Monodromy Properties of A-hypergeometric Functions

Thesis Committee:
Prof. dr. E.P. van den Ban, *Universiteit Utrecht.*
Dr. C. Berkesch, *University of Minnesota.*
Prof. dr. H.T. Koelink, *Radboud Universiteit.*
Prof. dr. S. Tanabe, *Galatasaray University.*
Prof. dr. W. Zudilin, *Radboud Universiteit.*

ISBN: 978-90-393-7365-1

Monodromy Properties of A-hypergeometric Functions

Monodromie eigenschappen van A-hypergeometrische functies

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de
Universiteit Utrecht
op gezag van de
rector magnificus, prof.dr. H.R.B.M. Kummeling,
ingevolge het besluit van het college voor promoties
in het openbaar te verdedigen op

maandag 8 februari 2021 des middags te 2.30 uur

door

Carlos Albertus Verschoor

geboren op 27 december 1990
te Groningen

Promotor:
Prof. dr. F. Beukers

Dit proefschrift werd (mede) mogelijk gemaakt met financiële steun van NWO

Contents

Chapter 1

Introduction

1.1 Hypergeometric Functions

Hypergeometric functions appear as generalizations of classical functions. Euler defined it as the following power series

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{1.1}$$

Here $(a)_n = a \cdot (a+1) \cdot \ldots \cdot (a+n-1)$ denoted the Pochhammer symbol.

It was noted by Euler that if we specialize the parameters a, b, c, we can obtain elementary functions. For example, taking $a = 1, b = 1, c = 1$ in (1.1) gives us the standard geometric series $\sum_{n=0}^{\infty} z^n$.

Taking $a = 1, b = 1, c = 2$ we see that the power series (1.1) is equal to that of $-\ln(1-z)/z$. Taking other specializations, we can find many more classical functions including $\sin, \arcsin, \arctan, \ldots$.

Gauss later published an article [Gau13] which studies the hypergeometric power series in more detail. This is where he introduced the rank 2 differential equation

$$ z(1-z)F'' + (c - (a+b+1)z)F' + abF = 0. \tag{1.2}$$

The hypergeometric function (1.1) is a solution to this differential equation. Due to the contributions of Gauss to hypergeometric functions, the function (1.1) is often known as Gauss' hypergeometric function.

Since then, mathematicians have studied several analogues and generalizations of hypergeometric functions. These generalizations include the generalized hypergeometric functions of the type ${}_nF_{n-1}$, Appell's hypergeometric functions F_1, F_2, F_3, F_4 [App80, App82], Horn's hypergeometric functions $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6, H_7$ [Hor89, Hor31] and Lauricella's hypergeometric functions F_A, F_B, F_C, F_D [Lau93].

As an example, the generalized hypergeometric function in one variable takes the form

$$ {}_nF_{n-1}\left(\begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1} \end{array} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_n)_k}{(b_1)_k (b_2)_k \ldots (b_{n-1})_k k!} z^k. \tag{1.3} $$

Appell's hypergeometric functions and Horn's hypergeometric functions are two variable generalizations of (1.1). For example

$$ F_4\left(\begin{array}{c} \alpha, \ \beta \\ \gamma, \ \gamma' \end{array} \middle| \, x, y \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n, $$

The Lauricella F_C is a multivariable analogue of Appell's F_4 hypergeometric function

$$ F_C\left(\begin{array}{c} a, \ b \\ c_1, \ldots, c_n \end{array} \middle| \, x_1, \ldots, x_n \right) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_1+\ldots+m_n} x_1^{m_1} \ldots x_n^{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n} m_1! \ldots m_n!}. $$

Notice that taking $n = 1$ we simply obtain Gauss' hypergeometric function, and taking $n = 2$ we get Appell's F_4.

All of these examples can be expressed in terms of A-hypergeometric functions as introduced by Gelfand, Kapranov and Zelevinsky in a series of papers [GGZ87, GKZ88, GKZ89, GKZ90]. This leads to a more general framework for studying hypergeometric functions.

1.2 Monodromy

It was known to Gauss that hypergeometric functions are multivalued. This problem can now be described in terms of monodromy. Monodromy describes the behaviour of solutions to linear differential equations under analytic continuation. In 1857, Riemann was the first to fully determine the monodromy group for Gauss' hypergeometric functions [Rie57]. Over the years, many more monodromy groups have been determined for generalizations of hypergeometric functions. The monodromy of the one-variable case ${}_nF_{n-1}$ can be found through Barnes integrals as defined in [Bar07]. But an easier method to find its monodromy is through Levelt's theorem in [BH89]; Monodromy of the multivariable Appell's F_1 and its generalization Lauricella's F_D are found in [Pic81, Ter83, DM86, Sas77]; Appell's F_2 and its generalization Lauricella's F_A in [Kat00, MY14]; Appell's F_4 and its generalization Lauricella's F_C in [Tak80, Kan81, HU08, Got14]. At the time of writing, no general approach to finding the full monodromy group of an A-hypergeometric system is known.

In [Beu16], Beukers gives a method of finding a subgroup of the full monodromy group of A-hypergeometric functions using Mellin-Barnes integral solutions of the associated differential system. This approach, however, only works under very restrictive conditions. These conditions are necessary to find a Mellin-Barnes basis of solutions. It is with respect to the basis of Mellin-Barnes solutions that we can find monodromy groups. In Chapter 2, we will introduce A-hypergeometric functions and Beukers' method to find monodromy groups.

The goal of Chapter 3 is to give an explicit construction of an invariant Hermitian form over such groups. In particular, these groups are monodromy groups with respect to a Mellin-Barnes basis constructed through Beukers' method. The main theorem of Chapter 3, Theorem 3.1.1, gives an explicit construction of this Hermitian form.

The construction of this Hermitian matrix does not explicitly use Mellin-Barnes integrals. This makes it possible to extend Beukers' construction without assuming the existence of Mellin-Barnes bases. Thus we want to construct a matrix group that has an invariant Hermitian form following the construction of Theorem 3.1.1. We call this invariant Hermitian form the virtual Hermitian form. It is often the case that the constructed matrix group, which we call the virtual monodromy group, is a subgroup of the full monodromy group. In Chapter 4 we give conditions and algorithms to determine whether such a virtual extension exists.

Chapter 5 is concerned with the signature of this Hermitian form. In Beukers [Beu10] a combinatoric criterion is given to determine whether all solutions of an A-hypergeometric system are algebraic. This criterion is based on the properties of apexpoints. In Chapter 5 we connect this criterion to the definiteness of the Hermitian form. The main theorem of this chapter, Theorem 5.2.1, is purely combinatoric and only involves apexpoints. Corollary 5.4.2 then gives a combinatoric criterion for the definiteness of the Hermitian form.

1.3 Factorizations

In Chapter 6 we turn to relations between classical hypergeometric functions. In 1933, Bailey [Bai33] published an identity where Appell's

F_4 factors into two Gauss hypergeometric function.

$$F_4\left(\begin{array}{c}a,\ b\\ c,\ a+b-c+1\end{array}\middle|\, x(1-y), y(1-x)\right)$$
$$= {}_2F_1\left(\begin{array}{c}a,\ b\\ c\end{array}\middle|\, x\right){}_2F_1\left(\begin{array}{c}a,\ b\\ a+b-c+1\end{array}\middle|\, y\right).$$

A similar identity is known, where Appell's F_2 decomposes into two Gauss hypergeometric functions [Zud19, Vid09]. It reads

$$F_2\left(\begin{array}{c}a+b-\frac{1}{2},\ a,\ b\\ 2a,\ 2b\end{array}\middle|\, \frac{4u(1-u)(1-2v)}{(1-2uv)^2}, \frac{4v(1-v)(1-2u)}{(1-2uv)^2}\right)$$
$$= (1-2uv)^{-1+2a+2b}{}_2F_1\left(\begin{array}{c}a+b-\frac{1}{2},\ a\\ 2a\end{array}\middle|\, 4u(1-u)\right){}_2F_1\left(\begin{array}{c}a+b-\frac{1}{2},\ b\\ 2b\end{array}\middle|\, 4v(1-v)\right).$$

Thus we can ask ourselves the following question. Given a solution H of a rank 4 hypergeometric system of Horn type, can we find a specialization of parameters and variables such that H factors into a product of two Gauss hypergeometric functions? We show that if the monodromy group of the hypergeometric system is a subgroup of the group of orthogonal similitudes $GO(4)$, then a Bailey-type factorization exists. We will give an algorithm that tells us which specializations of the parameters have this property. And for classical hypergeometric systems of rank 4, we give a table of specializations for which the monodromy group is contained in $GO(4)$. After identifying these specializations, we will construct Bailey-type factorizations for Horn's hypergeometric functions H_1, H_4 and H_5.

The first Horn function that we address is H_4. We show that a two-dimensional specialization of the parameters gives the following factorization

$$H_4\left(\begin{array}{c}q_0,\ q_1\\ 1+q_0-q_1,\ 2q_1\end{array}\middle|\, \frac{\left(s^2-1\right)\left(t^2-1\right)}{4\,(st-1)^2}, \frac{2st}{st-1}\right)$$
$$= (1-st)^{q_0}\,{}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_0-q_1+1\end{array}\middle|\, 1-s^2\right){}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_1+\frac{1}{2}\end{array}\middle|\, t^2\right).$$

This is an identity as powerseries in $(s,t)=(1,0)$. We show that a one-dimensional specialization of the parameters on the Horn function H_1 gives the following Bailey-type factorization

$$H_1\left(\begin{array}{c}q_0-\frac{1}{2},\ q_0,\ \frac{1}{2}\\ 2q_0\end{array}\middle|\, \phi(-1+2uv, 1+2v)\right) =$$

$$
\begin{aligned}
&(1+v-uv-2uv^2)^{2q_0}\left(\frac{1-u-2uv}{(1+v)(1+2v+2v^2)}\right)^{2q_0-1}\\
&\quad\times {}_2F_1\left(\left.\begin{matrix} 1-q_0,\ \frac{3}{4}-q_0 \\ 2-2q_0 \end{matrix}\,\right|\, -8v(1+v)(1+2v+2v^2)\right)\\
&\quad\times {}_2F_1\left(\left.\begin{matrix} q_0-\frac{1}{4},\ q_0 \\ 2q_0 \end{matrix}\,\right|\, 8uv(1-uv)(1-2uv+2u^2v^2)\right),
\end{aligned}
$$

where

$$
\phi(s,t)=\left(\frac{(s^4-1)(t^4-1)}{(s^2t^2-1)^2},\left(\frac{st+1}{st-1}\right)^2\right).
$$

This is an identity in powerseries in $(u,v)=(0,0)$.

And finally, we show that under a one-dimensional specialization of the parameters, the Horn function H_5 factors as

$$
H_5\left(\left.\begin{matrix} q,\ q-\frac{1}{2} \\ 2q \end{matrix}\,\right|\, \frac{x(y-1)^2}{4(3xy-1)^2},\frac{4(xy^2+1)(x+1)y}{(3xy-1)(y-1)^2}\right)=
$$

$$
(1-3xy)^q(1-y)^{2q-1}{}_2F_1\left(\left.\begin{matrix} \frac{3}{2}q-\frac{1}{2},\ \frac{3}{2}q \\ q+\frac{1}{2} \end{matrix}\,\right|\, -xy^2\right){}_2F_1\left(\left.\begin{matrix} \frac{1}{2}q,\ \frac{1}{2}q+\frac{1}{2} \\ \frac{3}{2}-q \end{matrix}\,\right|\, -x\right).
$$

Chapter **2**

Preliminaries

2.1 The A-hypergeometric system

In this section, we want to define A-hypergeometric functions. This chapter will not give a thorough introduction into A-hypergeometric functions. We will only discuss the aspects of A-hypergeometric functions that are needed in the context of this thesis. For a more thorough introduction into A-hypergeometric functions we refer to the notes from Beukers [Beu11], the notes from Stienstra [Sti07] or the original papers from Gelfand, Kapranov and Zelevinsky [GGZ87, GKZ88, GKZ89, GKZ90]. To motivate the definition of an A-hypergeometric function, we shall first start with an example.

Recall the classical Gauss hypergeometric function.

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n. \tag{2.1} $$

Here $(a)_n$ denotes the Pochhammer symbol, defined as

$$ (a)_n = a(a+1)\dots(a+n-1). $$

In terms of the Γ-function the Pochhammer symbol can be written as $(a)_n = \Gamma(a+n)/\Gamma(a)$. For reasons of convenience, we will assume $a, b, c \notin \mathbb{Z}$. We can rewrite (2.1) as

$$ {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} x^n. \tag{2.2} $$

Recall Euler's reflection formula

$$ \Gamma(z)\Gamma(1-z) = \sin(\pi z). $$

Using Euler's reflection formula we can now move the factor $\Gamma(a+n)$ and $\Gamma(b+n)$ from the numerator to the denominator in (2.2). Because $a, b \notin \mathbb{Z}$ this will not cause any problems. So we have the following proportionality relation

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| \, x \right) \propto \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1-a-n)\Gamma(1-b-n)\Gamma(c+n)\Gamma(n+1)}. $$

Here proportionality means that the left-hand side and right-hand side of the equation differ by a constant, non-zero factor. Now replace x by $\frac{z_3 z_4}{z_1 z_2}$. We get

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \middle| \, \frac{z_3 z_4}{z_1 z_2} \right) \propto \sum_{n=0}^{\infty} \frac{z_1^{-n}}{\Gamma(1-a-n)} \cdot \frac{z_2^{-n}}{\Gamma(1-b-n)} \cdot \frac{z_3^{n}}{\Gamma(c+n)} \cdot \frac{z_4^{n}}{\Gamma(n+1)}. $$

Now multiply the right hand side by $z_1^{-a} z_2^{-b} z_3^{c-1}$. We get

$$ \sum_{n=0}^{\infty} \frac{z_1^{-a-n}}{\Gamma(1-a-n)} \cdot \frac{z_2^{-b-n}}{\Gamma(1-b-n)} \cdot \frac{z_3^{c-1+n}}{\Gamma(c+n)} \cdot \frac{z_4^{n}}{\Gamma(n+1)}. \tag{2.3} $$

Consider the row vector $\boldsymbol{\gamma} = (-a, -b, c-1, 0)$ and consider the row vector $\boldsymbol{l} = (-1, -1, 1, 1)$ then we can rewrite (2.3) as

$$ \sum_{n=0}^{\infty} \prod_{j=1}^{4} \frac{z_j^{-\gamma_j + l_j n}}{\Gamma(1-\gamma_j + l_j n)}. $$

Since $\frac{1}{\Gamma(n+1)}$ is zero when $n \in \mathbb{Z}_{\leq -1}$, we can extend this sum to a summation over the integer lattice L generated by the row vector $(-1, -1, 1, 1)$. We obtain

$$ \sum_{\boldsymbol{l} \in L} \prod_{j=1}^{4} \frac{z_j^{-\gamma_j + l_j}}{\Gamma(1-\gamma_j + l_j)}. $$

It is more or less clear how to generalize this series expansion. Fix a positive integer N and let $\boldsymbol{\gamma} \in \mathbb{R}^N$. Let $L \subset \mathbb{Z}^N$ be a lattice of rank d which satisfies the following conditions.

1. L is contained in the hyperplane $\sum_{i=1}^{N} l_i = 0$.

2. L is saturated, i.e $(L \otimes \mathbb{R}) \cap \mathbb{Z}^N = L$.

Now define

$$ \Phi_{\boldsymbol{\gamma}}^{L} := \sum_{\boldsymbol{l} \in L} \prod_{j=1}^{N} \frac{z_j^{\gamma_j + l_j}}{\Gamma(\gamma_j + l_j + 1)}. $$

For the moment this is a formal series expansion. Notice that $\Phi^L_{\boldsymbol{\gamma}} = \Phi^L_{\boldsymbol{\gamma}+\boldsymbol{l}}$ for any $\boldsymbol{l} \in L$. Let $r = N - d$ and let A be an $r \times N$-matrix with integer entries such that L is the integer kernel of A. In our example we can choose

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is this "A" after which the A-hypergeometric functions are named. Let us define $\boldsymbol{\alpha} = A\boldsymbol{\gamma}^\intercal$. Notice that $A(\boldsymbol{\gamma} + \boldsymbol{l})^\intercal = \boldsymbol{\alpha}$ for any $\boldsymbol{l} \in L \otimes \mathbb{R}$. In our example the parameter vector reads $\boldsymbol{\alpha} = (a, b, c-1)^\intercal$. We call this the parameter vector of the A-hypergeometric system we will define. Because L is contained in the hyperplane $\sum_{i=1}^N l_i = 0$, there is a linear form $h : \mathbb{R}^r \to \mathbb{R}$ where $h(\boldsymbol{a}) = 1$ for all column vectors $\boldsymbol{a}$ of A.
In this thesis we will also speak about a B-matrix. This is simply an integer $d \times N$ matrix whose rows form a $\mathbb{Z}$-basis of L.
It turns out that $\Phi^L_{\boldsymbol{\gamma}}$ satisfies a system of partial differential equations. First of all, let $\boldsymbol{m} = (m_1, \ldots, m_N)$ be an integer row vector such that $\boldsymbol{m} \cdot \boldsymbol{l} = 0$ for all $\boldsymbol{l} \in L$. Then, for any $\lambda \in \mathbb{C}^*$, one easily sees that

$$\Phi^L_{\boldsymbol{\gamma}}(\lambda^{m_1} z_1, \ldots, \lambda^{m_N} z_N) = \lambda^{\boldsymbol{m}\cdot\boldsymbol{\gamma}} \Phi^L_{\boldsymbol{\gamma}}(z_1, \ldots, z_N).$$

Take the derivate with respect to λ and set $\lambda = 1$. Then we see that $\Phi^L_{\boldsymbol{\gamma}}$ is annihilated by the differential operator

$$m_1 z_1 \partial_{z_1} + \cdots + m_N z_N \partial z_N - \boldsymbol{m} \cdot \boldsymbol{\gamma}.$$

In particular, if we let $\boldsymbol{m}$ be the i-th row of $A = (A_{ij})$ we see that $\Phi^L_{\boldsymbol{\gamma}}$ is annihilated by the Euler operator

$$Z_i := A_{i1} z_1 \partial_{z_1} + \cdots + A_{iN} z_N \partial z_N - \alpha_i.$$

There is a second set of differential equations which arises from the observation

$$\partial^{\lambda_1}_{z_1} \cdots \partial^{\lambda_N}_{z_N} \Phi^L_{\boldsymbol{\gamma}} = \Phi^L_{\boldsymbol{\gamma}-\boldsymbol{\lambda}}$$

for any $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N_{\geq 0}$. Let now $\boldsymbol{\lambda} \in L$ and write $\boldsymbol{\lambda} = \boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-$, where $\boldsymbol{\lambda}^\pm$ are integer vectors with non-negative entries. Then,

$$\partial^{\boldsymbol{\lambda}^+} \Phi^L_{\boldsymbol{\gamma}} = \Phi^L_{\boldsymbol{\gamma}-\boldsymbol{\lambda}^+} = \Phi^L_{\boldsymbol{\gamma}-\boldsymbol{\lambda}^-} = \partial^{\boldsymbol{\lambda}^-} \Phi^L_{\boldsymbol{\gamma}}.$$

We use the notation $\partial^{\boldsymbol{\lambda}} = \partial^{\lambda_1}_{z_1} \cdots \partial^{\lambda_N}_{z_N}$ and the second step follows from the invariance of $\Phi^L_{\boldsymbol{\gamma}}$ when $\boldsymbol{\gamma}$ is shifted over vectors in L. Thus we find that Φ^L_γ is annihilated by the so-called box operators

$$\Box^{\boldsymbol{\lambda}} := \prod_{\lambda_i>0} \partial_{z_i}^{\lambda_i} - \prod_{\lambda_i<0} \partial_{z_i}^{-\lambda_i}$$

for all $\boldsymbol{\lambda} \in L$.

The *A***-hypergeometric system** $H_A(\boldsymbol{\alpha})$ is the system of differential equations generated by

1. The Euler operators

$$Z_j = A_{j1}\partial_{z_1} + \cdots + A_{jN}\partial_{z_N} - \alpha_j, \quad j = 1, \ldots, N-d. \tag{2.4}$$

2. The box operators

$$\Box^{\boldsymbol{\lambda}} = \partial^{\boldsymbol{\lambda}^+} - \partial^{\boldsymbol{\lambda}^-}, \quad \boldsymbol{\lambda} \in L. \tag{2.5}$$

An A-hypergeometric function is a holomorphic function in $z_1, \ldots, z_N$ which satisfies the equations in the A-hypergeometric system.

Either A together with a parameter vector $\boldsymbol{\alpha}$ or B with $\boldsymbol{\gamma}/L$ is enough to encode all the information about the A-hypergeometric system. Sometimes we prefer to use A and sometimes we prefer its so called Gale dual B. The columns of A are denoted by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ and the columns of B are denoted by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$.

Example 2.1.1. The hypergeometric function ${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\, z\right)$ can be obtained from solutions of the A-hypergeometric system where $B = (1\,1\,{-1}\,{-1})$, $z = \frac{x_1x_2}{x_3x_4}$ and parameter vector $\boldsymbol{\gamma} = (-a, -b, c-1, 0)$. To encode the same A-hypergeometric system using the matrix A and parameter vector $\boldsymbol{\alpha}$ we can use

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = (a, b, c-1)^{\mathsf{T}}.$$

Example 2.1.2. Appell's F_4 is the hypergeometric function defined by

$$F_4\left(\begin{array}{c} a,\ b \\ c,\ c' \end{array}\middle|\, x, y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!} x^m y^n.$$

Write out the Pochhammer symbols and use Euler's reflection formula on the Γ-functions appearing in the numerator of each summand. Then up to a constant factor we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^m y^n}{\Gamma(1-a-m-n)\Gamma(1-b-m-n)\Gamma(c+m)\Gamma(c'+n)\Gamma(m+1)\Gamma(n+1)}.$$

Substitute $x = \frac{z_3 z_5}{z_1 z_2}$ and $y = \frac{z_4 z_6}{z_1 z_2}$ and premultiply with $z_1^{-a} z_2^{-b} z_3^{c-1} z_4^{c'-1}$ to get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{z_1^{-a-m-n}}{\Gamma(1-a-m-n)} \cdot \frac{z_2^{-b-m-n}}{\Gamma(1-b-m-n)} \cdot \frac{z_3^{c-1+m}}{\Gamma(c+m)} \cdot \frac{z_4^{c'-1+n}}{\Gamma(c'+n)} \cdot \frac{z_5^{m}}{\Gamma(m+1)} \cdot \frac{z_6^{n}}{\Gamma(n+1)}.$$

Let L be the lattice generated by $(-1,-1,1,0,1,0)$ and $(-1,-1,0,1,0,1)$ and let $\boldsymbol{\gamma} = (-a,-b,c-1,c'-1,0,0)$ then this summation equals $\Phi^L_{\boldsymbol{\gamma}}$. In other words Appell's hypergeometric function $F_4\left(\begin{matrix} a, & b \\ c, & c' \end{matrix}\middle| x, y\right)$ can be obtained from the solutions of the A-hypergeometric system where

$$B = \begin{pmatrix} -1 & -1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{\gamma} = (-a,-b,c-1,c'-1,0,0).$$

The A-hypergeometric systems we are interested in are those that are irreducible and only depend on $\boldsymbol{\alpha}$ modulo $\mathbb{Z}$. We can achieve this by assuming the system is non-resonant.

Definition 2.1.3. An A-hypergeometric system $H_A(\boldsymbol{\alpha})$ is called **non-resonant** if the boundary of the cone $C(A) := \langle \boldsymbol{a}_1, \ldots, \boldsymbol{a}_N \rangle_{\mathbb{R}_{\geq 0}}$ does not intersect the translated lattice $\boldsymbol{\alpha} + \mathbb{Z}^r$.

Theorem 2.1.4 ([GKZ90, Theorem 2.11])**.** *A non-resonant A-hypergeometric system $H_A(\boldsymbol{\alpha})$ is irreducible.*

For reasons that will become clear in the next section we also want $\boldsymbol{\alpha}$ to be totally non-resonant.

Definition 2.1.5. An A-hypergeometric system $H_A(\boldsymbol{\alpha})$ is called **totally non-resonant** if for each $r-1$-independent subset $\{\boldsymbol{a}_{j_1}, \ldots, \boldsymbol{a}_{j_{r-1}}\}$ of A we have that $\langle \boldsymbol{a}_{j_1}, \ldots, \boldsymbol{a}_{j_{r-1}} \rangle_{\mathbb{R}_{\geq 0}}$ does not intersect the lattice $\boldsymbol{\alpha} + \mathbb{Z}^r$.

We will always assume that $\boldsymbol{\alpha}$ is chosen totally non-resonant in the remainder of this thesis unless otherwise stated.

Theorem 2.1.6 ([Ado94, Corollary 5.20])**.** *Let $Q(A)$ be the convex hull of the points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$. If the system $H_A(\boldsymbol{\alpha})$ is non-resonant then the holonomic rank of $H_A(\boldsymbol{\alpha})$ is equal to* $\mathrm{Vol}(Q(A))$. *Here the volume* Vol *is normalized such that a $r-1$-simplex has volume* 1.

Let us denote by D the holonomic rank of the A-hypergeometric system. Therefore when the system is non-resonant, then $D = \mathrm{Vol}(Q(A))$.

2.2 Power series solutions

Recall the formal powerseries expansion

$$\Phi_{\boldsymbol{\gamma}}^{L} = \sum_{\boldsymbol{l}\in L} \frac{\boldsymbol{z}^{\boldsymbol{\gamma}+\boldsymbol{l}}}{\boldsymbol{\Gamma}(\boldsymbol{\gamma}+\boldsymbol{l}+\mathbf{1})}. \tag{2.6}$$

Here and throughout this thesis we use the convention that for any vector $\boldsymbol{v} = (v_1, \ldots, v_N)$ the entity $\boldsymbol{\Gamma}(\boldsymbol{v})$ is written as $\prod_{i=1}^{N} \Gamma(v_i)$ and $\boldsymbol{z}^{\boldsymbol{v}} = \prod_{i=1}^{N} z_i^{v_i}$. And here $\mathbf{1}$ is the ones-vector. For a scalar c and vector $\boldsymbol{v}$ we let $c^{\boldsymbol{v}} = (c^{v_1}, \ldots, c^{v_N})$. We have seen in the previous section that $\Phi_{\boldsymbol{\gamma}}^{L}$ satisfies the A-hypergeometric system $H_A(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = A\boldsymbol{\gamma}^{\intercal}$. Notice that these equations do not change if we shift $\boldsymbol{\gamma}$ by a vector from $L\otimes\mathbb{R}$. Hence we get in principle an infinite dimensional space of formal solutions. However, we shall only be interested in those shifts of $\boldsymbol{\gamma}$ that yield Puiseux series solutions with a domain of convergence. They belong to the D-dimensional solution space mentioned in Theorem 2.1.6.

The question is now how to determine these shifts. To answer this question we will use that $1/\Gamma(x)$ is 0 if $x \in \mathbb{Z}_{\leq 0}$. Another observation is that if we let a basis for L be $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_d$, then even though we have N variables $z_1, \ldots, z_N$, effectively we are only using $x_1 = \boldsymbol{z}^{\boldsymbol{l}_1}, \ldots, x_d = \boldsymbol{z}^{\boldsymbol{l}_d}$. In this way we can rewrite $\Phi_{\boldsymbol{\gamma}}^{L}$ as

$$\Phi_{\boldsymbol{\gamma}} = \boldsymbol{z}^{\boldsymbol{\gamma}} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{\boldsymbol{x}^{\boldsymbol{k}}}{\boldsymbol{\Gamma}(\boldsymbol{\gamma}+\boldsymbol{k}B+\mathbf{1})},$$

where B is the $d \times N$-matrix with $\boldsymbol{l}_i$ as its i-th row and $\boldsymbol{k}$ is considered a row-vector.

To describe the shifts of $\boldsymbol{\gamma}$ we fix $\boldsymbol{\gamma}_0$ such that $\boldsymbol{\alpha} = A\boldsymbol{\gamma}_0^{\intercal}$ and parametrize all shifts by $\boldsymbol{\gamma}_0 + \boldsymbol{\mu}B$, where $\boldsymbol{\mu} \in \mathbb{R}^d$ is considered as row vector. Since $\Phi_{\boldsymbol{\gamma}+\boldsymbol{l}} = \Phi_{\boldsymbol{\gamma}}$ for all $\boldsymbol{l} \in L$, we can restrict $\boldsymbol{\mu}$ to the domain $[0,1)^d$. We can now rewrite $\Phi_{\boldsymbol{\gamma}} = \boldsymbol{z}^{\boldsymbol{\gamma}_0}\Psi_{\boldsymbol{\mu}}$ where

$$\Psi_{\boldsymbol{\mu}} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{\boldsymbol{x}^{\boldsymbol{k}+\boldsymbol{\mu}}}{\boldsymbol{\Gamma}(\boldsymbol{\gamma_0}+(\boldsymbol{k}+\boldsymbol{\mu})B+\mathbf{1})}.$$

We denote the columns of B by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N$.
Fix $I \subset \{1, 2, \ldots, N\}$ with cardinality d and $\boldsymbol{b}_i, i \in I$ linearly independent. We call such a set a **cotriangle**, the reason being that the vectors $\boldsymbol{a}_i, i \in I^c$ span a simplex (triangle) in the set A. Then choose $\boldsymbol{\mu} \in [0,1)^d$ such that $\boldsymbol{\gamma}_0 + \boldsymbol{\mu}B$ has integer components at the indices $i \in I$. Let B_I be the

submatrix of B consisting of the columns $\boldsymbol{b}_i, i \in I$ and let $\boldsymbol{\gamma}_{0I}$ be the sub-rowvector of $\boldsymbol{\gamma}_0$ consisting of the indices in I. Then we need to solve $\boldsymbol{\gamma}_{0I} + \boldsymbol{\mu} B_I \in \mathbb{Z}^d$ in $\boldsymbol{\mu} \in [0,1)^d$. This comes down to counting the number of shifted integral points in the fundamental parallelogram spanned by the rows of B_I. Clearly the number of solutions is independent of this shift and there are precisely Δ_I solutions, where $\Delta_I = |\det(B_I)|$. Having found such $\boldsymbol{\mu}$ we note that the sum in the definition of $\Psi_{\boldsymbol{\mu}}$ is restricted to the domain $\gamma_{0,i} + (\boldsymbol{k} + \boldsymbol{\mu})\boldsymbol{b}_i \geq 0$, $i \in I$. This is because $1/\Gamma(x+1) = 0$ if x is a negative integer.
Choose a point $\boldsymbol{\rho}$ in the interior of the positive cone spanned by the $\boldsymbol{b}_i, i \in I$. Then $\Psi_{\boldsymbol{\mu}}$ converges at the points $\boldsymbol{x}$ with $|x_i| = t^{\rho_i}$ for sufficiently small $t > 0$. See [Beu11] for the necessary estimates. We call $\boldsymbol{\rho}$ a convergence direction.
It is conceivable that besides I there is another index at which $\boldsymbol{\gamma}_0 + \boldsymbol{\mu} B$ has an integer component. Since

$$\boldsymbol{\alpha} = A\boldsymbol{\gamma}^\intercal = A\boldsymbol{\gamma}^\intercal + AB^\intercal \boldsymbol{\mu}^\intercal,$$

this means that $\boldsymbol{\alpha}$ can be written as a linear combination of the vectors $\boldsymbol{a}_i$ with fewer than $r = N - d$ non-integral coordinates. By our assumption of total non-resonance, see Definition 2.1.5, this situation cannot occur. We conclude that I is uniquely determined by $\boldsymbol{\mu}$.

Definition 2.2.1. We call $\boldsymbol{\mu} \in [0,1)^d$ a **solution point** and denote the corresponding set I by $I(\boldsymbol{\mu})$. Its corresponding parameter vector is denoted by $\boldsymbol{\gamma}^{\boldsymbol{\mu}} := \boldsymbol{\gamma}_0 + \boldsymbol{\mu} B$.

Let us reverse the situation and start with a convergence direction $\boldsymbol{\rho} \in \mathbb{R}^d$ not in the hyperplane spanned by any $d-1$ vectors $\boldsymbol{b}_i$. The set of cotriangles I such that $\boldsymbol{\rho}$ is contained in the positive cone generated by $\boldsymbol{b}_i, i \in I$ is denoted by $\mathcal{I}_{\boldsymbol{\rho}}$. Each cotriangle I contributes Δ_I solution points $\boldsymbol{\mu}$ and so we find $\sum_{I \in \mathcal{I}_{\boldsymbol{\rho}}} \Delta_I$ Laurent series solutions that converge around $\boldsymbol{\rho}$. We call $\mathcal{I}_{\boldsymbol{\rho}}$ a **cotriangulation** of B.
From [JRS10, Section 5.4] it follows that cotriangulations of B are in one-to-one correspondence with triangulations of A. The correspondence is given by associating a cotriangle I with a triangle (simplex) spanned by $\boldsymbol{a}_i, i \in I^c$. Furthermore, it follows from [Beu11, Lemma 14.2] that $\Delta_I = |\det(\boldsymbol{a}_i)_{i \in I^c}|$. Hence $\sum_{I \in \mathcal{I}_{\boldsymbol{\rho}}} \Delta_I$ equals $\mathrm{Vol}(Q(A))$, which is precisely the rank of our hypergeometric system $H_A(\boldsymbol{\alpha})$. Thus the Laurent series $\boldsymbol{z}^{\boldsymbol{\gamma}_0} \Psi_{\boldsymbol{\mu}}$ with $I(\boldsymbol{\mu}) \in \mathcal{I}_{\boldsymbol{\rho}}$ forms a basis of solutions with a common domain of convergence.

