that were developed to study complete metric spaces as models. In choosing which precursors we want to treat, we follow the list given in the introduction of [5].

2.1 Łukasiewicz infinite valued logic

The first mathematicians to give an axiomatisation of a logic with an infinite truth set were Łukasiewicz and Tarski in [15]. In this article, the authors collect and expand on results from propositional calculus. We note that the original article uses Polish notation. We will translate this notation to a contemporary equivalent for readability and to easier draw parallels with the other systems.

We follow the presentation given in the original article, and use modern notation and equivalent concepts when possible. We start by defining the set of propositions S.

Definition 2.1. The set S of all propositions is the smallest set containing all propositional variables which is closed under implication and negation.

So we see that the elements of the set S set are built up using denumerably many propositional variables $P, Q, R, \dots$, the binary logical connective implication and the unary logical connective negation. We will denote these connectives by C and N respectively.

Now that we have defined our set of sentences, we can look at consequences of sentences.

Definition 2.2. Let $X \subseteq S$ be a fixed set of propositions. We define *the set of consequences of* X, denoted by $Cn(X)$, as the smallest set containing X which is closed under substitution and modus ponens derivation.

Łukasiewicz and Tarski then set out to find sets X such that $Cn(X) = X$. To do so, Tarski introduced the concept of a logical matrix, which we will now define.

Definition 2.3. A *logical matrix* is a tuple $\mathfrak{M} = \langle A, B, f, g\rangle$, which consists of two disjoint sets A and B, a binary function $f : (A \cup B)^2 \to A \cup B$ and a unary function $g : A \cup B \to A \cup B$.

A logical matrix is called *normal* if we have that $\forall x \in B \forall y \in A : f(x, y) \in A$.

We will see that the set $A \cup B$ will play the role of the set of truth values. Although this set can be infinite, there still is a sort of discrete truth/false division visible in the form of the sets A and B. In particular, the set B will be interpreted the set of truth values that are 'close to truth' and A will be interpreted the set of truth values that are 'close to false'.

As with classical first order logic, we will need a way to assign a truth value to a given sentence. The following definition gives us just that.

Definition 2.4. A function $h : S \to A \cup B$ is called *a value function of a matrix* $\mathfrak{M}$ if we have for all $x, y \in S$ that $h(C(x, y)) = f(h(x), h(y))$ and $h(N(x)) = g(h(x))$.

Furthermore, we say that the proposition x is *satisfied* by $\mathfrak{M}$ if we have that $h(x) \in B$ for all possible value functions h.

So we see that the functions f and g serve as an interpretation of implication and negation, respectively. Also, we see that in a normal logical matrix we have that an implication of the form 'something close to truth implies something close to false' must evaluate to 'something close to false', which is what we would expect from the evaluation of implication.

To illustrate these concepts, we look at the following simple example.

Example 2.5. Let $\mathfrak{M} = \langle A, B, f, g\rangle$ where $A = \{0\}$, $B = \{1\}$, f is given by $f(0,0) = f(0,1) = f(1,1) = 1$ and $f(1,0) = 0$, and g is given by $g(0) = 1$ and $g(1) = 0$. ❏

We see that this example gives rise to an 'ordinary' two-valued propositional calculus.

In [15] Łukasiewicz and Tarski showed that there is a general way to expand the truth set to contain n elements, for any $n \in \mathbb{N}$. We leave that for now and immediately go to the infinite case.

Definition 2.6. *Łukasiewicz propositional logic* is the set of all sentences satisfied by the matrix $\mathfrak{M} = \langle A, B, f, g\rangle$, where $A = \mathbb{Q} \cap [0, 1)$, $B = \{1\}$, $f(x, y) = \min(1, 1 - x + y)$ and $g = 1 - x$.

So here we see that we have a logical system with an infinite truth set.

We can wonder whether or not Łukasiewicz propositional logic is axiomatizable. This turns out to be the case. The following four[1] axioms make up the axiom schema of Łukasiewicz logic. Here we choose to substitute $\rightarrow$ for C and $\neg$ for N to make the axioms more readable.

1. $P \rightarrow (Q \rightarrow P)$
2. $(P \rightarrow Q) \rightarrow ((Q \rightarrow R) \rightarrow (P \rightarrow R))$
3. $((P \rightarrow Q) \rightarrow Q) \rightarrow ((Q \rightarrow P) \rightarrow P)$
4. $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$

Furthermore, we write $\vdash P$ when P can be derived from these axioms using only substitution and modus ponens as derivation rules.

In [19] the ideas of the original paper are further developed. The authors introduce additional connectives derived from C and N. These include the following:

- $A(P, Q) = C(C(P, Q), Q) = (P \rightarrow Q) \rightarrow Q$
- $K(P, Q) = N(A(N(P), N(Q))) = \neg((\neg P \rightarrow \neg Q) \rightarrow \neg Q)$
- $B(P, Q) = C(N(P), Q) = \neg P \rightarrow Q$
- $L(P, Q) = N(C(P, N(Q))) = \neg(P \rightarrow \neg Q)$

The truth-value functions are calculated pointwise from the truth-value functions of C and N. We note that A and B both behave like $\vee$ when restricted to $\{0, 1\}$, and that K and L both behave like $\wedge$.

The remainder of [19] is spent on a proof that this system is actually complete. The proof is rather abstract and not very intuitive. In search of a more algebraic approach, C. C. Chang devised the concept of MV-algebras in [7]. Inspired by the fact that the Boolean prime ideal theorem implies the completeness of first order logic, he set out to find an equivalent proof for Łukasiewicz logic.

[1]In [15] the schema contains five axioms, but the fifth one was shown to be derivable from the others.

In [7], Chang first presents the definition of MV-algebras and some elementary results. We will present some of these later on in Section 6. After that he gives some examples of MV-algebras, of which the most prominent is called L. The elements of L are equivalence classes of formulas under the equivalence relation $\equiv$, given by $P \equiv Q \Leftrightarrow \vdash C(P, Q)$ and $\vdash C(Q, P)$ both hold.

Chang only succeeds partly in proving completeness, but manages to complete his argument in [8].

2.2 Fuzzy Logic

Another example of a logic that uses $[0, 1]$ as a truth space is fuzzy logic. Fuzzy logic originated in [20]. Here, Zadeh considered *fuzzy set theory*, which is in essence a method to introduce a concept of 'vagueness' in mathematics. The way he does that, is to give every element of a set X a number in $(0, 1]$ that indicates to which extent this element is 'inside' X. This simple concept eventually became the basis for a wide area of research. However, the machinery used in this research is largely unrelated to that used in logic.

In the beginning of the 1990s, some mathematicians began research into a solid logical foundation for fuzzy logic. In doing so, they discovered connections between fuzzy logic and already existing infinite valued logical systems, such as Łukasiewicz logic. Most of the results can be found in [12].

The basis of mathematical fuzzy logic is relatively simple; it is based on a small number of design choices, which are made to reflect fuzzy logic as introduced by Zadeh. The most fundamental choices are as follows:

- The real unit interval $[0, 1]$ is the standard truth space.
- Every n-ary connective c is semantically interpreted by a *truth function* $F_c : [0, 1]^n \to [0, 1]$, and the truth value of a formula is calculated 'pointwise' from its subformulas.
- The truth value of the connective $* : [0, 1]^2 \to [0, 1]$ that we will interpret as conjunction must satisfy the following natural conditions:
  - $x * y = y * x$
  - $(x * y) * z = x * (y * z)$
  - If $x \leq x'$ and $y \leq y'$ then $x * y \leq x' * y'$
  - $x * 1 = x$
  - $*$ is continuous on $[0, 1]^2$.

  Any such connective is called a *continuous t-norm*.

There are several examples of continuous t-norms. The most prominent examples are te following:

- The *Łukasiewicz t-norm*: $x *_L y = \max(x + y - 1, 0)$.
- The *Gödel t-norm*: $x *_G y = \min(x, y)$.
- The *product t-norm*: $x *_\Pi y = x \cdot y$.

The name 'Łukasiewicz t-norm' suggests that there is a connection between Łukasiewicz logic and fuzzy logic with this continuous t-norm. This is indeed the case. To make this connection clear, we need the following proposition.

Proposition 2.7. *For any continuous t-norm $*$ there is a unique binary operation $\Rightarrow_*$ on $[0,1]$ such that for all $x, y, z \in [0,1]$ we have $(z * x \leq y) \leftrightarrow (z \leq x \Rightarrow_* y)$ or, equivalently, $x \Rightarrow_* y = \sup\{z | x * z \leq y\}$. This connective is called the residuum of $*$.*

We see that the definition of a residuum reminds us very much of the definition of a Heyting implication in a Heyting algebra (see Definition 4.4). This observation is further strengthened when we see what the residua are for the t-norms we previously defined.

Example 2.8. Let $x, y \in [0,1]$ be given. We then have the following residua:

- The *Łukasiewicz implication*: $x \Rightarrow_L y = \begin{cases} \min(1, 1-x+y) & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$.
- The *Gödel implication*: $x \Rightarrow_G y = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$.
- The *product implication*: $x \Rightarrow_\Pi y = \begin{cases} y/x & \text{if } x > y \\ 1 & \text{otherwise} \end{cases}$.

❑

So now we see that the Łukasiewicz implication resembles the function f in the logical matrix $\mathfrak{M}$ of Łukasiewicz propositional logic, as we can recall from Definition 2.6.

From the connectives we have defined, we can make some more. Sometimes we can do this in a way that is universal for every continuous t-norm, as we see in the following proposition.

Proposition 2.9. *Let $*$ be a continuous t-norm and let $\Rightarrow_*$ be its residuum. Then the following two equalities hold:*

$$x * (x \Rightarrow_* y) = \min(x, y)$$

$$\min((x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x) = \max(x, y)$$

Looking at the way these connectives relate to the residuum, we see that min behaves like conjunction, but in a different way from $*$. To emphasize the difference, we introduce some notation in the following definition.

Definition 2.10. Let $*$ be a continuous t-norm. We will call call $*$ the *strong conjunction*, and denote it by $\otimes$. The connective $\min(x, y)$ will be referred to as the *weak conjunction* and denoted by $\wedge$.

Similarly, we will call $\max(x, y)$ the *weak disjunction*[2], and denote it by $\vee$.

Lastly, there is also the concept of negation. Like in the case of classical first order logic, negation will be the same as implying falsity. So we see that for our defined t-norms, this looks as follows:

- $\neg_L x = 1 - x$
- $\neg_G x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

[2] One could ask if there is also a strong disjunction. It turns out that this is not always the case, but MV-algebras, for example, do have one.

- $\neg_{\Pi} x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

Here we also see that the Łukasiewicz negation resembles the function g that we defined inDefinition 2.6.

Together with the constants 0 and 1, all the propositional connectives form a language. We make this rigorous in the following definition.

Definition 2.11. We define the *language of the propositional fuzzy logic* $\mathcal{L}$ to consist of propositional variables, denoted $x, y, z, \ldots$, the binary connectives $\otimes, \rightarrow, \wedge$ and $\vee$, the unary connective $\neg$ and the constants $\overline{1}$ and $\overline{0}$.

Formulas in $\mathcal{L}$ are formed in the usual way.

The last bit we want to introduce here is the concept of an evaluation of $\mathcal{L}$-formulas. In the end, we need to find a logical way to give formulas a truth value in the interval $[0, 1]$. This will be done as in the following definition.

Definition 2.12. A $[0, 1]$-*evaluation of propositional variables* is a mapping e from the set of variables to $[0, 1]$. For any continuous t-norm $*$, we can extend e to an evaluation of all $\mathcal{L}$-formulas, by the following recursion. Here, x is a variable and ϕ, ψ are formulas.

- $e_*(x) = e(x)$
- $e_*(\overline{0}) = 0$
- $e_*(\overline{1}) = 1$
- $e_*(\neg\phi) = \neg_*(e_*(\phi))$
- $e_*(\phi \otimes \psi) = e_*(\phi) * e_*(\psi)$
- $e_*(\phi \rightarrow \psi) = e_*(\phi) \Rightarrow_* e_*(\psi)$
- $e_*(\phi \wedge \psi) = \min(e_*(\phi), e_*(\psi))$
- $e_*(\phi \vee \psi) = \max(e_*(\phi), e_*(\psi))$

From the construction of the connectives, we see that this definition is somewhat superfluous, since all these cases follow from the cases of $\otimes$, $\rightarrow$ and $\overline{0}$. We did include them here to make the parallel with continuous logic more apparent. We show this parallel in Section 3.2. We will use a similar interpretation for continuous logic, as we will see in Definition 4.29.

2.3 Chang and Keisler's logic

The first true attempt at finding a logic specifically made for metric spaces was made by Chang and Keisler in [9]. In this book, the authors present a logic which can be seen as a generalisation of continuous logic. However, it turns out that the logic they present is too general, which makes it hard to concretely work with. We will not work out the details of this logic, but we note some differences.

The main difference between Chang and Keisler's logic and continuous logic is that in the framework of the former every compact Hausdorff space can be a truth space. An important consequence of this is that this logic needs to have an infinite number of quantifiers, since general compact Hausdorff spaces lack a natural linear ordering. This makes the Chang and Keisler's logic somewhat awkward to use.

Another consequence is that, despite the fact that the models are metric spaces, equality is still seen as a discrete concept. That is, in Chang and Keisler logic equality has 'values' true and false, whereas in continuous logic equality can take every value in the interval $[0, 1]$. Thus, in continuous logic equality becomes a kind of distance symbol, which seems to suit the given context in a better way.

2.4 Henson's logic for Banach structures

Henson's logic for Banach structures was, as the name suggests, introduced to research certain properties of Banach spaces. In particular, it was first devised to study the relation between a Banach space and its ultrapowers. We will highlight the relevant theory and concepts of this logic. We follow the presentation as given in [13], but with some definitions we choose to follow [14]. This will allow for a better comparison with (a variant of) continuous logic in Section 7.

We start by looking at exactly what structures we want to describe with our logic. These structures are defined as follows:

Definition 2.13. A *normed space structure* $\mathcal{M}$ consists of the following data:

- A family of normed vector spaces $(M^{(s)}|s \in S)$ called the *sorts* of $\mathcal{M}$, at least one of which is equal to $\mathbb{R}$.
- A collection of functions of the form $F : M^{(s_1)} \times \cdots \times M^{(s_n)} \to M^{(s_0)}$, all of which are uniformly continuous on every bounded subset of its domain. This collection should at least contain the following functions for every $s \in S$:
  - The norm of $M^{(s)}$, seen as a function from $M^{(s)} \times M^{(s)}$ to $M^{(s)}$.
  - The vector operations $+$ and $-$, seen as functions from $M^{(s)} \times M^{(s)}$ to $M^{(s)}$.
  - The additive identity 0, seen as 0-ary function into $M^{(s)}$.
  - Scalar multiplication seen as a function from $\mathbb{R} \times M^{(s)}$ to $M^{(s)}$.

Given a normed space structure $\mathcal{M}$, it is easy to see that it is possible to create a signature L based on $\mathcal{M}$. Such an L would have a collection of sorts S, at least one of which is interpreted as $\mathbb{R}$, and every designated function F is given a syntactic function symbol f, which is then interpreted as F in $\mathcal{M}$.

This presentation as given in [14] allows, as we can see, for a very broad class of structures. In order to better illustrate the properties of this logic, we will give an example. Our choice will be a real valued Banach space $(X, ||\cdot||)$. Our language L_X will then look as follows:

- The sort set S will have two elements: X and $\mathbb{R}$.
- The functions are the vector space operations on X, the additive identity 0_X and $||\cdot||$, together with the field operations on $\mathbb{R}$, the additive identity 0 and $|\cdot|$.

Given such a signature L for $\mathcal{M}$, we can define terms in the regular way: every variable of sort s is a term of sort s, and if $f : s_1 \times s_n \to s$ is a function symbol, and t_k is a term of sort s_k, then $f(t_1, \ldots, t_n)$ is a term of sort s. Furthermore if a term t is of sort s with $M(s) = \mathbb{R}$, we will call t a *real-valued term*.

Now, instead of looking at all the possible L-formulas, we will look at subset, defined as follows:

Definition 2.14. Let L be a signature for normed space structures. The set of *positive bounded L-formulas* is recursively defined as follows:

- If t is a real-valued term and r is a a rational number, then both $r \leq t$ and $t \leq r$ are positive bounded formulas.
- If ϕ and ψ are both positive bounded formulas, then so are $\phi \wedge \psi$ and $\phi \vee \psi$.
- If ϕ is a positive bounded formula, x is a variable and r is a positive rational number, then both $\exists x(||x|| \leq r \wedge \phi)$ and $\forall x(||x|| \leq r \rightarrow \phi)$ are positive bounded formulas.

Given a term or formula, we can easily define by recursion the interpretation of said term or formula in a normed space structure. And by another standard recursive definition, we can define when a normed space structure $\mathcal{M}$ satisfies a positive bounded formula $\phi(x_1, \ldots, x_n)$ at a certain tuple $\vec{m}$, which we denote by $\mathcal{M} \models \phi(m_1, \ldots, m_n)$.

One may wonder why we restrict ourselves to the set of positive bounded formulas. As it turns out, many properties of normed space structures can be checked using just positive bounded formulas. We have for example the following result[3].

Proposition 2.15. *(Theorem 1.13 in [13]) Let E and F be Banach spaces. Then the following are equivalent:*

1. *There is a Banach Space H such that each positive bounded sentence true in E or in F is true in H.*
2. *E and F have isometric ultrapowers.*

Henson then constructs a method to 'approximate' a given positive bounded formula ϕ. Intuitively, an approximation of ϕ is a formula ϕ' where the bounds on the norms are relaxed. We will give a definition using recursion.

Definition 2.16. Let ϕ be a positive bounded formula.

- If ϕ is of the form $r \leq t$, then the approximations of ϕ are all formulas of the form $r' \leq t$, where $r' < r$.
- If ϕ is of the form $r \geq t$, then the approximations of ϕ are all formulas of the form $r' \geq t$, where $r' > r$.
- If ϕ is of the form $\psi_0 \wedge \psi_1$, then the approximations of ϕ are all formulas of the form $\psi_0' \wedge \psi_1'$, where ψ_0' is an approximation of ψ_0 and ψ_1' is an approximation of ψ_1.
- If ϕ is of the form $\psi_0 \vee \psi_1$, then the approximations of ϕ are all formulas of the form $\psi_0' \vee \psi_1'$, where ψ_0' is an approximation of ψ_0 and ψ_1' is an approximation of ψ_1.
- If ϕ is of the form $\exists x(||x|| \leq r \wedge \psi)$, then the approximations of ϕ are all formulas of the form $\exists x(||x|| \leq r' \wedge \phi')$, where $r' > r$ and ϕ' is an approximation of ϕ.
- If ϕ is of the form $\forall x(||x|| \leq r \rightarrow \psi)$, then the approximations of ϕ are all formulas of the form $\forall x(||x|| \leq r' \rightarrow \phi')$, where $r' < r$ and ϕ' is an approximation of ϕ.

If ϕ' is an approximation of ϕ, we write $\phi' > \phi$.

[3] For those familiar with it, this result resembles the Keisler-Shelah isomorphism theorem.

With the concept of approximation of positive bounded formulas also comes the concept of approximate truth.

Definition 2.17. Let $\phi(x_1, \ldots, x_n)$ be a positive bounded L-formula. For each L-structure $\mathcal{M}$ and tuple $\vec{m}$ we define that $\mathcal{M} \models_A \phi(m_1, \ldots, m_n)$ if and only if we have for all $\phi' > \phi$ that $\mathcal{M} \models \phi'(m_1, \ldots, m_n)$

And with approximate truth also comes approximate equivalency.

Definition 2.18. Let $\mathcal{M}$ and $\mathcal{N}$ be L-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are *approximately elementary equivalent* if for each positive bounded sentence ϕ we have that $\mathcal{M} \models_A \phi \Leftrightarrow \mathcal{N} \models_A \phi$. We denote this by $\mathcal{M} \equiv_A \mathcal{N}$.

Over the years, Henson worked out the concept of positive bounded theories for bounded spaces to include many concepts known from first order logic. A full overview can be seen in [14].

2.5 Compact abstract theories

The last precursor of continuous logic we want to discuss is the concept of Compact Abstract Theories, or *cats* for short. Cats were first defined in [1], where the author presented a logic that generalizes first order logic using *positive formulas*, similar to Henson's approach. We will present the basic concepts, and state a few results.

We fix a signature $\mathcal{L}$, which we assume for simplicity to contain only relation symbols. We define formulas once again in a usual manner, and we write $\phi(\vec{x})$ if the variables $\vec{x}$ contain the free variables occurring in ϕ. We denote the set of $\mathcal{L}$-formulas as $\mathcal{L}_{\omega,\omega}$. The following definition will be the starting point for our presentation of cats.

Definition 2.19. A set $\Delta \in \mathcal{L}_{\omega,\omega}$ is a *positive fragment* of $\mathcal{L}$ if it contains all atomic formulas and is closed under conjunction, disjunction and sub-formulas. With this last condition we mean that whenever $\lambda(\phi_1, \ldots, \phi_n) \in \Delta$, where λ is *any* connective, so not just conjunction or disjunction, we must also have that $\phi_1, \ldots, \phi_n \in \Delta$.

Let Δ be a positive fragment. We denote $\Sigma(\Delta)$ for the closure of Δ under existential quantification, and we denote $\Pi(\Delta)$ to be the set $\{\neg\phi : \phi \in \Sigma(\Delta)\}$.

Lastly, let M and N be two $\mathcal{L}$-structures. We call a map $f : M \to N$ a Δ-homomorphism if $M \models \phi(\vec{m}) \Rightarrow N \models \phi(f(\vec{m}))$, for every n-ary formula $\phi(\vec{x}) \in \Delta$ and $\vec{m} \in M^n$.

For a given model M, we now want to look at whether there exists a $\phi(\vec{x})$ in some $\Sigma(\Delta)$ such that $\phi(\vec{m})$ holds for some $\vec{m} \in M^n$. If we have that $M \models \phi(\vec{m})$, then we can apply a Δ-homomorphism $f : M \to N$ and see that $N \models \phi(f(\vec{m}))$, and we can repeat this as often as we would like. If we have $M \not\models \phi(\vec{m})$, then we don't know what happens under the image of a Δ-homomorphism. To remedy this, we define a new concept, which allows us to give certain answers to our problem.

Definition 2.20. A Π-*theory* T is a set of $\Pi(\Delta)$ sentences, for a given positive fragment Δ.

A model $M \models T$ is *existentially closed* (or *e.c.*), if every Δ-homomorphism $f : M \to N$ is a $\Sigma(\Delta)$-embedding.

In other words, this means that for every $\phi(\vec{x}) \in \Sigma(\Delta)$ and $\vec{m} \in M^n$, we have that $N \models \phi(f(\vec{m})) \Rightarrow M \models \phi(\vec{m})$. The implication the other way around is given by the fact that f is a Δ-homomorphism.

By a model theoretical argument involving the Axiom of Choice, we can see that for every model $M \models T$ there is a Δ-homomorphism $f : M \to N$ where N is e.c.

In order to define cats, we need to look at types and type spaces for models of $\Pi(\Delta)$-theories. To do so more efficiently, we assume that every instance of T is a $\Pi(\Delta)$-theory for Δ, where Δ is a fixed positive fragment. Furthermore, we shorten $\Sigma(\Delta)$ and $\Pi(\Delta)$ to Σ and Π.

We start off with the definition of types of positive fragments.

Definition 2.21. Let $M \models T$ be given, along with $\vec{m} \in M^I$ for some index set I and let Ξ be an arbitrary[4] positive fragment. We define the Ξ-*type of* $\vec{m}$ as the set $\{\phi(\vec{x}|\phi \in \Xi, M \models \phi(\vec{m}))\}$. We denote this by $\text{tp}^M_\Xi(\vec{m})$.

As we may recall from first order model theory, type spaces can be endowed with a topology. In classical first order logic, this is also known as the Stone topology. We see that we can generalize this concept to Π-theories.

Definition 2.22. Let I be a set of indices and let $S_I(T) = \{\text{tp}^M_\Sigma(\vec{m}) | M \models T, \vec{m} \in M^I\}$. Additionally, for $\phi(\vec{x}) \in \Sigma$, we denote $\langle\phi\rangle = \{p \in S_I(T) \mid \phi \in p\}$.

Let Ξ be an arbitrary positive fragment. We define the Ξ-topology on $S_I(T)$ as the topology generated by $\{\langle\phi\rangle | \phi(\vec{x}) \in \Xi\}$ as closed sets. We call $S_I(T)$ with the Σ-topology the *space of I-types of T*.

If $f : n \to m$ is a function, then we define $f^* : S_m(T) \to S_n(T)$ by $f^*(p) = \{\phi(\vec{x}) : f_*(\phi) \in p\}$, where $f_*(\phi) = \phi(x_{f(0)}, \ldots, x_{f(n-1)})$ is a change of variable function.

In classical first order logic, the standard topology on type spaces makes such a space into a compact totally disconnected Hausdorff space. This result does not fully generalize to our situation, but we still have the following.

