Lens Sequences in Apollonian Circle Packings

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Contents

1 Introduction

Kocik showed in his paper [1] the existence of a formula for a sequence of curvatures of circles that fit within a bounding area defined by two other circles of equal size. He called this sequence a "Lens Sequence".

In this thesis we aim to show his results, and expand on some of them. We will also go more into how lens sequences appear in Apollonian packings than Kocik did. We also provide a few more conjectures, and provide a few more angles for future examination.

I'd like to personally thank Gunther Cornelissen for his help with all the steps in the process of writing this thesis, as well as thank Lyn Blackwell for her contributions in making sure the text was grammatically correct and flowed well.

2 Apollonian Packings

2.1 Descartes configurations

A Descartes configuration is a set of 4 mutually tangential circles in the plane. This includes considering straight lines as 'degenerate' circles with curvature 0, with their interior being an appropriate half-plane. We will more concretely define what we mean by "appropriate" later. There are 4 types of Descartes configurations, and they can be described as follows:

1. A set of three mutually tangent circles with the fourth circle placed in between the other three
2. A set of three mutually tangent circles with a fourth circle enclosing them, tangent to all three
3. A straight line with three differently-sized circles placed in tangent on one side of the line
4. A set of two parallel straight lines with two circles of equal radius placed tangentially between them

Theorem 2.1 (Descartes Circle Theorem). *Given curvatures a, b, c, d of four circles, where the curvature of a circle is the inverse of the radius, in a Descartes configuration, then*

$$a^2 + b^2 + c^2 + d^2 = \frac{1}{2}(a + b + c + d)^2$$

Descartes himself showed an equivalent result to this in 1643 for the first type of configuration, though he showed it in terms of the radii of the circles. To expand this so it also works for the other three types of configurations, we will have to make some distinctions.
For a circle with radius r and thus curvature $\frac{1}{r}$, we will define it's *oriented radius* and *oriented curvature* as follows.

- If the standard interior is still the interior of the circle, then the oriented radius and oriented curvature are still r and $\frac{1}{r}$ respectively.
- If, however, what we will call the interior of the circle is what is usually considered the exterior, then the oriented radius and oriented curvature are $-r$ and $-\frac{1}{r}$ respectively, as shown in Figure 2. For ease of use, whenever we talk about curvatures in this thesis, we are talking about the oriented curvature.

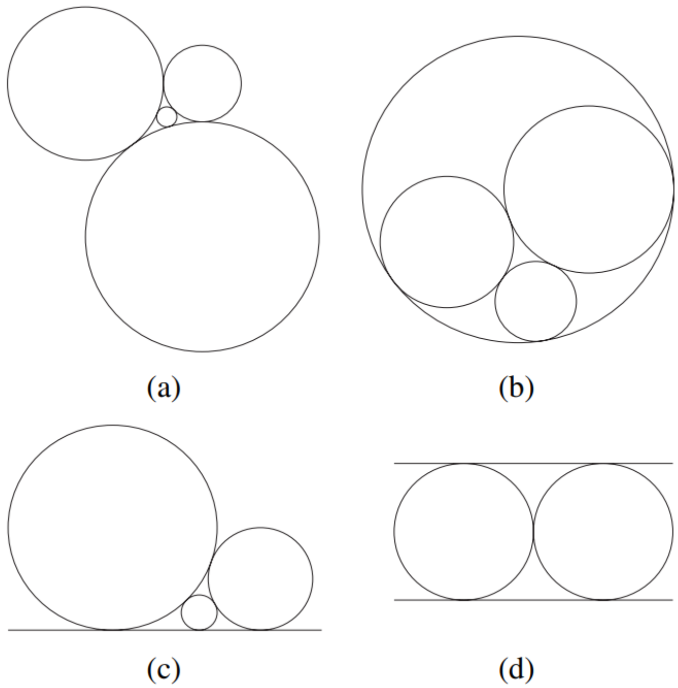

Figure 1: The 4 Descartes configurations, recreated from [2], Fig. 2.

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With these distinctions in place, we can take our Descartes configurations and turn them into *oriented Descartes configurations* by defining the orientation of each of the circles such that the interiors of all circles are disjoint. Notice that for case 3 and 4, we don't actually have to know whether the straight lines have the r or $-r$ type of radius, as the curvature is 0 either way; We can just define their interior to be the half-plane not containing any of the other circles.

As shown in [3] if we connect all the centers in the first Descartes configuration, we get 4 triangles, where one of them covers the same area as the other three combined. Using Heron's formula for the area of a triangle, one then goes on to

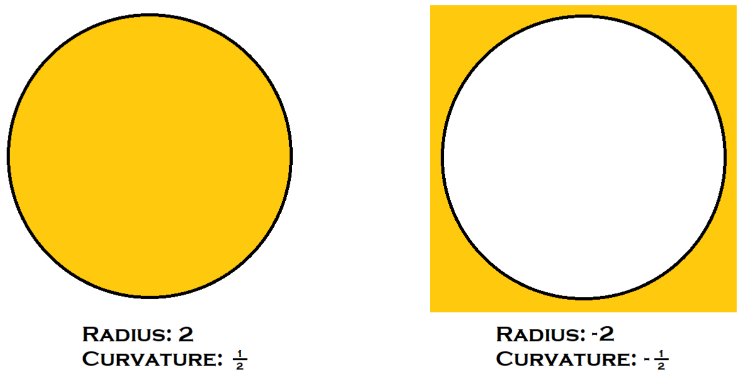

Figure 2: The interior for both regular and negative radius and curvature

prove the Descartes Circle Theorem from the following equation:

$$\sqrt{r_1r_2r_3(r_1+r_2+r_3)} = \sqrt{r_1r_2r_4(r_1+r_2+r_4)} + \sqrt{r_1r_4r_3(r_1+r_4+r_3)} + \sqrt{r_4r_2r_3(r_4+r_2+r_3)}$$

If we look at the second configuration, there are two options: Either the large circle's center lies within the triangle formed by the other three centers, or it lies outside it.
If it lies inside the triangle, we will get the following equation (using $-r_4$ as the radius for the largest circle)

$$\sqrt{r_1r_2r_3(r_1+r_2+r_3)} = \sqrt{-(r_4)r_1r_2(-r_1-r_2-r_4)} + \sqrt{-(r_4)r_1r_3(-r_1-r_4-r_3)} + \sqrt{-(r_4)r_2r_3(-r_4-r_2-r_3)}$$

Which is very clearly the same formula after factoring out (-1) terms where needed.
If it does not lie inside the triangle, the equation becomes:

$$\sqrt{r_1r_2r_3(r_1+r_2+r_3)} + \sqrt{-(r_4)r_1r_2(-r_1-r_2-r_4)} = \sqrt{-(r_4)r_1r_3(-r_1-r_4-r_3)} + \sqrt{-(r_4)r_2r_3(-r_4-r_2-r_3)}$$

While it is not the exact same equation, it is incredibly similar. We can follow the same procedure as in [3] still, and after the second step of squaring and re-arranging we would find that the only difference is a negative sign for the RHS, but this gets cancelled in the next step because we square it again, and of course $(-x)^2 = x^2$. This shows that the Descartes Circle Theorem at the very least holds for configuration 1 and 2.

