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Entropy of Rindler space

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Abstract

The black hole information paradox still stuns physicists until this day. There is no general consensus how the information can be conserved in a black hole and what microstates contribute to the entropy from a quantum point of view. By starting from a free scalar field Lagrangian in 2 dimensions one can calculate the entropy of Rindler space from the viewpoint of different observers. One finds that there is no unique vacuum state for the massless scalar field in Rindler space and that the entropy depends on the perspective of the observer. Lastly, one is interested how the back reaction influences the results. In the particular calculations done in this thesis, back reaction is not applicable. In general, for an arbitrary state and mass of the scalar field it would be interesting to study how the back reaction could influence the entropy.

Contents

1 Introduction

The existence of black holes is one of the first things that followed from the Einstein equations, even though it was only implicit at first. This is because of two reasons. Firstly, black holes were only given their name around the 1960's. Secondly, the existence of black holes was heavily doubted at that time. The history of black holes starts in 1916 when the Schwarzschild solution [1] was introduced. As mentioned before, the existence of this solution in nature was heavily doubted at that point in time. Consequently, the question whether black holes could form became important and this would be studied for almost the rest of the twentieth century. Another peculiar fact about this solution received a lot of attention. Namely that it has two singularities. One singularity occurs at the Schwarzschild radius and the other one occurs at the origin. This peculiarity was partially resolved by Eddington, who showed in 1924 that one of these singularities vanishes after a coordinate transformation [2]. In 1926 Eddington then mentioned the possibility that a star could collapse to a stellar object of the size of a Schwarzschild radius. This was then in the 1930's investigated by the names of Chandrasekhar and Oppenheimer [3], [4]. This led to the famous Chandrasekhar limit at $1.4M_{\text{solar}}$ and the Tolman-Oppenheimer-Volkoff limit estimated between $1.5M_{\text{solar}}$ and $3.0M_{\text{solar}}$. Current bounds on the Tolman-Oppenheimer-Volkoff limit lie around $2.17M_{\text{solar}}$. These bounds were also debated a lot at first but have now been established as valid theories. The debate ended when Oppenheimer wrote three different papers with students disproving Landau's theory that stars have neutron cores and showing that stars could collapse to a black hole [3].

A bit later followed the golden age for general relativity: the 1960's. First, the solutions for a rotating black hole was found by Kerr [5] and an extension to a charged rotating black hole was found by Kerr and Newman [6]. This was almost 50 years later than the original Schwarzschild solution and it was doubted by many that it could be done. This was followed by the Penrose Hawking Singularity theorems which proved that under reasonable conditions, collapse to a singularity was inevitable [7], [8]. It was at this point that physicists had to wait a little longer to also gather some evidence for black holes experimentally. This began with Martin Ryle, who gathered data about radio sources in the universe. This data led to the awareness about quasi-stellar radio sources, or quasars [9], [10]. Still the measurements on Active Galactic Nuclei could not be completely explained theoretically and this needed the introduction of black holes. Thus, in 1984 Martin Reese proposed that enormous black holes could power quasars in these galaxies to emit the measured radiation [11]. This was then confirmed by models that described how black holes could radiate this kind of energy.

At approximately the same time, black hole thermodynamics were also created. Classically, it was assumed that the black hole mass could not decrease, as nothing can travel faster than the speed of light and escape the black hole. This is why an analogy was made between the thermodynamical entropy of the black hole and its mass, both being thought of as non-decreasing quantities. Indeed, in 1973 Bekenstein showed using thermodynamics that the black hole entropy was related to its area and thus its mass [12]. With this formula, the generalized thermodynamical laws for black holes were also set up, completing the analogy between the entropy and the area of the black hole. These laws were now also valid for rotating and charged black hole, where for the former energy could be extracted by the means of a Penrose process [13], which was a bottleneck for this analogy. In that same time, Misner, Wheeler and Thorne postulated the no-hair theorem stating that a black hole can be completely characterized by the observables: mass, electric charge and angular momentum and that all other information is hidden inside the black hole [14]. The phrase 'Black holes have no hair' is often credited to Bekenstein. One year later Hawking made use of quantum field theory in curved spacetime to determine that black holes do radiate in a thermal manner [15]. The temperature of this black body radiation is given by:

$$T_{\text{Hawking}} = \frac{\hbar c^3}{8\pi G k_{\text{B}} M}. \tag{1}$$

This stirred up the physics community, as one of the postulates of quantum mechanics is that information is conserved. This is needed to have unitary evolution operators in quantum mechanics. If the radiation would indeed by merely thermal, one would lose a lot of the information about the infalling matter and would thus violate one of the core postulates of quantum mechanics. This is also known as the black hole information paradox.

There is still a way out of this problem, if one of the following assumptions appears not to be true [16]:

- A black hole can form as a prediction of General relativity.
- Around the horizon of the black hole, quantum field theory is valid in curved spacetimes.
- The Hawking radiation causes the black hole to decrease in mass.
- Physical spacetimes have Cauchy surfaces.

If these assumptions are true, one knows that black holes form and radiate away the information. With the first assumption it is not meant whether black holes can or cannot form in our universe, it is meant whether black holes can or cannot form as a prediction of General relativity. There are a lot of measurements that argue for the existence of black holes, as black holes would be great candidates for explaining these measurements. However, there are also proposals that state that black holes cannot form, because of an effect called pre-Hawking radiation [17], [18]. These papers argue that during a gravitational collapse pre-Hawking radiation prevents the creation of a horizon and that the collapse keeps going at a slower pace than the pre-Hawking evaporation. These models have been heavily criticised however [19]. The second statement ensures that the computation that Hawking did is valid. The third assumption is needed to let the black hole evaporate eventually. The first three assumptions were relatively logical, but the last assumption is a bit trickier. To explain the last assumption, consider the following situation: One forms a black hole from a pure state. Now one assumes that the first three assumptions are true and that thus the black hole radiates away energy. This radiation appeared to be thermal and as a mixed state. One has to be careful at this point. In quantum mechanics it is fine if a pure state evolves into a mixed state, as long as the final state is not on a Cauchy surface [20]. Thus, this last assumption is added, because one wants to find a Cauchy surface, Σ_{final}, at some time after the black hole has formed, such that one can obtain the information from the earlier Cauchy surface, Σ_{initial}, before the black hole was formed. One can thus also see the black hole information paradox as a retrodictability problem. Assuming that physical spacetimes have Cauchy surfaces, is usually a basic axiom of general relativity, however one can alleviate this assumption as is done by Tim Maudlin [21]. This specific proposal has led to technical problems as pointed out in [22], but solutions could also be sought in this direction.

Assuming that all previous assumptions are true, a black hole will form and radiate away, destroying the information about everything that went in. Having established this, the solution ought then to be sought for in corrections to the calculation or a violation of the postulates of a physical theory. In summary, it has been shown that the solution to this black hole information problem boils down to one of the following five postulates being violated (ignoring possible loopholes that have not yet been found) [23], [24]:

- There exists a unitary S-matrix describing the evolution from infalling matter to outgoing Hawking radiation. [23]
- The physics outside the horizon can be described in good approximation by the semi-classical field equations. [23]
- The number of microstates of a black hole of mass M scales as the exponential of the Bekenstein entropy. [23]
- The Einstein equivalence principle, i.e. an observer experiences nothing out of the ordinary at the horizon. [23]
- Remnants and baby universes do not exist. [24]

Solving the black hole information paradox thus boils down to letting go of one of the postulates of quantum mechanics, general relativity or statistical physics or a violation of the consequences of the theory. The first three postulates are together called complementarity, [25], and the fourth one is sometimes also called the nonexistence of firewalls at the horizon [23]. With this, there is meant that there are not infinitely many particles floating at the horizon. In other words, if an observer would be to fall into the black hole in an enclosed cabin, the observer should not be able to experience when he crosses the event horizon. As far as the observer is concerned, he could still be standing on a planet with a massive gravitational attraction; by the Einstein equivalence principle. The fifth item was not considered in the first paper by Almheiri, Marolf, Polchinski and Sully, [23], and can be seen as a loophole through all other postulates. There were proposed several solutions making use of this loophole that did not violate any of the postulates listed above [26],[27],[28]. One should realize however, that Hawking did not analyse the situation exactly. Hawking assumed that the geometry of the situation did not change as Hawking radiation was emitted. In other words, the energy of the black hole is not transferred into the energy of the radiation. To be more exact, Hawking took the limit where the mass of the black hole went to infinity. If one does not take this limit, then radiation can again influence the geometry and this influences the radiation once again, and so on. This is called back reaction. Hawking did not include this back reaction. Therefore, a solution could be that this back reaction could influence the radiation in such a way that the radiation is not thermal and does contain the information of the matter that fell into the black hole. This solution method became popular as a result of 't Hooft's brick wall model [29]. Other flaws to Hawking's calculation have not yet been found and there exist some theorems that claim that this and other small alterations do not change the result [24],[30],[31],[32].

As one can choose a lot of different postulates to violate, there exist a wide variety of methods to solve the black hole information paradox. Thus far no solution or solution method has become dominant. Several solutions try to incorporate back reaction in the theory to see whether the radiation gets correlations from which one could extract the information, a recent example [33]. Moreover, solutions can be sought using a quantum theory of gravity, e.g. string theory or loop quantum gravity. For example, Penington et al. make use of a replica trick that creates a replica black hole. The theory then connects the replicas with wormholes and this leads to a finite entropy [34]. Mathur proposes a solution where the black hole is regarded as a fuzzy ball of strings and not a singularity [35]. Ashtekar et al. propose a solution using loop quantum gravity, where the information is conserved, but only comes out at future infinity [36]. Another possibility within quantum gravity is that the black hole does have soft hair [37], thus violating the No-hair theorem. In this solution, the authors argue that if a theory has super translation symmetries, then the black hole needs to carry a large amount of soft hair. This soft hair can then be used to describe the microstates of the black hole and allow for a unitary evolution. Another solution identifies antipodal points on the horizon, which cause the evolution to be unitary [38]. Furthermore, solutions try to develop a theory that goes beyond monogamy of entanglement to solve the problem [39], [40]. This means that one tries to build up a mathematical construct in which one can capture entanglement between more than 2 particles, which is the case for the current mathematical construct. Solutions that advocate a remnant also exist [27], [41], [42]. These papers show that this remnant can leak out information by emitting very soft particles during a time of order m^4. In this scheme correlations are transferred via quantum gravity degrees of freedom, which can only happen in regions of high curvature. The other quantum gravity degrees of freedom cannot exit the smaller black hole and die off when the remnant is approximated as flat space. This would lead to a pure to mixed state transition and Unruh and Wald interpret this as a decoherence phenomenon [16], [20].

Moreover, the black hole information paradox has not only led to a variety of solutions, it has also led to a variety of fascinating ideas, for example holography [43], [44] and entanglement entropy [45]. It can be shown that with every Hawking pair that is created, the entanglement entropy increases with ln(2) [24], [46]. When half the black hole has radiated away, that means that now the entanglement entropy is at the Bekenstein bound. Thus, information must start coming out of the black hole at this point. This is known as the Page curve and is shown in Figure 1. In this figure the red line denotes the Bekenstein bound, the blue line the entropy following from Hawking's calculation and the green line is the Page curve. At the time that the black hole has evaporated completely, the final state is again pure, and the information is encoded in correlations. It is only not described how it is encoded in the correlations. The Page curve has a special status within string theory, being one of the things that led to the introduction of the AdS/CFT correspondence [47]. This correspondence tells us that Anti-de Sitter spacetimes which develop black holes, can be mapped onto a state of a conformal field theory on the boundary. As the boundary in this system is the horizon which is far away from the regions with high curvature, the quantum gravity degrees of freedom will never be exposed in this model. The evolution on this boundary will then also be unitary [16]. If this is true, one should conclude that quantum field theory in curved space time should be considered as the approximate theory. Another possible solution is that information escapes the black hole by quantum teleportation [48]. This solution assumes that information is transferred non-locally.

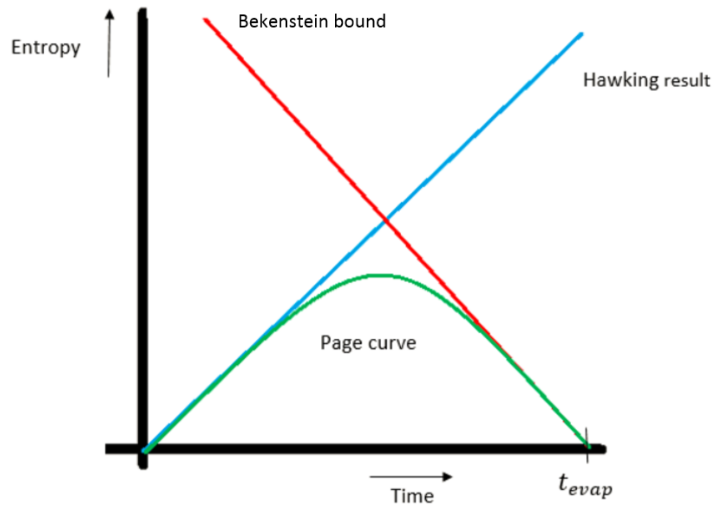

Figure 1: Sketch of the Page curve. The red line denotes the Bekenstein bound. The blue line denotes the result for the entropy that follows from Hawking's calculation and the green line shows the Page curve.

Some solutions take more exotic approaches and diverge further from standard physics. An example of this is, is the assumption that quantum gravity effects can grow and become relevant at larger scales, just as ordinary quantum mechanics does in situations like lasers, super fluids, superconductors etc [24]. These quantum gravity effects could let the black hole tunnel to a white hole in a time faster than the black hole would evaporate [49]. There also exist toy models that include non-local interactions in their theory to resolve the paradox. These non-local interactions then lead to non-local bursts of quantum information [50]. As one can see, the black hole information paradox has led to many solutions and a wide variety of new physical ideas that might at some point lead to a good theory of quantum gravity. Unfortunately, thus far no theory has given an entirely satisfactory answer on this problem yet and the search continues. Therefore, in this thesis one has tried to make a small contribution to all the knowledge that already exists about the black hole information paradox.

The aim of this thesis is to calculate the Von Neumann entropy in a near horizon situation and analyse how the back reaction influences this result. One could try this with the Schwarzschild geometry, but that geometry has a complicated form that makes calculations tough. One can argue that doing this for Rindler space yields comparable results. Firstly, this is because Rindler space describes an accelerating observer with respect to Minkowski. This is approximately the same for an observer hovering at a constant radius sufficiently close to the black hole horizon by the Einstein equivalence principle. More analogies were pointed out by Rindler [51] and the arguments will be summarized by in the following paragraphs. Consider the Schwarzschild metric and the Schwarzschild coordinates (r, t, θ, ϕ) and transform them to Kruskal (u, v, θ, ϕ) coordinates to analytically extend the Schwarzschild geometry to include the area inside the black hole. In formula they are given by:

$$\left(\frac{c^2 r}{2G_N m} - 1\right) \exp(\frac{c^2 r}{2G_N m}) = u^2 - v^2$$
$$t = 4\frac{G_N m}{c^2} \operatorname{arctanh}\left(\frac{v}{u}\right).$$

As a first argument, one sees that the Minkowski (x, y, z, t) and Rindler (x, y, ζ, τ) coordinates correspond to

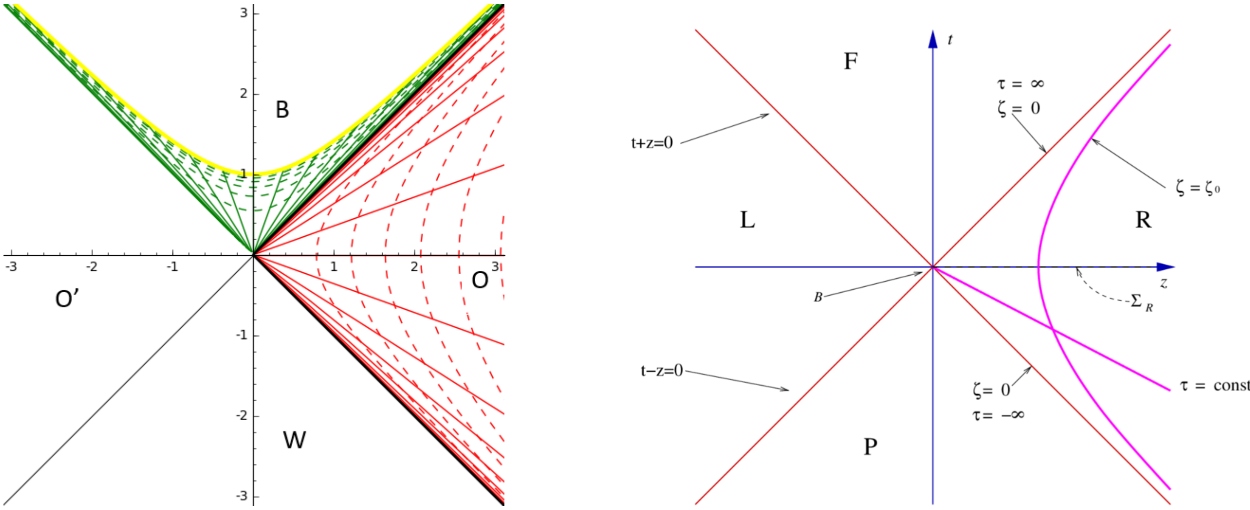

Figure 2: Diagrams of the Schwarzschild space expressed in Kruskal coordinates on the left, taken from [52], and the Minkowski space expressed into Rindler coordinates on the right, taken from [53]. On the left the yellow line denotes the singularity, $r = 0$, the green part denotes the region inside the black hole horizon and the red region corresponds to the region outside the black hole horizon. On the right, the region denoted with R is region described by the Rindler coordinates. The pink lines denote the lines of constant τ and constant ζ. The light cones in both figures are at 45 degrees.

Schwarzschild and Kruskal coordinates, because the coordinate transformations have a similar form:

$$\zeta^2 = x^2 - c^2t^2$$
$$\tau = \text{arctanh}\left(\frac{ct}{x}\right).$$

As a consequence, the Kruskal diagram and the diagram for the Rindler coordinates share many similarities, see Figure 2. On the left, one sees Schwarzschild space. The red region corresponds to all of the space outside of the horizon, the green part is the region inside the horizon and the yellow line denotes the singularity $r = 0$. On the right, one sees that the Rindler coordinates only describe the part of Minkowski space given by $z > |t|$, or the region denoted with R. The limits of the Rindler space coordinates are given and the lines of constant ζ and τ are denoted with the pink lines. In both these diagrams the light cones are at 45 degrees everywhere. One very often sees both these diagrams maximally extended to all 4 quadrants. This is because both transformations have the symmetry $u \to -u, v \to -v$ or $x \to -x, t \to -t$. This way one can also describe the region $z < -c|t|$, or L, from Minkowski in another set of Rindler coordinates, which are then glued together as is seen in the right part of the figure. In Schwarzschild this extension is a bit more subtle, as one already extended the solution to regions $r < 2\frac{G_N m}{c^2}$. Now the region $u < -|v|$, or region O', is regarded as another copy of the space outside the horizon. The B region is mirrored on the horizontal axis and the region in W corresponding to the green region in B is now seen as a white hole. In both cases the horizon is given by the line $u = v, z = ct$, because the light cones are at 45 degrees. No information can travel from the L or F region to the R region in the Rindler figure. For the same reason information cannot travel from the O' and B regions to the O region, the space outside the horizon in Schwarzschild.

Furthermore, applying Lorentz transformations to the Kruskal coordinates and the Minkowski coordinates does not change the metric and yields the same translations in respectively Schwarzschild coordinates and

Rindler coordinates. Namely applying the Lorentz transform:

$$\begin{aligned} x' &= x\,\cosh(\theta) - t\,\sinh(\theta) \\ t' &= -x\,\sinh(\theta) + t\,\cosh(\theta) \\ u' &= u\,\cosh(\theta) - v\,\sinh(\theta) \\ v' &= -u\,\sinh(\theta) + v\,\cosh(\theta) \end{aligned}$$

induces $\zeta' = \zeta$, $\tau' = \tau - \theta$ in Rindler and $r' = r, t' = t - 4m\theta$ in Schwarzschild. The last reason one can give for the similarity of Rindler to Schwarzschild coordinates is, that in the near horizon limit the Schwarzschild metric and the Rindler metric have the same form. The Schwarzschild and the Rindler metric are given by:

$$\begin{aligned} ds^2_{\text{Schwarzschild}} &= -(1 - \frac{2G_N m}{c^2 r})c^2 dt^2 + (1 - \frac{2G_N m}{c^2 r})^{-1} dr^2 + r^2 d\Omega^2 \\ ds^2_{\text{Rindler}} &= -(\alpha\zeta)^2 d\tau^2 + d\zeta^2 + r^2(\xi) d\Omega^2. \end{aligned}$$

Here α is a constant, m is considered as the mass of the black hole and $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. The $r^2(\xi)d\Omega^2$ is introduced to describe a two sphere with a certain radius as the remaining two coordinates in Rindler space. In order to get the analogy between the near Schwarzschild geometry and the Rindler geometry one has to take $r^2(\xi) = 1 \times \frac{4G_N^2 m^2}{c^4}$ as will be seen from the following equation. Significant deviations between the two metrics do occur far from the event horizon, however gravity is significantly weaker there to have a large impact on the propagation of perturbations. Thus, one has a sphere of radius $r = \frac{2G_N m}{c^2}$ for the angular part of the Rindler manifold $\mathbb{R}^2 \times S^2$ in four dimensions. Now doing the coordinate transformation $X^2 = 1 - \frac{2G_N m}{c^2 r}, T = 4\frac{\sqrt{G_N} m t}{\sqrt{\hbar c}}$ on the Schwarzschild coordinates yields $dr = \frac{4mG_N dX}{c^2(X^2-1)^2}$ and consequently:

$$\begin{aligned} ds^2_{\text{Schwarzschild}} &= -16\frac{G_N m^2}{\hbar c} X^2 dT^2 + \frac{16 G_N^2 m^2}{c^4(1-X^2)^4} dX^2 + \frac{16 G_N^2 m^2}{4c^4}\left(1 + X^2 + ...\right)^2 \left[d\theta^2 + \sin^2(\theta)d\phi^2\right] \\ &= 16\frac{G_N^2 m^2}{c^4}\left(-\frac{c^3}{G_N \hbar} X^2 dT^2 + (1 + 4X^2 + ...)dX^2 + \frac{1}{4}\left(1 + X^2 + ...\right)^2 \left[d\theta^2 + \sin^2(\theta)d\phi^2\right]\right). \end{aligned}$$

Now one assumes that one measures distances in Schwarzschild spacetime in units of $16\frac{G_N^2 m^2}{c^4}$ and Rindler time is measured in units α. In the near horizon limit the most significant terms are now given by the X^2 factor for the dT^2 and the 1 in front of the dX^2. Therefore, one can argue that there is an analogy between the Rindler and the Schwarzschild metric. All these arguments give a reason to believe that results obtained in a Rindler geometry will be similar to results that would be obtained by analysing the much more complicated Schwarzschild geometry. Therefore, in this thesis one analyses the Rindler geometry.

As mentioned before, certain parts of the spacetime are not accessible to the Rindler observer. One would now like to associate the loss of information of the L and the F part of the figure to an entropy. If this is done in a careful way, this might be similar to what happens in the Schwarzschild background. This could possibly lead to an insight in how one should analyse the entropy of a black hole. In this thesis, this is attempted by considering how a Minkowski observer sees the Rindler states. This can be done by the means of a Bogoliubov transformation. This transformation describes how the mode functions and the annihilation and the creation operators of different spacetimes are related. In this thesis the entanglement entropy is used to calculate the entropy. The reason for this is that Bekenstein showed that there was an upper limit to the amount of entropy S that can be contained within a volume [54]:

$$S \leq S_{\text{bound}} = \frac{2\pi k_B R E}{\hbar c}, \tag{2}$$

where k_B Boltzmann's constant, R the radius of a sphere that can enclose the system, E the total energy, $\hbar$ the reduced Planck constant and c is the speed of light. A simple thermodynamical calculation shows that 3 dimensional black holes exactly satisfy this bound:

$$S_{\text{BH}} = \frac{k_B A}{4 l_p^2} = \frac{4\pi k_B G m^2}{\hbar c} = \frac{2\pi k_B}{\hbar c}\left(\frac{2Gm}{c^2}\right)(mc^2) = S_{\text{bound}}$$

using that the Planck lenght $l_{\rm P}$ is given by $l_{\rm P} = \sqrt{\frac{\hbar G}{c^3}}$. The entanglement entropy is a measure for the entanglement between two systems. One expects this to be maximal when one completely traces out one system i.e. when no information is accessible at all. These two facts combined give a reason to believe that the entanglement entropy can be used to calculate the entropy of the black hole. The entanglement entropy is best explained by starting with the definition of the Von Neumann entropy [55]:

$$S_{\rm vN} = {\rm Tr}[-\hat{\rho}\ln(\hat{\rho})]. \tag{3}$$

In this equation one uses the density matrix, which is written in terms of the eigenstates of the system:

$$\hat{\rho} = \sum_i \rho_i \left|i\right\rangle \left\langle i\right|.$$

The density matrix is said to be pure, when all $\rho_i = 0$ except for one. In general, this can be a mixed state with arbitrary coefficients satisfying:

$$\sum_i \rho_i = 1.$$

Now, suppose our system can be divided into subsystems labelled as: $A, B, ...$ and so on. Suppose our system can be divided in 2 pieces, then one can define a reduced density matrix as:

$$\hat{\rho}_{\rm A} = {\rm Tr}_{\rm B}(\hat{\rho}),$$

where B is the subsystem that we want to trace out (e.g. because it is inaccessible to us). The entanglement entropy is then defined by [45]:

$$S_{\rm EE} = {\rm Tr}[-\hat{\rho}_{\rm A}\ln(\hat{\rho}_{\rm A})]. \tag{4}$$

One can give a small example to illustrate how the entropy can be generated tracing out a part of the system and using the entanglement entropy. The main idea thus is that we assume the Hilbert space of our Rindler system to be separable in two parts $H = H_L \otimes H_R$. One part is behind the Rindler horizon, H_L and the particles in the other part H_R can be measured by an observer. If this is the case, one can trace out the left part from the density matrix, because that is the part that will be inevitably lost. Suppose one has a system of 2 particles with either spin up or spin down and at some point, particle 2 will be inaccessible to the observer. The current quantum state is given by:

$$\hat{\rho} = \rho_1 \left|0\right\rangle_1 \left|0\right\rangle_2 \left\langle 0\right|_1 \left\langle 0\right|_2 + \rho_2 \left|0\right\rangle_1 \left|1\right\rangle_2 \left\langle 0\right|_1 \left\langle 1\right|_2 + \rho_3 \left|1\right\rangle_1 \left|0\right\rangle_2 \left\langle 1\right|_1 \left\langle 0\right|_2 + \rho_4 \left|1\right\rangle_1 \left|1\right\rangle_2 \left\langle 1\right|_1 \left\langle 1\right|_2.$$

with the amplitudes $\rho_{1,2,3,4}$ for the different states. Now, suppose the second particle fell behind the Rindler horizon and the information is entirely lost to the Rindler observer. One is still able to measure and know the state of particle one, therefore we trace out the second particle out of the density matrix:

$$\begin{aligned}\hat{\rho}_1 = Tr_2[\hat{\rho}] = \left\langle 0\right|_2 \hat{\rho} \left|0\right\rangle_2 + \left\langle 1\right|_2 \hat{\rho} \left|1\right\rangle_2 &= \rho_1 \left|0\right\rangle_1 \left\langle 0\right|_1 + \rho_3 \left|1\right\rangle_1 \left\langle 1\right|_1 + \rho_2 \left|0\right\rangle_1 \left\langle 0\right|_1 + \rho_4 \left|1\right\rangle_1 \left\langle 1\right|_1 \\ &= (\rho_1 + \rho_2) \left|0\right\rangle_1 \left\langle 0\right|_1 + (\rho_3 + \rho_4) \left|1\right\rangle_1 \left\langle 1\right|_1.\end{aligned}$$

Firstly, one immediately sees that the different ρ still sum up to 1. With these different density matrices one can now calculate the Von Neumann entropy as given in Equation 3. To do this, one goes from the bra ket notation to a vector notation. Let us define:

$$\left|0\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \left|1\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

and for the bra's we take the transposed versions of these vectors. Thus, our density matrix $\hat{\rho}_1$ is given by:

$$\hat{\rho}_1 = \begin{pmatrix}\rho_1 + \rho_2 & 0\\ 0 & \rho_3 + \rho_4\end{pmatrix}. \tag{5}$$

For the Von Neumann entropy we need to calculate the logarithm of this matrix. The logarithm of a matrix is defined the following way:

$$\ln(I+M) = \sum_{n=1}^{\infty}(-1)^{k+1}\frac{M^k}{k} \tag{6}$$

Using this definition one can calculate the logarithm of our density matrix. The entanglement entropy is then given by:

$$\begin{aligned} S_{EE,1} &= -\mathrm{Tr}[\hat{\rho}_1 \ln(\hat{\rho}_1)] \\ &= -\mathrm{Tr}[\begin{pmatrix} \rho_1+\rho_2 & 0 \\ 0 & \rho_3+\rho_4 \end{pmatrix}\begin{pmatrix} \ln(\rho_1+\rho_2) & 0 \\ 0 & \ln(\rho_3+\rho_4) \end{pmatrix}] \\ &= -(\rho_1+\rho_2)\ln(\rho_1+\rho_2) - (\rho_3+\rho_4)\ln(\rho_3+\rho_4) \end{aligned} \tag{7}$$

For the system of two particles, we have a four-dimensional basis given by:

$$|0\rangle_1 |0\rangle_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, |1\rangle_1 |0\rangle_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, |0\rangle_1 |1\rangle_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, |1\rangle_1 |1\rangle_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

With this basis, one sees that our density matrix for the two particles is then given by:

$$\rho = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_3 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_4 \end{pmatrix} \tag{8}$$

Taking again the logarithm and calculating the Von Neumann entropy for the system of two particles yields:

$$S_{vN1,2} = -\rho_1\ln(\rho_1) - \rho_2\ln(\rho_2) - \rho_3\ln(\rho_3) - \rho_4\ln(\rho_4). \tag{9}$$

Comparing this with Equation 7, one sees that these are in general not the same. One can show that the Von Neumann entropy of the system with two particles is always greater than or equal to the entanglement entropy of the system of 1 particle. One can give a geometrical proof of this when one only considers ρ_1 and ρ_2. The function $-x\ln(x)$ is a concave function for $x < 1$. Thus, the sum of $-x\ln(x)$ and $y\ln(y)$ is always larger than or equal to $-(x+y)\ln(x+y)$, for $x+y < 1$. This also holds for the part containing ρ_3 and ρ_4. Thus, one always has that Equation 9 is larger than Equation 7. This is shown more clearly in Figure 3. This is indeed what one expects, because the particle that fell behind the horizon increased the entropy of the black hole because it increased the black hole area via backreaction. The particle could come back via Hawking radiation, a process which would increase the entropy again back up to the original S_{vN} as one knows that $\frac{dS_{vN}}{dt} = 0$. Looking a bit more at the state that we started off with, suppose that one has a fully entangled set of two particles, i.e. $\rho_2 = \rho_3 = 0$. Then, one sees that $S_{vN1,2} = S_{EE,1}$, the maximum possible entropy. This makes sense, as no information is lost by losing the second particle. All the information that particle two has, can be found by measuring particle one. One would think one has a contradiction here now, as thermal Hawking radiation would increase the entropy. This would increase the Von Neumann entropy to a value that is larger than the original Von Neumann entropy, which contradicts $\frac{dS_{vN}}{dt} = 0$. This is the black hole information paradox in a simple example. From this example it is clear that there should be a bit more going on than just thermal Hawking radiation. There should be a lot of entanglement between the outgoing radiation and particle two and therefore also with particle one. On the other hand, one could have a situation where one ends up with a pure state for the first particle after tracing out particle two, e.g. $\rho_3 = \rho_4 = 0$. In this case the Von Neumann entropy of particle one is zero and this signifies that there is no entanglement in the system. The original state of particle 2 cannot be retrieved by measuring particle 1 and is thus lost. The entanglement entropy of particle 2 is exactly the same as the original entropy of the two particles $S_{EE,2} = S_{vN1,2}$. In this case there should be maximal entanglement between particle 2 and the Hawking radiation, otherwise $\frac{dS_{vN}}{dt} = 0$ would be violated. If this were not the case, the information would be lost, as particle 2 cannot be measured by an observer anymore.

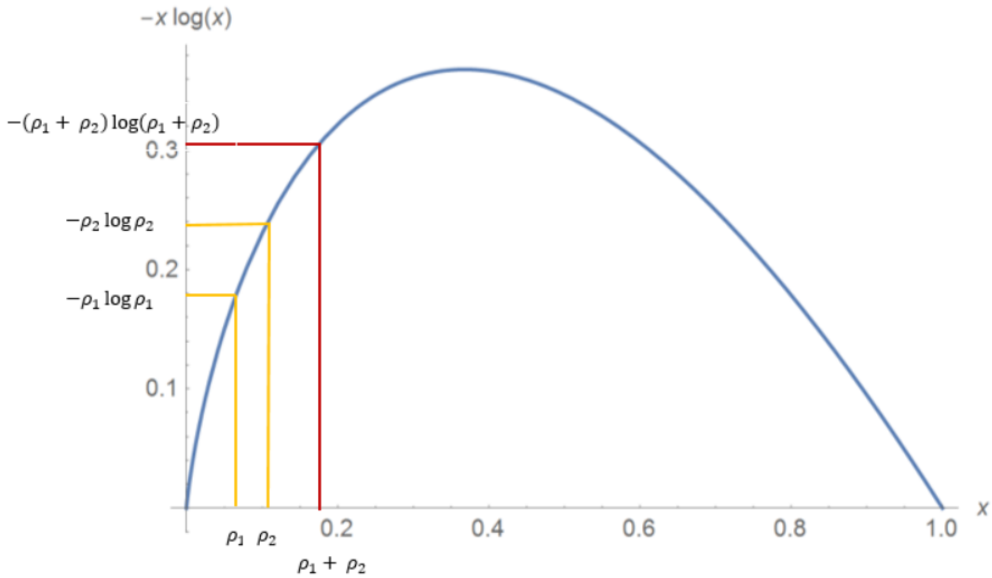

Figure 3: Geometrical proof of the Von Neumann entropy being larger than the entanglement entropy of the single particle. Because the function $-x\ln(x)$ is a concave function of x. One always has that the sum of $-x\ln(x)$ and $y\ln(y)$ is larger than $-(x+y)\ln(x+y)$, for $x+y<1$.

A last interesting example shows the effect of partitioning the system in N state partitions. In N state partitions the Von Neumann entropy is given by: $S_{vN} = -\sum_{i=1}^{N} \rho_i \ln(\rho_i) - \lambda(\sum_{i=1}^{N} \rho_i - 1)$, where λ a Lagrange multiplier, which ensures that $\mathrm{Tr}[\hat{\rho}] = 1$. Now, extremising with respect to ρ_i and λ gives a set of equations. The solution to this set of equations is given by $\rho_i = e^{-1-\lambda}$ and $N = e^{1+\lambda}$. This gives the following for the entropy: $S_{vN} = -\sum_{i=1}^{e^{1+\lambda}} e^{-1-\lambda}(1+\lambda) = 1+\lambda = \ln(N)$. For $N=1$, pure state, the entropy is 0. For $N=2$, partition into 2 states, the maximum entropy is given by $\ln(2)$. The maximum entropy is larger for more partitions and the entropy is a monotonic function of ρ_i (for $\rho_i < \frac{1}{N}$). This suggests also that increasing the number of partitions generally increases the entanglement entropy.

In this thesis one takes the action for a free real scalar field with a cosmological constant to renormalise the energy momentum tensor:

$$S = \int d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \Lambda \right). \tag{10}$$

One works with a metric with the signature $\mathrm{sign}(g_{\mu\nu}) = (-,+,+,+)$ and all physical constants are set equal to 1, $c = \hbar = G_N = k_{\mathrm{B}} = 1$. With this action and the use of perturbative quantum field theory one can derive the Feynman rules and calculate several quantities using the Feynman diagrams [56]. There will be worked with the Schwinger-Keldysh formalism. This formalism is explained more thoroughly in [57]. This formalism has the same Feynman rules as those of usual interacting quantum mechanical systems, with the exception that:

- Every vertex is assigned a polarity, either a plus (+) or a minus (-) polarity. $V \to V_\pm$
- a vertex V_+ scales as $-i \times \lambda$ and a vertex V_- scales as $i \times \lambda$. λ is the coupling constant corresponding to the vertex.
- Vertices can be connected using a two-point function that now has two extra indices denoting the polarities: $i\Delta^{ab}(x, x')$, where $a, b = \pm$ and x, x' denote the coordinates at which the vertices are inserted.
- For every internal vertex, one needs to sum over all polarities. All external polarities are kept fixed.

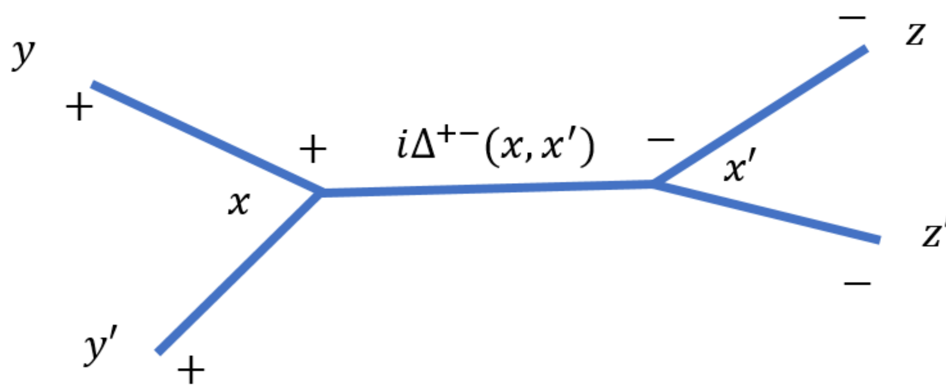

Figure 4: Example of a diagram in the Schwinger Keldysh formalism. In this particular diagram of $2 \to 2$ scattering one has 2 internal vertices which both have to take 2 polarities, thus one has to sum four diagrams for this amplitude (up to the order where one has zero loops). The polarities of the external legs are fixed. For this diagram one has to insert the $i\Delta^{+-}(x, x')$ propagator.

These rules mean that one has many more diagrams than in the usual Feynman perturbation theory. An example of a $2 \to 2$ scattering diagram is given in Figure 4. In this diagram one has 2 internal vertices at position x and x'. These vertices both take the plus and the minus polarity, when one calculates the full amplitude for this process. Thus, in total one has to sum four of these diagrams to calculate this amplitude (up to the order where one has zero loops). In the diagram in the figure one has to insert the $i\Delta^{+-}(x, x')$ propagator. The other propagators have to be inserted in the diagrams where one has the different polarities for the vertices. The following identifications can be made for the propagators:

$$i\Delta^{++}(x, x') = i\Delta_{\rm F}(x, x'), \tag{11}$$

where $i\Delta_{\rm F}(x, x')$ the Feynman propagator. Furthermore, one has

$$i\Delta^{--}(x, x') = i\Delta_{\rm D}(x, x'), \tag{12}$$

where $i\Delta_{\rm D}(x, x')$ the Dyson propagator. Lastly, the propagators with two different polarities are given by the positive and negative frequency Wightman functions, respectively $i\Delta^{+}(x, x')$ and $i\Delta^{-}(x, x')$.

$$i\Delta^{-+}(x, x') = i\Delta^{+}(x, x'), \tag{13}$$

$$i\Delta^{+-}(x, x') = i\Delta^{-}(x, x'). \tag{14}$$

One can make the following identification for the ++-propagator:

$$i\Delta^{++}(x, x') = i\Delta_{\rm F}(x, x') = \langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle\,, \tag{15}$$

where T is the time ordering symbol defined as:

$$T\left[\phi(t, \vec{x})\phi(t', \vec{x}')\right] = \begin{cases} \phi(t, \vec{x})\phi(t', \vec{x}') & \text{when } t > t' \\ \phi(t', \vec{x}')\phi(t, \vec{x}) & \text{when } t' > t. \end{cases}$$

The energy momentum tensor is the proper 1-point vertex $\Gamma_{\mu\nu}$, or the gravitational tadpole, which contributes to the effective action as, $\Gamma \sim -\frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}_{\pm}\Gamma^{\pm}_{\mu\nu}$, where $g^{\mu\nu}_{\pm}(x)$ is the metric tensor one-point function, which on shell evaluates to, $g^{\mu\nu}_{\pm}(x) = \langle \hat{g}_{\mu\nu}(x)\rangle$, and $\Gamma^{\pm}_{\mu\nu}(x) = T^{\pm}_{\mu\nu}(x)$ is usually referred to as the energy-momentum tensor. Now, when evaluated, one can get two types of energy momentum tensor, $\Gamma^{+}_{\mu\nu}(x) \equiv T^{+}_{\mu\nu}(x) = \langle T^{*}\left[\hat{T}_{\mu\nu}(x)\right]\rangle$, $\Gamma^{-}_{\mu\nu}(x) \equiv T^{-}_{\mu\nu}(x) = \langle \bar{T}^{*}\left[\hat{T}_{\mu\nu}(x)\right]\rangle$, where T^{*} ($\bar{T}^{*}$) stands for the special time ordering ordering (special anti-time ordering) operator. For this one introduces a special time ordering symbol that commutes with the derivatives such that $T^*\partial_\mu = \partial_\mu T$. Now it turns out that, when evaluated on shell, both $T^+_{\mu\nu}(x)$ and $T^-_{\mu\nu}(x)$ are real and equal to each other. From this discussion it is clear that one uses the ++-propagator for the + polarity energy momentum tensor because the ++-propagator uses the normal time ordering symbol. One can see the diagrammatic representation of the energy momentum tensor in Figure 5.