Proposition 2.23. *With the given topology, every $S_I(T)$ space is compact and T_1; that is, every two points both have an open neighbourhood not containing the other.*

Now we are ready to give a definition of a compact abstract theory. We explicitly say *a* definition, because in [1] the author presents three equivalent concepts, and collectively dubs these compact abstract theories. Here, we present one of these concepts.

Definition 2.24. A Π-theory T is called a *positive Robinson theory* if for every two models M_0 and M_1 of T and every two tuples $\vec{m}_0 \in M_0^I$ and $\vec{m}_1 \in M_1^I$ such that $\text{tp}^{M_0}_\Delta(\vec{m}_0) \subseteq \text{tp}^{M_1}_\Delta(\vec{m}_1)$, we have that $\text{tp}^{M_0}_\Sigma(\vec{m}_0) = \text{tp}^{M_1}_\Sigma(\vec{m}_1)$. So in other words, we have that Δ-types determine Σ-types.

To illustrate the relation between cats and continuous logic, we would like to refer to [2], where additional properties and results on cats are defined, including metric cats, which allow the internal notion of a metric in all of their models.

[4] In practice, this will always be Δ or Σ

3 Continuous logic

In this section, we will define continuous logic. After that, we show the relation between continuous logics and the precursors we have presented in the previous section. Lastly, we give an example of a theory in continuous logic.

3.1 Definition of continuous logic

We follow the presentation given in [6].

When defining continuous logic, we have to start from the very beginning. We will mimic the construction of first order logic, with the difference that our truth space will not be the discrete set $\{0,1\}$, but the continuous and compact interval $[0,1]$. One important change will be that we will regard the truth symbol T to be 0, and the false symbol F to be 1. Choosing this opposite assignment will prove to be more elegant.

We will see that the definition of a signature will be different, and that we have to rethink the concepts of connectives and quantifiers. First, we will define a continuous signature, but for now without the concept of equality, which we will add later.

Definition 3.1. A *non-metric continuous signature* consists of a set of constants, function symbols and relation symbols, where each function and relation symbol also has an arity.

As implied by the name, the equality in a continuous signature will take the form of a metric. Before we look at equality, we first look at the logical connectives.

In classical first order logic, a connective of arity n is a function from $\{0,1\}^n \to \{0,1\}$. In continuous logic, we define a connective of arity n to be a continuous function from $[0,1]^n \to [0,1]$. Some examples we may use are:

- Constants in $[0,1]$, which have arity 0.
- $\neg x = 1 - x$ and $\frac{x}{2}$, which have arity 1
- $x \wedge y = \min(x,y)$, $x \vee y = \max(x,y)$, $x \dot{-} y = (x-y) \vee 0$ and $|x-y|$, all of which have arity 2.

In first order logic, we use a well-known set of Boolean connectives, and it is known that this set is functionally complete, that is, using these connectives we can construct every function from $\{0,1\}^n \to \{0,1\}$. Naturally, we want the same for the continuous connectives. One way to do this would be to take every continuous function of the form $[0,1]^n \to [0,1]$ to be a connective. This would however become a problem when looking at the size of a language, since there are, in a sense, too many continuous functions. This problem can be circumvented by only looking at a countable set of connectives generated by a countable number of continuous functions from $[0,1]^n \to [0,1]$. The precise definition of this is as follows:

Definition 3.2. A *system of continuous connectives* is a set $\mathcal{F} = \bigcup_{n\in\mathbb{N}} F_n$, where F_n is a set of continuous functions $[0,1]^n \to [0,1]$.

We call a system of continuous connectives *closed* if it satisfies the following two conditions:

1. For every n and every $m < n$, the projection to the mth coordinate belongs to F_n
2. For any $f \in F_n$ and $g_0, \dots g_{n-1} \in F_m$, we have that the composition $f \circ \langle g_0, \dots, g_{n-1}\rangle$ belongs to F_m.

It is easy to see that every system of continuous connectives $\mathcal{F}$ can be extended to a closed system. A closed system generated by a system of continuous connectives $\mathcal{F}$ will be denoted by $\bar{\mathcal{F}}$.

We call a system of continuous connectives $\mathcal{F}$ *full* if for $\bar{\mathcal{F}}$ we have that for every $0 < n < \omega$ the set $\bar{F}_n$ is dense in the set of continuous functions from $[0,1]^n \to [0,1]$ with respect to the compact-open topology.

The following corollaries gives two such examples.

Corollary 3.3. *Let $C \subseteq [0,1]$ be a dense set of truth constants such that $1 \in C$ (the dyadic numbers for example). Then $F_0 = C$, $F_2 = \{\dot{-}\}$, and $F_n = \emptyset$ otherwise is a full system.*

Corollary 3.4. *The system with $F_1 = \{\neg, \frac{x}{2}\}$, $F_2 = \{\dot{-}\}$ and $F_n = \emptyset$ otherwise is full.*

The actual choice of a system of connectives will not matter, as long as the system is full. All other connectives can be approximated by the connectives in the full system. We will however mostly use the latter one, since this one has the advantage of being finite.

Looking at this system, we can try to find for each of the connectives a counterpart in classical first order logic. As notation suggests, the counterpart of the function $\neg$ is clearly classical negation. For $\dot{-}$, we see the following: $x \dot{-} y = 0$ whenever y is closer to 1 than x is. Since 1 is the value of absolute falsity, we could see this as x being at least as true as y. This should remind us of classical implication, since $x \to y$ holds whenever x is at least as true as y. The function $\frac{x}{2}$ does not have a direct counterpart in classical first order logic.

The next part of continuous logic we will consider are the quantifiers. The situation here is actually much simpler. We see that we have a very natural reinterpretation of the classical first order quantifiers $\forall x$ and $\exists x$. To illustrate this, we first look at properties of the quantifiers in discrete first order logic, and try to generalize that to the continuous case.

So we start with the discrete case. We look at an arbitrary n-ary relation symbol R. We can see R as a property of the variables $x_0, \dots x_{n-1}$. We define $j(R)$ to be the same relation as R, but with an extra dummy variable x_n. So $j(R)$ is $n+1$-ary. We now want to define for every $n+1$-ary relation symbol Q two new relations $\exists x_n Q$ and $\forall x_n Q$, and of course we want to have that these new relations behave as the quantifiers we know. We can see that the following properties are sufficient and necessary to determine these relations as we want to.

$$Q \to j(R) \Leftrightarrow (\exists x_n Q) \to R$$
$$j(R) \to Q \Leftrightarrow R \to (\forall x_n Q)$$

Remark 3.5. The reader familiar with categorical logic can see that these properties state that the quantifiers behave like adjoints to the functor j. One can compare this to the relevant results in Section 4.1, in particular the last bit of Proposition 4.5, where quantifiers are also seen as adjoints.

We now generalize this situation to the continuous case. So relation symbols will be seen as continuous n-ary functions into $[0,1]$. We can define j to be the same as above. We also have to translate $\to$ into the continuous case. Since we assumed truth to be 0 and falsity to be 1, we can see that $\geq$ is a suitable replacement. So for every n-ary function f and $n+1$-ary function g, we must have the following:

$$g \geq j(f) \Leftrightarrow (\exists x_n g) \geq f$$
$$j(f) \geq g \Leftrightarrow f \geq (\forall x_n g)$$

From this, we can see that there is in fact only a single possible reinterpretation of the quantifiers: we have that $\exists x_n g$ must be $\inf_{x_n} g$ and $\forall x_n g$ must be $\sup_{x_n} g$.

Now that we have defined a signature and our logical symbols, we can define formulas in the same way as in first order logic. And now that we have formulas, we can also look at theories. In the discrete case, a theory is a set of of sentences which we assume to be true. We can translate this to the continuous case as follows:

Definition 3.6. A *condition* is an expression of the form $\phi = 0$, where ϕ is a formula. We call a condition *sentential* if ϕ is a sentence. A *theory* is then a set of sentential conditions.

Before we can look at models of theories, we still have to consider the concept of equality. To do this, we will introduce a binary relation symbol d, much like the symbol $=$ in first order logic, which we will interpret as a distance symbol. That is, we no longer require that equality is a discrete concept taking only values in $\{0, 1\}$, but choose to extend it to the whole truth space.

We want that the symbol d behaves as much as $=$ as possible. To do so, we use the same tactic as we used with the quantifiers: we list some of the properties that discrete equality has, and generalise them to the continuous case.

The first property we want to reflect is that equality is an equivalence relation. So we must have the following axioms:

$$\forall x : x = x$$

$$\forall x \forall y : x = y \to y = x$$

$$\forall x \forall y \forall z : x = y \to (y = z \to x = z)$$

When we translate these sentences to continuous logic, we get the following:

$$\sup_x d(x, x) = 0$$

$$\sup_x \sup_y (d(x, y) \dot{-} d(y, x)) = 0$$

$$\sup_x \sup_y \sup_z ((d(x, z) \dot{-} d(y, z)) \dot{-} d(x, y)) = 0$$

If we look at what this actually means, we can see that these axioms tell us precisely that d should be a pseudo-metric, which justifies the notation we chose. We recall the definition of a pseudo-metric:

Definition 3.7. Let X be a set. We call a function $d : X \to X \to [0, \infty)$ a *pseudo-metric* if the following conditions hold:

- $\forall x \in X : d(x, x) = 0$
- $\forall x_0, x_1 \in X : d(x_0, x_1) = d(x_1, x_0)$
- $\forall x_0, x_1, x_2 \in X : d(x_0, x_1) \leq d(x_0, x_2) + d(x_2, x_1)$

Next to the fact that we want to reflect that $=$ is an equivalence relation, we also want to reflect that $=$ is a congruence relation for every function and relation symbol. That is, for every n-ary function symbol f and relation symbol P we have n axioms, one for each argument, of the following form:

$$\forall \vec{x} \forall \vec{y} \forall w : z = w \to (f(\vec{x}, z, \vec{y}) = f(\vec{x}, w, \vec{y}))$$

$$\forall\vec{x}\forall\vec{y}\forall w : z = w \to (P(\vec{x}, z, \vec{y}) \to P(\vec{x}, w, \vec{y}))$$

If we translate these schemata as we did with the previous axioms, we see that we get axioms stating that every function symbol f and relation symbol R should be 1-Lipschitz in every argument. We recall the definition of a 1-Lipschitz function:

Definition 3.8. Let (X, d_X) and (Y, d_Y) be two (pseudo-)metric spaces. We call a function $f : X \to Y$ *1-Lipschitz* if the following condition holds:

$$\forall x_0, x_1 \in X : d_Y(f(x_0), f(x_1)) \leq d_X(x_0, x_1)$$

We will see in Section 3.3 that this requirement is a bit too strict, so we will relax 1-Lipschitz to uniform continuity. We do that using the following definitions.

Definition 3.9. A *modulus of uniform continuity* is a function $\Delta : (0, 1] \to (0, 1]$.

Definition 3.10. Let (X, d) and (Y, d') be metric spaces, $f : X \to Y$ a function and Δ a modulus of uniform continuity. We say that f is *uniformly continuous with respect to* Δ whenever we have that $\forall\epsilon > 0 \forall x, y \in X : d(x, y) < \Delta(\epsilon) \Rightarrow d'(f(x), f(y)) < \epsilon$.

It is easy to see that f is uniformly continuous if and only if there is a modulus of uniform continuity Δ such that f is uniformly continuous with respect to Δ.

So now we will fix for every n-ary symbol s and each $i < n$ a modulus of uniform continuity $\Delta_{s,i}$, and formulate our axioms to say that s is uniformly continuous with respect to $\Delta_{s,i}$ when we see s as a function of its ith argument. So we see that for every n-ary function symbol f and relation symbol R, every $i < n$ and every[5] $\epsilon > 0$ an axiom of the following form:

$$\sup_{x_{<i}} \sup_{y_{<n-i-1}} \sup_{z} \sup_{w} (\Delta_{f,i}(\epsilon) \dot{-} d(z, w)) \wedge (d(f(\vec{x}, z, \vec{y}), f(\vec{x}, w, \vec{y})) \dot{-} \epsilon) = 0$$

$$\sup_{x_{<i}} \sup_{y_{<n-i-1}} \sup_{z} \sup_{w} (\Delta_{P,i}(\epsilon) \dot{-} d(z, w)) \wedge ((P(\vec{x}, z, \vec{y}) \dot{-} P(\vec{x}, w, \vec{y})) \dot{-} \epsilon) = 0$$

So we see that in order to accurately generalize equality to the continuous case, we will need to add more elements to our signature than just a distance symbol. That is what we do in the following definition.

Definition 3.11. A *(metric) continuous signature* is a non-metric continuous signature together with a binary relation symbol d, called the *distance symbol*, and for every symbol s of arity n and each $i < n$ a modulus of uniform continuity $\Delta_{s,i}$, called the *modulus of uniform continuity of s with respect to the ith argument*.

So now we finally have all the definitions and concepts ready to define structures.

Definition 3.12. Let L be a continuous signature. We define a *continuous L-pre-structure*, also simply called an *L-pre-structure*, as a set M together with an interpretation $c^M \in M$ for every constant c in L, a function $f^M : M^n \to M$ for every function symbol f of arity n in L and a function $R^M : M^n \to [0, 1]$ for every relation symbol R of arity n, with the additional requirement that d^M is interpreted as a pseudo-metric and every $\Delta_{s,i}$ as a modulus of uniform continuity of s with respect to the ith argument.

[5] We may also regard only rational ϵ, so that we only have countably many axioms.

Additionally, we define a *continuous L-structure*, or simply *L-structure*, as an L-pre-structure with the additional requirement that d^M is interpreted as a complete metric.

Note that since d is a relation symbol, we must have by definition that d^M is a function $d^M(x, y) : M^2 \to [0, 1]$. Therefore, all L-(pre-)structures M are metrically bounded by definition.

It is not immediately clear why we would require that d be a complete metric. There is a good reason for this, but this is beyond the scope of this introduction. We would like to refer [6] for more details.

Given a pre-structure M for some signature L, the next thing to look at is the interpretation of terms and formulas in M. As with discrete first order logic, we will do this recursively.

Definition 3.13. Let $\tau(\vec{x})$ be a term with n free variables and let M be an L-pre-structure. We define the *interpretation of $\tau(\vec{x})$ in M* as a function $\tau^M : M^n \to M$ as follows:

- If $\tau = x_i$, then $\tau^M(\vec{m}) = m_i$.
- If $\tau = f(\sigma_0, \dots, \sigma_{k-1})$, where the σ_i are all sub-terms, then $\tau^M(\vec{m}) = f^M(\sigma_0^M(\vec{m}), \dots, \sigma_{k-1}^M(\vec{m}))$

Definition 3.14. Let $\phi(\vec{x})$ be a formula with n free variables and let M be an L-pre-structure. We define the *interpretation of $\phi(\vec{x})$ in M* as a function $\phi^M : M^n \to [0, 1]$ as follows:

- If $\phi = P(\tau_0, \dots, \tau_{k-1})$ is atomic, then $\phi^M(\vec{m}) = P^M(\tau_0^M(\vec{m}), \dots, \tau_{k-1}^M(\vec{m}))$.
- If $\phi = \lambda(\psi_0, \dots, \psi_{k-1})$, where λ is a logical connective, then $\phi^M(\vec{m}) = \lambda(\psi_0^M(\vec{m}), \dots, \psi_{k-1}^M(\vec{m}))$.
- If $\phi = \inf_y \psi(y, \vec{x})$, then $\phi^M(\vec{m}) = \inf_{a \in M} \psi^M(a, \vec{m})$.
- If $\phi = \sup_y \psi(y, \vec{x})$, then $\phi^M(\vec{m}) = \sup_{a \in M} \psi^M(a, \vec{m})$.

3.2 Relation between continuous logic and other infinite valued logics

In this subsection, we look at the relation between continuous logic and the other logics that we discussed in the previous section.

Łukasiewicz and fuzzy logic

We recall that a Łukasiewicz propositional logic is the set of sentences freely generated by the connectives C and N, satisfied the logical matrix $\langle \mathbb{Q} \cap [0, 1), \{1\}, \min(1, 1 - x + y), 1 - x \rangle$.

Also recall the definition of the Łucasiewicz implication from fuzzy logic.

$$x \Rightarrow_L y = \begin{cases} \min(1, 1 - x + y) & \text{if } \; x > y \\ 1 & \text{otherwise} \end{cases}$$

In continuous logic, our implication looks different at first glance. The function $x \dot{-} y = \max(0, y - x)$ is nevertheless equivalent to the Łukasiewicz implication, which we see as follows:

We recall the convention that in continuous logic 0 represents truth and 1 represents falsity, whereas in Łucasiewicz fuzzy logic this is the other way around. To accommodate for this, we must swap the truth values of Łucasiewicz fuzzy logic around. This can be done by replacing every n-ary connective $\lambda(x_1, \dots, x_n)$ by $\neg_L \lambda(\neg_L x_1, \dots, \neg_L x_n)$. For example, for the implication this gives us:

$$\begin{aligned}\neg_L(\neg_L x \Rightarrow_L \neg_L y) &= 1 - \min(1, 1-(1-x)+(1-y)) = \max(1-1, 1-(1-(1-x)+(1-y))) \\ &= \max(0, 1-(1-1+x+1-y)) = \max(0, 1-1+1-x-1+y) \\ &= \max(0, y-x) = x \dot{-} y\end{aligned}$$

So we see that the context in which we are working in is similar. There is a small difference however. As we may recall, continuous propositional logic is the set of propositions freely generated by the connectives $\dot{-}$, $\neg$ and $\frac{1}{2}$. So compared to Łukasiewicz propositional logic, we have an additional important connective.

The difference is further described in the first part of [5]. Here, the authors give an axiomatisation of continuous propositional logic as follows:

- $(\phi \dot{-} \psi) \dot{-} \phi$
- $((\chi \dot{-} \phi) \dot{-} (\chi \dot{-} \psi)) \dot{-} (\psi \dot{-} \phi)$
- $(\phi \dot{-} (\phi \dot{-} \psi)) \dot{-} (\psi \dot{-} (\phi \dot{-} \phi))$
- $(\phi \dot{-} \psi) \dot{-} (\neg\psi \dot{-} \neg\phi)$
- $\frac{1}{2}\phi \dot{-} (\phi \dot{-} \frac{1}{2}\phi)$
- $(\phi \dot{-} \frac{1}{2}\phi) \dot{-} \frac{1}{2}\phi$.

The first four axioms are exactly the axioms for Łukasiewicz propositional logic, and the last two specify that the connective $\frac{1}{2}$ behaves as one expects.

From this, we immediately have that every model of continuous logic is therefore an MV-algebra, but more about that later.

Henson's logic

At first, it might seem hard to compare Henson's logic for Banach structures with continuous logic, first and foremost since Banach spaces are generally unbounded structures, whereas continuous logic only deals with metrically bounded structures. So we cannot directly define a theory of Banach spaces in continuous logic.

To work around this problem, at least temporarily, it is possible in continuous logic to define the theory T_{cvx} of closed convex subsets of Banach spaces of diameter at most 1. This theory can then further be expanded to a theory of which the models are precisely the units balls of Banach spaces. This theory can be found in examples 4.4 and 4.5 of [6]. It can then be shown that this approach has the same power of expression as Henson's logic.

Another approach can be found in [3], where the author presents a generalisation of continuous logic which allows a truth space of the form $[0, \infty)$. This generalisation is closer to Henson's logic for Banach spaces, and therefore allows for more structure to be translated between the two. We explore this connection in detail in Section 7.

Compact abstract theories

There is also a correspondence between continuous logic and a certain class of compact abstract theories. To state this, we need some additional definitions.

Definition 3.15. Let T be a compact abstract theory. We say that T is a *Hausdorff cat* if every type-space $S_I(T)$ is Hausdorff.

Definition 3.16. Let T be a Hausdorff cat. We say that T is *open* if for every injective map $f : n \to m$ the map $f^* : S_m(T) \to S_n(T)$ sends open subsets to open subsets.

We can now state the following result, which can also be found in section 4.5 of [6]:

Proposition 3.17. *Every continuous first order theory is an open Hausdorff cat.*

3.3 An application of continuous logic

In this subsection, we look at a practical example of how to apply continuous logic in model theory. We follow the presentation as given in [4], where there are also more examples.

One area where continuous logic has been applied, is probability theory. Using continuous logic, one can look at the model theory of probability spaces by viewing the measure algebra of a probability space as a metric space.

First we recall some definitions from probability theory. Recall that a probabilty space is a triple (Ω, Σ, μ), where Ω is a set, Σ is a σ-algebra on Ω and μ is a σ-additive measure such that $\mu(\Omega) = 1$.

We want to choose a suitable underlying set for our structure. The σ-algebra Σ looks promising, but it will behave better if we transform it a bit. To do that, we first define an equivalence relation $\sim_\mu$ on Σ as follows: for $A, B \in \Sigma$, we say that $A \sim_\mu B$ if and only if $\mu(A \triangle B) = 0$, where $A \triangle B$ stands for the symmetric difference $(A \cap B^c) \cup (B \cap A^c)$. We denote $[A]_\mu$ for the equivalence class of A under $\sim_\mu$. The collection of equivalence classes is denoted by $\widehat{\Sigma}$ and is called the *event set*. It is known that the event set together with complement, union and intersection forms a Boolean algebra. Furhtermore, it is easy to see that the event space is a σ-algebra again, and that μ is a well-defined measure on $\widehat{\Sigma}$.

Given a probability space (Ω, Σ, μ) we can, using these data, build a continuous structure. To do that, we first look at what our continuous signature looks like, keeping in mind that our underlying set will be the event space $\widehat{\Sigma}$.

- The constants are 0 and 1, which will be interpreted as the unique events with measure 0 and 1 respectively.
- The function symbols are the complement $\cdot^c$, the union $\cup$ and intersection $\cap$.
- The measure μ will serve as a unary relation symbol.
- The metric d will be defined as $d([A]_\mu, [B]_\mu) = \mu(A \triangle B)$, where $\triangle$ is the symmetric difference $(x \cap y^c) \cup (x^c \cap y)$. By definition of the event set, the function d is well-defined.
- The moduli of uniform convergence are as follows: $\Delta_{\cdot^c,0}(\epsilon) = \epsilon$, $\Delta_{\cup,0}(\epsilon) = \Delta_{\cup,1}(\epsilon) = \frac{1}{2}\epsilon$, $\Delta_{\cap,0}(\epsilon) = \Delta_{\cap,1}(\epsilon) = \frac{1}{2}\epsilon$, $\Delta_{\mu,0}(\epsilon) = \epsilon$.

Remark 3.18. We would like to note that the functions $\cap$ and $\cup$ are not 1-Lipschitz, and therefore that restricting function and relation symbols to 1-Lipschitz functions would have been too restrictive to properly formulate this language, let alone the following theory.

Given this signature, we can list the axioms of the theory of probability algebras.

Definition 3.19. We define the theory of probability algebras as follows:

1. The first part of axioms express the system $(\cap, \cup, \cdot^c)$ as a Boolean algebra. We will not explicitly write these out.
2. The second part of the axioms express that μ behaves as a measure[6]. This looks as follows:
   - $\mu(0) = 0$
   - $\mu(1) = 1$
   - $\sup_x \sup_y (\mu(x \cap y) \dot{-} \mu(x)) = 0$
   - $\sup_x \sup_y (\mu(x) \dot{-} \mu(x \cup y)) = 0$
   - $\sup_x \sup_y ((\mu(x) \dot{-} \mu(x \cap y)) \dot{-} (\mu(x \cup y) \dot{-} \mu(y))) = 0$
   - $\sup_x \sup_y ((\mu(x \cup y) \dot{-} \mu(y)) \dot{-} (\mu(x) \dot{-} \mu(x \cap y))) = 0$
3. The third part of the axioms exhibit the connection between the metric d and the measure μ.
   - $\sup_x \sup_y (d(x, y) \dot{-} \mu(x \triangle y)) = 0$
   - $\sup_y \sup_x (\mu(x \triangle y) \dot{-} d(x, y)) = 0$

We will denote this theory by PrA.

One interesting extension of this theory is the theory of atomless probability spaces. We first give a definition.

Definition 3.20. Let (Ω, Σ, μ) be a probability space. We say that (Ω, Σ, μ) is *atomless* if for every element $S \in \Sigma$ there is an element $S' \in \Sigma$ such that $S' \subseteq S$ and $0 < \mu(S') < \mu(S)$.

Remark 3.21. An equivalent characterization of (Ω, Σ, μ) being an atomless space is that for every element $S \in \Sigma$ and $r \in [0, 1]$ there is an an element $S' \in \Sigma$ such that $S' \subseteq S$ and $\mu(S') = r \cdot \mu(S)$. We will use this definition for our axioms.

We can now extend the theory PrA to become a theory of atomless probability algebras.

Definition 3.22. We define the theory APA as theory PrA with the following axioms added:

- $\sup_x \inf_y (\mu(x \cap y) \dot{-} \mu(x \cap y^c)) = 0$
- $\sup_x \inf_y (\mu(x \cap y^c) \dot{-} \mu(x \cap y)) = 0$

We see that this means that the event x can be 'split in half' by the event y. By the above remark, this is sufficient for a model of APA to be atomless.

