A general approach to almost structures in geometry

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* from left to right: Mole Antonelliana (Torino, Italy), tower of Lie groupoids, Domtoren (Utrecht, The Netherlands)

A general approach to almost structures in geometry

Een algemeen kader voor bijna structuren in de meetkunde

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. H.R.B.M. Kummeling, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op woensdag 26 februari 2020 des middags te 4.15 uur

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Introduction

Differential Geometry studies various kinds of geometric structures on "nice" spaces (manifolds). Such structures appear naturally, e.g. when trying to measure, or even make sense of, distances, areas, etc., or when formalising physical theories. For instance, to talk about distances on such nice spaces, one needs to look at the class of Riemannian structures; or, for the mathematical structure that supports Hamilton's formulation of classical mechanics, one looks at the class of symplectic (or even Poisson) structures.

For each kind of geometric structures, one is interested in studying the entire class of them - let us call it $\mathcal{S}$. While the main interest is on such "$\mathcal{S}$-structures", one can also talk about "almost $\mathcal{S}$-structures", which arise naturally as their "shadows". When saying "shadow", one should have in mind an "over-simplified version" - which gives a rough idea/first order approximation of the actual structure, but without going into intricate details. A fundamental question, known as the integrability problem, is to understand when such a "shadow" is actually real, i.e. when an almost $\mathcal{S}$-structure arises from an $\mathcal{S}$-structure.

There are various frameworks to make precise sense of $\mathcal{S}$-structures but, somehow, each framework has its own limitations: there are always more examples of geometric structures than the frameworks can accommodate. The situation is even more problematic when looking for a general theory allowing to handle also "almost $\mathcal{S}$-structures": the existing literature is always restricted to the so-called "transitive case" (when symmetries allow one to pass from one point to another).

This thesis presents a general framework for studying almost structures and for proving integrability results, which is not restricted to the "transitive case". The standard approach to (almost) geometric structures, at least in the transitive case, is via the theory of G-structures, for an appropriate group of symmetries G. Here is an important remark at the foundation of this thesis: in the non-transitive case, one has to allow for local symmetries as well. Accordingly, one should concentrate not only on groups G, but allow also more general, point-dependent, version of groups: groupoids $\mathcal{G}$ and pseudogroups Γ.

The key concept of our approach to almost structures is that of a "principal Pfaffian bundle". Roughly speaking, it consists of a principal action by a groupoid $\mathcal{G}$ on a space P, together with two differential forms, one on P and one on $\mathcal{G}$, which are compatible in an appropriate sense. To make this precise, we draw techniques and inspiration from Poisson geometry and the formal theory of PDEs.

In this introduction we provide some background and sketch the main results of this thesis in a bit more detail; at the end we will recap the main achievements and give an outline of its chapters.

Γ-structures The majority of objects commonly known as "geometric structures on a smooth manifold" can be defined by a smooth atlas with special changes of coordinates. For instance, let us look at the following examples:

- For a symplectic manifold M^{2n}, the changes of coordinates preserve the canonical symplectic form of $\mathbb{R}^{2n}$.

- For a flat Riemannian manifold M^n, the changes of coordinates preserve the canonical Euclidean metric of $\mathbb{R}^n$.

- For a complex manifold M^{2n}, the changes of coordinates preserve the canonical complex structure of $\mathbb{R}^{2n}$.

- For a contact manifold M^{2n+1}, the changes of coordinates preserve the canonical contact structure of $\mathbb{R}^{2n+1}$.

The sets of diffeomorphisms preserving the tensors on $\mathbb{R}^n$ mentioned above are examples of Lie pseudogroups. Intuitively, a pseudogroup is the right notion to describe local symmetries of an object, in a similar way that groups describe global symmetries. Indeed, the concept of pseudogroup originated from the pioneering studies of Sophus Lie on the local symmetries of differential equations, in the same way that the notion of a group arose from the studies of Galois on the symmetries of algebraic equations.