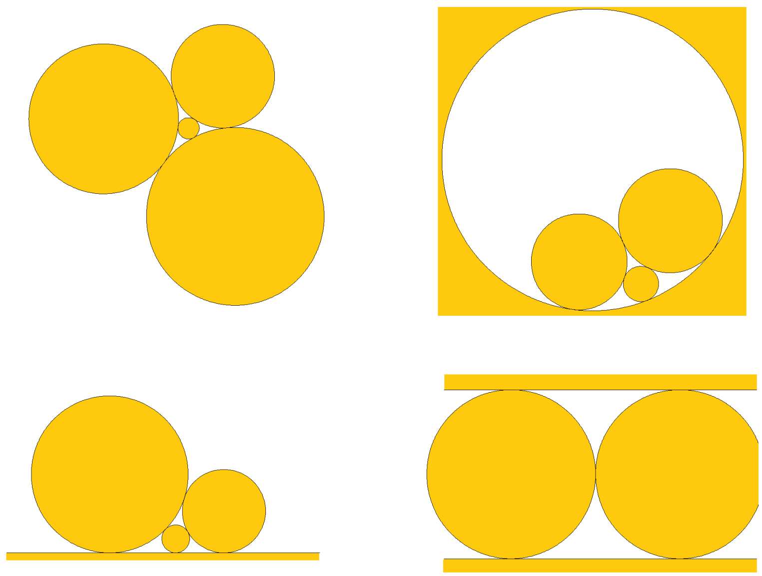

Figure 3: The 4 Descartes configurations with colored-in interiors

To prove configuration 3, let $a = \frac{1}{r_1}$, $b = \frac{1}{r_2}$, $c = \frac{1}{r_3}$, where r_1, r_2, r_3 are the radii of the three tangent circles.
Applying the Pythagorean theorem to right triangles in figure 4 gives us the following equations:

$$\begin{aligned} x^2 &= (r_1 + r_2)^2 - (r_1 - r_2)^2 & \rightarrow \quad & x = 2\sqrt{r_1 r_2} \\ y^2 &= (r_2 + r_3)^2 - (r_3 - r_2)^2 & \rightarrow \quad & y = 2\sqrt{r_2 r_3} \\ (x+y)^2 &= (r_1 + r_3)^2 - (r_3 - r_1)^2 & \rightarrow \quad & x + y = 2\sqrt{r_1 r_3} \end{aligned}$$

This gives us the following equality:

$$2\sqrt{r_1 r_2} + 2\sqrt{r_2 r_3} = 2\sqrt{r_1 r_3}$$

Dividing both sides by $2\sqrt{r_1 r_2 r_3}$ gives us:

$$\sqrt{c} + \sqrt{a} = \sqrt{b}$$

Squaring both sides and re-arranging gives us

$$a - b + c = -2\sqrt{ac}$$

Squaring both sides again and re-arranging gives us

$$a^2 + b^2 + c^2 = 2ac + 2ab + 2bc$$

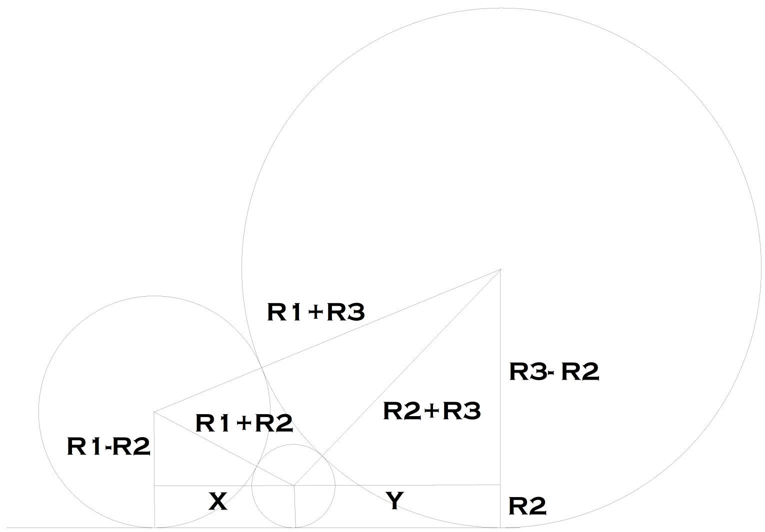

Figure 4: A wire-frame of configuration type 3, with some lengths marked

Now if we add $a^2 + b^2 + c^2$ to both sides, we can factor the right-hand side into $(a + b + c)^2$, giving us our final equation:

$$2(a^2 + b^2 + c^2) = (a + b + c)^2$$

Which is the same as the original equation in the Descartes Circle Theorem for $d = 0$, which is exactly what we should get. This proves the theorem for configuration 3.

Finally, configuration 4 is very trivial to prove: As the two lines are parallel, the two circles must be the same size, and $2(0^2+0^2+a^2+a^2) = (0+0+a+a)^2 = 4a^2$ for all $a \in \mathbf{R}$.
With this, the Descartes circle theorem works for all four oriented Descartes configurations, as stated in [2].

2.1.1 The Complex Descartes Theorem

There is an extension of Descartes' Circle Theorem which also encodes the centers of the circles. It is beautifully similar to the original theorem:

Theorem 2.2 (Complex Descartes Theorem)**.** *If four circles A, B, C, D form an oriented Descartes configuration, then considering their centers to be complex numbers z_i and their curvatures to be real numbers b_i, the following equality holds:*
$(b_1z_1)^2 + (b_2z_2)^2 + (b_3z_3)^2 + (b_4z_4)^2 = \frac{1}{2}(b_1z_1 + b_2z_2 + b_3z_3 + b_4z_4)^2$

We will not prove this theorem here, only use it, but for readers that want to see such a proof, one can be found in [2] or [4].
Instead, we will use it in the future to show the co-linearity of centers of circles by first working out their curvature with the original Descartes theorem, and then working out their centers using the complex Descartes Theorem.

The only issue with this is that it no longer works for Descartes configurations with a straight line involved. This is because the center of such a line can only really be defined as lying at infinity; But that would lead to one of the terms in the formula taking on the form $0 \cdot \infty$, which is of course nonsense. For these situations, however, you can generally use a different method to calculate the center of a circle given the other centers. We will elaborate on this further when we actually use the theorem.

2.2 Apollonian Packings

A circle packing is an arrangement of circles of equal or differing sizes such that no circle can be increased in radius without causing overlap. An Apollonian circle packing can be constructed by taking a set of mutually tangent circles and repeatedly filling the space in between with more tangent circles. This counts as a circle packing because every added circle is tangent to other circles, meaning the radius could not be increased without causing overlap. This leaves us with a fractal-like structure of smaller and smaller circles.

For this example we start with a circle of radius -1. This is similar to a circle of radius 1, but what would usually be considered the 'outside' of the circle is now the inside. This way we can take what would usually be considered the inside of the circle, and place more circles in there without it overlapping the area of our original circle. For example, we can take an original circle of radius -1, and then place two circles of radius $\frac{1}{2}$ such that all three circles are mutually tangent.
If we then look at a part of the plane that is not inside any of the circles, this is a curvilinear triangle which we will call a lune. We can see that each lune is bounded by three circle arcs, coming from mutually tangent circles. We can place new circles in these lunes such that the new circle is tangent to each of the three bounding circles. Repeating this process will create a well-known Apollonian packing called the Apollonian Window.

You can create an Apollonian packing by placing any 2 circles of any size inside another bigger circle, as long as these two circles are tangent to the original circle and tangent to each other. We, however, specifically care about the ones where every circle has an integer curvature. Luckily, we can easily construct such packings by taking an integer solution to the equation in the Descartes circle theorem.