In [4], the authors list some model theoretic results regarding APA, one of which is the following.

Proposition 3.23. *(Proposition 16.6 in [4]) The theory APA admits quantifier elimination.*

For more results regarding APA and other examples of theories of continuous logic, we refer to [4].

[6] A consequence of the last four axioms is that $\mu(x \cap y) + \mu(x \cup y) = \mu(x) + \mu(y)$, so combined with the first two axioms μ is indeed interpreted as a measure.

4 Figueroa's work

In this section, we look at the work by Figueroa as presented in his master's thesis [10]. To do so, we first start with a small introduction to categorical logic. After that, we list all the other necessary prerequisites and preliminary definitions. Next, we will look at the concept of hyperdoctrines, which will play a central role in this work. Lastly, we specifically look at Figueroa's approach to a categorical interpretation of continuous logic, and we will discuss this approach.

4.1 Categorical logic

In this subsection, we look at some elementary results of categorical logic. Categorical logic is the study of logical systems using category theory. As we will see, category theory can give us a very unified treatment of many logics, and we will see that continuous logic could be one of those logics. We will follow the presentation and results as given in [18].

We start off with the following definitions.

Definition 4.1. Let $\mathcal{C}$ be a category. We call $\mathcal{C}$ *locally small* if for every two objects C and D in $\mathcal{C}$ the collection of arrows $\{f|f : C \to D\}$ is a set.

Definition 4.2. Let $\mathcal{C}$ be a category. We call $\mathcal{C}$ *well-powered* if for every object C in $\mathcal{C}$ the collection of subobjects of C is a set.

Throughout the rest of this section, we assume that every category is both locally small and well-powered.

Next, we recall the definition of adjoints. For our intents and purposes, it suffices to restrict ourselves to the special case of morphisms in the category of preorders.

Definition 4.3. Recall $Preorder$ is the category whose objects are preorders and whose morphisms are functions that preserve the order. Let $f : X \to Y$ and $g : Y \to X$ be two morphisms in $Preorder$. We say that f is *left adjoint* to g, or equivalently that g is *right adjoint* to f, if we have for all $x \in X$ and $y \in Y$ that $x \leq g \circ f(x)$ and $f \circ g(y) \leq y$. We denote this by $f \dashv g$.

In what follows we will look at many kinds of preorders. We recall the following definition of a special kind of preorder.

Definition 4.4. Let $L = \langle \bot, \top, \wedge, \vee \rangle$ be a lattice. We say that L is a *Heyting algebra* if for every $x, y \in L$ there is an element denoted $x \to y$ with the following property for all $z \in L$:

$$z \wedge x \leq y \;\Leftrightarrow\; z \leq x \to y$$

The operation $\to$ is called the *Heyting implication*.

Recall that in every category $\mathcal{C}$ the subobjects of a given object X form a poset: for two subobjects $m : Y \to X$ and $n : Z \to X$, we say that $Y \leq Z$ if there is an arrow $s : Y \to Z$ such that $m = ns$. Depending on the category $\mathcal{C}$, this poset can be equipped with extra operators or structure. We have for example the following proposition.

Proposition 4.5. *Let $\mathcal{C}$ be a category and let X an object of the presheaf category $Set^{\mathcal{C}^{op}}$. Then the poset $Sub(X)$ is a Heyting algebra. For every map $f : X \to Y$ in $Set^{\mathcal{C}^{op}}$, the pullback map $f^* : Sub(Y) \to Sub(X)$ commutes with the Heyting structure $(\bot, \top, \wedge, \vee, \to)$. Moreover f^* has both a right adjoint $\forall_f$ and a left adjoint $\exists_f$.*

One can see that in this proposition we use practically all logical symbols we know from classical first-order logic. This is of course no coincidence. It turns out that there is a canonical way to interpret any logical language in, amongst others, a category of presheaves. The exact definition of the interpretation of a language in a category of presheaves is as follows:

Definition 4.6. Let L be a language and let $\mathcal{C}$ be a category. The *interpretation of L in $Set^{\mathcal{C}^{op}}$* is given by choosing for each sort S a presheaf $[\![S]\!]$, for each function symbol $f : S_1, \dots, S_n \to S$ of the language, an arrow $[\![S_1]\!] \times \dots \times [\![S_n]\!] \to [\![S]\!]$, and for every relation symbol $R \subseteq S_1, \dots, S_n$ a subobject $[\![R]\!]$ of $[\![S_1]\!] \times \dots \times [\![S_n]\!]$.

Given such an interpretation of a language L, we can extend it to an interpretation of L-terms and first-order L-formulas using the Heyting algebra structure given in $\mathrm{Set}^{\mathcal{C}^{op}}$ and the adjoints provided by Proposition 4.5. For the exact definition in this special case, we refer to [18]. We will see the definition of this in a more general setting later on. For now, we define the tool with which we want to accomplice this more general interpretation, and that is the hyperdoctrine.

Definition 4.7. Let $\mathcal{C}$ be a category. A *pseudofunctor into Preorder*[7] $F : \mathcal{C} \to \mathcal{D}$ consists of assignments $F(C)$ in $\mathcal{D}$ for every object in $\mathcal{C}$ and $F(f)$ in $\mathcal{D}$ for every morphism f in $\mathcal{C}$, such that for every object $C \in \mathcal{C}$ we have that $F(\mathrm{id}_C)$ is isomorphic to $\mathrm{id}_{F(C)}$, and for every pair of composable morphisms f and g be have that $F(f \circ g)$ is isomorphic to $F(f) \circ F(g)$.

Furthermore let F and G be pseudofunctors into *Preorder*. A *pseudonatural transformation in Preorder* $\eta : F \to G$ is an assignment of morphisms $\eta_C : F(C) \to G(C)$ such that for any arrow $f : C \to C'$ we have that $G(f) \circ \eta_C$ is isomorphic $\eta_{C'} \circ F(f)$.

In what follows, we will simply refer to these concepts as *pseudofunctors* and *pseudonatural transformations.*

Definition 4.8. Let $\mathcal{C}$ be a category with finite products. A *hyperdoctrine* is a pseudofunctor $\mathbf{P} : \mathcal{C}^{op} \to Preorder$.

One should think of $\mathcal{C}$ is being a category of types, or structures. The hyperdoctrine then assigns to every structure the preorder of propositions in said structure, and to every morphism a certain 'substitution of variables' operation. With this in mind, we can try to interpret logic in a hyperdoctrine. The definition as given here is too simple to do anything rigorous immediately, but we can nevertheless give an example that should make clear in what sense we want to use hyperdoctrines.

Example 4.9. Let $\mathcal{C}$ be a category. We can define a hyperdoctrine $\mathbf{P} : (\mathrm{Set}^{\mathcal{C}^{op}})^{op} \to Preorder$ by defining $\mathbf{P}(X) = \mathrm{Sub}(X)$ and $\mathbf{P}(f) = f^*$. ❑

We will treat the concept of hyperdoctrines and the interpretation of logic in a hyperdoctrine extensively in Section 4.3.

4.2 Prerequisites and preliminary definitions

We continue with a list of definitions and results we will need as prerequisites. These can all be found in [10]. First, we recall the definitions of certain kinds of categories.

Definition 4.10. Let $\mathcal{C}$ be a category with finite limits. Given a morphism $f : X \to Y$ in $\mathcal{C}$, we define $\mathrm{im}(f)$ as the smallest subobject of Y through which f factors, whenever such a subobject

[7] It is possible to define pseudofunctors whose codomain is arbitrary, but that is more involved, and since we won't use the general definition we only look at this simpler case.

exists. This subobject is called the *image* of f.

Definition 4.11. A *regular category* $\mathcal{C}$ is a category with finite limits and with the property that every morphism has an image which commutes with the pullback operator. That is, for every morphism f and g in $\mathcal{C}$ we have that $f^*(\text{im}(g)) = \text{im}(f^*(g))$. This is equivalent to the following: if

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

is a pullback in $\mathcal{C}$, then $f^*(\text{im}(g)) = \text{im}(g')$.

Definition 4.12. A *coherent category* $\mathcal{C}$ is a well-powered regular category in which for every object X in $\mathcal{C}$ we have that the poset of subobjects $\text{Sub}_{\mathcal{C}}(X)$ has binary joins and a bottom element, and for every morphism $f : X \to Y$ in $\mathcal{C}$ we have that $\text{Sub}_{\mathcal{C}}(f)$ preserves these joins and bottom element.

Definition 4.13. A *Heyting category* $\mathcal{C}$ is a coherent category such that the arrow $\text{Sub}_{\mathcal{C}}(f)$: $\text{Sub}_{\mathcal{C}}(Y) \to \text{Sub}_{\mathcal{C}}(X)$ has a right adjoint for every morphism $f : X \to Y$.

Next, we will define an important trio of categories, namely the categories of certain metric spaces.

Definition 4.14. We define the category $pMet_1$ as the category with objects pseudo-metric spaces with diameter at most 1, and morphisms uniformly continuous functions. Similarly, we define the categories Met_1 and $cMet_1$ as the full subcategories of $pMet_1$ on the metric spaces and complete metric spaces, respectively.

The following useful lemma says something about the existence of limits and colimits in these categories of metric spaces.

Lemma 4.15. *(Proposition 6 in [10]) The categories $pMet_1$, Met_1 and $cMet_1$ have finite limits, an initial object and binary coproducts. Moreover, $pMet_1$ and Met_1 also have coequalizers.*

Proof. We will list the limits and colimts that exist, and refer to [10] for the proofs.

The terminal object in $pMet_1$, Met_1 and $cMet_1$ is (X, d), where X is a one-element set, and d is the only possible metric.

We now show that $pMet_1$, Met_1 and $cMet_1$ have pullbacks. Let $f : (X, d_X) \to (Z, d_Z)$ and $g : (Y, d_Y) \to (Z, d_Z)$ be uniformly continuous functions. Then the pullback is given by the following diagram:

$$\begin{array}{ccc} (X \times_Z Y, d) & \xrightarrow{\pi_Y} & (Y, d_Y) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle g} \\ (X, d_X) & \xrightarrow{f} & (Z, d_Z) \end{array}$$

Here $X \times_Z Y$ is the usual pullback f along g in Set, and the (pseudo-)metric d on $X \times_Z Y$ is defined by $d((x, y)(x', y')) = \max(d_X(x, x'), d_Y(y, y'))$.

We note that as a consequence of this, we have that the product of two spaces (X, d_X) and (Y, d_Y) is given by $(X \times Y, d)$, where $X \times Y$ is the usual Cartesian product, and d is the same as above.

The initial object in $pMet_1$, Met_1 and $cMet_1$ is given by the empty space with the empty metric.

We now show that $pMet_1$, Met_1 and $cMet_1$ have coproducts. For spaces (X, d_X) and (Y, d_Y), the coproduct will be denoted by $(X+Y, d)$. Here $X+Y$ is the usual disjoint union, and d is the metric given as follows for all $(i, z), (i', z') \in X+Y$:

$$d((i,z),(i',z')) = \begin{cases} d_X(z,z') & \text{if } i = i' = 0 \\ d_Y(z,z') & \text{if } i = i' = 1 \\ 1 & \text{if } i \neq i' \end{cases}$$

Lastly we will show a construction for equalizers in $pMet_1$ and Met_1. So suppose we have two arrows $f, g : (X, d_X) \to (Y, d_Y)$. Their coequalizer is a morphism $c : (Y, d_Y) \to (Y/\sim, d'_Y)$, where $\sim$ is the smallest equivalence relation such that $f(x) \sim g(x)$ for all $x \in X$, and d'_Y is the (pseudo-)metric given by:

$$d'_Y([y],[y']) = \inf\{d_Y(y,q_1) + d_Y(p_2,q_2) + \cdots + d(p_n,y') | n \geq 2, \forall i \leq n-1 : q_i \sim p_{i+1}\}$$

The morphism c itself is then simply given by the quotient map.

❑

Lastly, we define two other categories that we will use later on. To do so, we first need another definition.

Definition 4.16. Let $(S, \leq)$ be a preorder. We define the *poset reflection of* $(S, \leq)$ as the poset of which the elements are equivalence classes of elements of S under the relation $\sim$, which is defined as $a \sim b \Leftrightarrow a \leq b \wedge b \leq a$.

It is clear that every order-preserving function f between preorders can easily be transformed into an order-preserving function g between the poset reflections of said preorders. This g will be called the *poset reflection of* f.

Definition 4.17. We define the category $DistPrelattice$ as the category whose objects are preorders whose poset reflections are distributive lattices, and whose morphisms are order-preserving functions f whose poset reflections additionally preserve meets, joins and the top and bottom element.

We define the category $HeytPre$ as the category whose objects are preorders whose poset reflections are Heyting algebras, and whose morphisms are order-preserving functions f whose poset reflections additionally preserve meets, joins, the top and bottom element and Heyting implications.

Now we will define what it means for a continuous logic to be interpreted in a categorical sense. This interpretation will be our aiming point when trying to construct a suitable hyperdoctrine, but more about that later.

Definition 4.18. For a fixed signature Σ, a *continuous interpretation* $[\![\cdot]\!]$ *of* Σ consist of assignments of the following form:

1. For each sort S in Σ we assign an object $[\![S]\!]$ of $cMet_1$.

2. For every function symbol $f : S_1 \times \cdots \times S_n \to S$ in Σ we assign a morphism $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$ in $cMet_1$.

3. For every relation symbol $R \subseteq (S_1, \ldots, S_n)$ in Σ we assign a morphism $[\![R]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to ([0,1], d)$ in $cMet_1$. The metric d here is defined as the standard metric on the unit interval $[0, 1]$.

Now that we have defined what a continuous interpretation $[\![\cdot]\!]$ of a signature Σ is, we can define what it means for a signature Σ to be interpreted by $[\![\cdot]\!]$. To do this more simply, we introduce some notation. For a context $\bar{x} = x_1^{S_1}, \ldots, x_n^{S_n}$, we define $\mathbf{s}(\bar{x})$ as $[\![S_1]\!] \times, \ldots, \times [\![S_n]\!]$. Furthermore, we write $[\![t]\!]_{\bar{x}}$ for the interpretation of a term t whose free variables are $\bar{x}$ and $[\![\phi]\!]_{\bar{x}}$ or the interpretation of a formula ϕ whose free variables are $\bar{x}$. Using this, we have the following definition:

Definition 4.19. For a fixed signature Σ and continuous interpretation $[\![\cdot]\!]$ of Σ, we define the *interpretation of* Σ *by* $[\![\cdot]\!]$ as follows:

1. $[\![x^{S_i}]\!]_{\bar{x}} = \pi_i$
2. $[\![f(t_1, \ldots, t_n)]\!]_{\bar{x}} = [\![f]\!] \circ \langle [\![t_1]\!]_{\bar{x}}, \ldots, [\![t_n]\!]_{\bar{x}} \rangle$
3. $[\![R(t_1, \ldots, t_n)]\!]_{\bar{x}} = [\![R]\!] \circ \langle [\![t_1]\!]_{\bar{x}}, \ldots, [\![t_n]\!]_{\bar{x}} \rangle$
4. $[\![s = t]\!]_{\bar{x}} = d_t \circ \langle [\![s]\!]_{\bar{x}}, [\![t]\!]_{\bar{x}} \rangle$
5. $[\![\top]\!]_{\bar{x}} = 0$
6. $[\![\bot]\!]_{\bar{x}} = 1$
7. $[\![\phi \wedge \psi]\!]_{\bar{x}} = \max([\![\phi]\!]_{\bar{x}}, [\![\psi]\!]_{\bar{x}})$
8. $[\![\phi \vee \psi]\!]_{\bar{x}} = \min([\![\phi]\!]_{\bar{x}}, [\![\psi]\!]_{\bar{x}})$
9. $[\![\exists y \phi]\!]_{\bar{x}} = \inf_y([\![\phi]\!]_{\bar{x}y})$
10. $[\![\forall y \phi]\!]_{\bar{x}} = \sup_y([\![\phi]\!]_{\bar{x}y})$
11. $[\![\phi \to \psi]\!]_{\bar{x}} = [\![\phi]\!]_{\bar{x}} \dot{-} [\![\psi]\!]_{\bar{x}}$

Here we have that 0 and 1 are constant functions, d_t is the metric on the space $\mathbf{s}(t)$, the function $\pi_i : \mathbf{s}(\bar{x}) \to [\![S_i]\!]$ is the projection, and the function $\inf_y([\![\phi]\!]_{\bar{x}y})$ sends some $a \in \mathbf{s}(\bar{x})$ to $\inf_{b \in \mathbf{s}(y)}\{[\![\phi]\!]_{\bar{x}y}(a, b)\}$. The function $\sup_y([\![\phi]\!]_{\bar{x}y})$ is defined similarly.

We want to stress that this definition does not readily give an interpretation for all logical connectives that exist in continuous logic. We can, however, easily extend such an interpretation to the full continuous language. If we have a connective λ with arity n and for every $i \leq n$ we have a formula in context $(\phi_i|\bar{x})$ then we can define that $[\![\lambda(\phi_1, \ldots, \phi_n)]\!]_{\bar{x}} = \lambda([\![\phi_1]\!]_{\bar{x}}, \ldots, [\![\phi_n]\!]_{\bar{x}})$.

4.3 Hyperdoctrines

Now we will have a look at the concept of hyperdoctrines.

Definition 4.8 is too general in most cases, so we want to put some extra structure on hyperdoctrines so that they become useful. To do so, we first need some definitions, which we will give now.

Definition 4.20. Let $\mathcal{C}$ be category with finite products and let $\mathbf{P} : \mathcal{C}^{op} \to Preorder$ be a pseudo-functor.

We say that $\mathbf{P}$ *has equality* whenever we have that the following two conditions hold:

1. For every object X of $\mathcal{C}$, there is an element $Eq_X \in \mathbf{P}(X \times X)$ such that for any $A \in \mathbf{P}(X \times X)$ we have that $\top_X \leq \mathbf{P}(\delta)(A) \Leftrightarrow Eq_X \leq A$, where $\delta : X \to X \times X$ is the diagonal.
2. Denoting $\pi_{13} : A \times B \times A \times B \to A \times A$ and $\pi_{24} : A \times B \times A \times B \to B \times B$ for the projections to the denoted components, we have that $\mathbf{P}(\pi_{13})(Eq_A) \wedge \mathbf{P}(\pi_{24})(Eq_B) \simeq Eq_{A \times B}$.

We say that $\mathbf{P}$ *has left adjoints* if we have that $\mathbf{P}(f)$ has a left adjoint $\exists_f^{\mathbf{P}}$ for every morphism f in $\mathcal{C}$. Similarly, we say that $\mathbf{P}$ *has right adjoints* if we have that $\mathbf{P}(f)$ has a right adjoint $\forall_f^{\mathbf{P}}$ for every morphism f in $\mathcal{C}$.

We say that the left adjoints in $\mathbf{P}$ *satisfy the Beck-Chevalley condition* if we have for any pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{l} & B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

in $\mathcal{C}$ it holds that $\exists_l^{\mathbf{P}} \circ \mathbf{P}(f) \simeq \mathbf{P}(g) \circ \exists_k^{\mathbf{P}}$. We also have a similar definition for right adjoints.

We say that $\mathbf{P}$ *satisfies the Frobenius condition* if $\mathbf{P}$ has left adjoints and for every $f : B \to A$ in $\mathcal{C}$, $\alpha \in \mathbf{P}(A)$ and $\beta \in \mathbf{P}(B)$ we have $\exists_f^{\mathbf{P}}(\mathbf{P}(f)(\alpha) \wedge \beta) \simeq \alpha \wedge \exists_f^{\mathbf{P}}(\beta)$.

We say that $\mathbf{P}$ *has left adjoints for projections* if we have that $\mathbf{P}(\pi)$ has a left adjoint $\exists_\pi^{\mathbf{P}}$ for every projection $\pi : X \times Y \to X$ in $\mathcal{C}$. Similarly, we say that $\mathbf{P}$ *has right adjoints* if we have that $\mathbf{P}(\pi)$ has a right adjoint $\forall_\pi^{\mathbf{P}}$ for every projection π in $\mathcal{C}$.

We say that the left adjoints for projections in $\mathbf{P}$ *satisfy the Beck-Chevalley condition* if we have for any pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow{\pi'} & D \end{array}$$

in $\mathcal{C}$, where π and π' are projections, it holds that $\exists_\pi^{\mathbf{P}} \circ \mathbf{P}(f) \simeq \mathbf{P}(g) \circ \exists_{\pi'}^{\mathbf{P}}$. We also have a similar definition for right adjoints.

We say that $\mathbf{P}$ *satisfies the Frobenius condition for projections* if $\mathbf{P}$ has left adjoints for projections and for every projection $\pi : B \to A$ in $\mathcal{C}$, $\alpha \in \mathbf{P}(A)$ and $\beta \in \mathbf{P}(B)$ we have $\exists_\pi^{\mathbf{P}}(\mathbf{P}(\pi)(\alpha) \wedge \beta) \simeq \alpha \wedge \exists_\pi^{\mathbf{P}}(\beta)$.

Whenever it is not ambiguous, we may write $\exists_f^{\mathbf{P}}$ as $\exists_f$ and $\forall_f^{\mathbf{P}}$ as $\forall_f$.

Remark 4.21. We would like to emphasize that the functor $\mathbf{P}$ is a contravariant functor. As a result, we have that every morphism $f : X \to Y$ in $\mathcal{C}$ becomes a morphism $\mathbf{P}(f) : \mathbf{P}(Y) \to \mathbf{P}(X)$ in the image of $\mathbf{P}$. If we have that $\mathbf{P}$ has left adjoints, then we have that $\exists_f$ is a morphism $\mathbf{P}(X) \to \mathbf{P}(Y)$. The same holds for $\forall_f$.

Now that we have defined all the preliminary concepts, we can define some specific hyperdoctrines which will be of importance.

Definition 4.22. Let $\mathcal{C}$ be a category with finite products.

- A *coherent hyperdoctrine* is a pseusofunctor $\mathcal{C}^{op} \to DistPrelattice$ that satisfies the Frobenius condition and has left adjoints that satisfy the Beck-Chevalley condition.

- A *quantifier hyperdoctrine* is a coherent hyperdoctrine which also has right adjoints.
- A *first-order hyperdoctrine* is a pseudofunctor $\mathcal{C}^{op} \to HeytPre$ that has left and right adjoints which both satisfy the Beck-Chevalley condition.
- A *coherent hyperdoctrine for projections* is a pseusofunctor $\mathcal{C}^{op} \to DistPrelattice$ that has equality, satisfies the Frobenius condition for projections and has left adjoints for projections that satisfy the Beck-Chevalley condition.
- A *quantifier hyperdoctrine for projections* is a coherent hyperdoctrine for projections which also has right adjoints for projections that satisfy the Beck-Chevalley condition.
- A *first-order hyperdoctrine for projections* is a pseudofunctor $\mathcal{C}^{op} \to HeytPre$ that has equality and that has left and right adjoints for projections which both satisfy the Beck-Chevalley condition.

The first result regarding hyperdoctrines shows us how to construct hyperdoctrines from other hyperdoctrines using a 'change-of-base' technique.

Theorem 4.23. *(Proposition 16 in [10]) Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a coherent hyperdoctrine for projections, a quantifier hyperdoctrine for projections, or a first-order hyperdoctrine for projections respectively. Let $\mathcal{D}$ be a category with finite products and let $F : \mathcal{D} \to \mathcal{C}$ be a functor that preserves finite products. Then $\mathbf{P} \circ F^{op}$ is a coherent hyperdoctrine for projections, or a quantifier hyperdoctrine for projections, or a first-order hyperdoctrine for projections respectively.*

Proof. It is clear that $\mathbf{P} \circ F^{op}$ is indeed a hyperdoctrine.

We look at objects $D, D' \in \mathcal{D}$. We denote π_D and $\pi_{D'}$ for the projections from $D \times D'$, and $\pi_{F(D)}$ and $\pi_{F(D')}$ for the projections from $F(D) \times F(D')$. Since F preserves finite products, we see now that $F(D \times D')$ is isomorphic to $F(D) \times F(D')$. From this, it easily follows that $\exists_{\pi_{F(D)}}$ is isomorphic to the left adjoint of $\mathbf{P} \circ F^{op}$. With some abuse of notation, we may say that $\exists_{\pi_{F(D)}}$ *is* the left adjoint to $\mathbf{P} \circ F^{op}$. This same argument also holds for right adjoints, should they exist.

We refer to [10] for the proof of the equality predicate, and Beck-Chevalley and Frobenius conditions. ❑

Using this theorem, we can define morphisms between hyperdoctrines.

Definition 4.24. Let $\mathbf{P}$ with base $\mathcal{C}$ and $\mathbf{P}'$ with base $\mathcal{D}$ be coherent hyperdoctrines. A *morphism of coherent hyperdoctrines* $\mathbf{P} \to \mathbf{P}'$ is a pair (F, η) where $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves finite products and $\eta : \mathbf{P} \to \mathbf{P}' \circ F^{op}$ is a pseudonatural transformation such that for every morphism $f : X \to Y$ in $\mathcal{C}$ we have that $\eta_X \circ \exists_f^{\mathbf{P}} \simeq \exists_f^{\mathbf{P}' \circ F^{op}} \circ \eta_Y$.