More precisely, a pseudogroup on a manifold X is a subset $\Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$ of locally defined diffeomorphisms on X, which is closed under composition, inversion, localisation and glueing: it can be seen as a combination of a group-like and a sheaf-like object. A Lie pseudogroup is a pseudogroup which arises as solutions of a PDE. In modern terms, one associates to any pseudogroup Γ the Ehresmann jet groupoids $J^k\Gamma$ [32]; they are the spaces of k-jets of elements of Γ, and the Lie condition on Γ means that all $J^k\Gamma$ have to be smooth.

A smooth atlas whose changes of coordinates belong to a given pseudogroup Γ is called a Γ-atlas, and an equivalence class of them is a Γ-structure; at this stage, the Lie condition is not required yet. For instance, the previous examples are reformulated as follows:

- For Γ the pseudogroup of local symplectomorphisms of the canonical symplectic structure of $\mathbb{R}^{2n}$, Γ-structures on M are symplectic structures.

- For Γ the pseudogroup of local isometries of the canonical Euclidean metric of $\mathbb{R}^n$, Γ-structures on M are flat Riemannian metrics.

- For Γ the pseudogroup of local bi-holomorphisms of the canonical complex structure of $\mathbb{R}^{2n}$, Γ-structures on M are complex structures.
- For Γ the pseudogroup of local contactomorphisms of the canonical contact structure of $\mathbb{R}^{2n+1}$, Γ-structures on M are contact structures.

It is easy to detect the following pattern: for every "canonical" structure on the "model" $\mathbb{R}^n$, the diffeomorphisms preserving such structure generate a pseudogroup Γ (which also happens to be Lie). Moreover, the corresponding Γ-structure is the geometric structure on M locally modelled on the geometric structure on $\mathbb{R}^n$, in the same way that the smooth structure of M is locally modelled on the canonical smooth structure of $\mathbb{R}^n$.

Almost Γ-structures The fundamental question motivating this entire thesis is the following:

what is an "almost structure" in the general framework of (Lie) pseudogroups?

The concept of "almost structure" is already clear in many examples; for instance, an almost symplectic structure is a 2-form which is non-degenerate (but not necessarily closed). Actually, one can give a precise answer to the question above for an entire class of Γ-structures on M^n: those for which the pseudogroup Γ is of the form

$$\Gamma_G := \{\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n) \mid d_x\phi \in G \quad \forall x \in \mathrm{dom}(\phi)\},$$

for some Lie subgroup $G \subseteq GL(n,\mathbb{R})$. The first three examples above belong to this class, and the relevant G is, respectively, $O(n)$, $Sp(n)$ and $GL(n,\mathbb{C})$.

The role of an "almost Γ_G-structure" is then taken over by the classical notion of a <u>G-structure</u>. This consists in having for each point $x \in M$ a frame of T_xM which is "adapted" to the geometric structure; for instance, for $G = Sp(n)$, a frame which makes T_xM into a symplectic vector space. More precisely, if $Fr(M)$ denotes the principal $GL(n,\mathbb{R})$-bundle of frames of M, a G-structure consists in a reduction of $Fr(M)$ to a subbundle with structure group G.

A G-structure can be interpreted as the first-order approximation of a Γ_G-structure. Indeed, the differential of the charts of a Γ_G-atlas defines the adapted frames of a G-structure. Conversely, we say that a G-structure is <u>integrable</u> when it comes from a Γ_G-structure.

The fourth example mentioned above - that of contact structures - does not belong to this class of integrable G-structures since the pseudogroup of local contactomorphisms is not of the form Γ_G. The same thing can be said for other common geometric structures, such as affine structures, analytic structures, Thurston (G,X)-structures, etc.

The problem of defining almost Γ-structures for a general pseudogroup Γ has been discussed e.g. in [44, 4], but those authors restrict to the specific case of a *transitive* Lie pseudogroup Γ over $\mathbb{R}^n$. This means that, for any two points

$x, y \in \mathbb{R}^n$, one requires the existence of an element $\phi \in \Gamma$ sending x to y. In this thesis, we assume no condition on our Lie pseudogroups and formulate the following new definition:

Definition 1. Let Γ be a Lie pseudogroup on X and denote by $J^1(X, M) \to M$ the bundle of 1-jets of diffeomorphisms between opens of X and M. An <u>almost Γ-structure</u> on M is a subbundle $P \subseteq J^1(X, M)$ which is a principal $J^1\Gamma$-bundle over M:

$$\begin{tikzcd} & P \arrow[dl] \arrow[drr] & \curvearrowleft & J^1\Gamma \arrow[d, shift left] \arrow[d, shift right] \\ M & & & X \end{tikzcd}$$

♦

Note that here we used the general notion of principal bundle for an action by a Lie groupoid. For the transitive Lie pseudogroup $\Gamma = \Gamma_G$, we interpret G as the isotropy group of $J^1\Gamma$ and we recover the classical definition of a G-structure.