Definition 2.2.2. A **chamber** is a fully dimensional cone constructed as an intersection of the form $\mathcal{C}_{\boldsymbol{\rho}} := \bigcap_{I\in\mathcal{I}_{\boldsymbol{\rho}}} C_I$, where C_I is the cone generated by the $\boldsymbol{b}_i$ for $i \in I$. It has the property that for each convergence direction $\boldsymbol{\rho}'$ we pick in the interior of the chamber that $\mathcal{C}_{\boldsymbol{\rho}'} = \mathcal{C}_{\boldsymbol{\rho}}$. In this way cotriangulations and chambers are in one-to-one correspondence. A **wall** is any face of a chamber that is of codimension 1. The polyhedral complex Σ_B generated by the chambers $\mathcal{C}_{\boldsymbol{\rho}}$ and all of their faces is called the **secondary fan** Σ_B.

2.3 Mellin-Barnes Integrals

Let notation be as above and choose a vector $\boldsymbol{\sigma} \in \mathbb{R}^d$. For any vector $\boldsymbol{s} = (s_1, \ldots, s_d)$ denote $\boldsymbol{ds} = ds_1 \wedge ds_2 \wedge \ldots \wedge ds_d$. Then consider the integral

$$M(\boldsymbol{z}) = M(z_1, \ldots, z_N) := \int_{\boldsymbol{\sigma}+i\mathbb{R}^d} \boldsymbol{\Gamma}(-\boldsymbol{\gamma}_0 - \boldsymbol{s}B)\boldsymbol{z}^{\boldsymbol{\gamma}_0+\boldsymbol{s}B}\boldsymbol{ds}.$$

This is a so-called Mellin-Barnes integral. When there is a basis of solutions for an A-hypergeometric system in terms of Mellin-Barnes integrals, then this will help us to find the monodromy group for these A-hypergeometric functions. A quick summary about Mellin Barnes integrals is given here, for a more thorough introduction see [Beu16].

Let us first introduce the variables $\boldsymbol{x} = \boldsymbol{z}^B$ and rewrite $M(\boldsymbol{z})$ as $\boldsymbol{z}^{\boldsymbol{\gamma}_0}\tilde{M}(\boldsymbol{x})$, where

$$\tilde{M}(\boldsymbol{x}) = \int_{\boldsymbol{\sigma}+i\mathbb{R}^d} \boldsymbol{\Gamma}(-\boldsymbol{\gamma_0} - \boldsymbol{s}B)\boldsymbol{x}^{\boldsymbol{s}}\boldsymbol{ds}.$$

Theorem 2.3.1 ([Beu16, Theorem 3.1])**.** *Suppose that $\gamma_{0,i} < -\boldsymbol{b}_i \cdot \boldsymbol{\sigma}$ for $i = 1, \ldots, N$ and that $M(\boldsymbol{z})$ converges. Then $M(\boldsymbol{z})$ satisfies the differential system $H_A(\boldsymbol{\alpha})$.*

Now not all systems admit a choice for $\boldsymbol{\gamma}_0$ where $\gamma_{0,i} < -\boldsymbol{b}_i \cdot \boldsymbol{\sigma}$. Using contiguity relations we can change $\boldsymbol{\alpha}$ without affecting the monodromy and we still have a freedom in $\boldsymbol{\sigma}$. In [Beu16] it is shown that we can choose $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ such that $\boldsymbol{\gamma}_0$ satisfies the conditions of Theorem 2.3.1 without affecting monodromy.

For convergence of Mellin-Barnes solutions we will define the open zonotope

$$Z_B^\circ = \left\{ \sum_{i=1}^N \nu_i \boldsymbol{b}_i \mid 0 < \nu_i < 1 \right\}.$$

Note that our definition of a zonotope is scaled with a factor two compared to its definition in [Beu16].

Theorem 2.3.2 ([Beu16, Corollary 4.2])**.** *Let* $\boldsymbol{\tau} = \frac{1}{2\pi}\operatorname{Arg}(\boldsymbol{x})$ *be a componentwise choice of argument of the vector* $\boldsymbol{x}$*. Then* $\tilde{M}(\boldsymbol{x})$ *converges absolutely if* $\boldsymbol{\tau} \in \frac{1}{2}Z_B^\circ$*.*

And lastly we quickly state how linearly independent solutions can be found. And thus how we can find a basis of solutions using Mellin-Barnes integrals. The following theorem tells us that choosing different $\boldsymbol{\tau} \in \frac{1}{2}Z_B^\circ$ we can obtain independent Mellin-Barnes solutions.

Theorem 2.3.3 ([Beu16, Proposition 4.6])**.** *Let* $H_A(\boldsymbol{\alpha})$ *be a non-resonant* A*-hypergeometric system of rank* D*. Let* $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_q \in \frac{1}{2}Z_B^\circ$ *be points whose coordinates differ by integers. Fix a point* $\boldsymbol{x}^0 \in (\mathbb{C}^\times)^d$ *and choose for each* $\boldsymbol{\tau}_i$ *the Mellin-Barnes integral* $\tilde{M}_i(\boldsymbol{x})$ *with this argument choice for* $\boldsymbol{x}^0$*. Then* $\tilde{M}_1, \ldots, \tilde{M}_q$ *are linearly independent in a neighbourhood of* $\boldsymbol{x}^0$*.*

In particular this implies that if $q = D$, then we have a basis of solutions of $H_A(\boldsymbol{\alpha})$ given by Mellin-Barnes integrals.

2.4 Monodromy

Given an A-hypergeometric system with solution space V. Any non-zero solution in V remains a non-zero solution if we analytically continue it around some cycle. This means that analytic continuation along some cycle c induces a linear map $\phi_c : V \to V$. All the possible elements ϕ_c give the monodromy group. Seeing the elements ϕ_c as matrices, then the monodromy group will depend on a choice of basis. In our case this basis will be a space of Mellin-Barnes solutions.

Let $\boldsymbol{n} \in \mathbb{Z}^d$ be a column vector and let $c(\boldsymbol{n})$ be the cycle

$$\{(e^{2\pi i n_1 t}x_1, \ldots, e^{2\pi i n_d t}x_d) \mid t \in [0,1]\}.$$

Analytic continuation of the Laurent series solution $\Psi_{\boldsymbol{\mu}}(\boldsymbol{x})$ along $c(\boldsymbol{n})$ gives $e^{2\pi i \boldsymbol{n}\cdot\boldsymbol{\mu}}\Psi_{\boldsymbol{\mu}}(\boldsymbol{x})$. This means that, given a convergence direction $\boldsymbol{\rho}$, and its corresponding basis of local Laurent series solutions $\Psi_{\boldsymbol{\mu}_1}, \ldots, \Psi_{\boldsymbol{\mu}_D}$, the monodromy elements $\phi_{c(\boldsymbol{n})}$ can be written in matrix form as

$$\chi_{\boldsymbol{\rho},\boldsymbol{n}} := \begin{pmatrix} e^{2\pi i \boldsymbol{\mu}_1 \boldsymbol{n}} & 0 & \cdots & 0 \\ 0 & e^{2\pi i \boldsymbol{\mu}_2 \boldsymbol{n}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i \boldsymbol{\mu}_D \boldsymbol{n}} \end{pmatrix}.$$

This gives a commutative subgroup of the monodromy group which is generated by the elements $\chi_{\boldsymbol{\rho},j} := \chi_{\boldsymbol{\rho},\boldsymbol{e}_j},\ j = 1,\ldots,d$.

Now suppose that $H_A(\boldsymbol{\alpha})$ has a Mellin-Barnes basis of solutions and therefore there exists a set $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_D$ such that $\boldsymbol{\tau}_i \in \frac{1}{2}Z_B^\circ$ are distinct and differ by integers. Denote the Mellin-Barnes integral corresponding to the argument choice $2\pi\boldsymbol{\tau}_j$ by M_j.

Consider the Mellin-Barnes basis near a point $\boldsymbol{x}^0$. Analytic continuation of M_1 along the path $c(\boldsymbol{\tau}_j - \boldsymbol{\tau}_1)$ changes M_1 into M_j. Note that this is independent of the choice of $\boldsymbol{x}^0$. If we write a local series expansion $\tilde{M}_1 = \sum_{k=1}^D \lambda_k \Psi_{\boldsymbol{\mu}_k}$ for some convergence direction $\boldsymbol{\rho}$, then analytic continuation along $c(\boldsymbol{\tau}_j - \boldsymbol{\tau}_1)$ will result in $\tilde{M}_j = \sum_{k=1}^D \lambda_k e^{2\pi i(\boldsymbol{\tau}_j - \boldsymbol{\tau}_1)\cdot\boldsymbol{\mu}_k}\Psi_{\boldsymbol{\mu}_k}$. If one of these λ_k's is zero, we see that $\tilde{M}_1,\ldots,\tilde{M}_D$ spans a space of dimension strictly less than D, which is in contradiction with $\tilde{M}_1,\ldots,\tilde{M}_D$ being linearly independent. Hence it must be that the λ_k's are all nonzero. We can then normalize the $\Psi_{\boldsymbol{\mu}_k}$ such that the λ_k's are 1 and obtain a transition matrix between Mellin-Barnes solutions to local power series solutions.

$$X_{\boldsymbol{\rho}} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i\boldsymbol{\mu}_1(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)} & e^{2\pi i\boldsymbol{\mu}_2(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)} & \ldots & e^{2\pi i\boldsymbol{\mu}_D(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)} \\ e^{2\pi i\boldsymbol{\mu}_1(\boldsymbol{\tau}_3-\boldsymbol{\tau}_1)} & e^{2\pi i\boldsymbol{\mu}_2(\boldsymbol{\tau}_3-\boldsymbol{\tau}_1)} & \ldots & e^{2\pi i\boldsymbol{\mu}_D(\boldsymbol{\tau}_3-\boldsymbol{\tau}_1)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i\boldsymbol{\mu}_1(\boldsymbol{\tau}_D-\boldsymbol{\tau}_1)} & e^{2\pi i\boldsymbol{\mu}_2(\boldsymbol{\tau}_D-\boldsymbol{\tau}_1)} & \ldots & e^{2\pi i\boldsymbol{\mu}_D(\boldsymbol{\tau}_D-\boldsymbol{\tau}_1)} \end{pmatrix}. \tag{2.7}$$

Such that

$$\begin{pmatrix} \tilde{M}_1 \\ \tilde{M}_2 \\ \tilde{M}_3 \\ \vdots \\ M_D \end{pmatrix} = X_{\boldsymbol{\rho}} \begin{pmatrix} \Psi_{\boldsymbol{\mu}_1} \\ \Psi_{\boldsymbol{\mu}_2} \\ \Psi_{\boldsymbol{\mu}_3} \\ \vdots \\ \Psi_{\boldsymbol{\mu}_D} \end{pmatrix}.$$

This means that the monodromy subgroup generated by $\chi_{\boldsymbol{\rho},j}$ with respect to a basis of local series expansions, can be transformed through $X_{\boldsymbol{\rho}}$ into a monodromy subgroup with respect to a basis of Mellin-Barnes solutions.

The matrices that generate this monodomy subgroup with respect to a basis of Mellin-Barnes solutions are defined as

$$M_{\rho,j} = X_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j}X_{\boldsymbol{\rho}}^{-1}.$$

By changing the convergence direction $\boldsymbol{\rho}$ we will therefore obtain multiple subgroups, which together will generate a larger subgroup of the monodromy group $\mathcal{M}$. Since it is unclear whether this generates the whole monodromy group, we will define a subgroup of the monodromy.

Definition 2.4.1. The Mellin-Barnes group $\mathcal{M}_{MB}$ is the group generated by the matrices $M_{\boldsymbol{\rho},j}$ for all $j = 1, \ldots, d$ and convergence directions $\boldsymbol{\rho}$.

Remark 2.4.2. The Mellin-Barnes group corresponds to the power series Ψ_μ, though we started out with the power series Φ_γ. These power series differ by a monomial factor. Hence their corresponding monodromy groups are the same upto multiplication by scalars.

Part I

The Hermitian Form

CHAPTER **3**

The Hermitian Form

3.1 The main theorem

In this chapter we adopt the notations from Chapter 2. In particular $X_{\boldsymbol{\rho}}$ are the transition matrices given in (2.7). Our goal is to prove the following theorem.

Theorem 3.1.1. *Let $H_A(\boldsymbol{\alpha})$ be a totally non-resonant A-hypergeometric system admitting a Mellin-Barnes basis of solutions. Then there exists a non-trivial Hermitian form H which is invariant under the group $\mathcal{M}_{MB}$. Furthermore given any convergence direction $\boldsymbol{\rho}$, this Hermitian form can be given explicitly as*

$$H = (\bar{X}_{\boldsymbol{\rho}}^{\intercal})^{-1}\Delta_{\boldsymbol{\rho}}X_{\boldsymbol{\rho}}^{-1} \tag{3.1}$$

where $\Delta_{\boldsymbol{\rho}}$ is the diagonal matrix

$$\operatorname{Diag}\left(\left\{\Delta_{I_k}\prod_{l\in I_k}(-1)^{\gamma_l^{\boldsymbol{\mu}_k}}\prod_{i\notin I_k}\sin(\pi\gamma_i^{\boldsymbol{\mu}_k})\right\}_{k=1,\ldots,D}\right) \tag{3.2}$$

and where $\boldsymbol{\mu}_k$ runs over all solutions points with $I_k := I(\boldsymbol{\mu}_k)\in\mathcal{I}_{\boldsymbol{\rho}}$.

3.2 Proof

Notation 3.2.1. Due to lack of space for certain formulas and equations, we sometimes use a different notation for matrices. In our case for a $M\times N$

matrix where M and N are known we use the notation

$$\{a_{rc}\}_{r,c} := \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{pmatrix}.$$

For diagonal matrices of fixed dimension N we may use the notation

$$\{a_r\}_{rr} := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_N \end{pmatrix}.$$

Proof. Fix a convergence direction $\boldsymbol{\rho}$ and consider $H_{\boldsymbol{\rho}} = (\bar{X}_{\boldsymbol{\rho}}^{\intercal})^{-1}\Delta_{\boldsymbol{\rho}} X_{\boldsymbol{\rho}}^{-1}$ where $X_{\boldsymbol{\rho}}$ is the transitition matrix given in (2.7) and $\Delta_{\boldsymbol{\rho}}$ is given in (3.2). We show that $H_{\boldsymbol{\rho}}$ is a Hermitian matrix for the monodromy matrices $M_{\boldsymbol{\rho},j} = X_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j}X_{\boldsymbol{\rho}}^{-1}$ defined in Chapter 2. This comes down to showing that:

$$\left(\overline{X_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j}X_{\boldsymbol{\rho}}^{-1}}\right)^{\intercal}(\bar{X}_{\boldsymbol{\rho}}^{\intercal})^{-1}\Delta_{\boldsymbol{\rho}}X_{\boldsymbol{\rho}}^{-1}X_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j}X_{\boldsymbol{\rho}}^{-1} = (\bar{X}_{\boldsymbol{\rho}}^{\intercal})^{-1}\Delta_{\boldsymbol{\rho}}X_{\boldsymbol{\rho}}^{-1}. \tag{3.3}$$

This simplifies to

$$\overline{\chi_{\boldsymbol{\rho},j}}^{\intercal}\Delta_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j} = \Delta_{\boldsymbol{\rho}}.$$

As all of these matrices are diagonal, and $\overline{\chi_{\boldsymbol{\rho},j}}^{\intercal}$, $\chi_{\boldsymbol{\rho},j}$ are each others inverse we see that the equality is true.

The remainder of the proof consists of showing that $H_{\boldsymbol{\rho}}$ is independent of the choice of $\boldsymbol{\rho}$. The resulting matrix H is then an invariant Hermitian form for all local monodromy matrices $M_{\boldsymbol{\rho},j}$.

As explained in Chapter 2 we associate to each convergence direction a set of solution points $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_D$ and cotriangles $I_k := I(\boldsymbol{\mu}_k)$.

To prove the independence of $H_{\boldsymbol{\rho}}$, we calculate $H_{\boldsymbol{\rho}}^{-1}$, where we denote $\tilde{\boldsymbol{\tau}}_l = \boldsymbol{\tau}_l - \boldsymbol{\tau}_1$.

$$
\begin{aligned}
H_{\boldsymbol{\rho}}^{-1} &= X_{\boldsymbol{\rho}} \Delta_{\boldsymbol{\rho}}^{-1} \bar{X}_{\boldsymbol{\rho}}^{\intercal} \\
&= \left\{ e^{2\pi i \boldsymbol{\mu}_c \tilde{\boldsymbol{\tau}}_r} \right\}_{r,c} \left\{ \frac{1}{\Delta_{I_r}} \prod_{l \in I_r} (-1)^{\gamma_l^{\boldsymbol{\mu}_r}} \prod_{i \notin I_r} \csc(\pi \gamma_i^{\boldsymbol{\mu}_r}) \right\}_{rr} \left\{ e^{-2\pi i \boldsymbol{\mu}_r \tilde{\boldsymbol{\tau}}_c} \right\}_{r,c} \\
&= \left\{ \frac{e^{2\pi i \boldsymbol{\mu}_c \tilde{\boldsymbol{\tau}}_r}}{\Delta_{I_c}} \prod_{l \in I_c} (-1)^{\gamma_l^{\boldsymbol{\mu}_c}} \prod_{i \notin I_c} \csc(\pi \gamma_i^{\boldsymbol{\mu}_c}) \right\}_{r,c} \left\{ e^{-2\pi i \boldsymbol{\mu}_r \tilde{\boldsymbol{\tau}}_c} \right\}_{r,c} \\
&= \left\{ \sum_{k=1}^{D} \frac{e^{2\pi i \boldsymbol{\mu}_k (\tilde{\boldsymbol{\tau}}_r - \tilde{\boldsymbol{\tau}}_c)} \prod_{l \in I_k} (-1)^{\gamma_l^{\boldsymbol{\mu}_k}}}{\Delta_{I_k} \prod_{i \notin I_k} \sin(\pi \gamma_i^{\boldsymbol{\mu}_k})} \right\}_{r,c} \\
&= (2i)^r \left\{ \sum_{k=1}^{D} \frac{e^{2\pi i \boldsymbol{\mu}_k (\boldsymbol{\tau}_r - \boldsymbol{\tau}_c)}}{\Delta_{I_k}} \prod_{l \in I_k} e^{\pi i \gamma_l^{\boldsymbol{\mu}_k}} \prod_{l \notin I_k} \frac{1}{e^{\pi i \gamma_l^{\boldsymbol{\mu}_k}} - e^{-\pi i \gamma_l^{\boldsymbol{\mu}_k}}} \right\}_{r,c} \\
&= (2i)^r \left\{ \sum_{k=1}^{D} \frac{e^{2\pi i \boldsymbol{\mu}_k (\boldsymbol{\tau}_r - \boldsymbol{\tau}_c)}}{\Delta_{I_k}} \prod_{l=1}^{N} e^{\pi i \gamma_l^{\boldsymbol{\mu}_k}} \prod_{l \notin I_k} \frac{1}{e^{2\pi i \gamma_l^{\boldsymbol{\mu}_k}} - 1} \right\}_{r,c} \\
&= (2i)^r \prod_{j=1}^{N} e^{\pi i \gamma_{0j}} \left\{ \sum_{k=1}^{D} \frac{e^{2\pi i \boldsymbol{\mu}_k (\boldsymbol{\tau}_r - \boldsymbol{\tau}_c)}}{\Delta_{I_k}} \prod_{l \notin I_k} \frac{1}{e^{2\pi i \gamma_l^{\boldsymbol{\mu}_k}} - 1} \right\}_{r,c}.
\end{aligned}
$$

Each component of the inner matrix will be linked to a sum of certain residues, which can be seen from Lemma 3.3.1 below. Using this and using $\boldsymbol{\tau}_r - \boldsymbol{\tau}_c \in Z_B^{\circ}$ it follows from Corollary 3.3.6 below that $H_{\boldsymbol{\rho}}$ is independent of the choice of $\boldsymbol{\rho}$. □

3.3 Residues

Define the following differential form

$$
\omega := \omega(\boldsymbol{\tau}, \boldsymbol{z}) = \frac{\boldsymbol{z}^{\boldsymbol{\tau}}}{(x_1 \boldsymbol{z}^{\boldsymbol{b}_1} - 1)(x_2 \boldsymbol{z}^{\boldsymbol{b}_2} - 1) \ldots (x_N \boldsymbol{z}^{\boldsymbol{b}_N} - 1)} \frac{d\boldsymbol{z}}{\boldsymbol{z}}, \tag{3.4}
$$

where $x_j = e^{2\pi i \gamma_{0,j}}$. Here $\frac{d\boldsymbol{z}}{\boldsymbol{z}}$ is short for $\frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_d}{z_d}$. And $\boldsymbol{z}^{\boldsymbol{b}}$ stands for $z_1^{b_1} \cdots z_d^{b_d}$. Certain residues of this form are special cases of so called **Binomial Residues** ([CDS02]), we will use properties of these binomial residues in Chapter 4. For any solution point $\boldsymbol{\mu}$, define the vector

$$
\boldsymbol{\zeta}^{\boldsymbol{\mu}} := e^{2\pi i \boldsymbol{\mu}},
$$

where we use the notation $e^{2\pi i \boldsymbol{v}} = (e^{2\pi i v_1}, \ldots, e^{2\pi i v_d})$.

Notice that

$$x_i(\boldsymbol{\zeta}^{\boldsymbol{\mu}})^{\boldsymbol{b}_i} = e^{2\pi i \gamma_{0,i}} e^{2\pi i \boldsymbol{\mu} \boldsymbol{b}_i} = e^{2\pi i \gamma_i^{\boldsymbol{\mu}}} = 1,$$

for all $i \in I(\boldsymbol{\mu})$ because $\gamma_i^{\boldsymbol{\mu}} \in \mathbb{Z}$ for all $i \in I(\boldsymbol{\mu})$. We thus see that $\boldsymbol{\zeta}^{\boldsymbol{\mu}}$ is a solution to the system of equations $x_i \boldsymbol{z}^{\boldsymbol{b}_i} - 1 = 0,\ i \in I(\boldsymbol{\mu})$ in $\boldsymbol{z}$.

Let $f_i = x_i \boldsymbol{z}^{\boldsymbol{b}_i} - 1$ for $i = 1, \ldots, N$. Following [GH78, p. 650] we may define the residue

$$\operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega = \pm \frac{(\boldsymbol{\zeta}^{\boldsymbol{\mu}})^{\boldsymbol{\tau}}}{J_I(\boldsymbol{\zeta}^{\boldsymbol{\mu}}) \prod_{j \in I^c} f_j(\boldsymbol{\zeta}^{\boldsymbol{\mu}})}, \tag{3.5}$$

where $I = I(\boldsymbol{\mu})$, where we choose the sign $\pm$ to be $\operatorname{sign}(\det(B_I))$ and where J_I is the Jacobian determinant given by

$$J_I = \left| \left\{ z_r \frac{\partial f_{I_c}}{\partial z_r} \right\}_{r,c} \right| .$$

Due to the simplicity of the functions f_i we can easily show that

$$J_I = \det(B_I) \prod_{j \in I} x_j \boldsymbol{z}^{\boldsymbol{b}_{I_j}} .$$

By definition of $\boldsymbol{\zeta}^{\boldsymbol{\mu}}$ we get $\prod_{j \in I} x_j (\boldsymbol{\zeta}^{\boldsymbol{\mu}})^{\boldsymbol{b}_{I_j}} = 1$, so as a consequence we get

$$\operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega = \frac{(\boldsymbol{\zeta}^{\boldsymbol{\mu}})^{\boldsymbol{\tau}}}{\Delta_I \prod_{j \in I^c} f_j(\boldsymbol{\zeta}^{\boldsymbol{\mu}})}. \tag{3.6}$$

Lemma 3.3.1 is now a direct consequence of 3.6.

Lemma 3.3.1. *Let $\boldsymbol{\mu}$ be a solution point then we have*

$$\operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = \frac{\boldsymbol{e}^{2\pi i \boldsymbol{\mu} \boldsymbol{\tau}}}{\Delta_I \prod_{j \in I^c} (\boldsymbol{e}^{2\pi i \gamma_j^{\boldsymbol{\mu}}} - 1)},$$

where $I = I(\boldsymbol{\mu})$.

Using these residues we can now write a typical entry of the matrix $H_{\boldsymbol{\rho}}^{-1}$ in the proof of Theorem 3.1.1 as

$$\sum_{\boldsymbol{\mu}: I(\boldsymbol{\mu}) \in \mathcal{I}_{\boldsymbol{\rho}}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega(\boldsymbol{\tau}_r - \boldsymbol{\tau}_c, \boldsymbol{z}).$$

It would be tempting to prove that such an entry is independent of $\boldsymbol{\rho}$, and hence the corresponding cotriangulation $\mathcal{I}$, by using general properties of multidimensional residues. Unfortunately we have been unable to do so. Instead we shall follow a local appoach where we show equality of these sums for neighbouring cotriangulations. In doing so we shall make use of residue calculus for one variable rational functions.

Recall Definition 2.2.2 of Chapter 2.

Definition 3.3.2. For any wall W of the chamber $C_{\mathcal{I}}$ we denote by $\mathcal{I}_W$ all the cotriangles $I \in \mathcal{I}$ whose cones C_I have W as a (sub)-face.

Definition 3.3.3. Two cotriangulations $\mathcal{I}$ and $\mathcal{J}$ are called adjacent if their corresponding chambers share the same wall. We call this wall the common wall between $\mathcal{I}$ and $\mathcal{J}$.

Given adjacent triangulations $\mathcal{I}$ and $\mathcal{J}$ with common wall W then a cotriangle $I \in \mathcal{I}_W$ is characterized by having $d-1$ indices $i_1, \ldots, i_{d-1}$ for which the cone generated by $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}$ contains W. The remaining index of I corresponds to a $\boldsymbol{b}_{i_d}$ being on either side of W. Conversely, given indices $i_1, \ldots, i_{d-1}$ for which the corresponding cone generated by $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}$ contains W and given an index i_d for which the $\boldsymbol{b}$-vector is not on the hyperplane $\mathrm{Hyp}(W)$. Then $I = (i_1, \ldots, i_d)$ is either in $\mathcal{I}_W$ or $\mathcal{J}_W$, depending on which side of the wall $\boldsymbol{b}_{i_d}$ lies.

Proposition 3.3.4. *Let $\mathcal{I}$ and $\mathcal{J}$ be two adjacent cotriangulations with common wall W and suppose $\boldsymbol{\tau} \in Z_B^\circ$ then*

$$\sum_{\boldsymbol{\mu}: I(\boldsymbol{\mu}) \in \mathcal{I}_W} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = \sum_{\boldsymbol{\nu}: I(\boldsymbol{\nu}) \in \mathcal{J}_W} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\nu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}).$$

Proof. Choose any $i_1, \ldots, i_{d-1}$ such that $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}$ are linearly independent and the cone spanned by them contains W. It suffices to prove our lemma in case the sums run over all $I \in \mathcal{I}_W, J \in \mathcal{J}_W$ which contain $i_1, \ldots, i_{d-1}$. The full lemma then follows after summation over all sets $i_1, \ldots, i_{d-1}$ such that the cone spanned by $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}$ contains W.

Choose coordinates in $\mathbb{Z}^d$ such that the d-th coordinates of $\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}$ are zero. In general we denote the d-th coordinate of $\boldsymbol{b}_i$ by β_i. Hence $\beta_i = 0$ for $i = i_1, \ldots, i_{d-1}$. Write $\boldsymbol{z}^{\boldsymbol{b}_i} = Q_i(z_1, \ldots, z_{d-1}) z_d^{\beta_i}$ where Q_i is a monomial in $z_1, \ldots, z_{d-1}$. Similarly we write $\boldsymbol{z}^{\boldsymbol{\tau}} = Q_0(z_1, \ldots, z_{d-1}) z_d^{\tau_d}$. Let δ be the determinant of $\left(\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}\right)$ where we remove the last row, which is zero. Then by construction we have that for any i the following holds

$$\det\left(\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_{d-1}}, \boldsymbol{b}_i\right) = \beta_i \delta.$$

The sign of β_i determines on which side of W the vector $\boldsymbol{b}_i$ lies. Choose an index i_d with $\beta_{i_d} \neq 0$ and let $\boldsymbol{\zeta}$ be a point such that $x_j \boldsymbol{\zeta}^{\boldsymbol{b}_j} = 1$ for $j \in I := \{i_1, \ldots, i_d\}$. Then Lemma 3.3.1 tells us that

$$\operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = \frac{Q_0(\zeta_1, \ldots, \zeta_{d-1})\zeta_d^{\tau_d}}{\Delta_I \prod_{j \notin \{i_1, \ldots, i_d\}} \left(x_j Q_j(\zeta_1, \ldots, \zeta_{d-1})\zeta_d^{\beta_j} - 1\right)}. \tag{3.7}$$

We like to write this as a one variable residue. The variable will be called w. Consider

$$\Omega(w) = \frac{Q_0(\zeta_1, \ldots \zeta_{d-1})w^{\tau_d}}{\prod_{j \notin \{i_1, \ldots, i_{d-1}\}} \left(x_j Q_j(\zeta_1, \ldots, \zeta_{d-1})w^{\beta_j} - 1\right)} \frac{dw}{w}.$$

Let w_0 be a pole of $\Omega(w)$ which is $\neq 0, \infty$. We associate the index $i(w_0)$ such that w_0 is a zero of $x_{i(w_0)} Q_{i(w_0)} w^{\beta_{i(w_0)}} - 1$ and we write $I(w_0) = \{i_1, \ldots, i_{d-1}, i(w_0)\}$. Furthermore we let $\boldsymbol{w}_0 = (\zeta_1, \ldots, \zeta_{i_d}, w_0)$. Take the residue at $w = w_0$,

$$\frac{Q_0(\zeta_1, \ldots \zeta_{d-1})w_0^{\tau_d}}{\prod_{j \notin I(w_0)} \left(x_j Q_j(\zeta_1, \ldots, \zeta_{d-1})w_0^{\beta_j} - 1\right)} \frac{1}{\beta_{i(w_0)}}.$$

When $w_0 = \zeta_d$ we see that this differs by a factor $\beta_{i_d}/\Delta_I = \operatorname{sign}(\beta_{i_d})$ from (3.7). Suppose that $\operatorname{sign}(\beta_i) > 0$ if $\{i_1, \ldots, i_d, i\} \in \mathcal{I}_W$.
Let P be the set of poles $\neq 0, \infty$ of $\Omega(w)$. We take the sum of the residues of $\Omega(w)$ over all poles in P. We get

$$\sum_{w_0 \in P} \operatorname*{Res}_{w=w_0} \Omega(w) = \sum_{w_0 \in P} \operatorname{sign}(\beta_{i(w_0)}) \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{w}_0} \omega(\boldsymbol{\tau}, \boldsymbol{z}).$$

Without loss of generality we can assume for all i that $\operatorname{sign}(\beta_i) > 0$ if and only if $\{i_1, \ldots, i_{d-1}, i\} \in \mathcal{I}_W$. Let $K = \{i_1, \ldots, i_{d-1}\}$ and let $\mathcal{I}_K = \{I \in \mathcal{I}_W : K \subset I\}$ and $\mathcal{J}_K = \{I \in \mathcal{J}_W : K \subset I\}$. Thus our summation becomes

$$\sum_{\boldsymbol{\mu}: I(\boldsymbol{\mu}) \in \mathcal{I}_K} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) - \sum_{\boldsymbol{\nu}: I(\boldsymbol{\nu}) \in \mathcal{J}_K} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\nu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}).$$

To complete our proof we need to show that $\sum_{w_0 \in P} \operatorname{Res}_{w=w_0} \Omega(w) = 0$. Since the sum of all residues of a one variable rational function is zero, it suffices to show that $\operatorname{Res}_{w=0} \Omega(w) + \operatorname{Res}_{w=\infty} \Omega(w) = 0$. We prove that both residues are 0. For the residue at $w = 0$ we expand $\Omega(w)$ in a Laurent series in w times $\frac{dw}{w}$. The support of this series in contained in the set of integers

$$\geq \tau_d + \sum_{j \notin K} \max(0, -\beta_j) = \tau_d - \sum_{j:\beta_j < 0} \beta_j.$$

Since $\boldsymbol{\tau}$ is in the interior of Z_B we know that there exist $\lambda_1, \ldots, \lambda_N \in (0,1)$ such that $\boldsymbol{\tau} = \sum_{j=1}^{N} \lambda_j \boldsymbol{b}_j$. Hence $\tau_d = \sum_{j=1}^{N} \lambda_j \beta_j$ and

$$\tau_d - \sum_{j:\beta_j<0} \beta_j = \sum_{j:\beta_j>0} \lambda_j \beta_j + \sum_{j:\beta_j<0} (\lambda_j - 1)\beta_j.$$

All terms in this summation are positive, hence the Laurent series expansion of $\Omega(w)$ is in fact a Taylor series with a zero constant term. Hence $\operatorname{Res}_{w=0} \Omega(w) = 0$. We deal similarly with $w = \infty$. □

Lemma 3.3.5. *Let $\mathcal{I}$ and $\mathcal{J}$ be two cotriangulations then there exists a sequence of cotriangulations $\mathcal{I}_1, \ldots, \mathcal{I}_N$ such that $\mathcal{I}_1 = \mathcal{I}$, $\mathcal{I}_N = \mathcal{J}$ and $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ are adjacent for all $i = 1, \ldots, N-1$.*

Proof. Let $\mathcal{I}_{\boldsymbol{\rho}}$ correspond to the cotriangulation with convergence direction $\boldsymbol{\rho}$ and $\mathcal{I}_{\boldsymbol{\rho}'}$ correspond to the cotriangulation with convergence direction $\boldsymbol{\rho}'$. Then make a continuous path $f : [0,1] \to \mathbb{R}^d$ such that $f(0) = \boldsymbol{\rho}$ and $f(1) = \boldsymbol{\rho}'$ which may only cross walls of the secondary fan in one point. It cannot cross lower dimensional faces of the secondary polytope. Consider the sequence $0 < t_0 < \ldots < t_N < 1$ which are all points such that $f(t_i)$ is on a wall. And consider the sequence of cotriangulations

$$\mathcal{I}_{f(0)}, \mathcal{I}_{f\left(\frac{t_0+t_1}{2}\right)}, \mathcal{I}_{f\left(\frac{t_1+t_2}{2}\right)}, \ldots, \mathcal{I}_{f\left(\frac{t_{N-1}+t_N}{2}\right)}, \mathcal{I}_{f(1)}.$$

Then each consecutive cotriangulation is adjacent by definition of the path. □

Corollary 3.3.6. *Let $\mathcal{I}$ and $\mathcal{J}$ be two different cotriangulations and suppose $\boldsymbol{\tau} \in Z_B^{\circ}$. Then*

$$\sum_{\boldsymbol{\mu}:I(\boldsymbol{\mu})\in\mathcal{I}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = \sum_{\boldsymbol{\nu}:I(\boldsymbol{\nu})\in\mathcal{J}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\nu}}} \omega(\boldsymbol{\tau}, \boldsymbol{z}).$$

Proof. Suppose $\mathcal{I}$ and $\mathcal{J}$ are adjacent cotriangulations with common wall W. For the cotriangles $I \in \mathcal{I}$ such that $I \in \mathcal{J}$, there is nothing to prove as the summands on both side cancel each other out. So we are left with sums over $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ for corresponding cotriangles in $\mathcal{I}_W$ and $\mathcal{J}_W$ respectively. Now we simply apply Proposition 3.3.4.