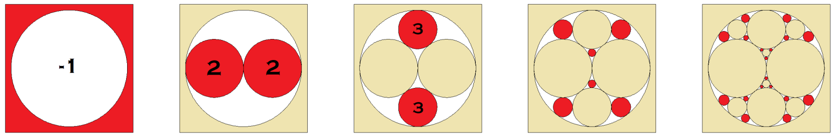

Figure 5: The first few steps of the Apollonian Window

If we take the circle theorem and solve for d, we get that

$$\begin{aligned}
a^2+b^2+c^2+d^2 &= \frac{1}{2}(a+b+c+d)^2\\
a^2+b^2+c^2+d^2 &= \frac{1}{2}((a+b+c)+d)^2\\
a^2+b^2+c^2+d^2 &= \frac{1}{2}((a+b+c)^2+2(a+b+c)d+d^2)\\
d^2 &= \frac{1}{2}(a+b+c)^2+(a+b+c)d+\frac{1}{2}d^2-a^2-b^2-c^2\\
\frac{1}{2}d^2-(a+b+c)d &= \frac{1}{2}(a^2+b^2+c^2+2ab+2bc+2ac)-a^2-b^2-c^2\\
\frac{1}{2}d^2-(a+b+c)d &= -\frac{1}{2}(a^2+b^2+c^2)+ab+bc+ac
\end{aligned}$$

$$d^2-2(a+b+c)d+(a^2+b^2+c^2-2ab-2bc-2ac)=0$$

Applying the quadratic formula here gives us

$$d=\frac{2(a+b+c)\pm\sqrt{4(a+b+c)^2-4(a^2+b^2+c^2-2ab-2bc-2ac)}}{2}$$

But by writing out the $(a+b+c)^2$ term and combining like terms, we can take this down to:

$$d=a+b+c\pm\frac{\sqrt{16(ab+bc+ca)}}{2}$$

Which then simplifies to

$$d=a+b+c\pm 2\sqrt{ab+bc+ca}$$

We see here that there are up to two solutions for d when given a, b, c. We can also see that if a, b, c, d are integers, then the other solution of this equation for d, which we can denote d', must necessarily be an integer as well.
This implies the following lemma:

Lemma 2.3. *Given an oriented Descartes configuration with all four circles having integral curvature, any circle that is tangent to three of these circles must have an integral curvature.* □

This lemma helps us prove the following theorem for integral Apollonian packings:

Theorem 2.4. *Given an Apollonian packing, if it contains a Descartes configuration with integral curvatures, then every circle in the packing will have integral curvature.*

Proof. By construction, every circle in the Apollonian packing is tangent to three other circles that are themselves mutually tangent as well. Clearly, the lunes inside the Descartes configuration are filled with circles of integral curvature by Lemma 2.3. However, this works the other way too; Each of the circles in the original Descartes configuration is itself part of a lune. Take one of these and call it circle A. Because the circles that make up the lune must be mutually tangent, two of them will be part of the original Descartes configuration as well. Call these circles B and C. The last circle that bounds the lune will be tangent to circles B and C by definition of the lune, and tangent to circle A by how A is placed in the lune. This means A, B, C and our new circle D form another Descartes configuration. However, because A, B and C are part of a Descartes configuration with integral curvature, by Lemma 2.3 we find that D must have integral curvature as well. Thus we can expand our configuration both inward and outward, and every circle in the packing must have integral curvature.

□

However, we don't have to start with a circle of negative curvature. We can also start with two parallel straight lines, which as stated before have curvature 0. Then we can place a circle in between such that the circle is tangent to both lines, and say this is a circle with curvature 1.

Now we have two areas that can't really be considered triangles (unless we consider them to be in the projective plane, but that's not necessary here). Instead, we can fill up the empty space with an infinite line of circles with curvature 1, each one tangent to the parallel lines and tangent to the circle to the left and right of it.

Now we see curvilinear triangles forming, albeit with one of the sides being a regular straight line. We can use these as our lunes, placing more circles into them like before. Repeating this procedure like before gives us another well-known Apollonian packing called the Apollonian Belt.

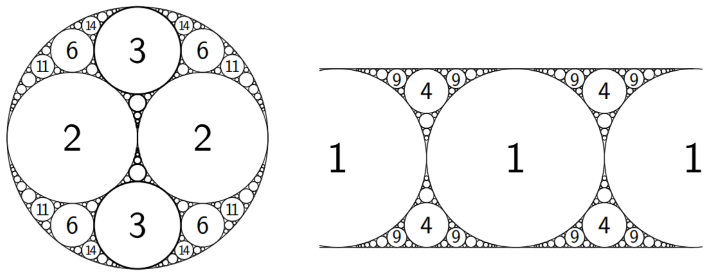

Figure 6: The Apollonian Window and Apollonian Belt side by side. Source: [5], Figure 1.

3 Lens Sequences

3.1 Lens Sequences in Apollonian Packings

If we look at the Apollonian Window we created before, we can visualize a line going down the middle of this fractal. On this line we can see an infinite chain of circles, seemingly bounded by the two circles of curvature 2. To follow the terminology put forth by Jerzy Kocik in [1], we will refer to such the sequence of curvatures of such a chain of circles as a *lens sequence*, named as such due to the similarity of the bounding shape to a lens. There are more such chains of circles in just this packing alone, for example the on horizontal line through the centers of the circles with curvature 3, which Kocik also displays in his paper.

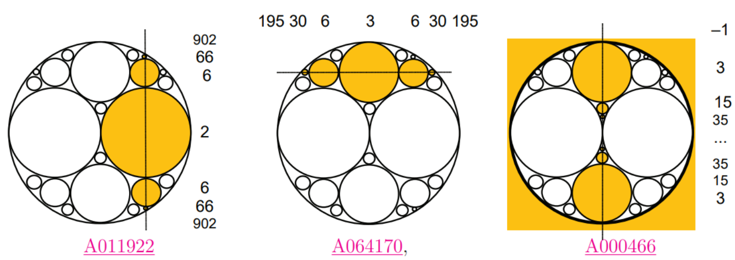

Figure 7: Lens sequences in the Apollonian Window, with their respective OEIS entries. Source: [1], Figure 3.

A sequence that Kocik did not display was the sequence that includes the curvatures 14, 6 and 39, the circles corresponding to which have been highlighted below. We will introduce a more formal concept of a lens sequence below, and there we will show that these curvatures do indeed lead to an integer lens sequence, as well as the fact that the next term in the sequence both forwards and backwards does indeed appear in the Apollonian Window.

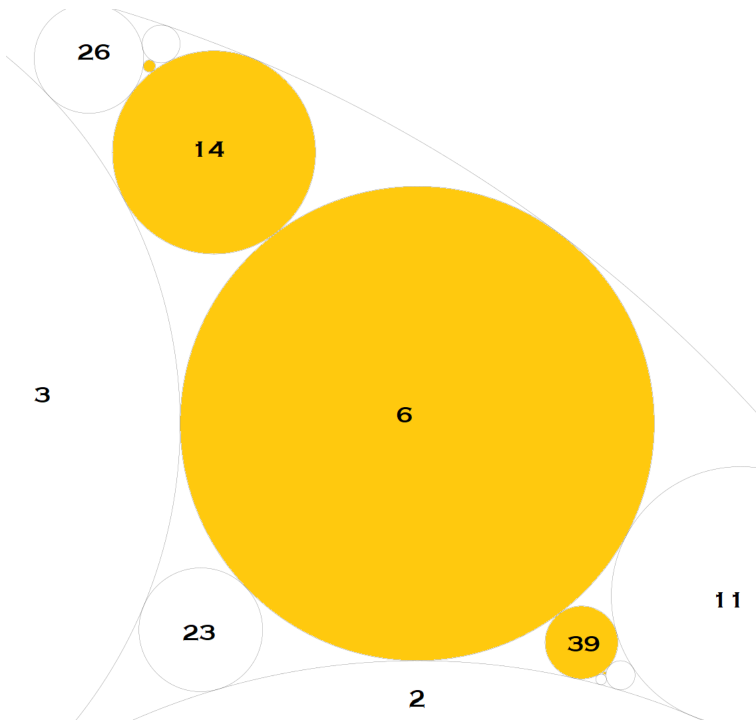

Figure 8: A diagonal lens sequence in the Apollonian Window

One can also spot lens sequences in the Apollonian Belt. One is quite obvious, it's a similar sequence to the first one we mentioned; Between any two circles of curvature 1, we see a vertical line of circles. After the formal definition of the lens sequences, we will show that it is in some way related to the vertical sequence found on the center line of the Apollonian Window.
A second lens sequence is slightly more hidden. In fact, the two horizontal lines can be considered as bounding circles, which makes the infinite chain of circles with curvature 1 into an almost trivial lens sequence. We will show later in chapter 3.3.2 that there are at least two more possible lens sequences, and likely many more than that.