Definition 4.25. Let $\mathbf{P}$ with base $\mathcal{C}$ and $\mathbf{P}'$ with base $\mathcal{D}$ be coherent hyperdoctrines for projections. A *morphism of coherent hyperdoctrines for projections* $\mathbf{P} \to \mathbf{P}'$ is a pair (F, η) where $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves finite products and $\eta : \mathbf{P} \to \mathbf{P}' \circ F^{op}$ is a pseudonatural transformation such that for every projection $\pi : X \to Y$ in $\mathcal{C}$ we have that $\eta_X \circ \exists_\pi^{\mathbf{P}} \simeq \exists_\pi^{\mathbf{P}' \circ F^{op}} \circ \eta_Y$ and where for every object X in $\mathcal{C}$ we have that $\eta_{X \times X}(Eq_X) \simeq \mathbf{P}'(\langle F(\pi_1), F(\pi_2) \rangle)(Eq_{F(x)})$, where $\pi_1, \pi_2 : X \times X \to X$ are the obvious projections.

Remark 4.26. The above definitions can be extended to the notion of quantifier and first-order hyperdoctrines by additionally requiring that $\eta_X \circ \forall_f^{\mathbf{P}} \simeq \forall_f^{\mathbf{P}' \circ F^{op}} \circ \eta_Y$. The same holds for quantifier and first-order hyperdoctrines for projections.

A special case of morphism of hyperdoctrines is given in the following definition.

Definition 4.27. Let $(F, \eta) : \mathbf{P} \to \mathbf{Q}$ be a morphism of hyperdoctrines as defined in one of the previous definitions. We say that (F, η) is *an embedding* if F is full and faithful and for each object X in the base category of $\mathbf{P}$ and every $A, B \in \mathbf{P}(X)$ we have that $\eta_X(A) \leq \eta_X(B) \Leftrightarrow A \leq B$.

The main reason we look at hyperdoctrines, is because of the fact that we can interpret first-order logic in a hyperdoctrine. It depends on the kind of hyperdoctrine exactly how much logic can be interpreted, as we will see in the following definitions.

Definition 4.28. Let $\mathbf{P} : \mathcal{C}^{op} \to Preorder$ be a hyperdoctrine and Σ be a signature. An *interpretation* $[\![\cdot]\!]$ *of* Σ *in* $\mathbf{P}$ is an assignment of the following form:

1. For each sort S in Σ we assign an object $[\![S]\!]$ of $\mathcal{C}$.
2. For every function symbol $f : S_1 \times \cdots \times S_n \to S$ in Σ we assign a morphism $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$ in $\mathcal{C}$.
3. For every relation symbol $R \subseteq (S_1, \ldots, S_n)$ in Σ we assign an element $[\![R]\!]$ of $\mathbf{P}([\![S_1]\!], \ldots, [\![S_n]\!])$.

We see that this definition already has many similarities to the definition of a continuous interpretation. Indeed, we will look at the case that $\mathcal{C} = cMet_1$ later. Now we look at the definition of the interpretation of terms and formulas. Where applicable, we use the notation that is familiar to us.

Definition 4.29. For a fixed signature Σ and interpretation $[\![\cdot]\!]$ of Σ in $\mathbf{P}$, we define the *interpretation of terms and formulas in context over* Σ *by* $[\![\cdot]\!]$ *in* $\mathbf{P}$ as follows:

1. $[\![x^S]\!]_{\bar{x}} = \pi'$
2. $[\![f(t_1, \ldots, t_n)]\!]_{\bar{x}} = [\![f]\!] \circ \langle [\![t_1]\!]_{\bar{x}}, \ldots, [\![t_n]\!]_{\bar{x}} \rangle$
3. $[\![R(t_1, \ldots, t_n)]\!]_{\bar{x}} = \mathbf{P}(\langle [\![t_1]\!]_{\bar{x}}, \ldots, [\![t_n]\!]_{\bar{x}} \rangle)([\![R]\!])$
4. $[\![s = t]\!]_{\bar{x}} = \mathbf{P}(\langle [\![s]\!]_{\bar{x}}, [\![t]\!]_{\bar{x}} \rangle)(Eq_{\mathbf{s}(\bar{x})})$
5. $[\![\top]\!]_{\bar{x}} = \top$
6. $[\![\bot]\!]_{\bar{x}} = \bot$
7. $[\![\phi \wedge \psi]\!]_{\bar{x}} = [\![\phi]\!]_{\bar{x}} \wedge [\![\psi]\!]_{\bar{x}}$
8. $[\![\phi \vee \psi]\!]_{\bar{x}} = [\![\phi]\!]_{\bar{x}} \vee [\![\psi]\!]_{\bar{x}}$
9. $[\![\exists y \phi]\!]_{\bar{x}} = \exists_\pi([\![\phi]\!]_{\bar{x}y})$
10. $[\![\forall y \phi]\!]_{\bar{x}} = \forall_\pi([\![\phi]\!]_{\bar{x}y})$
11. $[\![\phi \to \psi]\!]_{\bar{x}} = [\![\phi]\!]_{\bar{x}} \to [\![\psi]\!]_{\bar{x}}$

Here we have that $\pi' : \mathbf{s}(\bar{x}) \to [\![S]\!]$ and $\pi : \mathbf{s}(\bar{x}) \times \mathbf{s}(y) \to \mathbf{s}(\bar{x})$ are projections.

We see that we need a lot of structure on $\mathbf{P}$ to make sure that we can correctly interpret the logic as we have defined it. Luckily, this structure is given to us if $\mathbf{P}$ is a hyperdoctrine with some extra structure. More rigorously, we can easily check that a hyperdoctrine $\mathbf{P}$ has interpretations of terms and formulas along clauses 1 to 9 whenever $\mathbf{P}$ is a coherent hyperdoctrine (for projections). Furthermore, if $\mathbf{P}$ is a quantifier hyperdoctine, this also holds for clause 10. Clause 11 holds whenever $\mathbf{P}$ is a first-order hyperdoctrine.

We see that the interpretation of a term $(t|\bar{x})$ is a morphism $[\![t]\!]_{\bar{x}} : \mathbf{s}(\bar{x}) \to \mathbf{s}(t)$ in $\mathcal{C}$ and the interpretation of a formula $(\phi|\bar{x})$ is an element $[\![\phi]\!]_{\bar{x}}$ of $\mathbf{P}(\mathbf{s}(\bar{x}))$.

Given these interpretations, we say that a sequent $\phi \vdash_{\bar{x}} \psi$ in Σ *holds* with respect to $[\![\cdot]\!]$ if $[\![\phi]\!]_{\bar{x}} \leq [\![\psi]\!]_{\bar{x}}$ as elements of $\mathbf{P}(\mathbf{s}(\bar{x}))$. We also say that $\phi \dashv\vdash_{\bar{x}} \psi$ holds if $[\![\phi]\!]_{\bar{x}} \leq [\![\psi]\!]_{\bar{x}}$ and $[\![\psi]\!]_{\bar{x}} \leq [\![\phi]\!]_{\bar{x}}$ hold. Lastly, we see that a formula is *true* whenever we have that $\top \vdash_{\bar{x}} \phi$ holds.

As a final definition regarding hyperdoctrines, we define the image of an interpretation under a morphism of hyperdoctrines.

Definition 4.30. Let $\mathbf{P} : \mathcal{C}^{op} \to Preorder$ and $\mathbf{Q} : \mathcal{D}^{op} \to Preorder$ be hyperdoctrines and let a morphism of hyperdoctrines $(F, \eta) : \mathbf{P} \to \mathbf{Q}$ be given. Furthermore, suppose that we have a signature Σ and an interpretation $[\![\cdot]\!]$ of Σ in $\mathbf{P}$. We define the interpretation $[\![\cdot]\!]'$ of Σ in $\mathbf{Q}$ as follows:

1. For each sort S in Σ, we define $[\![S]\!]' = F([\![S]\!])$.
2. For every function symbol $f : S_1 \times \cdots \times S_n \to S$ in Σ we define $[\![f]\!]' = F([\![f]\!])$.
3. For every relation symbol $R \subseteq (S_1, \ldots, S_n)$ in Σ we define $[\![R]\!]' = \eta_{([\![S_1]\!], \ldots, [\![S_n]\!])}([\![R]\!])$.

We will also write $(F, \eta)[\![\cdot]\!]$ for $[\![\cdot]\!]'$.

We should note that, as in Theorem 4.23, we abuse notation slightly by implicitly identifying $F(S_1 \times \cdots \times S_n)$ with $F(S_1) \times \cdots \times F(S_n)$.

With this definition, we have the following theorem, which can be checked easily by induction on formulas. Of course, the theorem can easily be expanded to the other hyperdoctrines we have defined.

Theorem 4.31. *Let $(F, \eta) : \mathbf{P} \to \mathbf{Q}$ be a morphism of coherent hyperdoctrines for projections and let $[\![\cdot]\!]$ be an interpretation of a signature Σ in $\mathbf{P}$. Then for any coherent sequent S over Σ we have that $[\![S]\!]$ is valid in $[\![\cdot]\!]$ implies that $[\![S]\!]'$ is valid in $(F, \eta)([\![\cdot]\!])$. If we have that (F, η) is an embedding, then we also have the reverse implication.*

4.4 Figueroa's approach

In this subsection, we look at Figueroa's attempt to interpret continuous logic in a hyperdoctrine. To do so, we define a new category, and see that it comes with an interesting hyperdcotrine. We use this hyperdoctrine to construct additional hyperdcotrines that we will use to interpret continuous logic.

First, we define the notion of a partial equivalence relation over a hyperdoctrine, and see that these induce a category in which the monos are nicely characterized. Immediately after that, we define the notion of a strict relation on a partial equivalence relation, and show that strict relations are in a 1-1 correspondence to subobjects.

Definition 4.32. Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a coherent hyperdoctrine. A *partial equivalence relation over* $\mathbf{P}$ is a pair $(X, \sim)$ where X is an object of $\mathcal{C}$ and $\sim$ an element of $\mathbf{P}(X \times X)$ that is symmetric an transitive. That is, we have that the following two sequents are valid for $\sim$:

1. $x \sim y \vdash_{xy} y \sim x$
2. $x \sim y \wedge y \sim z \vdash_{xyz} x \sim z$

Definition 4.33. For a given coherent hyperdoctrine $\mathbf{P}$, we define $PER(\mathbf{P})$ as *the category of partial equivalence relations over* $\mathbf{P}$. The objects are partial equivalence relations over $\mathbf{P}$ and morphisms $(X, \sim) \to (Y, \sim)$ are given by isomorphism classes of functional relations F, which are elements of $\mathbf{P}(X \times Y)$ such that the following sequents hold:

1. $F(x, y) \vdash_{xy} x \sim x \wedge y \sim y$
2. $F(x, y) \wedge x \sim x' \wedge y \sim y' \vdash_{xx'yy'} F(x', y')$
3. $F(x, y) \wedge F(x, y') \vdash_{xyy'} y \sim y'$
4. $x \sim x \vdash_x \exists y F(x, y)$

For morphisms $f : (X, \sim) \to (Y, \sim)$ and $g : (Y, \sim) \to (Z, \sim)$ represented by functional relations F and G respectively, we can define the composition $g \circ f$ to be the isomorphism class of the functional relation $[\![\exists y(F(x, y) \wedge G(y, z))]\!]_{xz}$. Through some tedious work, it can be checked that is operation is associative. Furthermore, it is easy to see that for every object $(X, \sim)$ the identity morphism is given by the equivalence class of the relation $\sim$ itself.

Theorem 4.34. *Let* $\mathbf{P}$ *be a coherent hyperdoctrine for projections and let* $f : (X, \sim) \to (Y, \sim)$ *be a morphism in* $PER(\mathbf{P})$ *represented by a functional relation* F*. Then we have that the following are equivalent:*

- f *is a monomorphism*
- *The sequent* $F(x, y) \wedge F(x', y) \vdash_{xx'y} x \sim x'$ *holds.*
- f *is isomorphic to the morphism* $[\sim_F] : (Y, \sim_F) \to (Y, \sim)$*, where* $\sim_F$ *is the relation on* $\mathbf{P}(Y \times Y)$ *such that* $y \sim_F y' \dashv\vdash y \sim y' \wedge \exists x F(x, y)$*.*

Using this characterization of monomorphisms, we will see that subobjects in the category $PER(\mathbf{P})$ are equivalent to strict relations. First we, need to define what we mean by that.

Definition 4.35. Let $\mathbf{P}$ be a coherent hyperdoctrine for projections and let $(X, \sim)$ be an object in $PER(\mathbf{P})$. A *strict relation on* $(x, \sim)$ is an element of $\mathbf{P}(X)$ such that the following sequents hold:

1. $\phi(x) \vdash_x x \sim x$
2. $\phi(x) \wedge x \sim x' \vdash_{xx'} \phi(x')$

Theorem 4.36. *Let* $\mathbf{P}$ *be a coherent hyperdoctrine for projections and let* $(Y, \sim)$ *be an object in* $PER(\mathbf{P})$*. Then there is a 1-1 correspondence* $H_{(Y,\sim)}$ *between subobjects of* $(Y, \sim)$ *and isomorphism classes of strict relations on* $(Y, \sim)$*.*

Proof. Let $f : (X, \sim) \to (Y, \sim)$ be a monomorphism representing a subobject, and let F be its representing functional relation. It is straightforward to check from the definition that $[\![\exists x F(x, y)]\!]_y$ is a strict relation. We will denote this operation by $H_{(Y,\sim)}$.

Conversely, let $\phi \in \mathbf{P}(Y)$ be a strict relation on $(Y, \sim)$. It is then clear to see that the relation $\sim_1 = [\![y \sim y' \wedge \phi(y)]\!]_{yy'}$ is functional, and therefore gives rise to an equivalence class of morphisms $[\sim_1] : (Y, \sim_1) \to (Y, \sim)$. We can see that this is a monomorphism by Theorem 4.34, and therefore this morphism represents a subobject. We will denote this operation by $H^{-1}_{Y(,\sim)}$, although we have yet to prove that this operation is actually the inverse of $H_{(Y,\sim)}$.

Now we only have to show that this correspondence is 1-1. So let $\phi \in \mathbf{P}(Y)$ be a strict relation on $(Y, \sim)$. We see that:

$$H_{(Y,\sim)} \circ H^{-1}_{(Y,\sim)}(\llbracket \phi(y) \rrbracket_y) = H_{(Y,\sim)}(\llbracket y \sim y' \wedge \phi(y) \rrbracket_{yy'}) = \llbracket \exists y'(y \sim y' \wedge \phi(y)) \rrbracket_y$$

Since ϕ is a strict relation, we see that $\phi(y) \vdash_y y \sim y$, and then of course we also have that $\phi(y) \vdash_y \exists y'(y \sim y' \wedge \phi(y))$. Since we also have that $\exists y'(y \sim y' \wedge \phi(y)) \vdash_y \phi(y)$, obviously, we may conclude that $H_{(Y,\sim)} \circ H^{-1}_{(Y,\sim)}(\llbracket \phi(y) \rrbracket_y) = \llbracket \exists y'(y \sim y' \wedge \phi(y)) \rrbracket_y \simeq \llbracket \phi(y) \rrbracket_y$.

For the converse, let $f : (X, \sim) \to (Y, \sim)$ be a monomorphism represented by F. We see that:

$$H^{-1}_{(Y,\sim)} \circ H_{(Y,\sim)}(\llbracket F(x,y) \rrbracket_{xy}) = H^{-1}_{(Y,\sim)}(\llbracket \exists x F(x,y) \rrbracket_y) = \llbracket y \sim y' \wedge \exists x F(x,y) \rrbracket_{yy'}$$

By Theorem 4.34, $\llbracket y \sim y' \wedge \exists x F(x,y) \rrbracket_{yy'}$ is isomorphic to F. So we are done. ❑

We will use this function H when defining several useful functors. The first one follows now.

Definition 4.37. Let $\mathbf{P}$ be a coherent hyperdoctrine for projections. We define $Strict_{PER(\mathbf{P})} : PER(\mathbf{P}) \to Preorder$ as the functor which sends $(X, \sim)$ to the set of isomorphism classes of strict relations on $(X, \sim)$ and sends a morphism $F : (X, \sim) \to (Y, \sim)$ to the function $H_{(X,\sim)} \circ Sub_{PER(\mathbf{P})}(F) \circ H^{-1}_{(Y,\sim)}$. We define the ordering $\leq$ on $Strict_{PER(\mathbf{P})}$ such that for any $A, B \in Strict_{PER(\mathbf{P})}$ we have $A \leq B \Leftrightarrow H^{-1}_{X,\sim}(A) \leq' H^{-1}_{X,\sim}(B)$, where $\leq'$ is the ordering of $Sub_{PER(\mathbf{P})}(X, \sim)$.

To show that these definitions are useful, we have the following theorem.

Proposition 4.38. *Let $\mathbf{P}$ be a coherent hyperdoctrine for projections. We have that $Sub_{PER(\mathbf{P})}$ and $Strict_{PER(\mathbf{P})}$ are coherent hyperdoctrines and $(id_{cMet_1}, H) : Strict_{PER(\mathbf{P})} \to Sub_{PER(\mathbf{P})}$ is an isomorphism of coherent hyperdoctrines for projections.*

We also have the following nice result regarding the order on $Strict_{PER(\mathbf{P})}$.

Proposition 4.39. *Let $\mathbf{P}$ be a coherent hyperdoctrine for projections. Let $(X, \sim)$ be an object in $PER(\mathbf{P})$, and let $A, B \in Strict_{PER(\mathbf{P})}(X, \sim)$ be given, and let P_A and P_B be the strict relations that represent them. Denoting $\leq_{\mathbf{P}}$ for the ordering on $\mathbf{P}(X)$, we have the following:*

$$P_A \leq_{\mathbf{P}} P_B \Leftrightarrow A \leq B$$

So we see that order on $Strict_{PER(\mathbf{P})}$ closely resembles that of $\mathbf{P}$.

We could ask ourselves whether or not the result of Proposition 4.38 also holds if we replace coherent hyperdoctrine with quantifier hyperdoctrine. We can see in the following proposition that this is not always the case, but the following suffices for our work, as we will see.

Proposition 4.40. *Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a quantifier hyperdoctrine for projections, and suppose that $\pi : (Y \times Z, \sim) \to (Y, \sim)$ is a projection in $PER(\mathbf{P})$, such that $y \sim y \dashv\vdash \top$ holds in $\mathbf{P}$. Then $Strict_{PER(\mathbf{P})}(\pi)$ has a right adjoint. If $z \sim z \dashv\vdash \top$ holds for every $(Y, \sim)$, then $Strict_{PER(\mathbf{P})}$ is a quantifier hyperdoctrine for projections.*

Proof. We define $\forall_\pi([\phi(x,y)]) = [y \sim y \wedge \forall z \phi(y,z)]$. We can easily see that the right-hand side is a strict and order-preserving relation. To see that we have an adjunction, we look at the following for every strict relation P on $(Y \times Z, \sim)$ and Q on $(Y, \sim)$:

$$
\begin{aligned}
&(Strict_{PER(\mathbf{P})}(\pi)(Q))(y) \vdash_{yz} P(y,z) \\
\Leftrightarrow\ &\exists y'(Q(y) \wedge y' \sim y) \vdash_{yz} P(y,z) \\
\Leftrightarrow\ &Q(y) \wedge y \sim y \vdash_{yz} P(y,z) \\
\Leftrightarrow\ &Q(y) \wedge y \sim y \vdash_{y} \forall z P(y,z) \\
\Leftrightarrow\ &Q(y) \vdash_{y} y \sim y \wedge \forall z P(y,z) \\
\Leftrightarrow\ &Q(y) \vdash_{y} \forall_{\pi}(P(y,z))
\end{aligned}
$$

❑

Now we will define a certain hyperdoctrine which will be fundamental in defining the hyperdoctrine which will be used to interpret continuous logic. Before we do so, we need another definition.

Definition 4.41. Let X be a set, and let $f, g : X \to [0,1]$. We write $f \sqsubseteq g$ if we have for every $\epsilon > 0$ that there is a $\delta > 0$ such that $f(x) \leq \delta$ implies that $g(x) \leq \epsilon$ for all $x \in X$.

Lemma 4.42. *Let X be a set. The set of functions from X to $[0,1]$ with the ordering $\sqsubseteq$ gives rise to a preorder.*

Proof. First we prove reflexivity. Let $f : X \to [0,1]$ be given. For any $\epsilon > 0$ we choose $\delta = \epsilon$, and we see clearly that $f(x) \leq \delta$ gives that $f(x) \leq \epsilon$ for all $x \in X$. So we have that $f \sqsubseteq f$.

Now we prove transitivity. Let $f, g, h : X \to [0,1]$ be given such that $f \sqsubseteq g$ and $g \sqsubseteq h$. Let $\epsilon > 0$ be given. This gives us $\delta_0 > 0$ such that $g(x) \leq \delta_0$ implies that $h(x) \leq \epsilon$ for all $x \in X$. But this gives us a $\delta_1 > 0$ such that $f(x) \leq \delta_1$ implies that $g(x) \leq \delta_0$ for all $x \in X$, since $\delta_0 > 0$. So we see that $f(x) \leq \delta_1$ implies that $h(x) \leq \epsilon$. So we have that $f \sqsubseteq h$. ❑

The choice for the order $\sqsubseteq$ on the predicates is not immediately explained in [10], but in the more recent preprint [11] Van den Berg and Figueroa do give an explanation. They refer to [4], from which the following result is paraphrased.

Proposition 4.43. *(Proposition 7.14 in [4]) Let L be a continuous metric signature, and let $\phi(\vec{x})$ and $\psi(\vec{x})$ be L-formulas. Furthermore, let T be an L-theory and let $\mathcal{M}$ be an ω-saturated model of T. Then the following statements are equivalent:*

1. *For all $\vec{m} \in \mathcal{M}^n$, if $\phi_{\mathcal{M}}(\vec{m}) = 0$ then $\psi_{\mathcal{M}}(\vec{m}) = 0$.*
2. *$\forall \epsilon > 0 \exists \delta > 0 \forall \vec{m} \in \mathcal{M}^n : \phi_{\mathcal{M}}(\vec{m}) < \delta \Rightarrow \psi_{\mathcal{M}}(\vec{m}) \leq \epsilon$.*

The second condition is clearly the same as saying $\phi \sqsubseteq \psi$, and the first condition is a certain form of implication. Since this implication is the closest to the notion of consequence that the authors of [11] could get, they went with that. We recall from our introduction of continuous logic in Section 3 and the comparison with precursors in Section 3.2 that we suggested that there actually is a connective that plays the role of implication, namely $\dot{-}$. We will try to fit $\dot{-}$ in a hyperdoctrine in Section 6. For now, we will look at Figueroa's approach.

Lemma 4.42 gives rise to the following definition.

Definition 4.44. Let X be a set. We define $\mathbf{P}_{CL}(X)$ as the preorder $(\{X \to [0,1]\}, \sqsubseteq)$. For a function $f : X \to Y$, we let $\mathbf{P}_{CL}(f) : \mathbf{P}_{CL}(Y) \to \mathbf{P}_{CL}(X)$ be the function that sends an $\alpha \in \mathbf{P}_{CL}(Y)$ to $\alpha \circ f$.

We can easily check that this is assignment gives rise to a functor.

An important result is the following theorem.

Theorem 4.45. *The functor* $\mathbf{P}_{CL} : Set^{op} \to Preorder$*, which sends* X *to* $\mathbf{P}_{CL}(X)$ *and* f *to* $\mathbf{P}_{CL}(f)$*, is a quantifier hyperdoctrine.*

Proof. We give a list of the necessary properties.

- The top element of $\mathbf{P}_{CL}(X)$ is given by the constant function 0.
- The bottom element of $\mathbf{P}_{CL}(X)$ is any function f for which we have that $\inf_{x \in X} f(x) > 0$.
- Binary meets and joins of two functions f and g are given by $\max(f,g)$ and $\min(f,g)$ respectively.
- For any morphism $f : Y \to X$, we have a left adjoint $\exists_f : \mathbf{P}_{CL}(Y) \to \mathbf{P}_{CL}(X)$. For every morphism $\alpha : Y \to [0,1]$ and any $x \in X$, we define $(\exists_f \alpha)(x) = \inf\{\alpha(y) | y \in Y, f(y) = x\}$.
- For any morphism $f : Y \to X$, we have a left adjoint $\forall_f : \mathbf{P}_{CL}(Y) \to \mathbf{P}_{CL}(X)$. For every morphism $\alpha : Y \to [0,1]$ and any $x \in X$, we define $(\forall_f \alpha)(x) = \sup\{\alpha(y) | y \in Y, f(y) = x\}$.

❑

Remark 4.46. In general, we have that $\mathbf{P}_{CL}(X)$ is not a poset. For example, we can see that any function f for which we have that $\inf_{x \in X} f(x) > 0$ is isomorphic to the constant function 1.

Theorem 4.45 is the strongest result we can get, since we also have the following.