Now suppose $\mathcal{I}$ and $\mathcal{J}$ are not adjacent cotriangulations. Then by Lemma 3.3.5 there exists a sequence of adjacent cotriangulations between $\mathcal{I}$ and $\mathcal{J}$. We can now apply Proposition 3.3.4 to each pair of adjacent contriangulations in the sequence. □

3.4 Remarks

Remark 3.4.1. Corollary 3.3.6 together with Lemma 3.3.1 gives the final step in the proof of Theorem 3.1.1 which establishes the existence an invariant Hermitian form with respect to $\mathcal{M}_{MB}$. The question remains whether this Hermitian form is uniquely determined (up to a constant factor). As we know this uniqueness is equivalent to the irreducibility of the action of $\mathcal{M}_{MB}$. In all explicit examples we have seen so far, the Hermitian form is indeed unique.

Remark 3.4.2. Recent work by Saiei Matsubara-Heo and Yoshiaki Goto [GMH20, Theorem 3.3], confirms the signature computation of Theorem 3.1.1. Their work does not assume the existence of a Mellin-Barnes basis. In [GMH20, Theorem 3.3] they claim that the signature of the invariant Hermitian form for any A-hypergeometric function with totally non-resonant parameter vector $\boldsymbol{\alpha}$ with $h(\boldsymbol{\alpha}) \notin \mathbb{Z}$ is determined by the signature of

$$\sin\left(-\pi \sum_{i \notin I(\boldsymbol{\mu})} \gamma_i^{\boldsymbol{\mu}}\right) \prod_{i \notin I(\boldsymbol{\mu})} \sin(\pi \gamma_i^{\boldsymbol{\mu}}), \quad (\boldsymbol{\mu} : I(\boldsymbol{\mu}) \in \mathcal{I}), \tag{3.8}$$

In Theorem 3.1.1 we see that the signature corresponds to those of

$$\Delta_{I(\boldsymbol{\mu})} \prod_{l \in I(\boldsymbol{\mu})} (-1)^{\gamma_l^{\boldsymbol{\mu}}} \prod_{i \notin I(\boldsymbol{\mu})} \sin(\pi \gamma_i^{\boldsymbol{\mu}}), \quad (\boldsymbol{\mu} : I(\boldsymbol{\mu}) \in \mathcal{I}).$$

Since $\Delta_{I(\boldsymbol{\mu})} > 0$ we can ignore $\Delta_{I(\boldsymbol{\mu})}$. Also note that

$$-\sum_{i \notin I} \gamma_i^{\boldsymbol{\mu}} = -\sum_{i=1}^{N} \gamma_{0i} + \sum_{i \in I} \gamma_i^{\boldsymbol{\mu}}.$$

When $i \in I(\boldsymbol{\mu})$ then $\gamma_i^{\boldsymbol{\mu}} \in \mathbb{Z}$, hence these contribute to a sign change in the leftmost sin function in (3.8). This sign change is exactly the product

$$\prod_{l \in I(\boldsymbol{\mu})} (-1)^{\gamma_l^{\boldsymbol{\mu}}}.$$

So this means we can rewrite (3.8) to

$$\sin\left(-\pi \sum_{i=1}^{N} \gamma_{0i}\right) \prod_{l \in I(\boldsymbol{\mu})} (-1)^{\gamma_l^{\boldsymbol{\mu}}} \prod_{i \notin I(\boldsymbol{\mu})} \sin(\pi \gamma_i^{\boldsymbol{\mu}}), \quad (\boldsymbol{\mu} : I(\boldsymbol{\mu}) \in \mathcal{I}). \tag{3.9}$$

Note that the left-most factor equals $\sin(-\pi h(\boldsymbol{\alpha}))$. So when $h(\boldsymbol{\alpha}) \notin \mathbb{Z}$ we recover our result.

Chapter 4

The Virtual Hermitian Form

4.1 Introduction

In chapter 3 we have constructed a Hermitian form which is invariant under the subgroup of the monodromy group generated by local monodromy elements. For the construction we had assumed the existence of a basis of solutions of the A-hypergeometric system given by Mellin-Barnes integrals. Strictly speaking the proof of Theorem 3.1.1 does not explicitly use the Mellin-Barnes integrals, but only the vectors $\boldsymbol{\tau}_i \in \mathbb{Z}^d$ associated to them. The key to the proof of Theorem 3.1.1 is that we show independence of the choice of the a local basis that we use for the construction of the Hermitian matrix. In this chapter we extend this idea by choosing $\boldsymbol{\tau}_i$ not necessarily associated to Mellin-Barnes solutions and study to what extent the Hermitian matrix constructed is independent of the choice of local basis. If successful, we say that the $\boldsymbol{\tau}_i$ correspond to **virtual** Mellin-Barnes solutions and the Hermitian matrix a **virtual** Hermitian form. The hope is that the corresponding virtual monodromy matrices correspond to actual monodromy matrices without the knowledge of Mellin-Barnes solutions.

In Chapter 3 we showed how the Hermitian form emerges from certain residues of ω depending on a set of $\boldsymbol{\tau}$'s chosen inside the zonotope. It was defined as

$$\omega(\boldsymbol{\tau}, \boldsymbol{z}) := \frac{\boldsymbol{z}^{\boldsymbol{\tau}}}{(x_1 \boldsymbol{z}^{\boldsymbol{b}_1} - 1)(x_2 \boldsymbol{z}^{\boldsymbol{b}_2} - 1) \dots (x_n \boldsymbol{z}^{\boldsymbol{b}_N} - 1)} \frac{d\boldsymbol{z}}{\boldsymbol{z}}. \tag{4.1}$$

To define the virtual Hermitian form we will rely on these residues. First of all H contains only information about the difference between the $\boldsymbol{\tau}_i$'s that give Mellin-Barnes integrals. What we can do is forget about these Mellin Barnes integrals and let $\boldsymbol{\tau} \in \mathbb{Z}^d$ be a possible difference of two vectors. To ensure that H is well-defined, i.e. independent of the cotriangulation we can use the following condition.

Definition 4.1.1. A vector $\boldsymbol{\tau} \in \mathbb{Z}^d$ is called **cotriangulation independent** if for any two cotriangulations $\mathcal{I}$ and $\mathcal{J}$ we have

$$\sum_{\boldsymbol{\mu}:I(\boldsymbol{\mu})\in\mathcal{I}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\mu}}}(\omega(\boldsymbol{\tau},\boldsymbol{z})) = \sum_{\boldsymbol{\nu}:I(\boldsymbol{\nu})\in\mathcal{J}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{\zeta}^{\boldsymbol{\nu}}}(\omega(\boldsymbol{\tau},\boldsymbol{z})). \tag{4.2}$$

In Chapter 3 we have seen that we can restrict to adjacent cotriangulations to verify condition 4.1.1.

Lemma 4.1.2. *Let $\boldsymbol{\tau} \in \mathbb{Z}^d$. Then $\boldsymbol{\tau}$ is cotriangulation independent if and only if for any two adjacent cotriangulations $\mathcal{I}$ and $\mathcal{J}$ equation* (4.2) *holds.*

Proof.
$\Rightarrow$. Trivial.
$\Leftarrow$. By lemma 3.3.5, let $\mathcal{I} = \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_N = \mathcal{J}$ be any sequence of triangulations such that $\mathcal{I}_i$ and $\mathcal{I}_{i+1}$ are adjacent for $i = 1, \ldots, N-1$. Then equation (4.2) holds for $\mathcal{I} = \mathcal{I}_1$ and $\mathcal{J} = \mathcal{I}_N$. $\square$

Definition 4.1.3. The set of all cotriangulation independent $\boldsymbol{\tau} \in \mathbb{Z}^d$ is called the **Frobenius Cavity** and is denoted by $\mathscr{F}_B$. A cotriangulation independent set or **CI-set** is a set T of D vectors in $\mathbb{R}^d$ whose difference set $T - T := \{a - b \,|\, a, b \in T\}$ is a subset of $\mathscr{F}_B$.

Given a CI-set then we can construct a matrix H like in Theorem 3.1.1. We do this by taking $\boldsymbol{\tau}$ in this CI-set. In each component of H^{-1} we now get the same result for each choice of convergence direction as the differences of these $\boldsymbol{\tau}$ all give cotriangulation independent results. This brings us to the following definition.

Definition 4.1.4. A totally non-resonant A-hypergeometric system of holonomic rank D is said to admit a **Virtual Hermitian Form** or **VHF** if its Frobenius cavity contains at least one CI-set. The Virtual Hermitian Form corresponding to CI-set T is the matrix $\widetilde{H}_T$ as constructed in Theorem 3.1.1 by using elements in T as substitute for $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D$.

Example 4.1.5. If the A-hypergeometric system admits a Mellin-Barnes basis of solutions then it admits a VHF one of which is H itself. This is evident as every vector in $Z_B^\circ \cap \mathbb{Z}^d$ is cotriangulation independent and the $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D \in \frac{1}{2} Z_B^\circ$ that helped to create H will now form a CI-set.

Example 4.1.6. The Appell system F_4 does not allow a basis of Mellin-Barnes solutions. However, with the choice of the points $(0,0), (1,0), (0,1)$, $(1,1)$ as $\boldsymbol{\tau}_i$ in formula (2.7) we get a Hermitian matrix through formula (3.1). The Hermitian matrix turns out to be independent of the choice of cotriangulation. Recall the matrices $M_{\boldsymbol{\rho},j} = X_{\boldsymbol{\rho}}\chi_{\boldsymbol{\rho},j}X_{\boldsymbol{\rho}}^{-1}$ defined in Chapter 2. Using this choice of $\boldsymbol{\tau}_i$ we find that these $M_{\boldsymbol{\rho},j}$ generate part of the monodromy. See Example 4.4.5 for the complete example.

Our first goal is to give a characterization of $\mathscr{F}_B$. Unfortunately all we have is a conjectural description which is explained in Conjecture 4.4.3. In order to show that $\mathscr{F}_B$ can be larger than $Z_B^\circ \cap \mathbb{Z}^d$ at all we describe two algorithms in the next section.

The first one shows how you can generate more points in $\mathscr{F}_B$ using a known subset of $\mathscr{F}_B$. The second one shows how you can generate more points in its complement $\mathscr{F}_B^c$ using a known subset of $\mathscr{F}_B^c$ and $\mathscr{F}_B$.

We finally remark that in the many examples of $\mathscr{F}_B$ that we have computed a large proportion allows a choice of a CI-set, thus allowing a virtual Mellin-Barnes basis. Unfortunately there also exist some exotic cases where this is not possible, for instance in Example 4.4.7. In section 4.5 we try to shed some light on finding these CI-sets.

4.2 Zonotopal Propagation

The idea for this algorithm is that we start with a specific set of points that we know are cotriangulation independent and then use those to generate others.

Algorithm 4.2.1 (Zonotopal Propagation)**.** In this algorithm we start with a subset $U_0 \subset \mathscr{F}_B$ as input. The algorithm produces a subset of $\mathscr{F}_B$ which is hopefully larger than U_0.

Step 1 Given U_k we construct

$$T = \{\boldsymbol{\zeta} \pm \boldsymbol{b} \notin U_k | \boldsymbol{\zeta} \in U_k,\ \boldsymbol{b} \text{ column of } B\}.$$

Step 2 For each $\boldsymbol{\tau} \in T$ we let $B_{\boldsymbol{\tau}}^+$ be the set of columns $\boldsymbol{b}$ such that $\boldsymbol{\tau}+\boldsymbol{b} \in U_k$ and $B_{\boldsymbol{\tau}}^-$ the set of columns $\boldsymbol{b}$ such that $\boldsymbol{\tau} - \boldsymbol{b} \in U_k$. Finally we let $B_{\boldsymbol{\tau}}^0$ be the set of columns of B which are neither in $B_{\boldsymbol{\tau}}^+$ nor in $B_{\boldsymbol{\tau}}^-$. If $B_{\boldsymbol{\tau}}^-$ and $B_{\boldsymbol{\tau}}^+$ have a vector in common, set $\boldsymbol{\tau} \in U_{k+1}$.

Step 3 Let $S_{\boldsymbol{\tau}}$ be the $\mathbb{Z}$-span of $B_{\boldsymbol{\tau}}^0$. For each $\boldsymbol{\tau} \in T$ check whether $\boldsymbol{\tau} - \sum_{\boldsymbol{b} \in B_{\boldsymbol{\tau}}^-} \boldsymbol{b}$ is in $S_{\boldsymbol{\tau}}$. If not then $\boldsymbol{\tau} \in U_{k+1}$.

Step 4 If $U_k = U_{k+1}$ go to step 5, otherwise return to Step 1 after incrementing k.

Step 5 Output U_k.

In practice this algorithm will not give many new points if we take $U_0 = Z_B^\circ \cap \mathbb{Z}^d$. However, by explicitly testing the boundary points of Z_B for cotriangulation independence we can find $\mathscr{F}_B \cap Z_B$. We call this the **brute force method**. We then conjecture

Conjecture 4.2.2. *If we let $U_0 = \mathscr{F}_B \cap Z_B$ in Algorithm 4.2.1 then the algorithm fully reconstructs $\mathscr{F}_B$.*

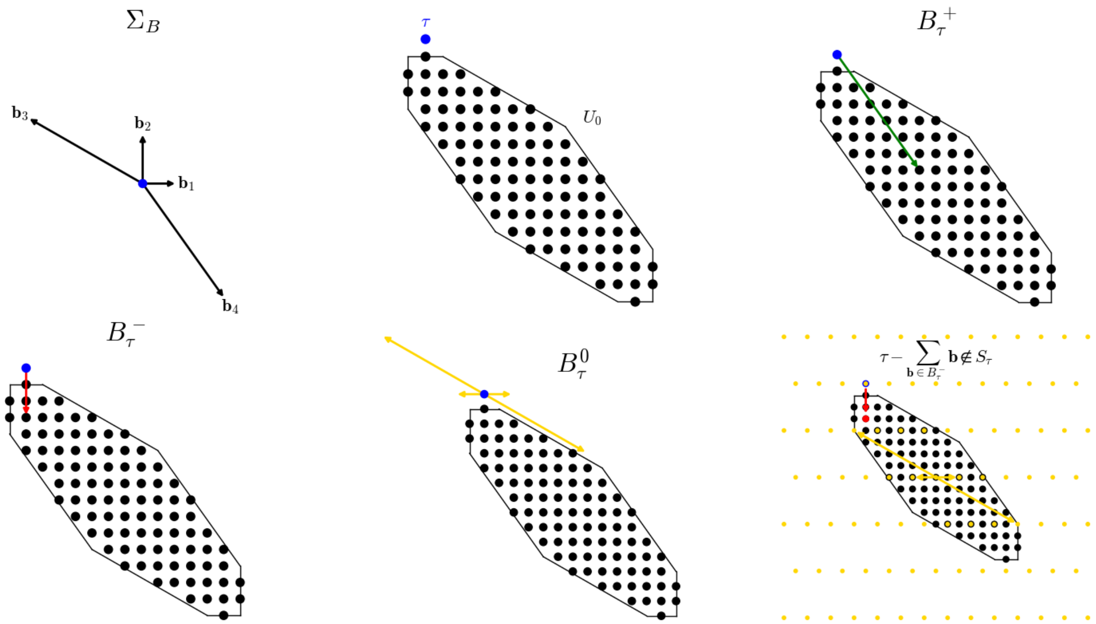
We illustrate our algorithm in Example 4.2.3.

In Example 4.3.2 we will see how complex these pictures may get when we iterate. In both examples we let $U_0 = \mathscr{F}_B \cap Z_B$, obtained via our brute force method.

Example 4.2.3. Consider the A-hypergeometric system where

$$B = \begin{pmatrix} 2 & 0 & -7 & 5 \\ 0 & 3 & 4 & -7 \end{pmatrix}.$$

When starting with initial conditions $U_0 = \mathscr{F}_B \cap Z_B$, we want to know whether $\boldsymbol{\tau} \in U_1$. Visually Algorithm 4.2.1 comes down to the following steps.



The first picture depicts the vectors $b_i, i = 1, \ldots, 4$. Then we choose a point $\boldsymbol{\tau}$ just above the zonotope in the second picture. In the following steps we shall verify that $\boldsymbol{\tau} \in U_1$. In the third picture we see that $\boldsymbol{\tau} + \boldsymbol{b}_4 \in U_0$ and $B^+_{\boldsymbol{\tau}} = \{\boldsymbol{b}_4\}$. In the fourth picture we see that $\boldsymbol{\tau} - \boldsymbol{b}_2 \in U_0$ and $B^-_{\boldsymbol{\tau}} = \{\boldsymbol{b}_2\}$.

The remaining $\boldsymbol{b}_i$-vectors yield $B^0_{\boldsymbol{\tau}} = \{\boldsymbol{b}_1, \boldsymbol{b}_3\}$. They are shown together with their opposites in yellow in the fifth picture. Now $S_{\boldsymbol{\tau}}$ is the $\mathbb{Z}$-span of $B^0_{\boldsymbol{\tau}}$ and we see in the last picture that $\boldsymbol{\tau} - \sum_{\boldsymbol{b} \in B^-_{\boldsymbol{\tau}}} \boldsymbol{b}$ is not in $S_{\boldsymbol{\tau}}$. So indeed $\boldsymbol{\tau} \in U_1$.

Algorithm 4.2.1 is obtained by investigating the differential properties of a binomial residue as defined in [CDS02]. We consider a specialization of the binomial residue where we put the β-parameters as used in [CDS02] equal to 1.

Choose a cotriangle I and let Γ_I be the finite set of solutions $(z_1, \ldots, z_d)$ to the equations $x_i + y_i \boldsymbol{z}^{\boldsymbol{b}_i} = 0, i \in I$. Then define the differential form

$$\hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z}) = \frac{\boldsymbol{z}^{\boldsymbol{\tau}}}{(x_1 + y_1 \boldsymbol{z}^{\boldsymbol{b}_1})(x_2 + y_2 \boldsymbol{z}^{\boldsymbol{b}_2}) \ldots (x_N + y_N \boldsymbol{z}^{\boldsymbol{b}_N})} \frac{d\boldsymbol{z}}{\boldsymbol{z}},$$

to define the binomial residue

$$R^{\boldsymbol{\tau}}_{\Gamma_I}(\boldsymbol{x}, \boldsymbol{y}) := \sum_{P \in \Gamma_I} \operatorname*{Res}_{\boldsymbol{z}=P} \hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z}). \tag{4.3}$$

Here $\operatorname{Res}_{\boldsymbol{z}=P} \hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z})$ is defined as in (3.5), where we take $f_j = x_i + y_i \boldsymbol{z}^{\boldsymbol{b}_i}$. For any formal linear combination $\Gamma = \sum_I a_I \Gamma_I$ we define by linear extension

$$R^{\boldsymbol{\tau}}_{\Gamma}(\boldsymbol{x}, \boldsymbol{y}) = \sum_I a_I R^{\boldsymbol{\tau}}_{\Gamma_I}(\boldsymbol{x}, \boldsymbol{y}).$$

We shall be interested in Γ of the form $\sum_{I \in \mathcal{I}} \Gamma_I - \sum_{J \in \mathcal{J}} \Gamma_J$, where $\mathcal{I}, \mathcal{J}$ are any two cotriangulations. Clearly we shall be interested in $\boldsymbol{\tau}$ such that $R^{\boldsymbol{\tau}}_{\Gamma}(\boldsymbol{x}, \boldsymbol{y}) = 0$ for such Γ.

Theorem 4.2.4 ([CDS02])**.** *The integral $R^{\boldsymbol{\tau}}_{\Gamma}(\boldsymbol{x}, \boldsymbol{y})$ is a rational solution to the B_L-hypergeometric system $H_{B_L}(-\mathbf{1}, -\boldsymbol{\tau})$, where $\mathbf{1}$ is the vector of length N containing only ones and*

$$B_L := \begin{pmatrix} Id_N & Id_N \\ 0 & B \end{pmatrix}.$$

In this setting B_L is called the **Lawrence lifting** of B. This theorem shows that $R^{\boldsymbol{\tau}}_{\Gamma}$ satisfies the following differential equations

$$(x_i \partial_{x_i} + y_i \partial_{y_i} + 1) R^{\boldsymbol{\tau}}_{\Gamma} = 0, \quad i = 1, \ldots, N. \tag{4.4}$$

$$\left(\tau_j + \sum_{i=1}^{N} \boldsymbol{b}_{i,j} y_i \partial_{y_i}\right) R_\Gamma^{\boldsymbol{\tau}} = 0, \quad j = 1, \ldots, d. \tag{4.5}$$

Another property can be observed by differentiating with respect to x_i and y_i, which comes down to shifting the parameter vector $\boldsymbol{\tau}$.

$$\partial_{y_i} R_\Gamma^{\boldsymbol{\tau}}(\boldsymbol{x}, \boldsymbol{y}) = \partial_{x_i} R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{b}_i}(\boldsymbol{x}, \boldsymbol{y}). \tag{4.6}$$

See the proof of [CDS02, Lemma 3.4] for more information.

Correctness of Algorithm 4.2.1. Let U be a set of vectors such that $R_\Gamma^{\boldsymbol{\tau}} = 0$ for all $\boldsymbol{\tau} \in U$ and some choice of Γ. Then what can we say about the values of $R_\Gamma^{\boldsymbol{\tau}}$ for other $\boldsymbol{\tau}$ outside of U? We only use the differential properties of $R_\Gamma^{\boldsymbol{\tau}}$ for this.

Let $\boldsymbol{\tau} \in \mathbb{Z}^d \setminus U$, then we want to know whether $R_\Gamma^{\boldsymbol{\tau}} = 0$.

Denote the set of those $\boldsymbol{b}_i$-vectors such that $\boldsymbol{\tau} - \boldsymbol{b}_i \in U$ by B^- and the set of those $\boldsymbol{b}_i$-vectors such that $\boldsymbol{\tau} + \boldsymbol{b}_i \in U$ by B^+ and the set of remaining $\boldsymbol{b}_i$-vectors by B^0. Now we can use (4.6). When $\boldsymbol{b}_i \in B^+$ this means $\partial_{y_i} R_\Gamma^{\boldsymbol{\tau}} = 0$ and when $\boldsymbol{b}_i \in B^-$ this means that by a shift of the parameter that $\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}} = 0$. Hence in the former $R_\Gamma^{\boldsymbol{\tau}}$ doesn't depend on y_i, and in the latter $R_\Gamma^{\boldsymbol{\tau}}$ doesn't depend on x_i.

Now we use (4.4), hence in our case when $\boldsymbol{b}_i \in B^+$ we have $x_i \partial_{x_i} R_\Gamma^{\boldsymbol{\tau}} = -R_\Gamma^{\boldsymbol{\tau}}$ which means $R_\Gamma^{\boldsymbol{\tau}} = \frac{1}{x_i} C$ where C doesn't depend on x_i or y_i. Similarly when $\boldsymbol{b}_i \in B^-$ then $y_i \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}} = -R_\Gamma^{\boldsymbol{\tau}}$ and hence $R_\Gamma^{\boldsymbol{\tau}} = \frac{1}{y_i} C$ where C doesn't depend on variables x_i or y_i. Note that as a consequence if B^+ and B^- have overlap, that automatically $R_\Gamma^{\boldsymbol{\tau}} = 0$ and we are done.

Therefore we are left with $R_\Gamma^{\boldsymbol{\tau}} = \prod_{\boldsymbol{b}_i \in B^+} \frac{1}{x_i} \cdot \prod_{\boldsymbol{b}_i \in B^-} \frac{1}{y_i} \cdot C$ where C only depends on x_i and y_i for those i that are indices of $\boldsymbol{b}_i \in B^0$. Finally we use (4.5) for $j = 1, \ldots, d$, this will give us

$$\begin{aligned}
\left(\tau_j + \sum_{i=1}^{N} \boldsymbol{b}_{i,j} y_i \partial_{y_i}\right) R_\Gamma^{\boldsymbol{\tau}} &= 0 \\
-\sum_{\boldsymbol{b}_i \in B^-} \boldsymbol{b}_{i,j} R_\Gamma^{\boldsymbol{\tau}} + \sum_{\boldsymbol{b}_i \in B^0} \boldsymbol{b}_{i,j} y_i \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}} &= -\tau_j R_\Gamma^{\boldsymbol{\tau}} \\
\sum_{\boldsymbol{b}_i \in B^0} \boldsymbol{b}_{i,j} y_i \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}} &= \left(-\tau_j + \sum_{\boldsymbol{b}_i \in B^-} \boldsymbol{b}_{i,j}\right) R_\Gamma^{\boldsymbol{\tau}}. \qquad (4.7)
\end{aligned}$$

As $R_\Gamma^{\boldsymbol{\tau}}$ is rational we can write it as a multivariate Laurent series $\sum_{\boldsymbol{v},\boldsymbol{w}} \lambda_{\boldsymbol{v},\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{v}} \boldsymbol{y}^{\boldsymbol{w}}$ where $\lambda_{\boldsymbol{v},\boldsymbol{w}} \in \mathbb{C}$ and $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^N$. Now we apply (4.7) to obtain

$$\sum_{\boldsymbol{v},\boldsymbol{w}} \left(\sum_{\boldsymbol{b}_i \in B^0} \boldsymbol{w}_i \boldsymbol{b}_{i,j} \right) \lambda_{\boldsymbol{v},\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{v}} \boldsymbol{y}^{\boldsymbol{w}} = \sum_{\boldsymbol{v},\boldsymbol{w}} \left(-\tau_j + \sum_{\boldsymbol{b}_i \in B^-} \boldsymbol{b}_{i,j} \right) \lambda_{\boldsymbol{v},\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{v}} \boldsymbol{y}^{\boldsymbol{w}}.$$

To make both sides equal for all $j = 1, \ldots, d$ we get for each $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{Z}^N$ seperately that

$$\lambda_{\boldsymbol{v},\boldsymbol{w}} \sum_{\boldsymbol{b}_i \in B^0} \boldsymbol{w}_i \boldsymbol{b}_i = \lambda_{\boldsymbol{v},\boldsymbol{w}} \left(-\boldsymbol{\tau} + \sum_{\boldsymbol{b}_i \in B^-} \boldsymbol{b}_i \right).$$

This means that for $R_\Gamma^{\boldsymbol{\tau}}$ to be certainly zero, $-\boldsymbol{\tau} + \sum_{\boldsymbol{b}_i \in B^-} \boldsymbol{b}_i$ cannot be in the $\mathbb{Z}$-module generated by the vectors in B^0. If this is the case, we conclude $R_\Gamma^{\boldsymbol{\tau}} = 0$. $\square$

4.3 Zonotopal Elimination

But there is more to this, given points not in $\mathscr{F}_B$ we can also find more points that are not in $\mathscr{F}_B$.

Theorem 4.3.1 (Zonotopal Elimination)**.** *Let $\boldsymbol{\tau} \in \mathbb{Z}^d$ be such that $R_\Gamma^{\boldsymbol{\tau}} \neq 0$. There exists a vector $\sigma \in \{\pm 1\}^N$ such that $R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{s}} \neq 0$ for all $\boldsymbol{s}$ in the semigroup S generated by $\{\sigma_i \boldsymbol{b}_i\}_{i=1,\ldots,N}$.*

Proof. For each $i = 1, \ldots, N$ we will find a possible σ_i separately. We say that a function $f(\boldsymbol{z})$ is polynomial in z_i if it can be written in the form $f_0 + f_1 z_i + \ldots + f_k z_i^k$ where f_j are functions that do not depend on z_i. We call k the degree of f in terms of z_i. The proof only uses equations (4.6) and (4.4) and these will be recalled when needed. There are four cases to consider

- $R_\Gamma^{\boldsymbol{\tau}}$ is not polynomial in y_i, then by repeatedly applying ∂_{y_i} to $R_\Gamma^{\boldsymbol{\tau}}$ we will never see it become 0. Hence by repeatedly applying the differential equation
$$\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{b}_i} = \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}}.$$
We see that we can take $\sigma_i = 1$.

- $R_\Gamma^{\boldsymbol{\tau}}$ is not polynomial in x_i, then by repeatedly applying ∂_{x_i} to $R_\Gamma^{\boldsymbol{\tau}}$ we will never see it become 0. Hence by repeatedly applying the differential equation
$$\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}} = \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}-\boldsymbol{b}_i}.$$
We see that we can take $\sigma_i = -1$.

- $R_\Gamma^{\boldsymbol{\tau}}$ is polynomial in y_i, then after repeatedly applying

  $$\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{b}_i} = \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}}$$

  means that for some $k > 0$ the function $R_\Gamma^{\boldsymbol{\tau}+k\boldsymbol{b}_i}$ does not depend on y_i, and hence either it already is 0 or it can be made zero by applying the same procedure after we replace $\boldsymbol{\tau}$ by $\boldsymbol{\tau} + k\boldsymbol{b}_i$ and we take a different $\boldsymbol{b}_j$ as a generator. Therefore this is something that we want to avoid. So pick $\sigma_i = -1$ then we have to show that $R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i}$ depends on y_i for all $k > 0$. Consider the differential equation

  $$x_i \partial_{x_i} R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i} + y_i \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i} + R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i} = 0.$$

  The part $y_i \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i} + R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i}$ is non-zero and still polynomial in y_i with the same degree as $R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i}$. Hence $\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i}$ is non-zero and depends polynomially on y_i. As a consequence

  $$\partial_{x_i} R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i} = \partial_{y_i} R_\Gamma^{\boldsymbol{\tau}-(k+1)\boldsymbol{b}_i}$$

  simply tells us that $R_\Gamma^{\boldsymbol{\tau}-(k+1)\boldsymbol{b}_i}$ is a nonzero polynomial in y_i with degree one more than $R_\Gamma^{\boldsymbol{\tau}-k\boldsymbol{b}_i}$.

- $R_\Gamma^{\boldsymbol{\tau}}$ is polynomial in x_i, then symmetrically to the previous case we can show that $\sigma_i = 1$.