<images of all these sequences >

3.2 Lens Sequences outside Apollonian Packings

Now that we've seen some examples, we will define a lens sequence an object on its own, rather than as a part of an Apollonian packing. Following the procedure found in [1], we can formulate the following definitions:

Definition. *A lens is the intersection of the interiors of two circles of equal radius and curvature, where:*

- *the interior is defined as usual if the radius and curvature are positive*

- *the interior is defined as the usual exterior if the radius and curvature are negative*

- *the interior is defined as the half-plane containing part or all of the other circle if the radius and curvature are 0*

the interior is defined as usual if the radius and curvature are positive
the interior is defined as the usual exterior if the radius and curvature are negative
the interior is defined as the half-plane containing part or all of the other circle if the radius and curvature are 0

We will call these two circles the bounding circles of the lens.

Definition. *A lens sequence is an infinite sequence of curvatures* (b_n) *of circles placed inside a lens, such that:*

- *Each circle has its center placed on the center line of the lens.*
- *Each circle is tangent to the two bounding circles of the lens.*
- *Each circle is tangent to the circles to the left and the right of itself.*

This leads to the obvious question: In which ways can two circles of equal curvature create such a lens? We will consider this on a case-wise basis depending on the sign of the curvature.

- If the bounding circles have positive curvature and
  - the circles do not overlap, or they are tangent to each other, there is no lens by lack of intersection of their interior, so there is no possible lens sequence.
  - the circles *do* overlap, then the area of overlap is where we have our lens. Kocik calls this a "converging lens".

- If the bounding circles have negative curvature and
  - the circles do not overlap, or they are tangent to each other, then the lens is the area outside of both circles. Kocik's list has this as "diverging lens (regular)", but for future text flow we will call this a regular diverging lens.
  - the circles *do* overlap, then the lens is still the area outside of both circles, but now rather than a continuous path along the center line of the lens we have 2 seperate parts. Kocik's list calls this a "diverging lens (corrupted)", but again for text flow reasons we will call this a corrupted diverging lens.
- Finally, when the circles have curvature 0, we effectively have two straight lines and there are two options:
  - The lines cross at some angle θ, which Kocik calls a "wedge"
  - The two lines are parallel, which Kocik calls a "slab"

This exhausts all possibilities for the bounding circles, so we can definitively say we have classified all types of lens sequences.

We will now state theorems and constants regarding lens sequences, all of which can be found in [1]. Page numbers will be listed, as well as theorem numbers where appropriate.

Theorem 3.1 ([1], page 3, Theorem 1)**.** *Let a, b and c be the curvatures of the initial three disks generating a lens sequence, b≠0.*
Then the sequence is determined by the following non-homogeneous three term recurrence formula:

$$b_n = \alpha b_{n-1} - b_{n-2} + \beta$$

where

$$\alpha = \frac{ab + bc + ca}{b^2} - 1 \text{ and } \beta = \frac{b^2 - ac}{b}.$$

We say that (a, b, c) is the seed *of the lens, and that this seed* generates *the lens sequence.*

Remark. *We can see that if α, β, a, b, c are all integers, then every value in the sequence will be an integer. If $\gcd(a, b, c) = 1$, we call the sequence primitive.*

We will be following the proof that Kocik provides in [1], section 5. To prove this theorem, we need the following things.

First, we need the Pedoe inner product on 2 circles C_1 and C_2, which is equal to:

$$\langle C_1, C_2 \rangle = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$$

A few important values to note: If the two circles intersect, their inproduct is equal to $\cos\theta$, where θ is their angle of intersection.
If the two circles are externally tangent, the inproduct is equal to 1. Note that this can also be viewed as an intersection angle of $0°$.
If the two circles are internally tangent, the inproduct is equal to -1. Note that this, similar to above, can be viewed as an intersection, just at an angle of $180\,\deg$ instead.
Second, we need a Descartes-like formula that Kocik found in [6], which gives us a matrix equation which functions similarly to the Descartes Circle Theorem. For any four circles C_i, with $i = 1, \cdots, 4$, we will define a matrix f with the entries $f_{ij} = \langle C_i, C_j \rangle$. Assume without loss of generality that f is invertible. We get that

$$\mathbf{b}^T F \mathbf{b} = 0$$

Where $F = f^{-1}$, and $\mathbf{b} = [b_1, b_2, b_3, b_4]^T$ is a vector made up of the curvatures of the initial circles C_i.
For a proof of this equation, we direct the reader to [6], section 3.

Now, for the actual proof of theorem 3.1, we will do this in two steps. First, we prove that a recurrence equation of the form $b_n = \alpha b_{n-1} - b_{n-2} + \beta$ exists, and then we prove that α and β can be defined in terms of the seed (a, b, c).

Proof. Let x, y be two consecutive curvatures in the lens sequence constructed with bounding circles of curvature A. Denote the inproduct of the bounding circles with K.
We organize the circles as $(C_i) = (A, A, x, y)$. This gives us, for converging and diverging lenses respectively:

$$f_{conv} = \begin{bmatrix} -1 & K & -1 & -1 \\ K & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \end{bmatrix}, f_{div} = \begin{bmatrix} -1 & K & 1 & 1 \\ K & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$F_{conv} = f_{conv}^{-1} = -\frac{1}{8} \begin{bmatrix} \frac{4}{K+1} & \frac{-4}{K+1} & 2 & 2 \\ \frac{-1}{K+1} & \frac{4}{K+1} & 2 & 2 \\ 2 & 2 & K+1 & K-3 \\ 2 & 2 & K-3 & K+1 \end{bmatrix}$$

$$F_{div} = f_{div}^{-1} = -\frac{1}{8} \begin{bmatrix} \frac{4}{K+1} & \frac{-4}{K+1} & -2 & -2 \\ \frac{-1}{K+1} & \frac{4}{K+1} & -2 & -2 \\ -2 & -2 & K+1 & K-3 \\ -2 & -2 & K-3 & K+1 \end{bmatrix}$$

We will continue this proof for the converging lens case, but as evident from their similar forms, the case for the diverging lens will go analogously, just with a different initial matrix.