Integrability problem Any Γ-structure induces an almost Γ-structure: it is enough to consider as P the first-order approximation of a Γ-atlas $\mathcal{A}$:

$$J^1\mathcal{A} := \{j^1_x\chi \mid \chi \in \mathcal{A}, x \in \mathrm{dom}(\chi)\}.$$

The natural question is then the converse, known as *integrability problem*:

when does an almost Γ-structure come from a Γ-structure?

The question makes sense of course also for $\Gamma = \Gamma_G$, and the integrability problem for G-structures has been indeed extensively studied, e.g. by [43, 89, 86, 56, 63, 58]. In particular, to every G-structure one can associate the sequence of the so-called <u>intrinsic torsions</u> $\{T^k_{intr}\}_{k\geq 1}$, whose vanishing is a necessary condition for the G-structure to be integrable. Borrowing this terminology from the theory of PDEs, we say that the structure is <u>formally integrable</u> when $T^k_{intr} = 0$ for all k. However, formal integrability is necessary but not sufficient for integrability:

integrable G-structure $\Rightarrow$ formally integrable G-structure $\Rightarrow$ G-structure.

For instance, an almost complex manifold (M, J) is formally integrable if and only if its Nijenhuis tensor $T^1_{intr} = N_J$ vanishes (here $T^k_{intr} = 0$ for all $k \geq 2$). A famous and difficult theorem due to Newlander-Nirenberg states that the vanishing of N_J is sufficient for the existence of a complex atlas, i.e. for (M, J) to be integrable.

Many authours, e.g. Guillemin, Sternberg, Singer and Kumpera, were of course aware that they were studying a special case - the transitive one - of a more general problem. Some of them even wrote a “Part 1”, see e.g. [86] and [63], and announced they will discuss the intransitive case in a second future publication, which unfortunately never appeared. This thesis fills this gap: our main contribution is a detailed study of the integrability problem in the framework of almost Γ-structures, i.e. understanding when the following arrows can be inverted:

$$\Gamma\textit{-structure} \Rightarrow \textit{formally integrable almost } \Gamma\textit{-structure} \Rightarrow \textit{almost } \Gamma\textit{-structure.}$$

More precisely, given an almost Γ-structure, the integrability problem is handled in two steps:

- Solving the formal integrability problem. This is quite algebraic in nature, having relations with the theory of Spencer cohomology and prolongations of Lie algebras.
- Studying when formal integrability implies integrability. This requires analytic tools, e.g. the Cartan-Kähler theorem [13] from the theory of exterior differential systems, based on the classical Cauchy-Kowalevski theorem.

The abstract framework In order to study the integrability problem, we observe first the situation from a more conceptual point of view. The mindset for what we discuss below can be comparable to the following: when dealing with a linear algebra problem on $\mathbb{R}^n$, it might be easier (as well as more elegant) to solve the same problem in the abstract context of real vector spaces of dimension n. Accordingly, we are going to understand what the abstract structure behind an almost Γ-structure is, solve the problem in this new framework, and then go back to almost Γ-structures.

Recall that any jet bundle is equipped with a special 1-form with coefficients, known as the Cartan form. The intuitive idea is that any PDE on a jet bundle $P \subseteq J^k X$ is characterised by the geometric properties of its Cartan form θ (see [62, 84]). A Pfaffian fibration $(P, \theta) \to M$ captures such "PDE structure" by forgetting the jet bundle where the PDE lived and keeping only the most relevant data.