Theorem 4.47. *Let* X *be an infinite set. Then we have that* $\mathbf{P}_{CL}(X)$ *does not have a Heyting implication.*

Proof. We will show that there exist functions $g, h \in \mathbf{P}_{CL}(X)$ such that for all functions $\alpha \in \mathbf{P}_{CL}(X)$ there is a function $f \in \mathbf{P}_{CL}(X)$ such that we *don't* have that $f \wedge g \sqsubseteq h \Leftrightarrow f \sqsubseteq \alpha$. Since X is infinite, we can look at a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_i = x_j \Rightarrow i = j$. We take h to be the constant function 1, and we define g as follows:

$$g(x) = \begin{cases} \frac{1}{n} & \text{if } \ x = x_n \\ 1 & \text{otherwise} \end{cases}$$

It is easy to see that $g \not\sqsubseteq h$, since $\inf_x g(x) = 0$.

Suppose first that $\alpha = \top$. Then we can take $f = g$, and therefore we have that $f \wedge g = g \not\sqsubseteq h$, but we do have that $f = g \sqsubseteq \alpha = \top$. So assume that $\alpha \neq \top$, which means that for some $x' \in X$ we have that $\alpha(x') > 0$. We can then define f such that $f(x') = 0$ and $f(x) = 1$ otherwise. We then see that $\inf_x (f \wedge g)(x) > 0$, so $f \wedge g \sqsubseteq \bot = h$. But we also clearly have that $f \not\sqsubseteq \alpha$. We conclude that there doesn't exist a Heyting implication on $\mathbf{P}_{CL}$. ❑

A result we will use later on is the following corollary of Proposition 4.40 and Theorem 4.45.

Corollary 4.48. *Let $\pi : (Y, \sim) \times (Z, \sim) \to (X, \sim)$ be a projection in $PER(\mathbf{P}_{CL})$ such that the sequent $z \sim z \dashv\vdash_z \top$ holds in $\mathbf{P}_{CL}$. Then there exists a morphism:*

$$\forall_\pi : Strict_{PER(\mathbf{P}_{CL})}(Y \times Z, \sim) \to Strict_{PER(\mathbf{P}_{CL})}(Y, \sim)$$

such that $\pi^* \dashv \forall_\pi$.

Another way to state this corollary is that $Strict_{PER(\mathbf{P}_{CL})}$ is almost a quantifier hyperdoctrine. We will see that this result is sufficient for our intents and purposes.

We have not yet made clear why we are looking at partial equivalence relations when the primary objects we regard are metric spaces. The following lemmas will make this correspondence clear.

Lemma 4.49. *Let (X, d) be an object of $pMet_1$. Then (X, d) can be regarded as an element of $PER(\mathbf{P}_{CL})$.*

Proof. We must check that d can be seen as a symmetric and transitive element of $\mathbf{P}_{CL}(X \times X)$.

First of all, we see that d is a function from $X \times X$ to the interval $[0, 1]$, since X was assumed to have diameter at most 1, so d is indeed an element of $\mathbf{P}_{CL}(X \times X)$.

Next, we see that $d(x, y) = d(y, x)$, so d is clearly a symmetric relation.

Lastly, by the triangle equality, we have for any three points $x, y, z \in X$ that $d(x, z) \leq d(x, y) + d(y, z)$, which implies that $d(x, z) \leq \max(d(x, y), d(y, z))$. So by choosing $\delta = \frac{1}{2}\epsilon$, we see that $\max(d(x, y), d(y, z)) < \delta \Rightarrow d(x, z) < \epsilon$, and therefore the sequent $d(x, y) \wedge d(y, z) \vdash_{xyz} d(x, z)$ holds. ❑

Lemma 4.50. *Let $f : (X, d) \to (Y, d')$ be a uniformly continuous function with modulus of uniform continuity Δ. Then the element of $\mathbf{P}_{CL}(X \times Y)$ that sends a pair (x, y) to $d'(f(x), y)$ is a functional relation.*

Proof. We check whether the sequents given in the definition of a functional relation hold.

First we check whether we have that $d'(f(x), y) \vdash_{xy} d(x, x) \wedge d'(y, y)$. So we must check whether we have for every $\epsilon > 0$ that there is some $\delta > 0$ such that $d'(f(x), y) < \delta$ implies that $\max(d(x, x), d'(y, y)) < \epsilon$. But this is trivially given, since $d(x, x) = d'(y, y) = 0$ in every (pseudo-)metric space.

Secondly, we look at the sequent $d'(f(x), y) \wedge d(x, x') \wedge d'(y, y') \vdash_{xx'yy'} d'(f(x'), y')$. So let $\epsilon > 0$ be given. We choose $\delta = \min(\Delta(\frac{1}{3}\epsilon), \frac{1}{3}\epsilon)$. This gives us that $d'(f(x'), y') \leq d'(f(x'), f(x)) + d'(f(x), y) + d'(y, y') \leq \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon$.

Thirdly, we look at the sequent $d'(f(x), y) \wedge d'(f(x), y') \vdash_{xyy'} d'(y, y')$. We choose $\delta = \frac{1}{2}\epsilon$. This gives us that $d'(y, y') \leq d'(y, f(x)) + d'(f(x), y') \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$.

Lastly, we look at the sequent $d(x, x) \vdash_x \exists y d'(f(x), y)$. Since $d(x, x) = 0 < \delta$ for every $\delta > 0$, we must have that $[\![\exists y d'(f(x), y)]\!]_x(x) = 0$ for every $x \in X$. We may assume X to be non-empty, since we have that the sequent is trivially true otherwise. We see that $[\![\exists y d'(f(x), y)]\!]_x(x) = \exists_\pi([\![d'(f(x), y)]\!]_{xy})(x)$. Recalling the definition of the adjoint $\exists_\pi$, we see that $\exists_\pi([\![d'(f(x), y)]\!]_{xy})(x) = \inf\{d'(f(x'), y') | y' \in Y, \pi_X(x', y') = x\}$, where $\pi_X : X \times Y \to X$ is the projection to the first coordinate. Since we assumed that x is non-empty, we can choose some $x_0 \in X$, and denoting y_0 for $f(x_0)$, we have that $d'(f(x_0), y_0) = d'(y_0, y_0) = 0$, so in particular we have that $\inf\{d'(f(x'), y') | y' \in Y, \pi_X(x', y') = x\} = 0$, as desired. ❑

We note that the proof given here works in every pseudo-metric space. We even have the following result.

Proposition 4.51. *For any object (X, d) of $pMet_1$, let $G'(X, d)$ be (X, d) considered as an object of $PER(\mathbf{P}_{CL})$. For any morphism $f : (X, d) \to (Y, d')$ in $pMet_1$, let $G'(f)$ be the isomorphism class of functional relations isomorphic to $d'(f(x), y)$. Then we have that G' is a functor $pMet_1 \to PER(\mathbf{P}_{CL})$. Moreover, the restriction $G : cMet_1 \to PER(\mathbf{P}_{CL})$ is also a well-defined functor.*

Proposition 4.52. *The functor $G : cMet_1 \to PER(\mathbf{P}_{CL})$ is full and faithful, preserves finite products and preserves finite coproducts.*

As a last prerequisite, we prove a theorem which will help us simplify proofs that certain hyperdoctrines are coherent, quantifier or first-order hyperdoctrines. In order to do that, we first need two definitions.

Definition 4.53. Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a coherent hyperdoctrine for projections, let $F : \mathcal{C}^{op} \to DistPrelattice$ be any functor, and let $\eta : F \to \mathbf{P}$ be a natural transformation of functors. We say that $\mathbf{P}$ *has η-restrictable adjoints* if for every projection $\pi : C \times C' \to C$ we have that the morphism $\exists_\pi^{\mathbf{P}} \circ \eta_{C\times C'}$ factors through η_C, and, should $\mathbf{P}$ be a quantifier hyperdoctrine, the morphism $\forall_\pi^{\mathbf{P}} \circ \eta_{C\times C'}$ factors through η_C. In other words, we have that there are arrows $f, g : F(C \times C') \to F(C)$ that make the following diagrams commute:

$$\begin{array}{ccc} F(C \times C') & \xrightarrow{f} & F(C) \\ {\scriptstyle \eta_{C\times C'}}\downarrow & & \downarrow{\scriptstyle \eta_C} \\ P(C \times C') & \xrightarrow{\exists_\pi^{\mathbf{P}}} & \mathbf{P}(C) \end{array}$$

$$\begin{array}{ccc} F(C \times C') & \xrightarrow{g} & F(C) \\ {\scriptstyle \eta_{C\times C'}}\downarrow & & \downarrow{\scriptstyle \eta_C} \\ P(C \times C') & \xrightarrow{\forall_\pi^{\mathbf{P}}} & \mathbf{P}(C) \end{array}$$

Definition 4.54. Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a coherent hyperdoctrine for projections, let $F : \mathcal{C}^{op} \to DistPrelattice$ be any functor, and let $\eta : F \to \mathbf{P}$ be a natural transformation of functors. We say that $\mathbf{P}$ *has η-restrictable equality predicates* if for every diagonal $\delta : C \to C \times C$ we have that there is a unique element $\phi \in F(C \times C)$, up to isomorphism, such that $\eta_{C\times C}(\phi) = Eq_C$, where Eq_C is the equality predicate of C in $\mathbf{P}$.

Theorem 4.55. *Let $\mathbf{P} : \mathcal{C}^{op} \to DistPrelattice$ be a coherent, quantifier or first-order hyperdoctrine for projections, let $F : \mathcal{C}^{op} \to Distprelattice$ be a functor, and $\eta : F \to \mathbf{P}$ a natural transformation such that every component preserves and reflects the order. If $\mathbf{P}$ has η-restrictable left adjoints, then F has left adjoints satisfying the Beck-Chevalley and Frobenius condition. If $\mathbf{P}$ has η-restrictable right adjoints, then F has right adjoints. If $\mathbf{P}$ has η-restrictable equality predicates, then F has equality.*

In particular, if we have that $\mathbf{P}$ has η-restrictable adjoints and η-restrictable equality predicates, then F is a coherent, quantifier or first-order hyperdoctrine for projections, respectively.

Proof. We will prove the last statement, and we can see that the statements before that follow from this proof. We will give a list of the necessary structures and prove all the properties.

We first note that every component $\eta_C : F(C) \to \mathbf{P}(C)$ is a morphism of prelattices or pre-Heyting algebras, and therefore perserves and reflects not only the order, but also binary joins, binary meets, the top element and bottom element, and, should it exist, the Heyting implication. Therefore, we can immediately look at adjoints.

Let $\pi : C \times C' \to C$ be a projection. We want to see whether $F(\pi)$ has a left adjoint. Since we assumed that $\mathbf{P}$ has η-restrictable adjoints, we see that $\exists^{\mathbf{P}}_\pi \circ \eta_{C\times C'} = \eta_C \circ f$, for some morphism $f : F(C \times C') \to F(C)$. We will put $\exists^F_\pi = f$. Note that we thus have that $\exists^{\mathbf{P}}_\pi \circ \eta_{C\times C'} = \eta_C \circ \exists^F_\pi$.

So we now have that $\eta_C(\exists^F_\pi(\phi)) = \exists^{\mathbf{P}}_\pi(\eta_{C\times C'}(\phi))$. Using this fact, we see that $\exists^F_\pi$ is indeed a left adjoint, since we have the following for every $\alpha \in F(C)$ and $\beta \in F(C \times C')$:

$$\begin{aligned}
&\beta \leq F(\pi)(\alpha)\\
\Leftrightarrow\ &\eta_{C\times C'}(\beta) \leq \eta_{C\times C'}(F(\pi)(\alpha))\\
\Leftrightarrow\ &\eta_{C\times C'}(\beta) \leq \mathbf{P}(\pi)(\eta_C(\alpha))\\
\Leftrightarrow\ &\exists^{\mathbf{P}}_\pi(\eta_{C\times C'}(\beta)) \leq \eta_C(\alpha)\\
\Leftrightarrow\ &\eta_C(\exists^F_\pi(\beta)) \leq \eta_C(\alpha)\\
\Leftrightarrow\ &\exists^F_\pi(\beta) \leq \alpha
\end{aligned}$$

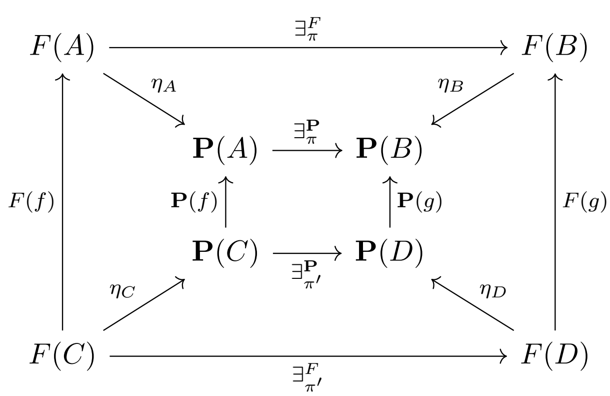
Here we used in the first and last step that every component of η is an order preserving and reflecting function, and in second-to-last step we used that $\exists^{\mathbf{P}}_\pi \circ \eta_{C\times C'} = \eta_C \circ \exists^F_\pi$, as discussed above. So F indeed has left adjoints.

By a similar argument as above, we can see that F also has right adjoints whenever $\mathbf{P}$ has them.

Now we show that F satisfies the Beck-Chevalley condition. Let the following diagram be a pullback diagram in $\mathcal{C}$.

$$\begin{array}{ccc}
A & \xrightarrow{\pi} & B\\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g}\\
C & \xrightarrow{\pi'} & D
\end{array}$$

Here π and π' are projections. We see that this gives rise to the following diagram in $DistPrelattice$:



We see that the inner square commutes because the Beck-Chevalley condition holds for $\mathbf{P}$. We see that the top and bottom square commute by definition of the left adjoints of F and because $\mathbf{P}$ has η-restrictable adjoints. Lastly, the left and right square commute since η is a natural transformation. So we see that the outer square commutes, and therefore the Beck-Chevalley condition holds for F.

Next, we will show that the Frobenius condition holds. So let $\pi : A \to B$ be a projection in $\mathcal{C}$, and let $\alpha \in F(A)$ and $\beta \in F(B)$ be given. We see that we have the following chain of equivalences:

$$\begin{aligned}
&\exists^F_\pi(F(\pi)(\beta) \wedge \alpha) \simeq \beta \wedge \exists^F_\pi(\alpha)\\
\Leftrightarrow\ &\eta_B(\exists^F_\pi(F(\pi)(\beta) \wedge \alpha)) \simeq \eta_B(\beta \wedge \exists^F_\pi(\alpha))\\
\Leftrightarrow\ &\exists^{\mathbf{P}}_\pi(\eta_A(F(\pi)(\beta))) \wedge \eta_A(\alpha) \simeq \eta_B(\beta) \wedge \eta_B(\exists^F_\pi(\alpha))\\
\Leftrightarrow\ &\exists^{\mathbf{P}}_\pi(\mathbf{P}(\pi)(\eta_B(\beta))) \wedge \eta_A(\alpha) \simeq \eta_B(\beta) \wedge \exists^{\mathbf{P}}_\pi(\eta_A(\alpha))
\end{aligned}$$

But we know that the last statement holds, since the Forbenius condition holds for $\mathbf{P}$, so the Frobenius condition also holds for F.

Lastly, we need to show that F has equality. So let $C \in \mathcal{C}$ be an object and let $Eq^{\mathbf{P}}_C$ be the equality predicate for C in $\mathbf{P}$. Since $\mathbf{P}$ has η-restrictable equality predicates, we see that there is an element $\phi \in F(C \times C)$ such that $\eta_{C\times C}(\phi) = Eq^{\mathbf{P}}_C$. It is then easy to see by the properties of η that this element ϕ is actually the equality predicate for C in F. ❑

Now we are ready to define the quantifier hyperdoctrine for projections which will interpret continuous logic.

Definition 4.56. For every object (X, d) of $cMet_1$ and morphism $f : (Y, d) \to (X, d)$ in $cMet_1$, we define $CMT((X, d))$ as the set of all uniformly continuous functions $X \to [0, 1]$, and we define $CMT(f) : CMT((X, d)) \to CMT((Y, d))$ as the function which takes $\alpha : X \to [0, 1]$ to $\alpha \circ f$. When we order the elements of every $CMT((X, d))$ with the ordering $\sqsubseteq$, which is the same as defined in Defintion 4.41, we see that $CMT : cMet_1^{op} \to Preorder$ is a functor.

We see that CMT bears resemblance to the functor $\mathbf{P}_{CL}$. We will use this resemblance to reuse the structure of $\mathbf{P}_{CL}(X)$ on $CMT((X, d))$.

To see that CMT is actually a quantifier hyperdoctrine for projections, we first define another quantifier hyperdoctrine for projections and a natural transformation between the two.

Definition 4.57. Recall that $Strict_{PER(\mathbf{P})} : PER(\mathbf{P}) \to Preorder$ is the functor which sends a partial equivalence class to the set of isomorphism classes of strict relations. Recall futhermore that $G : cMet_1 \to PER(\mathbf{P}_{CL})$ is the functor which interprets every complete metric space as a partial equivalence relation. We now define $CL = Strict_{PER(\mathbf{P}_{CL})} \circ G^{op} : cMet_1^{op} \to Preorder$. Since we know that $Strict_{PER(\mathbf{P}_{CL})}$ is a quantifier hyperdoctrine[8] and G preserves finite products, we see that CL is a quantifier hyperdoctrine for projections, by Theorem 4.23.

Before we define the natural transformation between CMT and CL, we first note the following.

[8]Recall that by 4.48 we know that $Strict_{PER(\mathbf{P}_{CL})}$ does not have all right adjoints, but as we will see the adjoints that exist suffice.

Lemma 4.58. *Let* $\alpha \in CMT((X,d))$ *be given for some* $(X,d) \in cMet_1$. *Then* α *can be seen as strict relation on* (X,d)*, when we regard* (X,d) *as a partial equivalence relation.*

Proof. We show that both sequents hold.

First we see that $\alpha(x) \vdash_x d(x,x)$. For this sequent to hold, we need that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\alpha(x) < \delta$ implies $d(x,x) < \epsilon$ for every $x \in X$. But this is trivially true.

Secondly we show that $\alpha(x) \wedge d(x,x') \vdash_{xx'} \alpha(x')$. For this sequent to hold, we need that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\max(\alpha(x), d(x,x')) < \delta$ implies $\alpha(x') < \epsilon$ for every $x \in X$. Denoting Δ for the modulus of uniform continuity of α, we choose $\delta = \Delta(\frac{1}{2}\epsilon)$. This gives us that $\max(\alpha(x), |\alpha(x) - \alpha(x')|) < \frac{1}{2}\epsilon$ which in turn gives us that $\alpha(x') < \epsilon$.

So we conclude that α is a strict relation on (X,d). ❑

Definition 4.59. We define $q : CMT \to CL$ as the natural transformation that sends a function α to its isomorphism class of strict relations. That is, we define the components as follows for every $(X,d) \in cMet_1$ and $\alpha \in CMT((X,d))$: $q_{(X,d)}(\alpha) = [\alpha]$.

Theorem 4.60. *The functor* $CMT : cMet_1^{op} \to Preorder$ *is a quantifier hyperdoctrine for projections.*

Proof. We give a list of the necessary structure. We see that we can copy a lot of the structure of $\mathbf{P}_{CL}$:

- The top element of $CMT((X,d))$ is given by the constant function 0.
- The bottom element of $CMT((X,d))$ is any function f for which we have that $\inf_{x\in X} f(x) > 0$.
- Binary meets and joints of two functions f and g are given by $\max(f,g)$ and $\min(f,g)$ respectively.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a left adjoint $\exists_\pi^{CMT} : CMT((X \times Y, d)) \to CMT((Y,d'))$. For every morphism $\alpha : X \times Y \to [0,1]$ and any $y \in Y$, we define $(\exists_\pi^{CMT}\alpha)(y) = (\exists_\pi \alpha)(y)$, where $\exists_\pi \alpha$ is the left adjoint of $\mathbf{P}_{CL}(\pi)$.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a right adjoint $\forall_\pi^{CMT} : CMT((X \times Y, d)) \to CMT((Y,d'))$. For every morphism $\alpha : X \times Y \to [0,1]$ and any $y \in Y$, we define $(\forall_\pi^{CMT}\alpha)(y) = (\forall_\pi \alpha)(y)$, where $\forall_\pi \alpha$ is the right adjoint of $\mathbf{P}_{CL}(\pi)$.

To see that this structure actually realises CMT as a quantifier hyperdoctrine for projections, we use Theorem 4.55.

First we note that the components of $q : CMT \to CL$ preserve and reflect the order by Proposition 4.39.

Next, we show that CL has q-restrictable equality predicates. To do so, we must first see what the equality predicates in CL are. We note that $CL = Strict_{PER(\mathbf{P}_{CL})} \circ G$, so we start by looking at the equality predicates in $Strict_{PER(\mathbf{P}_{CL})}$. For any $(X, \sim) \in PER(\mathbf{P}_{CL})$, it is easy to see that $[\sim] \in Strict_{PER(\mathbf{P}_{CL})}((X,\sim) \times (X,\sim))$ is the required equality predicate. From the proof of Theorem 4.23, we see that the equality predicate on $CL((X,d) \times (X,d))$ is therefore given by $[d]$. And $d : (X,d) \times (X,d) \to [0,1]$ is clearly a uniformly continuous function, so $d \in CMT((X,d) \times (X,d))$ and $q(d) = [d]$. So CL indeed has q-restrictable equality predicates.

Lastly, we show that CL has q-restrictable adjoints. We start with the right adjoints. We see from the proof of Proposition 4.40 and the fact that d is a reflexive relation that $\forall_\pi^{CL}([\phi]) = [d(y,y) \wedge \forall\pi\phi] = [\forall\pi\phi]$, for every $\phi \in CMT(X \times Y, d)$. So in particular we have that

$$(\forall_\pi^{CL} \circ q_{(X\times Y,d)})(\phi) = \forall_\pi^{CL}([\phi]) = [\forall\pi\phi] = (q_{(X.d)} \circ \forall_\pi^{CMT})(\phi)$$

So we see that $\forall_\pi^{CL} \circ q_{(X\times Y,d)}$ factors through $q_{(X,d)}$, as desired. With a similar argument, we have the same result for $\exists_\pi^{CL}$. So CL indeed has q-restrictable adjoints.

We may conclude by Theorem 4.55 that CMT is a quantifier hyperdoctrine for projections. ❑

Corollary 4.61. *Let $q : CMT \to CL$ be the natural transformation as defined earlier. Then $(id_{cMet_1^{op}}, q)$ is an embedding of quantifier hyperdoctrines for projections.*

We now have defined our hyperdoctrine that we can use to interpret continuous logic, or at least a part of it. We compare the interpretation of our hyperdoctrine with the already defined continuous interpretation of Definition 4.19. Since we have that CMT is a quantifier hyperdoctrine, we see that we cannot immediately interpret the 11th segment, so we already miss a significant part.

Since we already miss a suitable interpretation for one connective, we may wonder if there are other connectives that don't behave as we want. The following theorem gives an answer to that question.

Theorem 4.62. *Let u be a connective of arity $n \geq 1$, that is, a continuous function from $[0,1]^n \to [0,1]$, both with the standard metric for the respective spaces. Then the following statements are equivalent:*

1. *Let $\vec{x}, \vec{y} \in [0,1]^n$ be given such that $\forall i \leq n : x_i = 0 \Leftrightarrow y_i = 0$. Then we either have that $u(\vec{x}) = u(\vec{y}) = 0$, or $u(\vec{x}) > 0$ and $u(\vec{y}) > 0$.*
2. *For any non-empty pseudo-metric space (Y,d) and for any two sequences $(\phi_i)_{i\leq n}$ and $(\phi'_i)_{i\leq n}$ of uniformly continuous elements of $\mathbf{P}_{CL}(Y)$ such that $\forall i \leq n \phi_i \simeq \phi'_i$, we have that $u(\phi_1,\ldots,\phi_n) \simeq u(\phi'_1,\ldots,\phi'_n)$.*

So we see that we have an equivalent characterisation for connectives that preserve isomorphism. We can also see that there are many connectives that fail this prerequisite. For example, we see that $x \dot{-} y$ and $1 - x$ do not meet the prerequisite. This would imply that the interpretation of these connectives in CMT is far from optimal, although both to us and to Figueroa it is unknown how serious this limitation is.

Figueroa was aware of this shortcoming, and to resolve this, he extended the embedding q to an embedding into a subobject functor of a category of sheaves on a site, which is known to be an example of a first-order hyperdoctrine. We present this embedding now.

As said, this embedding has a codomain which is a category of subobjects of sheaves defined by a certain topology. In particular, we look at the following Grothendieck topology.

Definition 4.63. Let $\mathcal{C}$ be a coherent category. We define the *coherent Grothendieck topology J* on $\mathcal{C}$ as follows: a sieve $(f_i : X_i \to X | i \in I)$ is covering if it contains a finite subset of morphisms of which the union of their images is X.