We get the following matrix equation after multiplying both sides by a factor -8:

$$\begin{bmatrix} A & A & x & y \end{bmatrix} \begin{bmatrix} \frac{4}{K+1} & \frac{-4}{K+1} & 2 & 2 \\ \frac{-1}{K+1} & \frac{4}{K+1} & 2 & 2 \\ 2 & 2 & K+1 & K-3 \\ 2 & 2 & K-3 & K+1 \end{bmatrix} \begin{bmatrix} A \\ A \\ x \\ y \end{bmatrix} = 0$$

Multiplying this all out gives us the following quadratic equation:

$$(1+K)x^2 + (1+K)y^2 + 2(K-3)xy + 8Ax + 8Ay = 0$$

Treating A and x as constants lets us solve this for y using the quadratic formula:

$$y_{1,2} = \frac{(3-K)x - 4A \pm 2\sqrt{2(1-K)x^2 - 8Ax + 4A^2}}{1+K}$$

These two solutions correspond to the two curvatures for the circles that are tangent to the circle of curvature x and both of the bounding circles; This is exactly the curvatures before and after x in the lens sequence.
Adding these two solutions together gives:

$$y_1 + y_2 = \frac{6-2K}{1+K}x - \frac{8A}{1+K}$$

As these are three consecutive curvatures in the lens sequence, we can label these terms as follows:
$y_1 = b_{n-2}$
$y_2 = b_n$
$x = b_{n-1}$
$\alpha = \frac{6-2K}{1+K}$
$\beta = -\frac{8A}{1+K}$
Which gives us $b_n = \alpha b_{n-1} - b_{n-2} + \beta$, as desired.

Next, we need to prove that α and β can actually be defined in terms of a, b, c. For this, we need to show that both A and K can be defined in terms of a, b, c. To do this, take 3 consecutive circles in the chain with curvatures a, b, c, as well as one of the bounding circles of the associated lens with curvature A.
The only inproduct in this configuration we do not know is $\langle a, c\rangle$. However, because the circles in the chain have co-linear centers, we can easily calculate the distance between these two circles to be $(\frac{1}{a} + \frac{2}{b} + \frac{1}{c})$, which gives us:

$$\langle a, c\rangle = \frac{(\frac{1}{a} + \frac{2}{b} + \frac{1}{c})^2 - (\frac{1}{a})^2 - (\frac{1}{c})^2}{\frac{2}{ac}} = 2\frac{ab + bc + ca}{b^2} + 1$$

Letting $z = \frac{ab+bc+ca}{b^2}$ for brevity, and ordering the circles as (a, b, c, A), we get:

$$f = \begin{bmatrix} -1 & 1 & 2z+1 & -1 \\ 1 & -1 & 1 & -1 \\ 2z+1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

$$F = f^{-1} = \frac{1}{4} \begin{bmatrix} -\frac{1}{z+1} & 1 & \frac{1}{z+1} & -1 \\ 1 & -(z+1) & 1 & z-1 \\ \frac{1}{z+1} & 1 & -\frac{1}{z+1} & -1 \\ -1 & z-1 & -1 & -(z+1) \end{bmatrix}$$

Let $\mathbf{v} = [a, b, c, A]^T$, then solving $\mathbf{v}^T F \mathbf{v} = 0$ for A gives us:

$$A = \frac{b(ac - b^2)}{(a+b)(b+c)}$$

Now to find the value for K, we go back to our initial configuration; The two bounding disks which have the same curvature (that we now know the value of), and two consecutive circles in the chain.
With the ordering (a, b, A, A), we get:

$$f = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & -1 & K \\ -1 & -1 & K & -1 \end{bmatrix}$$

$$F = f^{-1} = -\frac{1}{8} \begin{bmatrix} K+1 & K-3 & 2 & 2 \\ K-3 & K+1 & 2 & 2 \\ 2 & 2 & \frac{4}{K+1} & -\frac{4}{K+1} \\ 2 & 2 & -\frac{4}{K+1} & \frac{4}{K+1} \end{bmatrix}$$

Let $\mathbf{v} = [a, b, A, A]^T$ and solve $\mathbf{v}^T F \mathbf{v} = 0$ for K to get:

$$K = \frac{8b^2}{(a+b)(b+c)} - 1$$

We can now substitute these values of A and K into $\alpha = \frac{6-2K}{1+K}$ and $\beta = -\frac{8A}{1+K}$ to get the desired outcome:

$$\alpha = \frac{ab+bc+ca}{b^2} - 1 \text{ and } \beta = \frac{b^2 - ac}{b}.$$

□

Theorem 3.2. *The values α and β of a given lens sequence are the same no matter which three consecutive terms are used to define it.*

Proof. We will prove this with a kind of induction.
Let α and β be defined with a triple (a, b, c). This is our base case. Then for our induction step, the next term in the sequence will be $d = \alpha c - b + \beta$.
Now we define α' and β' with our new consecutive triple (b, c, d):

$$\alpha' = \frac{bc + cd + db}{c^2} - 1 \text{ and } \beta' = \frac{c^2 - bd}{c}.$$

First, we will prove $\alpha = \alpha'$. To do this, we will substitute in our definition for d into our definition for α':

$$\begin{aligned}
\alpha' &= \frac{bc + cd + db}{c^2} - 1 \\
&= \frac{bc + c(\alpha c - b + \beta) + (\alpha c - b + \beta)b}{c^2} - 1 \\
&= \frac{bc + \alpha c^2 - bc + \beta c + \alpha bc - b^2 + \beta b}{c^2} - 1 \\
&= \frac{\alpha c^2 + \beta c + \alpha bc - b^2 + \beta b}{c^2} - 1 \\
&= \alpha + \frac{\beta(b + c) + \alpha bc - b^2}{c^2} - 1 \\
&= \alpha + \frac{\frac{b^2 - ac}{b}(b + c) + (\frac{ab + bc + ca}{b^2} - 1)bc - b^2}{c^2} - 1 \\
&= \alpha + \frac{b^2 - ac + \frac{b^2 c - ac^2}{b} + \frac{abc + bc^2 + c^2 a}{b} - bc - b^2}{c^2} - 1 \\
&= \alpha + \frac{b^2 - ac + bc - \frac{ac^2}{b} + ac + c^2 + \frac{c^2 a}{b} - bc - b^2}{c^2} - 1 \\
&= \alpha + \frac{c^2}{c^2} - 1 \\
&= \alpha
\end{aligned}$$

And for $\beta = \beta'$:

$$\begin{aligned}
\beta' &= \frac{c^2 - bd}{c} \\
&= \frac{c^2 - b(\alpha c - b + \beta)}{c} \\
&= \frac{c^2 - \alpha bc + b^2 - \beta b}{c} \\
&= \frac{c^2 - (\frac{ab+bc+ca}{b^2} - 1)bc + b^2 - b^2 + ac}{c} \\
&= \frac{c^2 - \frac{abc+bc^2+c^2a}{b} + bc + ac}{c} \\
&= \frac{c^2 - ac - c^2 - \frac{c^2a}{b} + bc + ac}{c} \\
&= \frac{c(b - \frac{ac}{b})}{c} \\
&= \frac{b^2 - ac}{b} \\
&= \beta
\end{aligned}$$

So our α and β do not depend on our initial choice of a generating triple. □

Remark. *A few more constants of lens functions:*

- $$\cos\phi = \frac{6-\alpha}{2+\alpha},$$

  where ϕ is the angle of intersection between the circles (if they intersect). For non-intersecting circles we will see that $\alpha < 2$, meaning there is no valid value for ϕ to be gleaned from this equation.

- $$L = -\frac{2\sqrt{\alpha^2 - 4}}{\beta},$$

  where L is the total length of the lens.

- $$R = \frac{\alpha + 2}{\beta},$$

  where R is the radius of the bounding circles of the lens. All three of these constants can be found in ([1], page 3), where the formulas are also proven to be correct.

All three of these constants can be found in ([1], page 3), where the formulas are also proven to be correct.