In this thesis, one calculates the Von Neumann entropy using the density matrix $\hat{\rho}$. Given a density matrix $\hat{\rho}$ one already saw in Equation 3, that one calculates the Von Neumann entropy as:

$$S_{\rm vN} = {\rm Tr}[-\hat{\rho}\ln(\hat{\rho})], \tag{16}$$

where Tr denotes that one takes the trace of the expression inside the brackets. The density matrix is also used to calculate expectation values of operators $\hat{O}(t)$:

$$\langle\hat{O}(t)\rangle = Tr\left[\hat{\rho}\hat{O}(t)\right]. \tag{17}$$

This formula holds for any operator $\hat{O}$. Examples of this are the positive and negative frequency Wightman functions:

$$i\Delta^+(x,x') = {\rm Tr}[\hat{\rho}\hat{\phi}(x)\hat{\phi}(x')] = \langle\hat{\phi}(x)\hat{\phi}(x')\rangle\,, \tag{18}$$

$$i\Delta^-(x,x') = {\rm Tr}[\hat{\rho}\hat{\phi}(x')\hat{\phi}(x)] = \langle\hat{\phi}(x')\hat{\phi}(x)\rangle \tag{19}$$

If the density matrix is Gaussian, then the density matrix has the following form:

$$\begin{aligned}\hat{\rho}(\phi,\phi';t) = N\exp\Big(-\int dx\int dy A(x,y)\phi(x)\phi(y) - \int dx\int dy B(x,y)\phi'(x)\phi'(y)\\ +\int dx\int dy 2C(x,y)\phi(x)\phi'(y)\Big)\,,\end{aligned} \tag{20}$$

where A, B and C are functions that can be expressed in terms of the two-point functions, when one has a Gaussian state. An example calculation of the A, B and C is done in [58]. When one has a system that is translation invariant, then a Wigner transform brings the density matrix in a diagonal basis. In a diagonal basis it is possible to calculate the Von Neumann entropy. The Wigner transform is defined as:

$$f(k) = \int_{-\infty}^{\infty}\frac{d(\xi-\xi')}{2\pi}f(\xi-\xi')e^{ik(\xi-\xi')}. \tag{21}$$

Moreover, the density matrix in a diagonal basis will be given by the product of simple oscillators for quantum mechanical particles:

$$\hat{\rho} = \prod_k \hat{\rho}_k, \tag{22}$$

with an energy given by: $\omega = \sqrt{k^2+m^2}$. In this thesis, one assumes the system to be in a Gaussian state. One can assume the state to be Gaussian in the case of a theory with weak interactions. If a theory only has weak interactions, then non Gaussianities remain small. Moreover, in some theories with strong interactions one can also get Gaussian states. An example of this is Quantum Chromodynamics, where if one goes to the bound states, in the form of mesons and baryons, then one also finds that the states are mostly Gaussian. Thus, the assumption that the system is in a Gaussian state is usually a valid assumption. If the density matrix is Gaussian, then one can prove that the Von Neumann entropy is given by:

$$S_{vN} = (\bar{n}(k,\tau)+1)\ln(\bar{n}(k,\tau)+1) - \bar{n}(k,\tau)\ln(\bar{n}(k,\tau)). \tag{23}$$

In this formula $\bar{n}$ is the statistical particle number. This is also known as the entropy formula for $\bar{n}$ bosonic non-interacting particles. If one has a Gaussian state one only requires the two-point functions in the calculations. The route to the statistical particle number is then as follows. First one calculates the statistical function, which is the sum of the positive and negative frequency Wightman functions:

$$\begin{aligned} F(\tau,\xi;\tau',\xi') &= \frac{1}{2}i\Delta^{+}(\tau,\xi;\tau',\xi') + i\Delta^{-}(\tau,\xi;\tau',\xi') \\ &= \frac{1}{2}\left[\langle\phi(\tau,\xi)\phi(\tau',\xi')\rangle + \langle\phi(\tau',\xi')\phi(\tau,\xi)\rangle\right]. \end{aligned} \tag{24}$$

From this it is already clear that one only requires the two-point functions to calculate the entropy. This statistical function will be Wigner transformed to Wigner space:

$$F(k,\tau,\tau',\xi+\xi') = \int_{-\infty}^{\infty}\frac{d(\xi-\xi')}{2\pi}F(\tau,\xi;\tau',\xi')e^{ik(\xi-\xi')}. \tag{25}$$

This Wigner transform only brings the system in a diagonal basis, if the statistical function is a function of only $\xi-\xi'$ and not of $\xi+\xi'$. Until Equation 28 one assumes that this is indeed the case. In this is the case one can calculate the Gaussian invariant as:

$$\Delta^2(k,\tau) = \lim_{\tau\to\tau'} 4\left[F(k,\tau,\tau')\partial_\tau\partial'_\tau F(k,\tau,\tau') - \partial_\tau F(k,\tau,\tau')\partial'_\tau F(k,\tau,\tau')\right]. \tag{26}$$

This Gaussian invariant is analogous to the Heisenberg uncertainty principle from quantum mechanics. However, for the Gaussian invariant one has that:

$$\Delta^2(k,\tau) \geq 1. \tag{27}$$

For pure states the Gaussian invariant equals 1 and for mixed states the Gaussian invariant is greater than 1. This relation can be visualised the same way as one can visualise the Heisenberg uncertainty principle in the phase space diagram, i.e. the position-momentum diagram. A bigger area in the phase space diagram corresponds to less information about the quantum mechanical oscillator. Similarly, a deviation from 1 of the Gaussian invariant denotes a loss of information of the system in quantum field theory. A visualisation of this and a more elaborate explanation is done by Koksma and Prokopec in [59]. The statistical particle number encapsulates the deviation from 1 and is defined as:

$$\bar{n}(k,\tau) = \frac{\Delta(k,\tau)-1}{2}. \tag{28}$$

This statistical particle number can then be inserted in Equation 23 to calculate the entropy per mode. Similarly, $\bar{n}\geq 0$ is the generalised Heisenberg uncertainty relation, where $\bar{n}=0$ denotes a pure state. This statistical particle number is a measure of entropy and should not be observer dependent. One will see later that $\bar{n}>0$, because one has a state that describes only a part of the space-time manifold. This then leads to entropy. The entropy from Equation 23 can then be integrated over all modes to find the total entropy of the system. This describes in short the outline of the calculation followed in this thesis. Variations are done by calculating the two-point functions via different methods and considering the two-point function in Rindler space from the viewpoint of a Minkowski observer.

The aim of this thesis is then to use the entanglement entropy to calculate this entropy in the case of a Rindler observer and possibly make some remarks about the Schwarzschild case. There can also be looked at how back reaction influences this process.

As the argumentation for the procedure in this thesis should now be mostly clear, this thesis is structured as follows: Firstly, in Section 2 the D-dimensional two-point function and energy momentum tensor will be calculated in Minkowski space. These will later be used in a variety of situations. Moving to Section 3, the Rindler space coordinates are introduced and the equations of motion for the fields are determined. These are then solved for the massless case and this field is then quantised. Then the two-point functions for the naive Rindler vacuum are calculated from the quantised field. The massive field is harder to solve using the equations of motion, but it can be solved making use of the space being maximally symmetric. This namely fixes the Feynman propagator to be a function of only the invariant distance. One then fixes the Feynman prescription and writes down the Feynman-, advanced and retarded propagators, and the Wightman and Hadamard (statistical) function. Lastly one determines the Bogoliubov coefficients between the quantised Minkowski fields and the quantised Rindler fields. These Bogoliubobv coefficients then tell us the relations between the mode functions of Minkowski and Rindler space and the creation and annihilation operators of Minkowski and Rindler space. This can then later be used to calculate how an observer in Minkowski space perceives the naive Rindler vacuum. Then, in Section 4 the Feynman propagators are used to calculate the energy momentum tensors for the naive Rindler vacuum and the Rindler invariant vacuum. Furthermore, the energy momentum tensors will be renormalised. After calculating the curvature tensors, one can calculate the back reaction on the geometry. In Section 5 one uses the Bogoliubov coefficients to calculate how a Minkowski observer perceives the naive Rindler vacuum. In Section 6 one then obtains the two-point function of the naive Rindler vacuum and the Rindler invariant vacuum to calculate the Von Neumann entropy. This thesis ends with a discussion of all obtained results in Section 7.

2 Calculations in Minkowski

In this section the calculation of the two-point function of the Minkowski vacuum in D dimensions is described. These calculations are done to get used to the quantum field theory calculations in a simple setting. This is added to this thesis to introduce important concepts and to give examples for dimensional regularisation in D-dimensional systems. One want to use dimensional regularisation, because that regularisation scheme does not violate Lorentz symmetry. In the first section, the calculation of the two-point function in the Minkowski vacuum is described and in the second subsection, it is used to calculate the renormalised energy momentum tensor from the semi-classical Einstein equations.

2.1 Two-point function of the Minkowski vacuum

The calculation starts following the standard procedure:

$$i\Delta^+(x,x') = \langle\hat\phi(x)\hat\phi(x')\rangle = \langle 0_M|\,\hat\phi(x)\hat\phi(x')\,|0_M\rangle \tag{29}$$

First one determines the expression for the field in Minkowski space in canonical quantisation. The field must satisfy the following equation of motion:

$$(-\partial^2+m^2)\phi(x)=0. \tag{30}$$

One Fourier transforms this equation to obtain:

$$(k^2+m^2)\tilde\phi(k)=0, \tag{31}$$

where $k^\mu=(k^0,k^i)$ and $k^2=k^\mu k_\mu$. The tilde on the ϕ denotes that the function is in Fourier space. This equation is solved by

$$\tilde\phi(k)=A(k)\delta(k^2+m^2)=A(k)\delta(-{k^0}^2+|\vec k|^2{+}m^2)=A(k)\left[\frac{\delta\left(k^0-\sqrt{|\vec k|^2{+}m^2}\right)}{2\sqrt{|\vec k|^2{+}m^2}}+\frac{\delta\left(k^0+\sqrt{|\vec k|^2{+}m^2}\right)}{2\sqrt{|\vec k|^2{+}m^2}}\right], \tag{32}$$

where A is any regular function of k^0 and $\vec k$. For spatially homogeneous states, such as Minkowski, one has that $A=A(k^0,|\vec k|)$. When k^0 is project onto the positive and negative frequency shell, it can be regarded as two different functions of $|\vec k|$, A_+ and A_-. This is only valid for spatially homogeneous waves. In the last step one made use of $\delta(f(x))=\sum_i\frac{\delta(x-x_i)}{|f'(x_i)|}$, where x_i the zeroes of the function $f(x)$. In position space this is given by:

$$\phi(x)=\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\left\{e^{i\vec k\cdot\vec x-i\omega t}A_+(|\vec k|)+e^{i\vec k\cdot\vec x+i\omega t}A_-(-|\vec k|)\right\}, \tag{33}$$

where $\omega=\sqrt{|\vec k|^2{+}m^2}$ and where everything depending on $|\vec k|$ is absorbed in $A(|\vec k|)$. Now one quantises the field by imposing the following commutation relations:

$$[\hat\phi(x,t),\hat\pi(x',t)]=i\delta^{D-1}(x-x'), \tag{34}$$

where $\hat\pi(x',t)$ the canonical momentum of $\hat\phi(x,t)$. In Minkowski space the canonical momentum is given by: $\hat\pi(x,t)=\partial_t\hat\phi(x,t)$. The commutator between the field and itself is zero as well as the commutator of the canonical momentum with itself. One also imposes

$$[\hat b(\vec k),\hat b^\dagger(\vec k)]=(2\pi)^{D-1}\delta(k-k'), \tag{35}$$

where $\hat b(\vec k),\hat b^\dagger(\vec k)$ denote the annihilation and the creation operator respectively. All other commutators vanish. Imposing these commutators yields the following expression for the field in D dimensions:

$$\hat\phi(x)=\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\left(\phi(\omega,t)\hat b(\vec k)e^{i\vec k\cdot\vec x}+\phi^*(\omega,t)\hat b^\dagger(\vec k)e^{-i\vec k\cdot\vec x}\right), \tag{36}$$

where the mode function $\phi(\omega, t)$ is given by $\phi(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$. As a consistency check one can calculate the Wronskian of the mode functions and check that it equals i. Calculating the Wronskian:

$$\begin{aligned} W[\phi(\omega,t),\phi^*(\omega,t)] &= \phi(\omega,t)\partial_t\phi^*(\omega,t) - \phi^*(\omega,t)\partial_t\phi(\omega,t) \\ &= \frac{e^{-i\omega t}}{\sqrt{2\omega}}\frac{i\omega e^{i\omega t}}{\sqrt{2\omega}} - \frac{e^{i\omega t}}{\sqrt{2\omega}}\frac{-i\omega e^{-i\omega t}}{\sqrt{2\omega}} \\ &= i \end{aligned} \tag{37}$$

Thus one indeed knows that Equation 36 describes the correct expression for the field in canonical quantisation. Inserting these definitions for the fields into Equation 29:

$$\begin{aligned} &\langle \hat{\phi}(x)\hat{\phi}(x')\rangle \\ &= \langle 0_M| \int \frac{d^{D-1}k d^{D-1}k'}{(2\pi)^{2D-2}} \left[\left(\phi(\omega,t)\hat{b}(\vec{k})e^{i\vec{k}\cdot\vec{x}} + \phi^*(\omega,t)\hat{b}^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\right) \left(\phi(\omega',t)\hat{b}(\vec{k}')e^{i\vec{k}'\cdot\vec{x}'} + \phi^*(\omega',t)\hat{b}^\dagger(\vec{k}')e^{-i\vec{k}'\cdot\vec{x}'}\right)\right] |0_M\rangle \end{aligned} \tag{38}$$

The annihilation operator acting on the vacuum yields zero and $\langle 0_M| \hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}') |0_M\rangle = (2\pi)^{D-1}\delta^{D-1}(\vec{k}-\vec{k}')$. Therefore only one term remains in the expression and is given by:

$$\langle \hat{\phi}(x)\hat{\phi}(x')\rangle = \langle 0_M| \int \frac{d^{D-1}k d^{D-1}k'}{(2\pi)^{2D-2}} \left[\phi(\omega,t)\phi^*(\omega',t')e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'}\hat{b}(\vec{k})\hat{b}^\dagger(\vec{k}')\right] |0_M\rangle = \tag{39}$$

$$\int \frac{d^{D-1}k d^{D-1}k'}{(2\pi)^{D-1}}\phi(\omega,t)\phi^*(\omega',t')\delta^{D-1}(\vec{k}-\vec{k}')\left[e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'}\right] = \int \frac{d^{D-1}k}{(2\pi)^{D-1}}\phi(\omega,t)\phi^*(\omega,t')e^{i\vec{k}(\vec{x}-\vec{x}')} \tag{40}$$

To solve this integral, one transforms the integral to (D-1)-dimensional spherical coordinates given by:

$$\begin{aligned} k_1 &= |\vec{k}|\cos(\phi_1) \\ k_2 &= |\vec{k}|\sin(\phi_1)\cos(\phi_2) \\ &. \\ &. \\ &. \\ k_{D-1} &= |\vec{k}|\sin(\phi_1)\sin(\phi_2)...\sin(\phi_{D-2}). \end{aligned}$$

Here, $\phi_{D-1} \in [0, 2\pi]$ and $\phi_1, \phi_2, ..., \phi_{D-2} \in [0, \pi]$. Spherical coordinates in D dimensions have the following volume element or Jacobian:

$$d^D V = |\vec{k}|^{D-1}\sin^{D-2}(\phi_1)\sin^{D-3}(\phi_2)...\sin(\phi_{D-2})d|\vec{k}|d\phi_1...d\phi_{D-1}. \tag{41}$$

Now rotating our coordinates system such that the angle between $\vec{k}$ and $\vec{x}$ can be merely described by the angle ϕ_1, i.e. $\vec{k}\cdot\vec{x} = |\vec{k}||\vec{x}|\cos(\phi_1)$. Then the integral is given by:

$$\begin{aligned} \langle \hat{\phi}(x)\hat{\phi}(x')\rangle = \frac{1}{(2\pi)^{D-1}} \int_0^\infty d|\vec{k}| \int_0^{2\pi} d\phi_{D-2} \int_0^\pi d\phi_{D-3}... \int_0^\pi d\phi_1 |\vec{k}|^{D-2}\phi(\omega,t)\phi^*(\omega',t') \times \\ \sin^{D-3}(\phi_1)\sin^{D-4}(\phi_2)...\sin(\phi_{D-3})e^{i|\vec{k}||\vec{x}-\vec{x}'|\cos(\phi_1)} \end{aligned} \tag{42}$$

The ϕ_1 integral can be found at equation 8.411.7 from the book: Table of integrals, products and series [60]:

$$\int_0^\pi d\phi_1 \sin^{D-3}(\phi_1)e^{ik|\vec{x}-\vec{x}'|\cos(\phi_1)} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{D-2}{2})}{\left(\frac{|\vec{k}||\vec{x}-\vec{x}'|}{2}\right)^{\frac{D-3}{2}}} J_{\frac{D-3}{2}}(|\vec{k}||\vec{x}-\vec{x}'|). \tag{43}$$

The integral over all the other angles is done, by considering the volume of the $(n-1)$-sphere

$$\begin{aligned} S_{D-1} &= \frac{D\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2}+1)} \\ &= \int_0^{2\pi} d\phi_{D-1} \int_0^\pi d\phi_{D-2}... \int_0^\pi d\phi_1 \sin^{D-2}(\phi_1)\sin^{D-3}(\phi_2)...\sin(\phi_{D-2}). \end{aligned} \tag{44}$$

Considering the integral over all angles, except of course the ϕ_1 angle:

$$\int_0^{2\pi} d\phi_{D-2} \int_0^{\pi} d\phi_{D-3} ... \int_0^{\pi} d\phi_2 \sin^{D-4}(\phi_2) \sin^{D-5}(\phi_3) ... \sin(\phi_{D-3}) \tag{45}$$

$$= \int_0^{2\pi} d\phi_{D-3} \int_0^{\pi} d\phi_{D-4} ... \int_0^{\pi} d\phi_1 \sin^{D-4}(\phi_1) \sin^{D-5}(\phi_2) ... \sin(\phi_{D-4}), \tag{46}$$

after renaming the integration variables to a variable with a subscript of exactly one lower. Comparing Equation 44 to 46, one sees that our integral is exactly the surface 'area' of S_{D-3}. Thus our integral over the angles is equal to

$$\int_0^{2\pi} d\phi_{D-3} \int_0^{\pi} d\phi_{D-4} ... \int_0^{\pi} d\phi_1 \sin^{D-4}(\phi_2) \sin^{D-5}(\phi_3) ... \sin(\phi_{D-3}) = S_{D-3} = \frac{(D-2)\pi^{\frac{D-2}{2}}}{\Gamma(\frac{D}{2})}. \tag{47}$$

Inserting all these results back into Equation 42, one has:

$$\langle \hat{\phi}(x)\hat{\phi}(x')\rangle = \int_0^{\infty} \frac{d|\vec{k}|}{2k^0(2\pi)^{D-1}} |\vec{k}|^{D-2} \frac{(D-2)\pi^{\frac{D-2}{2}}}{\Gamma(\frac{D}{2})} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{D-2}{2})}{\left(\frac{|\vec{k}||\vec{x}-\vec{x}'|}{2}\right)^{\frac{D-3}{2}}} J_{\frac{D-3}{2}}(|\vec{k}||\vec{x}-\vec{x}'|) e^{-ik^0(t-t')}. \tag{48}$$

Putting all the constant coefficients in front and making use of the property of the gamma function that $\Gamma(z+1) = z\Gamma(z)$, one obtains:

$$\langle \hat{\phi}(x)\hat{\phi}(x')\rangle = \frac{1}{2^{D-1}\pi^{\frac{D-1}{2}}} \int_0^{\infty} \frac{d|\vec{k}|}{k^0} |\vec{k}|^{D-2} \frac{J_{\frac{D-3}{2}}(|\vec{k}||\vec{x}-\vec{x}'|) e^{-ik^0(t-t')}}{(\frac{|\vec{k}||\vec{x}-\vec{x}'|}{2})^{\frac{D-3}{2}}}. \tag{49}$$

For the last integral, one makes use of Equation 6.645 from the book Table of integrals, products and series [60]:

$$\int_1^{\infty} dx (x^2-1)^{\frac{\nu}{2}} e^{-\alpha x} J_\nu(\beta\sqrt{x^2-1}) = \sqrt{\frac{2}{\pi}} \beta^\nu (\alpha^2+\beta^2)^{-\frac{\nu+\frac{1}{2}}{2}} K_{\nu+\frac{1}{2}}(\sqrt{\alpha^2+\beta^2}). \tag{50}$$

To get our expression in this form, one uses the substitution $y = \frac{k^0}{m} = \sqrt{\frac{|\vec{k}|^2}{m^2}+1}$, $dy = \frac{dk^0}{m} = \frac{\frac{2|\vec{k}|d|\vec{k}|}{m^2}}{2\sqrt{\frac{|\vec{k}|^2}{m^2}+1}}$. This can be rewritten as: $|\vec{k}|^2 = m^2(y^2-1)$ and $|\vec{k}|d|\vec{k}| = m^2 y dy$. Using this substitution yields:

$$\begin{aligned} \langle \hat{\phi}(x)\hat{\phi}(x')\rangle &= \frac{1}{2^{D-1}\pi^{\frac{D-1}{2}}} \int_1^{\infty} dy \frac{m^2 y}{my} \sqrt{m^2(y^2-1)}^{D-3} \frac{J_{\frac{D-3}{2}}(\sqrt{m^2(y^2-1)}|\vec{x}-\vec{x}'|) e^{-imy(t-t')}}{(\frac{\sqrt{m^2(y^2-1)}|\vec{x}-\vec{x}'|}{2})^{\frac{D-3}{2}}} \\ &= \frac{m^{\frac{D-1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}|\vec{x}-\vec{x}'|^{\frac{D-3}{2}}} \int_1^{\infty} dy (y^2-1)^{\frac{D-3}{4}} J_{\frac{D-3}{2}}(m|\vec{x}-\vec{x}'|\sqrt{y^2-1}) e^{-im(t-t')y}. \end{aligned} \tag{51}$$

From this one sees that one has $\alpha = im(t-t')$, $\beta = m|\vec{x}-\vec{x}'|$ and $\nu = \frac{D-3}{2}$. However, one still needs to include an $i\epsilon$ prescription to this, otherwise the integral is not convergent. So, instead one has $\alpha = im(t-t'-i\epsilon)$. Applying the formula then gives:

$$\begin{aligned} \langle \hat{\phi}(x)\hat{\phi}(x')\rangle &= \frac{m^{\frac{D-1}{2}}}{2^{\frac{D+1}{2}}\pi^{\frac{D-1}{2}}|x-x'|^{\frac{D-3}{2}}} \\ &\left[\sqrt{\frac{2}{\pi}}(m|\vec{x}-\vec{x}'|)^{\frac{D-3}{2}}(-m^2(t-t'-i\epsilon)^2+m^2|\vec{x}-\vec{x}'|^2)^{-\frac{D-2}{4}} K_{\frac{D-2}{2}}(\sqrt{-m^2(t-t'-i\epsilon)^2+m^2|\vec{x}-\vec{x}'|^2})\right] \\ &= \frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}} \frac{K_{\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}}, \end{aligned} \tag{52}$$

where $z = m\sqrt{-(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2} = m\sqrt{\Delta X_+^2}$, where $\Delta X_+^2 = -(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2$. The plus denotes that this distance is the argument for the positive Wightman function. From this one easily obtains the propagator with the coordinates switched:

$$\langle\hat{\phi}(x')\hat{\phi}(x)\rangle = \frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{K_{\frac{D-2}{2}}(z')}{z'^{\frac{D-2}{2}}}. \tag{53}$$

Here $z' = m\sqrt{-(-t+t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2} = m\sqrt{-(t-t'+i\epsilon)^2+|\vec{x}-\vec{x}'|^2} = m\sqrt{\Delta X_-^2}$. As one sees later, this different $i\epsilon$ prescription gives rise to the spectral function that describes the statistical behaviour of the system. The spectral function describes which states are available in the system. With these two results, one easily writes down the time ordered, or the Feynman propagator:

$$\begin{aligned} i\Delta_{\rm F} = \langle T[\phi(x)\phi(x')]\rangle &= \theta(t-t')\langle\hat{\phi}(x)\hat{\phi}(x')\rangle + \theta(t'-t)\langle\hat{\phi}(x')\hat{\phi}(x)\rangle = \theta(t-t')i\Delta^+(x,x') + \theta(t'-t)i\Delta^-(x,x') \\ &= \theta(t-t')\frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{K_{\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}} + \theta(t'-t)\frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{K_{\frac{D-2}{2}}(z')}{z'^{\frac{D-2}{2}}}. \end{aligned} \tag{54}$$

One can immediately verify that indeed $i\Delta^+ = (i\Delta^-)^*$. One wants to find the $D\to 2$ and massless limit of this expression to compare this with results obtained later in this thesis. For this one needs the following expression for the Bessel function:

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}\left(\sum_{k=0}^{\infty}\frac{1}{\Gamma(k-\nu+1)k!}\left(\frac{z}{2}\right)^{2k-\nu} - \sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\nu+1)k!}\left(\frac{z}{2}\right)^{2k+\nu}\right) \tag{55}$$

For the massless limit, i.e. $z = m\sqrt{-(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2}\to 0$, only the $k=0$ parts of the sum contribute. Thus one obtains:

$$\frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{K_{\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}} = \frac{1+(D-2)\ln(m)}{2\pi}\left(1-\frac{D-2}{2}\ln(z)\right)\frac{\pi}{2}\frac{1}{\sin(\pi\frac{D-2}{2})}\left[\frac{1-\frac{D-2}{2}\ln\left(\frac{z}{2}\right)}{\Gamma\left(1-\frac{D-2}{2}\right)} - \frac{1+\frac{D-2}{2}\ln\left(\frac{z}{2}\right)}{\Gamma\left(1+\frac{D-2}{2}\right)}\right]. \tag{56}$$

One expands the Gamma function as $\Gamma(z) = \frac{1}{z} - \gamma_{\rm E} + O(z)$ for small z and one uses that $\Gamma(z+1) = z\Gamma(z)$. Lastly, the Taylor series of $\frac{1}{1\pm x}$ for x near 0 is given by $\frac{1}{1\pm x} = 1\mp x + O(x^2)$ and the Taylor series of $\sin(x)$ is given by $x + O(x^3)$. This way one obtains:

$$\begin{aligned} \frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{K_{\frac{D-2}{2}}(z)}{z^{\frac{D-2}{2}}} &= \frac{1}{2\pi}\frac{\pi}{2}\frac{1}{\pi\frac{D-2}{2}}\left[\left(1-\frac{D-2}{2}\ln\left(\frac{z}{2}\right)\right)\left(1+\frac{D-2}{2}\gamma_{\rm E}\right) - \left(1+\frac{D-2}{2}\ln\left(\frac{z}{2}\right)\right)\left(1-\frac{D-2}{2}\gamma_{\rm E}\right)\right] \\ &= \frac{-1}{4\pi}\ln(z^2) + \frac{\gamma_{\rm E}}{2\pi} + \frac{\ln(2)}{2\pi} \\ &= \frac{-1}{4\pi}\ln\left(\frac{m^2}{\mu^2}\right) - \frac{1}{4\pi}\ln(\mu^2(\Delta X_+)^2) + \frac{\gamma_{\rm E}}{2\pi} + \frac{\ln(2)}{2\pi} \\ &= -\frac{1}{4\pi}\ln(\mu^2(\Delta X_+)^2) + \phi^2(\mu). \end{aligned} \tag{57}$$

Here one added a mass scale μ and a condensate $\phi^2(\mu)$ to renormalise the result. $\phi^2(\mu)$ is given by: $\phi^2(\mu) = -\frac{1}{4\pi}\ln\left(\frac{m^2}{\mu^2}\right) + \gamma_{\rm E} + \frac{\ln(2)}{2\pi}$ As one can calculate the result is independent of the mass scale μ, namely:

$$\mu\partial_\mu(i\Delta^+) = \mu\partial_\mu\left(-\frac{1}{4\pi}\ln\left[\mu^2(\Delta X_+)^2\right]\right) + \mu\partial_\mu\phi^2(\mu) = \frac{-\mu}{4\pi}\frac{2}{\mu} + \frac{\mu}{4\pi}\frac{2}{\mu} = 0. \tag{58}$$

Thus the result is indeed independent of μ. To calculate the energy momentum tensor from this Feynman propagator one takes a slightly different expansion of the Bessel function [61]:

$$\frac{K_\nu(z)}{z^\nu} = \frac{\Gamma(1-\frac{D}{2})}{2^{\frac{D}{2}}}\sum_{n=0}^{\infty}\left[\frac{(\frac{z}{2})^{2n}}{(\frac{D}{2})_n n!} + \frac{\Gamma(\frac{D}{2}-1)}{2^{\frac{D}{2}}}\frac{(\frac{z}{2})^{2n+2-D}}{(2-\frac{D}{2})_n n!}\right]. \tag{59}$$

Here the $(2-\frac{D}{2})_n$ is the Pochhammer symbol. In this expansion we have a sum with an exponent depending on the number of dimensions. This dimension is then analytically extended to a complex dimension, where the function is well defined. Then one analytically extends this complex dimension to a region where the function is not well-defined. A thorough investigation shows one that this procedure removes all power-law divergences. Therefore this became known as 'automatic subtraction' [61]. Thus our two-point function in D dimensions is given by:

$$\begin{aligned}\langle\hat{\phi}(x)\hat{\phi}(x')\rangle &= \frac{m^{D-2}}{2^{\frac{D}{2}}\pi^{\frac{D}{2}}}\frac{\Gamma(1-\frac{D}{2})}{2^{\frac{D}{2}}} \\ &\left[(m\sqrt{-(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2})^0+\frac{2}{2^2D}(m\sqrt{-(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2})^2+...\right] \\ &= \frac{m^{D-2}}{2^D\pi^{\frac{D}{2}}}\Gamma(1-\frac{D}{2})\left[1+\frac{m^2}{2D}(-(t-t'-i\epsilon)^2+|\vec{x}-\vec{x}'|^2)+...\right]\end{aligned} \tag{60}$$

It is enough to consider only the first two orders for the purpose of calculating the energy momentum tensor. This is because one sends t to t' and x to x', which only gives a contribution if one does not have terms scaling as $x-x'$ or $t-t'$. The energy momentum tensor takes two derivatives and therefore only terms of second order survive when taking the limit. This will be seen more explicitly in the next subsection.

2.2 Renormalised energy momentum tensor in Minkowski in D dimensions

This section is devoted to calculating the renormalised energy momentum tensor in D dimensions. This is the simplest calculation of the gravitational 1 loop tadpole. In this section one calculates the averaged +-polarity energy momentum tensor $\langle\hat{T}^+_{\mu\nu}\rangle$ in the context a free scalar field in D-dimensional Minkowski space. One will argue a bit later that this is enough to know the full energy momentum tensor. The calculation is done diagrammatically up to 1 loop contributions. The renormalised energy momentum tensor is given by the dimensionally regularised average energy momentum tensor:

$$\langle\hat{T}_{\mu\nu}\rangle = \langle 0_M|\,\hat{T}_{\mu\nu}\,|0_M\rangle + \text{counter terms.} \tag{61}$$

The classical energy momentum tensor is defined as:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{62}$$

which needs to obtain a quantum description. With the free scalar field action in Minkowski space, one thus gets:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\eta_{\mu\nu}\phi^2.$$

One can make an operator of the energy momentum tensor after quantising the fields ϕ. One now wants to squeeze this between a bra and a ket and use the two-point function to calculate this quantity. However, one only knows the two-point functions $\langle\hat{\phi}(x)\hat{\phi}(x')\rangle$ and $\langle\hat{\phi}(x')\hat{\phi}(x)\rangle$. One needs two ϕ's to obtain a non-vanishing result. Usually the two-point function only depends on the difference between the coordinates. One can thus get rid of the coordinate dependence by taking different space-time points for the different ϕ and taking the limit where the difference between the space-time points goes to 0. Furthermore, one wants to take the time ordered product of $\partial_\mu\partial'_\nu\phi(x)\phi(x')$. For this one introduces the special time ordering symbol again. Thus, one has:

$$\langle T^+_{\mu\nu}\rangle = \lim_{x\to x'}\left[\partial_\mu\partial'_\nu\langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\partial'_\beta\langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle - \frac{1}{2}m^2\eta_{\mu\nu}\langle T[\hat{\phi}(x)\hat{\phi}(x')]\rangle\right]. \tag{63}$$

One sees that one only needs the Feynman propagator $i\Delta^{++}$ for this contribution to the energy momentum tensor. Diagrammatically, the propagator is shown on the left in Figure 5 and the energy momentum tensor is shown on the right in Figure 5. Here the vertex $V^{(3)}_{\mu\nu}$ is the three-leg vertex where one leg is the metric, and two legs are the scalar field. This vertex is given by the second order variational derivative of the

Figure 5: On the left one sees the ++ propagator, which has been calculated in the previous subsection far. On the right the expression for the energy momentum tensor is shown. The wiggly line corresponds to the metric and the solid lines correspond to the scalar field. The red dot denotes the three point vertex $V^{(3)}_{\mu\nu}$. One obtains a loop in the diagram because one takes the limit $x \to x'$. In this thesis one only calculates the energy momentum tensor up to the tadpole order.

energy momentum tensor with respect to the scalar field: $V^{(3)}_{\mu\nu} = \frac{\delta^2 T_{\mu\nu}}{\delta\phi\delta\phi}$. One is only interested in the tadpole contribution and because one has an internal vertex, only $\langle T^+_{\mu\nu}\rangle$ and $\langle T^-_{\mu\nu}\rangle$ contribute. The $\langle T^+_{\mu\nu}\rangle$ contribution uses the Feynman propagator $i\Delta^{++}$ and the $\langle T^-_{\mu\nu}\rangle$ uses the Dyson propagator $i\Delta^{--}$. The Dyson propagator is defined as:

$$i\Delta^*_{\rm F} = i\Delta_D = i\Delta^{--} = \theta(t-t')i\Delta^-(x,x') + \theta(t'-t)i\Delta^+(x,x'). \tag{64}$$

Using that $i\Delta^- = [i\Delta^+]^*$, one easily concludes that $i\Delta^{++}$ and $i\Delta^{--}$ are each other's complex conjugate by comparing with Equation 54. Because $i\Delta^{++}$ and $i\Delta^{--}$ are each other's complex conjugate, one knows that $\langle T^+_{\mu\nu}\rangle$ and $\langle T^-_{\mu\nu}\rangle$ are each other's complex conjugate. As these are the only contributions to the total energy momentum tensor up to tadpole order one can always ignore the imaginary part and one can always just calculate either $\langle T^+_{\mu\nu}\rangle$ or $\langle T^-_{\mu\nu}\rangle$ to know the complete result.

Handling Equation 63 term by term, starting with the easiest term, using equation 60:

$$-\frac{1}{2}m^2\eta_{\mu\nu}\lim_{x\to x'}\langle T[\hat\phi(x)\hat\phi(x')]\rangle = -\eta_{\mu\nu}\frac{m^D}{2^D\pi^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right). \tag{65}$$

Then, one considers the first term:

$$\lim_{x\to x'}\partial_\mu\partial'_\nu\langle T[\hat\phi(x)\hat\phi(x')]\rangle = -2\eta_{\mu\nu}\frac{m^{D-2}}{2^{D-1}\pi^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\frac{m^2}{2D} = -\eta_{\mu\nu}\frac{m^D}{2^{D-1}\pi^{\frac{D}{2}}D}\Gamma\left(1-\frac{D}{2}\right). \tag{66}$$

One sees that one only gets a non-vanishing contribution by taking a t and a t' derivative or by taking an x and an x' derivative. This in total gives a factor $\frac{-2m^2}{2D}$ and an extra minus sign for the t and t' derivative. Taking a t and an x derivative will always yield 0 when the limit is taken. This can all be neatly described using $\eta_{\mu\nu}$. The second term takes the sum because of the Einstein summation convention and therefore gives a factor D. Thus one obtains for the second term:

$$\begin{aligned}-\frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\lim_{x\to x'}\partial_\alpha\partial'_\beta\langle T[\hat\phi(x)\hat\phi(x')]\rangle &= -\frac{-2D}{2}\eta_{\mu\nu}\frac{m^{D-2}}{2^{D-1}\pi^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\frac{m^2}{2D}\\ &= \eta_{\mu\nu}\frac{m^D}{2^D\pi^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right).\end{aligned} \tag{67}$$

Interestingly, the ϵ from the $i\epsilon$ prescription does not contribute to the energy momentum tensor up to order 2 in ϵ, which is of course a required result as the renormalisation should not affect the physics. One sees that the second and the third term cancel, Equation 65 cancels with Equation 67. Thus with all these terms combined, one sees that the averaged energy momentum tensor is given by:

$$\begin{aligned}\langle T^+_{\mu\nu}\rangle &= \eta_{\mu\nu}\frac{m^D}{(4\pi)^{\frac{D}{2}}}\Gamma\left(-\frac{D}{2}\right)\\ &= \eta_{\mu\nu}\left(\frac{m^2}{4\pi}\right)^{\frac{D}{2}}\Gamma\left(-\frac{D}{2}\right).\end{aligned} \tag{68}$$

To see this one makes use of the property of the gamma function $\Gamma(z+1) = z\Gamma(z)$. In dimensional regularization one now analytically extends the Dimension D to the complex plane and later one takes the limit back to $D=4$ or $D=2$. One introduces a mass scale μ to make everything dimensionally correct. One also has to expand x^y as: $x^y = e^{y\ln(x)} = 1 + y\ln(x)$. The analytic extension of the dimension to a complex number kills nearly all divergences that might arise, except divergences scaling as $\frac{1}{D-4}$ divergences. In this case one can add a cosmological constant counter term to renormalise the result. In this dimension, the Gamma function can be expanded as:

$$\begin{aligned}\Gamma\left(-2-\frac{D-4}{2}\right) &= -\frac{1}{2+\frac{D-4}{2}}\Gamma\left(-1-\frac{D-4}{2}\right) \\ &= \frac{1}{2+\frac{D-4}{2}}\frac{1}{1+\frac{D-4}{2}}\Gamma\left(-\frac{D-4}{2}\right),\end{aligned} \tag{69}$$

where $\Gamma(z) = z^{-1} - \gamma_{\mathrm{E}} + O(z)$. After expanding the fractions, one obtains:

$$\langle T^+_{\mu\nu}\rangle = \eta_{\mu\nu}\frac{\mu^{D-4}m^4}{16\pi^2}\left[1+\frac{D-4}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]\left(\frac{1}{2}-\frac{D-4}{8}\right)\left(1-\frac{D-4}{2}\right)\left(-\frac{2}{D-4}-\gamma_{\mathrm{E}}\right). \tag{70}$$

Working out the brackets then gives the result:

$$\begin{aligned}\langle T^+_{\mu\nu}\rangle &= -\eta_{\mu\nu}\frac{\mu^{D-4}m^4}{16\pi^2}\left[\frac{1}{D-4}+\frac{1}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\frac{1}{2}\gamma_{\mathrm{E}}-\frac{3}{4}\right] \\ &= -\eta_{\mu\nu}\frac{m^4}{16\pi^2}\left[\frac{\mu^{D-4}}{D-4}+\frac{1}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\frac{1}{2}\gamma_{\mathrm{E}}-\frac{3}{4}\right].\end{aligned} \tag{71}$$

In the last step there is used that the limit of $D-4$ to zero is well defined in every term, except the term with $\frac{1}{D-4}$. Thus μ^{D-4} only multiplies that term.