It is known that the coherent topology on such a category $\mathcal{C}$ is subcanonical, in such a sense that all representable presheaves on $\mathcal{C}$ are sheaves for the coherent topology. So the Yoneda embedding can be seen as an embedding in the category of sheaves. In addition, Figueroa gives us the following lemma and corollary:

Lemma 4.64. *Let $\mathcal{C}$ be a small Heyting Category and let J be a subcanonical Grothendiek topology on $\mathcal{C}$. Then the Yoneda embedding, seen as embedding into a category of sheaves, preserves universal quantification.*

Corollary 4.65. *If $\mathcal{C}$ is a small Heyting category and J is the coherent topology on $\mathcal{C}$ then the Yoneda embedding $Y : \mathcal{C} \to Sh(\mathcal{C}, J)$ is a Heyting functor.*

Now we want to apply this to the situation we have described in this section. That is, we want to find a suitable subobject topos of sheaves $\mathcal{D}$ and a full morphism of quantifier hyperdoctrines (for projections) $(F, \eta) : CMT \to \mathrm{Sub}_{\mathcal{D}}$. We want to do this by, once more, looking at the functor CL.

Unfortunately, we cannot do this directly. The previous lemma and corollary only work in the case that $\mathcal{C}$ is small, and $PER(\mathbf{P}_{CL})$ is not small, in general. So we have to work with a slightly adapted version of the notions introduced in the previous chapters. That is, we will look at some Grothendieck universe U that contains $[0,1]$ and restrict the constructions to U. This will be done as follows:

- We define $PER(\mathbf{P}_{CL})^U$ to be the full subcategory of $PER(\mathbf{P}_{CL})$ such that for every object $(X, \sim)$ of $PER(\mathbf{P}_{CL})$ we have that $X \in U$ and $\sim \in U$.
- There is still a correspondence between subobjects of objects of $PER(\mathbf{P}_{CL})^U$ and strict relations on objects of $PER(\mathbf{P}_{CL})^U$. So we have a ntural definition of $Strict^U_{PER(\mathbf{P}_{CL})}$ and H^U. In the same way, we can also define a functor G^U.
- We define $cMet^U_1$ as the full subcategory of $cMet^U_1$ such that $X \in U$ and $d \in U$.
- We define CMT^U as the restriction of CMT and we define CL^U as $G^{U^{op}} \circ Strict^U_{PER(\mathbf{P}_{CL})}$.

With these definitions, it is easy to check that $PER(\mathbf{P}_{CL})^U$ is a small coherent category and for every projection π in $PER(\mathbf{P}_{CL})^U$, we have that $Strict^U_{PER(\mathbf{P}_{CL})}(\pi)$ has a right adjoint.

So now we see that we can use the lemma and the proposition to conclude that the Yoneda embedding $Y : PER(\mathbf{P}_{CL}) \to \mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)$ is a coherent functor which preserves the right adjoint to $Strict^U_{PER(\mathbf{P}_{CL})}(\pi)$. We can also define the functor $Y_{\mathrm{Sub}} : \mathrm{Sub}_{PER(\mathbf{P}_{CL})^U} \to \mathrm{Sub}_{\mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)}$ as the natural transformation induced by Y. That is, for any element $(X', \sim')$ of $\mathrm{Sub}_{PER(\mathbf{P}_{CL})^U}(X, \sim)$, we look at $Y((X', \sim))$ as element of $\mathrm{Sub}_{\mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)} Y((X, \sim))$.

By repeating the same arguments as before, we immediately have that $q^U_{(X,d)} \circ \exists^{CMT^U}_\pi \simeq \exists^{CL^U}_\pi \circ q^U_{(Y,d)}$ and $q^U_{(X,d)} \circ \forall^{CMT^U}_\pi \simeq \forall^{CL^U}_\pi \circ q^U_{(Y,d)}$, for any two complete metric spaces (X, d) and (Y, d) with a projection $\pi : (Y, d) \to (X, d)$. So in particular, we have the morphism $(\mathrm{id}_{cMet^U_1}, q^U) : CMT^U \to CL^U$ of quantifier hyperdoctrines of projections. It is clear that q^U is still an embedding.

We can also define another morphism. From the previous discussion, we know that we have the functor $Y \circ G^U$. Since G^U is full and faithful and Y is an embedding, we see that $Y \circ G^U$ preserves finite products. We also have a natural transformation $Y_{\mathrm{Sub}} \circ H^U : CL^U \to \mathrm{Sub}_{\mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)}$. This is a natural transformation, since both components are natural transformations.

Now that we have these two morphsims, we can combine them to form a morphism $(Y \circ G^U, Y_{\mathrm{Sub}} \circ H^U) \circ (\mathrm{id}_{cMet^U_1}, q^U)$ with domain CMT^U and codomain $\mathrm{Sub}_{\mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)}$. We conclude that this is indeed an embedding of CMT into a first-order hyperdoctrine.

When we want to study this embedding and, perhaps more importantly, the image of the embedding in the first-order hyperdoctrine, we run into a bit of a problem. The embedding is not very well understood, and the Heyting implication in the codomain category $\mathrm{Sub}_{\mathrm{Sh}(PER(\mathbf{P}_{CL})^U, J)}$ is also unclear, and therefore most probably still not sufficient to be a true continuous interpretation as defined in Definition 4.19. In the next section, we will try to find a more direct approach at finding a first-order hyperdoctrine to interpret continuous logic.

5 An attempt at a first-order hyperdoctrine

In this section, we will adjust the hyperdoctrine $\mathbf{P}_{CL}$ to better suit continuous logic and, hopefully, find a suitable interpretation. The first step, as was also suggested by Figueroa, is to still look at $\mathbf{P}_{CL}$, but change the order on the codomain into a point-wise order. The following definition shows what we mean.

Definition 5.1. Let X be a set and let $f, g \in \mathbf{P}_{CL}(X)$ be given. We say that $f \sqsubseteq' g$ if and only if we have that $\forall x \in X : g(x) \leq f(x)$.

We can see immediately that this new order gives rise to not only a pre-order, but even more so a partial order. We will study $\mathbf{P}_{CL}$ with this new order. To avoid confusion with the earlier parts, we will use a new name.

Definition 5.2. Let X be a set. We define $\mathbf{Q}_{CL}(X)$ as the partial order $(\{X \to [0,1]\}, \sqsubseteq')$. For a function $f : X \to Y$, we let $\mathbf{Q}_{CL}(f) : \mathbf{Q}_{CL}(Y) \to \mathbf{Q}_{CL}(X)$ be the function that sends an $\alpha \in \mathbf{Q}_{CL}(Y)$ to $\alpha \circ f$.

We have already discussed that $\mathbf{Q}_{CL}(X)$ is indeed a partial order for every set X, and it is easy to see that this assignment gives rise to a functor. The first important question is whether or not $\mathbf{Q}_{CL}$ is a quantifier hyperdoctrine. The following theorem will show that not only is this the case, we also have something stronger.

Theorem 5.3. *The functor $\mathbf{Q}_{CL} : Set^{op} \to Poset$, which sends a set X to $\mathbf{Q}_{CL}(X)$ and a function $f : X \to Y$ to $\mathbf{Q}_{CL}(f)$, is a first-order hyperdoctrine.*

Proof. We first list the necessary structure. As expected from the fact that $\mathbf{P}_{CL}$ and $\mathbf{Q}_{CL}$ are very alike, we see that the structure is almost the same.

- The top element of $\mathbf{Q}_{CL}(X)$ is given by the constant function 0.
- the bottom element of $\mathbf{Q}_{CL}(X)$ is given by the constant function 1.
- Binary meets and joins of two functions f and g are given by $\max(f, g)$ and $\min(f, g)$ respectively.
- For any morphism $f : Y \to X$, we have a left adjoint $\exists_f : \mathbf{Q}_{CL}(Y) \to \mathbf{Q}_{CL}(X)$. For every morphism $\alpha : Y \to [0,1]$ and any $x \in X$, we define $(\exists_f \alpha)(x) = \inf\{\alpha(y) | y \in Y, f(y) = x\}$.
- For any morphism $f : Y \to X$, we have a right adjoint $\forall_f : \mathbf{Q}_{CL}(Y) \to \mathbf{Q}_{CL}(X)$. For every morphism $\alpha : Y \to [0,1]$ and any $x \in X$, we define $(\forall_f \alpha)(x) = \sup\{\alpha(y) | y \in Y, f(y) = x\}$.

❏

Now we want to define our replacement for CMT. However, we have to be cautious, since we want that our replacement is a first-order hyperdoctrine for projections. The following result shows that we have to do more than just change the order on the codomain of CMT.

Theorem 5.4. *Let (X, d) be an pseudo-metric space. Then the set of uniform continuous functions from (X, d) to $[0,1]$ does not have a Heyting implication.*

Proof. For simplicity, we take (X, d) to be $[0,1]$ with the standard metric. We can see that this argument can be extended to general metric spaces.

We want to find two uniformly continuous functions $g, h : [0,1] \to [0,1]$ such that for every uniformly continuous $\alpha : [0,1] \to [0,1]$ we have that there is a uniformly continuous function $f : [0,1] \to [0,1]$ such that $f \wedge g \sqsubseteq' h \Leftrightarrow f \sqsubseteq' \alpha$ does not hold.

We take $h = 1$ and $g = \min(2x, 1)$.

Without loss of generality, we may assume that $\forall x \leq \frac{1}{2} : \alpha(x) = 1$, otherwise we can take $f = \alpha$ and see that $\alpha \wedge g \not\sqsubseteq' h$ and $\alpha \sqsubseteq' \alpha$. But since α is uniformly continuous, there must be an open interval $(\frac{1}{2}, y)$ such that $\alpha > 0$ on that interval. It is clear that we can then find a function f such that $\forall x < \frac{1}{2} : f(x) = 1$ and $f < \alpha$ on the interval $(\frac{1}{2}, y)$. Such an f would then suffice, since then $f \wedge g = 1 = h$ and $f \not\sqsubseteq' \alpha$. ❑

So we have to do a bit more work to accurately translate CMT to this new situation. The way we will do this, is to not only change the codomain of CMT, but also change the domain. We define the new domain here.

Definition 5.5. We define the category $pMetL_1$ as the category whose objects are pseudo-metric spaces with diameter at most 1, and whose morphisms are 1-Lipschitz functions.

It is easy to see that $\mathrm{id}_{(X,d)}$ is the identity morphism on (X, d) and that the composition of two 1-Lipschitz functions is again 1-Lipschitz. Now to define our substitute for CMT.

Definition 5.6. For every object (X, d) of $pMetL_1$ and morphism $f : (Y, d) \to (X, d)$ in $pMetL_1$, we define $CMS((X, d))$ as the set of all 1-Lipschitz functions $X \to [0,1]$, and we define $CMS(f) : CMS((X, d)) \to CMS((Y, d))$ as the function which takes $\alpha : X \to [0,1]$ to $\alpha \circ f$. When we order the elements of every $CMS((X, d))$ with the ordering $\sqsubseteq'$, which is the same as defined earlier, we see that $CMS : pMetL_1^{op} \to Preorder$ is a functor.

We want to show that CMS is a first-order hyperdoctrine for projections. We do this by showing there is a natural transformation from CMS into a first-order hyperdoctrine for projections.

Theorem 5.7. *There exists a natural transformation $i : CMS \to \mathbf{Q}_{CL} \circ U^{op}$, where $U : pMetL_1 \to Set$ is the forgetful functor, and where we consider CMS and $\mathbf{Q}_{CL} \circ U^{op}$ as functors $pMetL_1^{op} \to Preorder$.*

Proof. The choice of the transformation i is the obvious one: since $CMS((X, d)) \subseteq (\mathbf{Q}_{CL} \circ U^{op})((X, d))$ for all $(X, d) \in pMetL_1$, we simply take $i_{(X,d)}$ to be the inclusion morphism $CMS((X, d)) \to (\mathbf{Q}_{CL} \circ U)((X, d))$. It is then easy to see that i is a natural transformation. It is also immediate that i is order-preserving and order-reflecting, since i sends every element $f \in CMS((X, d))$ to itself and the order on $CMS((X, d))$ coincides with that on $(\mathbf{Q}_{CL} \circ U^{op})((X, d))$. The result then follows by checking the prerequisites of Theorem 4.55. ❑

Remark 5.8. For the proof that CMS is a first-order hyperdoctrine for projections, we need to know whether we have that $\mathbf{Q}_{CL} \circ U^{op}$ is a first-order hyperdoctrine for projections. Since $\mathbf{Q}_{CL}$ is a first-order hyperdoctrine of projections, the result would follow from Theorem 4.23 if we have that U preserves finite products. We see that this is the case, since we have an adjunction $F \dashv U$, where $F : Set \to pMetL_1$ is the free functor which sends a set X to (X, d), with d being the discrete metric. So U, being a right adjoint, preserves all finite limits.

Now that we have this, we can prove the following theorem.

Theorem 5.9. *The functor CMS is a first-order hyperdoctrine for projections.*

Proof. We first give a list of the necessary properties. We see that we can copy a lot of the structure of $\mathbf{Q}_{CL}$:

- The top element of $CMS((X,d))$ is given by the constant function 0.
- The bottom element of $CMS((X,d))$ is the constant function 1.
- Binary meets and joints of two functions f and g are given by $\max(f,g)$ and $\min(f,g)$ respectively.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a left adjoint $\exists_\pi^{CMS} : CMS((X \times Y, d)) \to CMS((Y, d'))$. For every morphism $\alpha : X \times Y \to [0,1]$ and any $y \in Y$, we define $(\exists_\pi^{CMS}\alpha)(y) = (\exists_\pi \alpha)(y)$, where $\exists_\pi \alpha$ is the left adjoint of $\mathbf{Q}_{CL}(\pi)$.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a right adjoint $\forall_\pi^{CMS} : CMS((X \times Y, d)) \to CST((Y, d'))$. For every morphism $\alpha : X \times Y \to [0,1]$ and any $y \in Y$, we define $(\forall_\pi^{CMS}\alpha)(y) = (\forall_\pi \alpha)(y)$, where $\forall_\pi \alpha$ is the right adjoint of $\mathbf{Q}_{CL}(\pi)$.

We have already seen that these choices for joins, meets, top element and bottom element give rise to a distributive prelattice. So we indeed have a functor $CMS : cMet_1^{op} \to DistPrelattice$.

The Heyting implication, however, looks different. To define the Heyting implication, we first recall that, for two given functions $g, h \in CMS(X)$, we denote the set $\{x \in X \mid h(x) > g(x)\}$ as $\{h > g\}$. Also recall that we denote $\mathrm{bd}(X)$ for the boundary of a set, i.e. the closure of X minus the interior of X in the topology generated by d.

We now define the Heyting implication in the following lemma:

Lemma 5.10. *For any two functions $g, h \in CMS((X,d))$, we have that the Heyting implication is given as*

$$(g \to h)(x) = \begin{cases} h(x) & \text{if } h(x) > g(x) \\ \max(0, \sup_{b \in bd(\{h>g\})}\{h(b) - d(x,b)\}) & \text{otherwise} \end{cases}$$

Proof. First, we need to show that $(g \to h)$ as defined here is actually a 1-Lipschitz function. So let two elements $x, x' \in X$ be given. We want to show that $|(g \to h)(x) - (g \to h)(x')| \leq d(x,x')$.

If we have that $h(x) > g(x)$ and $h(x') > g(x')$, then $(g \to h)(x) = h(x)$ and $(g \to h)(x') = h(x')$ and the result follows from the fact that h is 1-Lipschitz.

Assume now that we have that $h(x') \leq g(x')$ and $h(x) > g(x)$. For all $b' \in \mathrm{bd}(\{h > g\})$, we have $h(x) \geq h(b') - d(x,b') \geq h(b') - d(x,x') - d(x',b')$, where we used that h is 1-Lipschitz in the first step. So we have that

$$h(x) + d(x,x') \geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x',b)\}$$

and since we also have that $h(x) + d(x,x') \geq 0$, we also have that

$$h(x) + d(x,x') \geq \max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x',b)\}).$$

And from that we may conclude that

$$d(x,x') \geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x',b)\} - h(x)$$

In other words, we have that $(g \to h)(x') - (g \to h)(x) \leq d(x, x')$. Similarly we can show that $(g \to h)(x) - (g \to h)(x') \leq d(x, x')$, so we indeed have that $|(g \to h)(x) - (g \to h)(x')| \leq d(x, x')$ in this case.

Now assume that $h(x') \leq g(x')$ and $h(x) \leq g(x)$. We have:

$$\begin{aligned}\max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\}) &\geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\} \\ &\geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', x) - d(x, b)\} \\ &\geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x, b)\} - d(x', x).\end{aligned}$$

So we also have that

$$\max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\}) + d(x, x') \geq \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x, b)\}.$$

And since we can see that $\max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\}) + d(x, x') \geq 0$, we also have that

$$\max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\}) + d(x, x') \geq \max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x, b)\}).$$

And from that we may conclude that

$$d(x, x') \geq \max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x, b)\}) - \max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x', b)\}).$$

In other words, $(g \to h)(x) - (g \to h)(x') \leq d(x, x')$. Similarly we can show that $(g \to h)(x') - (g \to h)(x) \leq d(x, x')$, so we indeed have that $|(g \to h)(x) - (g \to h)(x')| \leq d(x, x')$ in this case.

So overall, we may conclude that $(g \to h)(x)$ is indeed a 1-Lipschitz function.

So now we have to show that $(g \to h)(x)$ is a Heyting implication. This means that we must have for all $f, g, h \in CMS((X, d))$ that $f \wedge g \sqsubseteq' h \Leftrightarrow f \sqsubseteq' g \to h$. Writing out what this means, we get that we must have that

$$\forall x \in X : \max(f(x), g(x)) \geq h(x) \Leftrightarrow \forall x \in X : f(x) \geq (g \to h)(x).$$

So let $x_0 \in X$ be given and assume that $\max(f(x_0), g(x_0)) \geq h(x_0)$. If we assume that $h(x_0) > g(x_0)$, then we must have that $f(x_0) \geq h(x_0) = (g \to h)(x_0)$. So assume that $h(x_0) \leq g(x_0)$. Then we must show that $f(x_0) \geq \max(0, \sup_{b \in \mathrm{bd}(\{h>g\})} \{h(b) - d(x_0, b)\})$. Since $f(x_0) \geq 0$, it is sufficient to show that $\forall b' \in \mathrm{bd}(\{h > g\}) : f(x_0) \geq h(b') - d(x_0, b')$. To do this, we look at $f(b')$. Since we assumed that $b' \in \mathrm{bd}(\{h > g\})$ we have that

$$\forall \epsilon > 0 \exists c \in \{h > g\} : d(b', c) < \epsilon.$$

Since f is 1-Lipschitz, we see, using that $f(c) = h(c)$ since $h(c) > g(c)$, that

$$\forall \epsilon > 0 \exists c \in \{h > g\} : |f(b') - h(c)| = |f(b') - f(c)| \leq d(b', c) < \epsilon$$

So we must have that $f(b') = h(b')$. By the fact that f is 1-Lipschitz we see that

$$h(b') - f(x_0) \leq |h(b') - f(x_0)| = |f(b') - f(x_0)| \leq d(x_0, b')$$

So indeed we have that $\forall b' \in \mathrm{bd}(\{h > g\}) : f(x_0) \geq h(b') - d(x_0, b')$, and we may conclude that $f(x_0) \geq (g \to h)(x_0)$.

Suppose now that we have an $x_0 \in X$ such that $f(x_0) \geq (g \to h)(x_0)$. Assume that $h(x_0) > g(x_0)$. Then we can easily see that $\max(f(x_0), g(x_0)) \geq f(x_0) \geq (g \to h)(x_0) = h(x_0)$. So assume that $h(x_0) \leq g(x_0)$. Then we have immediately that $\max(f(x_0), g(x_0)) \geq g(x_0) \geq h(x_0)$.

So we see that $g \to h$ is actually the Heyting implication in $CMS(X)$. ❑

To show that $\forall_\pi^{CMS}$ and $\exists_\pi^{CMS}$ are actually right and left adjoints, we need the following lemma.

Lemma 5.11. *For every $\pi_X : (X \times Y, d') \to (X, d)$ we have that $i_{(X,d)} \circ \forall_{\pi_X}^{CMS} = \forall_{\pi_X}^{\mathbf{Q}_{CL} \circ U^{op}} \circ i_{(X \times Y, d')}$.*

Proof. Since U preserves products, we see that the right adjoint to $\mathbf{Q}_{CL} \circ U^{op}(\pi_X)$ is equal to the morphism $\forall_{U(\pi_X)}^{\mathbf{Q}_{CL}}$, as is given in the proof of 4.23.

Let $\phi \in CMS(X \times Y, d')$. Denoting ϕ' for $i_{(X \times Y, d')}(\phi)$ we see that $\forall (x, y) \in (X \times Y, d') : \phi(x, y) = \phi'(x, y)$. So in particular we see that:

$$\begin{aligned}\forall_{U(\pi_X)}^{\mathbf{Q}_{CL}}(\phi')(x) &= \sup_{(x',y') \in U(X \times Y, d')} \{\phi'((x', y')) | U(\pi_X)(x', y') = x\} \\ &= i_{(X,d)} \circ \sup_{(x',y') \in (X \times Y, d)} \{\phi((x', y')) | \pi_X(x', y') = x\} \\ &= i_{(X,d)} \circ \forall_{\pi_X}^{CMS}(\phi)(x)\end{aligned}$$

❑

With a similar proof, we can see that we also have that $i_{(X,d)} \circ \exists_{\pi_X}^{CMS} = \exists_{\pi_X}^{\mathbf{Q}_{CL} \circ U^{op}} \circ i_{(X \times Y, d')}$. As a direct corollary, we have that $\mathbf{Q}_{CL}$ has i-restrictable adjoints, as defined in Definition 4.53.

Lastly, we will show that CMS has equality. Let an object $(X, d_X) \in pMetL_1$ be given. We show that the equality predicate on $CMS((X, d_X) \times (X, d_X))$ is equal to the psuedo-metric d_X when seen as function from $X \times X \to [0, 1]$. Recall that the pseudo-metric on $(X, d_X) \times (X, d_X)$ is the function $d_{X \times X} : (X \times X) \times (X \times X) \to [0, 1]$ which sends $((x_0, x_1), (x_2, x_3)) \to \max(d_X(x_0, x_2), d_X(x_1, x_3))$.

The function d_X is 1-Lipschitz in both components. We show this for the first component as follows, and the case for the second component follows from symmetry.

$$\begin{aligned}|d_X(x_0, x_1) - d_X(x_0, x_2)| &\leq |d_X(x_0, x_2) + d_X(x_2, x_1) - d_X(x_0, x_2)| \\ &= d_X(x_1, x_2) \\ &= \max(d_X(x_0, x_0), d_X(x_1, x_2)) \\ &= d_{X \times X}((x_0, x_1), (x_0, x_2))\end{aligned}$$

First we show for all $f \in CMS(X \times X)$ that $\top_X \sqsubseteq' CMS(\delta)(f) \Leftrightarrow d_X \sqsubseteq' f$. Writing this out, we see that this means that $(\forall x \in X : (f \circ \delta)(x) = 0) \Leftrightarrow (\forall (x, x') \in X \times X : f(x, x') \leq d_X(x, x'))$. From right to left is easy; since $d_X(x, x) = 0$, we also must have that $(f \circ \delta)(x) = f(x, x) = 0$. Suppose now that there is a tuple (x, x') such that $f(x, x') > d_X(x, x')$. Since f is 1-Lipschitz, this means that $|f(x, x') - f(x, x)| \leq d_{X \times X}((x, x), (x', x)) = \max(0, d_X(x', x)) = d_X(x, x') < f(x, x')$. Clearly, this implies that $f(x, x) \neq 0$.

For the second condition we look at (X, d_X), (Y, d_Y) and $(X \times Y, d_{X \times Y})$. We take $d_{X \times Y}$ and write this metric in terms of the other ones. For every $(x, y, x', y') \in X \times Y \times X \times Y$ this looks as follows:

$$\begin{aligned}
&d_{X\times Y}((x,y),(x',y')) = \\
&\max(d_X(x,x'), d_Y(y,y')) = \\
&\max(d_X \circ \pi_{13}(x,y,x',y'), d_Y \circ \pi_{24}(x,y,x',y'))
\end{aligned}$$

But this is precisely equivalent to saying that $Eq_{X\times Y} = CMS(\pi_{13})(Eq_X) \wedge CMS(\pi_{24})(Eq_Y)$. So we see that d_X is indeed the equality predicate on $CMS((X,d) \times (X,d))$.

So by Theorem 4.55 we see that CMS is a first-order hyperdoctrine for projections. ❑

So we see that we actually do have a first-order hyperdoctrine. We now want to see if we are closer to correctly interpreting continuous logic in a hyperdoctrine.

One thing we can see is that we cannot run into the problem posed in Theorem 4.62, since for every (X, d), we have that $CMS((X, d))$ is a partial order, and therefore two $f, g \in CMS((X, d))$ are isomorphic if and only if they are equal.

The Heyting implication, however, is more of a reason for concern. We can immediately see that the implication defined in the previous theorem does not at all resemble $\dot{-}$, so our interpretation in CMS is still not how we want it to be. We could look at ways to overcome this problem, perhaps by choosing yet again a different order on $CMS((X, d))$. But the following theorem shows us that we need to do much more than that.