3.3 Returning to previous sequences

Kocik has shown in [1] that there are lens sequences to be found in the Apollonian Window, but all of his examples were strictly horizontal or vertical lines. This might lead to the conjecture that all the lens sequences in Apollonian packings must be either positioned horizontally or vertically. However, we have found the start of a sequence that runs at an angle. We will consider all the seeds we have mentioned before, and show some stronger evidence that supports our claim that these form lens sequences.

3.3.1 The (14,6,39) sequence

As stated above, the circles with curvature (14, 6, 39) appeared to be co-linear, so we will plug these into our definitions for α and β:

$$\alpha = \frac{14 \cdot 6 + 6 \cdot 39 + 39 \cdot 14}{6^2} - 1 = 23 \text{ and } \beta = \frac{6^2 - 14 \cdot 39}{6} = -85.$$

These are both integers, and so by our recurrence equation every term in the sequence will be integer. Specifically, the next curvature in the sequence will be $23 \cdot 39 - 6 - 85 = 806$ and the previous one (to the left of 14) will be $23 \cdot 14 - 6 - 85 = 231$. Through solving a series of Descartes equations, we can show that these values do actually appear in the Apollonian Window in the places they are supposed to:
To show 806 appears:

$$\begin{aligned}
39 + 11 + 2 \pm 2\sqrt{39 \cdot 11 + 11 \cdot 2 + 2 \cdot 39} &= \{6, 98\} \\
39 + 98 + 2 \pm 2\sqrt{39 \cdot 98 + 98 \cdot 2 + 2 \cdot 39} &= \{11, 267\} \\
39 + 98 + 267 \pm 2\sqrt{39 \cdot 98 + 98 \cdot 267 + 267 \cdot 39} &= \{2, 806\}
\end{aligned}$$

And to show 231 appears on the other side:

$$\begin{aligned}
14 + 3 - 1 \pm 2\sqrt{14 \cdot 3 + 14 \cdot (-1) + 3 \cdot (-1)} &= \{6, 26\} \\
14 + 26 - 1 \pm 2\sqrt{14 \cdot 26 + 14 \cdot (-1) + 26 \cdot (-1)} &= \{3, 75\} \\
14 + 26 + 75 \pm 2\sqrt{14 \cdot 26 + 26 \cdot 75 + 75 \cdot 14} &= \{-1, 231\}
\end{aligned}$$

Now to show that the 5 circles with curvatures $(231, 14, 6, 39, 806)$ actually have co-linear centers, we have to know their centers' position. Luckily the complex Descartes theorem gives us a way to do exactly that.
Rather than type out the whole computation, which would look almost identical to the one above, we will just give a list of circle curvatures and centers in the

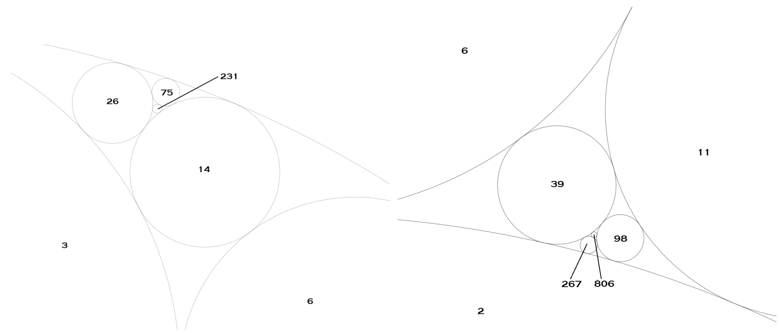

Figure 9: The location of the next terms of the sequence

form $\{k\} \leftrightarrow (x, y)$, where k is the curvature and (x, y) is the position of the center. All of these are obtained from setting the large circle of curvature -1 to be centered at the origin, which puts the two circles of curvature 2 at $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, 0)$ respectively, and using the complex Descartes theorem from that point. We also need to transform the Cartesian co-ordinates into complex numbers, this will simply be done with $(x, y) \equiv x + yi$. This list will contain the 5 circles we want to look at, as well as every other circle required in their construction.

$$
\begin{array}{lll}
\{-1\} \leftrightarrow (0,0) & \{14\} \leftrightarrow (\frac{5}{14}, \frac{6}{7}) & \{39\} \leftrightarrow (\frac{8}{13}, \frac{20}{39}) \\
\{2\} \leftrightarrow (-\frac{1}{2}, 0) & \{26\} \leftrightarrow (\frac{7}{26}, \frac{12}{13}) & \{98\} \leftrightarrow (\frac{9}{14}, \frac{24}{49}) \\
\{2\} \leftrightarrow (\frac{1}{2}, 0) & \{75\} \leftrightarrow (\frac{8}{25}, \frac{14}{15}) & \{267\} \leftrightarrow (\frac{56}{89}, \frac{130}{267}) \\
\{3\} \leftrightarrow (0, \frac{2}{3}) & \{231\} \leftrightarrow (\frac{24}{77}, \frac{212}{231}) & \{806\} \leftrightarrow (\frac{509}{806}, \frac{198}{403}) \\
\{6\} \leftrightarrow (\frac{1}{2}, \frac{2}{3}) & \{11\} \leftrightarrow (\frac{8}{11}, \frac{6}{11}) &
\end{array}
$$

If we now consider the co-ordinates for $\{231\}, \{14\}, \{6\}, \{39\}$ and $\{806\}$, after calculating a line equation from any two of these and testing the rest, we find that they all lie on the line $4x + 3y = 4$.

While this is not proof that the lens sequence continues on past this point (more on this in the conclusion of this thesis), this shows that these circles do in fact show up with both the right size to form the core of a lens sequence, as well as being co-linear. This is enough for us to state that this is likely to be another lens sequence and thus that non-horizontal and non-vertical lens sequences can in fact show up in the Apollonian Window.

Proposition 3.3. *The circles with curvatures 231, 14, 6, 39 and 806 in the*

Apollonian Window are co-linear.

Conjecture 3.4. *The lens sequence with seed (14,6,39) exists in full in the Apollonian Window.*

3.3.2 Sequences in the Apollonian Belt

We will also quickly look at the sequences found in the Apollonian Belt. As stated before, the two horizontal lines can be bounding circles, which together with the infinite sequence of circles with curvature 1 make a slab-type lens sequence, with $\alpha = 2$ and $\beta = 0$. The more interesting sequence is the one running straight down between two circles of curvature 1. If we take the first 3 non-zero entries in this sequence, we find:

$$1 + 1 + 0 \pm 2\sqrt{1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0} = \{0, 4\}$$
$$1 + 1 + 4 \pm 2\sqrt{1 \cdot 1 + 1 \cdot 4 + 1 \cdot 4} = \{0, 12\}$$
$$1 + 1 + 12 \pm 2\sqrt{1 \cdot 1 + 1 \cdot 12 + 1 \cdot 12} = \{4, 24\}$$

This gives us the following values for α and β:

$$\alpha = \frac{4 \cdot 12 + 12 \cdot 24 + 24 \cdot 4}{12^2} - 1 = 2 \text{ and } \beta = \frac{12^2 - 24 \cdot 4}{12} = 4.$$

Meanwhile if we look at the center vertical line in the Apollonian Window with the first three terms being (-1, 3, 15), we find:

$$\alpha = \frac{-1 \cdot 3 + (-1) \cdot 15 + 3 \cdot 15}{3^2} - 1 = 2 \text{ and } \beta = \frac{3^2 - (-1) \cdot 15}{3} = 8.$$

However, if we scale a whole sequence by a certain factor γ, α would stay the same because both the numerator and the denominator of the fraction would be scaled by a factor of γ^2, which cancels out.
However, β would change, as the numerator would be scaled by a factor γ^2 but the denominator would only be scaled by a factor γ.