A Pfaffian groupoid, in turn, is a Lie groupoid whose source map $s : (\mathcal{G}, \theta) \to M$ is a Pfaffian fibration compatible with the multiplicative structure of $\mathcal{G}$. The space of jets $J^k\Gamma$ of a pseudogroup Γ, together with its Cartan form, is the prototypical example of Pfaffian groupoid. Analogies can be made with the more classical notions of symplectic [21] and contact [54] groupoids: those are also Lie groupoids $\mathcal{G}$ with a differential form on it, which is either symplectic or contact, and which is compatible with the multiplicative structure of $\mathcal{G}$.

The notions of Pfaffian fibrations and of Pfaffian groupoids appeared in [81] just a few years ago. In this thesis, inspired by Hamiltonian actions of symplectic groupoids on symplectic manifolds, we introduce the concept of a Pfaffian action of a Pfaffian groupoid on a Pfaffian fibration, and of the related principal Pfaffian bundle, for which the differential forms on P and on $\mathcal{G}$ are asked to satisfy further compatibility relations with the groupoid action. The canonical example of a principal Pfaffian bundle is given by any almost Γ-structure $P \subseteq J^1(X, M)$, together with the Cartan forms of $J^1\Gamma$ and of P. On the other hand, principal Pfaffian $J^1\Gamma$-bundles can be viewed as "abstract almost Γ-structures", since there is a unique immersion

$$j : (P, \theta) \hookrightarrow (J^1(X, M), \theta_{\text{can}})$$

which is compatible with the form θ on P and the Cartan form θ_{can} of $J^1(X, M)$, i.e. such that $j(P)$ is an "actual" almost Γ-structure.

Obstructions to formal integrability With the machinery of principal Pfaffian bundles at our disposal, we can solve the formal integrability problem by means of recursive prolongations: this procedure is inspired from the classical theory of prolongations for PDEs on jet bundles.

More precisely, given an almost Γ-structure P, we want first to characterise its integrability up to order 1, i.e. the existence of a principal $J^2\Gamma$-bundle, which we denote by $P^{(1)} \subseteq J^2(X, M)$, such that $P^{(1)} \to P$ is a surjective submersion. This is done by passing to the abstract picture of principal Pfaffian bundles, defining a "candidate" for $P^{(1)}$, and investigating under which conditions $P^{(1)}$ satisfies the properties required above. If these are satisfied, we apply the same arguments again to look for a principal $J^3\Gamma$-bundle $P^{(2)}$, which projects onto $P^{(1)}$, and so on. In conclusion, we have:

Theorem 2. *Let P be an almost Γ-structure on M. Then there exists a sequence of vector bundles $\mathcal{H}^k$ over M and of sections (defined inductively)*

$$T^k_{intr} \in \Gamma(\mathcal{H}^k),$$

whose vanishing is equivalent to the formal integrability of P.

In the case $\Gamma = \Gamma_G$, the T^k_{intr} reduce to the intrinsic torsions for G-structures.

Obstructions to integrability After reaching formal integrability, one should take care of the last step (studying when formal integrability implies integrability), which, as we anticipated, has an "analytic" flavour. If the formal integrability problem can be compared to compute the Taylor series of a function, this last step amounts to understanding when such a series converges to the function. Accordingly, it is a much harder problem, for which even in the case of G-structures there are no general solutions. Our contribution has been to detect precisely where the "analysis" is concentrated:

Theorem 3. *Let Γ be a Lie pseudogroup and assume that its non-linear Spencer complex is exact. Then, if an almost Γ-structure is integrable up to order 1, it is integrable.*

By non-linear Spencer complex of Γ we mean a sequence of the kind:

$$\mathrm{Bis}_{\mathrm{loc}}(J^2\Gamma) \xrightarrow{F} \Omega^1(X, A) \xrightarrow{G} \Omega^2(X, E),$$

where A the Lie algebroid of $J^1\Gamma$ and E is the coefficient of its Cartan form. The main advantage of this point of view is that it allows us to rephrase integrability in more explicit PDE terms: the map G can be interpreted as the PDE which defines the holonomic sections of $(P, \theta) \to M$, and the map F as the symmetries given by the $J^1\Gamma$-action. Accordingly, one can investigate the exactness of the linearised PDE (which is much easier to handle), and see when this is sufficient for the exactness of the non-linear one.