Theorem 5.12. *The function* $\dot{-}\colon [0,1]^2 \to [0,1]$ *is not 1-Lipschitz.*

Proof. We can look at $(\frac{4}{5}, \frac{1}{5})$ and $(\frac{3}{5}, \frac{3}{5})$. We see the following:

$$\left|\left(\frac{4}{5} \dot{-} \frac{1}{5}\right) - \left(\frac{3}{5} \dot{-} \frac{3}{5}\right)\right| = \frac{3}{5} \not\leq \frac{2}{5} = \max\left(\left|\frac{4}{5} - \frac{3}{5}\right|, \left|\frac{1}{5} - \frac{3}{5}\right|\right)$$

❑

So although this newly created first-order hyperdoctrine may be interesting by itself, it will not help us any further in finding a suitable category theoretical interpretation of continuous logic.

In fact, we can wonder what the order on a given space must look like in order for $\dot{-}$ to be a Heyting implication. The following theorem gives us some insight.

Theorem 5.13. *Let X be a set of functions with codomain $[0,1]$ such that all constant functions are elements of X. Furthermore, let $\leq$ be a partial ordering of X, such that the associated conjunction $\wedge$ is equal to the function* max. *Then $\dot{-}$ cannot be a Heyting implication on X.*

Proof. We look at the functions $f(x) = \frac{1}{4}$, $g(x) = \frac{1}{3}$ and $h(x) = \frac{1}{2}$. We see that $\max(f, g) = g$. So since we have that $g \leq h$, and it follows that $\frac{1}{4} = f \leq h \dot{-} g = \frac{1}{6}$. But we should have that $\frac{1}{4} \geq \frac{1}{6}$, since $\max\left(\frac{1}{4}, \frac{1}{6}\right) = \frac{1}{4}$. ❑

So we see that restricting the domain and codomain of our hyperdoctrine even further will not give us the desired result. In the next section, we will look at a different approach to incorporating $\dot{-}$ in a hyperdoctrine.

6 MV-algebras and MV-hyperdoctrines

In this section, we look at a the concept of MV-algebras. We first give the definition and state some results. After that, we show that there is a natural definition of an MV-hyperdoctrine, and we see that this kind of hyperdoctrine can be used to interpret continuous logic.

6.1 MV-algebras

In this subsection, we look at MV-algebras.

By Theorem 5.13, $\dot{-}$ cannot be interpreted as the Heyting implication of max on $[0,1]$. However, we can find another function that realizes $\dot{-}$ as an implication in the following sense.

Theorem 6.1. *Let* $\otimes : [0,1] \times [0,1] \to [0,1]$ *be the function defined as* $x \otimes y = \min(1, x+y)$*. Then we have* $\forall x, y, z \in [0,1] : x \otimes y \geq z \Leftrightarrow x \geq z \dot{-} y$.

Proof. Assume first that $x \otimes y \geq z$. We first look at the case that $x + y \geq 1$. Then we immediately have that $x \geq 1 - y = 1 \dot{-} y \leq z \dot{-} y$. So now we look at the case that $x + y < 1$. So we have that $x+y \geq z$. Then we also have that $x \geq z-y$ and, since $x \geq 0$, we also have $x \geq \max(0, z-y) = z \dot{-} y$.

So assume now that $x \geq z \dot{-} y$. This certainly implies that $x \geq z - y$, so $x + y \geq z$. Since we also have that $1 \geq z$, we may conclude that $x \otimes y \geq z$. ❑

We can wonder whether this new connective $\otimes$ is somewhat related to the connective max, since they both are in a certain sense a conjunction. The theorem shows that there is a simple relation between the two.

Proposition 6.2. *For all* $x, y \in [0,1]$ *we have that* $\max(x, y) = x \otimes (y \dot{-} x)$.

In fact, it turns out that we can generalize structures that behave like this. To do so, we adopt the notion of an MV-algebra, which we already mentioned in Section 2. But since in our situation 0 is the value for absolute truth instead of 1, as was the case in the original definition of MV-algebras, we choose to make some adjustments. Besides these adjustments, we follow the presentation given in [17].

Definition 6.3. An *MV-algebra* $\langle X, \otimes, \neg, \perp \rangle$ is a set X equipped with a binary operation $\otimes$, a unary operation $\neg$ and an element $\perp$ satisfying the following equations:

1. $x \otimes (y \otimes z) = (x \otimes y) \otimes z$
2. $x \otimes y = y \otimes x$
3. $x \otimes \perp = \perp$
4. $\neg\neg x = x$
5. $x \otimes \neg\perp = x$
6. $\neg(x \otimes \neg y) \otimes \neg y = \neg(\neg x \otimes y) \otimes \neg x$

We refer to the operator $\otimes$ as the *strong conjunction* of the MV-algebra.

There are some simple examples of MV-algebras. All of these are easy to check.

- Any Boolean algebra $\langle X, \wedge, \vee, \neg, 0, 1 \rangle$ can be interpreted as an MV-algebra by taking $\otimes = \wedge$.

- Take $X = [0,1]$ together with $x \otimes y = \min(1, x+y)$, $\neg x = 1 - x$ and $\bot = 1$.
- For any MV-algebra $\langle X, \otimes, \neg, \bot \rangle$ and any set Y we have that the set of functions $\{f : Y \to X\}$ becomes an MV-algebra by defining the operators pointwise on the functions.

Using the operators of an MV-algebra, we can derive additional operators which help in describing the structure of MV-algebras

Definition 6.4. We define additional operators as follows:

- We write $\top$ for $\neg\bot$.
- We define the *strong disjunction* $\oplus$ by $x \oplus y = \neg(\neg x \otimes \neg y)$.
- We define the *implication* $\twoheadrightarrow$ by $x \twoheadrightarrow y = \neg x \oplus y$.
- We define the *weak conjunction* $\wedge$ by $x \wedge y = x \otimes (x \twoheadrightarrow y)$.
- We define the *weak disjunction* $\vee$ by $x \vee y = \neg(\neg x \wedge \neg y)$.

Note that the implication $\twoheadrightarrow$ is in general not a Heyting implication. We can see that this is the case whenever strong and weak conjunction (or equally, disjunction) do not coincide.

We can write down what this means for the MV-algebra we defined on $[0,1]$.

- $\top = 0$
- $x \oplus y = \max(0, 1 - x - y)$
- $x \twoheadrightarrow y = \max(0, y - x) = y \dot{-} x$
- $x \wedge y = \max(x, y)$
- $x \vee y = \min(x, y)$

We see that the weak conjunction and disjunction are exactly the conjunction and disjunction on $[0,1]$ when we see this as a lattice, with the usual ordering. The following theorem states that every MV-algebra has a similar lattice structure.

Theorem 6.5. *Let $\langle X, \otimes, \neg, \bot \rangle$ be an MV-algebra and let $\leq$ be an order defined as $x \leq y \Leftrightarrow x \otimes \neg y = \top$. Then $\langle X, \leq \rangle$ is a poset and $\langle X, \leq, \bot, \top, \wedge, \vee \rangle$ is a lattice.*

From the definition of an MV-algebra, it is not immediately clear whether or not we have for any given MV-algebra that the weak conjunction is commutative, like it is the case with $[0,1]$. The following theorem shows that this is the case, and the proof also serves as a nice example of how one should reason in the setting of MV-algebras.

Theorem 6.6. *Let $\langle X, \otimes, \neg, \bot \rangle$ be an MV-algebra. Then we have that $x \wedge y = y \wedge x$.*

Proof. We write out the definition.

$$x \wedge y = x \otimes (x \twoheadrightarrow y) = x \otimes (\neg x \oplus y) = x \otimes \neg(\neg\neg x \otimes \neg y) = \neg(\neg\neg x \otimes \neg y) \otimes \neg\neg x$$

Here in the last step, we used equation 2 and 4 of Definition 6.3. Now we see that we can use equation 6 of the same definition to obtain that:

$$\neg(\neg\neg x \otimes \neg y) \otimes \neg\neg x = \neg(\neg x \otimes \neg\neg y) \oplus \neg\neg y$$

And in the same way as before, we see that we have the following:

$$\neg(\neg x \otimes \neg\neg y) \oplus \neg\neg y = y \otimes \neg(\neg\neg y \otimes \neg x) = y \otimes (\neg y \oplus x) = y \otimes (y \twoheadrightarrow x) = y \wedge x$$

❑

Lastly, we state two additional nice properties of MV-algebras.

Proposition 6.7. *Let $\langle X, \otimes, \neg, \bot \rangle$ be an MV-algebra. Then the following equations are satisfied.*

- $(x \twoheadrightarrow y) \vee (y \twoheadrightarrow x) = \top$
- $(x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = \bot$

Proposition 6.8. *(Theorem 2.3(iii) in [16]) Let $\langle X, \otimes, \neg, \bot \rangle$ be an MV-algebra. Then the following equations are satisfied.*

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

6.2 MV-hyperdoctrines

In this subsection, we will define the concept of MV-hyperdoctrines and show that these are useful when trying to find an interpretation for continuous logic. We will define MV-hyperdoctrines in the same way as we did with the other kinds. The first step would be to define a suitable codomain for the hyperdoctrines, which is what we do in the following definition.

Definition 6.9. We define the category $MVAlg$ as the category whose objects are MV-algebras, and whose arrows are functions that preserve strong conjunction, negation and bottom element.

Using this, we can define a new kind of hyperdoctrine that closely resembles first-order hyperdoctrines.

Definition 6.10. Let $\mathcal{C}$ be a category with finite products.

- An *MV-hyperdoctrine* is a quantifier hyperdoctrine $\mathcal{C}^{op} \to MVAlg$.
- An *MV-hyperdoctrine for projections* is a quantifier hyperdoctrine for projections $\mathcal{C}^{op} \to MVAlg$.

Likewise, we can define a morphism of MV-hyperdoctrines (for projections) by copying the definition of quantifier hyperdoctrines (for projections).

We stress that the adjoints we mention here are defined on the weak conjunction of the MV-algebras, and not the strong conjunction.

As an easy result, we have the following theorem.

Theorem 6.11. *$\mathbf{Q}_{CL}$ is an MV-hyperdoctrine.*

Proof. We have already seen that $\mathbf{Q}_{CL}$ is a quantifier hyperdoctrine, so all we have to prove is that $\mathbf{Q}_{CL}(X)$ is an MV-algebra. But this follows from the fact that $\langle [0,1], \min(1, x+y), 1-x, 1 \rangle$ is an MV-algebra, and that for $\mathbf{Q}_{CL}(X)$ we can just apply the operators pointwise. ❑

Now, we can wonder whether we can interpret continuous logic in an MV-hyperdoctrine. The answer is that we can, in a very similar way as we defined before, but we need to make a slight modification regarding the implication.

Definition 6.12. Let $\mathbf{P}$ be an MV-hyperdoctrine. Let Σ be a fixed signature and $[\![\cdot]\!]$ an interpretation of Σ in $\mathbf{P}$. We define the *MV-interpretation of terms and formulas in context over* Σ *by* $[\![\cdot]\!]$ *in* $\mathbf{P}$ as in Definition 4.29, replacing the 11th line with $[\![\phi \to \psi]\!]_{\bar{x}} = [\![\phi]\!]_{\bar{x}} \twoheadrightarrow [\![\psi]\!]_{\bar{x}}$.

So we see that this MV-interpretation is the same as the previously defined interpretation, but at sequent 11 we replaced the Heyting implication, which will not always be at our disposal, with the MV-implication.

Now we can define another hyperdoctrine that is similar to CMT, and that will serve as our method of interpretation.

Definition 6.13. For every object (X, d) of $cMet_1$ and morphism $f : (Y, d) \to (X, d)$ in $pMet_1$, we define $CMV((X, d))$ as the set of all uniformly continuous functions $X \to [0, 1]$, and we define $CMV(f) : CMV((X, d)) \to CMV((Y, d))$ as the function which takes $\alpha : X \to [0, 1]$ to $\alpha \circ f$. When we order the elements of every $CMV((X, d))$ with the ordering $\sqsubseteq'$, which is the same as defined in Definition 5.1, we see that $CMV : cMet_1^{op} \to Preorder$ is a functor.

In a similar fashion as before, we will see that CMV is a quantifier hyperdoctrine, and in this case also a bit more.

Theorem 6.14. *The functor* $CMV : pMet_1^{op} \to Preorder$*, which sends* (X, d) *to* $CMV((X, d))$ *and* f *to* $CMV(f)$*, is an MV-hyperdoctrine.*

Proof. We give a list of the necessary properties.

- The top element of $CMV(X)$ is given by the constant function 0.
- The bottom element of $CMV(X)$ is is given by the constant function 1.
- Strong conjunction and disjunction of two functions f and g are given by $\min(1, f + g)$ and $\max(0, 1 - f - g)$.
- Weak conjunction and disjunction of two functions f and g are given by $\max(f, g)$ and $\min(f, g)$ respectively.
- The implication of two functions f and g is given by $f \dot{-} g$.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a left adjoint $\exists_f^{CMV} : CMV((X \times Y, d)) \to CMV((Y, d'))$. For every morphism $\alpha : X \times Y \to [0, 1]$ and any $y \in Y$, we define $(\exists_\pi^{CMV} \alpha)(y) = \inf\{\alpha(y) | y \in Y, f(y) = x\}$.
- For any projection $\pi : (X \times Y, d) \to (Y, d')$, we have a right adjoint $\forall_f^{CMV} : CMV((X \times Y, d)) \to CMV((Y, d'))$. For every morphism $\alpha : X \times Y \to [0, 1]$ and any $y \in Y$, we define $(\forall_\pi^{CMV} \alpha)(y) = \sup\{\alpha(y) | y \in Y, f(y) = x\}$.

It is clear to see that $\langle CMV(X), \max(0, 1 - f - g), 1 - f, 1\rangle$ is an MV-algebra, since $CMV(X)$ is a space of functions with codomain $\langle [0, 1], \max(0, 1 - x - y), 1 - x, 1\rangle$, which is itself an MV-algebra.

We leave it to the reader to check that the inclusion $i : CMV \to \mathbf{Q}_{CL} \circ U^{op}$ is an order preserving and reflecting natural transformation, where $U^{op} : cMet_1^{op} \to Set$ is the forgetful functor. We can then follow the same steps as in Theorem 5.9 to check that $\mathbf{Q}_{CL} \circ U^{op}$ has i-restrictable adjoints.

By a similar proof as in Theorem 5.9, we see that the metric d is the equality predicate. ❑

So we see that we indeed have found a way to fully interpret continuous logic in a hyperdoctrine.

Remark 6.15. As discussed in Section 3.2, we know that for every (X, d) we have that $CMV(X)$ is a special kind of MV-algebra. We have the element $f(x) = \frac{1}{2}x$, which plays the role of the connective $\frac{1}{2}$ in the axiomatisation of continuous propositional logic. Whether this full subcategory of $MVAlg$ has additional special properties is currently unknown to us.

7 Unbounded continuous logic

In this section, we will look at an unbounded variant of continuous logic.

As discussed in section 3.2, there is a parallel between Henson's logic for Banach structures and continuous logic. This allowed us to translate many model theoretic concepts from Henson's logic to continuous logic, despite the former having mostly unbounded structures and the latter being restricted to bounded structures. As discussed in the introduction of [3] this method has some drawbacks. To overcome these, the author of [3] presents a variant of continuous logic. This variant has the interval $[0, \infty)$ as a truth space, and is therefore aptly called *unbounded continuous logic*.

We follow the presentation as given in [3].

7.1 Definitions and properties

As we did with continuous logic, we will define unbounded continuous logic from scratch, starting with the signature. However, before we do that, we first define some additional structure, which will help us look at the bounded parts of an unbounded structure.

Definition 7.1. Let (X, d) be a metric space and $\nu : X \to \mathbb{R}$ a function.

- We define $X^{\nu \leq r} = \{x \in X \mid \nu(x) \leq r\}$ and $X^{\nu < r} = \{x \in X \mid \nu(x) < r\}$. We will call these the *closed* and *open* ν*-balls of radius* r, respectively.
- We call ν a *gauge* on (X, d) if ν is 1-Lipschitz with respect to d and every ν-ball with finite radius is bounded in d. Whenever ν is a gauge we will call (X, d, ν) a *gauged space.*

So we see that a gauge ν somewhat behaves like a norm. In fact, it is easy to see that for every $x_0 \in X$ we have that $\nu(x) = d(x, x_0)$ is a gauge.

We now want to generalise the notion of a uniform continuous function to the concept of gauged spaces. To do so, we first redefine the concept of a modulus of uniform continuity.

Definition 7.2. An *(unbounded) modulus of uniform continuity* is a left continuous increasing function $\Delta : (0, \infty) \to (0, \infty)$.

Definition 7.3. We say that a function $f : (X, d_X, \nu_X) \to (Y, d_Y, \nu_Y)$ *respects* Δ *under* ν if for all $\epsilon > 0$ we have that:

$$d_X(x, y) \leq \Delta(\epsilon) \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

$$\nu_X(x), \nu_X(y) < \frac{1}{\epsilon} \Rightarrow \nu_Y(f(x)), \nu_Y(f(y)) < \frac{1}{\Delta(\epsilon)}$$

We say that f is uniformly continuous under ν if it respects some δ under ν.

In what follows, we want to look at uniformly continuous functions f whose domain is a Cartesian product of gauged metric spaces. In order to do so, we need to equip this Cartesian product with a gauged metric structure. We do this as follows:

Definition 7.4. Let $(X_i, d_i, \nu_i)_{i<n}$ be a finite collection of gauged metric spaces. We equip the Cartesian product $\prod_{i<n} X_i$ with the metric $d(\vec{x}, \vec{y}) = \sup_{i<n} d_i(x_i, y_i)$ and the gauge $\nu(\vec{x}) = \sup_{i<n} \nu_i(x_i)$.

We are now ready to define the concept of a signature for unbounded continuous logic.

Definition 7.5. An *unbounded continuous signature* $\mathcal{L}$ is a set of sorts S, a set of function symbols and a set of relation symbols together with the following:

- For every sort binary relation symbol d_S.
- For every sort unary relation symbol ν_S.
- For each symbol s a modulus of uniform continuity Δ_s.

Now that we have a definition of a signature, we can look at the definition of structures.

Definition 7.6. Let $\mathcal{L}$ be an unbounded continuous signature, which we assume for simplicity to have a single sort. An *(unbounded)* $\mathcal{L}$*-structure* is a complete metric gauged space (M, d^M, ν^M) together with an interpretation $f^M : M^n \to M$ for every function symbol f of arity n and a function $R : M^n \to [0, \infty)$ for every relation symbol R of arity n, with the additional requirement that every symbol s respects Δ_s under ν.

We note that the space $[0, \infty)$ is equipped with the standard gauged metric structure $([0, \infty), |x - y|, |x|)$ and that the space M^n is equipped with the gauged metric structure as described in 7.4.

We can now look at some examples.

Example 7.7. Let L be a bounded continuous signature, as in Definition 3.11. Recall that in the bounded case, we chose to equip every n-ary symbol s with a modulus of continuity $\Delta_{s,i}$ for every argument.

Let $\mathcal{L}$ be the unbounded signature obtained from L by adding a gauge symbol ν, and taking $\Delta_s = \max(1, \max_{i<n}(\Delta_{s,i}(\frac{x}{n})))$. Then it is easy to see that every L-structure is also an $\mathcal{L}$-structure, since we can just interpret ν to be constantly 0. So we see that unbounded continuous logic indeed extends bounded continuous logic. ❑

Example 7.8. Since we are trying to find a parallel with Henson's logic for Banach structures, we would like to view Banach spaces as unbounded structures. We take $\mathcal{L} = \{0, +, m_r : r \in \mathbb{Q}\}$, where m_r is scalar multiplication. We take the gauge to be $\nu(x) = d(x, 0) = ||x||$. For the moduli of uniform continuity, we take $\Delta_0(x) = x$, $\Delta_+ = \frac{x}{2}$ and $\Delta_{m_r} = |r|x$. Then it is easy to see that every real Banach space is naturally an unbounded $\mathcal{L}$-structure. ❑

Now we can look at the syntax of unbounded continuous logic. We start of by looking at the logical connectives. We want the connectives to be all continuous functions from $[0, \infty)^n$ to $[0, \infty)$, but as with bounded continuous logic, this would give us a very large set of connectives. So we want to restrict ourselves to a full system of connectives. It turns out that the following will suffice.

Proposition 7.9. *The system with* $F_0 = \{1\}$*,* $F_1 = \{\frac{x}{2}\}$*,* $F_2 = \{\dot{-}, +\}$ *and* $F_n = \emptyset$ *otherwise is full.*

Note in particular that the notation 1 is slightly ambiguous, since 1 is both a connective and, since it is 0-ary, also a formula. In what follows we will explicitly point out the role of the symbol 1 if this is not clear from the context.

Terms and atomic formulas are defined in the usual way.

We need to take care, however, when looking at the quantifiers. When we take, for example, $\sup_x$ without further restrictions to be the equivalent of universal quantification, we could run into problems. It is very well possible that $\sup_x \phi$ is infinite. Furthermore, even if ϕ is bounded, we still need that $\sup_{\nu(x)\leq r} \phi$ converges uniformly to $\sup_x \phi$ as r goes to infinity, otherwise $\sup_x \phi$ might not be uniformly continuous under ν. We see that this last problem may also hold for $\inf_x$.

So we see that we cannot simply quantify over every available formula. We will define the properties that a formula must have in the following definitions.

Definition 7.10. Let ϕ be an $\mathcal{L}$-formula. We will call ϕ *bounded* if ϕ is of one of the following forms:

- $\phi = \lambda(\psi_1, \ldots, \psi_n)$, where λ is an n-ary connective and every ψ_i is bounded.
- $\phi = \psi_0 \dot{-} \psi_1$, where ψ_0 is bounded and ψ_1 is any formula.
- $\phi = \inf_x \psi$, where ψ is bounded.
- $\phi = \sup_x \psi$, where ψ is bounded.

Definition 7.11. Let ϕ be a formula and x be a variable. We say that ϕ is *eventually constant in* x whenever one of the following conditions hold:

- ϕ is atomic and x does not appear in ϕ.
- $\phi = \lambda(\psi_1, \ldots, \psi_n)$, where λ is an n-ary connective and every ψ_i is eventually constant in x.
- $\phi = \psi \dot{-} \nu(x)$, where ψ is bounded.
- $\phi = \inf_y \psi$, where ψ is eventually constant in x.
- $\phi = \sup_y \psi$, where ψ is eventually constant in x.

Remark 7.12. We would like to point out some immediate corollaries of these definitions.

- No atomic formula is bounded.
- The formula 1 consists of a connective with a bounded formula in every argument, and is therefore bounded and eventually constant in every variable. As a consequence, we have that every constant r of the form $\frac{k}{2^n}$ is bounded and eventually constant in every variable, and therefore the same holds for $\phi \wedge r = r \dot{-} (r \dot{-} \phi)$, for every formula ϕ and constant r.

Using these new properties, we can define the class of $\mathcal{L}$-formulas.

Definition 7.13. Let $\mathcal{L}$ be an unbounded continuous signature. We define the set of $\mathcal{L}$-formulas as the smallest set that contains all atomic formulas, is closed under logical connectives and is closed under eventually constant quantification; that is, if ϕ is a formula that is eventually constant in x, then $\sup_x \phi$ and $\inf_x \phi$ are also formulas.

The properties of boundedness and eventual constancy are thus far purely syntactical, but we will look at the semantic properties and we will see that the definitions make sense. Before we do that, we first look at the syntax a little bit more. We can construct for every bounded formula ϕ a syntactic bound in the following way.

Definition 7.14. Let ϕ be a bounded formula. We derive a syntactic bound $B_\phi \in [0, \infty)$ on ϕ by recursion.

- If $\phi = \lambda(\psi_1, \ldots, \psi_n)$, where every ψ_i is bounded, then $B_\phi = \sup_{\vec{x} \in \Pi[0, B_{\psi_i}]} \lambda(\vec{x})$, where B_{ψ_i} is the syntactic bound on ψ_i.
- If $\phi = \psi_0 \dot{-} \psi_1$, where ψ_0 is bounded, then $B_\phi = B_{\psi_0}$.

- If $\phi = \inf_x \psi$ or $\phi = \sup_x \psi$, where ψ in bounded, then $B_\phi = B_\psi$.

We have a similar construction for eventual constancy. For every formula $\phi(x, \vec{y})$ that is eventually constant in a variable x, we want a certain threshold C, dependent on ϕ and x, such that ϕ does not depend on x any more if $\nu(x) > C$. We will do that as follows:

Definition 7.15. Let $\phi(x, \vec{y})$ be a formula that is eventually constant in a variable x. We derive a formula $\phi(\infty, \vec{y})$ and a syntactic constancy threshold $C_{\phi,x} \in [0, \infty)$ on ϕ by recursion.