In general one wants to renormalise this result. There can be made several choices to renormalise this result. One of which is the situation in which one considers a quartic term in the Lagrangian $\mathcal{L} = -\frac{\lambda}{4!}\phi^4$ as is described by Mancha et al. [61]. For a certain choice of parameters λ, m^2, this gives a Mexican hat potential with a lowest energy state for which the field has a non-zero expectation value: $\langle\phi\rangle = \phi_0$. This will then lead to a mass depending on the expectation value of the field: $m^2 = \frac{\lambda\phi_0^2}{2}$. One can then renormalise this result by adding a counter-term to the action of the form:

$$S_{\mathrm{CT}} = \int d^D x\sqrt{-g}\left[-\frac{\delta\lambda}{4!}\phi^4\right].$$

For a certain value of $\delta\lambda$. This way, one can renormalise the diverging energy momentum tensor of the Minkowski vacuum. An easier way to renormalise this result is to just add a cosmological constant counter-term in 4 dimensions:

$$\mathcal{L}_\Lambda = \int d^4x\sqrt{-g}\Lambda. \tag{72}$$

To renormalise the result one now needs the cosmological constant to be:

$$\Lambda = \frac{m^4}{16\pi^2}\frac{\mu^{D-4}}{D-4}. \tag{73}$$

This renormalises the 4-dimensional energy momentum tensor. This brings us to the result of the calculation, the renormalised + energy momentum tensor:

$$\langle T^+_{\mu\nu}\rangle_{ren} = -\eta_{\mu\nu}\frac{m^4}{16\pi^2}\left[\frac{1}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\frac{1}{2}\gamma_{\mathrm{E}}-\frac{3}{4}\right] \tag{74}$$

Calculating $\langle\hat{T}^-_{\mu\nu}\rangle_{ren}$ would give exactly the same result. There is an underlying reason for this. Namely, $\langle\hat{T}^\pm_{\mu\nu}\rangle_{ren}$ couples to the graviton one-point function $g_{\mu\nu,\pm}$, which are equal on-shell. Taking an expectation value of the energy momentum tensor projects it on-shell. Thus all one-point functions, expectation value of the energy momentum tensors and expectation values of composite operators such as $\langle\hat{T}_{\mu\nu}\rangle$ are independent of the Keldysh polarities. This is also valid for all energy momentum tensors that will be calculated for Rindler space.

3 Calculations in Rindler space

In this section the Rindler coordinates are introduced and following upon that one can calculate the fields in Rindler space. With these fields one can determine the Bogoliubov coefficients between the Rindler space-time and the Minkowski space-time. Then the two-point functions for the naive Rindler vacuum are used to determine the Feynman propagator and the spectral and statistical function. These functions are useful to calculate the entropy, because they describe the states of the system and their occupation respectively. Furthermore, a solution method is devised to solve for the massive propagator. One takes the massless limit and this two-point function is not equal to previously obtained two-point function. One calls the last obtained vacuum the Rindler invariant vacuum. Lastly, the positive and negative frequency Wightman functions, the advanced and retarded propagators for the Rindler invariant vacuum are given as well as the statistical and the spectral function.

3.1 Definition of the Rindler coordinates

One considers the transformation from Minkowski space to Rindler space given by:

$$x = \frac{e^{g\xi}}{g}\cosh(g\tau) \tag{75}$$

$$t = \frac{e^{g\xi}}{g}\sinh(g\tau) \tag{76}$$

These coordinates describe a uniformly accelerating observer with acceleration g. The metric in these coordinates is written as:

$$ds^2 = e^{2g\xi}(-d\tau^2 + d\xi^2). \tag{77}$$

With this metric one can calculate the curvature tensors and show that Rindler space is flat. This is done in Appendix A. In this subsection the standard procedure for canonical quantisation is described in Rindler space. If one has the following Lagrangian with corresponding action:

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2, \tag{78}$$

$$S = \int d^2x\sqrt{-g}\mathcal{L} \tag{79}$$

the equation of motion for the massive field is determined by taking the variation of the action with respect to the field and is given by:

$$\begin{aligned}(\quad - m^2)\phi &= \left[\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - m^2\right]\phi \\ &= 0.\end{aligned} \tag{80}$$

In Rindler coordinates, using the expression for the metric from Equation 77, or by using the rules for partial derivatives, see Appendix B, this is given by:

$$(\quad - m^2)\phi = \left[e^{-2g\xi}(\partial_\xi^2 - \partial_\tau^2) - m^2\right]\phi = 0. \tag{81}$$

3.2 Quantisation of the Rindler fields

In this section the quantisation of the fields in Rindler coordinates is depicted. Because both the massless case and the massive case could be interesting for our situation, one finds solutions to both the massless and the massive case. However, the massive field is harder to solve for and will not be calculated directly. This calculation is done in Subsection 3.8.

For the massless real scalar field, the equation of motion, Equation 81, simply becomes

$$(\partial_\xi^2 - \partial_\tau^2)\phi(\tau, \xi) = 0. \tag{82}$$

One can solve this wave equation by using a Fourier transform:

$$\phi(\tau, \xi) = \int \frac{dk}{2\pi} e^{ik\xi} \phi(\tau, k). \tag{83}$$

Now one quantises the field by introducing a creation and annihilation operator. One also has the requirement that $\phi(\tau, \xi) = \phi^*(\tau, \xi)$, thus one can impose:

$$\hat{\phi}(\tau, \xi) = \int \frac{dk}{2\pi} e^{ik\xi} \left[\phi(\tau, k)\hat{a}(k) + \phi^*(\tau, k)\hat{a}^\dagger(-k)\right]. \tag{84}$$

The second $\phi^*(\tau, k)$ does not depend on $-k$, because of spatial homogeneity: $\phi(\tau, k) = \phi(\tau, -k)$. While the second operator does depend on $-k$ to satisfy $\phi = \phi^*$. The $\hat{a}^\dagger, \hat{a}$ are the creation and annihilation operator and they create the Fock basis of the states. The naive Rindler vacuum $|0_R\rangle$ is defined as the state annihilated by all annihilation operators:

$$\hat{a}(k) |0_R\rangle = 0. \tag{85}$$

Higher energy states can then be created or annihilated by acting with creation and annihilation operators respectively:

$$\hat{a}^\dagger(k) |N_k, N'_{k'}, ...\rangle = \sqrt{2\pi}\sqrt{N_k + 1}\, |(N+1)_k, N'_{k'}, ...\rangle \tag{86}$$

$$\hat{a}(k) |N_k, N'_{k'}, ...\rangle = \sqrt{2\pi}\sqrt{N_k}\, |(N-1)_k, N'_{k'}, ...\rangle, \tag{87}$$

where the factor $\sqrt{2\pi}$ comes from the canonical quantisation. In the canonical quantisation one imposes the following commutation relations:

$$[\hat{a}(k), \hat{a}^\dagger(k')] = 2\pi\delta(k - k') \tag{88}$$

$$[\hat{a}(k), \hat{a}(k')] = 0 \tag{89}$$

$$[\hat{a}^\dagger(k), \hat{a}^\dagger(k')] = 0 \tag{90}$$

$$[\hat{\phi}(\tau, \xi), \hat{\pi}(\tau, \xi')] = i\delta(\xi - \xi') \tag{91}$$

$$W(\phi(\tau, k), \phi^*(\tau, k)) = i. \tag{92}$$

All other commutators vanish. The last function, W, is called the Wronskian, which was already introduced in Equation 37 and is defined as:

$$W(f, g) = f(\partial_\tau g) - (\partial_\tau f)g. \tag{93}$$

If one canonically quantised properly, this Wronskian condition is automatically satisfied. This thus serves as a check on the canonical quantisation. Of course it also works the other way around, one can apply the Wronskian and check that the commutator between $\phi(\tau, \xi)$ and $\pi(\tau, \xi')$ gives $i\delta(\xi - \xi')$. This method is used here. Thus inserting Equation 84 into Equation 82 and applying the Wronskian leads to the following system of equations for $\phi(\tau, k)$:

$$(\partial_\tau^2 + k^2)\phi(\tau, k) = 0 \tag{94}$$

$$\phi(\tau, k)\partial_\tau\phi^*(\tau, k) - \partial_\tau\phi(\tau, k)\phi^*(\tau, k) = i. \tag{95}$$

To get to the first equation, one must use that $e^{ik\xi}$ is an orthonormal basis and thus the integrand should vanish for every k. Then one uses the fact that the two terms that are remaining are each other's complex conjugate and should thus both be satisfied separately to arrive at Equation 94. The first equation is readily solved by:

$$\phi(\tau,k) = \alpha(k)e^{-i\omega\tau} + \beta(k)e^{i\omega\tau}, \tag{96}$$

where $\omega = |k|$ because of spatial homogeneity. Therefore $\alpha(k) = \alpha(-k)$ and $\beta(k) = \beta(-k)$ too. The Wronskian condition then gives:

$$\begin{aligned} W(\phi(\tau,k),\phi^*(\tau,k)) &= \left[\alpha(k)e^{-i\omega\tau} + \beta(k)e^{i\omega\tau}\right] \times \left[i\omega\alpha^*(k)e^{i\omega\tau} - i\omega\beta^*(k)e^{-i\omega\tau}\right] \\ &\quad - \left[-i\omega\alpha(k)e^{-i\omega\tau} + i\omega\beta(k)e^{i\omega\tau}\right] \times \left[\alpha^*(k)e^{i\omega\tau} + \beta^*(k)e^{-i\omega\tau}\right] \\ &= 2i\omega|\alpha(k)|^2 - 2i\omega|\beta(k)|^2. \end{aligned} \tag{97}$$

Now one defines $\tilde{\alpha}(k) = \alpha(k)\sqrt{2\omega}$ and $\tilde{\beta}(k) = \beta(k)\sqrt{2\omega}$, such that $|\tilde{\alpha}(k)|^2 - |\tilde{\beta}(k)|^2 = 1$. This is a consequence of the unitarity of evolution. Any $\tilde{\alpha}, \tilde{\beta}$ that satisfy that condition yield the correct solution, thus one chooses a simple one where $\tilde{\alpha} = 1$ and $\tilde{\beta} = 0$ [62]. So far, the real massless scalar field in Rindler space is thus given by:

$$\hat{\phi}(\tau,\xi) = \int \frac{dk}{2\pi} \frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k) + \frac{e^{i\omega\tau-ik\xi}}{\sqrt{2\omega}}\hat{a}^\dagger(k), \tag{98}$$

after going from $-k \to k$ in the second term. Now one verifies that this solution satisfies Equation 91. Equation 98, tells one that the mode function is given by: $\phi(\tau,k) = \frac{e^{-i\omega\tau}}{\sqrt{2\omega}}$. From the action one determines the canonical momentum:

$$\pi = \frac{\delta S}{\delta \partial_\tau \phi} = -\sqrt{-g}g^{\tau\tau}\partial_\tau\phi = \partial_\tau\phi. \tag{99}$$

Thus the canonical momentum in this case is given by:

$$\hat{\pi}(\xi,\tau) = \int \frac{dk}{2\pi} \frac{-i\sqrt{\omega}e^{-i\omega\tau+ik\xi}}{\sqrt{2}}\hat{a}(k) + \frac{i\sqrt{\omega}e^{i\omega\tau-ik\xi}}{\sqrt{2}}\hat{a}^\dagger(k). \tag{100}$$

The commutator is then given by:

$$\begin{aligned} [\hat{\phi}(\tau,\xi),\hat{\pi}(\tau,\xi')] &= \int \frac{dk}{2\pi}\int\frac{dk'}{2\pi}\frac{i}{2}(-[\hat{a}(k),\hat{a}(k')]e^{-2i\omega\tau+ik\xi+ik'\xi'} - [\hat{a}^\dagger(k),\hat{a}(k')]e^{-ik\xi+ik'\xi'} \\ &\qquad + [\hat{a}(k),\hat{a}^\dagger(k')]e^{ik\xi-ik'\xi'} + [\hat{a}^\dagger(k),\hat{a}^\dagger(k')]e^{2i\omega\tau-ik\xi-ik'\xi'}) \\ &= \frac{1}{4\pi^2}\int dk\int dk'\frac{i}{2}\left[2\pi\delta(k'-k)e^{-ik\xi+ik'\xi'} + 2\pi\delta(k-k')e^{ik\xi-ik'\xi'}\right] \\ &= \frac{i}{2\pi}\int dk\frac{1}{2}\left[e^{-ik(\xi-\xi')} + e^{ik(\xi-\xi')}\right] \\ &= \frac{i}{2\pi}\frac{2\pi}{2}\left[\delta(\xi-\xi') + \delta(\xi-\xi')\right] \\ &= i\delta(\xi-\xi'). \end{aligned} \tag{101}$$

Thus to satisfy all conditions, the massless field and its canonical momentum look like:

$$\hat{\phi}(\tau,\xi) = \int\frac{dk}{2\pi}\left[\frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k) + \frac{e^{i\omega\tau-ik\xi}}{\sqrt{2\omega}}\hat{a}^\dagger(k)\right] = \int\frac{dk}{2\pi}\left[e^{ik\xi}\phi(\tau,k)\hat{a}(k) + e^{-ik\xi}\phi^*(\tau,k)\hat{a}^\dagger(k)\right], \tag{102}$$

$$\hat{\pi}(\xi,\tau) = -i\int\frac{dk}{2\pi}\left[\frac{\sqrt{\omega}e^{-i\omega\tau+ik\xi}}{\sqrt{2}}\hat{a}(k) - \frac{\sqrt{\omega}e^{i\omega\tau-ik\xi}}{\sqrt{2}}\hat{a}^\dagger(k)\right] = -i\int\frac{dk}{2\pi}\frac{\sqrt{\omega}}{\sqrt{2}}\left[e^{ik\xi}\phi(\tau,k)\hat{a}(k) - e^{-ik\xi}\phi^*(\tau,k)\hat{a}^\dagger(k)\right], \tag{103}$$

where one has that the mode function is given by: $\phi(\tau,k) = \frac{e^{-i\omega\tau}}{\sqrt{2\omega}}$.

3.3 Two-point function for the naive Rindler vacuum

Now that the fields have been determined in Rindler coordinates, one calculates the vacuum two-point functions in Rindler coordinates.

$$\begin{aligned}
&\langle\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')\rangle \\
&= \langle 0_R| \int \frac{dk}{2\pi}\int \frac{dk}{2\pi}\left[\frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k)+\frac{e^{i\omega\tau-ik\xi}}{\sqrt{2\omega}}\hat{a}^\dagger(k)\right]\left[\frac{e^{-i\omega'\tau'+ik'\xi'}}{\sqrt{2\omega'}}\hat{a}(k')+\frac{e^{i\omega'\tau'-ik'\xi'}}{\sqrt{2\omega'}}\hat{a}^\dagger(k')\right]|0_R\rangle \\
&= \langle 0_R| \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi}\left[\frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k)\right]\left[\frac{e^{i\omega'\tau'-ik'\xi'}}{\sqrt{2\omega'}}\hat{a}^\dagger(k')\right]|0_R\rangle \\
&= \frac{1}{2\pi}\int\frac{dk}{2\omega}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right] \\
&= \frac{1}{2\pi}\int_0^\infty\frac{dk}{2\omega}e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}+\frac{1}{2\pi}\int_{-\infty}^0\frac{dk}{2\omega}e^{-i\omega(\tau-\tau')+ik(\xi-\xi')} \\
&= \frac{1}{2\pi}\int_0^\infty\frac{dk}{2k}e^{-ik(\tau-\tau')+ik(\xi-\xi')}+\frac{1}{2\pi}\int_0^\infty\frac{dk}{2k}e^{-ik(\tau-\tau')-ik(\xi-\xi')}
\end{aligned} \tag{104}$$

This integral can be done in multiple ways. The easiest way is by taking a derivative first and integrating later. This is at the cost of an unknown integration constant at the end. This constant will be ignored for now, but will be determined later, by calculating the two-point function in a different way. One also has to add a small $i\epsilon$ to make the integral convergent, this is also known as the Feynman $i\epsilon$ prescription. Using this derivative trick and adding a small $i\epsilon$ yields:

$$\begin{aligned}
\partial_\tau\langle\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')\rangle &= \frac{1}{2\pi}\int_0^\infty\frac{-idk}{2}e^{-ik(\tau-\tau')+ik(\xi-\xi')}+\frac{1}{2\pi}\int_0^\infty\frac{-idk}{2}e^{-ik(\tau-\tau')-ik(\xi-\xi')} \\
&= \frac{1}{2\pi}\int_0^\infty\frac{-idk}{2}e^{-ik(\tau-\tau'-\xi+\xi'-i\epsilon)}+\frac{1}{2\pi}\int_0^\infty\frac{-idk}{2}e^{-ik(\tau-\tau'+\xi-\xi'-i\epsilon)} \\
&= \frac{-i}{4\pi}\left[\frac{e^{-ik(\tau-\tau'-\xi+\xi'-i\epsilon)}}{-i(\tau-\tau'-\xi+\xi'-i\epsilon)}+\frac{e^{-ik(\tau-\tau'+\xi-\xi'-i\epsilon)}}{-i(\tau-\tau'+\xi-\xi'-i\epsilon)}\right]_0^\infty \\
&= \frac{-1}{4\pi}\left[\frac{1}{\tau-\tau'-\xi+\xi'-i\epsilon}+\frac{1}{\tau-\tau'+\xi-\xi'-i\epsilon}\right].
\end{aligned} \tag{105}$$

Integrating this result over τ then yields the $\phi\phi$ two-point function.

$$\langle\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')\rangle = -\frac{1}{4\pi}\left[\ln(\tau-\tau'-\xi+\xi'-i\epsilon)+\ln(\tau-\tau'+\xi-\xi'-i\epsilon)+C\right]. \tag{106}$$

With a suitable choice of the integration constant C, one can bring this expression into the form: $-\frac{1}{4\pi}\ln\left\{\mu^2\Delta x^2_{+-}\right\}+\phi^2(\mu)$, where $\Delta x^2_{+-}=(\tau-\tau'-i\epsilon)^2-(\xi-\xi')^2$ and $\phi^2(\mu)$ depends on μ such that $\mu\partial_\mu\phi^2(\mu)=\frac{1}{2\pi}$ Because one has a minus sign here, one sees that one has high correlations on the light cone and anti-correlations on large distances. The next two-point function will be the $\phi\pi$ two-point function:

$$\begin{aligned}
&\langle 0_R| \int \frac{dk}{2\pi}\int \frac{dk}{2\pi}\left[\frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k)+\frac{e^{i\omega\tau-ik\xi}}{\sqrt{2\omega}}\hat{a}^\dagger(k)\right]\left[\frac{-i\omega' e^{-i\omega'\tau'+ik'\xi'}}{\sqrt{2\omega'}}\hat{a}(k')+\frac{i\omega' e^{i\omega'\tau'-ik'\xi'}}{\sqrt{2\omega'}}\hat{a}^\dagger(k')\right]|0_R\rangle \\
&= \frac{1}{2\pi}\int\frac{idk}{2}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right] \\
&= \frac{1}{2\pi}\int_0^\infty\frac{idk}{2}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right]+\int_{-\infty}^0\frac{idk}{2}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right] \\
&= \frac{i}{4\pi}\left[\int_0^\infty dk\left[e^{-ik(\tau-\tau')+ik(\xi-\xi')}\right]+\int_0^\infty dk\left[e^{-ik(\tau-\tau')-ik(\xi-\xi')}\right]\right].
\end{aligned} \tag{107}$$

To make this integral convergent, one adds an $i\epsilon$ prescription again and then one obtains:

$$\begin{aligned}\langle\hat{\phi}(\tau,\xi)\hat{\pi}(\tau',\xi')\rangle &= \frac{i}{4\pi}\left[\frac{e^{-ik(\tau-\tau'-\xi+\xi'-i\epsilon)}}{-i(\tau-\tau'-\xi+\xi'-i\epsilon)}+\frac{e^{-ik(\tau-\tau'+\xi-\xi'-i\epsilon)}}{-i(\tau-\tau'+\xi-\xi'-i\epsilon)}\right]_0^\infty \\ &= \frac{1}{4\pi}\left[\frac{1}{\tau-\tau'-\xi+\xi'-i\epsilon}+\frac{1}{\tau-\tau'+\xi-\xi'-i\epsilon}\right].\end{aligned} \tag{108}$$

This result is in accordance with Equation 99, as this two-point function should just be a τ' derivative of the first two-point function that was calculated in this subsection. Lastly, one calculates the $\pi\pi$ two-point function:

$$\begin{aligned}\langle\hat{\pi}(\tau,\xi)\hat{\pi}(\tau',\xi')\rangle &= \int\frac{dk\omega}{4\pi}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right] \\ &= \int_0^\infty\frac{dk\omega}{4\pi}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right]+\int_{-\infty}^0\frac{dk\omega}{4\pi}\left[e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}\right] \\ &= \int_0^\infty\frac{dkk}{4\pi}\left[e^{-ik(\tau-\tau')+ik(\xi-\xi')}\right]+\int_0^\infty\frac{dkk}{4\pi}\left[e^{-ik(\tau-\tau')-ik(\xi-\xi')}\right]\end{aligned} \tag{109}$$

This has the form of a gamma function when one makes the substitution $x = ik(\tau-\xi-\tau'+\xi'-i\epsilon)$ for the first part and a similar expression for the second part. Using an analytical extension, this integral then gives $\Gamma(2) = 1! = 1$. Therefore the result is:

$$\begin{aligned}\langle\hat{\pi}(\tau,\xi)\hat{\pi}(\tau',\xi')\rangle &= \frac{1}{4\pi}\left[\frac{1}{(i(\tau-\tau'-\xi+\xi'-i\epsilon))^2}+\frac{1}{(i(\tau-\tau'+\xi-\xi'-i\epsilon))^2}\right] \\ &= \frac{-1}{4\pi}\left[\frac{1}{(\tau-\tau'-\xi+\xi'-i\epsilon)^2}+\frac{1}{(\tau-\tau'+\xi-\xi'-i\epsilon)^2}\right].\end{aligned} \tag{110}$$

Again one sees that this should be the correct answer as this is the τ derivative of the $\phi\pi$ two-point function from Equation 108.

3.4 The propagator in Rindler space

The appealing thing of canonical quantisation is that in this quantisation method the propagator is satisfying the same differential equation, up to a delta function, as the field, namely:

$$\sqrt{-g}(\quad + m^2)i\triangle_F(\xi,\xi';\tau,\tau') = i\delta(\xi-\xi')\delta(\tau-\tau'). \tag{111}$$

Where the Feynman propagator is given by:

$$i\triangle_F(\tau,\xi;\tau',\xi') = \theta(\tau-\tau')\langle\Omega|\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle + \theta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')\hat{\phi}(\xi,\tau)|\Omega\rangle. \tag{112}$$

In the previous subsection one saw that the two-point function for the naive vacuum was given by: $\langle\Omega|\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle = -\frac{1}{4\pi}\ln\left\{\mu^2\Delta x_{+-}^2\right\} + \phi^2(\mu)$. Thus the Feynmann propagator is given by:

$$i\triangle_F(\xi,\xi';\tau,\tau') = -\frac{1}{4\pi}\ln\left\{\mu^2\Delta X_{++}^2\right\} + \phi^2(\mu),$$

where $\Delta X_{++} = -(|\tau-\tau'|-i\epsilon)^2 + (\xi-\xi')^2$. In Equation 81, there was determined that the Alembertian was given by:

$$\quad = e^{-2g\xi}(\partial_\xi^2 - \partial_\tau^2) \tag{113}$$

One can apply this Alembertian to Equation 111. The first time derivative is given by:

$$\begin{aligned}\partial_\tau i\triangle(\xi,\xi';\tau,\tau') &= \delta(\tau-\tau')\langle\Omega|\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle - \delta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')\hat{\phi}(\xi,\tau)|\Omega\rangle \\ &+ \theta(\tau-\tau')\langle\Omega|\partial_\tau\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle + \theta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')\partial_\tau\hat{\phi}(\xi,\tau)|\Omega\rangle.\end{aligned} \tag{114}$$

The first two terms cancel because of the property $\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0)$ and the fact that the equal time commutator $[\hat{\phi}(\tau,\xi), \hat{\phi}(\tau,\xi')]$ vanishes. Now, taking the second derivative yields:

$$\begin{aligned}\partial_\tau^2 i\triangle_F(\xi,\xi';\tau,\tau') &= \delta(\tau-\tau')\langle\Omega|\left[\partial_\tau\hat{\phi}(\xi,\tau),\hat{\phi}(\xi',\tau)\right]|\Omega\rangle \\ &+ \theta(\tau-\tau')\langle\Omega|\partial_\tau^2\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle + \theta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')\partial_\tau^2\hat{\phi}(\xi,\tau)|\Omega\rangle .\end{aligned} \tag{115}$$

The ξ derivative only acts on the terms inside the bra's and ket's and is therefore easy to write down at once:

$$\partial_\xi^2 i\triangle_F(\xi,\xi';\tau,\tau') = \theta(\tau-\tau')\langle\Omega|\partial_\xi^2\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle + \theta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')\partial_\xi^2\hat{\phi}(\xi,\tau)|\Omega\rangle . \tag{116}$$

The mass term is just a multiplication, thus adding this all together then yields:

$$\begin{aligned}&\sqrt{-g}\quad i\triangle_F(\xi,\xi';\tau,\tau') = e^{2g\xi}\Big[-e^{-2g\xi}\delta(\tau-\tau')\langle\Omega|[\partial_\tau\hat{\phi}(\xi,\tau),\hat{\phi}(\xi',\tau)]|\Omega\rangle \\ &+\theta(\tau-\tau')\langle\Omega|(e^{-2g\xi}(\partial_\xi^2-\partial_\tau^2)+m^2)\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')|\Omega\rangle+\theta(\tau'-\tau)\langle\Omega|\hat{\phi}(\xi',\tau')(e^{-2g\xi}(\partial_\xi^2-\partial_\tau^2)+m^2)\hat{\phi}(\xi,\tau)|\Omega\rangle\Big] .\end{aligned} \tag{117}$$

The last two terms are now zero, when $\hat{\phi}(\tau,\xi)$ satisfies the equation of motion. The first term depends on the commutator of $[\partial_\tau\hat{\phi}(x),\hat{\phi}(x')]$, but this is just the equal time commutator between ϕ and π, which in canonical quantisation was given to be: $[\hat{\pi}(\xi',\tau),\hat{\phi}(\xi,\tau)] = -i\delta(\xi-\xi')$. Thus indeed one obtains:

$$\sqrt{-g}\quad i\triangle_F(\xi,\xi';\tau,\tau') = i\delta(\xi-\xi')\delta(\tau-\tau'). \tag{118}$$

Thus with the choice in Equation 112 for the propagator, Equation 111 is automatically satisfied.

3.5 Calculation of the Bogoliubov coefficients

One knows that the choice of coordinates should not be affecting the physics and thus the mode functions in Rindler should be related to the mode functions from Minkowski space. This is also true for the annihilation and creation operators. The Bogoliubov coefficients show how the mode functions between two different (regions of) space-time(s) are related and how the creation and annihilation operator of the two (regions of) space-times(s) are related. This is also called the Bogoliubov transform. From Figure 2 it is clear that Rindler space only describes a quarter of Minkowski space. The Rindler R-wedge from Figure 2 is limited by the past and future Cauchy horizons, which do not have a special meaning for Minkowski observers. This global difference is responsible for the entanglement entropy that a Minkowski observer perceives when observing a Rindler vacuum state. This section is devoted to calculating the Bogoliubov coefficients between Rindler space and Minkowski space.

Suppose one has the field in a given coordinate system given by

$$\phi(x,t) = \sum_i f_i(x,t)\hat{a}_i + f_i^*(x,t)\hat{a}_i^\dagger, \tag{119}$$

where the f_i, f_i^* are the mode functions and their complex conjugates and the $\hat{a}_i, \hat{a}_i^\dagger$ are the creation and annihilation operator. Furthermore, suppose that one has a different coordinate system with mode functions g_j, g_j^* and creation and annihilation operators $\hat{b}_j, \hat{b}_j^\dagger$. Now one can relate these two fields to each other by the Bogoliubov transform which is given by [63]:

$$g_j = \sum_i(\alpha_{ji}f_i + \beta_{ji}f_i^*). \tag{120}$$

Here α_{ji}, β_{ji} are the so called Bogoliubov coefficients. The Bogoliubov coefficients α and β are given via the Klein-Gordon norm:

$$\alpha_{ji} = (g_j, f_i)_{KG}, \tag{121}$$

$$\beta_{ji} = -(g_j, f_i^*)_{KG}, \tag{122}$$

where the Klein-Gordon norm is given by

$$(f_i, g_j)_{KG} = -i \int_{\Sigma_t} (f_i \partial_t g_j^* - g_j^* \partial_t f_i) d^{n-1}x. \tag{123}$$

Here, Σ_t denotes a constant time hyper-surface. This procedure is very similar to creating an orthonormal basis following the Gram-Schmidt method in linear algebra, but now using a Hilbert space with an inner product defined by the Klein-Gordon norm. One can easily derive some properties of the Klein-Gordon norm using this definition:

$$\alpha_{ij}^* = (f_i, g_j)_{KG}^* = -(f_i^*, g_j^*)_{KG} = (g_j, f_i)_{KG} = \alpha_{ji} \tag{124}$$

$$\beta_{ij}^* = -(f_i, g_j^*)_{KG}^* = (f_i^*, g_j)_{KG} = -(g_j, f_i^*)_{KG} = \beta_{ji} \tag{125}$$

From this it is clear that α_{ij} and β_{ij} are Hermitian matrices. Calculating the Klein-Gordon norm is usually long and therefore often one sees a shortcut being used in the calculation of the Bogoliubov coefficients. That also happens in this thesis. One can also relate the annihilation and creation operator via these coefficients:

$$\hat{b}_j = \sum_i (\alpha_{ji}^* \hat{a}_i - \beta_{ji}^* \hat{a}_i^\dagger). \tag{126}$$

This directly shows the most known result from Bogoliubov transformations that spaces that appear empty for one observer could be filled with particles for another observer. This is because the annihilation operator in one system is expressed in the creation operator of the other system in a Bogoliubov transform.

The massless fields $\hat{\phi}$ in the different coordinate systems were given by Equations 36, 102:

$$\hat{\phi}(t,x) = \int \frac{dk}{2\pi} \left[\frac{e^{-i\omega t + ikx}}{\sqrt{2\omega}} \hat{b}(k) + \frac{e^{i\omega t - ikx}}{\sqrt{2\omega}} \hat{b}^\dagger(k) \right]. \tag{127}$$

$$\hat{\phi}(\tau,\xi) = \int \frac{dk}{2\pi} \left[\frac{e^{-i\omega\tau + ik\xi}}{\sqrt{2\omega}} \hat{a}(k) + \frac{e^{i\omega\tau - ik\xi}}{\sqrt{2\omega}} \hat{a}^\dagger(k) \right]. \tag{128}$$

As $\omega = |k|$ in the massless case, it is convenient to split the integral in a part from 0 to ∞ and a part from 0 to $-\infty$. Doing this and transforming $k \to -k$ for the integrals from 0 to $-\infty$ yields the following expression for the fields:

$$\hat{\phi}(x,t) = \int_0^\infty \frac{dk}{2\pi\sqrt{2k}} [\hat{b}(k)e^{-ik(t-x)} + \hat{b}^\dagger(k)e^{+ik(t-x)} + \hat{b}(-k)e^{-ik(t+x)} + \hat{b}^\dagger(-k)e^{+ik(t+x)}] \tag{129}$$

$$\hat{\phi}(\xi,\tau) = \int_0^\infty \frac{dk}{2\pi\sqrt{2k}} [\hat{a}(k)e^{-ik(\tau-\xi)} + \hat{a}^\dagger(k)e^{+ik(\tau-\xi)} + \hat{a}(-k)e^{-ik(\tau+\xi)} + \hat{a}^\dagger(-k)e^{+ik(\tau+\xi)}]. \tag{130}$$

In the rest of this thesis one denotes the Minkowski annihilation and creation operator by $\hat{b}, \hat{b}^\dagger$, while the Rindler creation and annihilation operator are denoted with $\hat{a}$ and $\hat{a}^\dagger$. The splitting of the integrals makes it easier to distinguish the left moving parts from the right moving parts. To abbreviate the notation, one introduces the following notation for the mode functions:

$$f_k = \frac{e^{-ik(t-x)}}{\sqrt{2k}}, \; g_k = \frac{e^{-ik(t+x)}}{\sqrt{2k}}, \tag{131}$$

$$h_k = \frac{e^{-ik(\tau-\xi)}}{\sqrt{2k}}, \; j_k = \frac{e^{-ik(\tau+\xi)}}{\sqrt{2k}} \tag{132}$$

There can be concluded that f_k and h_k describe the right moving waves and g_k and j_k describe the left moving waves and that the left and right moving waves decouple entirely in the sense of the Klein-Gordon norm, namely:

$$(f_k, f_{k'})_{KG} = -(f_k^*, f_{k'}^*)_{KG} = 2\pi\delta(k-k') \tag{133}$$

$$(g_k, g_{k'})_{KG} = -(g_k^*, g_{k'}^*)_{KG} = 2\pi\delta(k-k') \tag{134}$$

$$(f_k^*, f_{k'})_{KG} = (g_k^*, g_{k'})_{KG} = (f_k, g_{k'})_{KG} = (f_k, g_{k'}^*)_{KG} = 0 \tag{135}$$

This is shown in Appendix C.

The specific form of the mode functions gives a reason to go to the following coordinate systems:

$$\bar{u} = t - x, \bar{v} = t + x \tag{136}$$

for Minkowski and

$$u = \tau - \xi, v = \tau + \xi, \tag{137}$$

for Rindler. Because the Hamiltonian in Minkowski is separable in a left moving part and a right moving part, one can equate the right moving and left moving parts separately. So, according to the Bogoliubov transform one now knows [63]

$$h_{k'} = \frac{e^{-ik'u}}{\sqrt{2k'}} = \int_0^\infty dk[\alpha_{k'k} f_k + \beta_{k'k} f_k^*] \tag{138}$$

$$j_{k'} = \frac{e^{-ik'v}}{\sqrt{2k'}} = \int_0^\infty dk[\epsilon_{k'k} g_k + \sigma_{k'k} g_k^*] \tag{139}$$

$$f_{k'} = \frac{e^{-ik'\bar{u}}}{\sqrt{2k'}} = \int_0^\infty dk[\alpha_{kk'}^* h_k - \beta_{kk'} h_k^*] \tag{140}$$

$$g_{k'} = \frac{e^{-ik'\bar{v}}}{\sqrt{2k'}} = \int_0^\infty dk[\epsilon_{kk'}^* j_k - \sigma_{kk'} j_k^*] \tag{141}$$

Now one can use these expressions on the right moving part of the Minkowski field:

$$\int_0^\infty \frac{dk'}{2\pi}(\hat{b}(k')f_{k'} + \hat{b}^\dagger(k')f_{k'}^*) = \int_0^\infty \frac{dk'}{2\pi}\hat{b}(k')\int_0^\infty dk(\alpha_{kk'}^* h_k - \beta_{kk'} h_k^*) \tag{142}$$

$$+ \int_0^\infty \frac{dk'}{2\pi}\hat{b}^\dagger(k')\int_0^\infty dk(\alpha_{kk'} h_k^* - \beta_{kk'}^* h_k). \tag{143}$$

But one knows that this should be equal to the right moving part in Rindler space, thus this should be equal to

$$\phi_{right}(\tau,\xi) = \int_0^\infty \frac{dk}{2\pi}[\hat{a}(k)h_k + \hat{a}^\dagger(k)h_k^*] \tag{144}$$

Comparing these two equations one reads off that

$$\hat{a}(k) = \int_0^\infty dk'[\hat{b}(k')\alpha_{kk'}^* - \hat{b}^\dagger(k')\beta_{kk'}^*], \tag{145}$$

$$\hat{a}^\dagger(k) = \int_0^\infty dk'[\hat{b}^\dagger(k')\alpha_{kk'} - \hat{b}(k')\beta_{kk'}]. \tag{146}$$

Similarly one finds for the Minkowski operators:

$$\hat{b}(k') = \int_0^\infty dk[\hat{a}(k)\alpha_{kk'} + \hat{a}^\dagger(k)\beta^*_{kk'}], \tag{147}$$

$$\hat{b}^\dagger(k') = \int_0^\infty dk[\hat{a}^\dagger(k)\alpha^*_{k'k} + \hat{a}(k)\beta_{k'k}]. \tag{148}$$

Now one applies the same procedure to our expressions for right moving parts of the mode functions from Equations 129 and 130:

$$\int_0^\infty \frac{dk}{2\pi}\frac{1}{\sqrt{2k}}[\hat{b}(k)e^{-ik\bar{u}} + \hat{b}^\dagger(k)e^{+ik\bar{u}}] = \int_0^\infty \frac{dk'}{2\pi}\frac{1}{\sqrt{2k'}}[\hat{a}(k')e^{-ik'u} + \hat{a}^\dagger(k')e^{+ik'u}] \tag{149}$$

Multiplying both sides with $\int_{-\infty}^{\infty} du\ e^{ipu}$, and using the fact that:

$$\int_{-\infty}^\infty \frac{du}{2\pi}e^{i(p-k)u} = \delta(p-k). \tag{150}$$

With this one gets on the right-hand side the following:

$$\text{r.h.s} = \int_0^\infty \frac{dk'}{2\pi}\frac{2\pi}{\sqrt{2k'}}[\delta(k'-p)\hat{a}(k') + \delta(k'+p)\hat{a}^\dagger(k')] = \frac{2\pi}{\sqrt{2p}}\hat{a}(p), \tag{151}$$

since $p > 0$. After bringing the factor in front of $\hat{a}(p)$ to the left-hand side, Equation 149 becomes:

$$\int_{-\infty}^\infty du \int_0^\infty \frac{dk}{2\pi}\frac{\sqrt{2p}}{\sqrt{2k}}[\hat{b}(k)e^{-ik\bar{u}+ipu} + \hat{b}^\dagger(k)e^{+ik\bar{u}+ipu}] = \hat{a}(p). \tag{152}$$

Combining this result with the form that the annihilation operator should have in Rindler, Equation 145:

$$\hat{a}(p) = \int_0^\infty dk[\hat{b}(k)\alpha^*_{pk} - \hat{b}^\dagger(k)\beta^*_{pk}], \tag{153}$$

allows one to just read of the Bogoliubov transformation coefficients. These are given by:

$$\alpha^*_{pk} = \sqrt{\frac{p}{k}}\int_{-\infty}^\infty \frac{du}{2\pi}e^{i(pu-k\bar{u})} \tag{154}$$

and

$$\beta^*_{pk} = -\sqrt{\frac{p}{k}}\int_{-\infty}^\infty \frac{du}{2\pi}e^{i(pu+k\bar{u})} \tag{155}$$

Following this same procedure, one can also read off the Bogoliubov transformation coefficients for the left moving part. These are given by:

$$\epsilon_{pk} = \sqrt{\frac{p}{k}}\int_{-\infty}^\infty \frac{dv}{2\pi}e^{i(pv-k\bar{v})} \tag{156}$$

and

$$\sigma_{pk} = -\sqrt{\frac{p}{k}}\int_{-\infty}^\infty \frac{dv}{2\pi}e^{i(pv+k\bar{v})}. \tag{157}$$

These integrals can be evaluated and the result for the integral appearing in the α^*_{pk} Bogoliubov coefficient is given by, see Appendix D:

$$\int_{-\infty}^\infty \frac{du}{2\pi}e^{i(pu-k\bar{u})} = \frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{-ip}{g}}\Gamma(\frac{-ip}{g}). \tag{158}$$

The other integrals have similar results and thus one obtains the following Bogoliubov coefficients:

$$\alpha_{pk} = \sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}), \tag{159}$$

$$\beta_{pk} = -\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}), \tag{160}$$

$$\epsilon_{pk} = \sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}), \tag{161}$$

$$\sigma_{pk} = -\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}). \tag{162}$$

With these results, one concludes that

$$\alpha_{pk} = \epsilon_{pk} = -e^{\frac{p\pi}{g}}\beta_{pk} = -e^{\frac{p\pi}{g}}\sigma_{pk}. \tag{163}$$

These Bogoliubov coefficients have to satisfy certain conditions for them to be the correct transformation. According to Equations 138 and 140, one has:

$$\begin{aligned} h_{k'} &= \int_0^\infty dk\,(\alpha_{k'k}f_k + \beta_{k'k}f_k^*) \\ &= \int_0^\infty dk\left(\alpha_{k'k}\left[\int_0^\infty dp[\alpha_{pk}^*h_p - \beta_{pk}h_p^*]\right] + \beta_{k'k}\left[\int_0^\infty dp[\alpha_{pk}h_p^* - \beta_{pk}^*h_p]\right]\right) \\ &= \int_0^\infty dp\left(h_p\left[\int_0^\infty dk\alpha_{k'k}\alpha_{pk}^* - \beta_{k'k}\beta_{pk}^*\right] + h_p^*\left[\int_0^\infty dk\beta_{k'k}\alpha_{pk} - \alpha_{k'k}\beta_{pk}\right]\right). \end{aligned} \tag{164}$$

As this needs to be equal to the original mode function again, one has the following two constraints on the Bogoliubov coefficients:

$$C_1 = \int_0^\infty dk\left(\alpha_{k'k}\alpha_{pk}^* - \beta_{k'k}\beta_{pk}^*\right) = \delta(p-k') \tag{165}$$

$$C_2 = \int_0^\infty dk\,(\beta_{k'k}\alpha_{pk} - \alpha_{k'k}\beta_{pk}) = 0. \tag{166}$$

The second constraint, called C_2, is easily checked as:

$$\begin{aligned} C_2 = \int_0^\infty dk\,[\beta_{k'k}\alpha_{pk} - \alpha_{k'k}\beta_{pk}] &= \int_0^\infty dk\left[-\sqrt{\frac{k'}{k}}\frac{e^{-\frac{k'\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ik'}{g}}\Gamma\left(\frac{ik'}{g}\right)\right]\left[\sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma\left(\frac{ip}{g}\right)\right] \\ &\quad - \left[\sqrt{\frac{k'}{k}}\frac{e^{\frac{k'\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ik'}{g}}\Gamma\left(\frac{ik'}{g}\right)\right]\left[-\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma\left(\frac{ip}{g}\right)\right] \\ &= \frac{1}{(2\pi g)^2}\int_0^\infty dk\sqrt{\frac{k'p}{k^2}}(\frac{g}{k})^{\frac{i(p+k')}{g}}\Gamma\left(\frac{ik'}{g}\right)\Gamma\left(\frac{ip}{g}\right)\left(e^{\frac{(k'-p)\pi}{2g}} - e^{\frac{(p-k')\pi}{2g}}\right). \end{aligned} \tag{167}$$

Now one only looks at the k integral:

$$\int_0^\infty \frac{dk}{k}k^{-\frac{i(p+k')}{g}} \tag{168}$$

One can make a substitution here. Let $x = \ln(k)$, then $dx = \frac{dk}{k}$. If one then writes $k^{\frac{-i(p+k')}{g}} = e^{-\ln(k)\frac{i(p+k')}{g}}$. The boundaries will change accordingly to $-\infty$ to ∞. One can now see that the k part of the integral simplifies to:

$$\int_0^\infty \frac{dk}{k} k^{\frac{-i(p+k')}{g}} = \int_{-\infty}^{\infty} dx \;\; e^{\frac{-i(p+k')x}{g}} \\ = 2\pi g\delta(p+k'), \tag{169}$$

where again Equation 150 was used after changing variables to $y = \frac{x}{g}$. Together with the property of the Gamma function:

$$|\Gamma(bi)|^2 = \Gamma(bi)\Gamma(-bi) = \frac{\pi}{b\sinh(\pi b)}, \tag{170}$$

one obtains:

$$C_2 = \frac{1}{2\pi g}\sqrt{k'p}g^{\frac{i(p+k')}{g}}\Gamma\left(\frac{ik'}{g}\right)\Gamma\left(\frac{ip}{g}\right)\left(e^{\frac{(k'-p)\pi}{2g}} - e^{\frac{(p-k')\pi}{2g}}\right)\delta(p+k') \\ = \frac{i}{2\pi g}p\frac{\pi}{\frac{p}{g}\sinh(\frac{p\pi}{g})}2\sinh\left(\frac{p\pi}{g}\right)\delta(p+k') = i\delta(p+k') = 0. \tag{171}$$

The last step is made by realizing that both p and k should be greater than 0 and thus the delta function cannot be satisfied. Thus the second constraint is satisfied. One denotes the first constraint with C_1:

$$C_1 = \int_0^\infty dk\left(\alpha_{k'k}\alpha^*_{pk} - \beta_{k'k}\beta^*_{pk}\right) \\ = \int_0^\infty dk\left[\sqrt{\frac{k'}{k}}\frac{e^{\frac{k'\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ik'}{g}}\Gamma\left(\frac{ik'}{g}\right)\right]\left[\sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{-ip}{g}}\Gamma\left(\frac{-ip}{g}\right)\right] \\ - \left[-\sqrt{\frac{k'}{k}}\frac{e^{-\frac{k'\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ik'}{g}}\Gamma\left(\frac{ik'}{g}\right)\right]\left[-\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{-ip}{g}}\Gamma\left(\frac{-ip}{g}\right)\right] \\ = \int_0^\infty dk\frac{\sqrt{k'p}}{k}\frac{\Gamma\left(\frac{ik'}{g}\right)\Gamma\left(\frac{-ip}{g}\right)}{(2\pi g)^2}(\frac{g}{k})^{\frac{i(k'-p)}{g}}\left(e^{\frac{(k'+p)\pi}{2g}} - e^{\frac{-(k'+p)\pi}{g}}\right). \tag{172}$$

The same form of integral is appearing here and therefore one obtains:

$$C_1 = \frac{1}{2\pi g}\sqrt{k'p}g^{\frac{i(k'-p)}{g}}\Gamma\left(\frac{ik'}{g}\right)\Gamma\left(\frac{-ip}{g}\right)\left(e^{\frac{(k'+p)\pi}{2g}} - e^{-\frac{(k'+p)\pi}{2g}}\right)\delta(k'-p). \tag{173}$$

Applying $f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$, yields two times a sine hyperbolic from the term in the brackets. The gamma functions again give a 1 over sine hyperbolic and a factor of πg, thus one obtains:

$$C_1 = \frac{2\pi g}{2\pi g}p\frac{1}{p\sinh(\frac{p\pi}{g})}\sinh(\frac{p\pi}{g})\delta(k'-p) = \delta(k'-p). \tag{174}$$

Thus the found Bogoliubov coefficients indeed satisfy the constraints.