Thus, if we scale the sequence found in the Apollonian Belt by a factor 2, we would get two sequences where:
- The values for α and β are the same (2 and 8 respectively)
- The curvatures of the bounding circles are the same (2 in both cases)
- The type of lens sequence is the same (a regular diverging lens)

Yet the two sequences do not have the same terms: One has terms of the form $4n^2-1$ ([1], page 6), and the other one has terms of the form $4 \cdot n(n-1) = 4n^2 - 4n$ (A033996 from the OEIS). Not to mention, the sequence Kocik found is primitive whereas the other sequence isn't primitive (though it can be made primitive by scaling down by a factor $\frac{1}{8}$). While they are similar in setup, the two sequences are not the same; This shows that the bounding circles and values of α and β alone are not enough to uniquely determine a lens sequence.

We also promised a third and fourth sequence, which show the possibility of even more. If we consider the four circles of curvature 4 around a circle of curvature 1, we can draw a line connecting 2 diagonally opposite circles. By the symmetry, this line clearly passes through the center of the large circle. This would give us a lens sequence with seed (4,1,4), with $\alpha = 23$ and $\beta = -15$. This would mean the next circle on either side has curvature $23 \cdot 4 - 1 - 15 = 76$, and indeed:

$$4 + 9 + 0 \pm 2\sqrt{4 \cdot 9 + 4 \cdot 0 + 9 \cdot 0} = \{1, 25\}$$
$$4 + 9 + 25 \pm 2\sqrt{4 \cdot 9 + 9 \cdot 25 + 4 \cdot 25} = \{0, 76\}$$

Calculating the centers for this set was slightly different, as we can't use the complex Descartes equation on a configuration involving a straight line; After all, there's no meaningful answer as to the 'center' of that line. Instead, any circle tangent to the line will be calculated differently. We know the y-coordinate of any such circle is simply $1 - r$ or $-1 + r$ where r is the radius, depending on whether it is tangent to the top or bottom line. At that point we only have one unknown, which we can solve by calculating the length between its center and the center of a circle we already know, and then just solving for our unknown x-coordinate.
This gives us the following curvature-center pairs:

$$\{-1\} \leftrightarrow (0,0) \qquad \{9\} \leftrightarrow (-\frac{4}{3}, \frac{8}{9}) \qquad \{76\} \leftrightarrow (-\frac{23}{19}, \frac{69}{76})$$
$$\{4\} \leftrightarrow (-1, \frac{3}{4}) \qquad \{25\} \leftrightarrow (-\frac{6}{5}, \frac{24}{25})$$

We find that the circles $\{-1\}, \{4\}$ and $\{76\}$ all lie on the line $3x+4y = 0$, which again makes it likely to form a full lens sequence.

Proposition 3.5. *There are circles with curvature 1, 4, and 76 that are colinear in the Apollonian Belt.*

Conjecture 3.6. *The lens sequence with seed (4,1,4) exists in full in the Apollonian Belt.*

If we rotate the center line a little to cross two circles of curvature 9 instead, we get (9,1,9) as the seed which gives us $\alpha = 98$ and $\beta = -80$. This would result in the next circle having curvature 801, and after the following steps we see that a circle of curvature 801 does indeed touch the circle of curvature 9:

$$4 + 9 + 0 \pm 2\sqrt{4 \cdot 9 + 4 \cdot 0 + 9 \cdot 0} = \{1, 25\}$$
$$0 + 9 + 25 \pm 2\sqrt{9 \cdot 25 + 9 \cdot 0 + 25 \cdot 0} = \{4, 64\}$$
$$0 + 9 + 64 \pm 2\sqrt{9 \cdot 64 + 9 \cdot 0 + 64 \cdot 0} = \{25, 121\}$$
$$121 + 9 + 64 \pm 2\sqrt{9 \cdot 64 + 9 \cdot 121 + 64 \cdot 121} = \{0, 388\}$$
$$388 + 9 + 64 \pm 2\sqrt{9 \cdot 64 + 9 \cdot 388 + 64 \cdot 388} = \{121, 801\}$$

Same thing as above, these are the corresponding curvature-center pairs:

$$\{-1\} \leftrightarrow (0,0) \qquad \{25\} \leftrightarrow (-\frac{4}{5}, \frac{24}{25}) \qquad \{388\} \leftrightarrow (-\frac{71}{97}, \frac{381}{388})$$
$$\{4\} \leftrightarrow (-1, \frac{3}{4}) \qquad \{64\} \leftrightarrow (-\frac{3}{4}, \frac{63}{64}) \qquad \{801\} \leftrightarrow (-\frac{196}{267}, \frac{784}{801})$$
$$\{9\} \leftrightarrow (-\frac{2}{3}, \frac{8}{9}) \qquad \{121\} \leftrightarrow (-\frac{8}{11}, \frac{120}{121})$$

Here we find the line $4x + 3y = 0$, and $\{-1\}, \{9\}$ and $\{801\}$ all lie on that line. Again, this lends credence to the claim that $(9, 1, 9)$ is the seed for another lens sequence.

These last two sequences put together lead to the assumption that any of these symmetric lines through the big circle results in a lens sequence, which would then mean that there are infinitely many lens sequences in the Apollonian Belt just using the large circle of curvature 1.

Proposition 3.7. *There are circles with curvature 1, 9, and 801 that are co-linear in the Apollonian Belt.*

Conjecture 3.8. *The lens sequence with seed (9,1,9) exists in full in the Apollonian Belt.*

Conjecture 3.9. *Any line through a circle of curvature 1 and another circle tangent to it in the Apollonian Belt contains a lens sequence.*

4 Underground Sequences

Perhaps the most interesting thing Kocik found about these lens sequences is the fact that they lend themselves to a quite unique factorization, which he dubbed the "underground sequence". In effect, he found that each entry in a lens sequence can be split into 2 factors a and b such that two adjacent entries share a factor. This is such an unexpected finding that we will show much of what Kocik has found, to show off how wondrous this find truly is.
For example, one of the first sequences we talked about, the one with seed (14,6,39), we looked at 5 total terms: 231, 14, 6, 39, 806. To accentuate the beauty of this find better, we will extend the sequence on both sides by 2 more terms using the recurrence equation, giving us the following sequence:

$$119606,\ 5214,\ 231,\ 14,\ 6,\ 39,\ 806,\ 18414,\ 422631$$

Now if we factor each term:

$$757 \cdot 158,\ 158 \cdot 33,\ 33 \cdot 7,\ 7 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 13,\ 13 \cdot 62,\ 62 \cdot 297,\ 297 \cdot 1423$$

We see that another sequence emerges:

$$757,\ 158,\ 33,\ 7,\ 2,\ 3,\ 13,\ 62,\ 297\ 1423$$

This is what Kocik calls the underground sequence.

Theorem 4.1 (Factorization Theorem [1], theorem 22)**.** *For every integer lens sequence (b_n) there exists an integer sequence (f_n) like shown above such that $b_n = f_{n-1}f_n$. Furthermore, if the lens sequence is primitive, the factorization is unique up to signs, and we have $|f_n| = \gcd(b_n, b_{n+1})$.*

Proof. ([1], page 13-14)
To prove this, we first need a simple fact about underground sequences that Kocik shows in ([1], Proposition 21): $f_{n+2} + f_{n-2} = \alpha f_n$, where α is defined from the respective lens sequence (b_n) as shown before. He shows this using a secondary equation for α: $\alpha = \frac{b_{n-1}}{b_n} + \frac{b_{n+2}}{b_{n+1}}$, and then substituting in the factored versions ($b_n = f_{n-1}f_n$).