- If ϕ is atomic and x does not appear in ϕ, then:
  - $\phi(\infty, \vec{y}) = \phi(\vec{y})$.
  - $C_{\phi,x} = 0$.
- If $\phi = \lambda(\psi_1, \ldots, \psi_n)$, where every ψ_i is eventually constant in x, then:
  - $\phi(\infty, \vec{y}) = \lambda(\psi_1(\infty, \vec{y}), \ldots, \psi_1(\infty, \vec{y}))$.
  - $C_{\phi,x} = \bigvee C_{\psi_i,x}$.
- If $\phi = \psi \dot{-} \nu(x)$, where ψ is bounded, then:
  - $\phi(\infty, \vec{y}) = 0$.
  - $C_{\phi,x} = B_\psi$.
- If $\phi = \inf_z \psi$, where ψ is eventually constant in $x \neq z$, then:
  - $\phi(\infty, \vec{y}) = \inf_z \psi(\infty, z, \vec{y})$.
  - $C_{\phi,x} = C_{\psi,x}$.

  The same holds for $\phi = \sup_z \psi$.
- If $\phi = \inf_x \psi$, where ψ is eventually constant in x, then:
  - $\phi(\infty, \vec{y}) = \inf_x \psi(x, \vec{y}) = \phi(\vec{y})$.
  - $C_{\phi,x} = 0$.

  The same holds for $\phi = \sup_x \psi$.

Now that we have finally defined all of the syntactic concepts, we can look at the semantics. Once again, the situation here is very similar to bounded continuous logic, with exception of the quantifiers.

Definition 7.16. Let $\tau(\vec{x})$ be a term with n free variables and let M be an unbounded $\mathcal{L}$-structure. We define the *interpretation of* $\tau(\vec{x})$ *in* M as a function $\tau^M : M^n \to M$ as follows:

- If $\tau = x_i$, then $\tau^M(\vec{m}) = m_i$.
- If $\tau = f(\sigma_0, \ldots, \sigma_{k-1})$, where the σ_i are all sub-terms, then $\tau^M(\vec{m}) = f^M(\sigma_0^M(\vec{m}), \ldots, \sigma_{k-1}^M(\vec{m}))$

Definition 7.17. Let $\phi(\vec{x})$ be a formula with n free variables and let M be an unbounded $\mathcal{L}$-structure. We define the *interpretation of* $\phi(\vec{x})$ *in* M as a function $\phi^M : M^n \to [0, \infty)$ as follows:

- If $\phi = P(\tau_0, \ldots, \tau_{k-1})$ is atomic, then $\phi^M(\vec{m}) = P^M(\tau_0^M(\vec{m}), \ldots, \tau_{k-1}^M(\vec{m}))$.
- If $\phi = \lambda(\psi_0, \ldots, \psi_{k-1})$, where λ is a connective, then $\phi^M(\vec{m}) = \lambda(\psi_0^M(\vec{m}), \ldots, \psi_{k-1}^M(\vec{m}))$.

- If $\phi = \inf_x \psi(x, \vec{y})$, then $\phi^M(\vec{m}) = \inf_{a \in M \cup \{\infty\}} \psi^M(a, \vec{m})$.
- If $\phi = \sup_x \psi(x, \vec{y})$, then $\phi^M(\vec{m}) = \sup_{a \in M \cup \{\infty\}} \psi^M(a, \vec{m})$.

Using this definition, we can see that the concepts of syntactic bounds and constancy thresholds semantically behave as such:

Proposition 7.18. *Let ϕ be an $\mathcal{L}$-formula and let M be an unbounded $\mathcal{L}$-structure.*

If ϕ is bounded, then $\phi(\vec{m}) \leq B_\phi$ for all $\vec{m} \in M$.

If $\phi(x, \vec{y})$ is eventually constant in x, then $\phi^M(b, \vec{m}) = \phi^M(\infty, \vec{m})$ for all $\vec{m} \in M$ and $b \in M$ such that $\nu(b) \geq C_{\phi,x}$.

Now that we have defined all the necessary syntax and semantics, we may wonder whether or not we have a compactness-like theorem for unbounded continuous logic. It turns out we do have one, but we to make a modification that puts a bound on the free variables occurring in a set of sentences. To state the theorem, we need the following definition.

Definition 7.19. Let Σ be a family of conditions $\{\phi_i \leq r_i : i \in \kappa\}$. We say that Σ is *approximately finitely satisfiable* if for every finite $w \subseteq \kappa$ and $\epsilon > 0$ the family $\Sigma_0 = \{\phi_i \leq r_i + \epsilon : i \in w\}$ is satisfiable.

Proposition 7.20. *Let $\mathcal{L}$ be an unbounded continuous signature, $r \in [0, \infty)$, and let Σ be a family of conditions in n free variables $\vec{x}$. The $\Sigma \cup \{\nu(x_i) \leq r : i < n\}$ is satisfiable if and only if it is approximately finitely satisfiable.*

Lastly, for what follows, we also have to take a quick look at type spaces, which are defined in a standard way.

Definition 7.21. Let $\mathcal{L}$ be an unbounded continuous signature and let M the universe of an $\mathcal{L}$-structure $\mathcal{M}$.

- Given an n-tuple $\vec{m} \in M$, we define $\mathrm{tp}(\vec{m})$, called the *type of* $\vec{m}$, as the set of all $\mathcal{L}$-conditions in n free variables satisfied by $\vec{m}$ in $\mathcal{M}$. We denote types by $p(\vec{x})$.
- Let $p(\vec{x}) = \mathrm{tp}(\vec{m})$ be given. For any formula $\phi(\vec{x})$, we write $\phi^p = \phi(\vec{m})$.
- The set of all n-types is denoted by S_n. The set of all n-types containing a theory T is denoted $S_n(T)$.
- Let s be a condition with n free variables. We denote $[s]^{S_n(T)}$ for the set $\{p \in S_n(T) : s \in p\}$. The family of all sets of this form make a base of closed sets for a Hausdorff topology on $S_n(T)$.
- We define $S_n^{\nu \leq r}(T) = \bigcap_{i<n} [\nu(x_i) \leq r]$, that is, the set of all types containing the condition $\nu(x_i) \leq r$ for all $i < n$. By Proposition 7.20 all sets of the form $S_n^{\nu \leq r}(T)$ are compact.

7.2 Relation to Henson's logic

Now it is time to look at the relation between Henson's logic for Banach structures and unbounded continuous logic. Recall from Section 2.4 that a signature L in Henson's logic is a set of sorts S, with at least one of the sorts being $\mathbb{R}$, and a set of designated function symbols. For our intents and purposes, we modify this slightly by replacing $\mathbb{R}$ with $[0, \infty)$.

Given a signature L in Henson's logic, it is now rather easy to transform L into an unbounded continuous signature $\mathcal{L}$ by replacing all function symbols that have codomain $[0, \infty)$ by relation symbols. Since every sort S is assumed to have a complete norm, we can take $\nu_S(x) = ||x||_S$ and $d_S(x, y) = ||x - y||_S$. Since we also required in Henson's logic that every function was uniformly continuous on every bounded subset of its domain, we also have that each function symbol must satisfy some modulus of uniform continuity under $|| \cdot ||$. It is possible to write out an L-theory T_0 whose models are precisely the structures that respect the moduli of uniform continuity under $|| \cdot ||$.

We thus have that every model of T_0 is also an $\mathcal{L}$-structure, since we have the identification as just presented. So one would expect certain classes of structures to behave similarly with respect to both Henson's logic and unbounded continuous logic. The following definition and theorem shows that this is the case.

Definition 7.22. Let $\mathcal{K}$ be a class of structures. We call $\mathcal{K}$ an *elemenatary class* in a certain logic if there exists a theory T in this logic such that the elements of $\mathcal{K}$ are precisely the models of T.

Theorem 7.23. *(Theorem 4.1 in [3]) A class of structures $\mathcal{K}$ is elementary in Henson's logic if and only if it is elementary in unbounded continuous logic.*

Using this theorem, we can show an even deeper equivalence between Henson's logic and unbounded continuous logic.

Theorem 7.24. *Two n-tuples in a structure M have the same type in Henson's logic if and only if they have the same type in unbounded continuous logic. Furthermore, this identification induces a homeomorphism of topological spaces $S_n^L(T_0) \simeq S_n^{\mathcal{L}}$.*

Proof. Let $\vec{x}$ and $\vec{y}$ be the two n-tuples. Assume that $\mathrm{tp}(\vec{x}) = \mathrm{tp}(\vec{y})$ in Henson's logic. By definition, the classes $\{M|M \models \mathrm{tp}(\vec{x})\} = \{M|M \models \mathrm{tp}(\vec{y})\}$ are elementary. By the previous proposition, we see that these classes are also elementary in unbounded continuous logic. And therefore $\vec{x}$ and $\vec{y}$ have the same type in unbounded continuous logic. The argument for the converse implication is the same.

For the second part of the theorem, we need the following lemma. Here we identify the set $S_n^{||\cdot|| \leq r}$ as the Henson's logic equivalent of $S_n^{\nu \leq r}$.

Lemma 7.25. *A set $X \subseteq S_n^{||\cdot|| \leq r}$ is closed if and only if the class $\{(M, \vec{m})|tp(\vec{m}) \in X\}$ is elementary.*

Proof. Recall that the sets $[s]^{S_n}$ form a base of closed sets on S_n, which means that every closed set in the topology on S_n can be written as an intersection of sets of the form $[s]^{S_n}$.

So assume $X \subseteq S_n^{||\cdot|| \leq r}$ is closed. By the above, we can write $X = \bigcap_{s \in S} [s]^{S_n}$ for some collection of conditions S. Let T be the theory obtained from substituting new constants for the free variables in the conditions of S. Note that we can do this, since there is a bound on the norm of the constants because $X \subseteq S_n^{||\cdot|| \leq r}$. The class $\{(M, \vec{m})|(M, \vec{m}) \models T\}$ is by definition elementary. And we can also see that $(M, \vec{m}) \models T$ if and only if $M \models s(\vec{m})$ for all $s \in S$, and therefore $\mathrm{tp}(\vec{m}) \in X$. So $\{(M, \vec{m})|\mathrm{tp}(\vec{m}) \in X\} = \{(M, \vec{m}), (M, \vec{m}) \models T\}$ is elementary.

For the converse assume that $\{(M, \vec{m})|\mathrm{tp}(\vec{m}) \in X\}$ is elementary. So there is a theory T such that $(M, \vec{m}) \models T$ if and only if $\mathrm{tp}(\vec{m}) \in X$. Let S be the set of conditions that appear in some type of the form $\mathrm{tp}(\vec{m})$. It is then easy to see that we must have that $X = \bigcap_{s \in S} [s]^{S_n}$, and therefore X is closed. ❑

So, again by the previous theorem, we have as an immediate corollary of this lemma that $S_n^{\|\cdot\|\leq r}$ in Henson's logic is homeomorphic to $S_n^{\nu\leq r}$ in unbounded continuous logic.

Now by a standard topological argument, we can see that these local homeomorphisms can be put together to create a global one. ❑

As a direct corollary of this theorem, we have the following:

Corollary 7.26. *For every set $\Sigma(\vec{x})$ of L-formulas there exists a set $\Gamma(\vec{x})$ of $\mathcal{L}$-formulas, and for every set $\Gamma(\vec{x})$ of $\mathcal{L}$-formulas there exists a set $\Sigma(\vec{x})$ of L-formulas,, such that for every structure M and $\vec{m} \in M$ we have that $M \models_A \Sigma(\vec{m}) \Leftrightarrow M \models \Gamma(\vec{m})$.*

So we see that we have an equivalence between Henson's logic and unbounded continuous logic, keeping in mind that we made some minor adjustments at the beginning of this subsection.

We note that the proof as given here is, evidently, far from direct or concrete. The author of [3] acknowledges this, and shows one direction of this proof explicitly. We will show this part of the proof, and fill in the other direction.

In order to write down the explicit equivalence between Henson's logic and unbounded continuous logic, we would like to recover the concept of a bounded quantifier in unbounded continuous logic. This concept is not an inherent part of the language, but nonetheless we are able to find a decent analogue.

Let ϕ be a formula, and assume for the moment that it is bounded. Let B_ϕ be its syntactic bound and let k be the least integer greater than B_ϕ. First we note the following equivalence in formulas:

$$\begin{aligned}(\phi + (\nu(x) \wedge r)) \dot{-} \nu(x) &= (\phi + \min(\nu(x), r)) \dot{-} \nu(x)\\ &= \max(0, \phi + \min(\nu(x), r) - \nu(x))\\ &= \max(0, \phi + \min(0, r - \nu(x)))\\ &= \max(0, \phi - \max(0, \nu(x) - r))\\ &= \max(0, \phi - (\nu(x) \dot{-} r))\\ &= \phi \dot{-} (\nu(x) \dot{-} r)\end{aligned}$$

By definition, the formula $(\phi + (\nu(x) \wedge r)) \dot{-} \nu(x)$ is bounded and eventually constant in x, and therefore this also holds for $\phi \dot{-} (\nu(x) \dot{-} r)$. From this, it follows that $\phi \dot{-} m(\nu(x) \dot{-} r)$ is also bounded and eventually constant in x for every $m > 0$.

Let $r' > r > 0$ be given. Let m be the least number such that there is an l such that $r \leq l2^{-m} < (l+1)2^{-m} \leq r'$. Let s be the smallest possible value of $l2^{-m}$. We then define the following:

- $\phi \downarrow^{x\leq r,r'} = \phi \dot{-} k2^m(\nu(x) \dot{-} s)$
- $\phi \uparrow^{x\leq r,r'} = k \dot{-} (k \dot{-} \phi) \downarrow^{x\leq r,r'}$
- $\sup_x^{r,r'} \phi = \sup_x \phi \downarrow^{x\leq r,r'}$
- $\inf_x^{r,r'} \phi = \inf_x \phi \uparrow^{x\leq r,r'}$

We first note that we have that both $\phi \downarrow^{x\leq r,r'}$ and $\phi \uparrow^{x\leq r,r'}$ are bounded and eventually constant in x, by the discussion above, and therefore $\sup_x^{r,r'} \phi$ and $\inf_x^{r,r'} \phi$ are well-defined.

We now want to look at the behaviour of $\phi \downarrow^{x \leq r, r'}$. By construction, we immediately have that $\phi \downarrow^{x \leq r, r'} \leq \phi$. But we can say more than that. If we assume that $\nu(x) \leq r$, we have that $\nu(x) \dot{-} s \leq \nu(x) \dot{-} r = 0$. So in particular, we have that $\phi \downarrow^{x \leq r, r'} = \phi$. If we assume that $\nu(x) \geq r'$, we have that $\nu(x) \dot{-} s \geq r' - s \geq (l+1)2^{-m} - l2^{-m} = 2^{-m}$. So in particular $\phi \downarrow^{x \leq r, r'} = \phi \dot{-} k2^m(\nu(x) \dot{-} s) \leq \phi \dot{-} k2^m 2^{-m} = \phi \dot{-} k = 0$.

From all this we may conclude that $\sup_{\nu(x) \leq r} \phi \leq \sup_x^{r,r'} \phi \leq \sup_{\nu(x) < r'} \phi$. With a similar reasoning as above, we can also see that $\inf_{\nu(x) \leq r} \phi \geq \inf_x^{r,r'} \phi \geq \inf_{\nu(x) < r'} \phi$.

Recall that in all of this we assumed that the formula ϕ was bounded. Whenever this is not the case, we may define $\phi \downarrow^{x \leq r, r'} = (\phi \wedge 1) \dot{-} k2^m(\nu(x) \dot{-} s)$, thereby effectively putting a bound on ϕ, after which we can proceed as above and obtain the same results.

We are now ready to give an alternative and more concrete proof of Corollary 7.26.

Proof of Corollary 7.26. We start with the side that has already been sketched out in [3]. Let $\phi(\vec{y}) \in \Sigma(\vec{y})$ be an L-formula. We will construct a set $\Gamma(\vec{y})$ of $\mathcal{L}$-sentences fulfilling the conditions that are set in the corollary.

As is stated in [14], we may assume that $\phi(\vec{y})$ is in prenex normal form. If we shorten the expression $\forall x(||x|| \leq r \rightarrow \phi)$ as $\forall^{\leq r} x \phi$ and $\exists x(||x|| \leq r \wedge \phi)$ as $\exists^{\leq r} x \phi$, this means that we may assume that ϕ looks as follows:

$$\forall^{\leq r_0} x_0 \exists^{\leq r_1} x_1 \ldots \psi(\vec{x}, \vec{y})$$

Here $\psi(\vec{x}, \vec{y})$ is combination of conjunctions and disjunctions of atomic L-formulas. Recall that an atomic L-formula is of the form $t_i \geq r_i$ or $t_i \leq r_i$, where t_i is a real-valued term.

Let us first assume that $\psi(\vec{x}, \vec{y})$ is exactly of the form $t(\vec{x}, \vec{y}) \leq 0$, for some real-valued term t. The approximate satisfaction of ϕ in a certain structure M would then mean that M has to satisfy ϕ' for every $\phi' > \phi$. By using the definition, we see that any $\phi' > \phi$ is of the form:

$$\forall^{\leq r_0'} x_0 \exists^{\leq r_1'} x_1 \ldots t(\vec{x}, \vec{y}) \leq r,$$

where $r > 0$, $r_0' < r_0$, $r_1' > r_1$ etc.

We can see that for every structure M we have that M satisfies all of the ϕ' if and only if M satisfies the following $\mathcal{L}$-formula for all $r_0' < r_0$, $r_1' > r_1$ etc.

$$\sup_{x_0}^{r_0', r_0} \inf_{x_1}^{r_1, r_1'} \ldots \psi(\vec{x}, \vec{y}) = 0$$

Thus we can take all these formulas and put those in our set $\Gamma(\vec{y})$. We can then repeat this construction for the other formulas in $\Sigma(\vec{y})$.

Now all that is left to do is to proof that we can indeed assume without loss of generality that $\psi(\vec{x}, \vec{y})$ is of the form $t(\vec{x}, \vec{y}) \leq 0$. We proof this by induction on the complexity of ψ.

If $\psi(\vec{x}, \vec{y})$ is atomic, and therefore of the form $t(\vec{x}, \vec{y}) \leq r$ or $t(\vec{x}, \vec{y}) \geq r$, we see that ψ is equivalent to $t(\vec{x}, \vec{y}) \dot{-} r \leq 0$ or $r \dot{-} t(\vec{x}, \vec{y}) \leq 0$, respectively. We may therefore assume that ψ is a positive Boolean combination of formulas of the form $t \leq 0$.

Suppose now that $\psi(\vec{x}, \vec{y})$ is of the form $(t_0 \leq 0) \wedge (t_1 \leq 0)$. We can easily see that this is equivalent to saying that $(t_0 \vee t_1) \leq 0$. Note that the conjunction is replaced with a disjunction, since 0 is the false value in Henson's logic, but the truth value in (unbounded) continuous logic.

Since we have a similar argument as above for the case $(t_0 \leq 0) \vee (t_1 \leq 0)$, we see that we can indeed find a single $t(\vec{x}, \vec{y})$ such that $\psi(\vec{x}, \vec{y})$ is equivalent to $t(\vec{x}, \vec{y}) \leq 0$. This completes the proof of the first implication.

Now for the converse implication. Let $\Gamma(\vec{y})$ be a set of $\mathcal{L}$-formulas and let $\phi(\vec{y}) \in \Gamma(\vec{y})$ be given. We will construct a set $\Sigma(\vec{y})$ of L-formulas fulfilling the requirement. We prove this by induction on the complexity of ϕ. By a discussion in [3], we may assume that ϕ is in prenex normal form.

Assume first that ϕ is atomic, and therefore of the form $R(\vec{y}) = 0$, for some predicate symbol R. By construction of the language $\mathcal{L}$, we know that R corresponds to some real-valued function symbol t in L. From this observation it follows that ϕ is equivalent to the L-formula $t(\vec{y}) \leq 0$.

Now we look at the case that ϕ is of the form $\lambda(\psi_1, \ldots, \psi_n) = 0$ for some n-ary connective λ. We may assume that $\lambda \in \{1, \frac{1}{2}x, x + y, x \dot{-} y\}$, since this system of connectives can be shown to be full, as is done in [3].

The case $\lambda = 1$ is trivial.

The case $\lambda = \frac{1}{2}x$ or $\lambda = x + y$ are also easy. In the first case we see that $\frac{1}{2}\psi(\vec{y}) = 0 \Leftrightarrow \psi(\vec{y}) = 0$, and in the second case we see that $\psi_0 + \psi_1 = 0 \Leftrightarrow \psi_0 = 0 \wedge \psi_1 = 0$. We are then reduced to a situation which is of lower complexity, and therefore we can apply the induction hypothesis.

We have to take a little bit of care when looking at the case $\lambda = x \dot{-} y$. When we look at ϕ of the form $\psi_0(\vec{y}) \dot{-} \psi_1(\vec{y}) = 0$, we could be tempted to simply state that the equivalent should be $\psi_0(\vec{y}) - \psi_1(\vec{y}) \leq 0$, but the term $\psi_0(\vec{y}) - \psi_1(\vec{y})$ should still take values in the interval $[0, \infty)$, and clearly that is not the case in general. Therefore, we must also truncate at 0 to get the desired effect. So the equivalent is therefore $(\psi_0(\vec{y}) - \psi_1(\vec{y})) \wedge 0 \leq 0$.

We now look at the case that ϕ is of the form $\exists x \psi(x, \vec{y})$. By construction, we must have that $\psi(x, \vec{y})$ is eventually constant in x, and therefore there exists a constancy threshold $C_{\psi,x}$. We know by Proposition 7.18 that $\phi^M(b, \vec{m}) = \phi(\infty, \vec{m})$ for all $b \in M$ with $\nu(b) \geq C_{\psi,x}$. We can therefore see that the satisfaction of ϕ is equivalent to approximate satisfaction of $\exists^{\leq C_{\psi,x}} x \psi(\infty, \vec{y}) \vee \psi(x, \vec{y})$.

Since we have a similar argument as above for the case that ϕ is of the form $\forall x \psi(x, \vec{y})$, we may conclude that the reverse implication also holds, and therefore the corollary is proven. ❑

8 Discussion

We have suggested a novel approach for interpreting continuous logic. Building on Figueroa's results, we introduced a new hyperdoctrine: CMV. Using the fact that propositional continuous logic is a special case of Łukasiewicz propositional logic, we have taken the codomain of CMV to be $MVAlg$. This implied that CMV is not a first-order hyperdoctrine, but an MV-hyperdoctrine instead. We have seen that this gives us a possibility to interpret first-order languages with a Heyting implication in CMV. Furthermore, we also see that Theorem 4.62 has no equivalent for CMV, since $CMV(X)$ is always a poset. Using CMV to interpret continuous logic in a hyperdoctrine therefore eliminates the two main concerns that arose when using CMT.

There are still some grey areas regarding our results. For example, we had to invent the concept of an MV-hyperdoctrine. It is currently unknown to us whether there are any other interesting examples of MV-hyperdoctrines. Furthermore, although we succeeded in overcoming the shortcomings of CMT, we also lose some interesting results. We can see that the morphism that embeds $cMet_1$ into $Strict_{PER(\mathbf{P}_{CL})}$ is no longer an embedding when we replace $\mathbf{P}_{CL}$ with $\mathbf{Q}_{CL}$. Therefore, we cannot embed CMV into a hyperdoctrine of subobjects in a topos of sheaves.

Suggestions for further research

- In Section 5 we have constructed a first order hyperdoctrine by restricting continuous logic to 1-Lipschitz functions. Although the resulting hyperdoctrine is not suitable to interpret continuous logic, it is perhaps an interesting object of study. For example, one could look at the fragment of continuous logic that can be interpreted naturally in this hyperdoctrine, and then study in what way this part relates to the interpretation of the rest of continuous logic.
- In [10], Figueroa shows that there is an embedding of the category $cMet_1$ into the category $PER(\mathbf{P}_{CL})$, and as a consequence of that there is also an embedding of hyperdoctrines. We may wonder if we have a similar result when we look at the partial equivalence relations over $\mathbf{Q}_{CL}$, although we probably have to take a different approach than in the former case, since the embedding between $cMet_1$ and $PER(\mathbf{P}_{CL})$ we defined earlier does not reflect the order of $\mathbf{Q}_{CL}$.
- The category of MV-algebras is known to have many interesting properties. For example, in [17], the author shows an equivalence Γ between the category of MV-algebras and the category of *abelian l-groups with order unit*. In our earlier discussions we saw that the MV-algebras in the codomain of the MV-hyperdoctrine CMV all contain the special element $\frac{1}{2}$. We could wonder if this special property is preserved under Γ, and what it entails in the category of abelian l-groups with order unit.
- In Section 7 we discussed a variant of continuous logic with $[0, \infty)$ as a truth space. We could wonder if it is possible to find a hyperdoctrine in which we can interpret unbounded continuous logic. One approach could be to look at sets of uniformly continuous functions with codomain $[0, \infty)$, but we need to be careful, as such sets endowed with the order $\sqsubseteq'$ do not have a bottom element. Furthermore, since we only have that $\exists x\phi$ and $\forall x\phi$ are formulas if ϕ is eventually constant in x, these algebraic structures probably need not have all adjoints. That would mean that the concept of a quantifier hyperdoctrine is too restrictive to interpret unbounded continuous logic.

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