3.6 Calculation of the two-point function for the naive Rindler vacuum

In the last subsection, the field was split in a left moving and a right moving part. To be sure that this is correct, one calculates its value again and compares it to the previously obtained result from 106. Then one determines the undetermined constant and one works out the $i\epsilon$ prescription using the principal sheet of the logarithm. This prescription will give rise to the statistical and spectral function which will be described in the next subsection.

One defines the naive vacuum of Rindler space by

$$\hat{a}(k)\left|0_R\right\rangle = 0, \tag{175}$$

for all values of k. The two-point function for the Rindler vacuum is then given by:

$$\begin{aligned}\left\langle 0_R\right| \hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')\left|0_R\right\rangle = \left\langle 0_R\right| \int_0^\infty \frac{dk}{2\pi}\int_0^\infty \frac{dk'}{2\pi}\frac{1}{2\sqrt{kk'}} \quad \times \\ [\hat{a}(k)e^{-ik(\tau-\xi)} + \hat{a}^\dagger(k)e^{+ik(\tau-\xi)} + \hat{a}(-k)e^{-ik(\tau+\xi)} + \hat{a}^\dagger(-k)e^{+ik(\tau+\xi)}] \quad \times \\ [\hat{a}(k')e^{-ik'(\tau'-\xi')} + \hat{a}^\dagger(k')e^{+ik'(\tau'-\xi')} + \hat{a}(-k')e^{-ik'(\tau'+\xi')} + \hat{a}^\dagger(-k')e^{+ik'(\tau'+\xi')}]\left|0_R\right\rangle\end{aligned} \tag{176}$$

Because one is dealing with the Rindler vacuum here, one only wants $\hat{a}^\dagger$ to the right and $\hat{a}$ to the left. One also assumes orthonormal states $\left\langle 0_R\right| \hat{a}(k)\hat{a}^\dagger(k')\left|0_R\right\rangle = 2\pi\delta(k-k')$. Additionally, since in the splitting all momenta are greater than zero, $k>0$, only a few terms remain in the entire product, namely:

$$\begin{aligned}&\left\langle 0_R\right| \hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')\left|0_R\right\rangle = \\ &\quad \left\langle 0_R\right| \int_0^\infty \frac{dk}{2\pi}\int_0^\infty \frac{dk'}{2\pi}\frac{1}{2\sqrt{kk'}}[\hat{a}(k)\hat{a}^\dagger(k')e^{-ik(\tau-\xi)}e^{ik'(\tau'-\xi')} \\ &\quad + \hat{a}(-k)\hat{a}^\dagger(-k')e^{-ik(\tau+\xi)}e^{ik'(\tau'+\xi')}\left|0_R\right\rangle \\ &\quad = \int_0^\infty \frac{dk}{4\pi k}\left[e^{-ik(\tau-\tau'-\xi+\xi')} + e^{-ik(\tau-\tau'+\xi-\xi')}\right],\end{aligned} \tag{177}$$

where one goes from the second line to the third line with the orthonormality of the states. It is neat to go to the coordinates u,v in Rindler and $\bar{u},\bar{v}$ in Minkowski as they are easily expressed into each other. Namely:

$$\bar{u} = t - x = \frac{1}{g}e^{g\xi}(\sinh(g\tau) - \cosh(g\tau)) = -\frac{1}{g}e^{(g(\xi-\tau))} = -\frac{1}{g}e^{-gu}, \tag{178}$$

$$\bar{v} = t + x = \frac{1}{g}e^{g\xi}(\sinh(g\tau) + \cosh(g\tau)) = \frac{1}{g}e^{(g(\xi+\tau))} = +\frac{1}{g}e^{gv}. \tag{179}$$

Thus one transforms our 2 point function according to:

$$\begin{aligned}\int_0^\infty \frac{dk}{4\pi k}e^{-ik(\tau-\tau'-\xi+\xi')}) &= \int_0^\infty \frac{dk}{4\pi k}e^{-ik(u-u')} \\ &= \int_0^\infty \frac{dk}{4\pi k}e^{\frac{ik}{g}ln(-g\bar{u})}e^{\frac{-ik}{g}ln(-g\bar{u}')} \\ &= \int_0^\infty \frac{dk}{4\pi k}\left(\frac{\bar{u}}{\bar{u}'}\right)^{\frac{ik}{g}}.\end{aligned} \tag{180}$$

This integral has a singularity at 0, however this singularity in the two-point function indicates that there is no Lorentz-invariant vacuum. The physical procedure to regulate this result would be to put the theory in a finite box. Instead of solving the integral over k, one replaces the integral with a discrete sum: $\int \frac{dk}{2\pi} \to \frac{1}{L}\sum_{n=-\infty}^{\infty}$. However, usually sums are pretty hard to solve. Doing the sum would give a divergence as $\lambda_1 \ln(L)$, regular terms, and terms of order $O\left(\frac{1}{L}\right)$. One could also introduce an IR momentum cut-off δ. Then one would obtain a divergence as $\lambda_2 \ln(\delta)$. Then, one finds that $\lambda_1 = -\lambda_2$ if one sets $L = \frac{2\pi}{\delta}$. Thus, these two ways of regulating the result give the same answer up to terms of order $O\left(\frac{1}{L}\right)$. Therefore one introduces an IR momentum cut-off delta here.

Realising that in these coordinates our integral has the same form as an upper incomplete gamma function

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}, \tag{181}$$

where one introduces a small imaginary part, $i\epsilon$ to ensure convergence. This yields:

$$\lim_{\delta \to 0} \int_{\delta}^{\infty} \frac{dk}{4\pi k} e^{-ik(u-u'-i\epsilon)} = \frac{1}{4\pi} \lim_{\delta \to 0} \Gamma(0, i(u-u'-i\epsilon)\delta).$$

In this limit, the gamma function can be expanded to give a more explicit form in terms of δ and yields:

$$\lim_{\delta \to 0} \Gamma(0, i(u-u'-i\epsilon)\delta) = \lim_{\delta \to 0} (-\gamma_E - \ln(\delta) - \ln[i(u-u'-i\epsilon)]) \tag{182}$$

where γ_E the Euler-Mascheroni constant. Also adding the left moving part yields:

$$\begin{aligned} \lim_{\delta \to 0} \int_{\delta}^{\infty} \frac{dk}{4\pi k} e^{-ik(v-v'-i\epsilon')} &= \frac{1}{4\pi} \lim_{\delta \to 0} \Gamma(0, i(v-v'-i\epsilon')\delta) \\ &= \frac{1}{4\pi} \lim_{\delta \to 0} (-\gamma_E - \ln(\delta) - \ln[i(v-v'-i\epsilon')]). \end{aligned} \tag{183}$$

Adding up Equations 182, 183 now yields the final expression for the two-point function. This result diverges as the IR momentum cut-off goes to 0. One adds a scalar phi^2 condensate depending on μ to absorb this infinity. An Unruh detector will know about the infinity of the condensate. For the reader unfamiliar with the concept of an Unruh detector, there is an explanation and an example of this concept in Section 6. Furthermore one has to consider the principal sheet to get a unique result for the complex logarithm[1]. The Euler-Mascheroni constant is absorbed in the condensate. This condensate ensures that all the physical observables do not have a μ dependence. Thus the expression becomes:

$$\begin{aligned} &\langle 0_R | \hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau') |0_R\rangle \\ &\quad = -\frac{1}{4\pi}\ln\left(\mu i(u-u'-i\epsilon)\right) - \frac{1}{4\pi}\ln\left(\mu i(v-v'-i\epsilon)\right) + \phi^2(\mu) \\ &\quad = -\frac{1}{4\pi}\ln\left(\mu i[\Delta\tau - i\epsilon - \Delta\xi]\right) - \frac{1}{4\pi}\ln\left(\mu i[\Delta\tau - i\epsilon + \Delta\xi]\right) + \phi^2(\mu) \\ &\quad = -\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau - i\epsilon)^2 + \Delta\xi^2\right]\right) + \phi^2(\mu). \end{aligned} \tag{184}$$

Here, and in the rest of this thesis, $\Delta\tau = \tau - \tau'$ and $\Delta\xi = \xi - \xi'$. Here the condensate is given by $\phi^2(\mu) = \frac{1}{4\pi}\left[-2\gamma_{\mathrm{E}} - \ln\left(\frac{\delta^2}{\mu^2}\right)\right]$. Just as in the previous Section, it is clear that this solution is independent of μ. Namely:

$$\mu\partial_\mu \langle 0_R | \hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau') |0_R\rangle = \frac{-2\mu}{4\pi\mu} + \frac{2\mu}{4\pi\mu} = 0. \tag{185}$$

One can also see from Equation 184 that this solution does not respect all the Rindler isometries. The explicit form of the Rindler Killing vectors is given in Appendix E. This will become more apparent when one analyses the massive Rindler field. This result can be coordinate transformed to Minkowski coordinates

[1] One uses the principal sheet to get an expression for the complex part of the two-point function. For this one takes the complex logarithm $\ln(z) = \ln(|z|) + i\mathrm{Arg}(z)$ with its argument taking values in $(-\pi, \pi)$, with a branch cut on the negative axis. The argument of z will be π for values $z = -a + i\epsilon$ and $-\pi$ for values $z = a - i\epsilon$ for $a > 0$ and $\epsilon > 0$. The calculation then goes as:

$$\begin{aligned} \langle 0_R | \hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau') |0_R\rangle &= -\frac{1}{4\pi}\ln\left(\mu i(u-u'-i\epsilon)\right) - \frac{1}{4\pi}\ln\left(\mu i(v-v'-i\epsilon)\right) + \phi^2(\mu) \\ = -\frac{1}{4\pi}\ln\left(\mu|u-u'|\right) - \frac{i\theta(u-u')}{8} + \frac{i\theta(u'-u)}{8} &- \frac{1}{4\pi}\ln\left(\mu|v-v'|\right) - \frac{i\theta(v-v')}{8} + \frac{i\theta(v'-v)}{8} + \phi^2(\mu) \\ = -\frac{1}{4\pi}\ln\left(\mu^2|u-u'||v-v'|\right) - \frac{i}{8}&\left[\theta(u-u') - \theta(u'-u) + \theta(v-v') - \theta(v'-v)\right] + \phi^2(\mu) \\ = -\frac{1}{4\pi}\ln\left(\mu^2|\Delta\tau^2 - \Delta\xi^2|\right) - \frac{i}{8}&\left[\theta(\Delta\tau - \Delta\xi) - \theta(\Delta\xi - \Delta\tau) + \theta(\Delta\tau + \Delta\xi) - \theta(-\Delta\tau - \Delta\xi)\right] + \phi^2(\mu). \end{aligned}$$

It is not obvious from this expression, but one will see later that this only takes non vanishing values inside the light cone. This shows that our 2D theory is strongly interacting, because normally this would only be non-zero exactly on the light cone.

using the inverse of Equation 178. This yields:

$$\begin{aligned}&\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle\\&=-\frac{1}{4\pi}\ln\left[\mu^2(u-u'-i\epsilon)(v-v'-i\epsilon)\right]+\phi^2(\mu)\\&=-\frac{1}{4\pi}\ln\left[\frac{\mu^2}{g^2}\left(\ln\left(\frac{\bar{u}'}{\bar{u}}\right)+ig\epsilon\right)\left(\ln\left(\frac{\bar{v}}{\bar{v}'}\right)-ig\epsilon\right)\right]+\phi^2(\mu)\\&=-\frac{1}{4\pi}\ln\left[\frac{\mu^2}{g^2}\left(\ln\left(\frac{t-x}{t'-x'}\right)+ig\epsilon\right)\left(\ln\left(\frac{t+x}{t'+x'}\right)-ig\epsilon\right)\right]+\phi^2(\mu).\end{aligned}\tag{186}$$

The principal sheet can again be used to determine the complex part of this expression[2].

3.7 The Feynman propagator and the Wightman functions in the naive Rindler vacuum

In subsection 3.6 the two-point function was calculated, Equation 184. In this section one uses this to write down the Feynman propagator for the naive Rindler vacuum. Moreover, the spectral function, the statistical function, the advanced propagator, and the retarded propagator are determined for the naive Rindler vacuum.

Firstly, the Feynman propagator is defined as [57]:

$$\begin{aligned}i\Delta_{\mathrm{F}}(\tau,\xi;\tau',\xi')&=\langle T[\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')]\rangle\\&=\theta(\tau-\tau')\,\langle\phi(\xi,\tau)\phi(\xi',\tau')\rangle+\theta(\tau'-\tau)\,\langle\phi(\xi',\tau')\phi(\xi,\tau)\rangle\\&=\theta(\tau-\tau')i\Delta^+(\tau,\xi;\tau',\xi')+\theta(\tau'-\tau)i\Delta^-(\tau,\xi;\tau',\xi').\end{aligned}\tag{187}$$

The $i\Delta^\pm$ are called the Wightman functions. The Wightman functions have to satisfy $[i\Delta^\pm]^*=i\Delta^\mp$. In Equation 184:

$$\langle 0_R|\,\hat{\phi}(\xi,\tau)\hat{\phi}(\xi',\tau')\,|0_R\rangle=-\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau-i\epsilon)^2+\Delta\xi^2\right]\right)+\phi^2(\mu).\tag{188}$$

One can see that an exchange of ξ,τ with ξ',τ' is the same as taking a complex conjugate. Thus the complex conjugation condition is satisfied. From this one can easily write down the Feynman propagator to be:

$$\begin{aligned}i\Delta_{\mathrm{F}}(\xi,\tau;\xi',\tau')&=-\frac{1}{4\pi}\ln\left(\mu^2\left[-\left(|\Delta\tau|-i\epsilon\right)^2+\Delta\xi^2\right]\right)+\phi^2(\mu)\\&=-\frac{1}{4\pi}\ln\left(\mu^2\Delta X_{++}^2\right)+\phi^2(\mu),\end{aligned}\tag{189}$$

where $\Delta X_{++}^2=-\left(|\Delta\tau|-i\epsilon\right)^2+\Delta\xi^2$. The spectral function is defined as the difference between the two Wightman functions. The spectral function then describes the causal behaviour of the system. The spectral function records which states are available in the system, but does not tell which states are occupied. From the definition of the spectral function it is clear that it equals two times the complex part of the two-point function. The footnotes on the previous page calculated the complex part of the two-point functions in terms of θ-functions. Thus the spectral function is given by:

$$\begin{aligned}i\Delta^c(\tau,\xi,\tau',\xi')&=i\Delta^+(\xi,\tau;\xi',\tau')-i\Delta^-(\xi,\tau;\xi',\tau')\\&=-\frac{i}{4}\left[\theta(\Delta\tau-\Delta\xi)-\theta(\Delta\xi-\Delta\tau)+\theta(\Delta\tau+\Delta\xi)-\theta(-\Delta\tau-\Delta\xi)\right].\end{aligned}\tag{190}$$

[2] Transforming the θ-functions from the previous expression one obtains:

$$\theta(u-u')=\theta\left(\frac{1}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)\right)=\theta(\bar{u}'-\bar{u})=\theta(\Delta x-\Delta t).$$

In the second step there is used that the logarithm is a strictly increasing function. This can be done in the same way for all θ-functions. Thus one gets the same form of θ-functions in Minkowski coordinates, thus one again concludes that the result is only non-vanishing inside the light cone.

It seems that the spectral function has support outside of the light cone, which would be non-physical. However, one can show that this is equal to, see Appendix F:

$$i\Delta^c(\tau,\xi,\tau',\xi') = \frac{i}{2}\left[\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2) - \theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)\right]. \tag{191}$$

From this form it is very clear that the spectral function has support only inside the light cone. This is still quite a difference with a 4-dimensional theory where the spectral function only has non-zero values exactly on the light cone. Therefore, 2-dimensional theories are strongly interacting, contrary to 4-dimensional theories. From the spectral function one obtains the retarded and the advanced propagators as:

$$i\Delta^r(\tau,\xi,\tau',\xi') = -\theta(\Delta\tau)i\Delta^c(\tau,\xi,\tau',\xi'), \tag{192}$$
$$i\Delta^a(\tau,\xi,\tau',\xi') = \theta(-\Delta\tau)i\Delta^c(\tau,\xi,\tau',\xi'). \tag{193}$$

Moreover, one can define the statistical function (or Hadamard) as [57]:

$$\begin{aligned} F(\tau,\xi;\tau',\xi') &= \frac{1}{2}(i\Delta^+(\tau,\xi;\tau',\xi') + i\Delta^-(\tau,\xi;\tau',\xi')) \\ &= -\frac{1}{4\pi}\ln\left|\mu^2(|\Delta\tau|^2-\Delta\xi^2)\right| + \phi^2(\mu). \end{aligned} \tag{194}$$

The statistical function tells one how the states are populated and is later used to calculate the entropy of the system. From the definition one also sees that the statistical function encapsulates the real part of the two-point function, because the positive and the negative Wightman functions are each other's complex conjugate.

3.8 Canonical quantisation of a massive real scalar field in Rindler space

The canonical quantisation of a massive real scalar field in Rindler yields quite some problems, therefore one takes a shortcut. In a maximally symmetric space, there exists a special state that is invariant under all Rindler isometries. This means that there exists a state for which the two-point function only depends on the invariant geodesic distance. An example where they use this in de Sitter space is given in [64]. The equation of motion for the propagator then reduces to an ordinary differential equation and can then be solved.

A flat D-dimensional space-time can be embedded in a D-dimensional space, which is not generally true for curved space-times. Thus one can take D-dimensional Minkowski as the embedding space and calculate the invariant distance of the embedding space. The invariant geodesic distance in Rindler space, should then just be the coordinate transformed invariant distance from Minkowski space. The invariant distance in Minkowski space can be calculated to be:

$$l(x_1,x_2,t_1,t_2) = \int_{x_1,t_1}^{x_2,t_2}\sqrt{-dt^2+dx^2} = \int_{t_1}^{t_2} dt\sqrt{-1+\left(\frac{dx}{dt}\right)^2}. \tag{195}$$

Applying the Euler-Lagrange equation yields:

$$\frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\sqrt{-1+\left(\frac{dx}{dt}\right)^2}}\right) = 0. \tag{196}$$

This equation can be integrated and then $\frac{dx}{dt}$ can be isolated. This yields $\frac{dx}{dt}$ = constant. Applying the initial conditions to this, one finds that $x(t) = \frac{x_1-x_2}{t_1-t_2}(t-t_2)+x_2$. Inserting this back into 195 gives one the invariant distance expressed in Minkowski coordinates:

$$\begin{aligned} l(x_1,x_2,t_1,t_2) &= \int_{t_1}^{t_2} dt\sqrt{-1+\left(\frac{x_1-x_2}{t_1-t_2}\right)^2} \\ &= (t_2-t_1)\sqrt{-1+\left(\frac{x_1-x_2}{t_1-t_2}\right)^2} \\ &= \sqrt{-(t_1-t_2)^2+(x_1-x_2)^2}. \end{aligned} \tag{197}$$

Transforming l^2 to Rindler coordinates using 75 gives the following expression for the squared invariant geodesic distance:

$$\begin{aligned} l^2(x_1,x_2,t_1,t_2) &= -(t_1-t_2)^2+(x_1-x_2)^2 \\ &= -\left(\frac{e^{g\xi_1}}{g}\sinh(g\tau_1)-\frac{e^{g\xi_2}}{g}\sinh(g\tau_1)\right)^2+\left(\frac{e^{g\xi_1}}{g}\cosh(g\tau_1)-\frac{e^{g\xi_2}}{g}\cosh(g\tau_1)\right)^2 \\ &= \frac{1}{g^2}\left[e^{2g\xi_1}+e^{2g\xi_2}-2e^{g(\xi_1+\xi_2)}\cosh(g(\tau_1-\tau_2))\right] \\ &= r(\xi_1,\xi_2,\tau_1,\tau_2). \end{aligned} \tag{198}$$

This $r(\xi_1,\xi_2,\tau_1,\tau_2)$ describes the invariant distance squared in Rindler space. Now, one assumes the propagator to be a function of this invariant distance $i\Delta(\xi_1,\xi_2,\tau_1,\tau_2)=i\Delta(r)$, while letting it satisfy the homogeneous differential equation for the propagator:

$$(e^{-2g\xi}(\partial_\xi^2-\partial_\tau^2)-m^2)i\triangle(\xi,\xi';\tau,\tau')=0\,. \tag{199}$$

Later, one imposes an $i\epsilon$ prescription and a separation in positive and negative frequency Wightman functions such that one gets a two-dimensional delta function on the right-hand side of this equation instead. Transforming this equation to an equation that only depends on the invariant distance:

$$\partial_\tau=\frac{\partial r}{\partial\tau}\frac{\partial}{\partial r}=-\frac{2}{g}e^{g(\xi+\xi')}\sinh\left(g(\tau-\tau')\right)\frac{\partial}{\partial r}, \tag{200}$$

$$\partial_\xi=\frac{\partial r}{\partial\xi}\frac{\partial}{\partial r}=\left[\frac{2}{g}e^{2g\xi}-\frac{2}{g}e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right]\frac{\partial}{\partial r}. \tag{201}$$

These first order derivatives lead to the following second order derivatives:

$$\begin{aligned} \partial_\tau^2 &= \frac{\partial}{\partial\tau}\left[-\frac{2}{g}e^{g(\xi+\xi')}\sinh\left(g(\tau-\tau')\right)\frac{\partial}{\partial r}\right] \\ &= -2e^{2g(\xi+\xi')}\cosh(g(\tau-\tau'))\frac{\partial}{\partial r}+\frac{4}{g^2}e^{2g(\xi+\xi')}\sinh^2(g(\tau-\tau'))\frac{\partial^2}{\partial r^2} \end{aligned} \tag{202}$$

$$\begin{aligned} \partial_\xi^2 &= \frac{\partial}{\partial\xi}\left[\left(\frac{2}{g}e^{2g\xi}-\frac{2}{g}e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right)\frac{\partial}{\partial r}\right] \\ &= \left[4e^{2g\xi}-2e^{g(\xi+\xi')}\right]\frac{\partial}{\partial r}+\left[\frac{2}{g}e^{2g\xi}-\frac{2}{g}e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right]^2\frac{\partial^2}{\partial r^2}. \end{aligned} \tag{203}$$

This does not look to promising yet, however when considering the whole differential operator, this simplifies neatly:

$$\begin{aligned} e^{-2g\xi}(\partial_\xi^2-\partial_\tau^2) &= e^{-2g\xi}\Bigg(\left[2e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))+4e^{2g\xi}-2e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right]\frac{\partial}{\partial r} \\ &\quad+\Bigg[-\frac{4}{g^2}e^{2g(\xi+\xi')}\sinh^2(g(\tau-\tau'))+\frac{4}{g^2}e^{4g\xi}+\frac{4}{g^2}e^{2g(\xi+\xi')}\sinh^2(g(\tau-\tau')) \\ &\quad-\frac{8}{g^2}e^{g(3\xi+\xi')}\cosh(g(\tau-\tau'))\Bigg]\frac{\partial^2}{\partial r^2}\Bigg) \\ &= 4\frac{\partial}{\partial r}+\left[\frac{4}{g^2}e^{2g\xi}-\frac{8}{g^2}e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right]\frac{\partial^2}{\partial r^2} \\ &= 4\frac{\partial}{\partial r}+4r\frac{\partial^2}{\partial r^2}. \end{aligned} \tag{204}$$

Since all the dependence on τ and ξ is gone, this is a good sign that this is indeed the correct invariant geodesic distance. Thus the new differential equation that the propagator has to satisfy is:

$$\left(4\frac{\partial}{\partial r}+4r\frac{\partial^2}{\partial r^2}-m^2\right)i\Delta(r)=0. \tag{205}$$

This equation can be written in the Bessel differential equation by making the substitution $s=m\sqrt{r}$. Here s is the dimensionless invariant distance. This substitution yields

$$\frac{d}{dr}=\frac{ds}{dr}\frac{d}{ds}=\frac{m}{2\sqrt{r}}\frac{d}{ds}=\frac{m^2}{2s}\frac{d}{ds}. \tag{206}$$

This changes the differential equation into:

$$\left(4\frac{d}{dr}+4r\frac{d^2}{dr^2}-m^2\right)i\Delta(r)=\left(4\frac{m^2}{2s}\frac{d}{ds}+4\frac{s^2}{m^2}\left(\frac{m^4}{4s^2}\frac{d^2}{ds^2}-\frac{m^4}{4s^3}\frac{d}{ds}\right)-m^2\right)i\Delta(s)=0. \tag{207}$$

Simplifying these terms, then exactly yields the Bessel equation:

$$\left(d_s^2+\frac{1}{s}d_s-1\right)i\Delta(s)=0. \tag{208}$$

The solutions to this equation are given by the modified Bessel functions of the first and the second kind, $I_0(s), K_0(s)$. Thus, the result of our propagator is given by:

$$i\Delta(m\sqrt{r})=AI_0(m\sqrt{r})+BK_0(m\sqrt{r}). \tag{209}$$

Now one still needs to apply boundary conditions in order to arrive at a solution without integration constants. First of all, one notices that A incorporates non-physical correlations as the Bessel I_0 function goes to infinity as the invariant distance goes to infinity. Even though, correlations could be found at large distances, these are a result of the field being highly excited. There are no signals travelling to places outside the light cone and therefore one can set $A=0$. Secondly, one needs to impose an $i\epsilon$ prescription. This does not appear via this method as one skips doing all the integrals. The $i\epsilon$-prescription is uniquely determined by the δ-function source on the right hands side of the propagator equation, Equation 111, by the form of the Feynman propagator in terms of the Wightman functions, and by using the fact that the positive and negative frequency Wightman functions are the homogeneous solutions to the propagator equation. One imposes the $i\epsilon$ prescription by subtracting the $i\epsilon$ from $\Delta\tau$. One verifies later that this is the correct $i\epsilon$ prescription. Lastly, to determine the constant B one requires the mass to 0 limit of this expression to be equal to the massless Minkowski limit of Equation 58 from Section 2. For small arguments, one can expand the Bessel function as:

$$K_0(z)=-\gamma_{\rm E}-\ln(\frac{z}{2})+O(z).$$

As the argument of the Bessel function goes to zero as the mass goes to 0, one obtains (including the $i\epsilon$ prescription):

$$\begin{aligned}
&K_0\left(\frac{m}{g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{\cosh(g\Delta\xi)-\cosh(g(\Delta\tau-i\epsilon))}\right)\\
&=-\gamma_{\rm E}-\ln(m)-\ln\left(\frac{1}{2g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{\cosh(g\Delta\xi)-\cosh(g(\Delta\tau-i\epsilon))}\right)\\
&=-\gamma_{\rm E}-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{1}{2}\ln\left(\frac{\mu^2}{2g^2}e^{g(\xi+\xi')}\left[\cosh(g\Delta\xi)-\cosh(g(\Delta\tau-i\epsilon))\right]\right)\\
&=-\gamma_{\rm E}-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{1}{2}\ln\left(\mu^2(\Delta X_{i\epsilon})^2\right),
\end{aligned} \tag{210}$$

where in the last step one recognised the invariant distance squared $(\Delta X)^2=\frac{1}{2g^2}e^{g(\xi+\xi')}\left[\cosh(g\Delta\xi)-\cosh(g\Delta\tau-ig\epsilon)\right]$ from Equation 198 up to the imposed $i\epsilon$ prescription. Thus one knows that this is equal to the massless Minkowski limit, Equation 58, if $B=\frac{1}{2\pi}$. Thus one knows that $B=\frac{1}{2\pi}$.

One obtains the Feynman propagator from this by changing $\Delta\tau \to |\Delta\tau|$. This encapsulates how the positive and negative frequency Wightman functions, which are each other's complex conjugate, add up.

$$\begin{aligned} i\Delta_{F,\mathrm{RI},m}(m\sqrt{r}) &= \frac{1}{2\pi}K_0\left(\frac{m}{g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{\cosh(g\Delta\xi)-\cosh(g|\Delta\tau|-ig\epsilon)}\right) \\ &= \frac{1}{2\pi}\left\{\theta(\Delta\tau)K_0\left(\frac{m}{g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{\cosh(g\Delta\xi)-\cosh(g\Delta\tau-ig\epsilon)}\right)\right. \\ &\left.+\theta(-\Delta\tau)K_0\left(\frac{m}{g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{\cosh(g\Delta\xi)-\cosh(g\Delta\tau+ig\epsilon)}\right)\right\} \\ &= \theta(\Delta\tau)i\Delta^+_m(\tau,\xi.\tau',\xi')+\theta(-\Delta\tau)i\Delta^-_m(\tau',\xi') \\ &= \theta(\Delta\tau)\langle 0_{\mathrm{RI}}|\,\hat\phi_m(\tau,\xi)\hat\phi_m(\tau',\xi')\,|0_{\mathrm{RI}}\rangle+\theta(-\Delta\tau)\langle 0_{\mathrm{RI}}|\,\hat\phi_m(\tau',\xi')\hat\phi_m(\tau,\xi)\,|0_{\mathrm{RI}}\rangle\,, \end{aligned} \tag{211}$$

where $\hat\phi_m$ is the quantised field for a massive scalar field and the subscript RI is used to denote the invariant Rindler vacuum. Now one can read off the positive and negative Wightman for the massive scalar field function. One of course needs to have that the advanced propagator is complex conjugate of the retarded propagator, which is automatically satisfied in this way. Then one wants to compare the massless two-point function to the massless limit of the massive two-point function, in other words, Equation 211 with Equation 189:

$$i\Delta_{\mathrm{F}}(\xi,\tau;\xi',\tau') = -\frac{1}{4\pi}\ln\left(\mu^2\Delta X^2_{++}\right)+\phi^2(\mu)$$

One has to work out the imaginary part of the massless limit of the massive Feynman propagator using the principal sheet of the complex logarithm. One starts with the $i\epsilon$ prescription from the positive frequency Wightman function:

$$\cosh(g(\Delta\tau-i\epsilon)) = \frac{e^{g\Delta\tau}(1-i\epsilon g)+e^{-g\Delta\tau}(1+i\epsilon g)}{2} = \cosh(g\Delta\tau)-i\epsilon g\sinh(g\Delta\tau)$$

yields:

$$\begin{aligned} &\lim_{m\to 0}\langle 0_{\mathrm{RI}}|\,\hat\phi_m(\xi,\tau)\hat\phi_m(\xi',\tau')\,|0_{\mathrm{RI}}\rangle = \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}}-\ln(m)-\frac{1}{2}\ln\left(\frac{e^{g(\xi+\xi')}}{2g^2}\right)\right. \\ &\left.-\frac{1}{2}\ln\left[\cosh(g\Delta\xi)-\cosh(g\Delta\tau)+i\epsilon g\sinh(g\Delta\tau)\right]\right] \\ &= \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}}-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{g}{2}(\xi+\xi')+\frac{1}{2}\ln\left(\frac{\mu^2}{g^2}\right)-\frac{1}{2}\ln\left(|\cosh(g\Delta\xi)-\cosh(g\Delta\tau)|\right)\right. \\ &\left.-\frac{i\pi}{2}\left[\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)-\theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)\right]\right]. \end{aligned} \tag{212}$$

Where in the last step, one uses the fact that the cosh-function is a symmetric increasing function of its argument therefore one can transform the theta functions to the difference of the squares. From this one can already calculate the spectral function for the Rindler invariant vacuum state. This spectral function is only non-zero inside the light-cone with a value of $\pm\frac{i}{2}$ by:

$$i\Delta^c_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi') = -\frac{i}{2}\left[\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)-\theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)\right]. \tag{213}$$

The spectral function for this state does not resemble the spectral function of the massless case: Equation 190:

$$i\Delta^c(\tau,\xi;\tau',\xi') = -\frac{i}{4}\left[\theta(\Delta\tau-\Delta\xi)-\theta(\Delta\xi-\Delta\tau)+\theta(\Delta\tau+\Delta\xi)-\theta(-\Delta\tau-\Delta\xi)\right]. \tag{214}$$

However it can be shown that these combinations of θ-functions are the same. This is done in Appendix F. The statistical function of the Rindler invariant vacuum is given by:

$$F_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi') = \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}}-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{g}{2}(\xi+\xi')+\frac{1}{2}\ln\left(\frac{\mu^2}{g^2}\right)-\frac{1}{2}\ln\left(|\cosh(g\Delta\xi)-\cosh(g\Delta\tau)|\right)\right]. \tag{215}$$

In Appendix F it is shown that in the limit where the coordinates are small, these statistical functions are equal. This is not particularly surprising, as for small values of the coordinates, everything will be in close approximation to the light cone. Close to the light cone, as $m^2\Delta X^2\|1$, one sees that the Rindler invariant distance reduces to $-\Delta\tau^2+\Delta\xi^2$. Therefore, one expects that the two solitions coincide close enough to the light cone. From this and the fact that $i\Delta^+=[i\Delta^-]^*$, one calculates the Feynman propagator to be:

$$\lim_{m\to 0} i\Delta^{\mathrm{F}}_{\mathrm{RI},m\to 0}=\frac{1}{2\pi}\left[-\gamma_{\mathrm{E}}-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)-\frac{g}{2}(\xi+\xi')+\frac{1}{2}\ln\left(\frac{\mu^2}{g^2}\right)-\frac{1}{2}\ln\left(|\cosh(g\Delta\xi)-\cosh(g\Delta\tau)|\right)\right.$$
$$\left.-\frac{i\pi}{2}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\theta(\Delta\tau^2-\Delta\xi^2)\right]. \tag{216}$$

One easily sees from this that the Feynman propagator has no imaginary part in the coincident limit, the limit where $t\to t'$. By comparing Equation 189 with Equation 216, one sees that the Feynman propagators for the different states are not equal, even though their contribution to the energy momentum tensor are both 0, as will be seen later. This raises the question how large the space of all states is in Rindler space that yield a vanishing contribution to the energy momentum tensor. In other words, how degenerate is the vacuum state in Rindler space. One comes back to this question later. One first has to deal with the formally divergent two-point function for the Rindler invariant vacuum. In the vacuum that was picked, the limit $m\to 0$ is singular. So one needs to regulate this vacuum. This is an important difference between the two states. In the naive Rindler vacuum one can remove the IR modes by introducing an IR momentum cut-off δ. This is not possible in the Rindler invariant vacuum.

Introducing a condensate in Equation 216:

$$\lim_{m\to 0} i\Delta^{\mathrm{F}}_{\mathrm{RI},m\to 0}=-\frac{1}{4\pi}\ln\left(\mu^2\left[\cosh(g\Delta\xi)-\cosh(g|\Delta\tau|-ig\epsilon)\right]\right)+\phi^2(\xi,\xi',\mu). \tag{217}$$

where the condensate is given by:

$$\phi^2(\xi,\xi',\mu)=\frac{1}{2\pi}\left[-\gamma_{\mathrm{E}}-\ln\left(\frac{m^2}{\mu^2}\right)-\frac{g}{2}\left(\xi+\xi'\right)-\ln\left(\sqrt{2g}\right)\right] \tag{218}$$

This condensate is spatially dependent. This is an interesting result. Moreover it seems to be impossible to write the condensate as a scalar condensate. As already remarked before, this condensate does not contribute to the energy momentum tensor because of its linear dependence on ξ and ξ'. Furthermore, one sees that this condensate does not depend on the time coordinates and does therefore not contribute to the entropy as will be seen in Section 6. Further remarks and conclusions will be made in the discussion, Section 7. In addition, one could add homogeneous solutions to the massless limit of the massive Feynman propagator in Equation 217. There are many possible homogeneous solutions to the massless equation. For example one could subtract the massless solution from this solution and one would get the ratio of the invariant distance and the naive distance in Rindler space. This specific choice would cancel the Hadamard singularity.