Let (b_n) be a primitive lens sequence. We will handle the case for if (b_n) is not primitive second.

Consider a triple (a, b, c) in the sequence (b_n).
Define $f_1 = \gcd(a, b)$. Then by using $b_n = f_{n-1}f_n$, we can define $f_0 = \frac{a}{f_1} = \frac{a}{\gcd(a,b)}$ and $f_2 = \frac{b}{f_1} = \frac{b}{\gcd(a,b)}$. All three of these are very clearly integers.
Now consider $f_3 = \frac{c}{f_2} = \frac{c\gcd(a,b)}{b}$. To show this is an integer, we can start with the formula for β, which is $\beta = \frac{b^2-ac}{b}$. We know that for a primitive lens sequence, β is an integer, which trivially leads to $\frac{ac}{b}$ being an integer as well.
Now we only have to replace a by $\gcd(a, b)$ to show integrality for f_3. Realize

now that performing this replacement only removes a factor $a^* = \frac{a}{\gcd(a,b)}$ from the numerator, but this factor has no prime factors in common with b. This means that $a^* | \frac{ac}{b}$, and so $\frac{ac}{b(a^*)} = \frac{c \gcd(a,b)}{b} = f_3$ must be an integer as well.

Now we can use the fact that $f_{n+2} + f_{n-2} = \alpha f_n$ alongside the integrality of the four terms shown above, which gives us that $f_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

It's quite simple to make this process work for non-primitive integer lens sequences (b_n).
First, convert the lens sequence to a primitive sequence by factoring out the common factor k, leaving us with a new sequence (b_n^*).
Next, perform the process above on this new sequence, giving us an underground sequence (f_n^*). We can now take every term with an even index in this underground sequence and multiply it by k to give us the sequence (f_n).
Now we can see that $b_n = k \cdot b_n^* = k \cdot f_n^* f_{n-1}^*$
For any n, exactly one of these two terms has an even index, so we can absorb the factor k into the product to give us
$b_n = k \cdot f_n^* f_{n-1}^* = f_n f_{n-1}$.
Which shows that (f_n) is an integer factorization of (b_n), just as we wanted.
To show the uniqueness of (f_n) for a primitive lens sequence (b_n), assume instead that there are two different integer factorizations with the respective initial terms (f_0, f_1, f_2, f_3) and (g_0, g_1, g_2, g_3) for a primitive sequence (b_n). We know that because g_0 and f_0 are both integers, that $\frac{g_0}{f_0}$ can be written as a rational number in simplest form $\frac{p}{q}$. Because $f_0 \neq g_0$, at least one of p and q is not 1. If $q = 1$, we can just flip which sequence is (f_n) and which is (g_n) such that $q \neq 1$, so we can assume without loss of generality that $q \neq 1$.

Now, because $g_i g_{i+1} = b_{i+1} = f_i f_{i+1}$, we can rewrite the initial terms of (g_n) as follows:

$$(g_0, g_1, g_2, g_3) = (\frac{p}{q} f_0, \frac{q}{p} f_1, \frac{p}{q} f_2, \frac{q}{p} f_3)$$

Because these have to be integers and because $\gcd(p, q) = 1$, we see that $q | f_0$ and $q | f_2$. However, this means that $q | f_0 f_1 = b_1$, $q | f_1 f_2 = b_2$, and $q | f_2 f_3 = b_3$, proving that (b_n) is not primitive; So by contradiction, a primitive lens sequence (b_n) must have a unique integer factorization sequence. □

By Kocik's theorem on the structure of underground sequences ([1], Theorem 24) we see that just like lens sequences, the underground sequence follows a three-term linear recurrence equation. However unlike the lens sequence, there are 2 recurrence equations, and which one you apply depends on the parity of n:
For $k, s \in \mathbb{Z}$, we can define a sequence (f_n) by

$$f_n = \begin{cases} k f_{n-1} - f_{n-2}, & \text{for } n \text{ even} \\ s f_{n-1} - f_{n-2}, & \text{for } n \text{ odd} \end{cases}$$

If we then define $b_n = f_{n-1}f_n$, we get a new sequence (b_n). This then forms a lens sequence, with

$$\alpha = ks - 2 \text{ and } \beta = kf_1^2 + sf_0^2 - ksf_0f_1$$

which Kocik goes on to prove in [1], pages 15 and 16.

One of the important uses of the underground sequence is being able to label and define lens sequences. In [1], definition 28, Kocik defines the *symbol* of a lens sequence as follows:
"A *symbol* of a lens sequence is the quadruple

$${}^{s}(p,q)^{k}$$

which defines the underground sequence (f_i) with $f_0 = p$, $f_1 = q$, and with constants s and k as in (21)"
where (21) was where Kocik defined the recurrence equations for f_n.

This is only a unique symbol for a given lens sequence if we somehow remove the ambiguity within the sequence itself, after all ${}^{s}(f_0, f_1)^k$ would be just as valid a symbol as ${}^{s}(f_2, f_3)^k$ for example. Kocik proposes we remove this ambiguity by requiring that f_0 has the smallest absolute value in the sequence and that $|f_{-1}| > f_0 \leq f_1$.

Kocik then goes on to show that the lens sequence defined by this symbol has the seed $((sp - q)p, pq, q(kq - p))$, which is trivial to show using the fact that $b_n = f_{n-1}f_n$, calculating f_{-1} and f_2, and then using those four values in the underground sequence to calculate out b_0, b_1, b_2.
He also shows that a primitive sequence with seed (a, b, c) has the following symbol:

$${}^{\frac{a+b}{p^2}}[\gcd(a,b), \gcd(b,c)]^{\frac{b+c}{q^2}}$$

Furthermore, he shows the following theorem and proof:

Theorem 4.2. *Given a lens sequence (b_n), it is only primitive if and only if its associated symbol ${}^{s}(p,q)^{k}$ fulfills the following criterion:*

$$\gcd(p,q) = \gcd(p,k) = \gcd(q,s) = 1$$

Proof. We know that $\gcd(b_0, b_1, b_2) = 1$ must hold for a primitive sequence. This implies $\gcd(p,q) = 1$, $\gcd(p,k) = 1$ and $\gcd(q,s) = 1$; After all, if any of these were not equal to 1, then all three terms (b_0, b_1, b_2) could be divided by that factor which would make (b_n) not primitive. Conversely, if all three of these gcds are 1, then there is no common factor among the three terms, so the sequence must be primitive. ∎

So in the end, we see that not only does the underground sequence give us another view into the structure of primitive lens sequences, it even provides us with a great way to categorize and label lens sequences.

5 Conclusion

Lens sequences make for a very interesting subject. While currently there has not been any practical use for them, they could lead to better understanding of the structure in Apollonian packings. Besides that, they could help get more younger people interested in mathematics, because fitting circles against each other is something most kids with an interest in mathematics will have played with before.

There's plenty to look into for future papers too though. We haven't been able to find a way to prove that full lens sequences show up in an Apollonian packing. One of the main issues is that it's not a set number of steps to find the next circle in the packing; Sometimes it takes 3 steps with Descartes' Theorem, sometimes it takes 5. However, there might be an opportunity to find a pattern using computer programs to generate many steps.
Once it's shown that all the entries in a lens sequence appear in the Apollonian packing, it should be easier to show that they are all co-linear using the complex Descartes' Theorem.

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