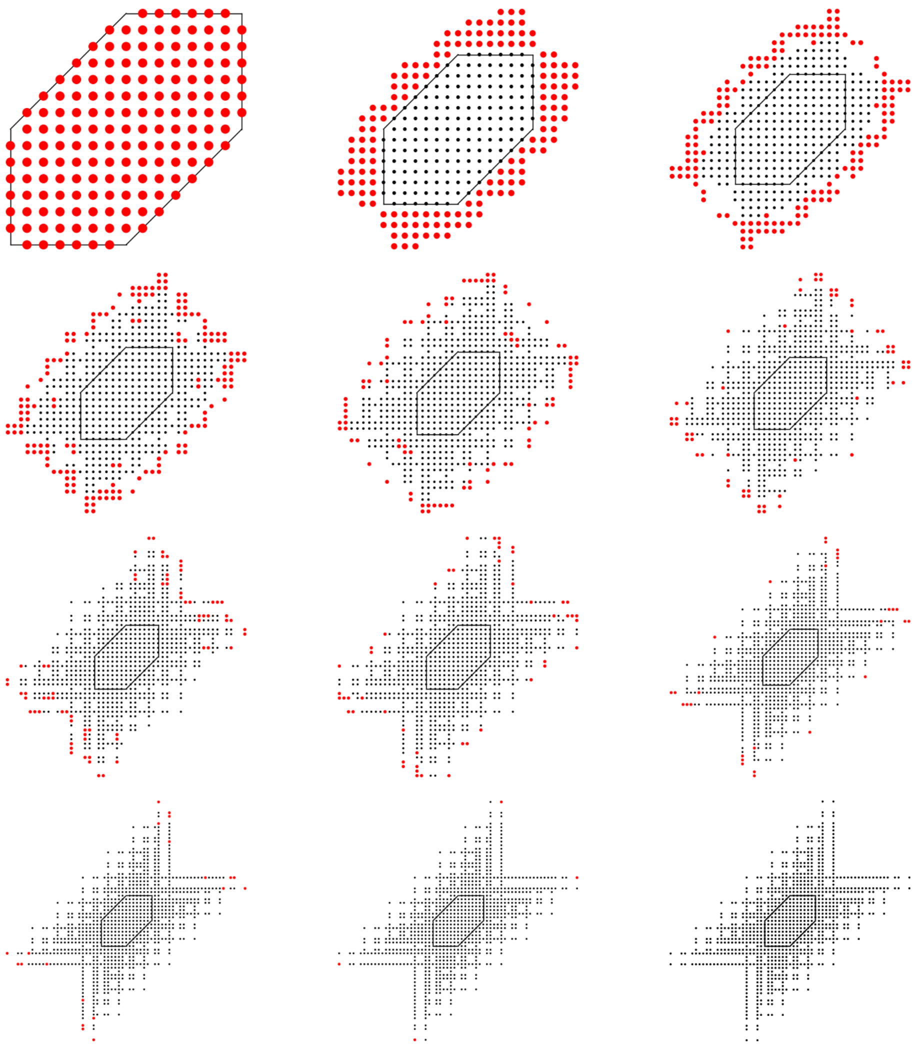
Now for $\boldsymbol{s} \in S$ we only have to keep track of the degrees of the x_i and y_i which are polynomial in $R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{s}}$ depending on whether we chose $\sigma_i = 1$, respectively $\sigma_i = -1$. These degrees can only increase as we move further away from $\boldsymbol{\tau}$ by choice of $\boldsymbol{\sigma}$. And in each move that we do, we can ensure by the discussion above that $R_\Gamma^{\boldsymbol{\tau}+\boldsymbol{s}} \neq 0$. □

Together with Algorithm 4.2.1 this gives us a very effective toolbox in generating $\mathscr{F}_B$. In most cases a handful of strategically picked $\boldsymbol{\tau}$ have to be checked by brute force to find $\mathscr{F}_B$. If we find that $\boldsymbol{\tau} \in \mathscr{F}_B$, use Algorithm 4.2.1 and if not then use Theorem 4.3.1 try to find a shifted semigroup of $\boldsymbol{\tau}$'s not contained in $\mathscr{F}_B$. In many cases for $\boldsymbol{\tau} \in \mathscr{F}_B^c$ the semigroup S is easy to verify and follows straight from the combinatorics of $\mathscr{F}_B$. Sometimes there are multiple possible choices for S for a given $\boldsymbol{\tau}$ and we have to take a closer look at $R_\Gamma^{\boldsymbol{\tau}}$ before we can assess which S can be eliminated.

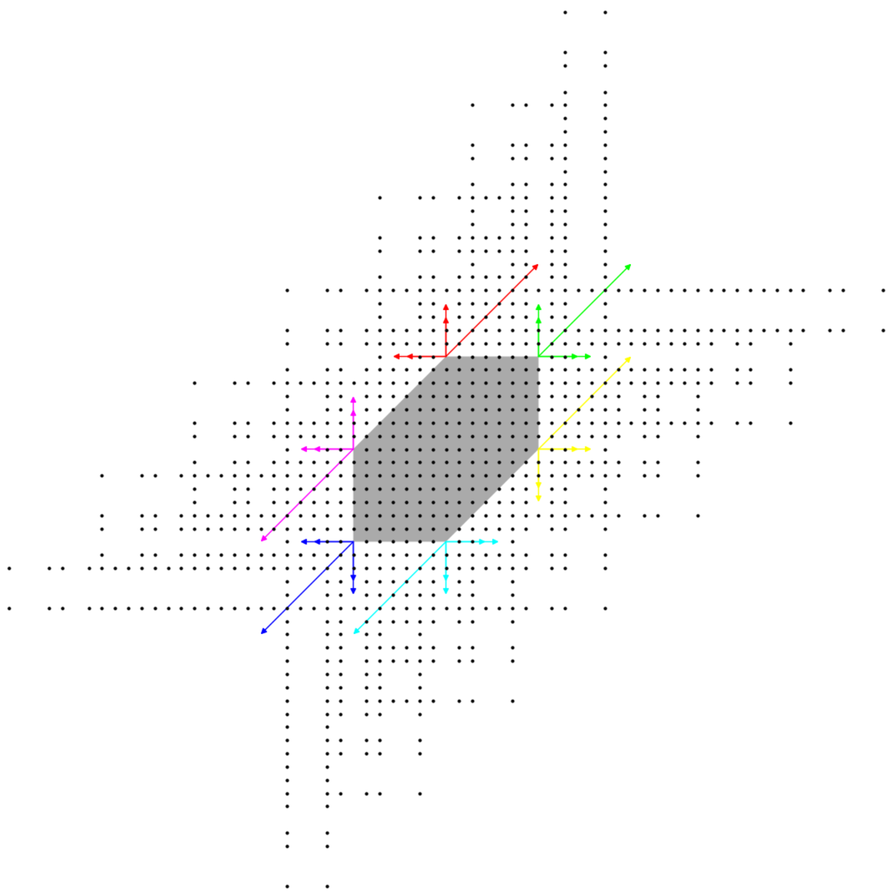
Example 4.3.2. Consider the A-hypergeometric system where $D = 49$ and

$$B = \begin{pmatrix} 3 & 4 & 0 & 0 & -7 \\ 0 & 0 & 3 & 4 & -7 \end{pmatrix}.$$

Then zonotopal propagation looks as follows. In the first picture we have used brute force to verify the points on the boundary of Z_B.



By zonotopal elimination we can show that this is exactly $\mathscr{F}_B$. The complement of $\mathscr{F}_B$ is the union of six shifted semigroups in this case, each one starting from a vertex of Z_B and the generating $\boldsymbol{b}_i$-vectors all pointing outward of the zonotope. For each of these semigroups the generators are given by the arrows. More about this structure is explained in the next section.



4.4 The Frobenius cavity

In this section we like to give a combinatorial description of the set $\mathscr{F}_B$. This description is conjectural and given in Conjecture 4.4.3. The only support we have for this conjecture is the many examples we tested and a heuristic argument.

The approach to finding the structure of $\mathscr{F}_B$ is to connect the definition of cotriangulation independence to a condition on residues of $\omega(\boldsymbol{\tau}, \boldsymbol{z})$ at points at 0 and ∞. The main idea was to use the following theorem

Theorem 4.4.1 (Gelfond-Khovanskii, [GK02, Theorem 2], [Kar18, Theorem 3.1]).

Let $D_1, \ldots, D_d$ and $D'_1, \ldots, D'_d$ be two sets of divisors on a compact analytic d-dimensional manifold M, each having a 0-dimensional intersection. Assume $D_i \cap D'_i = \emptyset$ for every i and put $Z = \bigcup_{i=1}^d D_i \cup \bigcup_{i=1}^d D'_i$. Then for any holomorphic $\omega \in \Omega^d(M \setminus Z)$ we have:

$$\sum_{p \in D_1 \cap \ldots \cap D_d} Res_p\, \omega = (-1)^d \sum_{q \in D'_1 \cap \ldots \cap D'_d} Res_q\, \omega.$$

If we take the sequence of divisors $D'_i = \{\boldsymbol{z} \in \mathbb{P}_1^d | z_i = 0 \text{ or } z_i = \infty\}$ for $i = 1, \ldots, d$. We let $\mathcal{T}$ be the set of cotriangles and

$$V = \{\boldsymbol{z} \in \mathbb{P}_1^d | \text{ There exists } I \in \mathcal{T} \text{ such that } x_{I_1}\boldsymbol{z}^{\boldsymbol{b}_{I_1}} - 1 = \ldots = x_{I_d}\boldsymbol{z}^{\boldsymbol{b}_{I_d}} - 1 = 0\}.$$

And then we have the set of divisors

$$D_i = \{\boldsymbol{z} \in \mathbb{P}_1^d | \text{ There exists a } \boldsymbol{v} \in V \text{ such that } v_i = z_i\},$$

for $i = 1, \ldots, d$. And in this way one checks that $D_i \cap D'_i = \emptyset$ and $\omega(\boldsymbol{\tau}, \boldsymbol{z})$ is holomorphic over $\mathbb{P}_1^d \setminus Z$, where $Z = \bigcup_{i=1}^d D_i \cup \bigcup_{i=1}^d D'_i$. In this way Theorem 4.4.1 tells us that

$$\sum_{I \in \mathcal{T}} \sum_{\boldsymbol{t} \in D_{I_1} \cap \ldots \cap D_{I_d}} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{t}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = 0 \tag{4.8}$$

if and only if

$$\sum_{\boldsymbol{t} \in D'_1 \cap \ldots \cap D'_d} \operatorname*{Res}_{\boldsymbol{z}=\boldsymbol{t}} \omega(\boldsymbol{\tau}, \boldsymbol{z}) = 0. \tag{4.9}$$

The problem with this approach is that the relation (4.8) is not necessarily a linear combination of the relations coming from cotriangulation independence. Furthermore if (4.8) holds and it is a linear combination of equations of the type (4.2) then it does not necessarily imply cotriangulation independence. So we are stuck here. However we use it as an indication on how the structure of $\mathscr{F}_B$ looks like and as motivation for Definition 4.4.2.

Definition 4.4.2. Let $\boldsymbol{\rho} \in \mathbb{R}^d$ be a convergence direction then the **Zonotopal Boundary Point** corresponding to $\boldsymbol{\rho}$ is

$$P_{\boldsymbol{\rho}} := \sum_{\substack{i=1 \\ \boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0}}^{N} \boldsymbol{b}_i.$$

The **Zonotopal Semigroup** corresponding to $\boldsymbol{\rho}$ is the semigroup generated by the $\boldsymbol{b}_i$ in the following fashion:

$$M_{\boldsymbol{\rho}} := \langle \{\boldsymbol{b}_i | \boldsymbol{b}_i \cdot \boldsymbol{\rho} \geq 0\} \cup \{-\boldsymbol{b}_i | \boldsymbol{b}_i \cdot \boldsymbol{\rho} < 0\} \rangle .$$

The **Zonotopal Span** of B, denoted by $\mathscr{Z}_B$ is the union of translated semigroups corresponding to B, i.e.

$$\mathscr{Z}_B := \bigcup_{\boldsymbol{\rho} \in S^{d-1}} P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}.$$

We denote by $\mathscr{Z}_B^c$ its complement in $\mathbb{Z}^d$.

Supported by experimental data and an heuristic argument we get the following conjecture.

Conjecture 4.4.3. *The Frobenius Cavity is equal to the complement of the Zonotopal span in $\mathbb{Z}^d$, i.e.*

$$\mathscr{F}_B = \mathscr{Z}_B^c.$$

The definition of $\mathscr{Z}_B^c$ is also the source of the name "Frobenius cavity" for $\mathscr{F}_B^c$. $\mathscr{Z}_B$ is nothing more than a multidimensional analogue of a numerical semigroup. In the complement of a numerical semigroup there are points called gaps and Frobenius numbers being the largest of these gaps. Also the Diophantine equation which determines whether a point is inside the numerical semigroup is called a Frobenius equation. In our case we have a union of multidimensional semigroups. When we take the complement in $\mathbb{Z}^d$ the points in the zonotope are always gaps. And we get some more gaps coming from the dynamics of these semigroups. In the experiments that we did we also obtain a union of semigroups by zonotopal elimination. This union has always corresponded to $\mathscr{Z}_B$. In addition, heuristically we can see that this happens. The following lemma shows how $\mathscr{Z}_B^c$ can be generated from $\hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z})$ by using certain residues around points at 0 and ∞.

Lemma 4.4.4. *Let $\Gamma_{\boldsymbol{\rho}}$ be the cycle $t^{\rho_1}\mathbb{S}^1 \times \ldots \times t^{\rho_d}\mathbb{S}^1$ where $t > 0$ is chosen small enough and let $|x_i/y_i| \neq 1$. Then $\frac{1}{(2\pi i)^d} \oint_{\Gamma_{\boldsymbol{\rho}}} \hat{\omega}(\boldsymbol{\tau}) = 0$ for all real $\boldsymbol{\rho} \in \mathbb{S}^{d-1}$ if and only if $\boldsymbol{\tau} \in \mathscr{Z}_B^c$.*

Proof. First assume $0 < |x_i/y_i| < 1$.

Depending on whether $\boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0$ or $\boldsymbol{b}_i \cdot \boldsymbol{\rho} \leq 0$ we can do the following:

$$\hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z}) = \boldsymbol{z}^{\boldsymbol{\tau}} \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} \geq 0} \frac{1/y_i}{\boldsymbol{z}^{\boldsymbol{b}_i} + x_i/y_i} \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} < 0} \frac{\boldsymbol{z}^{-\boldsymbol{b}_i}/x_i}{y_i/x_i + \boldsymbol{z}^{-\boldsymbol{b}_i}} \frac{d\boldsymbol{z}}{\boldsymbol{z}}$$

And this makes it possible to take a Laurent series expansion which converges when we take $|z_i^{\rho_i}|$ small enough and handle the case with $\boldsymbol{b}_i \cdot \boldsymbol{\rho} = 0$ separately using $|x_i/y_i| < 1$.

$$\begin{aligned}\hat{\omega}(\boldsymbol{\tau}, \boldsymbol{z}) &= \boldsymbol{z}^{\boldsymbol{\tau}} \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0} \frac{1}{y_i} \sum_{j=0}^{\infty} \frac{(-1)^j \boldsymbol{z}^{j\boldsymbol{b}_i}}{(x_i/y_i)^{j+1}} \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} < 0} \frac{1}{x_i} \sum_{j=0}^{\infty} \frac{(-1)^j (x_i/y_i)^{j+1}}{\boldsymbol{z}^{(j+1)\boldsymbol{b}_i}} \\ &\quad \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} = 0} \frac{1}{y_i} \sum_{j=0}^{\infty} \frac{(-1)^j (x_i/y_i)^j}{\boldsymbol{z}^{(j+1)\boldsymbol{b}_i}} \frac{d\boldsymbol{z}}{\boldsymbol{z}} \\ &= \boldsymbol{z}^{\boldsymbol{\tau}} \prod_{i=1}^{N} \frac{1}{x_i} \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0} \sum_{j=0}^{\infty} \frac{(-1)^j \boldsymbol{z}^{j\boldsymbol{b}_i}}{(x_i/y_i)^j} \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} \le 0} \sum_{j=0}^{\infty} \frac{(-1)^j (x_i/y_i)^{j+1}}{\boldsymbol{z}^{(j+1)\boldsymbol{b}_i}} \frac{d\boldsymbol{z}}{\boldsymbol{z}}.\end{aligned}$$

Hence when we want to find its residue we are looking for the constant term of the expression

$$\boldsymbol{z}^{\boldsymbol{\tau}} \prod_{i=1}^{N} \frac{1}{x_i} \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0} \sum_{j=0}^{\infty} \frac{(-1)^j \boldsymbol{z}^{j\boldsymbol{b}_i}}{(x_i/y_i)^j} \cdot \prod_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} \le 0} \sum_{j=0}^{\infty} \frac{(-1)^j (x_i/y_i)^{j+1}}{\boldsymbol{z}^{(j+1)\boldsymbol{b}_i}}.$$

This constant term is non-zero if and only if for some $\boldsymbol{j} \in \mathbb{Z}_{\ge 0}^N$

$$\boldsymbol{\tau} + \sum_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} > 0} j_i \boldsymbol{b}_i - \sum_{\boldsymbol{b}_i \cdot \boldsymbol{\rho} \le 0} (j_i + 1) \boldsymbol{b}_i = 0.$$

Hence we get $\boldsymbol{\tau} \notin P_{-\boldsymbol{\rho}} + M_{-\boldsymbol{\rho}}$ if and only if

$$\frac{1}{(2\pi i)^d} \oint_{\Gamma_{\boldsymbol{\rho}}} \hat{\omega} = 0.$$

When $|x_i/y_i| > 1$, analogously to the above we get the symmetric condition $-\boldsymbol{\tau} \notin P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$.

If $\frac{1}{(2\pi i)^d} \oint_{\Gamma_{\boldsymbol{\rho}}} \hat{\omega} = 0$ for both $|x_i/y_i| > 1$ and $0 < |x_i/y_i| < 1$ it will be zero for $|x_i/y_i| = 1$ by analytic continuation. As this is true for all $\boldsymbol{\rho}$, we find that $\frac{1}{(2\pi i)^d} \oint_{\Gamma_{\boldsymbol{\rho}}} \hat{\omega} = 0$ for all $\boldsymbol{\rho} \in \mathbb{C}^d \setminus \{0\}$ exactly when $\boldsymbol{\tau} \notin \mathscr{Z}_B$. □

Example 4.4.5. Appell's F_4 system does not admit a Mellin-Barnes basis of solutions, however it does admit a VHF. To see this we first construct $\mathscr{Z}_B$, take its complement in $\mathbb{Z}^d$ and then check whether we can find a CI-set in this complement. One can check that any point $\mathscr{Z}_B^c$ is cotriangulation independent. A possible Gale dual for F_4 is

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix}.$$

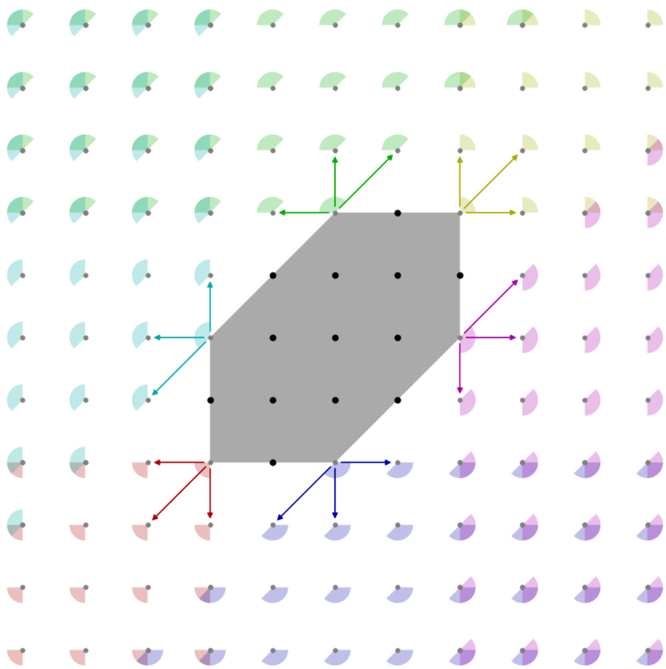


Figure 4.1: $\mathscr{Z}_B$ where B is a Gale dual belonging to F_4. The gray area is Z_B, the arrows show the generators of $M_{\boldsymbol{\rho}}$ starting at $P_{\boldsymbol{\rho}}$. Now in each zonotopal semigroup we created an unique arc and color such that we know which point belongs to which semigroup, note that some arcs will overlap. The 13 black dots that do not belong to a zonotopal semigroup represent $\mathscr{F}_B$.

From this information we can generate $\mathscr{Z}_B$ as seen in Figure 4.1.

In this case by doing the calculations one can show that $\mathscr{F}_B = \mathscr{Z}_B^c$ and it contains 13 dots and we have $D = 4$. A possible CI-set would be $\{(0,0),(1,0),(0,1),(1,1)\}$. Its difference set is inside $\mathscr{F}_B$ and so we are able to make a VHF H using this information.

Example 4.4.6. For examples of small dimension and small rank, it is possible to make these Frobenius Cavities very easily with $\mathscr{Z}_B^c$. However, as soon as we increase the dimension or increase the rank, things start to get complex very fast. For example, take the system of rank $D = 29$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 29 & 15 & 21 \end{pmatrix}$$

and has Gale dual

$$B = \begin{pmatrix} 2 & 0 & -7 & 5 \\ 0 & 3 & 4 & -7 \end{pmatrix}.$$

In Figure 4.2 we show $\mathscr{F}_B$, as this is only 2 dimensions there are still lots of restrictions on where the points can go. When $d > 2$ there will be even more freedom and the process of creating $\mathscr{Z}_B$ and then taking a complement becomes inefficient. Similarly the figures in Example 4.3.2

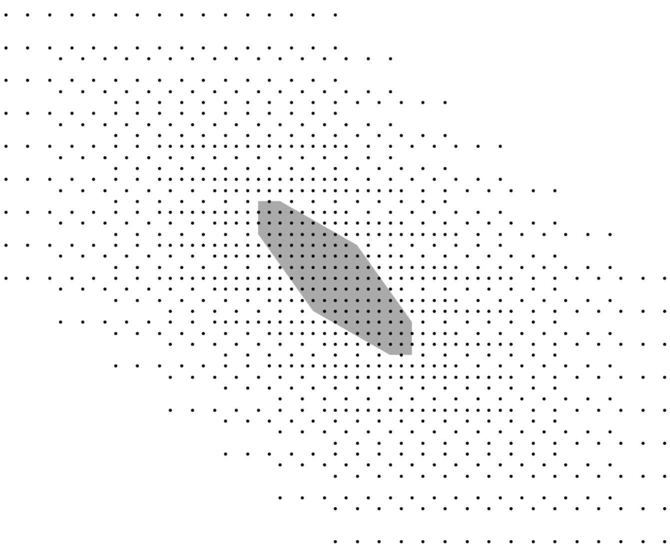


Figure 4.2: $\mathscr{F}_B$ and Z_B of the system of rank 29 in Example 4.4.6

show a similar complex structure appearing.

Example 4.4.7. The A-hypergeometric system of holonomic rank 25 where

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 & 2 & -7 \\ 0 & 1 & 2 & -2 & 1 & -1 & -1 \end{pmatrix},$$

is a case where $\mathscr{Z}_B^c = Z_B \cap \mathbb{Z}^d$. One can show that $\mathscr{F}_B = Z_B \cap \mathbb{Z}^d$ and now one checks that this system does not admit a Mellin-Barnes basis, hence it contains no difference set of rank 25. Thus this is a case where we fail to extend our set of $\boldsymbol{\tau}$'s and we cannot find a CI-set either.

Theorem 4.4.8. *$\mathscr{Z}_B^c$ is a finite set*

Proof. $\mathscr{Z}_B$ is the union of semigroups $P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$. Let $C_{\boldsymbol{\rho}}$ denote the cone starting at the origin with the same generators as $M_{\boldsymbol{\rho}}$. Now because the $\mathbb{Z}$-span of the $\boldsymbol{b}_i$'s is $\mathbb{Z}^d$ we notice that there is a point $S_{\boldsymbol{\rho}} \in P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$ such that

$(S_{\boldsymbol{\rho}}+C_{\boldsymbol{\rho}})\cap\mathbb{Z}^d \subset P_{\boldsymbol{\rho}}+M_{\boldsymbol{\rho}}$. Now pick a facet F of the zonotope Z_B and let $\boldsymbol{\rho}_1,\dots,\boldsymbol{\rho}_k$ be all convergence directions such that $P_{\boldsymbol{\rho}_1},\dots,P_{\boldsymbol{\rho}_k}$ are on F. And suppose there is a line L passing through F (or the hyperplane which extends F) which does not pass through $(S_{\boldsymbol{\rho}_1}+C_{\boldsymbol{\rho}_1})\cup\ldots\cup(S_{\boldsymbol{\rho}_k}+C_{\boldsymbol{\rho}_k})$. The zonotope is convex and $P_{\boldsymbol{\rho}}+C_{\boldsymbol{\rho}}$ contains the cone $P_{\boldsymbol{\rho}}+\widetilde{C}_{\boldsymbol{\rho}}$, which is the cone where we extend the edges of the zonotope outwards from $P_{\boldsymbol{\rho}}$. Hence the line L must be parallel to all facets adjacent to F for this line to exist. This means that Z_B is a parallelotope. The sum of columns of B is 0, so for every column vector of B there must be a parallel column vector of B pointing in the other direction. Hence $C_{\boldsymbol{\rho}}$ is a half space for any $\boldsymbol{\rho}$. So the line L must have intersection with $(S_{\boldsymbol{\rho}_1}+C_{\boldsymbol{\rho}_1})\cup\ldots\cup(S_{\boldsymbol{\rho}_k}+C_{\boldsymbol{\rho}_k})$, which is a contradiction.

As any line starting at 0 moving through a facet F eventually hits a saturated cone and from there it stays inside that cone. Cones are given by linear inequalities, so we see that the complement of $\bigcup_{\boldsymbol{\rho}\in\mathbb{R}^d\setminus\{0\}}(S_{\boldsymbol{\rho}_1}+C_{\boldsymbol{\rho}_1})$ in $\mathbb{R}^d$ is bounded. Hence the complement of $\bigcup_{\boldsymbol{\rho}\in\mathbb{R}^d\setminus\{0\}}(S_{\boldsymbol{\rho}_1}+C_{\boldsymbol{\rho}_1}\cap\mathbb{Z}^d)$ in $\mathbb{Z}^d$ is finite. $\square$

Thus far finding a direct connection between $\mathscr{Z}_B^c$ and $\mathscr{F}_B$ seems to be a hard task. Indirectly there is a vague connection found in the way we calculate residues. And we see that both structures exclude certain shifted semigroups. It becomes interesting when we try to see if Algorithm 4.2.1 is capable of finding $\mathscr{Z}_B^c$ given a specific initial configuration.

Theorem 4.4.9. *Algorithm 4.2.1 generates $\mathscr{Z}_B^c$ when the starting configuration is $U_0=\mathscr{Z}_B^c\cap Z_B$.*

Proof. We split the proof into two parts. The first part shows that the algorithm produces points in $\mathscr{Z}_B^c$. The second part shows that the algorithm fully reconstructs $\mathscr{Z}_B^c$.

Given a vector $\boldsymbol{\tau}$ and set $U\subset\mathbb{Z}^d$ denote the set of $\boldsymbol{b}_i$ such that $\boldsymbol{\tau}-\boldsymbol{b}_i\in U$ by B^- and the set of $\boldsymbol{b}_i$ such that $\boldsymbol{\tau}+\boldsymbol{b}_i\in U$ by B^+ and the set of remaining $\boldsymbol{b}_i$-vectors by B^0.

For the first part of the proof we suppose that the algorithm produces a point in $\mathscr{Z}_B$. Let $\boldsymbol{\tau}$ be the first such point. Denote the set constructed thus far by U. As $\boldsymbol{\tau}$ is inside a zonotopal semigroup for some $\boldsymbol{\rho}$, let $\boldsymbol{\tau}=P_{\boldsymbol{\rho}}+B\boldsymbol{k}$ for some $\boldsymbol{k}\in\mathbb{Z}^N$.

Recall that $P_{\boldsymbol{\rho}}=\sum_{\boldsymbol{b}_i\cdot\boldsymbol{\rho}>0}\boldsymbol{b}_i$ and

$$M_{\boldsymbol{\rho}}:=\langle\{\boldsymbol{b}_i|\boldsymbol{b}_i\cdot\boldsymbol{\rho}\geq 0\}\cup\{-\boldsymbol{b}_i|\boldsymbol{b}_i\cdot\boldsymbol{\rho}<0\}\cup\{\boldsymbol{b}_i|\boldsymbol{b}_i\cdot\boldsymbol{\rho}=0\}\rangle\,.$$

If the i-th index of $\boldsymbol{k}$ is non-zero then both $\boldsymbol{\tau}+\boldsymbol{b}_i$ and $\boldsymbol{\tau}-\boldsymbol{b}_i$ are in $\mathscr{Z}_B$, hence $\boldsymbol{\tau}\pm\boldsymbol{b}_i \notin U$. Since $\boldsymbol{\tau}$ was the first point to be found in the zonotopal span, this contradicts $\boldsymbol{b}_i \in B^+ \cup B^-$. Hence $\boldsymbol{b}_i \in B^0$ and $\boldsymbol{\tau} - P_{\boldsymbol{\rho}} \in \langle B^0\rangle_{\mathbb{Z}}$.

If $\boldsymbol{\rho}\cdot\boldsymbol{b}_i > 0$ then $\boldsymbol{\tau}+\boldsymbol{b}_i \in \mathscr{Z}_B$ so $\boldsymbol{\tau} \in B^0 \cap B^-$. Similarly if $\boldsymbol{\rho}\cdot\boldsymbol{b}_i = 0$ then $\boldsymbol{\tau}+\boldsymbol{b}_i \in \mathscr{Z}_B$ so $\boldsymbol{\tau} \in B^0 \cap B^-$. Thus we see that all summands of $P_{\boldsymbol{\rho}}$ are in $B^0 \cup B^-$. If $\boldsymbol{\rho}\cdot\boldsymbol{b}_i < 0$ then $\boldsymbol{\tau}-\boldsymbol{b}_i \in \mathscr{Z}_B$ and so $\boldsymbol{b}_i \notin B^-$. This means none of the elements from B^+ are in the summands of $P_{\boldsymbol{\rho}}$.

It thus follows from $\boldsymbol{\tau} - P_{\boldsymbol{\rho}} \in \langle B^0\rangle_{\mathbb{Z}}$ that $\boldsymbol{\tau} - \sum_{\boldsymbol{b}_i\in B^-} \boldsymbol{b}_i \in \langle B^0\rangle_{\mathbb{Z}}$. This contradicts the fact that $\boldsymbol{\tau}$ is a newly found point by Algorithm 4.2.1, which shows the first part of our proof.

Consider the sequence $U_0 \subset U_1 \subset \ldots \subset U_n$ found by zonotopal propagation and where the algorithm stops at U_n. We will now show that $\mathscr{Z}_B^c \subseteq U_n$. First we need a notion of height on the lattice $\mathbb{Z}^d$. So we define the height of a point $\boldsymbol{\zeta} \in \mathbb{Z}^d$ by $h(\boldsymbol{\zeta}) := \min_{c\in\mathbb{R}_{\geq 0}}[\boldsymbol{\zeta} \in cZ_B]$. The face $F_{\boldsymbol{\zeta}}$ is the face of minimal dimension of $h(\boldsymbol{\zeta})Z_B$ that contains $\boldsymbol{\zeta}$. Now note that this notion of height is also a notion of height for the cones generated by $M_{\boldsymbol{\rho}}$, in the sense that $h(\xi) = 0$ for $\xi \in \operatorname{Cone}(M_{\boldsymbol{\rho}})$ if and only if $\xi = 0$.

Pick a $\boldsymbol{\tau} \in \mathscr{Z}_B^c \setminus U_n$ of minimal height, if there are more choices pick the one whose face $F_{\boldsymbol{\tau}}$ has the lowest dimension. We know for all the points with $h(\boldsymbol{\tau}) \leq 1$ whether they are in $\mathscr{Z}_B$ or not. Therefore we can assume that $h(\boldsymbol{\tau}) > 1$. Suppose $\boldsymbol{b}_i$ is not parallel to the face $F_{\boldsymbol{\tau}}$. Then one of $\boldsymbol{\tau}\pm\boldsymbol{b}_i$ has height less than $h(\boldsymbol{\tau})$ or it must lie on a face containing $F_{\boldsymbol{\tau}}$. For the former case let $\sigma \in \{\pm 1\}$ such that $h(\boldsymbol{\tau}-\sigma\boldsymbol{b}_i) < h(\boldsymbol{\tau})$. If $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \in \mathscr{Z}_B$ then note that it is in a minimal shifted cone containing $P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$. The height function h works on this cone and either $-\boldsymbol{b}_i$ or $\boldsymbol{b}_i$. Any generator of this cone always moves outward from the zonotope. Because $h(\boldsymbol{\tau}-\sigma\boldsymbol{b}_i) < h(\boldsymbol{\tau})$ this means that $\sigma\boldsymbol{b}_i$ is a generator. Hence $\boldsymbol{\tau} \in P_{\boldsymbol{\rho}}+M_{\boldsymbol{\rho}}$ which is a contradiction. Hence $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \notin \mathscr{Z}_B$. By minimality of $\boldsymbol{\tau}$, this $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \in U_n$, hence $\boldsymbol{b}_i \in B^- \cup B^+$.

As a consequence B^0 only contains vectors parallel to faces that contain $F_{\boldsymbol{\tau}}$. Now suppose that one of $\boldsymbol{\tau}\pm\boldsymbol{b}_i$ has equal height and that $\boldsymbol{b}_i$ is not parallel to the face $F_{\boldsymbol{\tau}}$, but to a face that contains $F_{\boldsymbol{\tau}}$. Let $\sigma \in \{\pm 1\}$ be such that $h(\boldsymbol{\tau}-\sigma\boldsymbol{b}_i) = h(\boldsymbol{\tau})$. So assume that $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \in \mathscr{Z}_B$. In this case $h(\boldsymbol{\tau}+\sigma\boldsymbol{b}_i) > h(\boldsymbol{\tau})$ because the face $F_{\boldsymbol{\tau}-\sigma\boldsymbol{b}_i}$ is of a higher dimension than $F_{\boldsymbol{\tau}}$ and so $\boldsymbol{\tau}$ is an endpoint of the line segment in $F_{\boldsymbol{\tau}-\sigma\boldsymbol{b}_i}$ parallel to $\boldsymbol{b}_i$. So if $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \in \mathscr{Z}_B$ then $\sigma\boldsymbol{b}_i$ points outward of the zonotope. Now note that $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i$ is contained in a shifted semigroup $P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$. A generator of this semigroup cannot point inward to the zonotope. As a consequence $\boldsymbol{\tau} \in \mathscr{Z}_B$. This is a contradiction. So $\boldsymbol{\tau}-\sigma\boldsymbol{b}_i \in \mathscr{Z}_B^c$, as our $\boldsymbol{\tau}$ was chosen minimal and on a face of minimal dimension $\boldsymbol{\tau} - \sigma \boldsymbol{b}_i \in U_n$, which means that $\boldsymbol{b}_i \in B^- \cup B^+$.