Lastly, one calculates the retarded and advanced propagators. From the spectral function in Equation 213 one can immediately write down the advanced and the retarded propagator, which only depend on the part coming from the $i\epsilon$ prescription and not from the homogeneous solution. The advanced and retarded propagator for the massless limit of the massive case are given by:

$$\lim_{m\to 0} i\Delta^{r}_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi')=\frac{i}{2}\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2) \tag{219}$$
$$\lim_{m\to 0} i\Delta^{a}_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi')=\frac{i}{2}\theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2). \tag{220}$$

4 Quantum back reaction

As was seen in Section 3, the massless propagator and the massless limit of the massive propagator are not equal, therefore it appears that the Rindler vacuum is not unique. To see what the true vacuum is, one needs to look at the state with the lowest energy. To solve this one can calculate the energy momentum tensor of both vacua. Furthermore, one calculates the energy momentum tensor to see whether back reaction becomes applicable in this thesis. In the introduction it was explained how the energy momentum tensor looked diagrammatically, right part of Figure 5. From this one concluded that up to tadpole order the only contributions, $\langle T^{-}_{\mu\nu}\rangle$ and $\langle T^{+}_{\mu\nu}\rangle$, were each other's complex conjugates. Thus one only needed to calculate one of them, and one could ignore the imaginary parts. As a side remark, in this specific case the $\langle T^{+}_{\mu\nu}\rangle$ and the $\langle T^{-}_{\mu\nu}\rangle$ have an imaginary part that already drops out in the limit where $x \to x'$ for both contributions separately. For the interested reader, this is shown in Appendix G.

4.1 Energy momentum tensor for the naive Rindler vacuum

For the massless case in curved spacetime one has, as already seen from Equation 63:

$$\begin{aligned}\langle T^{+}_{\mu\nu}\rangle &= \lim_{x\to x'}\left[\partial_\mu\partial'_\nu \langle T[\hat\phi(x)\hat\phi(x')]\rangle - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\partial'_\beta \langle T[\hat\phi(x)\hat\phi(x')]\rangle\right]\\ &= \lim_{x\to x'}\left[\partial_\mu\partial'_\nu \langle T[\hat\phi(x)\hat\phi(x')]\rangle - \frac{1}{2}g_{\mu\nu}e^{-2g\xi}\left(-\partial_\tau\partial'_\tau + \partial_\xi\partial'_\xi\right)\langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle\right].\end{aligned} \tag{221}$$

When one is only calculating $\langle T^{+}_{\mu\nu}\rangle$, only the Feynman propagator is relevant. Recalling from Equation 189, that the Feynman propagator of the naive Rindler vacuum was given by:

$$\begin{aligned} i\Delta_{\mathrm{F}}(\xi,\tau;\xi',\tau') &= \theta(\Delta\tau)\left[-\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau - i\epsilon)^2 + \Delta\xi^2\right]\right) + \phi^2(\mu)\right]\\ &\quad + \theta(-\Delta\tau)\left[-\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau + i\epsilon)^2 + \Delta\xi^2\right]\right) + \phi^2(\mu)\right]\\ &= \frac{-1}{4\pi}\ln\left(\mu^2\left[-(|\Delta\tau| - i\epsilon)^2 + \Delta\xi^2\right]\right) + \phi^2(\mu)\end{aligned} \tag{222}$$

As argued, one can ignore the imaginary part. Ignoring the $i\epsilon$ removes the absolute value signs around $\Delta\tau$. The first order derivatives to the primed coordinates then yield:

$$\partial'_\tau \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = \frac{1}{2\pi}\frac{\Delta\tau}{\Delta\tau^2 - \Delta\xi^2}, \tag{223}$$

$$\partial'_\xi \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = \frac{1}{2\pi}\frac{-\Delta\xi}{\Delta\tau^2 - \Delta\xi^2}. \tag{224}$$

Taking another derivative with respect to the non-primed coordinates yields 4 options:

$$\partial_\tau\partial'_\tau \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = -\frac{1}{2\pi}\frac{\Delta\tau^2 + \Delta\xi^2}{(\Delta\tau^2 - \Delta\xi^2)^2} \tag{225}$$

$$\partial_\xi\partial'_\tau \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = \frac{1}{\pi}\frac{\Delta\xi\Delta\tau}{(\Delta\tau^2 - \Delta\xi^2)^2} \tag{226}$$

$$\partial_\xi\partial'_\xi \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = -\frac{1}{2\pi}\frac{\Delta\tau^2 + \Delta\xi^2}{(\Delta\tau^2 - \Delta\xi^2)^2} \tag{227}$$

$$\partial_\tau\partial'_\xi \langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\rangle = \frac{1}{\pi}\frac{\Delta\xi\Delta\tau}{(\Delta\tau^2 - \Delta\xi^2)^2}. \tag{228}$$

Plugging these results back into Equation 221 then gives the following for the energy momentum tensor:

$$\begin{aligned}\langle T^{+}_{\tau\tau}\rangle &= \lim_{x\to x'}\left[\frac{1}{2}\partial_\tau\partial'_\tau\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle+\frac{1}{2}\partial_\xi\partial'_\xi\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle\right]\\ &= \lim_{x\to x'}\left[-\frac{1}{2\pi}\frac{\Delta\tau^2+\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2}\right],\end{aligned} \tag{229}$$

$$\begin{aligned}\langle T^{+}_{\xi\xi}\rangle &= \lim_{x\to x'}\left[\frac{1}{2}\partial_\tau\partial'_\tau\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle+\frac{1}{2}\partial_\xi\partial'_\xi\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle\right]\\ &= \lim_{x\to x'}\left[-\frac{1}{2\pi}\frac{\Delta\tau^2+\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2}\right],\end{aligned} \tag{230}$$

$$\begin{aligned}\langle T^{+}_{\xi\tau}\rangle = \langle T^{+}_{\tau\xi}\rangle = \lim_{x\to x'}\left[\partial_\tau\partial'_\xi\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle\right] &= \lim_{x\to x'}\left[\partial_\xi\partial'_\tau\left\langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\right\rangle\right]\\ &= \lim_{x\to x'}\left[\frac{1}{\pi}\frac{\Delta\xi\Delta\tau}{(\Delta\tau^2-\Delta\xi^2)^2}\right].\end{aligned} \tag{231}$$

This obviously a diverging energy momentum tensor when the limit is taken of x going to x', except for the off diagonal terms. The off diagonal terms depend on how one takes the limits. Thus one should consider the renormalisation of this quantity. Before that one could still consider what happens in dimensional regularization in the 2-dimensional case. The 2-dimensional massless case was calculated in Section 2. One is allowed to use this result in the massless case, because the Rindler fields and the Minkowski fields are the same in the massless case. However the metric of course changes to the Rindler metric One has seen in Equation 68 that the D-dimensional energy momentum tensor was given by:

$$\langle T^{+}_{\mu\nu}\rangle = g_{\mu\nu}\frac{m^D}{(4\pi)^{\frac{D}{2}}}\Gamma(\frac{-D}{2}). \tag{232}$$

Using dimensional regularisation and analytically extending D the same way as in Section 2, one obtains:

$$\langle T^{+}_{\mu\nu}\rangle = g_{\mu\nu}\frac{m^2}{4\pi}\left[\frac{\mu^{D-2}}{D-2}+\frac{1}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_{\mathrm{E}}-1\right]. \tag{233}$$

Then, in the minimal subtraction scheme, one has to add a cosmological constant term to the action:

$$\Delta S = -\int d^2x\sqrt{-g}\Lambda_{\mathrm{2DRindler}} \tag{234}$$

$$\Lambda_{\mathrm{2DRindler}} = \frac{m^2}{4\pi}\frac{\mu^{D-2}}{D-2}. \tag{235}$$

The renormalised version of the + energy momentum tensor is thus given by:

$$\langle T^{+}_{\mu\nu}\rangle_{\mathrm{ren}} = g_{\mu\nu}\frac{m^2}{4\pi}\left[\frac{1}{2}\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_{\mathrm{E}}-1\right] \tag{236}$$

In the massless case this just becomes 0. One wants to write the expressions from Equations 229 and 230, in a form that only depends on the invariant distance $\sigma = \sqrt{\frac{2}{g^2}e^{g(\xi+\xi')}\left[\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right]}$. If this is the case, one can apply point splitting regularisation and subtract a σ dependent counter term such that all divergences coming from x going to x' are cancelled [65]. The infinitesimal distance limit of the invariant distance is given by: $(d\sigma)^2 = e^{g(\xi+\xi')}\left[\Delta\xi^2-\Delta\tau^2\right]$ To determine what kind of counter terms one needs, one first writes:

$$\begin{aligned}\langle T^{+}_{\tau\tau}\rangle &= \langle T^{+}_{\xi\xi}\rangle\\ &= \frac{1}{2\pi}\lim_{x\to x'}\left[-\frac{\Delta\tau^2+\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2}\right]\\ &= \frac{1}{2\pi}\lim_{x\to x'}\left[\frac{\Delta\tau^2+\Delta\xi^2}{e^{g(\xi+\xi')}(\Delta\xi^2-\Delta\tau^2)}\frac{1}{e^{g(\xi+\xi')}(\Delta\xi^2-\Delta\tau^2)}e^{2g(\xi+\xi')}\right]\\ &= \frac{1}{2\pi}\lim_{x\to x'}\left[\frac{e^{2g(\xi+\xi')}\left(\Delta\tau^2+\Delta\xi^2\right)}{(d\sigma)^4}\right]\end{aligned} \tag{237}$$

The last expression is not Lorentz covariant. It depends on how you take the limits. Thus it seems like point splitting cannot be applied here.

4.2 Energy momentum tensor for the Rindler invariant vacuum

Then the energy momentum tensor for the Rindler invariant vacuum. Recall that the Feynman propagator for the Rindler invariant vacuum was given by Equation 217:

$$\lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = -\frac{1}{4\pi}\ln\left(\mu^2\left[\cosh(g\Delta\xi) - \cosh(g|\Delta\tau| - ig\epsilon)\right]\right) + \phi^2(\xi,\xi',\mu). \tag{238}$$

The specific form of the spatial condensate makes one realise that the spatially dependent condensate does not survive two derivatives. The condensate does thus not contribute to the energy momentum tensor. As before, one ignores the imaginary part and one thus has $\Delta\tau$ without the absolute value signs. The first order derivatives to the primed coordinates are then given by:

$$\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = \frac{g}{4\pi}\left[-1 + \frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi) - \cosh(g\Delta\tau)}\right], \tag{239}$$

$$\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = -\frac{g}{4\pi}\frac{\sinh(g\Delta\tau)}{\cosh(g\Delta\xi) - \cosh(g\Delta\tau)}. \tag{240}$$

The second order derivatives to the non-primed coordinates again give 4 options:

$$\partial_\xi\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = \frac{-g^2}{4\pi}\frac{(-1 + \cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi) - \cosh(g\Delta\tau))^2} \tag{241}$$

$$\partial_\tau\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = -\frac{g^2}{4\pi}\frac{\sinh(g\Delta\xi)\sinh(g\Delta\tau)}{(\cosh(g\Delta\xi) - \cosh(g\Delta\tau))^2}, \tag{242}$$

$$\partial_\tau\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = \frac{-g^2}{4\pi}\frac{(-1 + \cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi) - \cosh(g\Delta\tau))^2} \tag{243}$$

$$\partial_\xi\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') = -\frac{g^2}{4\pi}\frac{\sinh(g\Delta\xi)\sinh(g\Delta\tau)}{(\cosh(g\Delta\xi) - \cosh(g\Delta\tau))^2}. \tag{244}$$

For the massive case one has the following expression for the energy momentum tensor 63:

$$\begin{aligned}\langle T^+_{\mu\nu}\rangle_{\mathrm{RI},m\to 0} = \lim_{x\to x'}\Bigg[\partial_\mu\partial'_\nu \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m} - \frac{1}{2}g_{\mu\nu}e^{-2g\xi}\left(-\partial_\tau\partial'_\tau + \partial_\xi\partial'_\xi\right)\lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') \\ - \frac{1}{2}\lim_{m\to 0} m^2 i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi')\Bigg].\end{aligned} \tag{245}$$

Before one goes any further, it is appropriate to first calculate the limit $\lim_{m\to 0} m^2\,\log(m)$. This is easily done using l'Hôpital's Rule and the substitution $m = \frac{1}{t}$:

$$\lim_{m\to 0} m^n\,\log(m) = \lim_{t\to\infty}\frac{-\ln(t)}{t^n} = \lim_{t\to\infty}\frac{\frac{-1}{t}}{nt^{n-1}} = \lim_{t\to\infty}\frac{-1}{nt^n} = 0. \tag{246}$$

With this piece of information one can calculate the energy momentum tensor to be:

$$\begin{aligned}\langle T^+_{\tau\tau}\rangle_{\mathrm{RI},m\to 0} &= \lim_{x\to x'}\left[\frac{1}{2}\partial_\tau\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') + \frac{1}{2}\partial_\xi\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi')\right] \\ &= \lim_{x\to x'}\left[\frac{-g^2}{4\pi}\frac{(-1 + \cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi) - \cosh(g\Delta\tau))^2}\right],\end{aligned} \tag{247}$$

$$\begin{aligned}\langle T^{+}_{\xi\xi}\rangle_{\mathrm{RI},m\,\to 0} &= \lim_{x\to x'}\left[\frac{1}{2}\partial_\xi\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi') + \frac{1}{2}\partial_\tau\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi')\right] \\ &= \lim_{x\to x'}\left[\frac{-g^2}{4\pi}\frac{(-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2}\right],\end{aligned} \tag{248}$$

$$\begin{aligned}\langle T^{+}_{\xi\tau}\rangle_{\mathrm{RI},m\to 0} = \langle T^{+}_{\tau\xi}\rangle_{\mathrm{RI},m\to 0} &= \lim_{x\to x'}\left[\partial_\tau\partial'_\xi \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi')\right] = \lim_{x\to x'}\left[\partial_\xi\partial'_\tau \lim_{m\to 0} i\Delta^{\mathrm{F}}_{RI,m}(\tau,\xi;\tau',\xi')\right] \\ &= \lim_{x\to x'}\left[\frac{g^2}{4\pi}\frac{\sinh(g\Delta\xi)\sinh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2}\right].\end{aligned} \tag{249}$$

Again here, the off diagonal terms depend on how one takes the limit. For the diagonal terms, one can easily see that the taking the limit of x to x' diverges at the same pace for both the massless limit of the massive case and the massless case, because for small arguments the expansions for the hyperbolic functions go as: $\sinh(x) \approx x, \cosh(x) \approx 1 + \frac{x^2}{2}$. So one concludes that the massless limit of the massive energy momentum tensor and the massless energy momentum tensor diverge in the same way. One could still wonder whether taking the mass to 0 limit this early yields the same result, however it has been checked using Wolfram Mathematica that taking this limit at the end yields the same results. Again one tries to write these results in terms of the invariant distance $\sigma = \sqrt{\frac{2}{g^2}e^{g(\xi+\xi')}\left[\cosh(g\Delta\xi) - \cosh(g\Delta\tau)\right]}$. One writes:

$$\begin{aligned}\langle T^{+}_{\tau\tau}\rangle_{\mathrm{RI},m\,\to 0} &= \langle T^{+}_{\xi\xi}\rangle_{\mathrm{RI},m\to 0} \\ &= \frac{-g^2}{4\pi}\frac{(-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau))}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\ &= \frac{-1}{\pi g^2}\frac{-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau)}{\sigma^4}e^{2g(\xi+\xi')}\end{aligned} \tag{250}$$

Again here, one finds an expression that is not Lorentz covariant. Thus again one cannot apply point splitting here. One hoped to be able to renormalise by adding counter terms only depending on σ, but it does not seem to be possible here. It is unclear why the point splitting method is not applicable here.

Nevertheless, one can use the result from the dimensional regularisation to conclude that the renormalised energy momentum tensor of the 2-dimensional Rindler vacuum and the invariant 2-dimensional Rindler vacuum is indeed 0. As mentioned in the previous Section, this raises the question of how many states yield a vanishing energy momentum tensor. One comes back to this in the Discussion and conclusion 7.

5 Bogoliubov calculations

In this section one uses the previously obtained Bogoliubov coefficients from section 3, Equations 159, 160, 161 and 162 to calculate how a Minkowski observer perceives the naive Rindler vacuum from a quantum field theory point of view. One squeezes two Minkowski fields between the massless Rindler vacuum bra and ket and then one Bogoliubov transforms the Minkowski creation and annihilation operator to a Rindler creation and annihilation operator. This calculation shows how a Minkowski observer experiences the Rindler vacuum.

5.1 Calculation of the two-point function for the naive Rindler vacuum for a Minkowski observer

The most interesting thing to calculate with the Bogoliubov coefficients is how a Minkowski observer perceives the Rindler vacuum. It is a commonly known result that the Minkowski vacuum appears as a state filled with a thermal distribution to a Rindler observer. Here, one goes a bit further and calculates the two-point function of the Rindler vacuum with respect to the Minkowski field. The two-point function, or positive frequency Wightman function $i\Delta^+_{\text{Rindler}}(t,x;t',x')$, for Rindler vacuum with a Minkowski observer is given by:

$$\begin{aligned}\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \langle 0_R|\ &\int_0^\infty \frac{dk}{2\pi}\int_0^\infty \frac{dk'}{2\pi}\frac{1}{2\sqrt{kk'}} \quad\times \\ &[\hat{b}(k)e^{-ik\bar{u}} + \hat{b}^\dagger(k)e^{+ik\bar{u}} + \hat{b}(-k)e^{-ik\bar{v}} + \hat{b}^\dagger(-k)e^{+ik\bar{v}}] \quad\times \\ &[\hat{b}(k')e^{-ik'\bar{u}'} + \hat{b}^\dagger(k')e^{+ik'\bar{u}'} + \hat{b}(-k')e^{-ik'\bar{v}'} + \hat{b}^\dagger(-k')e^{+ik'\bar{v}'}]\,|0_R\rangle\,.\end{aligned} \tag{251}$$

The Bogoliubov transform for the creation and annihilation of Minkowski space is given by:

$$\hat{b}(k) = \int_0^\infty dk'\,\left[\alpha_{k'k}\hat{a}(k') + \beta^*_{k'k}\hat{a}^\dagger(k')\right] \tag{252}$$

$$\hat{b}^\dagger(k) = \int_0^\infty dk'\,\left[\alpha^*_{k'k}\hat{a}^\dagger(k') + \beta_{k'k}\hat{a}(k)\right]. \tag{253}$$

Now one realises that in the first Minkowski field one can only have annihilation operators and for the second Minkowski field one can only have creation operators. Then one uses the orthonormality of states to immediately obtain:

$$\begin{aligned}\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle &= \int_0^\infty dk\int_0^\infty dk'\int_0^\infty dp\frac{1}{4\pi\sqrt{kk'}}\left[(\alpha_{pk}f_k + \beta_{pk}f_k^*)\left(\beta^*_{pk'}f_{k'} + \alpha^*_{pk'}f^*_{k'}\right)\right. \\ &\left.+ (\alpha_{p\ -k}g_k + \beta_{p\ -k}g_k^*)\left(\beta^*_{p\ -k'}g_{k'} + \alpha^*_{p\ -k'}g^*_{k'}\right)\right] \\ &= \int_0^\infty dk\int_0^\infty dk'\int_0^\infty dp\frac{p\Gamma\left(\frac{ip}{g}\right)\Gamma\left(-\frac{ip}{g}\right)}{4\pi kk'(2\pi g)^2}\left(\frac{k'}{k}\right)^{\frac{ip}{g}} \\ &\left[e^{-ik\bar{u}-ik'\bar{u}'} + e^{\frac{p\pi}{g}}e^{-ik\bar{u}+ik'\bar{u}'} + e^{-\frac{p\pi}{g}}e^{ik\bar{u}-ik'\bar{u}'} + e^{ik\bar{u}+ik'\bar{u}'}\right] \\ &\left[e^{-ik\bar{v}-ik'\bar{v}'} + e^{\frac{p\pi}{g}}e^{-ik\bar{v}+ik'\bar{v}'} + e^{-\frac{p\pi}{g}}e^{ik\bar{v}-ik'\bar{v}'} + e^{ik\bar{v}+ik'\bar{v}'}\right] \\ &= \int_0^\infty dk\int_0^\infty dk'\int_0^\infty dp\frac{1}{16\pi^2 gkk'\,\sinh(\frac{p\pi}{g})}\left(\frac{k'}{k}\right)^{\frac{ip}{g}} \\ &\left[e^{-ik\bar{u}-ik'\bar{u}'} + e^{\frac{p\pi}{g}}e^{-ik\bar{u}+ik'\bar{u}'} + e^{-\frac{p\pi}{g}}e^{ik\bar{u}-ik'\bar{u}'} + e^{ik\bar{u}+ik'\bar{u}'}\right] \\ &\left[e^{-ik\bar{v}-ik'\bar{v}'} + e^{\frac{p\pi}{g}}e^{-ik\bar{v}+ik'\bar{v}'} + e^{-\frac{p\pi}{g}}e^{ik\bar{v}-ik'\bar{v}'} + e^{ik\bar{v}+ik'\bar{v}'}\right].\end{aligned} \tag{254}$$

From this point on there are several choices to regulate the result. Therefore one divides this into subsubsections and explore the different options. However, all three methods do not look promising. One finds the calculations for these three methods below. Because this calculation does not work out, one cannot calculate the entropy of the naive Rindler vacuum from the viewpoint of a Minkowski observer.

5.1.1 Regulating via two small parameters

The integrals over k and k' can be solved using:

$$\int_0^\infty dk \frac{1}{k} k^a e^{-ikx} = (ix)^{-a}\Gamma(a). \tag{255}$$

For this one needs to have that $\mathrm{Im}(x) < 0$ and $\mathrm{Re}(a) > 0$, therefore one adds two small parameters ϵ_1,$\epsilon_2 > 0$ for the k-integral and another two for the k' integral.

$$\int_0^\infty dk \frac{1}{k} k^{\frac{ip}{g}+\epsilon_1} e^{-ik(\bar{u}-i\epsilon_2)} = (i(\bar{u}-i\epsilon_2))^{-\frac{ip}{g}-\epsilon_1}\Gamma(\frac{ip}{g}+\epsilon_1) \tag{256}$$

$$\int_0^\infty dk' \frac{1}{k'} k'^{\frac{-ip}{g}+\epsilon_3} e^{-ik'(\bar{u}'-i\epsilon_4)} = (i(\bar{u}'-i\epsilon_4))^{\frac{ip}{g}-\epsilon_3}\Gamma(-\frac{ip}{g}+\epsilon_3). \tag{257}$$

The ϵ-prescription in different terms is different because the minus signs in front of the p variable change throughout. The part for $\bar{v}$ is exactly the same once again, therefore that contribution will be written as $(\bar{u} \to \bar{v})$. This then yields:

$$\begin{aligned}
\langle 0_R | \hat{\phi}(x,t)\hat{\phi}(x',t') | 0_R \rangle = \frac{1}{16\pi^2 g}\int_0^\infty dp \frac{1}{\sinh(\frac{p\pi}{g})}\Big[& i^{-\epsilon_1-\epsilon_3}(\bar{u}-i\epsilon_2)^{-\frac{ip}{g}-\epsilon_1}\Gamma(\frac{ip}{g}+\epsilon_1)(\bar{u}'-i\epsilon_4)^{\frac{ip}{g}-\epsilon_3}\Gamma(-\frac{ip}{g}+\epsilon_3) \\
&+ e^{\frac{p\pi}{g}} i^{-\epsilon_1-\epsilon_3}(\bar{u}-i\epsilon_6)^{-\frac{ip}{g}-\epsilon_1}\Gamma(\frac{ip}{g}+\epsilon_1)(\bar{u}'+i\epsilon_4)^{\frac{ip}{g}-\epsilon_3}\Gamma(-\frac{ip}{g}+\epsilon_3) \\
&+ e^{\frac{-p\pi}{g}} i^{-\epsilon_1-\epsilon_3}(\bar{u}+i\epsilon_1)^{-\frac{ip}{g}-\epsilon_1}\Gamma(\frac{ip}{g}+\epsilon_1)(\bar{u}'-i\epsilon_4)^{\frac{ip}{g}-\epsilon_3}\Gamma(-\frac{ip}{g}+\epsilon_3) \\
&+ i^{-\epsilon_1-\epsilon_3}(\bar{u}+i\epsilon_2)^{-\frac{ip}{g}-\epsilon_1}\Gamma(\frac{ip}{g}+\epsilon_1)(\bar{u}'+i\epsilon_4)^{\frac{ip}{g}-\epsilon_3}\Gamma(-\frac{ip}{g}+\epsilon_3)\Big] + (\bar{u}\to\bar{v})
\end{aligned} \tag{258}$$

First, do the calculation where one neglects all epsilons. One does this just to see what one can expect. As will be seen later, this method does not work out, and a lot of hassle has been saved by not considering the ϵ's. The expression without all ϵ's is given by:

$$\begin{aligned}
&\langle 0_R | \hat{\phi}(x,t)\hat{\phi}(x',t') | 0_R \rangle = \frac{1}{16\pi^2 g}\int_0^\infty dp \frac{1}{\sinh(\frac{p\pi}{g})}\Big[\bar{u}^{-\frac{ip}{g}}\Gamma(\frac{ip}{g})\bar{u}'^{\frac{ip}{g}}\Gamma(-\frac{ip}{g}) \\
&+ e^{\frac{p\pi}{g}}\bar{u}^{-\frac{ip}{g}}\Gamma(\frac{ip}{g})\bar{u}'^{\frac{ip}{g}}\Gamma(-\frac{ip}{g}) + e^{\frac{-p\pi}{g}}\bar{u}^{-\frac{ip}{g}}\Gamma(\frac{ip}{g})\bar{u}'^{\frac{ip}{g}}\Gamma(-\frac{ip}{g}) + \bar{u}^{-\frac{ip}{g}}\Gamma(\frac{ip}{g})\bar{u}'^{\frac{ip}{g}}\Gamma(-\frac{ip}{g})\Big] + (\bar{u}\to\bar{v}) \\
&= \frac{1}{16\pi}\int_0^\infty dp \frac{1}{p\,\sinh^2(\frac{p\pi}{g})}\left(\frac{\bar{u}'}{\bar{u}}\right)^{\frac{ip}{g}}\Big[2 + e^{\frac{p\pi}{g}} + e^{\frac{-p\pi}{g}}\Big] + (\bar{u}\to\bar{v}) \\
&= \frac{1}{16\pi}\int_0^\infty dp \frac{1}{p\,\sinh^2(\frac{p\pi}{2g})} e^{\frac{ip}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)} + (\bar{u}\to\bar{v}).
\end{aligned} \tag{259}$$

One now rewrites the $\frac{1}{\sinh^2(\frac{a}{2})}$ by making use of an expansion of $\frac{1}{(1-x)^2} = \sum_0^\infty n x^{n-1}$. In this case, one has:

$$\begin{aligned}
\frac{1}{\sinh^2(\frac{a}{2})} &= \frac{e^{-a}}{(1-e^{-a})^2} \\
&= e^{-a}\sum_{n=0}^\infty n e^{-a(n-1)} \\
&= \sum_{n=0}^\infty n e^{-an}.
\end{aligned} \tag{260}$$

Inserting this back into Equation 259 and introducing the IR cut off δ yields:

$$\begin{aligned}
\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle &= \frac{1}{16\pi}\sum_{n=0}^{\infty} n\int_{\delta}^{\infty}\frac{dp}{p}e^{p(\frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)-\frac{\pi}{g}n)} + (\bar{u}\to\bar{v})\\
&= \frac{1}{16\pi}\sum_{n=0}^{\infty} n\Gamma\left(0,(\frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)-\frac{\pi}{g}n)\delta\right) + (\bar{u}\to\bar{v})\\
&= \frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[-\gamma_{\mathrm{E}}-\ln(\delta)-\ln\left(\frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)-\frac{\pi}{g}n\right)\right] + (\bar{u}\to\bar{v})\\
&= \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[-\gamma_{\mathrm{E}}-\ln(\delta)-\frac{\left(\frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)-\frac{\pi}{g}n\right)^y}{y}+\frac{1}{y}\right] + (\bar{u}\to\bar{v})\\
&= \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\mathrm{E}}-\ln(\delta)\right]-\lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{\left(\frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right)-\frac{\pi}{g}n\right)^y}{y}\right] + (\bar{u}\to\bar{v}).
\end{aligned} \tag{261}$$

Now one uses a small rewriting to write the second sum as two Hurwitz zeta functions:

$$\frac{1}{b}(a-bn)(a-bn)^y = \frac{a}{b}(a-bn)^y - n(a-bn)^y,$$

which implies:

$$n(a-bn)^y = \frac{a}{b}(a-bn)^y - \frac{1}{b}(a-bn)^{1+y}. \tag{262}$$

One then defines:

$$a = \frac{i}{g}\ln\left(\frac{\bar{u}'}{\bar{u}}\right), \tag{263}$$

$$b = \frac{\pi}{g}. \tag{264}$$

Thus, after using Equation 262 one obtains:

$$\begin{aligned}
\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle &= \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\mathrm{E}}-\ln(\delta)\right]\\
&-\lim_{y\to 0}\frac{1}{16\pi y}\sum_{n=0}^{\infty}\left[\frac{a}{b}\left(a-bn\right)^y-\frac{1}{b}\left(a-bn\right)^{y+1}\right] + (\bar{u}\to\bar{v})\\
&= \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\mathrm{E}}-\ln(\delta)\right]\\
&-\lim_{y\to 0}\frac{1}{16\pi y}\sum_{n=0}^{\infty}\left[\frac{a(-b)^y}{b}\left(n-\frac{a}{b}\right)^y-\frac{(-b)^{y+1}}{b}\left(n-\frac{a}{b}\right)^{y+1}\right] + (\bar{u}\to\bar{v})\\
&= \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\mathrm{E}}-\ln(\delta)\right]\\
&-\lim_{y\to 0}\frac{1}{16\pi y}\left[\frac{a(-b)^y}{b}\zeta\left(-y,-\frac{a}{b}\right)-\frac{(-b)^{y+1}}{b}\zeta\left(-y-1,-\frac{a}{b}\right)\right].
\end{aligned} \tag{265}$$

One can make a series expansion of the Hurwitz zeta function near 0 as:

$$\zeta(s,a) = \zeta(0,a) + s\zeta^{(1,0)}(0,a) + ... \tag{266}$$

where $\zeta^{(1,0)}$ denotes a derivative with respect to s. The zeroth order term is given by: $\zeta(0,a) = \frac{1}{2} - a$. One knows that the first order term scales as:

$$\zeta^{(1,0)}(0,a) = -\frac{1}{2}\ln(2\pi) + \ln\left(\Gamma(a)\right). \tag{267}$$

The second Hurwitz zeta function has to be expanded near 1 and is expanded as

$$\zeta(-1-y, -\frac{a}{b}) = \zeta(-1, -\frac{a}{b}) - \zeta^{(1,0)}(-1, -\frac{a}{b})y + O(y^2), \tag{268}$$

where the zeroth order term can be found by an exponential Fourier series

$$\begin{aligned}\zeta(-1,a) &= \frac{2}{(2\pi)^2}\Gamma(2)\left[\sin\left(\frac{-\pi}{2}\right)\sum_{k=1}^{\infty}\frac{\cos(2\pi ak)}{k^2} + \cos\left(\frac{-\pi}{2}\right)\sum_{k=1}^{\infty}\frac{\sin(2\pi ak)}{k^2}\right] \\ &= \frac{-1}{2\pi^2}\left[\mathrm{Li}_2(e^{-2i\pi a}) + \mathrm{Li}_2(e^{2i\pi a})\right],\end{aligned} \tag{269}$$

and the second order term:

$$\begin{aligned}\zeta^{(1,0)}(-1,-\frac{a}{b}) = \frac{-1}{12}&\left[-6\left(\frac{a}{b}\right)^2 - 18\frac{a}{b} - 12\left(-\frac{a}{b}-1\right)\ln\left(-\frac{a}{b}-1\right) + 12\ln(A) + 6\left(-\frac{a}{b}-1\right)\ln(2\pi) - 13\right] \\ &+ \psi^{(-2)}\left(-\frac{a}{b}-1\right) \\ &+ \left\{\sum_{i=0}^{\lfloor Re\left(\frac{a}{b}\right)\rfloor}\left[(-\frac{a}{b}+i)\ln(-\frac{a}{b}+i) - \frac{1}{2}\sqrt{\left(-\frac{a}{b}+i\right)^2}\ln\left((-\frac{a}{b}+i)^2\right)\right]\right\}\theta(\lfloor Re(\frac{a}{b})\rfloor).\end{aligned} \tag{270}$$

Here, the functions $\mathrm{Li}_2(a), \psi^{(-2)}(a)$ are respectively the polylogarithm and the poly gamma function. The constant A is the Glaisher constant, $A = 1.2824...$ The first sum from Equation 265 is regularised by using the analytic extension of the ζ-function: $\zeta(-1) = -\frac{1}{12}$. Inserting the expansions for the Hurwitz zeta functions then yields:

$$\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \frac{1}{16\pi}\Bigg\{-\frac{1}{12}\left[\frac{1}{y}-\gamma_{\mathrm{E}}-\ln(\delta)\right]$$
$$-\lim_{y\to 0}\frac{1+y\ln(-b)}{y}\Bigg\{\frac{a}{b}\left[\left(\frac{1}{2}+\frac{a}{b}\right)+\ln\left(\Gamma\left(-\frac{a}{b}\right)\right)y-\frac{1}{2}\ln(2\pi)y\right]+\frac{-1}{2\pi^2}\left[\mathrm{Li}_2(e^{-2i\pi a})+\mathrm{Li}_2(e^{2i\pi a})\right]$$
$$+\frac{y}{12}\left[-6\left(\frac{a}{b}\right)^2-18\frac{a}{b}-12\left(-\frac{a}{b}-1\right)\ln\left(-\frac{a}{b}-1\right)+12\ln(A)+6\left(-\frac{a}{b}-1\right)\ln(2\pi)-13\right]+\psi^{(-2)}\left(-\frac{a}{b}-1\right)y$$
$$+\left\{\sum_{i=0}^{\lfloor Re(\frac{a}{b})\rfloor}\left[(-\frac{a}{b}+i)\ln(-\frac{a}{b}+i)-\frac{1}{2}\sqrt{\left(-\frac{a}{b}+i\right)^2}\ln\left((-\frac{a}{b}+i)^2\right)\right]\right\}\theta(\lfloor Re(\frac{a}{b})\rfloor)y\Bigg\}\Bigg\}$$
$$=\frac{1}{16\pi}\Bigg\{\frac{1}{12}\left[\gamma_{\mathrm{E}}+\ln(\delta)\right]-\frac{1}{y}\left[\frac{1}{12}+\frac{a}{b}\left(\frac{1}{2}+\frac{a}{b}\right)-\frac{1}{2\pi^2}\left[\mathrm{Li}_2(e^{-2i\pi a})+\mathrm{Li}_2(e^{2i\pi a})\right]\right]$$
$$-\ln(-b)\left[\frac{a}{b}\left(\frac{1}{2}+\frac{a}{b}\right)-\frac{1}{2\pi^2}\left[\mathrm{Li}_2(e^{-2i\pi a})+\mathrm{Li}_2(e^{2i\pi a})\right]\right]-\ln\left(\Gamma\left(-\frac{a}{b}\right)\right)+\frac{1}{4}\ln(2\pi)\left(\frac{a}{b}\right)^2+\frac{3}{2}\frac{a}{b}$$
$$+\left(-\frac{a}{b}-1\right)\ln\left(-\frac{a}{b}-1\right)-\ln(A)-\frac{1}{2}\left(-\frac{a}{b}-1\right)\ln(2\pi)-\frac{13}{12}-\frac{-\frac{a}{b}-1}{2}\Bigg[-\gamma_{\mathrm{E}}(-\frac{a}{b}-3)-2\ln(-\frac{a}{b}-1)+2(1-\gamma_{\mathrm{E}})$$
$$+(-\frac{a}{b}-1)^3\sum_{k=1}^{\infty}\frac{1}{k(k-\frac{a}{b}-1)}{}_2\hat{F}_1\left(1,2;4,\frac{-\frac{a}{b}-1}{k-\frac{a}{b}-1}\right)\Bigg]$$
$$+\left\{\sum_{i=0}^{\lfloor Re(\frac{a}{b})\rfloor}\left[(-\frac{a}{b}+i)\ln(-\frac{a}{b}+i)-\frac{1}{2}\sqrt{\left(-\frac{a}{b}+i\right)^2}\ln\left((-\frac{a}{b}+i)^2\right)\right]\right\}\theta(\lfloor Re(\frac{a}{b})\rfloor)\Bigg\}. \tag{271}$$

Even though this is a result in terms of all analytic functions, one also has terms scaling as $\frac{1}{y}$, which do not cancel. This is problematic. To resum on replaced a logarithm by a function with a small parameter y and one hoped that the final result would not depend on y and $\frac{1}{y}$. This does not happen and a $\frac{1}{y}$ with a non-trivial space time dependence emerges in the final result. Some consideration of the previous steps might reveal the flaw. In Equation 260, one expanded the inverse sinus hyperbolic squared. This function is ill defined only in the point $a=0$ on the whole complex plane. While the sum is only defined for a real and $a>0$. One then later regularizes this sum from $n=0$ to infinity multiplying the $\frac{1}{y}$ term and that causes some problems. This causes the expansion parameters to not cancel out against each other.

5.1.2 Regulating via the Hurwitz Zeta function three small momentum cut-offs

The previous way of regulating did not work so let us go back to Equation 254 and try a different regulation process:

$$\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \int_0^\infty dk\int_0^\infty dk'\int_0^\infty dp\frac{1}{16\pi^2 gkk'\sinh(\frac{p\pi}{g})}\left(\frac{k'}{k}\right)^{\frac{ip}{g}}$$
$$\left[e^{-ik\bar{u}-ik'\bar{u}'}+e^{\frac{p\pi}{g}}e^{-ik\bar{u}+ik'\bar{u}'}+e^{-\frac{p\pi}{g}}e^{ik\bar{u}-ik'\bar{u}'}+e^{ik\bar{u}+ik'\bar{u}'}\right]$$
$$\left[e^{-ik\bar{v}-ik'\bar{v}'}+e^{\frac{p\pi}{g}}e^{-ik\bar{v}+ik'\bar{v}'}+e^{-\frac{p\pi}{g}}e^{ik\bar{v}-ik'\bar{v}'}+e^{ik\bar{v}+ik'\bar{v}'}\right]. \tag{272}$$

This time one uses a regulator ϵ and one introduces an IR cut-off δ already in the k integrals. This then gives the expression for an incomplete gamma function:

$$\begin{aligned}\int_\delta^\infty k^{\frac{-ip}{g}-1}e^{-ik(\bar{u}-i\epsilon)} &= (i\bar{u}+\epsilon)^{\frac{ip}{g}}\Gamma\left(\frac{-ip}{g},(i\bar{u}+\epsilon)\delta\right)\\ &= (i\bar{u}+\epsilon)^{\frac{ip}{g}}\left(\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\left(\frac{-ig(i\bar{u}+\epsilon)^{\frac{-ip}{g}}}{p}\right)\right)\\ &= (i\bar{u}+\epsilon)^{\frac{ip}{g}}\Gamma\left(\frac{-ip}{g}\right)-\frac{ig\delta^{\frac{-ip}{g}}}{p}.\end{aligned} \tag{273}$$

Inserting these results back into Equation 272, one obtains:

$$\begin{aligned}&\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \int_0^\infty dp\frac{1}{16\pi^2 g\sinh(\frac{p\pi}{g})}\Bigg\{ \qquad (274)\\ &\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)^{\frac{ip}{g}}\left[\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\left(\frac{-ig(i\bar{u}+\epsilon_1)^{\frac{-ip}{g}}}{p}\right)\right]\left[\Gamma\left(\frac{ip}{g}\right)+\delta'^{\frac{ip}{g}}\left(\frac{ig(i\bar{u}'+\epsilon_1')^{\frac{ip}{g}}}{p}\right)\right]\\ &+e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)^{\frac{ip}{g}}\left[\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\left(\frac{-ig(i\bar{u}+\epsilon_2)^{\frac{-ip}{g}}}{p}\right)\right]\left[\Gamma\left(\frac{ip}{g}\right)+\delta'^{\frac{ip}{g}}\left(\frac{ig(-i\bar{u}'+\epsilon_2')^{\frac{ip}{g}}}{p}\right)\right]\\ &+e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)^{\frac{ip}{g}}\left[\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\left(\frac{-ig(-i\bar{u}+\epsilon_3)^{\frac{-ip}{g}}}{p}\right)\right]\left[\Gamma\left(\frac{ip}{g}\right)+\delta'^{\frac{ip}{g}}\left(\frac{ig(i\bar{u}'+\epsilon_3')^{\frac{ip}{g}}}{p}\right)\right]\\ &+\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)^{\frac{ip}{g}}\left[\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\left(\frac{-ig(-i\bar{u}+\epsilon_4)^{\frac{-ip}{g}}}{p}\right)\right]\left[\Gamma\left(\frac{ip}{g}\right)+\delta'^{\frac{ip}{g}}\left(\frac{ig(-i\bar{u}'+\epsilon_4')^{\frac{ip}{g}}}{p}\right)\right]\Bigg\}\\ &+(\bar{u}\to\bar{v})\\ &=\int_0^\infty dp\frac{1}{16\pi^2 g\sinh(\frac{p\pi}{g})}\Bigg\{\Gamma\left(\frac{-ip}{g}\right)\Gamma\left(\frac{ip}{g}\right)\left[\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)^{\frac{ip}{g}}+e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)^{\frac{ip}{g}}+e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)^{\frac{ip}{g}}+\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)^{\frac{ip}{g}}\right]\\ &+\frac{ig}{p}\Gamma\left(\frac{-ip}{g}\right)\delta'^{\frac{ip}{g}}\left[(i\bar{u}+\epsilon_1)^{\frac{ip}{g}}+e^{\frac{p\pi}{g}}(i\bar{u}+\epsilon_2)^{\frac{ip}{g}}+e^{\frac{-p\pi}{g}}(-i\bar{u}+\epsilon_3)^{\frac{ip}{g}}+(-i\bar{u}+\epsilon_4)^{\frac{ip}{g}}\right]\\ &-\frac{ig}{p}\Gamma\left(\frac{ip}{g}\right)\delta^{\frac{-ip}{g}}\left[(i\bar{u}'+\epsilon_1')^{\frac{-ip}{g}}+e^{\frac{p\pi}{g}}(-i\bar{u}'+\epsilon_2')^{\frac{-ip}{g}}+e^{\frac{-p\pi}{g}}(i\bar{u}'+\epsilon_3')^{\frac{-ip}{g}}+(-i\bar{u}'+\epsilon_4')^{\frac{-ip}{g}}\right]\\ &+\frac{g^2}{p^2}\left(\frac{\delta'}{\delta}\right)^{\frac{ip}{g}}\left[1+e^{\frac{p\pi}{g}}+e^{\frac{-p\pi}{g}}+1\right]\Bigg\}+(\bar{u}\to\bar{v})\end{aligned}$$

The problem here is that the terms are of order p^{-3} and higher. This means that one has divergences of order p^{-2} and higher. These divergences are intertwined and are tough to separate. The first and the last line can be integrated after introducing another cut-off δ_p, however, the third and the second line can only be done in approximation. One sees that these terms have a scaling with respect to δ as $e^{\frac{-ip}{g}\ln(\delta)}$. When δ goes to zero this will start oscillating very rapidly and thus the contribution coming from large p are expected to go to zero. Thus one argues that only the smaller values of p contribute to the middle two integrals. This procedure is executed in Appendix H and the result is given by:

$$\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \phi^2(\mu) + \lim_{y\to 0}\frac{1}{4\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\rm E}\right] \tag{275}$$
$$-\lim_{y\to 0}\frac{1}{16\pi y}\left[\frac{c_1(-b_1)^y}{b_1}\zeta\left(-y,-\frac{c_1}{b_1}\right)-\frac{(-b_1)^{y+1}}{b_1}\zeta\left(-y-1,-\frac{c_1}{b_1}\right)+\frac{c_2(-b_2)^y}{b_2}\zeta\left(-y,-\frac{c_2}{b_2}\right)-\frac{(-b_2)^{y+1}}{b_2}\zeta\left(-y-1,-\frac{c_2}{b_2}\right)\right.$$
$$\left.+\frac{c_3(-b_3)^y}{b_3}\zeta\left(-y,-\frac{c_3}{b_3}\right)-\frac{(-b_3)^{y+1}}{b_3}\zeta\left(-y-1,-\frac{c_3}{b_3}\right)+\frac{c_4(-b_4)^y}{b_4}\zeta\left(-y,-\frac{c_4}{b_4}\right)-\frac{(-b_4)^{y+1}}{b_4}\zeta\left(-y-1,-\frac{c_4}{b_4}\right)\right]$$
$$+\frac{1}{16\pi^2}\sum_{i=1}^{4}\left[-\frac{g^2}{\pi}\left(\frac{3}{4}a'^2_{i,r}\right)-\frac{ig\gamma_{\rm E}}{\pi}\left(a'_{i,r}(1-\gamma_{\rm E})\right)-\frac{1}{6\pi}\left(3\gamma_{\rm E}^2+\frac{\pi^2}{2}\right)\gamma_{\rm E}-\frac{\pi\gamma_{\rm E}}{6}\right]$$
$$+\frac{1}{16\pi^2}\sum_{i=1}^{4}\left[-\frac{g^2}{\pi}\left(\frac{3}{4}a^2_{i,r}\right)+\frac{ig\gamma_{\rm E}}{\pi}\left(a_{i,r}(1-\gamma_{\rm E})\right)-\frac{1}{6\pi}\left(3\gamma_{\rm E}^2+\frac{\pi^2}{2}\right)\gamma_{\rm E}-\frac{\pi\gamma_{\rm E}}{6}\right]-\frac{1}{4}\left[\sum_{n=0}^{\infty}(2n+1)(1+\gamma_{\rm E})\right],$$

where the $a_{i,r}, b_i, c_i$ are given by:

$$a_{1,r} = -\frac{i}{g}\ln(i\bar{u}'+\epsilon_1'), a_{2,r} = -\frac{i}{g}\ln(-i\bar{u}'+\epsilon_2')+\frac{pi}{g}, \tag{276}$$
$$a_{3,r} = -\frac{i}{g}\ln(i\bar{u}'+\epsilon_3')-\frac{pi}{g}, a_{4,r} = -\frac{i}{g}\ln(-i\bar{u}'+\epsilon_4'), \tag{277}$$

$$c_1 = \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right),\ b_1 = -\frac{2\pi}{g}, \tag{278}$$
$$c_2 = \frac{\pi}{g}+\frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right),\ b_2 = -\frac{2\pi}{g}, \tag{279}$$
$$c_3 = \frac{-\pi}{g}+\frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right),\ b_3 = -\frac{2\pi}{g}, \tag{280}$$
$$c_4 = \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right),\ b_4 = -\frac{2\pi}{g}. \tag{281}$$

What one again sees here, is that the expansion coefficient y again mixes with the sum that needs to be regulated. One hoped that contributions with these terms from the different parts would cancel against each other, however, there only appears one term with $\sum \frac{n}{y}$. Even if all the $y's$ in the Hurwitz zeta function would cancel, this would give the same problem as before. Therefore one has to resort to yet another way of doing the integral.

5.1.3 Regulating via three small momentum cut-offs

The last way of regulating introduces three different momentum cut-offs and expands the integrand from Equation 274 around small values of p. In Equation 272 one sees that one has only a logarithmic divergence in δ_p. However, in Equation 274, it seems like one has a $\frac{1}{\delta_p^2}$ divergence. This oddity can be fixed by expanding the Γ-function for small p. One is allowed to do this, because the terms in the integral scale as $e^{\frac{\pm ip}{g}\ln(\delta)}$. This only contributes significantly for small p when δ goes to zero. Thus one writes:

$$\begin{aligned}\left[\Gamma\left(\frac{-ip}{g}\right)+\delta^{\frac{-ip}{g}}\frac{-ig(i\bar{u}+\epsilon_1)^{\frac{-ip}{g}}}{p}\right] &= \frac{ig}{p}-e^{-\frac{ip}{g}\ln(\delta(i\bar{u}+\epsilon_1))}\frac{ig}{p}\\ &= \frac{ig}{p}\left[1-\left(1-\frac{ip}{g}\ln(\delta(i\bar{u}+\epsilon_1))\right)\right]\\ &= -\ln(\delta)-\ln(i\bar{u}+\epsilon_1).\end{aligned} \tag{282}$$

One might doubt the small p expansion of the exponent as one has that $\ln(\delta)$ goes to infinity. One argues that this is okay, as one knows that $\lim_{x\to 0} x\ln(x)$ goes to 0 as seen in Equation 246 in the previous Section. Thus, one only needs p going to 0 just as quick as δ going to 0. This is not a mathematical proof of given fact, but it does give a good indication that this might be valid. If this assumption breaks down, then the calculation below is invalid. Inserting this result back in the first part of Equation 274 then shows that one only has a $\ln(\delta_p)$ divergence, as was expected. One now solves the integrals as:

$$\begin{aligned}\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle == \int_0^\infty dp\frac{1}{16\pi^2 g\sinh(\frac{p\pi}{g})}\Bigg\{\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)^{\frac{ip}{g}}\left[\ln(\delta)+\ln(i\bar{u}+\epsilon_1)\right]\left[\ln(\delta')+\ln(i\bar{u}'+\epsilon_1')\right]\\ + e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)^{\frac{ip}{g}}\left[\ln(\delta)+\ln(i\bar{u}+\epsilon_2)\right]\left[\ln(\delta')+\ln(-i\bar{u}'+\epsilon_2')\right]\\ + e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)^{\frac{ip}{g}}\left[\ln(\delta)+\ln(-i\bar{u}+\epsilon_3)\right]\left[\ln(\delta')+\ln(i\bar{u}'+\epsilon_3')\right]\\ + \left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)^{\frac{ip}{g}}\left[\ln(\delta)+\ln(-i\bar{u}+\epsilon_4)\right]\left[\ln(\delta')+\ln(-i\bar{u}'+\epsilon_4')\right]\Bigg\} + (\bar{u}\to\bar{v}).\end{aligned} \tag{283}$$

These integrals are very similar to integrals that have been seen already. They are slightly different however, because they are missing a factor $\frac{1}{p}$. This makes these integrals easier to solve. These integrals are given by:

$$\begin{aligned}\int_{\delta_p}^\infty dp\frac{e^{p\left(\frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)\right)}}{\sinh(\frac{p\pi}{g})} = \frac{g}{\pi}\Bigg[i\sqrt{\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}}\pi\frac{1}{\cosh\left(\frac{1}{2}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)\right)} - \gamma_{\mathrm{E}} - i\pi - \ln\left(\frac{2\pi}{g}\right) - \ln(\delta_p)\\ - \psi^{(0)}\left(\frac{1}{2}+\frac{i}{2\pi}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)\right)\Bigg]\end{aligned} \tag{284}$$

$$\begin{aligned}\int_{\delta_p}^\infty dp\frac{e^{p\left(\frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)+\frac{\pi}{g}\right)}}{\sinh(\frac{p\pi}{g})} = \frac{g}{\pi}\Bigg[i\sqrt{\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}}\pi\frac{1}{\sinh\left(\frac{1}{2}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)\right)} - \gamma_{\mathrm{E}} - i\pi - \ln\left(\frac{2\pi}{g}\right) - \ln(\delta_p)\\ - \psi^{(0)}\left(\frac{i}{2\pi}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)\right) + \frac{2i}{\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)}\Bigg]\end{aligned} \tag{285}$$

$$\int_{\delta_p}^\infty dp\frac{e^{p\left(\frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)-\frac{\pi}{g}\right)}}{\sinh(\frac{p\pi}{g})} = \frac{g}{\pi}\left[-\gamma_{\mathrm{E}} - i\pi - \ln\left(\frac{2\pi}{g}\right) - \ln(\delta_p) - \psi^{(0)}\left(\frac{i}{2\pi}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)\right) + \frac{2\pi i\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)}{\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)-1}\right] \tag{286}$$

$$\begin{aligned}\int_{\delta_p}^\infty dp\frac{e^{p\left(\frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)\right)}}{\sinh(\frac{p\pi}{g})} = \frac{g}{\pi}\Bigg[i\sqrt{\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}}\pi\frac{1}{\cosh\left(\frac{1}{2}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)\right)} - \gamma_{\mathrm{E}} - i\pi - \ln\left(\frac{2\pi}{g}\right) - \ln(\delta_p)\\ - \psi^{(0)}\left(\frac{1}{2}+i\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)\right)\Bigg]\end{aligned} \tag{287}$$

Bringing all this together then gives an expression for the two-point function:

$$\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \frac{1}{16\pi^3}\Bigg\{[\ln(\delta)+\ln(i\bar{u}+\epsilon_1)]\,[\ln(\delta')+\ln(i\bar{u}'+\epsilon_1')]\left[i\sqrt{\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}}\pi\frac{1}{\cosh\left(\frac{1}{2}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)\right)}\right. \tag{288}$$
$$\left.-\gamma_{\mathrm{E}}-i\pi-\ln\left(\frac{2\pi}{g}\right)-\ln(\delta_p)-\psi^{(0)}\left(\frac{1}{2}+\frac{i}{2\pi}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)\right)\right]+[\ln(\delta)+\ln(i\bar{u}+\epsilon_2)]\,[\ln(\delta')+\ln(-i\bar{u}'+\epsilon_2')]$$
$$\left[i\sqrt{\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}}\pi\frac{1}{\sinh\left(\frac{1}{2}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)\right)}-\gamma_{\mathrm{E}}-i\pi-\ln\left(\frac{2\pi}{g}\right)-\ln(\delta_p)-\psi^{(0)}\left(\frac{i}{2\pi}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)\right)+\frac{2i}{\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)}\right]$$
$$+[\ln(\delta)+\ln(-i\bar{u}+\epsilon_3)]\,[\ln(\delta')+\ln(i\bar{u}'+\epsilon_3')]\left[-\gamma_{\mathrm{E}}-i\pi-\ln\left(\frac{2\pi}{g}\right)-\ln(\delta_p)-\psi^{(0)}\left(\frac{i}{2\pi}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)\right)+\frac{2\pi i\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)}{\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)-1}\right]$$
$$+[\ln(\delta)+\ln(-i\bar{u}+\epsilon_4)]\,[\ln(\delta')+\ln(-i\bar{u}'+\epsilon_4')]\left[i\sqrt{\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}}\pi\frac{1}{\cosh\left(\frac{1}{2}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)\right)}-\gamma_{\mathrm{E}}-i\pi-\ln\left(\frac{2\pi}{g}\right)-\ln(\delta_p)\right.$$
$$\left.\left.-\,\psi^{(0)}\left(\frac{1}{2}+i\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)\right)\right]\right\}+(\bar{u}\to\bar{v})\,.$$

Unfortunately this third way of regulating again gives a hopelessly divergent answer with a complicated spacetime dependence. One sees by comparing Equation 288 with Equation 275 that even if one neglects all unregular parts, the remaining regular parts are inequal. Moreover, no matter how one regulates, one gets a coordinate dependence in the divergent terms. Different regularisations even lead to different space time dependences. This signifies that one needs a physical reason to regulate a certain way. The author of this paper does not know the answer to this problem and the only thing that can be concluded is that the Rindler vacuum appears to be filled with infinitely many IR particles from the viewpoint of a Minkowski observer.

6 Von Neumann entropy

This section is dedicated to calculating the entropy for the naive Rindler vacuum and the Rindler invariant vacuum. One first starts with the naive Rindler vacuum and in the next subsection one calculates the entropy for the Rindler invariant vacuum. Both cases consider a massless field.

6.1 Entropy for the naive Rindler vacuum

After having obtained the two-point functions, one can calculate the entropy in the following way. One first defines:

$$i\Delta^+ = \langle 0_R| \phi(\tau,\xi)\phi(\tau',\xi') |0_R\rangle\,,\; i\Delta^- = [i\Delta^+]^* \tag{289}$$

This calculation has already been done in Section 3, however for this calculation it is easier to not do the splitting of the integral in a positive and a negative part as was done in Section 3. Not doing this splitting gives:

$$\begin{aligned} i\Delta^+ &= \langle 0_R| \int \frac{dkdk'}{4\pi^2}\left[\frac{e^{-i\omega\tau+ik\xi}}{\sqrt{2\omega}}\hat{a}(k)+\frac{e^{i\omega\tau-ik\xi}}{\sqrt{2\omega}}\hat{a}^\dagger(k)\right]\left[\frac{e^{-i\omega'\tau'+ik'\xi'}}{\sqrt{2\omega'}}\hat{a}(k')+\frac{e^{i\omega'\tau'-ik'\xi'}}{\sqrt{2\omega'}}\hat{a}^\dagger(k')\right]|0_R\rangle \\ &= \int\frac{dk}{2\pi}\left[\frac{e^{-i\omega(\tau-\tau')+ik(\xi-\xi')}}{2\omega}\right]. \end{aligned}$$

One remembers that the definition of $\phi(\tau,k)=\frac{e^{-i\omega\tau}}{\sqrt{2\omega}}$, thus our result can be rewritten as:

$$i\Delta^+(\xi,\xi';\tau,\tau') = \int\frac{dk}{2\pi}\phi(\tau,k)\phi^*(\tau',k)e^{ik(\xi-\xi')}.$$

As this result is only a function of $\xi-\xi'$, one applies a Fourier (or Wigner) transform defined in the following way:

$$A(\xi-\xi';\tau,\tau') = \int\frac{dk}{2\pi}e^{ik(\xi-\xi')}\tilde{A}(k;\tau,\tau') \tag{290}$$

$$\tilde{A}(k;\tau,\tau') = \int d(\xi-\xi')e^{-ik(\xi-\xi')}A(\xi-\xi';\tau,\tau'). \tag{291}$$

If one considers $\phi(\tau,k)\phi^*(\tau',k)$ as the $\tilde{A}(k,\tau,\tau')$, one sees that the inverse is given by:

$$\phi(\tau,k)\phi^*(\tau',k) = \int d(\xi-\xi')e^{-ik(\xi-\xi')}i\Delta^+(\xi-\xi',\tau,\tau') = i\tilde{\Delta}^+(k,\tau,\tau'). \tag{292}$$

Similarly, the 2-point function $i\tilde{\Delta}^-$ is now given by:

$$\phi(\tau,k)^*\phi(\tau',k) = \int d(\xi-\xi')e^{-ik(\xi-\xi')}i\Delta^-(\xi-\xi',\tau,\tau') = i\tilde{\Delta}^-(k,\tau,\tau') \tag{293}$$

One now defines:

$$F(k,\tau,\tau') = \frac{1}{2}[i\tilde{\Delta}^+(k,\tau,\tau')+i\tilde{\Delta}^-(k,\tau,\tau')], \tag{294}$$

which at equal time is just given by $|\phi(\tau,k)|^2$. The derivatives to τ and τ' are now given by:

$$\partial_\tau F(k,\tau,\tau') = \frac{1}{2}[\partial_\tau\phi(\tau,k)\phi^*(\tau',k)+\partial_\tau\phi^*(\tau,k)\phi(\tau',k)] \overset{\tau\to\tau'}{=} \frac{1}{2}\partial_\tau|\phi(\tau,k)|^2 \tag{295}$$

$$\partial_\tau\partial_{\tau'}F(k,\tau,\tau') = \frac{1}{2}[\partial_\tau\phi(\tau,k)\partial_\tau'\phi^*(\tau',k)+\partial_\tau\phi^*(\tau,k)\partial\tau'\phi(\tau',k)] \overset{\tau\to\tau'}{=} |\partial_\tau\phi(\tau,k)|^2. \tag{296}$$

From quantum mechanics one knows that the Heisenberg uncertainty principle is given by: $4\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle - \langle\{\hat{x},\hat{p}\}\rangle \geq 1$. This can be generalized to field theory and to Wigner space as:

$$\begin{aligned}\Delta^2(\tau,k) &= \lim_{\tau\to\tau'} 4\left[F(k,\tau,\tau')\partial_\tau\partial'_\tau F(k,\tau,\tau') - \partial_\tau F(k,\tau,\tau')\partial'_\tau F(k,\tau,\tau')\right] \\ &= 4|\phi(\tau,k)|^2|\partial_\tau\phi(\tau,k)|^2 - 4[\phi(\tau,k)\partial_\tau\phi^*(\tau,k) + \phi^*(\tau,k)\partial_\tau\phi(\tau,k)]^2.\end{aligned} \tag{297}$$

For the massless Rindler field, with mode function $\phi(\tau,k) = \frac{e^{-i\omega\tau}}{\sqrt{2k}}$, this is given by:

$$\begin{aligned}\Delta^2(\tau,k) &= \frac{e^{-i\omega\tau}e^{+i\omega\tau}}{2\omega}\frac{(-i\omega)(+i\omega)e^{-i\omega\tau}e^{+i\omega\tau}}{2\omega} - [\frac{-i\omega e^{-i\omega\tau}e^{+i\omega\tau}}{2\omega} + \frac{i\omega e^{-i\omega\tau}e^{+i\omega\tau}}{2\omega}] \\ &= 1.\end{aligned} \tag{298}$$

In this calculation the entropy is given by [57]:

$$S_{vN} = (\bar{n}(k,\tau)+1)\ln(\bar{n}(k,\tau)+1) - \bar{n}(k,\tau)\ln(\bar{n}(k,\tau)), \tag{299}$$

where $\bar{n}$ is the statistical particle number given by: $\bar{n}(k,\tau) = \frac{\Delta-1}{2}$. Thus one concludes that the naive Rindler vacuum state appears as having 0 entropy for the Rindler observer.

$$\bar{n}(k,\tau) = 1, \tag{300}$$

$$S_{vN} = 0. \tag{301}$$

6.2 Entropy for the Rindler invariant vacuum

The result from the previous subsection raises the question whether the entropy of the Rindler invariant vacuum is also 0. Let us recall the propagator for the Rindler invariant vacuum:

$$i\Delta^+_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi') = \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \ln\left(\frac{1}{2g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{|\cosh(g\Delta\xi) - \cosh(g\Delta\tau - ig\epsilon)|}\right)\right]. \tag{302}$$

The statistical function is the sum of two complex conjugate quantities and is thus real. Thus the statistical function is given by:

$$\begin{aligned}\lim_{m\to 0} F_{\mathrm{RI},m\to 0}(\tau,\xi;\tau',\xi') &= \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \ln\left(\frac{1}{2g}\sqrt{2e^{g(\xi+\xi')}}\sqrt{|\cosh(g\Delta\xi) - \cosh(g\Delta\tau)|}\right)\right] \\ &= \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \frac{1}{2}\ln\left(\frac{e^{2g\Xi}}{2g^2}\right) - \frac{1}{2}\ln|\cosh(g\Delta\xi) - \cosh(g\Delta\tau)|\right].\end{aligned} \tag{303}$$

Here, Ξ is defined as $\Xi = \frac{\xi+\xi'}{2}$. The Wigner transformed statistical function is now given by:

$$\begin{aligned}\lim_{m\to 0} F_{\mathrm{RI},m\to 0}(k,\tau,\tau',\Xi) &= \frac{1}{2\pi}\int d(\xi-\xi')e^{-ik(\xi-\xi')}\left[-\gamma_{\mathrm{E}} - \ln(m) - \frac{1}{2}\ln\left(\frac{e^{2g\Xi}}{2g^2}\right)\right. \\ &\qquad\left. - \frac{1}{2}\ln|\cosh(g\Delta\xi) - \cosh(g\Delta\tau)|\right] \\ &= \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \frac{1}{2}\ln\left(\frac{e^{2g\Xi}}{2g^2}\right)\right]\delta(k) + I\end{aligned} \tag{304}$$

where I is given by:

$$I = -\int\frac{d(\xi-\xi')}{4\pi}e^{-ik(\xi-\xi')}\ln|\cosh(g\Delta\xi) - \cosh(g\Delta\tau)| \tag{305}$$

This integral over $\xi-\xi'$ is a hard integral to do. One starts by splitting the integral up into three parts, from $-\infty$ to $-\Delta\tau$, $-\Delta\tau$ to $\Delta\tau$ and from $\Delta\tau$ to ∞. This way one gets rid of the absolute sign of the logarithm and one can also introduce small $i\epsilon$'s to yield the answer finite. One then also introduces small δ's away from the singular points at $\Delta\tau$ and $-\Delta\tau$. Then, doing partial integration gives:

$$\begin{aligned}
I &= -\frac{1}{2}\int_{\Delta\tau+\delta}^{\infty}\frac{d(\Delta\xi)}{2\pi}e^{-i(k-i\epsilon)(\Delta\xi)}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\\
&\quad -\frac{1}{2}\int_{-\Delta\tau+\delta'}^{\Delta\tau-\delta}\frac{d(\Delta\xi)}{2\pi}e^{-ik(\Delta\xi)}\ln\left(\cosh(g\Delta\tau)-\cosh(g\Delta\xi)\right)\\
&\quad -\frac{1}{2}\int_{-\infty}^{-\Delta\tau-\delta'}\frac{d(\Delta\xi)}{2\pi}e^{-i(k+i\epsilon)(\Delta\xi)}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\\
&= -\frac{1}{4\pi}\left[\frac{1}{i(k-i\epsilon)}e^{-i(k-i\epsilon)\Delta\xi}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\right]_{\Delta\tau+\delta}^{\infty}\\
&\quad -\frac{1}{4\pi}\left[\frac{1}{ik}e^{-ik\Delta\xi}\ln\left(\cosh(g\Delta\tau)-\cosh(g\Delta\xi)\right)\right]_{-\Delta\tau+\delta'}^{\Delta\tau-\delta}\\
&\quad -\frac{1}{4\pi}\left[\frac{1}{i(k+i\epsilon)}e^{-i(k+i\epsilon)\Delta\xi}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\right]_{-\infty}^{-\Delta\tau-\delta'}\\
&\quad +\frac{g}{4\pi}\int_{\Delta\tau+\delta}^{\infty}d\Delta\xi\frac{1}{i(k-i\epsilon)}e^{-i(k-i\epsilon)\Delta\xi}\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\\
&\quad +\frac{g}{4\pi}\int_{-\Delta\tau+\delta'}^{\Delta\tau-\delta}d\Delta\xi\frac{1}{ik}e^{-ik\Delta\xi}\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\\
&\quad +\frac{g}{4\pi}\int_{-\infty}^{-\Delta\tau-\delta'}d\Delta\xi\frac{1}{i(k+i\epsilon)}e^{-i(k+i\epsilon)\Delta\xi}\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}.
\end{aligned} \tag{306}$$

The boundary terms are readily filled in and get regularized by the $i\epsilon$ prescriptions. They namely give:

$$\begin{aligned}
&-\frac{1}{4\pi}\left\{\left[\frac{1}{i(k-i\epsilon)}e^{-i(k-i\epsilon)\Delta\xi}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\right]_{\Delta\tau+\delta}^{\infty}\right.\\
&\quad -\left[\frac{1}{ik}e^{-ik\Delta\xi}\ln\left(\cosh(g\Delta\tau)-\cosh(g\Delta\xi)\right)\right]_{-\Delta\tau+\delta'}^{\Delta\tau-\delta}\\
&\quad \left.-\left[\frac{1}{i(k+i\epsilon)}e^{-i(k+i\epsilon)\Delta\xi}\ln\left(\cosh(g\Delta\xi)-\cosh(g\Delta\tau)\right)\right]_{-\infty}^{-\Delta\tau-\delta'}\right\}\\
&= \frac{-1}{4\pi}\left[\frac{-1}{i(k-i\epsilon)}e^{-i(k-i\epsilon)(\Delta\tau+\delta)}\ln(g\delta\sinh(g\Delta\tau))+\frac{1}{ik}e^{-ik(\Delta\tau-\delta)}\ln(g\delta\sinh(g\Delta\tau))\right.\\
&\quad \left.-\frac{1}{ik}e^{-ik(-\Delta\tau+\delta')}\ln(g\delta'\sinh(g\Delta\tau))+\frac{1}{i(k+i\epsilon)}e^{-i(k+i\epsilon)(-\Delta\tau-\delta')}\ln(g\delta'\sinh(g\Delta\tau))\right]
\end{aligned} \tag{307}$$

One should regard these terms in the distributional sense. This situation is quite similar to the representation of the δ-function: $\delta(x)=\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}$. Which is 0 for $x\neq 0$ and infinite for $x=0$. As long as $k\neq 0$, these terms cancel as all three regulators approach 0. The last three integrals are still hard, but with some rewriting one can solve them.

$$\begin{aligned}
\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)} &= \frac{e^{2g\Delta\xi}-1}{e^{2g\Delta\xi}+1-e^{g(\Delta\tau+\Delta\xi)}-e^{g(-\Delta\tau+\Delta\xi)}}\\
&= \frac{e^{2g\Delta\xi}-1}{(e^{g\Delta\xi}-e^{g\Delta\tau})(e^{g\Delta\xi}-e^{-g\Delta\tau})}\\
&= \frac{1-e^{-2g\Delta\xi}}{(1-e^{g(\Delta\tau-\Delta\xi)})(1-e^{-g(\Delta\tau+\Delta\xi)})}.
\end{aligned} \tag{308}$$

From the second line one makes a coordinate change to $x = e^{g\Delta\xi}$ with $dx = ge^{g\Delta\xi}d\Delta\xi$. The integrals are now of the form

$$\int \frac{dx}{x} x^a \frac{x^2-1}{(x-b)(x-\frac{1}{b})}$$

with $a = -\frac{i}{g}(k \pm i\epsilon)$ and $b = e^{g\Delta\tau}$. The boundaries change after the coordinate transformation and will now be from: 0 to $e^{-g\Delta\tau}e^{-g\delta}$, from $e^{-g\Delta\tau}e^{g\delta}$ to $e^{g\Delta\tau}e^{-g\delta}$ and from $e^{g\Delta\tau}e^{g\delta}$ to ∞. To obtain an expression in terms of analytic functions, one first has to split the second fraction into three terms

$$\begin{aligned}\frac{x^2-1}{(x-b)(x-\frac{1}{b})} &= \frac{x^2-bx-\frac{x}{b}+1}{(x-b)(x-\frac{1}{b})} + \frac{bx-1}{(x-b)(x-\frac{1}{b})} + \frac{\frac{x}{b}-1}{(x-b)(x-\frac{1}{b})} \\ &= 1 + \frac{b}{x-b} + \frac{\frac{1}{b}}{x-\frac{1}{b}}.\end{aligned} \tag{309}$$

Inserting this back into the original integral then gives the sum of three integrals. Two of which are very similar and the third one is easy. Starting with the easy integral for the upper limits:

$$\begin{aligned}\int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{x} x^a &= \left[\frac{x^a}{a}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\ &= -\frac{e^{-\frac{i}{g}(k-i\epsilon)(g\Delta\tau+\delta)}}{-\frac{i}{g}(k-i\epsilon)}\end{aligned} \tag{310}$$

Note that the $i\epsilon$ prescription ensured the convergence of the upper bound. Now for the second integral:

$$\begin{aligned}\int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{bdx}{x} \frac{x^a}{x-b} &= -\int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{x} \frac{x^a}{1-\frac{x}{b}} \\ &= -\sum_{n=0}^{\infty} \left(\frac{1}{b}\right)^n \int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{x} x^{a+n} \\ &= -\sum_{n=0}^{\infty} \left(\frac{1}{b}\right)^n \left[\frac{x^{a+n}}{a+n}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\ &= \left[-x^a \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \frac{\Gamma(a+n)}{\Gamma(a+n+1)}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\ &= \left[-x^a \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \frac{\Gamma(a)}{\Gamma(a+1)} \frac{(a)_n}{(a+1)_n} \frac{\Gamma(n+1)}{n!}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\ &= \left[\frac{-x^a}{a} \sum_{n=0}^{\infty} \left(\frac{x}{b}\right)^n \frac{(a)_n}{(a+1)_n} \frac{(1)_n}{n!}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\ &= \left[\frac{-x^a}{a} {}_2F_1\left(a, 1, a+1, \frac{x}{b}\right)\right]_{e^{g(\Delta\tau+\delta)}}^{\infty}.\end{aligned} \tag{311}$$

One opted to go the Hypergeometric functions, because they can be expanded well and they can be analytically extended. The other similar integral can be done the same way:

$$\begin{aligned}
\int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{bx}\frac{x^a}{x-\frac{1}{b}} &= -\int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{x}\frac{x^a}{1-bx} \\
&= -\sum_{n=0}^{\infty} b^n \int_{e^{g(\Delta\tau+\delta)}}^{\infty} \frac{dx}{x} x^{a+n} \\
&= -\sum_{n=0}^{\infty} b^n \left[\frac{x^{a+n}}{a+n}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\
&= \left[-x^a \sum_{n=0}^{\infty} (bx)^n \frac{\Gamma(a+n)}{\Gamma(a+n+1)}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\
&= \left[-x^a \sum_{n=0}^{\infty} (bx)^n \frac{\Gamma(a)}{\Gamma(a+1)}\frac{(a)_n}{(a+1)_n}\frac{\Gamma(n+1)}{n!}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\
&= \left[\frac{-x^a}{a} \sum_{n=0}^{\infty} (bx)^n \frac{(a)_n}{(a+1)_n}\frac{(1)_n}{n!}\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\
&= \left[\frac{-x^a}{a}\, {}_2F_1\left(a,1,a+1,bx\right)\right]_{e^{g(\Delta\tau+\delta)}}^{\infty}.
\end{aligned} \tag{312}$$

Once again, the $i\epsilon$ prescription ensures the convergence of the upper bounds. With these solutions to the integrals, the total expression for I will be:

$$\begin{aligned}
I = \frac{1}{4\pi}\Bigg\{ &\frac{1}{i(k-i\epsilon)}\left[\frac{x^{a_-}}{a_-}\left[1-{}_2F_1\left(a_-,1,a_-+1,\frac{x}{b}\right)-{}_2F_1\left(a_-,1,a_-+1,bx\right)\right]\right]_{e^{g(\Delta\tau+\delta)}}^{\infty} \\
&+\frac{1}{ik}\left[\frac{x^{a_0}}{a_0}\left[1-{}_2F_1\left(a_0,1,a_0+1,\frac{x}{b}\right)-{}_2F_1\left(a_0,1,a_0+1,bx\right)\right]\right]_{e^{-g(\Delta\tau-\delta')}}^{e^{g(\Delta\tau-\delta)}} \\
&+\frac{1}{i(k+i\epsilon)}\left[\frac{x^{a_+}}{a_+}\left[1-{}_2F_1\left(a_+,1,a_++1,\frac{x}{b}\right)-{}_2F_1\left(a_+,1,a_++1,bx\right)\right]\right]_{0}^{e^{-g(\Delta\tau+\delta')}}\Bigg\}
\end{aligned} \tag{313}$$

One important remark here is that the a's in the different lines all have a different $i\epsilon$ prescription. In the first line one has: $a_- = -\frac{i}{g}(k-i\epsilon)$, in the second line $a_0 = -\frac{i}{g}k$ and in the third line $a_+ = -\frac{i}{g}(k+i\epsilon)$. Inserting the boundary conditions and filling in the different a's then gives:

$$\begin{aligned}
I = \frac{g}{4\pi}\Bigg\{&\frac{1}{(k-i\epsilon)^2}\Bigg[(-1)^{\frac{i}{g}(k-i\epsilon)}\Gamma\left(1+\frac{i}{g}(k-i\epsilon)\right)\Gamma\left(1-\frac{i}{g}(k-i\epsilon)\right)\left(e^{i(k-i\epsilon)\Delta\tau}+e^{-i(k-i\epsilon)\Delta\tau}\right) \\
&-e^{-i(\Delta\tau+\delta)(k-i\epsilon)}\left(1-{}_2F_1\left(1,-\frac{i(k-i\epsilon)}{g},1-\frac{i(k-i\epsilon)}{g},e^{g(2\Delta\tau+\delta)}\right)-{}_2F_1\left(1,-\frac{i(k-i\epsilon)}{g},1-\frac{i(k-i\epsilon)}{g},e^{g\delta}\right)\right)\Bigg] \\
&+\frac{1}{k^2}\Bigg[e^{-i(\Delta\tau-\delta)k}\left(1-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta}\right)-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta}\right)\right) \\
&-e^{i(\Delta\tau-\delta')k}\left(1-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g(2\Delta\tau-\delta')-g\Delta\tau}\right)-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g\delta'}\right)\right)\Bigg]+ \\
&\frac{1}{(k+i\epsilon)^2}\Bigg[\left(e^{i(\Delta\tau+\delta')(k+i\epsilon)}\right)\left(1-{}_2F_1\left(1,-\frac{i(k+i\epsilon)}{g};1-\frac{i(k+i\epsilon)}{g};e^{-g(2\Delta\tau+\delta')}\right)-{}_2F_1\left(1,-\frac{i(k+i\epsilon)}{g};1-\frac{i(k+i\epsilon)}{g};e^{g\delta'}\right)\right) \\
&\Bigg]\Bigg\}.
\end{aligned} \tag{314}$$

This equation contains Hypergeometric functions evaluated at 1 when one takes the limit of regulators going to 0. These expressions reduce to simple Gamma functions. This explains why the result in the end becomes quite short. One can find the intermediate results for the expansions of the regulators in Appendix I. Inserting the result back into Equation 304, gives the following for the statistical function

$$\begin{aligned}
&F_{\mathrm{RI},m\to 0}(k,\tau,\tau',\Xi) = \\
&\quad = \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \frac{1}{2}\ln\left(\frac{e^{2g\Xi}}{2g^2}\right)\right]\delta(k) - \frac{ge^{-\frac{\pi k}{g}+i\Delta\tau k}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)}{4\pi k^2} \\
&\quad - \frac{ge^{-\frac{\pi k}{g}-i\Delta\tau k}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)}{4\pi k^2} - \frac{e^{-i\Delta\tau k}}{4k} - \frac{e^{i\Delta\tau k}}{4k} \\
&\quad = \frac{1}{2\pi}\left[-\gamma_{\mathrm{E}} - \ln(m) - \frac{1}{2}\ln\left(\frac{e^{2g\Xi}}{2g^2}\right)\right]\delta(k) - \frac{\cos(k\Delta\tau)\cosh(\frac{k\pi}{g})}{k\sinh(\frac{k\pi}{g})}.
\end{aligned} \tag{315}$$

The first thing one notes is that the term multiplying the delta function is what used to be the condensate in the massless case. This condensate depends linearly on Ξ. One sees that when calculating the entropy, one only takes derivatives with respect to τ and τ', so this condensate does not contribute to the entropy. The question is then, how one could observe this condensate, as it is multiplied by a delta function of zero momentum. It cannot be seen in the energy momentum tensor because it is linear in $\xi+\xi'$. Taking two derivatives will always kill the term. Also the Minkowski observer has a vanishing energy momentum tensor. This follows from taking the massless limit of the D dimensional energy momentum tensor from Equation 68. Thus this condensate cannot be measured by measuring the entropy or the energy momentum tensor. It can however be measured using an Unruh detector. An Unruh detector is a particle with different energy levels that couples to the scalar field by a monopole interaction. One assumes that the detector 'ticks' once it undergoes an energy transition. To do this one adds a term of the following form to the Lagrangian: $\lambda_\mu = \zeta\phi\chi$. χ is the field that performs the measurement for the Unruh detector. One now calculates the probability of this transition: $\langle 0_R, E|\, m(\lambda)\hat{\phi}(\tau(\lambda),\xi(\lambda))\hat{\phi}(\tau'(\lambda),\xi'(\lambda))\,|0_R, E_0\rangle$. In the Heisenberg picture the evolution of the monopole is given by: $m(\lambda) = e^{i\hat{H}_0\lambda}m(0)e^{-i\hat{H}_0\lambda}$, where $\hat{H}_0|E\rangle = E|E\rangle$, where λ is the proper time of the detector. Thus the transition amplitude is then given by, see Birrell and Davies [65]:

$$\begin{aligned}
&\langle 0_{\mathrm{RI}}, E|\, m(\lambda)\hat{\phi}(\tau(\lambda),\xi(\lambda))\hat{\phi}(\tau'(\lambda),\xi'(\lambda))\,|0_{\mathrm{RI}}, E_0\rangle \\
&= i\langle E|\, m(0)\,|E_0\rangle \int d\lambda e^{i(E-E_0)\lambda}\langle 0_R|\,\hat{\phi}(\tau(\lambda),\xi(\lambda))\hat{\phi}(\tau'(\lambda),\xi'(\lambda))\,|0_R\rangle \\
&= i\langle E|\, m(0)\,|E_0\rangle \int d\lambda e^{i(E-E_0)\lambda}\left[-\gamma_{\mathrm{E}} - \ln(m) - \ln\left(\frac{1}{2g}\sqrt{2e^{g(\xi(\lambda)+\xi'(\lambda))}}\sqrt{|\cosh(g\Delta\xi(\lambda)) - \cosh(g\Delta\tau(\lambda) - ig\epsilon)|}\right)\right].
\end{aligned} \tag{316}$$

For a detector that is standing still with respect to Rindler coordinates, one would get a $\delta(E-E_0)$ multiplied by the whole expression, and because $E > E_0$ there would be no transition possible. One would still not be able to measure the Ξ dependence of the condensate. Taking a more complicated trajectory of the detector would however yield a result with a possible transition. As an example trajectory, let one take a geodesic in Rindler space. To find a geodesic in Rindler space, one would usually need to solve the geodesic equation: $\frac{d^2x^\mu}{d\lambda^2} - \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$. One can make its life easier by finding a constant of the motion. If one has a Killing vector K, then $E = -K_\mu\frac{dx^\mu}{d\lambda}$ is a constant of the motion for a free particle. Taking the time like Killing vector $K = \partial_\tau$. This means that $K^\mu = (1,0,0,0)$ and $K_\mu = (e^{2g\xi},0,0,0)$. Now, one determines the constant of the motion for a free particle:

$$\begin{aligned}
E &= -K_\mu\frac{dx^\mu}{d\lambda} \\
&= e^{2g\xi}\frac{d\tau}{d\lambda}.
\end{aligned} \tag{317}$$

One knows that for a geodesic one also has $d\lambda^2 = -ds^2$, thus one obtains:

$$\begin{aligned}
1 &= e^{2g\xi}\left(\frac{d\tau}{d\lambda}\right)^2 - e^{2g\xi}\left(\frac{d\xi}{d\lambda}\right)^2 \\
&= E^2e^{-2g\xi} - e^{2g\xi}\left(\frac{d\xi}{d\lambda}\right)^2.
\end{aligned} \tag{318}$$

Going to a new variable $y = e^{2g\xi}$ and rearranging some terms:

$$\pm d\lambda = \frac{dy}{2g\sqrt{E^2 - y}}. \tag{319}$$

Integrating this is easy and the right hand side simply gives $-\frac{1}{g}\sqrt{E^2 - y}$. One solves to obtain an expression for $\xi(\lambda)$:

$$\xi(\lambda) = \frac{1}{2g}\ln\left(E^2 - [\pm g\lambda - gC_\xi]^2\right), \tag{320}$$

where C_ξ is the integration constant coming from the left-hand side of Equation 319. One now determines the constant by imposing the initial condition $\xi(\lambda = 0) = \xi_0$. One finds that $C = \pm\frac{1}{g}\sqrt{E^2 - e^{2g\xi_0}}$. One is free to choose E. A neat choice for E is $E = e^{2g\xi_0}$. This gives the following trajectory for the ξ coordinate:

$$\xi(\lambda) = \frac{1}{2g}\ln\left(e^{2g\xi_0} - g^2\lambda^2\right). \tag{321}$$

Now one solves for $\tau(\lambda)$. One starts with the Equation 317. Rewriting this equation yields:

$$\frac{e^{g\xi_0}}{g^2}\frac{d\lambda}{\frac{e^{2g\xi_0}}{g^2} - \lambda^2} = d\tau. \tag{322}$$

This is a known integral and this has as a solution:

$$\tau(\lambda) = \frac{1}{g}\mathrm{arctanh}\left(ge^{-g\xi_0}\lambda\right) - C_\tau. \tag{323}$$

One chooses the initial condition $\tau(\lambda = 0) = 0$ to set the constant $C_\tau = 0$. Then the integral that one needs to solve for the transition amplitude is given by:

$$\begin{aligned}
&\langle 0_{\mathrm{RI}}, E|\, m(\lambda)\hat{\phi}(\tau(\lambda), \xi(\lambda))\hat{\phi}(\tau'(\lambda), \xi'(\lambda))\, |0_{\mathrm{RI}}, E_0\rangle = i\,\langle E|\, m(0)\,|E_0\rangle \int d\lambda e^{i(E-E_0)\lambda} \\
&\Bigg\{ -\gamma_{\mathrm{E}} - \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right) - \frac{1}{2}\ln\left(\frac{\mu^2}{2g^2}\right) - \frac{\ln\left(e^{2g\xi_0} - g^2\lambda^2\right) + \ln\left(e^{2g\xi_0'} - g^2\lambda^2\right)}{2g} \\
&- \frac{1}{2}\ln\left|\cosh\left[\mathrm{arctanh}\left(e^{g\xi_0}g\lambda\right) - \mathrm{arctanh}\left(e^{g\xi_1}g\lambda\right)\right] - \cosh\left[\frac{1}{2}\left(\log\left(e^{2g\xi_0} - g^2\lambda^2\right) - \log\left(e^{2g\xi_1} - g^2\lambda^2\right)\right)\right]\right|\Bigg\}.
\end{aligned} \tag{324}$$

One ignores everything that is not part of the spatially dependent condensate, because this example is to show how an Unruh detector could measure the spatially dependent condensate. Thus one needs to solve the integral, let one call it I_{Unruh}:

$$I_{\mathrm{Unruh}} = -\int_{\lambda_1}^{\lambda_2} d\lambda \frac{\ln\left(e^{2g\xi_0} - g^2\lambda^2\right) + \ln\left(e^{2g\xi_0'} - g^2\lambda^2\right)}{2g}. \tag{325}$$

This integral can be solved, and the result is given in Appendix J. The answer is expressed in terms of the Exponential integral function. This amplitude establishes that one can observe the difference between the two Rindler states.