Finally this means that B^0 can only contain vectors parallel to $F_{\boldsymbol{\tau}}$. Hence $S^0 = \langle B^0 \rangle_{\mathbb{Z}}$ is parallel to $F_{\boldsymbol{\tau}}$ as well. Therefore, for any $s \in S^0$ we see that $h(\boldsymbol{\tau} + s) \geq h(\boldsymbol{\tau})$. However, on the other hand, there is a $\boldsymbol{k} \in \mathbb{Z}^N$ such that $\boldsymbol{\tau} - \sum_{\boldsymbol{b} \in B^-} \boldsymbol{b} + B\boldsymbol{k} = 0$ for which $k_i = 0$ if $\boldsymbol{b}_i \notin B^0$. Hence $h(\boldsymbol{\tau} - \sum_{\boldsymbol{b} \in B^-} \boldsymbol{b} + B\boldsymbol{k}) = 0$ implies that $h(\boldsymbol{\tau} + B\boldsymbol{k}) \leq 1$, which then implies that $h(\boldsymbol{\tau}) \leq 1$. This contradicts our initial assumption that $h(\boldsymbol{\tau}) > 1$. $\square$

Corollary 4.4.10. *If $\mathscr{F}_B \cap Z_B = \mathscr{Z}_B^c \cap Z_B$ then $\mathscr{Z}_B^c \subseteq \mathscr{F}_B$.*

Proof. Apply Theorem 4.4.9 and Zonotopal Propagation (Algorithm 4.2.1) to get $\mathscr{Z}_B^c \subseteq \mathscr{F}_B$. $\square$

Experimental data and the structure of $\mathscr{Z}_B$ suggests that the converse of Corollary 4.4.10 is also true. Namely if $\mathscr{F}_B \cap Z_B = \mathscr{Z}_B^c \cap Z_B$ then from the definition of $\mathscr{Z}_B$ the points of $\mathscr{Z}_B \cap Z_B$ contain all the points $P_{\boldsymbol{\rho}}$. And of course $P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$ gives us a shifted semigroup and then we can use Zonotopal Elimination (Theorem 4.3.1) to show that this semigroup is in $\mathscr{F}_B^c$. The only problem with this approach is that the vector $\boldsymbol{\sigma}$ in the statement of Theorem 4.3.1 is not always uniquely determined for these points $P_{\boldsymbol{\rho}}$.

Example 4.4.11. In Figure 4.1 we see that Appell's F_4 satisfies $\mathscr{F}_B \cap Z_B = \mathscr{Z}_B^c \cap Z_B$. As a consequence $\mathscr{Z}_B^c \subseteq \mathscr{F}_B$. Also for each point $\boldsymbol{\tau} = P_{\boldsymbol{\rho}}$ there is exactly one vector $\boldsymbol{\sigma}$ in Theorem 4.3.1 that we can pick for zonotopal elimination. Thus the shifted semigroups that we eliminate have to be $P_{\boldsymbol{\rho}} + M_{\boldsymbol{\rho}}$. So we get $\mathscr{F}_B = \mathscr{Z}_B^c$.

4.5 Finding CI-sets

In this section we want to know how we can find CI-sets in $\mathscr{Z}_B^c$ and whether the size of difference sets in $\mathscr{Z}_B^c$ can exceed the rank D of our system. Experimental data suggest the following conjectures

Conjecture 4.5.1. *Let $U \subset \mathbb{Z}^d$ such that $U - U \subseteq \mathscr{F}_B$ then $\#U \leq D$.*

and its counterpart

Conjecture 4.5.2. *Let $U \subset \mathbb{Z}^d$ such that $U - U \subseteq \mathscr{Z}_B^c$ then $\#U \leq D$.*

We have algorithms to create $\mathscr{F}_B$ and $\mathscr{Z}_B^c$ and we want to focus on finding CI-sets in these sets. For $\mathscr{F}_B$ there is not much we can say but hope that it is contained in $\mathscr{Z}_B^c$. This is why we focus on finding CI-sets in $\mathscr{Z}_B^c$ instead. Of course one can try to do this the hard way by brute forcing or backtracking. However this is totally unreasonable if there is no smart trick to it. The first thing we can do is try to use algorithms that are used in graph theory.

Definition 4.5.3. The covariograph $\mathscr{G}_B$ is the graph whose vertices are the points in $\mathscr{Z}_B^c$. And (v_1, v_2) is an edge in $\mathscr{G}_B$ if and only if $v_2 \in \{v_1 + w : w \in \mathscr{Z}_B^c\}$ and $v_1 \neq v_2$.

The name covariograph comes from the term covariogram, which is the function $g_K(x) = |K \cap (K + \{x\})|$ where $K \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$.The covariograph is the graph where $x \in \mathscr{Z}_B^c$ has an edge with every point in $\mathscr{Z}_B^c \cap (\mathscr{Z}_B^c + \{x\})$, so the degree of x in $\mathscr{G}_B$ is equal to $g_{\mathscr{Z}_B^c}(x) - 1$. This difference of one is because we do not allow self-edges. The only difference is that the domain of $g_K(x)$ covers all of $\mathbb{Z}^d$, where vertices of $\mathscr{G}_B$ only cover $\mathscr{Z}_B^c$.

Now the problem of finding a CI-set is equivalent to finding a clique of size D inside $\mathscr{G}_B$. This is much more efficient than backtracking or brute forcing, but it does not use any of the properties that $\mathscr{Z}_B^c$ has.

For a "better" algorithm we first give a heuristic that will find a CI-set in many occasions.

Definition 4.5.4. Let $S \subset \mathscr{Z}_B^c$ be any set of points and let $v \in \mathscr{Z}_B^c \setminus S$ then $w \in \mathscr{Z}_B^c \setminus (S \cup \{v\})$ is a S-neighbor of v if the difference set of $S \cup \{v, w\}$ is in the Frobenius cavity. The number of S-neighbors of v is denoted by $\mathscr{N}_S(v)$.

Heuristic 4.5.5.

initialize Start with the set $S_0 = \{\mathbf{0}\}$ and $k = 0$ and an exclusion set $X = \emptyset$

Step 1 For each $v \in \mathscr{Z}_B^c \setminus (S_k \cup X)$ calculate $\mathscr{N}_{S_k}(v)$.

Step 2 If for some v the number $\mathscr{N}_{S_k}(v)$ is smaller than $D - 1 - \#S_k$ we can exclude it and add it to X.

Step 3 Pick a v where $\mathscr{N}_{S_k}(v)$ is maximal and let $S_{k+1} = S_k \cup \{v\}$ and increment k, then go back to step 1. If this is not possible or $k = D$ (after incrementing) stop the algorithm. In the former case we failed to find a CI-set in the latter case we found a CI-set.

Now this Heuristic lends itself to a backtracking algorithm where at each step we sort the vertices that we want to visit in order of the size of $\mathscr{N}_{S_k}(v)$. The problem is that if $\mathscr{Z}_B^c$ does not have a CI-set this is just as bad as an ordinary backtracking algorithm. Another thing we can do is try this Heuristic probabilistically, vertices with high values of $\mathscr{N}_{S_k}(v)$ will then have high chances of being selected, this gives some randomization to the process which may lead to an improvement. Still, the worst case is just equal to doing a brute force if there is no CI-set at all.

Now one may also look at this problem from a geometrical perspective to give an upper bound on the number of CI-sets in $\mathscr{Z}_B^c$. Which can also be used to accelerate the above algorithms.

Theorem 4.5.6. *Let $\mathcal{L} \subset \mathbb{Z}^d$ be a lattice such that $\mathcal{L} \setminus \{0\}$ is a subset of $\mathscr{Z}_B$ and consider the canonical module homomorphism $m_{\mathcal{L}} : \mathbb{Z}^d \to \mathbb{Z}^d/\mathcal{L}$ and let $T \subset \mathbb{Z}^d$ be a CI-set. Then the restriction of $m_{\mathcal{L}}$ to T is injective.*

Proof. For any $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in T$ we have $\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \in \mathscr{Z}_B^c$, hence if $\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \in \mathcal{L}$ it must be that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. As a consequence $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ have to be unique upto translation by $\mathcal{L}$. As a consequence the map $m_{\mathcal{L}}$ restricted to T is injective. $\square$

Let $\Pi_{\mathcal{L}}$ be the fundamental parallellogram of $\mathcal{L}$. If we have a lattice $\mathcal{L}$ as in Theorem 4.5.6 and the volume $\mathrm{Vol}(\Pi_{\mathcal{L}})$ is as small as possible, this gives many constraints that will improve algorithms and it gives upper bounds for the number of $\boldsymbol{\tau}$ that can be in a CI-set.

Corollary 4.5.7. *Let $\mathcal{L} \subset \mathbb{Z}^d$ be a lattice such that $\mathcal{L} \setminus \{0\}$ is a subset of $\mathscr{Z}_B$ and such that $Vol(\Pi_{\mathcal{L}}) = D$ then Conjecture 4.5.2 holds.*

Corollary 4.5.8. *If the A-hypergeometric system has a cotriangulation $\mathcal{I}$ whose parallelograms $\{\Pi_I\}_{I \in \mathcal{I}}$ are all equiangular, then Conjecture 4.5.2 holds.*

Proof. The parallelograms $\{\Pi_I\}_{I \in \mathcal{I}}$ can be stacked together to form a big parallelogram Π of volume D, let $\mathcal{L}$ be the lattice with this fundamental parallelotope. Then the points of $\mathcal{L}$ that intersect with $\mathscr{Z}_B^c$ is the set $\{\mathbf{0}\}$. $\square$

Example 4.5.9 (Central Zonotopal Subdivisions)**.** Finding a suitable lattice $\mathcal{L}$ sometimes goes by trial and error. And in some cases it is possible to use some structure that we see in Z_B. For example one can decompose Z_B into smaller parallelotopes such as is done in [She74, Expression (56)] and its corollary which is accredited to McMullen. It says that Z_B attains a subdivision of parallelotopes such that each parallelotope corresponds to a unique cotriangle I and vice versa. For $d = 2$ we can choose our decomposition to start in the origin, this turns out to be a great source of lattices $\mathcal{L}$ if they exist. And of course they create great pictures as well.

Definition 4.5.10. Given a convergence direction $\boldsymbol{\rho}$ and cotriangulation $\mathcal{I}_{\boldsymbol{\rho}}$ and let Π_I denote the parallelogram generated by columns of B_I where I is a cotriangle and C_I the cone generated by columns of B_I. Moreover let the operation $X \oplus Y$ be interpreted as "parallelogram X is translated by a vector Y". The scale $S_{\boldsymbol{\rho}}$ is the union of the parallelograms

$$\bigcup_{I \in \mathcal{I}_{\boldsymbol{\rho}}} \Pi_I \oplus \left(\sum_{\boldsymbol{b}_i \in C_I^\circ} \boldsymbol{b}_i + \sum_{j \in I} \sum_{\substack{i<j \\ \boldsymbol{b}_i = k\boldsymbol{b}_j \\ k>0}} \boldsymbol{b}_i \right)$$

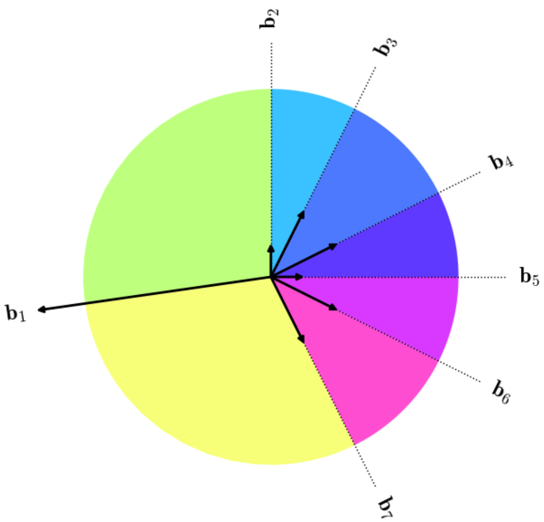
The union of all scales is called the **Central Zonotopal Subdivision** which we denote by Π_B.

If a scale $S_{\boldsymbol{\rho}}$ tiles the plane, then this tiling gives us a lattice $\mathcal{L}$ with volume of the fundamental parallelogram exactly D. And the lattice has all the right properties for Theorem 4.5.6. In simple examples scales like this are often found. And these tell us that the number of points in a CI-set is bounded above by D.

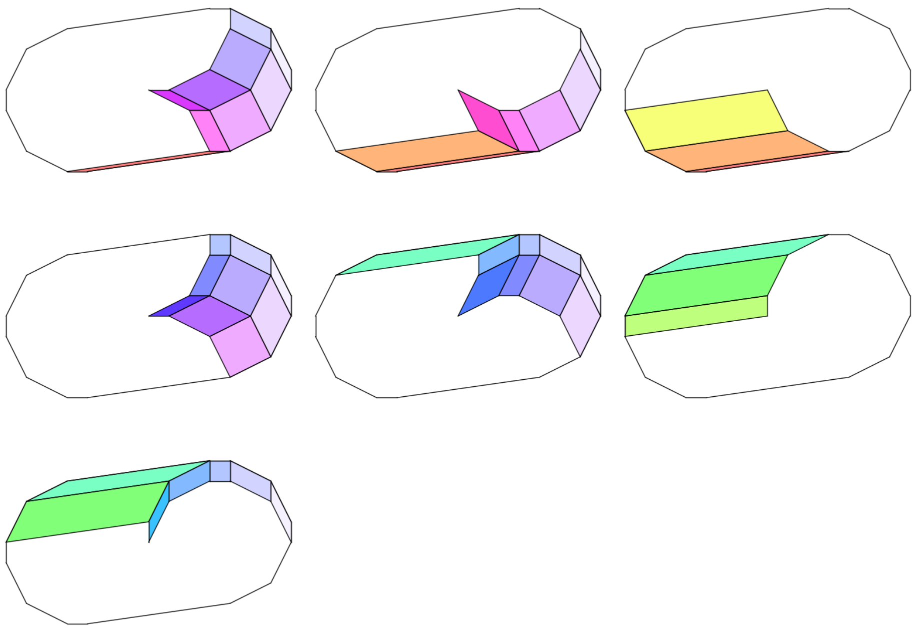
For example take the A-hypergeometric system of holonomic rank 25 where

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 & 2 & -7 \\ 0 & 1 & 2 & -2 & 1 & -1 & -1 \end{pmatrix},$$

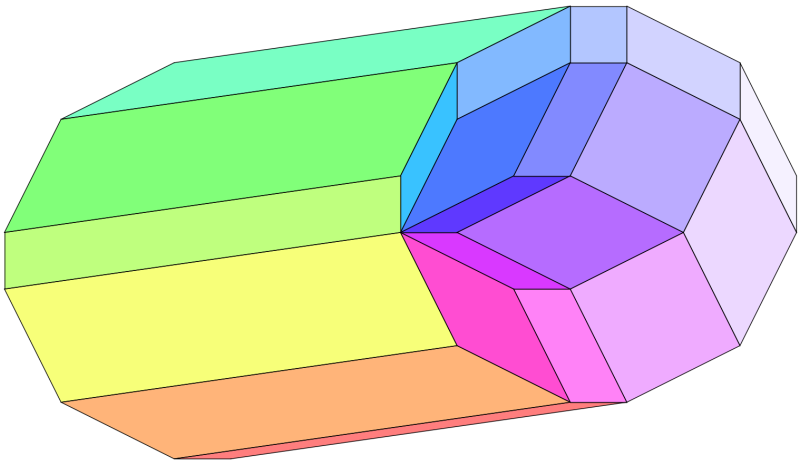
from Example 4.4.7. Let us color the secondary fan, depending on where a chamber is located. A similar coloring is used to create Π_B. Note that we have sorted the columns of B in clock-wise order.



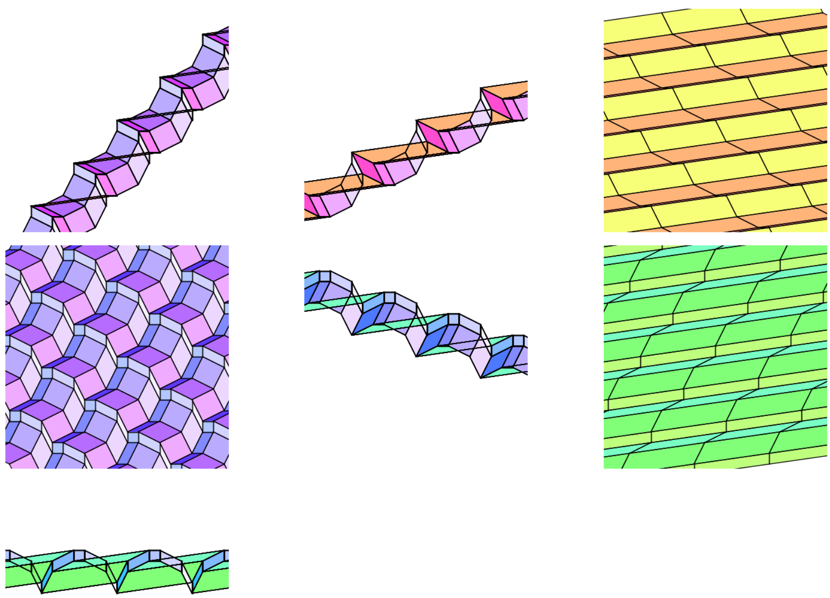
Now create scales for a $\boldsymbol{\rho}$ in each of these chambers. We draw the zonotope around it to give some perspective.



And then we can merge it together to form Π_B

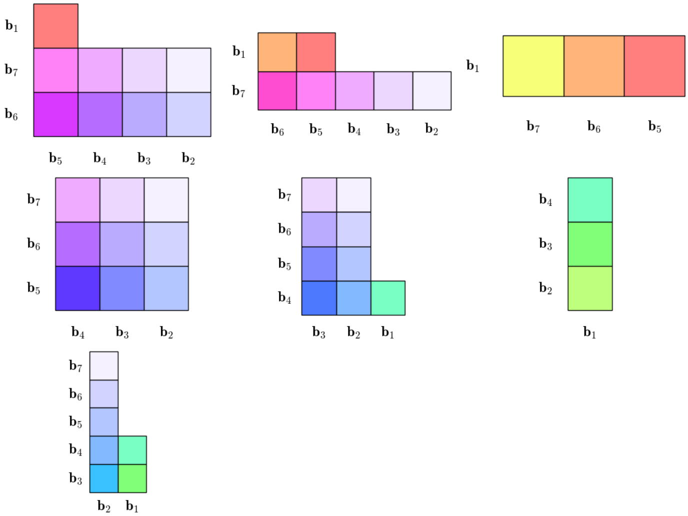


Only three of the scales can be used to tile the plane. The other four scales always overlap in some way or cannot tile the full plane. This is illustrated by the following picture



If the scales do not give us a nice upper bound, then there are more tricks that can help us by looking at these pictures. Namely, we may also tile the plane using other shapes inside Π_B and try to see how this affects the number points in a CI-set. Another way to represent these scales is as a

Young-diagram. We do this by deforming the parallelograms into squares. We keep the colors so you can compare them. On the right and bottom of each diagram the corresponding $\boldsymbol{b}$-vectors are given. If the diagram is a rectangle then this means the corresponding scale tiles the plane.



Chapter 5

Apexpoints

5.1 Introduction

Determining whether A-hypergeometric functions are algebraic has been a well studied problem. The first results on this problem for ${}_2F_1$ are given by Schwarz in [Sch73] where a list of parameter vectors $\boldsymbol{\alpha}$ is given such that every solution of the Gauss hypergeometric system is algebraic. In a paper by Beukers and Heckman [BH89] this result is extended to generalized hypergeometric functions ${}_nF_{n-1}$ using an interlacing condition on its parameters as described in Theorem 5.1.2.

Definition 5.1.1. Two finite sets $U = \{u_1, \ldots, u_n\} \subset [0, 1)$ and $V = \{v_1, \ldots, v_n\} \subset [0, 1)$ are said to **interlace**, if $U \cap V = \emptyset$ and the increasing sequence $w_1 < \ldots < w_{2n}$, where $w_i \in U \cup V$, $i = 1, \ldots, 2n$ has the property that either $w_j \in U$ and $w_{j+1} \in V$, or $w_j \in V$ and $w_{j+1} \in U$ for all $j = 1, \ldots, 2n - 1$.

Theorem 5.1.2. *Let $a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{Q}$, $b_n = 1$ and let M be their common denomimator. Then all solutions to the generalized hypergeometric equation with parameters $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$ are algebraic functions if and only if the sets $\{\{ka_1\}, \ldots, \{ka_n\}\}$ and $\{\{kb_1\}, \ldots, \{kb_n\}\}$ interlace for all integers k, where $\gcd(k, M) = 1$ and $1 \leq k < M$. Here $\{x\}$ is defined as the fractional part of x.*

Later, Beukers [Beu10] generalizes Theorem 5.1.2 to A-hypergeometric functions by replacing the interlacing condition by a condition on apexpoints.

Recall that $C(A)$ is the cone generated by the columns of A. And the cone $C(A_{I^c})$ is the cone generated by the vectors $\boldsymbol{a}_i$ for $i \in I^c$. Let $\boldsymbol{\alpha} \in \mathbb{R}^r$ be our parameter vector of an A-hypergeometric system and define $K_{\boldsymbol{\alpha}} = (\boldsymbol{\alpha} + \mathbb{Z}^r) \cap C(A)$. Then $\mathfrak{p} \in K_{\boldsymbol{\alpha}}$ is called an **apexpoint** if $\mathfrak{p} - \mathfrak{q} \notin C(A)$ for all $\mathfrak{q} \in K_{\boldsymbol{\alpha}}$ and $\mathfrak{q} \neq \mathfrak{p}$. The number of apexpoints for the system $H_A(\boldsymbol{\alpha})$ is denoted by $\sigma(A, \boldsymbol{\alpha})$.

Lemma 5.1.3. *There are at most D apexpoints, i.e. $\sigma(A, \boldsymbol{\alpha}) \leq D$.*

Proof. Given a cotriangulation $\mathcal{I}$, recall that $Q(A)$ can be triangulated with triangulation $\mathcal{I}^c$. Hence we can exploit the fact that an apexpoint must lie inside a simplicial cone $C(A_{I^c})$ for some $I \in \mathcal{I}$. Furthermore if this apexpoint $\mathfrak{p}$ is not in the fundamental parallelotope $\Pi_{I^c} := A_{I^c}[0,1)^r$ then notice that there is a column vector $\boldsymbol{a}$ of A_{I^c} such that $\mathfrak{p} \in \mathfrak{p} - \boldsymbol{a} + C(A)$, which is a contradiction. As a consequence there can be at most D apexpoints and they are contained in the fundamental paralellotopes Π_{I^c} where $I \in \mathcal{I}$. $\square$

Inspired by Lemma 5.1.3 we say that the number of apexpoints is maximal when $\sigma(A, \boldsymbol{\alpha}) = D$.

Theorem 5.1.4 (Beukers [Beu10])**.** *Let $\boldsymbol{\alpha} \in \mathbb{Q}^r$. Let M be the common denominator of the coordinates of $\boldsymbol{\alpha}$. Then the solution set of the A-hypergeometric system consists of algebraic solutions if and only if $\sigma(A, k\boldsymbol{\alpha})$ is maximal for all $k \in 1, \ldots, M$ and* $\gcd(k, M) = 1$.

In [BH89, Corollary 4.7] it is shown that interlacing of the parameters is equivalent to the monodromy-invariant Hermitian form being definite. In this section we like to establish a similar connection in the A-hypergeometric setting. Our main result is given in Corollary 5.4.2. Before we start we like to remark that if $\sigma(\boldsymbol{\alpha}, A) < D$, it does not seem possible to draw any conclusions about the signature of the invariant Hermitian form from the value of $\sigma(\boldsymbol{\alpha}, A)$.

For a number $x \in \mathbb{R}$ let $\{x\}$ denote the fractional part of x and $\lfloor x \rfloor$ denote the floor of x. For vectors we will let these operations work component-wise.

Theorem 5.1.5. *Fix a cotriangulation $\mathcal{I}$. Let $\mathfrak{p} \in K_{\boldsymbol{\alpha}}$ be an apexpoint then there exists a solution point $\boldsymbol{\mu}$ such that $\mathfrak{p} = A\{\boldsymbol{\gamma}^{\boldsymbol{\mu}}\}^\intercal$ and $I(\boldsymbol{\mu}) \in \mathcal{I}$.*

Proof. Choose $I \in \mathcal{I}$ such that $\mathfrak{p} \in C(A_{I^c})$ and write $\mathfrak{p} = A\boldsymbol{v}$ where $\boldsymbol{v} \in \mathbb{R}^N$ and $v_i = 0$ for all $i \in I$. Notice that $\boldsymbol{v}$ is uniquely determined in this way and since $\mathfrak{p} \in C(A_{I^c})$ we have $v_i \geq 0$ for all $i \in I^c$. Since $\mathfrak{p}$ is an apexpoint we have $\mathfrak{p} - \boldsymbol{a}_i \notin C(A)$ for all i. In particular we must have that $v_i < 1$ for all $i \in I^c$. Since $\boldsymbol{\alpha} \equiv \mathfrak{p} \pmod{\mathbb{Z}^r}$ and the vectors in A generate $\mathbb{Z}^r$ there must be a solution point $\boldsymbol{\mu}$, with $I(\boldsymbol{\mu}) = I$, such that $\boldsymbol{v}^\intercal \equiv \boldsymbol{\gamma}^{\boldsymbol{\mu}} \pmod{\mathbb{Z}^r}$. Hence $\boldsymbol{v} = \{\boldsymbol{\gamma}^{\boldsymbol{\mu}}\}^\intercal$. $\square$

The points $A\{\boldsymbol{\gamma}^{\boldsymbol{\mu}}\}^\intercal$, where $\boldsymbol{\mu}$ runs over all solution points with $I(\boldsymbol{\mu}) \in \mathcal{I}$, are called **Candidate apexpoints**. Each candidate apexpoint of $K_{\boldsymbol{\alpha}}$ is a point in a set $K_I := (\boldsymbol{\alpha} + \mathbb{Z}^r) \cap C(A_{I^c})$ for some $I \in \mathcal{I}$. This set K_I can be written as the disjoint union of shifted semigroups

$$K_{\boldsymbol{\mu}} := \left\{A\{\boldsymbol{\gamma}^{\boldsymbol{\mu}}\}^\intercal + A_{I^c}\boldsymbol{v} : \boldsymbol{v} \in \mathbb{Z}^r_{\geq 0}\right\},$$

Here $\boldsymbol{\mu}$ runs over all solution points with $I = I(\boldsymbol{\mu}) \in \mathcal{I}$. This disjoint union K_I consists of Δ_I sets.

Recall the linear form $h : \mathbb{R}^r \to \mathbb{R}$ where $h(\boldsymbol{a}) = 1$ for all column vectors $\boldsymbol{a}$ of A. This will give us a notion of height for points in the cone $C(A)$. In particular $h(\mathfrak{p}) = 0 \iff \mathfrak{p} = \mathbf{0}$ for any $\mathfrak{p} \in C(A)$.

5.2 Main theorem

Theorem 5.2.1. *Given an A-hypergeometric system $H_A(\boldsymbol{\alpha})$. Then the following three statements are equivalent:*

1. *All candidate apexpoints have the same height.*
2. *$K_{\boldsymbol{\alpha}}$ has a maximal number of apexpoints, i.e. $\sigma(A, \boldsymbol{\alpha}) = D$.*
3. *All candidate apexpoints have the same height modulo $2\mathbb{Z}$.*

5.3 Proof

Before we discuss the proof of this theorem we need a lemma. This lemma uses Radon's theorem to generate two triangulations of a point set of size $n+2$ in an n-dimensional space. We use it on the vectors of A. Note that these points all lie on the hyperplane defined by h. By a restriction to that hyperplane we may put $n = r - 1$ in the following theorem.

Lemma 5.3.1 ([Law86])**.** *For any point set $\mathscr{P}$ of size $n+2$ in $\mathbb{R}^n$ not all on the same hyperplane, there are exactly 2 triangulations (not necessarily admissable) that triangulate $Q(\mathscr{P})$.*

Lemma 5.3.1 also exactly tells what the triangulations are. First notice that a triangle contains $n+1$ points of $\mathscr{P}$. And so its corresponding cotriangle is determined by the remaining point. Let the points $\mathscr{P} = \{p_1, \ldots, p_{n+2}\}$ and consider the matrix

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ p_1 & p_2 & \ldots & p_{n+2} \end{pmatrix}.$$

Its kernel is one-dimensional and generated by a vector $\boldsymbol{l} \in \mathbb{R}^{n+2}$. The indices with positive entries in $\boldsymbol{l}$ correspond to one cotriangulation and all indices with negative entries in $\boldsymbol{l}$ correspond to the other cotriangulation.

Proof of theorem 5.2.1.

1 ⇒ 3. Trivial.

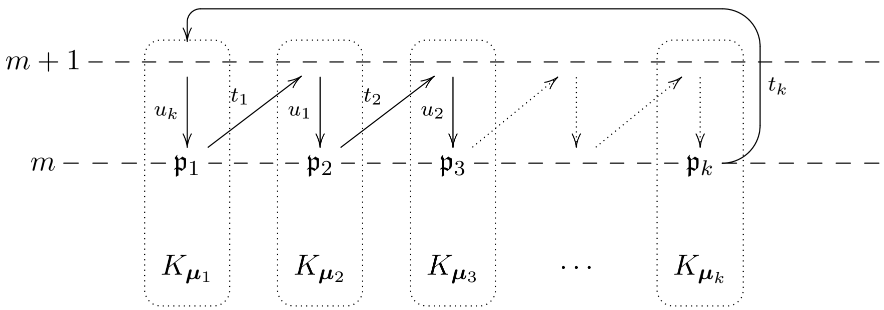
1 ⇒ 2. All candidate apexpoints have the same height, hence no apexpoint (or no point at all in $K_{\boldsymbol{\alpha}}$) can exist of lower height. Let $\mathfrak{p}$ be a candidate apexpoint and suppose it is not an apexpoint, then $\mathfrak{p} = \mathfrak{q} + \boldsymbol{r}$ for some $\mathfrak{q} \in K_{\boldsymbol{\alpha}}$ and some $\boldsymbol{r} \in C(A)$ where $\boldsymbol{q} \neq \boldsymbol{p}$. Height-wise this means $h(\mathfrak{p}) = h(\mathfrak{q} + \boldsymbol{r}) = h(\mathfrak{q}) + h(\boldsymbol{r})$. Now $\mathfrak{p}$ is of minimal height in $K_{\boldsymbol{\alpha}}$ and so $h(\mathfrak{q}) \geq h(\mathfrak{p})$. And $\boldsymbol{r} \in C(A)$ so $h(\boldsymbol{r}) \geq 0$. This is only possible when $h(\mathfrak{q}) = h(\mathfrak{p})$ and $h(\boldsymbol{r}) = 0$. This means that $\boldsymbol{r} = \boldsymbol{0}$, so $\mathfrak{p} = \mathfrak{q}$, which is a contradiction. We conclude that all candidate apexpoints are apexpoints.

2 ⇒ 1. Let us first explain the strategy for this proof. First, we pick any apexpoint $\mathfrak{p}$ of minimal height. It is of course contained in some $C(A_{I^c})$ for some cotriangle $I \in \mathcal{I}$. Then we expand the triangle I^c by adding and index $j \in I$. Let $S = I^c \cup \{j\}$. The cone $C(A_S)$ contains all apexpoints of $C(A_{I^c})$ and maybe some more. Now our goal is to find all of them in $C(A_S)$ by a procedure and show that they have the same height. And then by picking the S's differently we can cover all of $C(A)$. I sectioned each part of the proof to improve readability.

The process: Pick a triangulation $\mathcal{I}^c$ with at least two triangles and pick $I^c \in \mathcal{I}^c$ such that $C(A_{I^c})$ contains an apexpoint $\mathfrak{p}_1$ of minimal height m. Now pick another triangle J^c of $\mathcal{I}^c$ such that the corresponding triangles I^c and J^c are adjacent, i.e. $C(A_{I^c})$ and $C(A_{J^c})$ share a codimension 1 face. Let $S = I^c \cup J^c$. We triangulate A_S such that it contains I^c and J^c. Denote this triangulation by $\mathcal{I}^c_S$. By Lemma 5.3.1 A_S has another triangulation, denote this triangulation $\mathcal{J}^c_S$. Consider first the case where the $\mathbb{Z}$-span of columns of A_S is $\mathbb{Z}^r$.

We have chosen $\mathfrak{p}_1 \in C(A_{I_1^c})$ where $I_1 = I$. There exists a solution point $\boldsymbol{\mu}_1$ such that $I(\boldsymbol{\mu}_1) = I_1$ and $\mathfrak{p}_1 \in K_{\boldsymbol{\mu}_1}$. Now as $C(A_S)$ is generated by $r+1$ column vectors of A, there is exactly one vector $\boldsymbol{a}_{t_1} \in A_S$ which is not in $A_{I_1^c}$. Thus $\mathfrak{p}_1 + \boldsymbol{a}_{t_1} \notin K_{\boldsymbol{\mu}_1}$ hence it must be in another shifted semigroup, say $K_{\boldsymbol{\mu}_2}$ for some solution point $\boldsymbol{\mu}_2$ with $I_2 = I(\boldsymbol{\mu}_2) \in \mathcal{I}_S$. Clearly $\mathfrak{p}_1 + \boldsymbol{a}_{t_1}$ is not an apexpoint and so not a candidate apexpoint, however it has height $m+1$, so in $K_{\boldsymbol{\mu}_2}$ there must be an apexpoint $\mathfrak{p}_2$ of minimal height m which can be obtained by subtracting a vector $\boldsymbol{a}_{u_1}$ for some $u_1 \in I_2^c$.