As the condensate does not contain any dependence on the temporal variables, one knows that the $k = 0$, part of the expression does not contribute to the entropy. Thus one can treat the $k = 0$ separately from the $k \neq 0$ modes. For all other modes, the entropy can be calculated per mode k. To do this one calculates the Gaussian invariant for the $k \neq 0$ part of the expression. This is given by:

$$\Delta^2_{\mathrm{RI},m\to 0}(k,\tau,\tau') = 4\lim_{\tau\to\tau'}\left[F_{\mathrm{RI},m\to 0}(k,\tau,\tau')\partial_\tau\partial_\tau' F_{\mathrm{RI},m\to 0}(k,\tau,\tau') - \partial_\tau F_{\mathrm{RI},m\to 0}(k,\tau,\tau')\partial_\tau' F_{\mathrm{RI},m\to 0}(k,\tau,\tau')\right]. \tag{326}$$

It is not hard to see from Equations 315 and 326 that the Gaussian invariant is given by:

$$\Delta^2_{\mathrm{RI},m\ \to 0}(k) = \frac{\cosh^2(\frac{k\pi}{g})}{\sinh^2(\frac{k\pi}{g})}. \tag{327}$$

From this quantity one then again calculates the particle number expectation value, which is given by:

$$\begin{aligned}\bar{n}_{\mathrm{RI},m\ \to 0}(k) &= \frac{\Delta(k)-1}{2} \\ &= \frac{1}{2}\left(\left|\frac{\cosh(\frac{k\pi}{g})}{\sinh(\frac{k\pi}{g})}\right| - 1\right) \\ &= \frac{1}{e^{\frac{2|k|\pi}{g}}-1}.\end{aligned} \tag{328}$$

This is a thermal distribution of particles with temperature $T = \frac{g}{2\pi}$. This is a nice result as one would have also obtained this result by considering the naive Bogoliubov transformation of the annihilation and creation operators:

$$\begin{aligned}n(k) &= \langle 0_R|\, \hat{n}_M(k)\, |0_R\rangle \\ &= \langle 0_R|\, \hat{b}^\dagger(k)\hat{b}(k)\, |0_R\rangle \\ &= \langle 0_R| \left(\int_0^\infty dp[\hat{a}^\dagger(p)\alpha^*_{pk'} + \hat{a}(p)\beta_{pk'}]\right)\left(\int_0^\infty dk[\hat{a}(k)\alpha_{kk'} + \hat{a}^\dagger(k)\beta^*_{kk'}]\right)|0_R\rangle \\ &= \int_0^\infty dp\beta_{pk'}\beta^*_{pk'}.\end{aligned} \tag{329}$$

This particle number is not the same as the statistical particle number that was calculated in for example Equation 328. This number is of relevance when one studies how much energy is stored in some state. In the case of a thermal state they are equal, because a thermal density matrix is written as $\hat{\rho} = \frac{1}{Z}e^{-\beta\hat{H}}$. This diagonalises simultaneously with the Hamiltonian and then one thus has $n = \bar{n}$. As one can see in Equation 160, this gives rise to a $\frac{1}{\sinh(\frac{p\pi}{g})}$ and is thus divergent at $p = 0$. There can be worked around this by starting from Equation 165 and using Equation 163. This then gives:

$$\begin{aligned}\delta(p-k') &= \int_0^\infty dk(\alpha_{k'k}\alpha^*_{pk} - \beta k' k\beta^*_{pk}) \\ &= \left(e^{\frac{(k'+p)\pi}{g}} - 1\right)\int_0^\infty dk\beta_{k'k}\beta^*_{pk}.\end{aligned} \tag{330}$$

Letting p go to k and inserting back into Equation 328 then gives:

$$n(k) = \frac{\delta(0)}{e^{\frac{2k\pi}{g}}-1}. \tag{331}$$

This famous result was already derived by Unruh and is now called the Unruh effect [66]. This $\delta(0)$ is of course infinite, but this can be renormalised by putting the theory in a finite sized box of size L. The way this works is one rescales the creation and annihilation operator with a factor of $\sqrt{L}$. Instead of a Dirac delta distribution, one gets a Kronecker delta as the result of the commutator:

$$[\hat{a}, \hat{a}^\dagger] = \delta_{k,k'}, \tag{332}$$

where $k = \frac{2\pi n}{L}$ and $k' = \frac{2\pi n'}{L}$. Another way to obtain a finite result is by introducing a coarse-grained number operator. This is necessary because $\hat{a}^\dagger(k)\hat{a}(k)$ is a composite operator and thus does not have a finite expectation value. One defines the coarse grained number operator as:

$$N(k) = \int dkW(k,k')\hat{a}^\dagger(k)\hat{a}(k'), \tag{333}$$

where $W(k,k')$ is a window function satisfying $\int dkW(k,k)=1$. This way the $\delta(0)$ from 331 is renormalised and one obtains:

$$n_{ren}(k)=\frac{1}{e^{\frac{2k\pi}{g}}-1} \tag{334}$$

From this statistical particle number for $k\neq 0$, Equation 328, one obtains the entropy per mode:

$$\begin{aligned}S_{vN,\mathrm{RI},m\to 0}&=(\bar{n}_{\mathrm{RI},m\to 0}(k)+1)\ln(\bar{n}_{\mathrm{RI},m\to 0}(k)+1)-\bar{n}_{\mathrm{RI},m\to 0}(k)\ln(\bar{n}_{\mathrm{RI},m\to 0}(k))\\&=\frac{1}{2}\left(\coth\left(\frac{\pi k}{g}\right)-1\right)\left[-\log\left(\coth\left(\frac{\pi k}{g}\right)-1\right)+e^{\frac{2\pi k}{g}}\log\left(\frac{1}{2}\coth\left(\frac{\pi k}{g}\right)+\frac{1}{2}\right)+\log(2)\right].\end{aligned} \tag{335}$$

Thus in this section one concludes that the two different vacua that one constructed have a different entropy. The vacuum that is invariant under the symmetries of Rindler space contains a thermal distribution of particles while the vacuum that violates the Rindler symmetries appears to be empty. One can calculate the entropy from this by integrating 335 over all different modes:

$$S_{vN,\mathrm{RI},m\to 0}=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left[(\bar{n}_{\mathrm{RI},m\to 0}(k)+1)\ln(\bar{n}_{\mathrm{RI},m\to 0}(k)+1)-\bar{n}_{\mathrm{RI},m\to 0}(k)\ln(\bar{n}_{\mathrm{RI},m\to 0}(k))\right]. \tag{336}$$

This integral is symmetric in k and one can thus change it to two times the integral from 0 to infinity. One changes the integration variable to $x=e^{\beta k}$ where $\beta=\frac{2\pi}{g}$. Then, $dx=\beta e^{\beta k}dk$. This way one obtains:

$$\begin{aligned}S_{vN,\mathrm{RI},m\to 0}&=\frac{1}{\pi\beta}\int_1^{\infty}\frac{dx}{x}\left\{\frac{x}{x-1}\left[\ln(x)-\ln(\mathrm{x}-1)\right]+\frac{\ln(x-1)}{x-1}\right\}\\&=\frac{1}{\pi\beta}\int_1^{\infty}dx\left[\frac{\ln(x)}{x-1}-\frac{\ln(x-1)}{x}\right]\\&=\frac{1}{\pi\beta}\int_0^{\infty}dy\left[\frac{\ln(y+1)}{y}-\frac{\ln(y)}{y+1}\right]\\&=\frac{1}{\pi\beta}\left\{\int_0^{\infty}dy\left[\frac{\ln(y+1)}{y}\right]-\left[\ln(y)\ln(y+1)\right]_0^{\infty}+\int_0^{\infty}dy\frac{\ln(y+1)}{y}\right\}.\end{aligned} \tag{337}$$

Where in the last step one has partially integrated the second term. One sees that the first and the last term are the same now and that one needs to introduce two cut-offs for the boundary terms to converge. Therefore one introduces an IR cut-off $\delta << 1$ and a UV cut-off $\Delta >> 1$. Furthermore, one realises that the integral that has to be solved is the dilogarithm:

$$\mathrm{Li}_2(z)=-\int_0^z dt\frac{\ln(1-t)}{t}=-\int_0^{-z}\frac{\ln(1+t)}{t}. \tag{338}$$

With some manipulations on the integral corresponding to the dilogarithm, one then finds:

$$S_{vN,\mathrm{RI},m\to 0}=\frac{1}{\pi\beta}\left\{-\ln(\Delta)\ln(\Delta+1)+\ln(\delta)\ln(\delta+1)-2\left[\mathrm{Li}_2(-\Delta)-\mathrm{Li}_2(-\delta)\right]\right\}. \tag{339}$$

One knows an asymptotic expansion of the dilogarithm as $\mathrm{Li}_2\to -\frac{1}{2}\ln^2(-z)-\frac{\pi^2}{6}$ when $|z|\to\infty$. From the definition of the dilogarithm in Equation 338 one also sees that the dilogarithm approaches 0 for small arguments. Thus one can expand Equation 339 as:

$$S_{vN,\mathrm{RI},m\to 0}=\frac{1}{\pi\beta}\left\{-\ln^2(\Delta)-\ln(\Delta)+\frac{\pi^2}{3}+\ln(\delta)\right\}. \tag{340}$$

From this one sees that the entropy is both UV and IR divergent. It is common practice in literature to look at this in the following way. One introduces a maximal momentum $k_{\max}$. One does this by setting $\Delta=e^{\beta k_{\max}}-1$. From this one knows that $\beta k_{\max}=\ln(\Delta+1)\approx\ln(\Delta)$. Here the constant β is given by

$\beta = \frac{2\pi}{g}$ Moreover one introduces a minimal momentum $k_{\min}$ such that $\delta = e^{\beta k_{\min}} \approx \beta k_{\min}$. With these maximal and minimal momenta one obtains:

$$S_{vN,\mathrm{RI},m\,\to 0} \approx \frac{\beta k_{\max}^2}{\pi}\left\{1 - \frac{1}{\beta k_{\max}} + \frac{\beta k_{\min} + \frac{\pi^2}{3}}{(\beta k_{\max})^2}\right\} + \text{smaller order terms.} \tag{341}$$

In this expressions one takes $k_{\max}$ to be roughly of order Planck mass $m_{\mathrm{P}} = \sqrt{\frac{c\hbar}{G_{\mathrm{N}}}}$ and $k_{\min}$ as approximately: $k_{\min} \approx \frac{c^2}{2G_{\mathrm{N}}M}$, where M the black hole mass. Now the entropy scales as m_{P}^2, which is roughly the right answer. One now gets the total entropy by multiplying this entropy density with the volume, which in this case is just the length of the system. One can take this to be approximately the radius of the event horizon.

7 Discussion and conclusion

The aim of this thesis was to calculate the Von Neumann entropy in a near horizon analogue situation and analyse how the back reaction influences this result. This aim appeared to be a bridge too far, but a lot of work towards this aim has been carried out. In summary, the D-dimensional massless propagator was first calculated to get used to the methodology in Section 2. Then in Section 3, the massless field in Rindler space was quantised by solving the field equations. With these fields the massless Bogoliubov coefficients were determined and the consistency checks showed that these were the correct Bogoliubov coefficients. By defining a naive vacuum, the vacuum which vanishes by acting with all annihilation operators, the two-point function and the corresponding spectral and statistical function were calculated for this vacuum state. This naive vacuum violated the Rindler isometries, which became obvious when one calculated the massless limit of massive propagator. The massive propagator was determined using the fact that in every maximally symmetric space there exists a state in which the propagator only depends on the invariant distance. The massless limit of the propagator of this state is invariant under the Rindler isometries and contains a spatially dependent Bose condensate. Moving to Section 4, there was shown that the thus far considered states had a vanishing energy momentum tensor and that back reaction was thus not applicable. In Section 5, there is attempted to calculate the two-point function for the naive Rindler vacuum from the point of view of a Minkowski observer. This calculation contained many divergences with nontrivial space time dependences. Different ways of regulating the results did not lead to the same regular parts. One still needs to think of a physical reason for regulating in a certain way. Lastly, in Section 6 the Von Neumann entropy is calculated for the naive vacuum that breaks Rindler isometries and for the Rindler invariant vacuum. The naive Rindler vacuum appears to be empty, and has zero entropy. The Rindler invariant vacuum contains a thermal distribution of particles with corresponding entropy.

In this thesis a couple of peculiarities have been found. First of all, the spatially dependent Bose condensate that was obtained in the Rindler invariant vacuum does not contribute to the energy momentum tensor and neither to the entropy. This condensate breaks the Rindler isometries. This is a phenomenon called symmetry non-inheritance. This phenomenon tells us that matter fields do not have to satisfy the same symmetries as the space they live in. In the past couple of years constraints have been put on the symmetry non-inheritance phenomenon, [67], [68], [69], [70] and a Kerr Black hole model with scalar hair has been developed using this phenomenon [71]. This symmetry non-inheritance is possibly a way around the Bekenstein no-hair theorem and could thus possibly be an interesting avenue for future research.

Secondly, it appears that the Rindler vacuum state is degenerate. There has been calculated that both the naive Rindler vacuum and the Rindler invariant vacuum have a vanishing energy momentum tensor. This raises the question how big the space is of all Rindler states with a vanishing energy momentum tensor. From a first consideration, one can see from Equation 97 that one fixed two complex parameters $\alpha(k)$ and $\beta(k)$. These two complex parameters are thus four real degrees of freedom, however they still need to satisfy one constraint and one symmetry of the system. This symmetry is the symmetry that the phase of the mode function does not affect the physics: $\tilde{\phi}(\tau, k) = e^{i\theta}\phi(\tau, k)$. From this one concludes that there is already an infinite number of vacua that can be described by two real degrees of freedom. Moreover one has the vacuum that is invariant under Rindler isometries. The Rindler invariant vacuum does not seem to contain such a parametrisation, but to fully investigate this one would need to calculate the massive, quantised fields. A first estimate would then be that the space of Rindler vacua can at least be parametrised by two real parameters with one outlier that is not parametrised by the two parameters. An interesting follow up question would then be: how could a mass decrease this set of vacua? In other words, is the set of vacua with energy $m^2\left[\frac{1}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma_{\mathrm{E}}-1\right]$ smaller than the set of massless vacua. Of course, back reaction would become applicable in this situation. The existence of a non-vanishing energy momentum tensor would induce curvature in the system, effectively turning the system in some kind of curved Rindler space. Since the fields would have to satisfy a different equation now, this would entirely change the set of vacua.

What has been done in this thesis is primarily a toy model for a near horizon observer. One related the radial coordinate in Schwarzschild to the spatial Rindler coordinate. However, Schwarzschild also contains the spherical coordinates. These coordinates should be incorporated in the toy model as well. Possibly, one could analyse the situation the same way in the space of $\mathbb{R}^2 \times S^2$. One could take a metric in the form of

$$ds^2 = -e^{2g\xi}\left(d\tau^2 - d\xi^2\right) + r^2(\xi)d\Omega^2. \tag{342}$$

In the introduction there was shown how the 2-dimensional Rindler and the radial and temporal part of the Schwarzschild metric are equal in the limit where the observer is close to the horizon. This can be extended to the angular coordinates. Applying the same transformation as in the introduction without all constants this time, $X^2 = 1 - \frac{2m}{r}, T = 4mt$, yields:

$$ds^2 = 16m^2\left(-X^2dT^2 + (1 + 4X^2 + ...)dX^2 + \frac{1}{4}\left(1 + X^2 + ...\right)^2\left[d\theta^2 + \sin^2(\theta)d\phi^2\right]\right). \tag{343}$$

In the near horizon limit, the last part of this expression is exactly the 2-sphere with a Schwarzschild radius radius. So one can try to solve the field equations for the metric given in 342 with $r^2(\xi) = \frac{2G_\mathrm{N}m}{c^2}$. This would be a neat way to incorporate the angular dependence in a still somewhat easier metric than the Schwarzschild metric.

Lastly, it is important to consider how the obtained results can eventually help solving the black hole information paradox. In the Bogoliubov calculation from Section 5, a lot of infinities were found that depended non-trivially on the spacetime coordinates. This could be interpreted as a Minkowski observer seeing infinitely many infrared particles in the Rindler invariant vacuum. An interesting question is then whether adding a mass in the theory would regulate all these infinities. For this one needs to solve and quantise the massive equation of motion for Rindler space, and then calculate the Bogoliubov coefficients between the massive Minkowski field and the massive Rindler field. Doing this teaches us whether there are many highly energetic particles at the horizon. In other words, this could give insight in whether firewalls exist. This calculation could be done for all states and maybe it is possible to conclude from these results how the firewalls could depend on the states of the system.

Appendices

A Flatness of Rindler space

To show that Rindler is a flat spacetime, let one first start with the Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left[\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right]. \tag{344}$$

With the Rindler metric given by Equation 77, all non-zero Christoffel symbols are given by:

$$\Gamma^{\xi}_{\tau\tau} = g, \; \Gamma^{\xi}_{\xi\xi} = g, \; \Gamma^{\tau}_{\xi\tau} = g. \tag{345}$$

As the Christoffel symbols are constant throughout space and time, one can calculate the Riemann tensor as:

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{346}$$

The only unique component in 2 dimensions is R^{1}_{212}, which is easily calculated from the Christoffel symbols and yields 0. This was expected as one only did a coordinate transformation from Minkowski space. This means that Rindler space is flat.

B Equation of motion in Rindler coordinates

One wants to obtain the equation of motion that the fields have to satisfy in the Rindler coordinates. For this one makes use of the equations that the describe the transition between Minkowski coordinates and Rindler coordinates:

$$x = \frac{e^{g\xi}}{g} \cosh(g\tau) \tag{347}$$

$$t = \frac{e^{g\xi}}{g} \sinh(g\tau), \tag{348}$$

and their inverse:

$$\xi = \frac{1}{g} \ln\left(\sqrt{g^2(x^2 - t^2)} \right) \tag{349}$$

$$\tau = \frac{1}{g} \operatorname{arctanh}\left(\frac{t}{x} \right) \tag{350}$$

With this transformation the infinitesimal distance elements are given by:

$$dt = e^{g\xi}\sinh(g\tau)d\xi + e^{g\xi}\cosh(g\tau)d\tau \tag{351}$$

$$dx = e^{g\xi}\cosh(g\tau)d\xi + e^{g\xi}\sinh(g\tau)d\tau. \tag{352}$$

Thus the line element transforms as:

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 \\ &= e^{2g\xi} \left(-\sinh^2(g\tau)d\xi^2 - \cosh^2(g\tau)d\tau^2 - 2\cosh(g\tau)\sinh(g\tau)d\xi d\tau \right. \\ &\qquad \left. + \cosh^2(g\tau)d\xi^2 + \sinh^2(g\tau)d\tau^2 + 2\cosh(g\tau)\sinh(g\tau)d\xi d\tau \right) \\ &= e^{2g\xi} \left(-d\tau^2 + d\xi^2 \right). \end{aligned} \tag{353}$$

To transform all derivatives in the equation of motion one needs to apply the chain rule: $\partial_t = \sum_i \frac{\partial x^i}{\partial t}\partial_{x^i}$. The derivative with respect to t is given by:

$$\begin{aligned}\partial_t &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} \\ &= \frac{1}{2g}\frac{-2t}{x^2-t^2}\partial_\xi + \frac{1}{g}\frac{\frac{1}{x}}{1-\frac{t^2}{x^2}}\partial_\tau \\ &= \frac{1}{g}\frac{-t}{x^2-t^2}\partial_\xi + \frac{1}{g}\frac{x}{x^2-t^2}\partial_\tau \\ &= -e^{-g\xi}\sinh(g\tau)\partial_\xi + e^{-g\xi}\cosh(g\tau)\partial_\tau.\end{aligned} \tag{354}$$

The derivative with respect to x is given by:

$$\begin{aligned}\partial_x &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} \\ &= \frac{1}{2g}\frac{2x}{x^2-t^2}\partial_\xi + \frac{1}{g}\frac{\frac{-t}{x^2}}{1-\frac{t^2}{x^2}}\partial_\tau \\ &= \frac{1}{g}\frac{x}{x^2-t^2}\partial_\xi + \frac{1}{g}\frac{-t}{x^2-t^2}\partial_\tau \\ &= e^{-g\xi}\cosh(g\tau)\partial_\xi - e^{-g\xi}\sinh(g\tau)\partial_\tau.\end{aligned} \tag{355}$$

These two expressions lead to the following second derivatives:

$$\begin{aligned}\partial_t^2 &= \left(-e^{-g\xi}\sinh(g\tau)\partial_\xi + e^{-g\xi}\cosh(g\tau)\partial_\tau\right)\left(-e^{-g\xi}\sinh(g\tau)\partial_\xi + e^{-g\xi}\cosh(g\tau)\partial_\tau\right) \\ &= e^{-2g\xi}\sinh^2(g\tau)\partial_\xi^2 + e^{-2g\xi}\cosh^2(g\tau)\partial_\tau^2 - ge^{-2g\xi}\sinh^2(g\tau)\partial_\xi \\ &\quad + ge^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau - e^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\xi\partial_\tau - ge^{-2g\xi}\cosh^2(g\tau)\partial_\xi \\ &\quad - e^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau\partial_\xi + ge^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau,\end{aligned} \tag{356}$$

$$\begin{aligned}\partial_x^2 &= \left(e^{-g\xi}\cosh(g\tau)\partial_\xi - e^{-g\xi}\sinh(g\tau)\partial_\tau\right)\left(e^{-g\xi}\cosh(g\tau)\partial_\xi - e^{-g\xi}\sinh(g\tau)\partial_\tau\right) \\ &= e^{-2g\xi}\cosh^2(g\tau)\partial_\xi^2 + e^{-2g\xi}\sinh^2(g\tau)\partial_\tau^2 - ge^{-2g\xi}\cosh^2(g\tau)\partial_\xi \\ &\quad + ge^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau - e^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\xi\partial_\tau - ge^{-2g\xi}\sinh^2(g\tau)\partial_\xi \\ &\quad - e^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau\partial_\xi + ge^{-2g\xi}\sinh(g\tau)\cosh(g\tau)\partial_\tau.\end{aligned} \tag{357}$$

Thus now the partial derivatives in the equation of motion is given by:

$$-\partial_t^2 + \partial_x^2 = e^{-2g\xi}\left(-\partial_\tau^2 + \partial_\xi^2\right). \tag{358}$$

C Klein Gordon norm of Minkowski mode functions

In the main text one saw the Klein Gordon norm of the different Minkowski mode functions were given by:

$$(f_k, f_{k'})_{KG} = -(f_k^*, f_{k'}^*)_{KG} = 2\pi\delta(k-k') \tag{359}$$

$$(g_k, g_{k'})_{KG} = -(g_k^*, g_{k'}^*)_{KG} = 2\pi\delta(k-k') \tag{360}$$

$$(f_k^*, f_{k'})_{KG} = (g_k^*, g_{k'})_{KG} = (f_k, g_{k'})_{KG} = (f_k, g_{k'}^*)_{KG} = 0 \tag{361}$$

Here, one will carry out one of these products and all other ones can be done in a similar fashion.

$$(f_k, f_{k'})_{KG} = \frac{-i}{2\sqrt{kk'}}\int_{\Sigma_t} e^{-ik(t-x)}\partial_t(e^{ik'(t-x)}) - e^{ik'(t-x)}\partial_t(e^{-ik(t-x)})dx \tag{362}$$

$$= \frac{-i}{2\sqrt{kk'}}(ik'+ik)e^{-i(k-k')t}\int_{-\infty}^{\infty} e^{i(k-k')x}dx = \frac{-i\pi}{\sqrt{kk'}}(ik'+ik)e^{-i(k-k')t}\delta(k-k') = 2\pi\delta(k-k') \tag{363}$$

Here one used the fact that

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{ikx} = \delta(k), \tag{364}$$

and that $\delta(x-x_0)f(x) = f(x_0)\delta(x-x_0)$. For the functions with a vanishing Klein Gordon norm one has:

$$(f_k, g_{k'})_{KG} = \frac{-i}{2\sqrt{kk'}} \int_{\Sigma_t} e^{-ik(t-x)}\partial_t(e^{ik'(t+x)}) - e^{ik'(t-x)}\partial_t(e^{-ik(t+x)})dx \tag{365}$$

$$= \frac{-i}{2\sqrt{kk'}}(ik' + ik)e^{-i(k-k')t}\int_{-\infty}^{\infty} e^{i(k+k')x}dx = \frac{-i\pi}{\sqrt{kk'}}(ik'+ik)e^{-i(k-k')t}\delta(k+k') = 0. \tag{366}$$

The last step is made with the property of the delta function that $\delta(x-x_0)f(x) = f(x_0)\delta(x-x_0)$.

D Evaluation of the Bogoliubov coefficients integrals

In the main text, one stumbled upon the following integral:

$$A = \int_{-\infty}^{\infty} \frac{du}{2\pi} e^{i(pu-k\bar{u})}. \tag{367}$$

To solve this integral, one first remembers the definition for $\bar{u}$ and rewrite it in terms of u,

$$\bar{u} = t - x = \frac{1}{g}e^{g\xi}(sinh(g\tau) - cosh(g\tau)) = -\frac{1}{g}e^{(g(\xi-\tau))} = -\frac{1}{g}e^{-gu}. \tag{368}$$

Thus one needs to solve the following integral:

$$\int_{-\infty}^{\infty} \frac{du}{2\pi} e^{ipu} e^{i\frac{k}{g}exp(-ug)} \tag{369}$$

One can now do a variable change from $-ug$ to $ln(x)$. With this substitution one has

$$du = \frac{-1}{g}\frac{dx}{x}.$$

The integral boundaries will change to ∞ till 0 as a result of this substitution. Thus one obtains:

$$A = \frac{1}{2\pi}\int_{\infty}^{0} \left(-\frac{1}{g}\frac{dx}{x}\right) e^{\frac{-ip}{g}ln(x)} e^{i\frac{k}{g}exp^{ln(x)}} \tag{370}$$

$$= \frac{1}{2\pi g}\int_{0}^{\infty} dx\ x^{\frac{-ip}{g}-1} e^{\frac{ik}{g}x}. \tag{371}$$

Now one has to realise that the definition of the gamma function is given by:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt. \tag{372}$$

To work towards this form of an integral, one first makes a substitution of $z = \frac{-ik}{g}x$, which gives $dz = \frac{-ik}{g}dx$. This yields the following:

$$A = \frac{1}{2\pi g}\frac{g}{-ik}\int_0^{-i\infty} \left(\frac{g}{-ik}\right)^{\frac{-ip}{g}-1} z^{\frac{-ip}{g}-1}e^{-z}dz. \tag{373}$$

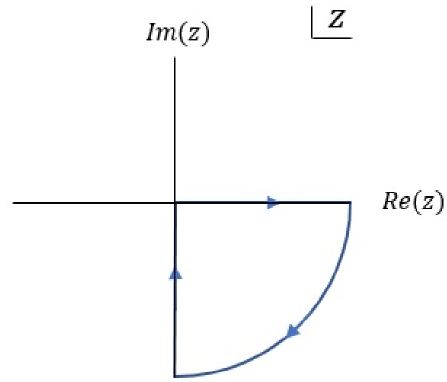

Figure 6: The integration contour as described in the text.

Now one has to make use of contour integration to see that this integral equals the integral of the same argument but with boundaries 0 and ∞. This is because one can close a contour around the bottom half corner of the complex z-plane. Here the contribution from the infinite quarter circle will vanish because of the e^{-z}. This is depicted in figure 6. Since there are no poles in the contour, one has the following equation for the contour integral:

$$0 = \int_{contour} = \int_{quartercircle} + \int_{-i\infty}^{0} + \int_{0}^{\infty}. \tag{374}$$

Thus equation 373 can be rewritten in the form of the gamma function and thus the result one obtains for our integral is:

$$A = \frac{1}{2\pi g}\frac{g}{-ik}\frac{-ik}{g}\left(\frac{g}{-ik}\right)^{\frac{-ip}{g}}\Gamma(\frac{-ip}{g}) = \frac{1}{2\pi g}\left(\frac{ig}{k}\right)^{\frac{-ip}{g}}\Gamma(\frac{-ip}{g}) \tag{375}$$

The only thing that one wants to get rid of now, is the i^i in the expression. One knows that $i^i = e^{i\ ln(i)} = e^{i(log(1)+arg(i))} = e^{-\frac{\pi}{2}}$. Using this in equation 375, then yields

$$A = \frac{e^{\frac{p\pi}{2g}}}{2\pi g}\left(\frac{g}{k}\right)^{\frac{-ip}{g}}\Gamma(\frac{-ip}{g}) \tag{376}$$

The other integrals can be carried out in roughly the same way. Thus giving us the following results for the Bogoliubov coefficients:

$$\alpha_{pk} = \sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}), \tag{377}$$

$$\beta_{pk} = -\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}), \tag{378}$$

$$\epsilon_{pk} = \sqrt{\frac{p}{k}}\frac{e^{\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}) \tag{379}$$

$$\sigma_{pk} = -\sqrt{\frac{p}{k}}\frac{e^{-\frac{p\pi}{2g}}}{2\pi g}(\frac{g}{k})^{\frac{ip}{g}}\Gamma(\frac{ip}{g}). \tag{380}$$

E Killing vectors in Rindler

In Appendix B, one saw that the partial derivatives were given by:

$$\partial_t = -e^{-g\xi}\sinh(g\tau)\partial_\xi + e^{-g\xi}\cosh(g\tau)\partial_\tau. \tag{381}$$

$$\partial_x = e^{-g\xi}\cosh(g\tau)\partial_\xi - e^{-g\xi}\sinh(g\tau)\partial_\tau. \tag{382}$$

These directly translate to the Killing vectors in Rindler space, because in Minkowski space the Killing vectors are given by: ∂_t, ∂_x and $x\partial_t + t\partial_x$. This means that the Rindler Killing vectors are given by:

$$K_1 = -e^{-g\xi}\sinh(g\tau)\partial_\xi + e^{-g\xi}\cosh(g\tau)\partial_\tau \tag{383}$$

$$K_2 = e^{-g\xi}\cosh(g\tau)\partial_\xi - e^{-g\xi}\sinh(g\tau)\partial_\tau \tag{384}$$

$$K_3 = \partial_\tau \tag{385}$$

F Equality of the spectral functions for the different states and equality of the statistical functions in a certain limit

The different spectral functions were given by, 213 190:

$$i\Delta^c_m(\tau,\xi;\tau',\xi') = -\frac{i}{2}\left[\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2) - \theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)\right]. \tag{386}$$

$$i\Delta^c(\tau,\xi;\tau',\xi') = -\frac{i}{4}\left[\theta(\Delta\tau-\Delta\xi) - \theta(\Delta\xi-\Delta\tau) + \theta(\Delta\tau+\Delta\xi) - \theta(-\Delta\tau-\Delta\xi)\right]. \tag{387}$$

To see that the spectral functions of the different states are equal, one can just look at the values the combination of θ-function take in different regions of the $\Delta\tau - \Delta\xi$ plane. Consider the Regions I,II,III,IV in Figure 7. Considering the θ-functions with quadratic arguments. One concludes that $\theta(\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)$ is non-zero only in region II and has a value +1 there and that -$\theta(-\Delta\tau)\theta(\Delta\tau^2-\Delta\xi^2)$ is non-zero only in region IV and has a value -1 there. The θ-functions of the massless case are given by:

$$\theta(\Delta\tau-\Delta\xi) = \begin{cases} 1, & \text{in regions II and III} \\ 0, & \text{in regions I and IV} \end{cases} \tag{388}$$

$$\theta(\Delta\xi-\Delta\tau) = \begin{cases} 1, & \text{in regions I and IV} \\ 0, & \text{in regions II and III} \end{cases} \tag{389}$$

$$\theta(\Delta\tau+\Delta\xi) = \begin{cases} 1, & \text{in regions I and II} \\ 0, & \text{in regions III and IV} \end{cases} \tag{390}$$

$$\theta(-\Delta\tau-\Delta\xi) = \begin{cases} 1, & \text{in regions III and IV} \\ 0, & \text{in regions I and II.} \end{cases} \tag{391}$$

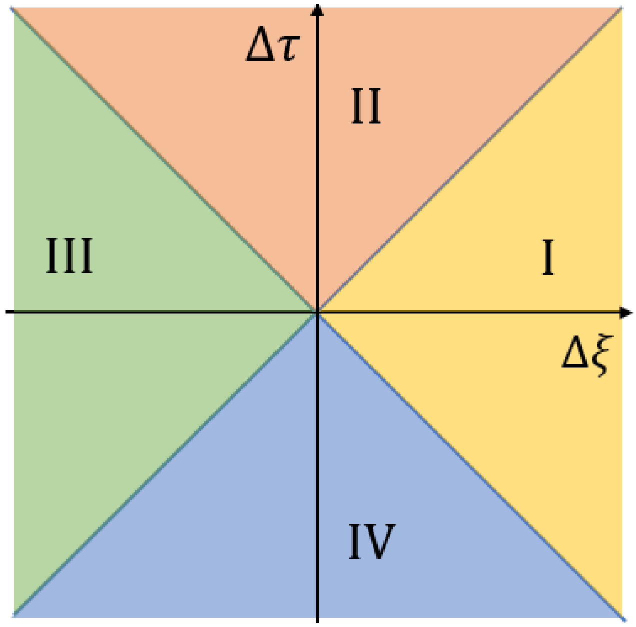

Figure 7: The $\Delta\xi - \Delta\tau$ plane divided in four different regions.

So the specific combination that appears in the spectral function yields the following values for the spectral function:

$$i\Delta^c(\tau,\xi;\tau',\xi') = \begin{cases} -\frac{i}{2}, & \text{in region II} \\ \frac{i}{2}, & \text{in region IV} \\ 0, \text{in regions I and III} \end{cases} \tag{392}$$

Thus one indeed has that the spectral functions in the different cases are equal and that the spectral function is only non-zero inside the light cone.

The statistical functions also appear to be inequal, however one can take the limit where ξ, ξ', τ, τ' are small and in that limit the statistical functions are equal:

$$\begin{aligned} F_{m\rightarrow 0}(\tau,\xi;\tau',\xi') &= -\frac{1}{4\pi}\ln\left(\frac{\mu^2}{g^2}\left[e^{2g\xi}+e^{2g\xi'}-2e^{g(\xi+\xi')}\cosh(g(\tau-\tau'))\right]\right)+\phi^2(\mu) \\ &= -\frac{1}{4\pi}\ln\left(\frac{\mu^2}{g^2}\left[1+2g\xi+2g^2\xi^2+1+2g\xi'+2g^2\xi'^2\right.\right. \\ &\qquad \left.\left. -2\left(1+g(\xi+\xi')+\frac{g^2}{2}(\xi+\xi')^2\right)\left(1+\frac{g^2}{2}\Delta\tau^2\right)\right]\right)+\phi^2(\mu) \\ &= -\frac{1}{4\pi}\ln\left(\mu^2\left[2\xi^2+2\xi'^2-\xi^2-2\xi\xi'-\xi'^2-\Delta\tau^2\right]\right)+\phi^2(\mu) \\ &= -\frac{1}{4\pi}\ln\left(\mu^2(\Delta\xi^2-\Delta\tau^2)\right)+\phi^2(\mu). \end{aligned} \tag{393}$$

This is indeed equal to the previously obtained spectral function.