Continue in this fashion with the new point until we get a path of apexpoints $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k, \ldots$, because we only have finitely many apexpoints this path is eventually periodic. So without loss of generality assume that $\mathfrak{p}_1 = \mathfrak{p}_{k+1}$ for some $k \geq 1$. This means we found an element of the kernel of A_S by describing the path we took as a vector $\boldsymbol{l}$ such that $A_S \boldsymbol{l} = 0$. This vector $\boldsymbol{l}$ is a sum of sparse vectors, each of which contains one 1 at position t_i and one -1 at position u_i and the rest of the values zero. Each summand represents a piece of the path. This process is also described in the following diagram where we want to show that $k = \widetilde{D}$ where $\widetilde{D} = \mathrm{Vol}(Q(A_S))$.



Non-triviality of $\boldsymbol{l}$: Consider the first step

$$\mathfrak{p}_1 \to \mathfrak{p}_1 + \boldsymbol{a}_i \to \mathfrak{p}_1 + \boldsymbol{a}_i - \boldsymbol{a}_j = \mathfrak{p}_2.$$

Here we have that i is uniquely determined by the index in $S \setminus I_1^c$. In this sense i corresponds to the cotriangle I_1 in $\mathcal{I}_S$. Let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ be two solution points with $J_1 = I(\boldsymbol{\nu}_1) \in \mathcal{J}_S$ and $J_2 = I(\boldsymbol{\nu}_2) \in \mathcal{J}_S$ such that $\mathfrak{p}_1 \in K_{\boldsymbol{\nu}_1}$ and $\mathfrak{p}_2 \in K_{\boldsymbol{\nu}_2}$.

Because $\mathfrak{p}_1 \neq \mathfrak{p}_2$ we have that $K_{\boldsymbol{\nu}_1} \neq K_{\boldsymbol{\nu}_2}$. Since $i \in I_1$ we have that $i \in J_1^c$ hence $\mathfrak{p}_1 + \boldsymbol{a}_i \in K_{\boldsymbol{\nu}_1}$. Hence $\mathfrak{p}_2 + \boldsymbol{a}_j \in K_{\boldsymbol{\nu}_1}$. The fact that $\mathfrak{p}_2 \in K_{J_2}$ and $K_{J_1} \neq K_{J_2}$ means that j is the unique index in $S \setminus J_2^c$. In this sense j corresponds to the cotriangle J_2 in $\mathcal{J}_S$.

We conclude that the upward steps correspond to vectors $\boldsymbol{a}_i$ with $i \in \mathcal{I}_S$ and the downward vectors to $\boldsymbol{a}_j$ with $j \in \mathcal{J}_S$. In particular, since $\mathcal{I}_S$ and $\mathcal{J}_S$ are disjoint, we find that $\boldsymbol{l}$ is non-trivial.

The kernel of A_S is known. It is generated by a single vector $\boldsymbol{l}$ whose positive entries add up to $\widetilde{D}$, these entries correspond to upward moves. Negative entries of $\boldsymbol{l}$ add up to $-\widetilde{D}$ and correspond to downward moves. As a consequence $\boldsymbol{l}$ must be a non-zero multiple of $\boldsymbol{l}$. This means that the path created from the process visits $\widetilde{D}$ distinct semigroups $K_{\boldsymbol{\mu}_k}$ with $I(\boldsymbol{\mu}_k) \in \mathcal{I}_S$. Hence in each of these semigroups we find an apexpoint of height m.

In $C(A)$ this means we have found $\widetilde{D}$ apexpoints of the same height m (which is minimal). In particular all the apexpoints coming from triangles I^c and J^c. As this process works as well for every adjacent triangle to I^c and J^c, we can cover up the whole triangulation by going through all the possible pairs of adjacent triangles.

The special case. We assumed that the $\mathbb{Z}$-span of columns of A_S spans $\mathbb{Z}^r$, however this is not always the case. If this is not the case then the greatest common divisor $g(S)$ among the minors of A_S is greater than 1. Our process still gives us a cycle through all triangles of a triangulation of A_S. However it may not cover $\widetilde{D}$ apexpoints. Instead we are only sure to cover a multiple of $\widetilde{D}/g(S)$ apexpoints.

Let I be a cotriangle in a cotriangulation $\mathcal{I}_S$ of A_S. Then if we start at an apexpoint of minimal height in $K_{\boldsymbol{\mu}}$, the cycle given by our process is fixed. It produces a vector $\boldsymbol{l} \in \ker(A_S)$. The positive entries of $\boldsymbol{l}$ tell us the upward moves we make, but it also tells us exactly how many times the path visits each cone $C(A_{J^c})$ for $J \in \mathcal{I}_S$. As $\ker(A_S)$ is one dimensional this means that the cone $C(A_{I^c})$ is visited a multiple of $\Delta_I/g(S)$ by the process.

Choose any set S' different from S with $r+1$ indices and a cotriangulation $\mathcal{I}'$ of $A_{S'}$ which contains I. Then the same holds and $C(A_{I^c})$ is visited a multiple of $\Delta_I/g(S')$ times by the process.

Now if we use the process for S on apexpoints of minimal height in $C(A_{I^c})$ we may find new apexpoints of minimal height in this cone, and we can continue until we have a multiple of $\Delta_I/g(S)$ apexpoints. If we do the same for S' we can continue until we find a multiple of $\Delta_I/g(S')$ apexpoints. Combining the two means we obtain a multiple of $\mathrm{lcm}(\Delta_I/g(S), \Delta_I/g(S')) = \Delta_I/(\gcd(g(S), g(S'))$ apexpoints in $C(A_{I^c})$ just by using the two paths continuously.

The number of visits to a triangle in one of these paths is only governed by multiples of $\boldsymbol{l}$ and $\boldsymbol{l}'$. Thus for all triangles J^c visited by a path from S or S' the same density of apexpoints have to be found. Hence in $C(A_{J^c})$ we find $\Delta_J/(\gcd(g(S), g(S'))$ apexpoints of minimal height.

We can just continue this procedure by picking a triangle that we already used. Choose an adjacent triangle that we haven't used before. And in this way we get a new set S'' and a new path and with this path new apexpoints of minimal height may be found.

Extend until the full triangulation $\mathcal{I}^c$ is complete. Again the density argument holds and we are now able to find $\Delta_I/(\gcd_{J\in\mathcal{I}}(\Delta_J))$ apexpoints of minimal height for any $I \in \mathcal{I}$.

Now we can vary the cotriangulation $\mathcal{I}$ to create new paths from the process and merge those together as well. And again the density argument holds. So in each cone $C(A_{I^c})$ we find $\Delta_I/(\gcd_J(\Delta_J))$ apexpoints of minimal height, where J runs over all cotriangles. $\gcd_J(\Delta_J)$ is the greatest common divisor of all d-minors of B. Because B is a basis for the saturated lattice L this greatest common divisor is equal to 1. As a consequence we have visited Δ_I apexpoints of minimal height for each cone $C(A_{I^c})$. In particular for the triangulation $\mathcal{I}^c$ we have found exactly D apexpoints of minimal height.

$3 \Rightarrow 1$. *(Sketch).* This part is completely analogous to the proof of step $2 \Rightarrow 1$ so we will just sketch how the process works. Choose adjacent triangles I^c and J^c create the set $S = I^c \cup J^c$ and A_S. And make sure there is an apexpoint $\mathfrak{p}_1$ of minimal height m in K_I. This means there is a solution point $\boldsymbol{\mu}_1$ such that $\mathfrak{p}_1 \in K_{\boldsymbol{\mu}_1}$. Now let t_1 be the index in $S \setminus I^c$ then $\mathfrak{p}_1 + \boldsymbol{a}_{t_1}$ is in a different shifted semigroup $K_{\boldsymbol{\mu}_2}$ for some solution point $\boldsymbol{\mu}_2$. The point $\mathfrak{p}_1 + \boldsymbol{a}_{t_1}$ has height $m+1$, but all the heights of candidate apexpoints are supposed to be in $m + 2\mathbb{Z}_{\geq 0}$. So clearly $\mathfrak{p}_1 + \boldsymbol{a}_{t_1}$ is not an candidate apexpoint. Hence we are able to go down in $K_{\boldsymbol{\mu}_2}$ and find a candidate apexpoint $\mathfrak{p}_2$ of height m in $K_{\boldsymbol{\mu}_2}$. In this way we create a path of candidate apexpoints analogous to the path created in $2 \Rightarrow 1$. Again, every semigroup the process visits gives us a candidate apexpoint of height m. The remainder of the proof is now analogous to $2 \Rightarrow 1$. □

5.4 Consequences

Now we want to convert this result into a condition based on the properties of a VHF $\widetilde{H}$.

Lemma 5.4.1. *Given an A-hypergeometric system $H_A(\boldsymbol{\alpha})$ which admits a VHF $\widetilde{H}$ then the signature of $\widetilde{H}$ is given by the signs of $(-1)^{\sum_{i=1}^{N}\lfloor\gamma_i^{\boldsymbol{\mu}}\rfloor}$ for all solution points $\boldsymbol{\mu}$ such that $I(\boldsymbol{\mu}) \in \mathcal{I}$*

Proof. The diagonal of the diagonalization of $\widetilde{H}$ has entries

$$\Delta_I \prod_{i\in I}(-1)^{\gamma_i^{\boldsymbol{\mu}}} \prod_{i\in I^c} \sin(\pi\gamma_i^{\boldsymbol{\mu}}),\ \text{solution points } \boldsymbol{\mu} : I = I(\boldsymbol{\mu}) \in \mathcal{I}.$$

See Theorem 3.1.1. Now Δ_I is always positive and $\mathrm{sign}(\sin(\pi\gamma_i^{\boldsymbol{\mu}})) =$

$(-1)^{\lfloor \gamma_i^{\boldsymbol{\mu}} \rfloor}$. Hence

$$\begin{aligned}\operatorname{sign}\left(\Delta_I \prod_{i\in I}(-1)^{\gamma_i^{\boldsymbol{\mu}}} \prod_{i\in I^c}\sin(\pi\gamma_i^{\boldsymbol{\mu}})\right) &= \prod_{i\in I}(-1)^{\gamma_i^{\boldsymbol{\mu}}} \prod_{i\in I^c}(-1)^{\lfloor \gamma_i^{\boldsymbol{\mu}} \rfloor} \\ &= \prod_{i=1}^{N}(-1)^{\lfloor \gamma_i^{\boldsymbol{\mu}} \rfloor} \\ &= (-1)^{\sum_{i=1}^{N}\lfloor \gamma_i^{\boldsymbol{\mu}} \rfloor}.\end{aligned}$$

□

Lemma 5.4.1 implies that $\widetilde{H}$ is definite if and only if all heights $h(A\{\boldsymbol{\gamma}^{\boldsymbol{\mu}}\}^{\mathsf{T}})$ are the same modulo $2\mathbb{Z}$. Application of criterion 3 of Theorem 5.2.1 now implies the following corollary.

Corollary 5.4.2. *Suppose the A-hypergeometric system $H_A(\boldsymbol{\alpha})$ admits a VHF $\widetilde{H}$. Then $\widetilde{H}$ is definite if and only if $K_{\boldsymbol{\alpha}}$ has a maximal number of apexpoints.*

Following the results from Matsubara-Heo and Goto about the signature of the Hermitian form, see Remark 3.4.2 and [GMH20], we can also give the following corollary.

Corollary 5.4.3. *Given an A-hypergeometric system $H_A(\boldsymbol{\alpha})$ where $h(\alpha) \notin \mathbb{Z}$. Then its invariant Hermitian form H is definite if and only if $K_{\boldsymbol{\alpha}}$ has a maximal number of apexpoints.*

Part II

Bailey-type factorizations

Chapter 6

Bailey-type factorizations

6.1 Introduction

In this chapter we want to find factorizations of certain A-hypergeometric functions which are similar to Bailey's identity [Bai33]

$$F_4\left(\begin{matrix} a,\ b \\ c,\ a+b-c+1 \end{matrix}\middle| x(1-y), y(1-x)\right)$$
$$= {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} a,\ b \\ a+b-c+1 \end{matrix}\middle| y\right). \tag{6.1}$$

Here F_4 is Appell's F_4 function. Other identities similar to (6.1) for Appell's F_2 have been found independently by Beukers [Zud19] and Vidunas [Vid09]. It reads,

$$F_2\left(\begin{matrix} a+b-\frac{1}{2},\ a,\ b \\ 2a,\ 2b \end{matrix}\middle| \frac{4u(1-u)(1-2v)}{(1-2uv)^2}, \frac{4v(1-v)(1-2u)}{(1-2uv)^2}\right)$$
$$= (1-2uv)^{-1+2a+2b} {}_2F_1\left(\begin{matrix} a+b-\frac{1}{2},\ a \\ 2a \end{matrix}\middle| 4u(1-u)\right) {}_2F_1\left(\begin{matrix} a+b-\frac{1}{2},\ b \\ 2b \end{matrix}\middle| 4v(1-v)\right). \tag{6.2}$$

For Appell's F_3 identities it is noted in [Vid09] that these follow directly from the observation that the two functions

$$F_2\left(\begin{matrix} a\ b_1,\ b_2 \\ c_1,\ c_2 \end{matrix}\middle| x, y\right),$$

$$x^{-b_1}y^{-b_2}F_3\left(\begin{matrix} 1+b_1-c_1\ 1+b_2-c_2,\ ,b_1,\ b_2 \\ 1+b_1+b_2-a \end{matrix}\middle| \frac{1}{x}, \frac{1}{y}\right)$$

satisfy the same differential equations.

The aim of this chapter is to generalize (6.1) to the classical two-variable hypergeometric functions. Let ${}_{2,2}F_{1,1}(x,y)$ be such a function. Then we look for factorizations of the form

$$ {}_{2,2}F_{1,1}(\phi(s,t),\psi(s,t)) = Q(s,t)^{\lambda}{}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\,\rho(s)\right){}_2F_1\left(\begin{array}{c} a',\ b' \\ c' \end{array}\middle|\,\sigma(t)\right), \tag{6.3} $$

where ϕ,ψ,Q are rational functions in two variables and ρ,σ rational functions in one variable. Analytic continuation of the right hand side in s,t will generate a space of functions of dimension $2\times 2=4$. Hence we expect the function on the left to be a solution of a rank 4 equation. Which is the case for Appell F_2,F_4.

In this chapter we are interested in Bailey-type factorizations for Horn functions of rank 4, in particular H_1,H_4 and H_5. These are defined as the hypergeometric series

$$ H_1\left(\begin{array}{c} a,\ b,\ c \\ d \end{array}\middle|\,x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m m!n!}x^m y^n, \tag{6.4} $$

$$ H_4\left(\begin{array}{c} a,\ b \\ c,\ d \end{array}\middle|\,x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n m!n!}x^m y^n, \tag{6.5} $$

$$ H_5\left(\begin{array}{c} a,\ b \\ c \end{array}\middle|\,x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_n m!n!}x^m y^n. \tag{6.6} $$

6.2 Monodromy

The monodromy group of ${}_2F_1(x)$ is contained in GL(2). Hence the monodromy group of ${}_{2,2}F_{1,1}$ is contained in GL(2) $\times$ GL(2). Over the complex numbers the group GL(2)$\times$GL(2) maps injectively into the group of orthogonal similitudes GO(4) by the map

$$ (M_1,M_2)\mapsto\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\mapsto M_1^{\intercal}\begin{pmatrix} a & b \\ c & d\end{pmatrix}M_2\right). $$

The GO(4) semi-invariant form is given by the determinant $ad-bc$. Thus to find Bailey-like factorizations we should look for rank 4 system whose monodromy is contained in GO(4). To find candidates for such hypergeometric systems, we write down the set of monodromy matrices

$M_1, \ldots, M_k$ as constructed in Chapter 2. Let Q be a candidate non-singular 4×4 matrix and test whether $M_i^T Q M_i = \lambda_i Q$ for some scalar λ_i. The requirement that we should find a non-trivial Q gives us a large set of restrictions on the hypergeometric parameters $\alpha_1, \ldots, \alpha_r$, or more correctly, $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_r}$. If the group generated by $M_1, \ldots, M_k$ acts irreducibly, the resulting matrix Q will be either symmetric or anti-symmetric. This follows from the fact that Q is uniquely determined up to a scalar and both Q and Q^T are a solution. In the first case $\langle M_1, \ldots, M_k\rangle$ will be contained in GO(4), in the second case in GSp(4), the group of symplectic similitudes.

First we give a short summary of how the algorithm works. The implementation follows straightforwardly from the algorithm, so we will omit that part.

The monodromy matrices given to us by Definition 2.4.1 have the form $M_{\boldsymbol{\rho},j} := X_{\boldsymbol{\rho}} \chi_{\boldsymbol{\rho},j} X_{\boldsymbol{\rho}}^{-1}$ where $\chi_{\boldsymbol{\rho},j}$ is a diagonal matrix.

Now we want to find a (skew)-symmetric matrix Q and a scalar $\lambda_{\boldsymbol{\rho},j} \in \mathbb{C}^*$ such that

$$M_{\boldsymbol{\rho},j}^{\intercal} Q M_{\boldsymbol{\rho},j} = \lambda_{\boldsymbol{\rho},j} Q.$$

This can then be written as

$$(X_{\boldsymbol{\rho}}^{\intercal})^{-1} \chi_{\boldsymbol{\rho},j} X_{\boldsymbol{\rho}}^{\intercal} Q X_{\boldsymbol{\rho}} \chi_{\boldsymbol{\rho},j} X_{\boldsymbol{\rho}}^{-1} = \lambda_{\boldsymbol{\rho},j} Q.$$

And then it can be written as

$$\chi_{\boldsymbol{\rho},j} X_{\boldsymbol{\rho}}^{\intercal} Q X_{\boldsymbol{\rho}} \chi_{\boldsymbol{\rho},j} = \lambda_{\boldsymbol{\rho},j} X_{\boldsymbol{\rho}}^{\intercal} Q X_{\boldsymbol{\rho}}.$$

So let $Q_{\boldsymbol{\rho}} = X_{\boldsymbol{\rho}}^{\intercal} Q X_{\boldsymbol{\rho}}$, then we have

$$\chi_{\boldsymbol{\rho},j} Q_{\boldsymbol{\rho}} \chi_{\boldsymbol{\rho}_j} = \lambda_{\boldsymbol{\rho},j} Q_{\boldsymbol{\rho}}.$$

So let $\chi_{\boldsymbol{\rho},j} = \mathrm{Diag}(\chi_1, \ldots, \chi_D)$ then for each row-column entry (r, c) where $Q_{\boldsymbol{\rho}}$ is non-zero, we need to have $\chi_r \chi_c = \lambda_{\boldsymbol{\rho},j}$.

These conditions result in restrictions on the parameters $\alpha_1, \ldots, \alpha_r$, or more correctly $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_r}$.

This turns out to be a reasonably fast approach to compute which specializations give us a monodromy group contained in GO(4) or GSp(4) for the classical cases. We need to be careful though, because the restrictions hold for $e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_r}$ we find $\boldsymbol{\alpha}$-vectors upto shifts in $\mathbb{Z}^r$. Table 6.1 gives the results of our implementation of the algorithm described. We feed it an A-hypergeometric system and it computes for which parameter vectors $\boldsymbol{\alpha}$ the system is either in GO(4) or GSp(4). The specialization given in Table 6.1 correspond to the classical parameters and so not to the

A-hypergeometric $\boldsymbol{\alpha}$-vector. Additionally note that $\boldsymbol{\alpha}$ may not be totally non-resonant, this is given in the third column. Here TNR means Totally Non-Resonant and NR means Non-Resonant. Note that this may give us an inconsistency, because the computation of the monodromy matrices was made under the assumption of TNR. However, NR occurs only in the symplectic case, which will not be considered any further. The fourth column tells us whether the monodromy for the corresponding specialization is contained in GO(4) or GSp(4).

Among the classical two-variable hypergeometric equations of rank 4 the cases F_3, H_2, H_7 are missing from the table. However, F_3 and H_2 are related to F_2 (they have the same A-polytope) and H_7 is related to H_4.

Table 6.1

System	Specialization (mod $\mathbb{Z}^r$)	Res.	Mon.
F_2	$(q_0, q_1, q_0 - q_1 + \frac{1}{2}, 2q_1, 2q_0 - 2q_1)$	TNR	GO(4)
	$(q_0, q_1, q_0 - q_1, 2q_1, 2q_0 - 2q_1)$	NR	GSp(4)
F_4	$(q_0, q_1, q_2, q_0 + q_1 - q_2)$	TNR	GO(4)
H_1	$(q + 1, q + \frac{1}{2}, \frac{1}{2}, 2q)$	TNR	GO(4)
	$(q, q, -\frac{1}{2}, 2q)$	NR	GSp(4)
H_4	$(q_0, q_1, q_0 - q_1, 2q_1)$	TNR	GO(4)
	$(q_0 - \frac{1}{2}, q_1, q_0 - q_1, 2q_1)$	TNR	GSp(4)
H_5	$(q + \frac{1}{2}, q, 2q)$	TNR	GO(4)
	$(q, q, 2q)$	NR	GSp(4)

Looking at F_4 and Bailey's identity (6.1) we see that the parameter vectors $(q_0, q_1, q_2, q_0+q_1-q_2)$ and its classical parameter vector $(a, b, c, a+b-c+1)$ are the same modulo $\mathbb{Z}^4$ if we take $q_0 = a, q_1 = b, q_2 = c$. The same can be said for Bailey's identity for F_2 in (3). Looking at this table one could now wonder whether H_1, H_4 and H_5 also admit a Bailey type decomposition.

To find a Bailey type factorization we follow an approach from [Vid09] who finds such identities for F_2 and F_4. Consider (6.3) and write $F(x, y) = {}_{2,2}F_{1,1}(x, y)$. Fix s and note that $F(\phi(s,t), \psi(s,t))$, as a function of t, satisfies an ordinary second order differential equation with rational function coefficients, because the right hand side of (6.3) does. The problem we like to solve is to find (rational) functions $x(t), y(t)$ such that

$f(t) := F(x(t), y(t))$ satisfies an ordinary second order differential equation. For general $x(t), y(t)$ such a function would satisfy a fourth order equation, so the second order restriction does give us restrictions on $x(t), y(t)$.

Suppose $f(t)$ is the solution to a second order differential system of the form

$$\frac{d^2 f}{dt^2} + c_1 \frac{df}{dt} + c_2 f = 0.$$

To find the relation with F and its differential equations we apply the chain rule and product rule multiple times to $f(t) = F(x(t), y(t))$.

$$\frac{df}{dt} = \dot{x} F_x + \dot{y} F_y \tag{6.7}$$

$$\frac{d^2 f}{dt^2} = \dot{x}^2 F_{xx} + 2\dot{x}\dot{y} F_{xy} + \dot{y}^2 F_{yy} + \ddot{x} F_x + \ddot{y} F_y \tag{6.8}$$

Here we denote

$$\dot{x} = \frac{dx}{dt}, \ \ \dot{y} = \frac{dy}{dt}.$$

6.3 Horn's H_4

The first system we want to investigate is Horn's H_4 hypergeometric function (6.5). This one is interesting because we can see from Table 6.1 that it's corresponding system has a monodromy group in GO(4) and the corresponding specialization of the parameters is two dimensional. A system of partial differential equations for H_4 can be found at [DG02, p.817]. It is

$$\begin{aligned} & x(1-4x)F_{xx} - 4xyF_{xy} - y^2 F_{yy} \\ & \quad + (c - (4a+6)x)F_x - 2(a+1)yF_y - a(a+1)F = 0, \end{aligned} \tag{6.9}$$

$$\begin{aligned} & y(1-y)F_{yy} - 2xyF_{xy} \\ & \quad + (d - (a+b+1)y)F_y - 2bxF_x - abF = 0. \end{aligned} \tag{6.10}$$

Now we would like to parameterize x and y with variable t such that we get an equation of the form

$$\frac{d^2 f}{dt^2} + c_1 \frac{df}{dt} + c_2 f = 0.$$

To achieve this we take (6.8) and eliminate partial derivatives on the righthand side terms of (6.8). First we can eliminate F_{xx} and F_{yy} using equations (6.9) and (6.10). Then we set the coefficient of F_{xy} to be zero. This coefficient is equal to

$$\frac{-2y(y-2)\dot{x}^2+2(y-1)(4x-1)\dot{x}\dot{y}-2x(4x-1)\dot{y}^2}{(y-1)(4x-1)}. \tag{6.11}$$

If we specialize this by

$$(x(u,v),y(u,v)) = \left(\frac{1-\left(\frac{v-\frac{1}{v}}{2}\frac{u-\frac{1}{u}}{2}\right)^2}{4}, 1+\frac{v+\frac{1}{v}}{2}\right),$$

we can factor (6.11) as

$$\frac{(v^2\dot{u}+2u\dot{v}-\dot{u})(v^2\dot{u}-2u\dot{v}-\dot{u})(u^2+1)^2(v^2-1)^2}{64(v^2+1)u^4v^3}.$$

This is equal to zero when $v=\pm 1$ or $u=\pm i$, but we skip those cases and focus on one of the two differential equations that emerge (the other one gives a similar result). Thus consider the following differential equation:

$$(v^2-1)\dot{u}=-2u\dot{v}.$$

Which can be simplified to

$$\frac{-1}{2u}du=\frac{1}{v^2-1}dv.$$

After integration we obtain

$$-\frac{1}{2}\log(u)+C=\frac{1}{2}(\log(1-v)-\log(1+v)).$$

And thus we get the solutions

$$u=C'\frac{1+v}{1-v},\ C' \text{ constant}.$$

Now set $v=t$, $u=C'\frac{1+t}{1-t}$ and let $C'=\frac{1+s}{1-s}$ with s another constant. This gives us a parameterization

$$x(s,t)=-\frac{(st^2+s+2t)(2st+t^2+1)s}{4(s+1)^2(s-1)^2t^2},\quad y(s,t)=1+\frac{t+\frac{1}{t}}{2},$$

which ensures that (6.11) is 0.

Now we have eliminated F_{xx}, F_{xy}, F_{yy} from (6.8). Equation (6.8) has now acquired the form

$$\frac{d^2 f}{dt^2} = c_3 F_x(x(t), y(t)) + c_4 F_y(x(t), y(t)) + c_5 f. \tag{6.12}$$

The coefficients c_3, c_4, c_5 are quite cumbersome to write down and can be found in Appendix A.1.

Then, what we would like to see is that the vector (c_3, c_4) is a multiple, say $r_2(t)$, of $(\dot{x}, \dot{y})$. In such a case we have

$$c_3 F_x + c_4 F_y = r_2(t)(\dot{x} F_x + \dot{y} F_y) = r_2 \frac{df}{dt}$$

in virtue of equation (6.7). It is a small miracle that this indeed happens for the parameter choices $a = q_0, b = q_1, c = 1 + q_0 - q_1, d = 2q_1$. This choice has been motivated by Table 6.1. This means that for fixed s the function

$$H_4\left(\begin{array}{c} q_0,\ q_1 \\ 1 + q_0 - q_1,\ 2q_1 \end{array} \middle| -\frac{\left(st^2 + s + 2\,t\right)\left(2\,st + t^2 + 1\right)s}{4\left(s^2 - 1\right)^2 t^2}, \frac{(t+1)^2}{2\,t} \right).$$

satisfies an ordinary Fuchsian equation of the form

$$\frac{d^2 F}{dt^2} + r_2 \frac{dF}{dt} + r_3 F. \tag{6.13}$$

Here r_2 and r_3 are some rational coefficients in q_0, q_1, s and t which can be found in Appendix A.1. It is a Fuchsian equation which we like to identify with a transform of the Gaussian hypergeometric equation. To that end we investigate its local exponents.

The following singularities with corresponding local exponents were found:

Singularity	Exponent 1	Exponent 2
$t = 1$	0	2
$t = -1$	0	$2 - 4q_1$
$t = 0$	q_0	$q_0 + 1$
$t = \infty$	q_0	$q_0 + 1$
Roots of $t^2 + 2t/s + 1 = 0$	0	$q_1 - q_0$
Roots of $t^2 + 2ts + 1 = 0$	0	$q_1 - q_0$

At $t = 0$ and $t = \infty$ we notice there is a difference of 1 in the local exponents, hence they may be apparent singularities. As the system for

${}_2F_1$ only has local exponents at $0, 1$ and ∞, we might want to map the differential equations for ${}_2F_1(z)$ under the covering

$$z \mapsto \frac{(s+1)^2}{(s-1)^2}\frac{(t+1)^4}{(t-1)^4}.$$

This is because $t = 1$ corresponds to $z = \infty$ with multiplicity 4; And $t = -1$ corresponds to $z = 0$ with multiplicity 4; If t is a root of $t^2+2t/s+1 = 0$ or $t^2+2ts+1 = 0$ this corresponds to $z = 1$ each with multiplicity 1. And $t = 0$ and $t = \infty$ correspond to regular points.

Recall that the Riemann scheme of ${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle| z\right)$ is equal to

Singularity	Exponent 1	Exponent 2
$z = 0$	0	$1-c$
$z = 1$	0	$c-a-b$
$z = \infty$	a	b

Hence the Riemann scheme of ${}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array}\middle| \frac{(s+1)^2}{(s-1)^2}\frac{(t+1)^4}{(t-1)^4}\right)$ becomes

Singularity	Exponent 1	Exponent 2
$t = 1$	$4a$	$4b$
$t = -1$	0	$4-4c$
$t = 0$	0	1
$t = \infty$	0	1
Roots of $t^2+2t/s+1 = 0$	0	$c-a-b$
Roots of $t^2+2ts+1 = 0$	0	$c-a-b$

Upto translation these exponents should be equal to the local exponents we found for H_4, hence the differences between local exponents form a linear set of equations. $4b-4a = 2$, $4-4c = 2-4q_1$ and $c-a-b = q_1-q_0$. This can be solved for $a = \frac{1}{2}q_0+\frac{1}{2}$, $b = \frac{1}{2}q_0$ and $c = q_1+\frac{1}{2}$. Indeed we can check that under these transformations from the ${}_2F_1$ system we obtain equation (6.13).

Now we need to translate the local exponents. For this we need to multiply by an additional factor of $(1-t)^{-2q_0}t^{q_0}$. This makes the local exponents at $t = 0$ and $t = \infty$ both q_0 and q_0+1 and the local exponents at $t = 1$ now become 0 and 2.

We conclude that

$$H_4\left(\begin{array}{c} q_0,\ q_1 \\ 1+q_0-q_1,\ 2q_1 \end{array}\middle| -\frac{\left(st^2+s+2\,t\right)\left(2\,st+t^2+1\right)s}{4\left(s^2-1\right)^2t^2}, \frac{(t+1)^2}{2\,t}\right)$$

satisfies the same second order ordinary differential equation in t as

$$(1-t)^{-2q_0}t^{q_0}{}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_1+\frac{1}{2}\end{array}\middle|\ \frac{(s+1)^2(t+1)^4}{(s-1)^2(t-1)^4}\right).$$

Now we want to transform this such that it becomes symmetric in the arguments s and t in the sense that swapping s and t does not change the system, this can be done with the following transformation:

$$s \to -\frac{s^2+1}{s^2-1},\ \ t \to \frac{st+1}{st-1}.$$

In this way we obtain

$$H_4\left(\begin{array}{c}q_0,\ q_1\\ 1+q_0-q_1,\ 2q_1\end{array}\middle|\ \frac{\left(s^4-1\right)\left(t^4-1\right)}{4\left(s^2t^2-1\right)^2},\frac{2s^2t^2}{s^2t^2-1}\right). \tag{6.14}$$

And as s is a constant it should satisfy the same differential equation in t as

$$\left(\frac{1}{1-st}\right)^{-2q_0}\left(\frac{st+1}{st-1}\right)^{q_0}{}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_1+\frac{1}{2}\end{array}\middle|\ s^4\right){}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_1+\frac{1}{2}\end{array}\middle|\ t^4\right). \tag{6.15}$$

By the symmetry in s,t we can also say that (6.14) and (6.15) satisfy the same second order ordinary differential equation in s.

To establish a Bailey-like identity we consider (6.14) near the point $(s,t)=(1,0)$. It is holomorphic there, so (6.14) is $(1-s^2t^2)^{q_0}$ times a hypergeometric series holomorphic near $s=1$ times a hypergeometric series holomorphic near $t=0$.