G Calculation of the energy momentum tensor without neglecting the imaginary part

In this Appendix one proves that the imaginary part in $\langle \hat{T}^{++}_{\mu\nu}\rangle$ drops out in the limit when $x \to x'$. First for the naive Rindler vacuum and afterwards also for the Rindler invariant vacuum. Recalling from Equation

189, that the Feynman propagator for the naive Rindler vacuum was given by:

$$\begin{aligned} i\Delta_{\mathrm{F}}(\xi,\tau;\xi',\tau') &= \theta(\Delta\tau)\left[-\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau-i\epsilon)^2+\Delta\xi^2\right]\right)+\phi^2(\mu)\right] \\ &+\theta(-\Delta\tau)\left[-\frac{1}{4\pi}\ln\left(\mu^2\left[-(\Delta\tau+i\epsilon)^2+\Delta\xi^2\right]\right)+\phi^2(\mu)\right] \\ &= \frac{-1}{4\pi}\ln\left(\mu^2\left[-(|\Delta\tau|-i\epsilon)^2+\Delta\xi^2\right]\right)+\phi^2(\mu) \end{aligned} \tag{394}$$

The first derivative to the primed coordinates yields

$$\begin{aligned} 2\pi\partial'_\tau\left\langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\right\rangle &= \frac{\Delta\tau}{\Delta\tau^2-\Delta\xi^2}-\frac{i\pi}{4}\theta(\Delta\tau)\left[-2\delta(\Delta\tau-\Delta\xi)-2\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{4}\theta(-\Delta\tau)\left[-2\delta(\Delta\tau-\Delta\xi)-2\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{2}\delta(\Delta\tau)\left[\theta(\Delta\tau-\Delta\xi)-\theta(\Delta\xi-\Delta\tau)+\theta(\Delta\tau+\Delta\xi)-\theta(-\Delta\tau-\Delta\xi)\right] \\ &= \frac{\Delta\tau}{\Delta\tau^2-\Delta\xi^2}+\frac{i\pi}{2}\theta(\Delta\tau)\left[\delta(\Delta\tau-\Delta\xi)+\delta(\Delta\tau+\Delta\xi)\right] \\ &-\frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta(\Delta\tau-\Delta\xi)+\delta(\Delta\tau+\Delta\xi)\right] \\ &= \frac{\Delta\tau}{\Delta\tau^2-\Delta\xi^2}+\frac{i\pi}{2}\theta(\Delta\xi)\left[\delta(\Delta\tau-\Delta\xi)-\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{2}\theta(-\Delta\xi)\left[\delta(\Delta\tau+\Delta\xi)-\delta(\Delta\tau-\Delta\xi)\right], \end{aligned} \tag{395}$$

$$\begin{aligned} 2\pi\partial'_\xi\left\langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\right\rangle &= \frac{-\Delta\xi}{\Delta\tau^2-\Delta\xi^2}-\frac{i\pi}{4}\theta(\Delta\tau)\left[2\delta(\Delta\tau-\Delta\xi)-2\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{4}\theta(-\Delta\tau)\left[2\delta(\Delta\tau-\Delta\xi)-2\delta(\Delta\tau+\Delta\xi)\right] \\ &= \frac{\Delta\tau}{\Delta\tau^2-\Delta\xi^2}-\frac{i\pi}{2}\theta(\Delta\tau)\left[\delta(\Delta\tau-\Delta\xi)-\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta(\Delta\tau-\Delta\xi)-\delta(\Delta\tau+\Delta\xi)\right] \\ &= \frac{\Delta\tau}{\Delta\tau^2-\Delta\xi^2}-\frac{i\pi}{2}\theta(\Delta\xi)\left[\delta(\Delta\tau-\Delta\xi)+\delta(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{2}\theta(-\Delta\xi)\left[\delta(\Delta\tau-\Delta\xi)+\delta(\Delta\tau+\Delta\xi)\right]. \end{aligned} \tag{396}$$

For these calculations one used the expression for the Feynman propagator with the θ-functions, as it is a bit easier. This expression is obtained using the principle sheet of the logarithm. In addition to that, one wrote down 2π times the propagator to avoid having to write $\frac{1}{2\pi}$ very often. Some of the complex terms cancel already because of the property of the Dirac-δ-function: $\delta(x-x_0)f(x)=\delta(x-x_0)f(x_0)$. Moreover, one has used this Dirac-δ-function property to make taking second derivatives easier. This way one can choose the θ functions which do not depend on the variable the derivative is taken to. Taking another derivative with respect to the non-primed coordinates yields 4 options:

$$\begin{aligned} 2\pi\partial_\tau\partial'_\tau\left\langle T[\hat\phi(\tau,\xi)\hat\phi(\tau',\xi')]\right\rangle &= \frac{\Delta\tau^2-\Delta\xi^2-2\Delta\tau^2}{(\Delta\tau^2-\Delta\xi^2)^2}+\frac{i\pi}{2}\theta(\Delta\xi)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)-\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &+\frac{i\pi}{2}\theta(-\Delta\xi)\left[\delta^{(1)}(\Delta\tau+\Delta\xi)-\delta^{(1)}(\Delta\tau-\Delta\xi)\right] \\ &= -\frac{\Delta\tau^2+\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2}+\frac{i\pi}{2}\theta(\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &-\frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \end{aligned} \tag{397}$$

$$2\pi\partial_\xi\partial'_\tau \langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\rangle = \frac{2\Delta\xi\Delta\tau}{(\Delta\tau^2-\Delta\xi^2)^2} - \frac{i\pi}{2}\theta(\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)-\delta^{(1)}(\Delta\tau+\Delta\xi)\right] + \frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)-\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \tag{398}$$

$$\begin{aligned} 2\pi\partial_\xi\partial'_\xi \langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\rangle &= \frac{-\Delta\tau^2+\Delta\xi^2-2\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2} - \frac{i\pi}{4}\theta(\Delta\tau)\left[-2\delta^{(1)}(\Delta\tau-\Delta\xi)-2\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &+ \frac{i\pi}{4}\theta(-\Delta\tau)\left[-2\delta^{(1)}(\Delta\tau-\Delta\xi)-2\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &= \frac{-\Delta\tau^2+\Delta\xi^2-2\Delta\xi^2}{(\Delta\tau^2-\Delta\xi^2)^2} + \frac{i\pi}{2}\theta(\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &- \frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \end{aligned} \tag{399}$$

$$\begin{aligned} 2\pi\partial_\tau\partial'_\xi \langle T[\hat{\phi}(\tau,\xi)\hat{\phi}(\tau',\xi')]\rangle &= \frac{2\Delta\xi\Delta\tau}{(\Delta\tau^2-\Delta\xi^2)^2} - \frac{i\pi}{2}\theta(\Delta\xi)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &+ \frac{i\pi}{2}\theta(-\Delta\xi)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)+\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &= \frac{2\Delta\xi\Delta\tau}{(\Delta\tau^2-\Delta\xi^2)^2} - \frac{i\pi}{2}\theta(\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)-\delta^{(1)}(\Delta\tau+\Delta\xi)\right] \\ &+ \frac{i\pi}{2}\theta(-\Delta\tau)\left[\delta^{(1)}(\Delta\tau-\Delta\xi)-\delta^{(1)}(\Delta\tau+\Delta\xi)\right]. \end{aligned} \tag{400}$$

In all expression it is now quite clear that the imaginary part goes to zero as $\Delta\tau \to 0$. Thus one indeed obtains the result that was given in the main text for the energy momentum tensor, Equations 229, 230, 231.

Now one looks at the Feynman propagator for the Rindler invariant vacuum.

$$\lim_{m\to 0} i\Delta^{\mathrm{F}}_m(\tau,\xi;\tau',\xi') = -\frac{1}{4\pi}\ln\left(\mu^2\left[\cosh(g\Delta\xi)-\cosh(g|\Delta\tau|-ig\epsilon)\right]\right) + \phi^2(\xi,\xi',\mu). \tag{401}$$

Again one takes the result for the Feynman propagator where the principle sheet is used to obtain θ-functions. The first order derivatives to the primed coordinates, while not ignoring the imaginary part, are given by:

$$\begin{aligned} 2\pi\partial'_\xi \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= \frac{g}{2}\left[-1+\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\right] \\ &- i\pi\Delta\xi\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau^2-\Delta\xi^2), \end{aligned} \tag{402}$$

$$\begin{aligned} 2\pi\partial'_\tau \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= -\frac{g}{2}\frac{\sinh(g\Delta\tau)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)} \\ &+ i\pi\Delta\tau\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau^2-\Delta\xi^2) \\ &+ i\pi\left[\delta(\Delta\tau)\theta(\Delta\tau)-\delta(-\Delta\tau)\theta(-\Delta\tau)-\delta(\Delta\tau)\theta(-\Delta\tau)\right. \\ &\left.+\delta(-\Delta\tau)\theta(\Delta\tau)\right]\theta(\Delta\tau^2-\Delta\xi^2). \end{aligned} \tag{403}$$

Again one sees that the last term in the τ derivative expression drops out because of the because of the property of the Dirac-δ-function: $\delta(x-x_0)f(x) = \delta(x-x_0)f(x_0)$. The derivatives can again be simplified such that one has to take fewer derivatives. This is done via $\delta(f(x)) = \frac{\sum_{x_0}\delta(x-x_0)}{|f'(x_0)|}$, where x_0 all zero crossings of the function $f(x)$. Doing this gives the following expressions for the first derivatives:

$$\begin{aligned}
2\pi\partial'_\xi \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= \frac{g}{2}\left[-1+\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\right] \\
&\quad-\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta(\Delta\tau-\Delta\xi) \\
&\quad-\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta(\Delta\tau+\Delta\xi) \\
&= \frac{g}{2}\left[-1+\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\right] \\
&\quad-\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau-\Delta\xi) \\
&\quad+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau+\Delta\xi),
\end{aligned} \tag{404}$$

and

$$\begin{aligned}
2\pi\partial'_\tau \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= -\frac{g}{2}\frac{\sinh(g\Delta\tau)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)} \\
&\quad+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau-\Delta\xi) \\
&\quad+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta(\Delta\tau+\Delta\xi) \\
&= -\frac{g}{2}\frac{\sinh(g\Delta\tau)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)} \\
&\quad+\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta(\Delta\tau-\Delta\xi) \\
&\quad-\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta(\Delta\tau+\Delta\xi).
\end{aligned} \tag{405}$$

The second order derivatives are given by:

$$\begin{aligned}
2\pi\partial_\xi\partial'_\xi \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= \frac{-g^2}{2}\frac{(-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\
&\quad+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&\quad+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau+\Delta\xi),
\end{aligned} \tag{406}$$

$$
\begin{aligned}
2\pi\partial_\tau\partial'_\xi \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= -\frac{g^2}{2}\frac{\sinh(g\Delta\xi)\sinh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\
&= \frac{g}{2}\left[-1+\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\right] \\
&\quad -\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&\quad -\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta^{(1)}(\Delta\tau+\Delta\xi) \\
&= \frac{g}{2}\left[-1+\frac{\sinh(g\Delta\xi)}{\cosh(g\Delta\xi)-\cosh(g\Delta\tau)}\right] \\
&-\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau+\Delta\xi),
\end{aligned}
\tag{407}
$$

$$
\begin{aligned}
2\pi\partial_\tau\partial'_\tau \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= \frac{-g^2}{2}\frac{(-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\
&\quad +\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&-\frac{i\pi}{2}\frac{\Delta\xi}{|\Delta\xi|}\left[\theta(\Delta\xi)^2+\theta(-\Delta\xi)^2-2\theta(\Delta\xi)\theta(-\Delta\xi)\right]\delta^{(1)}(\Delta\tau+\Delta\xi) \\
&= \frac{-g^2}{2}\frac{(-1+\cosh(g\Delta\xi)\cosh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\
&+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&+\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau+\Delta\xi)
\end{aligned}
\tag{408}
$$

$$
\begin{aligned}
2\pi\partial_\xi\partial'_\tau \lim_{m\to 0}\langle\hat{\phi}_m(\tau,\xi)\hat{\phi}_m(\tau',\xi')\rangle &= -\frac{g^2}{2}\frac{\sinh(g\Delta\xi)\sinh(g\Delta\tau)}{(\cosh(g\Delta\xi)-\cosh(g\Delta\tau))^2} \\
&\quad -\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau-\Delta\xi) \\
&\quad +\frac{i\pi}{2}\frac{\Delta\tau}{|\Delta\tau|}\left[\theta(\Delta\tau)^2+\theta(-\Delta\tau)^2-2\theta(\Delta\tau)\theta(-\Delta\tau)\right]\delta^{(1)}(\Delta\tau+\Delta\xi).
\end{aligned}
\tag{409}
$$

Here, one notes that all the complex terms in this expression drop out in the coincident limit, the limit where $t \to t'$. Thus one indeed obtains the result for the energy momentum tensor from the main text, Equations 247, 248, 249.

H Integral from the Bogoliubov calculation

In the main text one encountered the following integral:

$$\begin{aligned}
&\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \int_0^\infty dp\frac{1}{16\pi^2 g\sinh(\frac{p\pi}{g})}\Bigg\{\Gamma\left(\frac{-ip}{g}\right)\Gamma\left(\frac{ip}{g}\right)\left[\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon_1'}\right)^{\frac{ip}{g}}+e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon_2'}\right)^{\frac{ip}{g}}\right.\\
&\left.+e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon_3'}\right)^{\frac{ip}{g}}+\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon_4'}\right)^{\frac{ip}{g}}\right]\\
&+\frac{ig}{p}\Gamma\left(\frac{-ip}{g}\right)\delta'^{\frac{ip}{g}}\left[(i\bar{u}+\epsilon_1)^{\frac{ip}{g}}+e^{\frac{p\pi}{g}}(i\bar{u}+\epsilon_2)^{\frac{ip}{g}}+e^{\frac{-p\pi}{g}}(-i\bar{u}+\epsilon_3)^{\frac{ip}{g}}+(-i\bar{u}+\epsilon_4)^{\frac{ip}{g}}\right]\\
&-\frac{ig}{p}\Gamma\left(\frac{ip}{g}\right)\delta^{\frac{-ip}{g}}\left[(i\bar{u}'+\epsilon_1')^{\frac{-ip}{g}}+e^{\frac{p\pi}{g}}(-i\bar{u}'+\epsilon_2')^{\frac{-ip}{g}}+e^{\frac{-p\pi}{g}}(i\bar{u}'+\epsilon_3')^{\frac{-ip}{g}}+(-i\bar{u}'+\epsilon_4')^{\frac{-ip}{g}}\right]\\
&+\frac{g^2}{p^2}\left(\frac{\delta'}{\delta}\right)^{\frac{ip}{g}}\left[1+e^{\frac{p\pi}{g}}+e^{\frac{-p\pi}{g}}+1\right]\Bigg\}+(\bar{u}\to\bar{v})\\
&=I_1+I_2+I_3+I_4+(\bar{u}\to\bar{v}).
\end{aligned} \tag{410}$$

The I_1 is given by the first term, the term where one multiplies the two Γ-functions. I_2 is given by the third line of Equation 410, where one has $\Gamma\left(\frac{-ip}{g}\right)\delta'^{\frac{ip}{g}}$. I_3 is given by the 4th line of Equation 410, where one has $\Gamma\left(\frac{ip}{g}\right)\delta^{\frac{-ip}{g}}$. Lastly, I_4 is given by the fifth line of Equation 410, the term with the two δ's. One sees that these terms have a scaling with respect to δ as $e^{\frac{-ip}{g}\ln(\delta)}$. When δ goes to zero this will start oscillating very rapidly and thus the contribution coming from large p are expected to go to zero. One thus argues that only the smaller values of p contribute in I_2 and I_3. Thus one expands the gamma function for the first couple of orders. These terms scale as p^{-2}, thus one stops expanding the gamma function at the second order:

$$\Gamma\left(\frac{ip}{g}\right)=\frac{-ig}{p}-\gamma_{\mathrm{E}}+\frac{1}{6}\left(3\gamma_{\mathrm{E}}^2+\frac{\pi^2}{2}\right)\left(\frac{ip}{g}\right)+\left(-\frac{\zeta(3)}{3}-\frac{\gamma_{\mathrm{E}}^3}{6}-\frac{\gamma_{\mathrm{E}}\pi^2}{12}\right)\left(\frac{ip}{g}\right)^2. \tag{411}$$

As will be seen later, terms of order p^2 and higher in this expansion do not contribute. Inserting this expansion in the original integral yields the following integral:

$$\begin{aligned}
I_3=\int_{\delta_p}^\infty\frac{1}{16\pi^2 g\sinh\left(\frac{p\pi}{g}\right)}&\left[-\frac{g^2}{p^2}+\frac{ig\gamma_{\mathrm{E}}}{p}+\frac{1}{6}\left(3\gamma_{\mathrm{E}}^2+\frac{\pi^2}{2}\right)\right]\left[(i\bar{u}'+\epsilon_1')^{\frac{-ip}{g}}+e^{\frac{p\pi}{g}}(-i\bar{u}'+\epsilon_2')^{\frac{-ip}{g}}\right.\\
&\left.+e^{\frac{-p\pi}{g}}(i\bar{u}'+\epsilon_3')^{\frac{-ip}{g}}+(-i\bar{u}'+\epsilon_4')^{\frac{-ip}{g}}\right]e^{-\frac{ip}{g}\ln(\delta)}.
\end{aligned} \tag{412}$$

Here the lower bound has been adjusted to δ_p to avoid divergent integrals in hope of renormalising $\ln(\delta)$ terms later. This integral is still quite troublesome, therefore one first makes an approximation for the sinh function as well for small p:

$$\begin{aligned}
\frac{1}{\sinh(\frac{p\pi}{g})}&=\frac{1}{\left(\frac{p\pi}{g}+\frac{1}{6}\left(\frac{p\pi}{g}\right)^3\right)}\\
&=\frac{g}{p\pi}\frac{1}{1+\frac{1}{6}\left(\frac{p\pi}{g}\right)^2}\\
&=\frac{g}{p\pi}\left[1-\frac{1}{6}\left(\frac{p\pi}{g}\right)^2\right]\\
&=\frac{g}{p\pi}-\frac{1}{6}\frac{p\pi}{g}
\end{aligned} \tag{413}$$

Later, one will see that inserting terms up to this order is more than enough. After inserting this, one obtains:

$$I_3 = \int_{\delta_p}^{\infty} \frac{1}{16\pi^2 g}\left[\frac{g}{p\pi} - \frac{1}{6}\frac{p\pi}{g}\right]\left[-\frac{g^2}{p^2} + \frac{ig\gamma_{\mathrm{E}}}{p} + \frac{1}{6}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)\right]\left[(i\bar{u}' + \epsilon_1')^{\frac{-ip}{g}} + e^{\frac{p\pi}{g}}(-i\bar{u}' + \epsilon_2')^{\frac{-ip}{g}} + e^{\frac{-p\pi}{g}}(i\bar{u}' + \epsilon_3')^{\frac{-ip}{g}} + (-i\bar{u}' + \epsilon_4')^{\frac{-ip}{g}}\right] e^{-\frac{ip}{g}\ln(\delta)}. \tag{414}$$

Every term now has the form of an incomplete gamma function, which have δ_p divergences. The divergences are of order $\frac{1}{\delta_p^2}$, $\frac{1}{\delta_p}$ and $\ln(\delta_p)$. Solving the integrals and expanding for small δ_p, one gets:

$$\begin{aligned} I_3 = \frac{1}{16\pi^2}\sum_{i=1}^{4}\Bigg[&-\frac{g^2}{\pi}\left(\frac{1}{2\delta_p^2} + \frac{a_i}{\delta_p} + \frac{1}{4}\left[3a_i^2 - 2a_i^2\ln(-a_i) - 2a_i^2\ln(\delta_p)\right]\right) \\ &+ \frac{ig\gamma_{\mathrm{E}}}{\pi}\left(\frac{1}{\delta_p} + a_i - a_i\gamma_{\mathrm{E}} - a_i\ln(-a_i) - a_i\ln(\delta_p)\right) + \frac{1}{6\pi}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)(-\gamma_{\mathrm{E}} - \ln(-a_i) - \ln(\delta_p)) \\ &+ \frac{\pi}{6}(-\gamma_{\mathrm{E}} - \ln(-a_i) - \ln(\delta_p)) - \frac{i\gamma_{\mathrm{E}}}{g}\left(-\frac{1}{a_i}\right) - \frac{\pi}{36g^2}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)\frac{1}{a_i^2}\Bigg]. \end{aligned} \tag{415}$$

One now realizes that in the limit that δ to 0 the last two terms vanish. Thus the divergent structure contains terms containing $\ln(\ln(\delta))$, $\ln(\delta)$, $\ln(\delta_p)$ and multiplications of those divergences. In this expression the a_i are given by:

$$\begin{aligned} a_1 &= -\frac{i}{g}\ln(\delta) - \frac{i}{g}\ln(i\bar{u}' + \epsilon_1') \\ a_2 &= -\frac{i}{g}\ln(\delta) - \frac{i}{g}\ln(-i\bar{u}' + \epsilon_2') + \frac{pi}{g} \\ a_3 &= -\frac{i}{g}\ln(\delta) - \frac{i}{g}\ln(i\bar{u}' + \epsilon_3') - \frac{pi}{g} \\ a_4 &= -\frac{i}{g}\ln(\delta) - \frac{i}{g}\ln(-i\bar{u}' + \epsilon_4'). \end{aligned} \tag{416}$$

Similarly one can calculate the second line of Equation 410:

$$\begin{aligned} I_2 &= \int_{\delta_p}^{\infty} \frac{1}{16\pi^2 g\sinh\left(\frac{p\pi}{g}\right)}\frac{ig}{p}\Gamma\left(\frac{-ip}{g}\right)\delta'^{\frac{ip}{g}}\left[(i\bar{u} + \epsilon_1)^{\frac{ip}{g}} + e^{\frac{p\pi}{g}}(i\bar{u} + \epsilon_2)^{\frac{ip}{g}} + e^{\frac{-p\pi}{g}}(-i\bar{u} + \epsilon_3)^{\frac{ip}{g}} + (-i\bar{u} + \epsilon_4)^{\frac{ip}{g}}\right] \\ &= \int_{\delta_p}^{\infty} \frac{1}{16\pi^2 g}\left[\frac{g}{p\pi} - \frac{1}{6}\frac{p\pi}{g}\right]\left[-\frac{g^2}{p^2} - \frac{ig\gamma_{\mathrm{E}}}{p} + \frac{1}{6}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)\right]\left[(i\bar{u} + \epsilon_1)^{\frac{ip}{g}} + e^{\frac{p\pi}{g}}(i\bar{u} + \epsilon_2)^{\frac{ip}{g}} \right. \\ &\qquad \left. + e^{\frac{-p\pi}{g}}(-i\bar{u} + \epsilon_3)^{\frac{ip}{g}} + (-i\bar{u} + \epsilon_4)^{\frac{ip}{g}}\right] e^{\frac{ip}{g}\ln(\delta')} \\ &= \frac{1}{16\pi^2}\sum_{i=1}^{4}\Bigg[-\frac{g^2}{\pi}\left(\frac{1}{2\delta_p^2} + \frac{a_i'}{\delta_p} + \frac{1}{4}\left[3a_i'^2 - 2a_i{}'^2\ln(-a_i') - 2a_i'^2\ln(\delta_p)\right]\right) \\ &\qquad - \frac{ig\gamma_{\mathrm{E}}}{\pi}\left(\frac{1}{\delta_p} + a_i' - a_i'\gamma_{\mathrm{E}} - a_i'\ln(-a_i') - a_i'\ln(\delta_p)\right) + \frac{1}{6\pi}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)(-\gamma_{\mathrm{E}} - \ln(-a_i') - \ln(\delta_p)) \\ &\qquad + \frac{\pi}{6}(-\gamma_{\mathrm{E}} - \ln(-a_i') - \ln(\delta_p)) + \frac{i\gamma_{\mathrm{E}}}{g}\left(-\frac{1}{a_i'}\right) - \frac{\pi}{36g^2}\left(3\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{2}\right)\frac{1}{a_i'^2}\Bigg]. \end{aligned} \tag{417}$$

The a'_i in this expression are very similar to the previous a_i, namely:

$$\begin{aligned}a'_1 &= \frac{i}{g}\ln(\delta') + \frac{i}{g}\ln(i\bar{u}+\epsilon_1)\\ a'_2 &= \frac{i}{g}\ln(\delta') + \frac{i}{g}\ln(i\bar{u}+\epsilon_2) + \frac{\pi}{g}\\ a'_3 &= +\frac{i}{g}\ln(\delta') + \frac{i}{g}\ln(-i\bar{u}+\epsilon_3) - \frac{\pi}{g}\\ a'_4 &= \frac{i}{g}\ln(\delta') + \frac{i}{g}\ln(-i\bar{u}+\epsilon_4)\end{aligned} \tag{418}$$

The integral on the first line of Equation 410 can be solved as:

$$\begin{aligned}I_1 &= \int_{\delta_p}^{\infty} dp \frac{1}{16\pi p \sinh^2(\frac{p\pi}{g})}\left[\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon'_1}\right)^{\frac{ip}{g}} + e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon'_2}\right)^{\frac{ip}{g}} + e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon'_3}\right)^{\frac{ip}{g}} + \left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon'_4}\right)^{\frac{ip}{g}}\right]\\ &= \frac{1}{16\pi}\sum_{n=0}^{\infty} n \int_{\delta_p}^{\infty} \frac{dp}{p} e^{-n\frac{p\pi}{g}}\left[\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon'_1}\right)^{\frac{ip}{g}} + e^{\frac{p\pi}{g}}\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon'_2}\right)^{\frac{ip}{g}} + e^{\frac{-p\pi}{g}}\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon'_3}\right)^{\frac{ip}{g}} + \left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon'_4}\right)^{\frac{ip}{g}}\right]\\ &= \frac{1}{16\pi}\sum_{n=0}^{\infty} n \left[\Gamma\left(0, \left[-2n\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon'_1}\right)\right]\delta_p\right) + \Gamma\left(0, \left[-(2n-1)\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon'_2}\right)\right]\delta_p\right)\right.\\ &\qquad \left. + \Gamma\left(0, \left[-(2n+1)\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon'_3}\right)\right]\delta_p\right) + \Gamma\left(0, \left[-2n\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon'_4}\right)\right]\delta_p\right)\right].\end{aligned} \tag{419}$$

One has expanded this function already, so one can expand this function the same way again. This way one obtains:

$$\begin{aligned}I_1 &= \lim_{y\to 0}\frac{1}{4\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y} - \gamma_{\mathrm{E}} - \ln(\delta_p)\right] - \lim_{y\to 0}\frac{1}{16\pi}\sum_{n=0}^{\infty} n\left[\frac{\left(-2n\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon'_1}\right)\right)^y}{y}\right.\\ &\quad \left. + \frac{\left(-(2n-1)\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon'_2}\right)\right)^y}{y} + \frac{\left(-(2n+1)\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon'_3}\right)\right)^y}{y} + \frac{\left(-2n\frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon'_4}\right)\right)^y}{y}\right]\\ &= \lim_{y\to 0}\frac{1}{4\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y} - \gamma_{\mathrm{E}} - \ln(\delta_p)\right] - \lim_{y\to 0}\frac{1}{16\pi y}\left[\frac{c_1(-b_1)^y}{b_1}\zeta\left(-y, -\frac{c_1}{b_1}\right) - \frac{(-b_1)^{y+1}}{b_1}\zeta\left(-y-1, -\frac{c_1}{b_1}\right)\right.\\ &\quad + \frac{c_2(-b_2)^y}{b_2}\zeta\left(-y, -\frac{c_2}{b_2}\right) - \frac{(-b_2)^{y+1}}{b_2}\zeta\left(-y-1, -\frac{c_2}{b_2}\right) + \frac{c_3(-b_3)^y}{b_3}\zeta\left(-y, -\frac{c_3}{b_3}\right)\\ &\quad \left. - \frac{(-b_3)^{y+1}}{b_3}\zeta\left(-y-1, -\frac{c_3}{b_3}\right) + \frac{c_4(-b_4)^y}{b_4}\zeta\left(-y, -\frac{c_4}{b_4}\right) - \frac{(-b_4)^{y+1}}{b_4}\zeta\left(-y-1, -\frac{c_4}{b_4}\right)\right],\end{aligned} \tag{420}$$

where

$$c_1 = \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_1}{i\bar{u}'+\epsilon'_1}\right),\ b_1 = -\frac{2\pi}{g} \tag{421}$$

$$c_2 = \frac{\pi}{g} + \frac{i}{g}\ln\left(\frac{i\bar{u}+\epsilon_2}{-i\bar{u}'+\epsilon'_2}\right),\ b_2 = -\frac{2\pi}{g} \tag{422}$$

$$c_3 = \frac{-\pi}{g} + \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_3}{i\bar{u}'+\epsilon'_3}\right),\ b_3 = -\frac{2\pi}{g} \tag{423}$$

$$c_4 = \frac{i}{g}\ln\left(\frac{-i\bar{u}+\epsilon_4}{-i\bar{u}'+\epsilon'_4}\right),\ b_4 = -\frac{2\pi}{g}. \tag{424}$$

The Hurwitz zeta functions in Equation 420 can again be expanded and they will yield terms as in Equation 271. This is quite long, thus one refrains from doing it, until one is sure that the answer can be used. Lastly the fourth line in Equation 410 is given by:

$$\begin{aligned} I_4 &= \int_{\delta_p}^{\infty} dp \frac{1}{16\pi^2 g \sinh(\frac{p\pi}{g})} \frac{g^2}{p^2} \left(\frac{\delta'}{\delta}\right)^{\frac{ip}{g}} \left[1 + e^{\frac{p\pi}{g}} + e^{\frac{-p\pi}{g}} + 1\right] \\ &= \frac{g}{8\pi^2} \int_{\delta_p}^{\infty} dp \frac{\coth\left(\frac{p\pi}{2g}\right)}{p^2} \left(\frac{\delta'}{\delta}\right)^{\frac{ip}{g}} . \end{aligned} \tag{425}$$

Unfortunately, one cannot use the same argument as before, that only the small values of p contribute, because this time both δ and δ' go to zero. Thus now one expands the coth function as:

$$\begin{aligned} \coth\left(\frac{p\pi}{2g}\right) &= \frac{1 + e^{-\frac{p\pi}{g}}}{1 - e^{\frac{p\pi}{g}}} \\ &= \left(1 + e^{-\frac{p\pi}{g}}\right) \sum_{n=0}^{\infty} e^{-\frac{np\pi}{g}} \\ &= \sum_{n=0}^{\infty} \left(e^{-\frac{np\pi}{g}} + e^{-(n+1)\frac{p\pi}{g}}\right) . \end{aligned} \tag{426}$$

With this expansion, I_4 gives the following:

$$\begin{aligned} I_4 &= \frac{g}{8\pi^2} \sum_{n=0}^{\infty} \int_{\delta_p}^{\infty} \frac{dp}{p^2} \left[e^{p(\frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)} + e^{p(\frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)} \right] \\ &= \frac{g}{8\pi^2} \sum_{n=0}^{\infty} \left[\frac{1}{\delta_p} + \frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right) - \left[\frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right] \gamma_{\mathrm{E}} \right. \\ &- \left(\frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln\left(\frac{n\pi}{g} - \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) - \left(\frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln(\delta_p) \\ &+ \frac{1}{\delta_p} + \frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right) - \left[\frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right] \gamma_{\mathrm{E}} \\ &\left. - \left(\frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln\left(\frac{(n+1)\pi}{g} - \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) - \left(\frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln(\delta_p) \right] \\ &= \frac{g}{8\pi^2} \sum_{n=0}^{\infty} \left[\frac{2}{\delta_p} + \frac{-(2n+1)\pi}{g} + \frac{2i}{g}\ln\left(\frac{\delta'}{\delta}\right) - \left[\frac{-(2n+1)\pi}{g} + \frac{2i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right] \gamma_{\mathrm{E}} \right. \\ &- \left(\frac{-n\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln\left(\frac{n\pi}{g} - \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) - \left(\frac{-(n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln\left(\frac{(n+1)\pi}{g} - \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \\ &\left. - \left(\frac{-(2n+1)\pi}{g} + \frac{i}{g}\ln\left(\frac{\delta'}{\delta}\right)\right) \ln(\delta_p) \right] \end{aligned} \tag{427}$$

Now one puts all delta divergences in a condensate and then one arrives at a result for the two-point function. With all these results, one knows that the two-point function from the perspective of a Minkowski observer is given by:

$$\langle 0_R|\,\hat{\phi}(x,t)\hat{\phi}(x',t')\,|0_R\rangle = \phi^2(\mu) + \lim_{y\to 0}\frac{1}{4\pi}\sum_{n=0}^{\infty} n\left[\frac{1}{y}-\gamma_{\mathrm{E}}\right] \tag{428}$$
$$-\lim_{y\to 0}\frac{1}{16\pi y}\left[\frac{c_1(-b_1)^y}{b_1}\zeta\left(-y,-\frac{c_1}{b_1}\right)-\frac{(-b_1)^{y+1}}{b_1}\zeta\left(-y-1,-\frac{c_1}{b_1}\right)+\frac{c_2(-b_2)^y}{b_2}\zeta\left(-y,-\frac{c_2}{b_2}\right)-\frac{(-b_2)^{y+1}}{b_2}\zeta\left(-y-1,-\frac{c_2}{b_2}\right)\right.$$
$$\left.+\frac{c_3(-b_3)^y}{b_3}\zeta\left(-y,-\frac{c_3}{b_3}\right)-\frac{(-b_3)^{y+1}}{b_3}\zeta\left(-y-1,-\frac{c_3}{b_3}\right)+\frac{c_4(-b_4)^y}{b_4}\zeta\left(-y,-\frac{c_4}{b_4}\right)-\frac{(-b_4)^{y+1}}{b_4}\zeta\left(-y-1,-\frac{c_4}{b_4}\right)\right]$$
$$+\frac{1}{16\pi^2}\sum_{i=1}^{4}\left[-\frac{g^2}{\pi}\left(\frac{3}{4}a_{i,r}'^2\right)-\frac{ig\gamma_{\mathrm{E}}}{\pi}\left(a_{i,r}'(1-\gamma_{\mathrm{E}})\right)-\frac{1}{6\pi}\left(3\gamma_{\mathrm{E}}^2+\frac{\pi^2}{2}\right)\gamma_{\mathrm{E}}-\frac{\pi\gamma_{\mathrm{E}}}{6}\right]$$
$$+\frac{1}{16\pi^2}\sum_{i=1}^{4}\left[-\frac{g^2}{\pi}\left(\frac{3}{4}a_{i,r}^2\right)+\frac{ig\gamma_{\mathrm{E}}}{\pi}\left(a_{i,r}(1-\gamma_{\mathrm{E}})\right)-\frac{1}{6\pi}\left(3\gamma_{\mathrm{E}}^2+\frac{\pi^2}{2}\right)\gamma_{\mathrm{E}}-\frac{\pi\gamma_{\mathrm{E}}}{6}\right]-\frac{1}{4}\left[\sum_{n=0}^{\infty}(2n+1)(1+\gamma_{\mathrm{E}})\right].$$

Here $a_{i,r}$ and $a_{i,r}$ are the same constants as in Equations 416 and 418, but without the $\ln(\delta)$ and $\ln(\delta')$.

I Massive entropy intermediate integral result

In this appendix the intermediate results of expanding the Hypergeometric functions in Equation 314 are shown. The result was given by:

$$I=\frac{g}{4\pi}\left\{\frac{1}{(k-i\epsilon)^2}\left[(-1)^{\frac{i}{g}(k-i\epsilon)}\Gamma\left(1+\frac{i}{g}(k-i\epsilon)\right)\Gamma\left(1-\frac{i}{g}(k-i\epsilon)\right)\left(e^{i(k-i\epsilon)\Delta\tau}+e^{-i(k-i\epsilon)\Delta\tau}\right)\right.\right. \tag{429}$$
$$\left.-e^{-i(\Delta\tau+\delta)(k-i\epsilon)}\left(1-{}_2F_1\left(1,-\frac{i(k-i\epsilon)}{g},1-\frac{i(k-i\epsilon)}{g},e^{g(2\Delta\tau+\delta)}\right)-{}_2F_1\left(1,-\frac{i(k-i\epsilon)}{g},1-\frac{i(k-i\epsilon)}{g},e^{g\delta}\right)\right)\right]$$
$$+\frac{1}{k^2}\left[e^{-i(\Delta\tau-\delta)k}\left(1-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta}\right)-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta}\right)\right)\right.$$
$$\left.-e^{i(\Delta\tau-\delta')k}\left(1-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\left(2\Delta\tau-\delta'\right)-g\Delta\tau}\right)-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g\delta'}\right)\right)\right]+$$
$$\frac{1}{(k+i\epsilon)^2}\left[\left(e^{i(\Delta\tau+\delta')(k+i\epsilon)}\right)\left(1-{}_2F_1\left(1,-\frac{i(k+i\epsilon)}{g};1-\frac{i(k+i\epsilon)}{g};e^{-g(2\Delta\tau+\delta')}\right)-{}_2F_1\left(1,-\frac{i(k+i\epsilon)}{g};1-\frac{i(k+i\epsilon)}{g};e^{g\delta'}\right)\right)\right.$$
$$\left.\left.\vphantom{\frac{1}{1}}\right]\right\}.$$

One first lets $\epsilon\to 0$, this yields:

$$\lim_{\epsilon\to 0} I=\frac{g}{4\pi k^2}\left\{\left[e^{(-ik)(\delta'-\Delta\tau)}\left({}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g(\delta'-2\Delta\tau)}\right)+{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g\delta'}\right)-1\right)\right.\right.$$
$$\left.-e^{(ik)(\delta-\Delta\tau)}\left({}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{(2g)\Delta\tau-g\delta}\right)+{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta}\right)-1\right)\right]$$
$$-\left(e^{-i(\delta k+\Delta\tau k)}\right)\left[-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g(\delta+2\Delta\tau)}\right)-{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g\delta}\right)\right. \tag{430}$$
$$\left.+\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)\left(1+e^{\Delta\tau(2ik)}\right)e^{-\frac{\pi k}{g}+\delta(ik)}+1\right]$$
$$\left.-\left(e^{i(\delta'k+\Delta\tau k)}\right)\left[{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g(\delta'+2\Delta\tau)}\right)+{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta'}\right)-1\right]\right\}.$$

Then one sends δ to zero:

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0} I = \frac{-1}{4\pi k^2}\Bigg\{e^{-\frac{\pi k}{g}+(-ik)(\delta'-\Delta\tau)-\Delta\tau(ik)}\Bigg[ge^{\frac{\pi k}{g}+(ik)(\delta'-\Delta\tau)+\delta'(ik)+\Delta\tau(2ik)}\,{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g\delta'}\right)$$
$$+ge^{\frac{\pi k}{g}+(ik)(\delta'-\Delta\tau)+\delta'(ik)+\Delta\tau(2ik)}\,{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{-g(\delta'+2\Delta\tau)}\right)$$
$$-ge^{\frac{\pi k}{g}+(ik)(\delta'-\Delta\tau)+\delta'(ik)+\Delta\tau(2ik)}-ge^{\frac{\pi k}{g}+\Delta\tau(ik)}\,{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g\delta'}\right)$$
$$-ge^{\frac{\pi k}{g}+\Delta\tau(ik)}\,{}_2F_1\left(1,-\frac{ik}{g};1-\frac{ik}{g};e^{g(\delta'-2\Delta\tau)}\right)+\pi ke^{\frac{\pi k}{g}+(ik)(\delta'-\Delta\tau)}$$
$$+ge^{(ik)(\delta'-\Delta\tau)}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)+ge^{(ik)(\delta'-\Delta\tau)+\Delta\tau(2ik)}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)$$
$$+ge^{\frac{\pi k}{g}+\Delta\tau(ik)}\Bigg]\Bigg\}. \tag{431}$$

Lastly one sends δ' to zero:

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0} I = \left(-\frac{ge^{-\frac{\pi k}{g}+i\Delta\tau k}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)}{4\pi k^2}-\frac{ge^{-\frac{\pi k}{g}-i\Delta\tau k}\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)}{4\pi k^2}\right.$$
$$\left.-\frac{e^{-i\Delta\tau k}}{4k}-\frac{e^{i\Delta\tau k}}{4k}\right). \tag{432}$$

This expression can now be simplified using that

$$\Gamma\left(\frac{ik}{g}+1\right)\Gamma\left(1-\frac{ik}{g}\right)=\frac{\pi\frac{k}{g}}{\sinh\left(\frac{\pi k}{g}\right)},$$

that $\frac{e^{ikx}+e^{-ikx}}{2}=\cos(kx)$ and that $\frac{e^{kx}+e^{-kx}}{2}=\cosh(kx)$. Using all this will then yield the result given in the main text:

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0} I = -\frac{\cos(k\Delta\tau)\cosh(\frac{k\pi}{g})}{k\sinh(\frac{k\pi}{g})} \tag{433}$$

J Answer to the integral involving the Unruh detector

The result to Equation 325 is quite long and not of use in the main text and is therefore stated here.

$$
\begin{aligned}
I_{\text{Unruh}} & \\
&= -\int_0^\infty d\lambda \frac{\ln\left(e^{2g\xi_0} - g^2\lambda^2\right) + \ln\left(e^{2g\xi_0'} - g^2\lambda^2\right)}{2g} \\
&= \Bigg[\frac{i}{E - E_0} i e^{\frac{i(E_0-E)\left(e^{g\xi_0} + e^{g\xi_0'} - g\lambda\right)}{g}} \Bigg\{ \text{Ei}\left(\frac{i(E_0 - E)\left(e^{g\xi_0} - g\lambda\right)}{g}\right) \exp\left(-\frac{i(E_0 - E)\left(2e^{g\xi_0} + e^{g\xi_0'} - g\lambda\right)}{g}\right) \\
&\qquad + \text{Ei}\left(\frac{i(E_0 - E)\left(e^{g\xi_0'} - g\lambda\right)}{g}\right) \exp\left(-\frac{i(E_0 - E)\left(e^{g\xi_0} + 2e^{g\xi_0'} - g\lambda\right)}{g}\right) \\
&\qquad + e^{-\frac{i(E_0-E)\left(e^{g\xi_0} + e^{g\xi_0'}\right)}{g}} \Bigg[e^{\frac{i(E_0-E)\left(e^{g\xi_0} + g\lambda\right)}{g}} \text{Ei}\left(-\frac{i(E_0 - E)\left(g\lambda + e^{g\xi_0}\right)}{g}\right) \\
&\quad + e^{\frac{i(E_0-E)\left(e^{g\xi_0'} + g\lambda\right)}{g}} \text{Ei}\left(-\frac{i(E_0 - E)\left(g\lambda + e^{g\xi_0'}\right)}{g}\right) - \log\left(e^{2g\xi_0} - g^2\lambda^2\right) - \log\left(e^{2g\xi_0'} - g^2\lambda^2\right) \Bigg] \Bigg\} \Bigg]_{\lambda_0}^{\lambda_1},
\end{aligned}
\tag{434}
$$

where one has λ_0 and λ_1 as the boundary values for λ. One will not give much more thought to these boundary values and the numerical value for this amplitude, because experiments on this are still far away.

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