After setting the constant terms both equal to 1 we conclude that

$$H_4\left(\begin{array}{c}q_0,\ q_1\\ 1+q_0-q_1,\ 2q_1\end{array}\middle|\ \frac{\left(s^4-1\right)\left(t^4-1\right)}{4\left(s^2t^2-1\right)^2},\frac{2s^2t^2}{s^2t^2-1}\right)$$

$$=\left(1-s^2t^2\right)^{q_0}{}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_0-q_1+1\end{array}\middle|\ 1-s^4\right){}_2F_1\left(\begin{array}{c}\frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0\\ q_1+\frac{1}{2}\end{array}\middle|\ t^4\right).$$

Notice that only squared variables are used, so substitute $s\to\sqrt{s}$ and $t\to\sqrt{t}$ to obtain:

$$H_4\left(\begin{array}{c}q_0,\ q_1\\ 1+q_0-q_1,\ 2q_1\end{array}\middle|\ \frac{\left(s^2-1\right)\left(t^2-1\right)}{4\left(st-1\right)^2},\frac{2st}{st-1}\right)$$

$$= (1-st)^{q_0}\, {}_2F_1\left(\begin{matrix} \frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0 \\ q_0-q_1+1 \end{matrix}\,\middle|\, 1-s^2\right) {}_2F_1\left(\begin{matrix} \frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0 \\ q_1+\frac{1}{2} \end{matrix}\,\middle|\, t^2\right). \tag{6.16}$$

After a search in the literature it turned out that there is another way to get the same identity. In [Erd48, eq. 7.6, p. 382] we find the identity

$$H_4\left(\begin{matrix} \alpha,\beta \\ \gamma, 2\beta \end{matrix}\,\middle|\, x,y\right) = (1-y/2)^{-\alpha} F_4\left(\begin{matrix} \frac{1}{2}\alpha, \frac{1}{2}\alpha+\frac{1}{2} \\ \gamma, \beta+\frac{1}{2} \end{matrix}\,\middle|\, \frac{16x}{(2-y)^2}, \frac{y^2}{(2-y)^2}\right).$$

Using this identity we obtain

$$\begin{aligned} &H_4\left(\begin{matrix} q_0, q_1 \\ 1+q_0-q_1, 2q_1 \end{matrix}\,\middle|\, \frac{(s^2-1)(t^2-1)}{4(st-1)^2}, \frac{2st}{st-1}\right) \\ &= (1-st)^{q_0} F_4\left(\begin{matrix} \frac{1}{2}q_0, \frac{1}{2}q_0+\frac{1}{2} \\ 1+q_0-q_1, q_1+\frac{1}{2} \end{matrix}\,\middle|\, (1-s^2)(1-t^2), s^2t^2\right). \end{aligned}$$

We can now apply Bailey's factorization (6.1) to get the right hand side of (6.16).

6.4 Horn's H_1

A system of partial differential equations for Horn's H_1 function (6.4) can be found at [DG02, p.817]. It is given by

$$\begin{aligned} &x(1-x)F_{xx} + y^2 F_{yy} \\ &\quad + (d-(a+b+1)x)F_x - (a-b-1)yF_y - abF = 0, \end{aligned} \tag{6.17}$$

$$\begin{aligned} &y(1+y)F_{yy} - x(1-y)F_{xy} \\ &\quad + (a-1-(b+c+1)y)F_y - cxF_x - bcF = 0. \end{aligned} \tag{6.18}$$

We can use (6.17) and (6.18) to eliminate F_{xx} and F_{xy} from (6.8). The following coefficient for F_{yy} remains:

$$\frac{(y^3-y^2)\dot{x}^2 + (-2xy^2-2xy+2y^2+2y)\dot{x}\dot{y} + (x^2y-x^2-xy+x)\dot{y}^2}{(x-1)x(y-1)}. \tag{6.19}$$

Using the specialization

$$x(u,v) = 1 - \frac{(v-1/v)^2(u-1/u)^2}{16}, \quad y(u,v) = v^2,$$

the coefficient (6.19) factors as follows

$$\frac{4(v^2\dot{u}+2u\dot{v}-\dot{u})(v^2\dot{u}-2u\dot{v}-\dot{u})(u^2+1)^2v^4}{((uv+u+v-1)(uv+u-v+1)(uv-u+v+1)(uv-u-v-1)u^2)}.$$

Just like in the H_4 case we can make this zero by taking the parametrization

$$u = C\frac{1+t}{1-t}, \quad v = t.$$

And again we will pick the integration constant C to be

$$C = \frac{1+s}{1-s}.$$

Now in equation (6.8) we eliminated the coefficients F_{xx}, F_{xy} and F_{yy}, and we are left to eliminate F_x and F_y using (6.7). Call the coefficients for F_x and F_y respectively c_3 and c_4 and the constant coefficient c_5. These coefficients can be found in Appendix A.2. Then what we would like to see is that $c_4/c_3 = \dot{y}/\dot{x}$. Again miraculously this happens when $a = q_0 - \frac{1}{2}, b = q_0, c = \frac{1}{2}, d = 2q_0$, which matches the entry in Table 6.1 modulo $\mathbb{Z}^4$. This means that for fixed s the function

$$H_1\left(\begin{array}{c} q_0-\frac{1}{2},\ q_0,\ \frac{1}{2} \\ 2q_0 \end{array} \middle| -\frac{\left(st^2+s+2\,t\right)\left(2\,st+t^2+1\right)s}{\left(s^2-1\right)^2 t^2}, t^2\right)$$

satisfies an ordinary Fuchsian equation of the form

$$\frac{d^2F}{dt^2} + r_2\frac{dF}{dt} + r_3 F.$$

The coefficients r_2 and r_3 can be found at Appendix A.2. Let us analyze the local exponents of this differential equation.

Singularity	Exponent 1	Exponent 2
$t = 0$	$2q_0 - 1$	$2q_0$
$t = \infty$	$2q_0$	$2q_0 + 1$
Roots of $t^2 + 2t/s + 1 = 0$	0	$1 - 2q_0$
Roots of $t^2 + 2ts + 1 = 0$	0	$1 - 2q_0$

We see that in this case $t = 0$ and $t = \infty$ may be apparent singularities, but $t = 1$ and $t = -1$ are regular. Assume just like in the H_4 case that we can go from ${}_2F_1$ to this by the covering $z \mapsto \frac{(s+1)^2(t+1)^4}{(s-1)^2(t-1)^4}$ now we still want $t = 1$ ($z = \infty$ with multiplicity 4) and $t = -1$ ($z = 0$ with multiplicity 4) to be regular. Hence $1 - c = \frac{1}{4}$, so $c = \frac{3}{4}$. And $c - a - b = 1 - 2q_0$ and $b - a = \frac{1}{4}$ so $a = q_0 - \frac{1}{4}$ and $b = q_0$. Now we need to translate the local exponents such that $t = 0$ and $t = \infty$ become apparent singularities and $t = 1$ becomes regular, for this we need to multiply by the function by $(t-1)^{1-4q_0} t^{2q_0 - 1}$

We obtain that

$$H_1\left(\begin{array}{c} q_0-\frac{1}{2},\ q_0,\ \frac{1}{2} \\ 2q_0 \end{array}\middle|\ -\frac{\left(st^2+s+2\,t\right)\left(2\,st+t^2+1\right)s}{(s^2-1)^2t^2}, t^2\right)$$

satisfies the same second order differential equation in t as

$$(t-1)^{1-4q_0}t^{2q_0-1}{}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ \frac{3}{4} \end{array}\middle|\ \frac{(s+1)^2(t+1)^4}{(s-1)^2(t-1)^4}\right).$$

Now we want to make the arguments symmetric again. We can do this by the substitution

$$s\to-\frac{s^2+1}{s^2-1},\quad t\to\frac{st+1}{st-1}.$$

As a consequence

$$H_1\left(\begin{array}{c} q_0-\frac{1}{2},\ q_0,\ \frac{1}{2} \\ 2q_0 \end{array}\middle|\ \frac{(s^4-1)(t^4-1)}{(s^2t^2-1)^2}, \left(\frac{st+1}{st-1}\right)^2\right)$$

satisfies the same differential equation in t as

$$(st-1)^{4q_0-1}\left(\frac{st+1}{st-1}\right)^{2q_0-1}{}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ \frac{3}{4} \end{array}\middle|\ s^4\right){}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ \frac{3}{4} \end{array}\middle|\ t^4\right).$$

Hence by symmetry this is true both in s and t. To find a Bailey-like identity, which is an identity of two variable power series, we must look at the function

$$\phi:(s,t)\mapsto\left(\frac{(s^4-1)(t^4-1)}{(s^2t^2-1)^2}, \left(\frac{st+1}{st-1}\right)^2\right).$$

Then we wish to choose a point (s_0,t_0) with $s_0^4,t_0^4\in\{0,1\}$ such that ϕ maps an open neighbourhood of (s_0,t_0) to a neighbourhood of $(0,0)$. Unfortunately this is impossible. Such points must satisfy $s_0^4=t_0^4=1$ and $s_0t_0=-1$. For example, $s_0=-1, t_0=1$. But ϕ is not continuous in $(-1,1)$. In order to get a meaningful identity we could restrict ϕ to a neighbourhood U of $(-1,1)$ of the form $s=-1+2uv, t=1+2v$ with u,v small. One verifies that

$$\phi(-1+2uv,1+2v)=(-4u+\text{h.o.t.}\,,\ v^2+\text{h.o.t.}),$$

where 'h.o.t.' means 'higher order terms'. So ϕ is well-defined on U and its image is in a neighbourhood of $(0,0)$.

Therefore

$$H_1\left(\begin{array}{c} q_0-\frac{1}{2},\ q_0,\ \frac{1}{2} \\ 2q_0 \end{array}\middle|\,\phi(-1+2uv,1+2v)\right) \tag{6.20}$$

is holomorphic near the point $(u,v)=(0,0)$. The only function in the space spanned by the products of Gauss hypergeometric functions can be

$$\begin{aligned}&(st-1)^{2q_0}(st+1)^{2q_0-1}(1-t^4)^{1-2q_0}\\ &\times {}_2F_1\left(\begin{array}{c} 1-q_0,\ \frac{3}{4}-q_0 \\ 2-2q_0 \end{array}\middle|\,1-t^4\right) {}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ 2q_0 \end{array}\middle|\,1-s^4\right).\end{aligned}$$

After the substitution $s\to -1+2uv, t\to 1+2v$ this becomes, after normalization,

$$\begin{aligned}&(1+v-uv-2uv^2)^{2q_0}\left(\frac{1-u-2uv}{(1+v)(1+2v+2v^2)}\right)^{2q_0-1}\\ &\quad\times {}_2F_1\left(\begin{array}{c} 1-q_0,\ \frac{3}{4}-q_0 \\ 2-2q_0 \end{array}\middle|\,-8v(1+v)(1+2v+2v^2)\right)\\ &\quad\times {}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ 2q_0 \end{array}\middle|\,8uv(1-uv)(1-2uv+2u^2v^2)\right),\end{aligned} \tag{6.21}$$

which is also holomorphic at $(u,v)=(0,0)$. Since the constant terms of (6.20) and (6.21) are equal these power series expansions must be equal.

6.5 Horn's H_5

The system of differential equations corresponding to Horn's H_5 function (6.6), given in [DG02, p.817], is

$$\begin{aligned}&x(1+4x)F_{xx}-y(4x-1)F_{xy}-y^2F_{yy}\\ &\quad+(1-b+(4a-6)x)F_x+2(a+1)yF_y+a(a+1)F=0 \end{aligned}\tag{6.22}$$

$$\begin{aligned}&y(1-y)F_{yy}-xyF_{xy}+2x^2F_{xx}\\ &\quad+(c-(a+b+1)y)F_y+(2+a-2b)xF_x-abF=0.\end{aligned} \tag{6.23}$$

Again we would like to parameterize x and y with variable t such that we get an equation of the form

$$\frac{d^2f}{dt^2}+c_1\frac{df}{dt}+c_2f=0.$$

First we can eliminate F_{xx} and F_{xy} from (6.8) using equations (6.22) and (6.23). Then we set the coefficient of F_{yy} to be zero. This coefficient is equal to

$$\frac{(3\,xy^2-4\,xy-y^2+y)\dot{x}^2+(-12\,x^2y+8\,x^2-2\,xy+2\,x)\dot{x}\dot{y}+(12\,x^3-x^2)\dot{y}^2}{(12\,x-1)x^2}. \tag{6.24}$$

Instead of factoring (6.24) such that we only have to solve linear differential equations. It is easier to solve this right away, thanks to a suggestion from Wadim Zudilin. Namely let x be dependent on y and try to find an algebraic relation between x and y using power series solutions of $x(y)$. Rewrite the numerator of (6.24), then we want to solve

$$(3\,xy^2-4\,xy-y^2+y)\frac{d^2x}{dy^2}+(-12\,x^2y+8\,x^2-2\,xy+2\,x)\frac{dx}{dy}+(12\,x^3-x^2)=0. \tag{6.25}$$

Now by trying out different constant terms for $x(y)$ we can generate power series solutions from the recurrences of (6.25) upto a certain degree M. Suppose we have found power series solutions $x_1(y),\ldots,x_N(y)$ all of degree M. Then we can make a matrix Q_i where each row corresponds to the coefficient vector of $y^v x_i^u + O(y^M)$ and where the rows run over a sufficient number of pairs $(u,v)\in\mathbb{Z}^2_{\geq 0}$.

In our case it was enough to let $M=30$, the constant terms of $x(y)$ were chosen to be the twenty integers $\{-10,\ldots,-1\}\cup\{1,\ldots,10\}$, and we let $(u,v)\in\{(a,b): a=0,\ldots,3,\ b=0,\ldots,2\}$. This is because we don't expect the algebraic relation between x and y to be very complex.

The left kernel of each Q_i now corresponds to an algebraic relations between x and y. By our choice of $x_1(y),\ldots,x_{20}(y)$ and the choice of (u,v) this kernel turns out to be 1-dimensional for each of these Q_i. Let the algebraic relation that generates the left kernel of Q_i be denoted by f_i.

Then we interpolate the coefficients of $\{f_i\}_{i=1,\ldots,N}$ separately using variable a to form an algebraic relation

$$f(a,x,y):=\sum_{u=0}^{3}\sum_{v=0}^{2}c_{u,v}(a)x^u y^v. \tag{6.26}$$

The full relation is given in Appendix A.3. Solving $f(a,x,y)=0$ for y gives two solutions

$$y_1=\frac{4\left(144\,a^2x-144\,a^{\frac{3}{2}}x^{\frac{3}{2}}+4\,a^2-36\,a^{\frac{3}{2}}\sqrt{x}+36\,ax-4\sqrt{a}x^{\frac{3}{2}}+a-\sqrt{a}\sqrt{x}\right)(4\,a+1)}{(12\,a-1)^3x},$$

$$y_2 = \frac{4\left(144\,a^2x + 144\,a^{\frac{3}{2}}x^{\frac{3}{2}} + 4\,a^2 + 36\,a^{\frac{3}{2}}\sqrt{x} + 36\,ax + 4\,\sqrt{a}x^{\frac{3}{2}} + a + \sqrt{a}\sqrt{x}\right)(4\,a+1)}{(12\,a-1)^3 x}.$$

Pick $y = y_2$ and substitute $x \to t^2$ and $a \to s^2$ to get the specialization

$$x(s,t) = t^2,\ y(s,t) = \frac{4\left(144s^2t^2 + 4s^2 + 32st + 4t^2 + 1\right)\left(4\,s^2+1\right)(s+t)s}{(12\,s^2-1)^3t^2}.$$

One checks that (6.24) is annihilated by these choices of x and y.

This means in (6.8) we eliminated the coefficients F_{xx}, F_{xy} and F_{yy}, and we would like to eliminate F_x and F_y using (6.7). Again we call the coefficients for F_x and F_y respectively c_3 and c_4 and the constant coefficient c_5. These coefficients can be found in Appendix A.3. And miraculously we find $c_4/c_3 = \dot{y}/\dot{x}$ when $a = q, b = q - \frac{1}{2}, c = 2q$. This means that for fixed s the function

$$H_5\left(\begin{matrix} q,\ q-\frac{1}{2} \\ 2q \end{matrix} \middle|\, t^2, \frac{4\left(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\right)\left(4\,s^2+1\right)(s+t)s}{(12\,s^2-1)^3t^2}\right).$$

satisfies an ordinary Fuchsian equation of the form

$$\frac{d^2F}{dt^2} + r_2\frac{dF}{dt} + r_3F.$$

The coefficients r_2 and r_3 can be found in Appendix A.3. The local exponents for this differential equation look like

Singularity	Exponent 1	Exponent 2
$t=0$	$2q-1$	$2q$
$t=\infty$	q	$q+1$
$t=-s$	0	$1-2q$
Roots of $144s^2t^2 + 4s^2 + 32st + 4t^2 + 1 = 0$	0	$1-2q$

Consider the covering

$$z \to -\frac{4(s+t)^2}{(12st+1)^2}.$$

The singularity $t = -s$ now corresponds to $z = 0$ with multiplicity 2. The singularity at a root of $144s^2t^2 + 4s^2 + 32st + 4t^2 + 1 = 0$ now corresponds to $z = 1$ with multiplicity 1. And lastly $z = \infty$ corresponds to the regular point $t = -\frac{1}{12s}$ with multiplicity 2. This means the local exponents of ${}_2F_1(z)$ under this covering will look like

Singularity	Exponent 1	Exponent 2
$t=0$	0	1
$t=\infty$	0	1
$t=-s$	0	$2(1-c)$
Roots of $144s^2t^2+4s^2+32st+4t^2+1=0$	0	$c-a-b$
$t=-\frac{1}{12s}$	2a	2b

Comparing the two tables we want to solve the equations $2(1-c) = 1-2q, 2(b-a)=1$ and $c-a-b=1-2q$. Which comes down to

$$a=\frac{3}{2}q-\frac{1}{2},\ b=\frac{3}{2}q,\ q+\frac{1}{2}.$$

Making the last table

Singularity	Exponent 1	Exponent 2
$t=0$	0	1
$t=\infty$	0	1
$t=-s$	0	$1-2q$
Roots of $144s^2t^2+4s^2+32st+4t^2+1=0$	0	$1-2q$
$t=-\frac{1}{12s}$	3q-1	3q

Hence we need to multiply by a factor $(12st+1)^{1-3q}t^{2q-1}$ to make the tables for H_5 and ${}_2F_1$ equal. As a consequence we get that

$$H_5\left(\begin{array}{c} q,\ q-\frac{1}{2} \\ 2q \end{array}\middle|\, t^2, \frac{4\left(144\,s^2t^2+4\,s^2+32\,st+4\,t^2+1\right)\left(4\,s^2+1\right)(s+t)s}{\left(12\,s^2-1\right)^3t^2}\right). \tag{6.27}$$

satisfies the same differential equation as the one coming from

$$(12st+1)^{1-3q}t^{2q-1}{}_2F_1\left(\begin{array}{c} \frac{3}{2}q-\frac{1}{2},\ \frac{3}{2}q \\ q+\frac{1}{2} \end{array}\middle|\, -\frac{4(s+t)^2}{(12st+1)^2}\right). \tag{6.28}$$

Next we want the arguments of H_5 in (6.27) to be symmetric, i.e. we want to find a parameterization of $s(u,v)$ and $t(u,v)$ such that

$$H_5(\phi(s(u,v),t(u,v)),\psi(s(u,v),t(u,v)))=H_5(\phi(s(v,u),t(v,u)),\psi(s(v,u),t(v,u))).$$

And we want the argument of ${}_2F_1$ in (6.28) to be independent of s. If we can achieve this then we can swap roles of u and v to make the Bailey identity by a symmetry argument. There are two approaches to this. The first approach is to make H_5 symmetric by plugging in a power series for t with variable coefficients. Then determine these coefficients and try to figure out which rational function belongs to this power series. When we perform this calculation where we fix $s(u,v)=\frac{u}{2}$, we find that by a miracle the argument of ${}_2F_1$ becomes independent of s. This approach is succesful but the result suggests a more intuitive approach which I will present here.

We want to find the inverse function of

$$g_s(t)=\frac{2(s+t)}{12st+1}.$$

This is simply

$$g_s^{-1}(t)=\frac{2t-s}{12st-2}.$$

Hence if we take $s(u,v)=\frac{u}{2}$ and $t(u,v)=g_{u/2}^{-1}(v)$ this means (6.28) becomes

$$(3uv+1)^q\left(1-3\,u^2\right)^{1-3q}\left(-\frac{u+v}{2}\right)^{2q-1}{}_2F_1\left(\begin{matrix}\frac{3q-1}{2},\ \frac{3q}{2}\\ q+\frac{1}{2}\end{matrix}\,\middle|\,-v^2\right). \tag{6.29}$$

Miraculously (6.27) becomes symmetric under this parameterization,

$$H_5\left(\begin{matrix}q,\ q-\frac{1}{2}\\ 2q\end{matrix}\,\middle|\,\frac{(u+v)^2}{4\,(3\,uv+1)^2},\frac{4\left(u^2+1\right)\left(v^2+1\right)uv}{(3\,uv+1)(u+v)^2}\right). \tag{6.30}$$

Multiply (6.29) by the constant

$$(-2)^{1-2q}(1-3u^2)^{3q-1}{}_2F_1\left(\begin{matrix}\frac{3q-1}{2},\ \frac{3q}{2}\\ q+\frac{1}{2}\end{matrix}\,\middle|\,-u^2\right).$$

This means that (6.29) turns into something symmetric:

$$(3uv+1)^q\left(-\frac{u+v}{2}\right)^{2q-1}{}_2F_1\left(\begin{matrix}\frac{3q-1}{2},\ \frac{3q}{2}\\ q+\frac{1}{2}\end{matrix}\,\middle|\,-u^2\right){}_2F_1\left(\begin{matrix}\frac{3q-1}{2},\ \frac{3q}{2}\\ q+\frac{1}{2}\end{matrix}\,\middle|\,-v^2\right).$$

By this symmetry we note that it satisfies the same differential equations as (6.30) in both u and v.

Consider the function

$$\eta:(u,v)\mapsto\left(\frac{(u+v)^2}{4\,(3\,uv+1)^2},\frac{4\left(u^2+1\right)\left(v^2+1\right)uv}{(3\,uv+1)(u+v)^2}\right).$$

Want want to pick a point (u_0, v_0) with $-u_0^2, -v_0^2 \in \{0,1\}$, such that η maps an open neighborhood of (u_0, v_0) to an open neighborhood of $(0,0)$. Just as we saw with H_1, this is impossible. To get a meaningful Bailey identity we will restrict η to a neighborhood of $(0,0)$ of the form $u = x, v = -x \cdot y$. By comparing the solution spaces we find and making both constant terms equal we see that

$$H_5\left(\begin{array}{c} q,\ q-\frac{1}{2} \\ 2q \end{array}\middle|\ \frac{x^2(y-1)^2}{4(3x^2y-1)^2}, \frac{4(x^2y^2+1)(x^2+1)y}{(3x^2y-1)(y-1)^2}\right) =$$

$$(1-3x^2y)^q(1-y)^{2q-1}{}_2F_1\left(\begin{array}{c} \frac{3}{2}q-\frac{1}{2},\ \frac{3}{2}q \\ q+\frac{1}{2} \end{array}\middle|\ -x^2y^2\right){}_2F_1\left(\begin{array}{c} \frac{1}{2}q,\ \frac{1}{2}q+\frac{1}{2} \\ \frac{3}{2}-q \end{array}\middle|\ -x^2\right).$$

Note that this identity only depends on x^2 and y. So we may substitute $x \to \sqrt{x}$ to obtain the Bailey-type identity

$$H_5\left(\begin{array}{c} q,\ q-\frac{1}{2} \\ 2q \end{array}\middle|\ \frac{x(y-1)^2}{4(3xy-1)^2}, \frac{4(xy^2+1)(x+1)y}{(3xy-1)(y-1)^2}\right) =$$

$$(1-3xy)^q(1-y)^{2q-1}{}_2F_1\left(\begin{array}{c} \frac{3}{2}q-\frac{1}{2},\ \frac{3}{2}q \\ q+\frac{1}{2} \end{array}\middle|\ -xy^2\right){}_2F_1\left(\begin{array}{c} \frac{1}{2}q,\ \frac{1}{2}q+\frac{1}{2} \\ \frac{3}{2}-q \end{array}\middle|\ -x\right). \tag{6.31}$$

Appendices

Appendix **A**

Additional Coefficients

In chapter 6 we left out some of the coefficients to improve readability.

A.1 Horn's H_4

The following coefficients belong to those of equation (6.12).

$$c_3 = \big(2\,bs^4t^6 + cs^4t^6 + 2\,as^3t^7 + bs^3t^7 + as^2t^8 - 4\,bs^4t^5 - 6\,bs^3t^6 - 2\,bs^2t^7 \\ +2\,s^3t^7 + s^2t^8 + 4\,bs^4t^4 - 2\,cs^4t^4 - 2\,as^3t^5 + 15\,bs^3t^5 + 4\,as^2t^6 + 8\,bs^2t^6 \\ -2\,cs^2t^6 + 2\,ast^7 + bst^7 - 4\,bs^4t^3 - 20\,bs^3t^4 - 2\,s^4t^4 - 22\,bs^2t^5 - 5\,s^3t^5 \\ -6\,bst^6 + 3\,s^2t^6 + 2\,st^7 + 2\,bs^4t^2 + cs^4t^2 - 2\,as^3t^3 + 15\,bs^3t^3 - 10\,as^2t^4 \\ +32\,bs^2t^4 + 4\,cs^2t^4 - 2\,ast^5 + 15\,bst^5 + 2\,bt^6 + ct^6 - 6\,bs^3t^2 - 2\,s^4t^2 \\ -22\,bs^2t^3 - 12\,s^3t^3 - 20\,bst^4 - 21\,s^2t^4 - 4\,bt^5 - 5\,st^5 + 2\,as^3t + bs^3t \\ +4\,as^2t^2 + 8\,bs^2t^2 - 2\,cs^2t^2 - 2\,ast^3 + 15\,bst^3 + 4\,bt^4 - 2\,ct^4 - 2\,bs^2t \\ -s^3t - 6\,bst^2 - 7\,s^2t^2 - 4\,bt^3 - 12\,st^3 - 2\,t^4 + as^2 + 2\,ast + bst + 2\,bt^2 \\ +ct^2 - st - 2\,t^2\big)\,s/\Big(\big(st^2 + s + 2\,t\big)\big(2\,st + t^2 + 1\big)\big(s^2 - 1\big)^2t^4\Big)$$

$$c_4 = -\,\big(as^2t^5 + bs^2t^5 + ast^6 - 2\,ds^2t^4 - 2\,bst^5 + s^2t^5 + st^6 - 2\,as^2t^3 \\ -2\,bs^2t^3 + 4\,ds^2t^3 - ast^4 + 4\,dst^4 + at^5 + bt^5 - 2\,ds^2t^2 + 4\,bst^3 - 8\,dst^3 \\ -4\,s^2t^3 - 2\,dt^4 - 2\,st^4 + t^5 + as^2t + bs^2t - ast^2 + 4\,dst^2 - 2\,at^3 - 2\,bt^3 \\ +4\,dt^3 - 2\,bst - s^2t - 2\,dt^2 - 7\,st^2 - 4\,t^3 + as + at + bt - t\big) \\ /\,\big(\big(st^2 + s + 2\,t\big)\big(2\,st + t^2 + 1\big)t^3\big)$$

$$c_5 = -\frac{\big(2\,bs^2t + ast^2 + 2\,ast - 4\,bst + st^2 + as + 2\,bt + 2\,st + s\big)a(t-1)^2}{(st^2 + s + 2\,t)(2\,st + t^2 + 1)t^2}.$$

The following coefficients belong to those of equation (6.13).

$$\begin{aligned}
r_2 = 2\ \big(&q_0s^2t^5 + q_1s^2t^5 + q_0st^6 - 4\,q_1s^2t^4 - 2\,q_1st^5 + s^2t^5 + st^6 - 2\,q_0s^2t^3\\
&+6\,q_1s^2t^3 - q_0st^4 + 8\,q_1st^4 + q_0t^5 + q_1t^5 - 4\,q_1s^2t^2 - 12\,q_1st^3 - 4\,s^2t^3\\
&-4\,q_1t^4 - 2\,st^4 + t^5 + q_0s^2t + q_1s^2t - q_0st^2 + 8\,q_1st^2 - 2\,q_0t^3 + 6\,q_1t^3\\
&-2\,q_1st - s^2t - 4\,q_1t^2 - 7\,st^2 - 4\,t^3 + q_0s + q_0t + q_1t - t\big)\\
&/\,\Big(\big(st^2+s+2\,t\big)\big(2\,st+t^2+1\big)(t+1)(t-1)t\Big)
\end{aligned}$$

$$r_3 = \frac{\big(2\,q_1s^2t + q_0st^2 + 2\,q_0st - 4\,q_1st + st^2 + q_0s + 2\,q_1t + 2\,st + s\big)q_0(t-1)^2}{(st^2+s+2\,t)(2\,st+t^2+1)t^2}.$$

A.2 Horn's H_1

Similarly we show the coefficients corresponding to the same equations, but now for Horn's H_1.

$$\begin{aligned}
c_3 = 2\ \big(&8\,cs^4t^6 + 2\,ds^4t^6 + 4\,as^3t^7 + 4\,bs^3t^7 + 12\,cs^3t^7 + 2\,as^2t^8 + 2\,bs^2t^8\\
&+4\,cs^2t^8 + 2\,s^3t^7 + s^2t^8 + 8\,cs^4t^4 - 4\,ds^4t^4 - 4\,as^3t^5 - 4\,bs^3t^5 + 40\,cs^3t^5\\
&+8\,as^2t^6 + 8\,bs^2t^6 + 44\,cs^2t^6 - 4\,ds^2t^6 + 4\,ast^7 + 4\,bst^7 + 12\,cst^7 - 4\,s^4t^4\\
&-8\,s^3t^5 + 2\,s^2t^6 + 2\,st^7 + 2\,ds^4t^2 - 4\,as^3t^3 - 4\,bs^3t^3 + 12\,cs^3t^3 - 20\,as^2t^4\\
&-20\,bs^2t^4 + 44\,cs^2t^4 + 8\,ds^2t^4 - 4\,ast^5 - 4\,bst^5 + 40\,cst^5 + 8\,ct^6 + 2\,dt^6\\
&-4\,s^4t^2 - 22\,s^3t^3 - 32\,s^2t^4 - 8\,st^5 + 4\,as^3t + 4\,bs^3t + 8\,as^2t^2 + 8\,bs^2t^2\\
&+4\,cs^2t^2 - 4\,ds^2t^2 - 4\,ast^3 - 4\,bst^3 + 12\,cst^3 + 8\,ct^4 - 4\,dt^4 - 4\,s^3t\\
&-18\,s^2t^2 - 22\,st^3 - 4\,t^4 + 2\,as^2 + 2\,bs^2 + 4\,ast + 4\,bst + 2\,dt^2 - s^2 - 4\,st - 4\,t^2\big)\,s\\
&/\,\Big(\big(st^2+s+2\,t\big)\big(2\,st+t^2+1\big)(s+1)^2(s-1)^2t^4\Big)
\end{aligned}$$

$$\begin{aligned}
c_4 = -2\ \big(&4\,bs^2t^3 + 4\,cs^2t^3 + 2\,ast^4 + 2\,bst^4 + 4\,cst^4 + 2\,s^2t^3 + st^4 - 4\,as^2t\\
&-8\,ast^2 + 8\,bst^2 + 4\,cst^2 + 4\,bt^3 + 4\,ct^3 + 2\,s^2t + 6\,st^2 + 2\,t^3\\
&-2\,as - 2\,bs - 4\,at + s + 2\,t\big)\,/\,\Big(\big(st^2+s+2\,t\big)\big(2\,st+t^2+1\big)\Big)
\end{aligned}$$

$$c_5 = -\frac{4\,\big(2\,cs^2t^3 + ast^4 + 2\,cst^4 - 2\,ast^2 + 2\,cst^2 + 2\,ct^3 + as\big)b}{(st^2+s+2\,t)(2\,st+t^2+1)t^2}$$

$$\begin{aligned}
r_2 = 2\ \big(&2\,q_0s^2t^3 + 2\,q_0st^4 + 2\,s^2t^3 + st^4 - 2\,q_0s^2t + 2\,q_0t^3\\
&+2\,s^2t + 6\,st^2 + 2\,t^3 - 2\,q_0s - 2\,q_0t + s + 2\,t\big)\\
&/\,\Big(\big(st^2+s+2\,t\big)\,\big(2\,st+t^2+1\big)\,t\Big)
\end{aligned}$$

$$r_3 = \frac{2\,\big(2\,q_0st^4 + 2\,s^2t^3 + st^4 - 4\,q_0st^2 + 4\,st^2 + 2\,t^3 + 2\,q_0s - s\big)q_0}{(st^2+s+2\,t)(2\,st+t^2+1)t^2}$$

A.3 Horn's H_5

The algebraic relation (6.26) is given by

$$
\begin{aligned}
f(a,x,y) =& \left(2985984a^6 - 1492992a^5 + 311040a^4 - 34560a^3 + 2160a^2 - 72a + 1\right)x^2y^2 \\
&- 256\left(20736\,a^5 + 11520\,a^4 + 1888\,a^3 + 80\,a^2 + a\right)x^3 \\
&- 288\left(27648\,a^6 + 6912\,a^5 - 1152\,a^4 - 160\,a^3 + 28\,a^2 - a\right)x^2y \\
&+ 128\left(41472\,a^6 + 20736\,a^5 + 4032\,a^4 + 704\,a^3 + 82\,a^2 - a\right)x^2 \\
&- 8\left(27648\,a^6 + 6912\,a^5 - 1152\,a^4 - 160\,a^3 + 28\,a^2 - a\right)xy \\
&+ 16\left(18432\,a^6 - 2304\,a^5 - 4608\,a^4 - 736\,a^3 - 8\,a^2 - a\right)x \\
&+ 4096\,a^6 + 4096\,a^5 + 1536\,a^4 + 256\,a^3 + 16\,a^2.
\end{aligned}
$$

The remaining coefficients appearing in the section 6.5 are given by

$$
\begin{aligned}
c_3 = -2&\left(192\,as^3t^2 - 192\,bs^3t^2 + 288\,as^2t^3 + 192\,s^3t^2 + 288\,s^2t^3 - 8\,as^3\right. \\
&-8\,bs^3 - 72\,bs^2t + 48\,ast^2 - 48\,bst^2 + 8\,at^3 + 4\,s^3 + 36\,s^2t + 48\,st^2 + 8\,t^3 \\
&\left.-2\,as - 2\,bs - 2\,bt + s + t\right) / \left(\left(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\right)(s+t)\right)
\end{aligned}
$$

$$
\begin{aligned}
c_4 = 4&\left(\left(13824\,as^6t^4 + 27648\,bs^6t^4 - 20736\,cs^6t^4 - 27648\,as^5t^5 + 27648\,bs^5t^5\right.\right. \\
&-41472\,as^4t^6 + 13824\,s^6t^4 - 27648\,s^5t^5 - 41472\,s^4t^6 - 1920\,as^6t^2 \\
&-1536\,bs^6t^2 + 1728\,cs^6t^2 + 4608\,bs^5t^3 - 3456\,as^4t^4 + 13824\,bs^4t^4 \\
&+5184\,cs^4t^4 - 7680\,as^3t^5 + 7680\,bs^3t^5 - 2304\,as^2t^6 + 1536\,s^6t^2 \\
&+13824\,s^5t^3 + 6912\,s^4t^4 - 7680\,s^3t^5 - 2304\,s^2t^6 - 64\,as^6 - 64\,bs^6 \\
&-576\,as^5t - 576\,bs^5t - 960\,as^4t^2 - 768\,bs^4t^2 - 432\,cs^4t^2 + 1280\,bs^3t^3 \\
&+288\,as^2t^4 + 1728\,bs^2t^4 - 432\,cs^2t^4 - 192\,ast^5 + 192\,bst^5 - 32\,at^6 \\
&+32\,s^6 + 576\,s^5t + 3360\,s^4t^2 + 3840\,s^3t^3 + 864\,s^2t^4 - 192\,st^5 - 32\,t^6 \\
&-32\,as^4 - 32\,bs^4 - 160\,as^3t - 160\,bs^3t - 120\,as^2t^2 - 96\,bs^2t^2 + 36\,cs^2t^2 \\
&+32\,bst^3 - 8\,at^4 + 12\,ct^4 + 16\,s^4 + 160\,s^3t + 240\,s^2t^2 + 96\,st^3 \\
&\left.\left.-4\,as^2 - 4\,bs^2 - 4\,ast - 4\,bst - ct^2 + 2\,s^2 + 4\,st + 2\,t^2\right)\left(4\,s^2+1\right)s\right) \\
&/\left(\left(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\right)\left(12\,s^2 - 1\right)^3(s+t)t^4\right)
\end{aligned}
$$

$$
\begin{aligned}
c_5 = -4a&\left(12\,as^3t^2 - 48bs^3t^2 + 36as^2t^3 + 12s^3t^2 + 36s^2t^3\right. \\
&\left.+4bs^3 + 3ast^2 - 12bst^2 + at^3 + 3st^2 + t^3 + bs\right) \\
&/\left(\left(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\right)(s+t)\,t^2\right)
\end{aligned}
$$

$$r_2 = 2\,\big(144\,qs^2t^3 + 144\,s^3t^2 + 144\,s^2t^3 - 8\,qs^3 - 36\,qs^2t$$
$$+4\,qt^3 + 4\,s^3 + 36\,s^2t + 36\,st^2 + 4\,t^3 - 2\,qs - qt + s + t\big)$$
$$/\,\big(\big(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\big)\,(s+t)\,t\big)$$

$$r_3 = -2\,\big(\big(72\,qs^3t^2 - 72\,qs^2t^3 - 72\,s^3t^2 - 72\,s^2t^3 - 8\,qs^3$$
$$+18\,qst^2 - 2\,qt^3 + 4\,s^3 - 18\,st^2 - 2\,t^3 - 2\,qs + s\big)\,q\big)$$
$$/\,\big(\big(144\,s^2t^2 + 4\,s^2 + 32\,st + 4\,t^2 + 1\big)\,(s+t)\,t^2\big)\,.$$

Samenvatting

Hypergeometrische Functies

Klassieke functies als $\sin^{-1}(x)$, $\tan^{-1}(x)$, $\log(x)$ of $\sqrt{x}$ duiken overal op in de wiskunde, in het dagelijks leven en we zijn ze vast wel eens tegengekomen op school. Hoewel deze functies allemaal verschillende toepassingen hebben en regels voor de ene functie niet op lijken te gaan voor de andere functie, zou je je kunnen afvragen of er misschien eigenschappen zijn die deze functies verbinden. Zo zouden we ons kunnen afvragen of er misschien een algemene formule bestaat die ze allemaal beschrijft. Euler vond precies zo'n formule, de zogenaamde hypergeometrische functie. De hypergeometrische functie is de volgende oneindige som

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x \right) = 1 + \frac{a \cdot b}{c \cdot 1} x + \frac{a \cdot (a+1) \cdot b \cdot (b+1)}{c \cdot (c+1) \cdot 1 \cdot 2} x^2 + \\ + \frac{a \cdot (a+1) \cdot (a+2) \cdot b \cdot (b+1) \cdot (b+2)}{c \cdot (c+1) \cdot (c+2) \cdot 1 \cdot 2 \cdot 3} x^3 + \dots . $$

In plaats van deze som steeds volledig uit te schrijven, kunnen we dit compact op schrijven als

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n. \tag{1} $$

Hier is $(a)_n = a \cdot (a+1) \cdot \ldots \cdot (a+n-1)$ de stijgende faculteit.

Door de waardes van de parameters a, b, c slim te kiezen krijgen we nu klassieke functies. Hier volgen een aantal voorbeelden.

$$ \sin^{-1}(x) = x\, {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \,\middle|\, -x^2 \right), $$

$$\log(x) = (x-1)\,{}_2F_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\, 1-x\right),$$

$$x^n = (x-1)\,{}_2F_1\left(\begin{matrix}-n,\ m\\ m\end{matrix}\,\middle|\, 1-x\right).$$

De hypergeometrische functie beschikt over meer interessante eigenschappen. Zo zag Gauss dat de hypergeometrische functie een oplossing is van de differentiaalvergelijking

$$x(1-x)F''(x) + (c-(a+b+1)x)F'(x) + abF(x) = 0. \tag{2}$$

De hypergeometrische functie (1) is echter niet de enige oplossing van dit differentiaal systeem. Een andere oplossing is bijvoorbeeld

$$G(x) = x^{1-c}{}_2F_1\left(\begin{matrix}1+a-c,\ 1+b-c\\ 2-c\end{matrix}\,\middle|\, x\right).$$

Lineaire combinaties van $G(x)$ en $F(x) = {}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\, x\right)$ geven de volledige oplossingsruimte van het differentiaal systeem (2). Dat wil zeggen alle oplossingen zijn van de vorm $kF(x) + lG(x)$ met k en l constanten. Om onderscheid te maken tussen verschillende hypergeometrische functies zullen we ${}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\, x\right)$ Gauss' hypergeometrische functie noemen.

Na het veralgemeniseren van een aantal klassieke functies tot hypergeometrische functies, zouden we ook hypergeometrische functies verder kunnen uitbreiden. Een voordehandliggende uitbreiding is om het aantal stijgende faculteiten in de teller en noemer van (1) te variëren. Dit noemen we de gegeneraliseerde hypergeometrische functie en deze is als volgt gedefinieerd

$${}_nF_{n-1}\left(\begin{matrix}a_1,\ldots,a_n\\ b_1,\ldots,b_{n-1}\end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\ldots(a_n)_k}{(b_1)_k(b_2)_k\ldots(b_{n-1})_k k!}z^k.$$

Maar er zijn meer generalisaties mogelijk. Zo zouden we ook het aantal variabelen kunnen veranderen.

Voorbeelden hiervan zijn Appell's hypergeometrische functies F_1, F_2, F_3, F_4, Horn's hypergeometrische functies $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6, H_7$ en Lauricella's hypergeometrische functies F_A, F_B, F_C, F_D. Hier is een voorbeeld van F_4,

$$F_4\left(\begin{matrix}\alpha,\ \beta\\ \gamma,\ \gamma'\end{matrix}\,\middle|\, x,y\right) = \sum_{m=0}^{\infty}\sum_{n=0}\frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!n!}x^m y^n.$$

Lauricella's F_C is op haar beurt weer een generalisatie van F_4 en is als volgt gedefinieerd

$$F_C\left(\begin{array}{c} a,\ b \\ c_1,\ldots,c_n \end{array}\middle| x_1,\ldots,x_n\right) = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}(b)_{m_1+\ldots+m_n}x_1^{m_1}\ldots x_n^{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}m_1!\ldots m_n!}.$$

We zouden ons ook in dit geval kunnen afvragen of al deze hypergeometrische functies niet allemaal op een algemenere manier te schrijven zijn. Dit is mogelijk door middel van A-hypergeometrische functies, geïntroduceerd door Gelfand, Kapranov en Zelevinsky. Het idee is dat je begint met een configuratie matrix A en een parameter vector α. De vector α komt overeen met de parameters van de hypergeometrische functie en de matrix A specificeert het type hypergeometrische functie. De keuze van matrix A en parameter vector α geeft ons nu een stelsel van differentiaal vergelijkingen en een bijbehorende basis van machtreeks oplossingen. Alle genoemde voorbeelden kunnen zo op deze manier beschreven worden als A-hypergeometrische functies.

Monodromie

Een probleem van hypergeometrische functies is dat ze meerwaardig zijn. Eenvoudige voorbeelden van functies met meerwaardigheidsproblematiek zijn $\sqrt{z}$ en $\log(z)$. Het is gebruikelijk dat $\sqrt{1} = 1$. Neem een kleine omgeving rondom $z = 1$ in de complexe ruimte als je domein. We gaan nu het domein uitbreiden langs de complexe eenheidscirkel. En als we dan terug komen bij $z = 1$ heeft onze functie $\sqrt{z}$ plots de waarde -1.

Voor $\log(z)$ is het gebruikelijk dat $\log(1) = 0$. Dus neem een kleine omgeving rondom $z = 1$ in de complexe ruimte als domein voor $\log(z)$. We gaan dit domein ook nu langs de eenheidscirkel uitbreiden. Zodra we terugkeren bij $z = 1$ vinden we twee mogelijke nieuwe waarden voor $\log(1)$, namelijk $-2\pi i$ of $2\pi i$. Deze waarden zijn afhankelijk van of je met de klok mee of tegen de klok in de complexe eenheidscirkel af gaat.

Het probleem van meerwaardigheid bij oplossingen van lineaire differentiaal vergelijkingen, zoals bij A-hypergeometrische functies het geval is, wordt tegenwoordig bestudeert in de vorm van monodromie. Monodromie kan als volgt worden beschreven. Stel we hebben een basis van oplossingen $f_1(z_1,\ldots,z_n),\ldots,f_r(z_1,\ldots,z_n)$ van een systeem van lineaire differentiaal vergelijkingen. Dan kunnen we de variabelen $z_1,\ldots,z_n$ in een lus laten bewegen en kijken hoe de oplossingen daar onder veranderen via analytische voortzetting. Omdat dit nog steeds oplossingen zijn van het zelfde stelsel differentiaal vergelijkingen kunnen we elke nieuwe oplossing schrijven als lineaire combinatie van onze basis $f_1(z_1, \ldots, z_n), \ldots, f_r(z_1, \ldots, z_n)$. Op deze manier ontstaan er lineaire afbeeldingen ten opzichte van de gekozen basis. Het geheel van deze lineaire afbeeldingen wordt ook wel de monodromie groep genoemd. Deze lineaire afbeeldingen noteren we in de vorm van matrices.

In dit proefschrift zijn we vooral geïnteresseerd in de eigenschappen van deze monodromie groep. En hoe deze ons kunnen helpen in het vinden van eigenschappen voor hypergeometrische functies.

In Hoofdstuk 2 worden de concepten A-hypergeometrische functie en monodromie verder ingeleid. We laten zien, hoe we onder strenge voorwaarden, een deel van de monodromie groep van een A-hypergeometrisch systeem kunnen genereren door middel van een methode door Frits Beukers. De strenge voorwaarden zijn nodig om een basis van zogenaamde Mellin-Barnes integralen te garanderen, die voldoen aan de A-hypergeometrische vergelijkingen. De resulterende monodromie groep is ten opzichte van een basis van deze Mellin-Barnes integralen.

Invariante Hermitische Vormen

Eén eigenschap van de monodromie groep waar we geïnteresseerd in zijn is de bijbehorende invariante Hermitische vorm. Heel concreet is deze Hermitische vorm een matrix H zodanig dat voor elke matrix M in de monodromie groep het volgende geldt:

$$\overline{M}^{\intercal} H M = H.$$

Hier is $\overline{M}^{\intercal}$ de geconjugeerde getransponeerde matrix van M. Verder is H een Hermitische matrix, dat wil zeggen $\overline{H}^{\intercal} = H$.

De hoofdstelling uit Hoofdstuk 3 geeft een expliciete constructie van een invariante Hermitische vorm H ten op zichte van de monodromie groep die door middel van de methode van Frits Beukers geconstrueerd wordt.

De constructie van de Hermitische vorm uit Hoofdstuk 3 gebruikt de Mellin-Barnes integralen niet expliciet. Dat maakt het mogelijk om het bestaan van deze Mellin-Barnes integralen achterwege te laten en de constructie in Hoofdstuk 3 virtueel uit te breiden. Heel specifiek willen we een matrix groep construeren die de constructie van de invariante Hermitische vorm zoals beschreven in Hoofdstuk 3 mogelijk maakt. Deze matrix groep is in veel gevallen een subgroep van de volledige monodromie groep. In Hoofdstuk 4 geven we voorwaarden en algoritmes die het vinden van zo'n virtuele monodromie groep mogelijk maken.

Een Hermitische matrix $H \in \mathbb{C}^{n \times n}$ is positief-definiet als voor alle vectoren $\boldsymbol{x} \in \mathbb{C}^n \setminus \{\boldsymbol{0}\}$ geldt dat $\boldsymbol{x}^{\intercal} H \boldsymbol{x} > 0$. En soortgelijk is een Hermitische matrix $H \in \mathbb{C}^{n\times n}$ negatief-definiet als voor alle vectoren $\boldsymbol{x} \in \mathbb{C}^n \backslash \{\boldsymbol{0}\}$ geldt dat $\boldsymbol{x}^\intercal H\boldsymbol{x} < 0$. Tot slot noemen we H definiet als deze ofwel positief-definiet dan wel negatief-definiet is. Hoofdstuk 5 geeft een criterium waarvoor de Hermitische vorm uit Hoofdstuk 3 definiet is. Dit criterium maakt gebruik van zogenaamde apexpunten. Apexpunten vinden hun oorsprong in de theorie van algebraïciteit van oplossingen van A-hypergeometrische functies. Om te bepalen of alle oplossingen van een A-hypergeometrisch systeem algebraïsch zijn heeft Frits Beukers een criterium gegeven op basis van deze apexpunten. De hoofstelling van Hoodstuk 5 verbind het criterium voor algebraïciteit van oplossingen van het A-hypergeometrische systeem aan de definietheid van de matrix H.

Bailey factorisaties

In Hoofdstuk 6 gaan we op zoek naar relaties tussen klassieke hypergeometrische functies. In 1933 publiceerde Bailey een identiteit waar Appell's F_4 wordt opgesplitst in twee Gauss' hypergeometrische functies

$$\begin{aligned} &F_4\left(\begin{matrix} a,\ b \\ c,\ a+b-c+1 \end{matrix}\,\middle|\, x(1-y), y(1-x)\right) \\ &= {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) {}_2F_1\left(\begin{matrix} a,\ b \\ a+b-c+1 \end{matrix}\,\middle|\, y\right). \end{aligned}$$

Ook voor Appell's F_2 is later zo'n type factorisatie gevonden

$$\begin{aligned} &F_2\left(\begin{matrix} a+b-\frac{1}{2},\ a,\ b \\ 2a,\ 2b \end{matrix}\,\middle|\, \frac{4u(1-u)(1-2v)}{(1-2uv)^2}, \frac{4v(1-v)(1-2u)}{(1-2uv)^2}\right) \\ &= (1-2uv)^{-1+2a+2b} {}_2F_1\left(\begin{matrix} a+b-\frac{1}{2},\ a \\ 2a \end{matrix}\,\middle|\, 4u(1-u)\right) {}_2F_1\left(\begin{matrix} a+b-\frac{1}{2},\ b \\ 2b \end{matrix}\,\middle|\, 4v(1-v)\right). \end{aligned} \tag{3}$$

We zouden ons nu kunnen afvragen hoe we zulke factorisaties kunnen vinden bij andere klassieke hypergeometrische functies zoals Horn's hypergeometrische functies.

We laten zien hoe eigenschappen van de monodromie gebruikt kunnen worden om vast te stellen voor welke specialisaties van de parameters een Horn functie van rang 4 opsplitst in twee Gauss hypergometrische functies. Nadat deze specialisatie is vastgesteld gaan we de Horn hypergeometrische functies van rang 4 daadwerkelijk factoriseren.

Dit resulteerde in de volgende factorisatie voor Horn's H_4

$$H_4\left(\begin{array}{c} q_0,\ q_1 \\ 1+q_0-q_1,\ 2q_1 \end{array}\middle|\ \frac{(s^2-1)(t^2-1)}{4(st-1)^2}, \frac{2st}{st-1}\right)$$

$$= (1-st)^{q_0}\, {}_2F_1\left(\begin{array}{c} \frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0 \\ q_0-q_1+1 \end{array}\middle|\ 1-s^2\right) {}_2F_1\left(\begin{array}{c} \frac{1}{2}q_0+\frac{1}{2},\ \frac{1}{2}q_0 \\ q_1+\frac{1}{2} \end{array}\middle|\ t^2\right).$$

Dit is een gelijkheid in machtreeksen ontwikkeld in het punt $(s,t)=(1,0)$. Voor Horn's H_1 vinden we de factorisatie

$$H_1\left(\begin{array}{c} q_0-\frac{1}{2},\ q_0,\ \frac{1}{2} \\ 2q_0 \end{array}\middle|\ \phi(-1+2uv, 1+2v)\right) =$$

$$(1+v-uv-2uv^2)^{2q_0}\left(\frac{1-u-2uv}{(1+v)(1+2v+2v^2)}\right)^{2q_0-1}$$
$$\times\, {}_2F_1\left(\begin{array}{c} 1-q_0,\ \frac{3}{4}-q_0 \\ 2-2q_0 \end{array}\middle|\ -8v(1+v)(1+2v+2v^2)\right)$$
$$\times\, {}_2F_1\left(\begin{array}{c} q_0-\frac{1}{4},\ q_0 \\ 2q_0 \end{array}\middle|\ 8uv(1-uv)(1-2uv+2u^2v^2)\right).$$

Hier nemen we

$$\phi(s,t) = \left(\frac{(s^4-1)(t^4-1)}{(s^2t^2-1)^2}, \left(\frac{st+1}{st-1}\right)^2\right).$$

Dit is een gelijkheid in machtreeksen ontwikkeld in het punt $(u,v)=(0,0)$.

En tot slot laten we zien dat H_5 af splitst als

$$H_5\left(\begin{array}{c} q,\ q-\frac{1}{2} \\ 2q \end{array}\middle|\ \frac{x(y-1)^2}{4(3xy-1)^2}, \frac{4(xy^2+1)(x+1)y}{(3xy-1)(y-1)^2}\right) =$$

$$(1-3xy)^q(1-y)^{2q-1}{}_2F_1\left(\begin{array}{c} \frac{3}{2}q-\frac{1}{2},\ \frac{3}{2}q \\ q+\frac{1}{2} \end{array}\middle|\ -xy^2\right) {}_2F_1\left(\begin{array}{c} \frac{1}{2}q,\ \frac{1}{2}q+\frac{1}{2} \\ \frac{3}{2}-q \end{array}\middle|\ -x\right).$$

Acknowledgements

First and foremost, I would like to thank my promotor Frits Beukers for trusting me with this position, for giving me the space to work on my own ideas and for teaching me new techniques and skills in both mathematical research and academical work. It was a pleasure working with you and your feedback was always very useful.

Secondly, I want to thank the thesis committee, Erik van den Ban, Christine Berkesch, Erik Koelink, Susumu Tanabe, and Wadim Zudilin for thoroughly reading my thesis and for giving me valuable feedback. In particular, I want to thank Wadim Zudilin for his useful suggestions on finding Bailey-type identities for Horn's H_5 in section 6.5.

Additionally, I want to explicitly thank my former office mates, Andrei and Jens, for creating a pleasant working atmosphere; for the mathematical conversations and the wonderful times at the office during these four years. I also want to thank the colleagues from the Utrecht Geometry Center for the conversations during lunch and for interesting seminars and discussions. I further want to thank the administrative staff of the department for the help and taking care of all administrative duties.

I am also grateful for the many friends that gave me the needed distractions from math. I want to thank my friends from scouting for the social activities I participated in during the first years of my research. In addition, thanks to all my musical friends and people that I had the privilege to play music with. In particular, I am thankful that Sebastiaan and Willem agreed to be my Paranymphs. Our friendships go back many years and the conversations, musical opportunities, and good laughs helped clear my mind and kept me motivated.

Finally, I want to thank my family for their support and that I can ask them to help me out whenever its needed.

Curriculum Vitae

Carlo Verschoor was born on 27 december 1990 in Groningen. In 2009 he graduated from Rölingcollege Belcampo with a HAVO-degree. Moreover, in 2009, Carlo Verschoor won the Jan Kommandeur prize for best high school thesis about finding large prime numbers. That same year he received a gold medal in the International Conference of Young Scientists after presenting his thesis.

In 2010 he got his propaedeutic diploma at the Noordelijke Hogeschool Leeuwarden in Business Mathematics. That same year he started his bachelor's in mathematics at the University of Groningen which he completed in 2013. In 2015 he got his master's degree at the University of Groningen with his thesis about Twists of the Klein Curve and of the Fricke Macbeath Curve.

Carlo Verschoor started his PhD in 2016 at the University of Utrecht under supervision of Frits Beukers. During this time he visited several international conferences and conducting several exercise classes for bachelor and master courses.

Glossary

General Symbols

$\{a_{rc}\}_{r,c}$	Alternative notation for matrices	22
B_I	Submatrix of a matrix B filtered by index set I	13
$\{x\}$	Fractional part of x, works component-wise for vectors	54
I^c	Complement of a discrete set I	12
$C(A)$	Cone spanned by columns of A	11
$\mathrm{Diag}(\boldsymbol{v})$	Diagonal matrix where $\boldsymbol{v}$ is on the diagonal	21
$\lfloor x \rfloor$	Floor of x, works component-wise for vectors	54
$\Gamma(z)$	The Gamma function	7
$\mathrm{GO}(n)$	Group of orthogonal similitudes	64
$\mathrm{GSp}(n)$	Group of symplectic similitudes	65
$\mathrm{Hyp}(W)$	The hyperplane that extends from W	25
$\Pi_{\mathcal{L}}$	Fundamental parallellogram of a lattice $\mathcal{L}$	48
$(a)_n$	Pochhammer Symbol	1
$Q(A)$	Convex Hull of a set A	11
Res	Residue	24
$\mathrm{Vol}(Q)$	Normalized volume of Q	11

A-hypergeometric Symbols

$\boldsymbol{a}$	Column vector of A	9

A	Structure matrix	9
$\boldsymbol{\alpha}$	A-hypergeometric parameter vector	9
$\boldsymbol{b}$	Column vector of B	10
B	Gale dual of A	9
B_L	Lawrence lifting of B	33
$\Box^{\boldsymbol{\lambda}}$	Box operator, see equation (2.5)	10
$\chi_{\boldsymbol{\rho},j}$	Local monodromy matrices	16
C_I	Cone corresponding to a cotriangle I	14
$\mathcal{C}_{\boldsymbol{\rho}}$	Chamber	14
D	Rank of the A-hypergeometric System	11
Δ_I	Absolute determinant of B_I	13
Δ_ρ	Hermitian Diagonal Matrix, see equation (3.2)	21
$\mathscr{F}_B$	The Frobenius Cavity	30
$\boldsymbol{\gamma}^{\boldsymbol{\mu}}$	Dual parameter vector corresponding to solution point $\boldsymbol{\mu}$	13
$\mathscr{G}_B$	Covariograph	47
H	Hermitian Matrix, see equation (3.1)	21
h	Linear form for which $h(\boldsymbol{a}) = 1$ for all column vectors $\boldsymbol{a}$ of A.	9
$H_A(\boldsymbol{\alpha})$	The A-hypergeometric system	10
I	Cotriangle	12
$\mathcal{I}_{\boldsymbol{\rho}}$	Cotriangulation	13
$\mathcal{I}_W$	The cotriangles in $\mathcal{I}$ whose cones have W as (sub)-face	25
$K_{\boldsymbol{\alpha}}$	Short for $(\boldsymbol{\alpha} + \mathbb{Z}^r) \cap C(A)$	54
K_I	Shifted semigroup inside $K_{\boldsymbol{\alpha}}$ corresponding to a cotriangle I	55
$K_{\boldsymbol{\mu}}$	Shifted semigroup inside $K_{\boldsymbol{\alpha}}$ corresponding to a solution point $\boldsymbol{\mu}$	55
L	Lattice of relations	8
$\mathcal{M}$	Full Monodromy Group	17
$\mathcal{M}_{MB}$	Mellin Barnes Monodromy Group	17
$M_{\boldsymbol{\rho}}$	Zonotopal Semigroup	40
$M_{\boldsymbol{\rho},j}$	Mellin Barnes monodromy matrices	17

Hypergeometric Functions

Abbreviations

Bibliography

[Ado94] Alan Adolphson. Hypergeometric functions and rings generated by monomials. *Duke Mathematical Journal*, 73(2):269–290, 1994. doi:10.1215/s0012-7094-94-07313-4.

[App80] P. Appell. Sur les séries hypergéométriques de deux variables et sur dés équations différentielles linéaires aux dérivés partielles. *C. R. Acad. Sci., Paris*, 90:296–299, 731–735, 1880.

[App82] P. Appell. Sur les fonctions hypergéométriques de deux variables. *Journal de Mathématiques Pures et Appliquées*, pages 173–216, 1882. URL: http://eudml.org/doc/234110.

[Bai33] W. N. Bailey. A reducible case of the fourth type of appells hypergeometric functions of two variables. *The Quarterly Journal of Mathematics*, os-4(1):305–308, 1933. doi:10.1093/qmath/os-4.1.305.

[Bar07] E. W. Barnes. The Asymptotic Expansion of Integral Functions Defined by Generalised Hypergeometric Series. *Proceedings of the London Mathematical Society*, s2-5(1):59–116, 1907. doi:10.1112/plms/s2-5.1.59.

[Beu10] Frits Beukers. Algebraic a-hypergeometric functions. *Inventiones mathematicae*, 180(3):589–610, 2010. arXiv:0812.1134, doi:10.1007/s00222-010-0238-y.

[Beu11] Frits Beukers. Notes on A-hypergeometric functions. *Arithmetic and Galois theories of differential equations, Sémin. Congr*, 23:25–61, 2011.

[Beu16] Frits Beukers. Monodromy of A-hypergeometric functions. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2016(718), 2016. arXiv:1101.0493v2, doi:10.1515/crelle-2014-0054.

[BH89] F. Beukers and G. Heckman. Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Inventiones Mathematicae*, 95(2):325–354, jun 1989. doi:10.1007/bf01393900.

[CDS02] Eduardo Cattani, Alicia Dickenstein, and Bernd Sturmfels. Binomial residues. *Annales de l'institut Fourier*, 52(3):687–708, 2002. doi:10.5802/aif.1898.

[DG02] Amédée Debiard and Bernard Gaveau. Hypergeometric symbolic calculus. i – systems of two symbolic hypergeometric equations. *Bulletin des Sciences Mathématiques*, 126(10):773–829, dec 2002. doi:10.1016/s0007-4497(02)01143-0.

[DM86] P. Deligne and G. D. Mostow. Monodromy of hypergeometric functions and non-lattice integral monodromy. *Publications mathématiques de l'IHÉS*, 63(1):5–89, 1986. doi:10.1007/bf02831622.

[Erd48] A. Erdélyi. XXXIX.—transformations of hypergeometric functions of two variables. *Proceedings of the Royal Society of Edinburgh. Section A. Mathematical and Physical Sciences*, 62(3):378–385, 1948. doi:10.1017/s0080454100006774.

[Gau13] CF Gauss. Disquisitiones generales circa seriem infinitam $1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}xx +$ etc.. *Commentationes societatis regiae scientiarum Gottingensis recentiores*, 2:1–46, 1813.

[GGZ87] I.M. Gelfand, M.I. Graev, and A.V. Zelevinsky. Holonomic systems of equations and series of hypergeometric type. *Doklady Akad. Nauk SSSR*, 295:14–19, 1987. (in Russian).

[GH78] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry (Pure and Applied Mathematics)*. Wiley-Interscience, 1978.

[GK02] O. A. Gelfond and A. G. Khovanskii. Toric geometry and grothendieck residues. *Moscow Math.*, 2(1):99–112, 2002.

[GKZ88] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Equations of hypergeometric type and Newton polytopes. *Doklady Akad. Nauk SSSR*, 300:529–534, 1988. (in Russian).

[GKZ89] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Hypergeometric functions and toric varieties. *Funktsional. Anal. i Prilozhen.*, 23:12–26, 1989. (in Russian).

[GKZ90] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. Generalized Euler integrals and A-hypergeometric functions. *Advances in Mathematics*, 84(2):255–271, 1990. `doi:10.1016/0001-8708(90)90048-r`.

[GMH20] Yoshiaki Goto and Saiei-Jaeyeong Matsubara-Heo. On the signature of monodromy invariant hermitian form. Unpublished, 2020.

[Got14] Yoshiaki Goto. The monodromy representation of Lauricella's hypergeometric function F_C. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, 16, 03 2014.

[Hor89] J. Horn. Ueber die Convergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen. *Mathematische Annalen*, 34(4):544–600, 1889. `doi:10.1007/bf01443681`.

[Hor31] J. Horn. Hypergeometrische Funktionen zweier Veränderlichen. *Mathematische Annalen*, 105(1):381–407, 1931. `doi:10.1007/bf01455825`.

[HU08] Yoshishige Haraoka and Youichi Ueno. Rigidity for Appell's Hypergeometric Series F_4. *Funkcialaj Ekvacioj*, 51(1):149–164, 2008. `doi:10.1619/fesi.51.149`.

[JRS10] A. De Loera Jesus, J. Rambau, and F. Santos. *Triangulations: structures for algorithms and applications*. Springer, 2010.

[Kan81] Jyoichi Kaneko. Monodromy group of Appell's system (F_4). *Tokyo J. Math.*, 4:35–54, 1981.

[Kar18] Roman Karasev. Residues and the combinatorial nullstellensatz. *Periodica Mathematica Hungarica*, 78(2):157–165, 2018. `doi:10.1007/s10998-018-00272-z`.

[Kat00] Mitsuo Kato. Appell's hypergeometric systems F_2 with finite irreducible monodromy groups. *Kyushu Journal of Mathematics*, 54(2):279–305, 2000. doi:10.2206/kyushujm.54.279.

[Lau93] G. Lauricella. Sulle funzioni ipergeometriche a piu variabili. *Rendiconti del Circolo Matematico di Palermo*, 7(S1):111–158, 1893. doi:10.1007/bf03012437.

[Law86] C.L. Lawson. Properties of n-dimensional triangulations. *Computer Aided Geometric Design*, 3:231–246, 1986.

[MY14] Keiji Matsumoto and Masaaki Yoshida. Monodromy of Lauricella's hypergeometric F_A-system. *Annali Della Scuola Normale Superiore Di Pisa-classe Di Scienze*, 13:551–577, 2014.

[Pic81] Émile Picard. Sur une extension aux fonctions de deux variables du problème de Riemann relatif aux fonctions hypergéométriques. *Annales scientifiques de l'École normale supérieure*, 10:305–322, 1881. doi:10.24033/asens.203.

[Rie57] Bernhard Riemann. Beiträge zur Theorie der durch die Gauss'sche Reihe $F(\alpha, \beta, \gamma, x)$ darstellbaren Functionen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen*, 7:3–22, 1857.

[Sas77] Takeshi Sasaki. On the finiteness of the monodromy group of the system of hypergeometric differential equations (F_D). *J. Fac. Sci., Univ. Tokyo, Sect. I A*, 24:565–573, 1977.

[Sch73] H.A. Schwarz. Ueber diejenigen fälle, in welchen die gaussische hypergeometrische reihe eine algebraische function ihres vierten elementes darstellt. *Journal für die reine und angewandte Mathematik*, 75:292–335, 1873. URL: http://eudml.org/doc/148203.

[She74] G. C. Shephard. Combinatorial properties of associated zonotopes. *Canadian Journal of Mathematics*, XXVI(2):302–321, 1974.

[Sti07] Jan Stienstra. GKZ Hypergeometric Structures. In Rolf-Peter Holzapfel, A. Muhammed Uludağ, and Masaaki Yoshida, editors, *Arithmetic and Geometry Around Hypergeometric Functions*, volume 260, pages 313–371. Birkhäuser Basel, 2007. doi:10.1007/978-3-7643-8284-1_12.

[Tak80] Kyoichi Takano. Monodromy group of the system for Appell's F_4. *Funkc. Ekvacioj, Ser. Int.*, 23:97–122, 1980.

[Ter83] Toshiaki Terada. Fonctions hypergéométriques F_1 et fonctions automorphes I. *Journal of the Mathematical Society of Japan*, 35(3):451–475, 1983. doi:10.2969/jmsj/03530451.

[Vid09] Raimundas Vidūnas. Specialization of appells functions to univariate hypergeometric functions. *Journal of Mathematical Analysis and Applications*, 355(1):145–163, 2009. doi:10.1016/j.jmaa.2009.01.047.

[Zud19] Wadim Zudilin. Hypergeometric heritage of W. N. Bailey. *Notices of the International Congress of Chinese Mathematicians*, 7(2):32–46, 2019. doi:10.4310/iccm.2019.v7.n2.a4.