Morita invariance of geometric structures A third question we devoted our attention to is the following:

when do "equivalent" Lie pseudogroups describe "equivalent" geometries?

First of all, one translates the problem from Lie pseudogroups to their associated (jet) Lie groupoids. It becomes important then to understand when two Lie groupoids can be considered "equivalent": unlike for Lie groups, the naïve notion of isomorphism between Lie groupoids is too strict. The "correct" notion to consider is that of Morita equivalence, which is more flexible and, in the case of Lie groups, becomes the usual isomorphism.

We then prove directly that Morita equivalent jet groupoids give rise to the same almost structures. However, the integrability conditions are, in general, not preserved under this equivalence, i.e. integrability is not a Morita invariant property. The most transparent counterexample is given by symplectic foliations and contact structures, which share their first-order approximation but are clearly different at the integrable level.

We provide two general results to understand further this phenomenon. Restricting to transitive Lie pseudogroups on $\mathbb{R}^n$, we show that a sufficient condition for integrability to be Morita invariant is the presence of translations inside the pseudogroup. More generally, as we did for the formal integrability problem, we can move to the abstract picture by defining the notion of Pfaffian Morita equivalence: in this framework, we have isolated a special class of Morita equivalences (which we call integrable) that preserve integrability.

Main achievements of this thesis

- A new definition of almost Γ-structure for any (possibly non-transitive) Lie pseudogroup Γ (Sections 5.2-5.5)
- An equivalent condition for formal integrability of almost Γ-structures (Section 6.1)
- A theory of Morita equivalence in the context of Pfaffian groupoids, including Morita invariant properties and several examples (Section 5.3-5.4)

As byproducts of this project, we also obtained several other minor results, such as a new understanding of the parallelism between contact structures and symplectic foliations via Morita equivalence.

Sketch of the chapters in this thesis

- Chapter 1: This chapter, based on the preprint [19], contains some background on jet bundles, PDEs and Spencer cohomology, which will be used throughout the entire thesis. Moreover, we present original results on the prolongations and integrability of Pfaffian fibrations, which will come into play starting from chapter 4.

- Chapter 2: Here we review the general theory of (classical, abstract and higher order) G-structures. Some results will not be used directly, but they constitute the basis for our intuition when working with almost Γ-structures in chapter 5. Other propositions will become particular cases of more general theorems proved later in the thesis.

- Chapter 3: We review the basics of Lie groupoids, Lie algebroids and Morita equivalences in order to study multiplicative forms with coefficients. These are differential forms on Lie groupoids compatible with the multiplication, and can be understood either from a global or an infinitesimal point of view: we recall some known results and prove new ones, to be used in the computations of the next chapters.

- Chapter 4: After reviewing the notion of (Lie) pseudogroups, we present a weaker revisitation of the concept of Pfaffian groupoid, motivated by the example of the structure group of a G-structure. We explain then how the integrability of Pfaffian fibrations from chapter 1 reduces in this multiplicative setting to (easier) Lie theoretic results.

- Chapter 5: This long chapter is the core of the thesis. First we introduce the notions of Γ-structure and almost Γ-structure. Then we develop the necessary machinery to study them from an abstract point of view, namely principal Pfaffian bundles and Pfaffian Morita equivalences. As a by-product, we obtain a couple of applications of principal Pfaffian bundles to generalised pseudogroups and to Cartan geometries (based on the preprint [18]). We conclude by developing the theory of prolongations for principal Pfaffian bundles and studying the obstructions for their formal integrability.

- Chapter 6: In this chapter we present the main theorems on the integrability of almost Γ-structures, relying heavily on the definitions and the results from chapter 5. In particular, we show necessary and sufficient conditions for formal integrability, recast the “analytic step” of the integrability problem into a more conceptual problem concerning the exactness of the non-linear Spencer sequence, and investigate the interplay between integrability and Morita equivalence.

Notations All manifolds and maps are smooth unless explicitly stated otherwise. Einstein convention and British spelling are adopted.

Chapter 1

1

Pfaffian fibrations

Throughout this thesis we will be using in several occasions the formalism of jets and various aspects related to them; we collect the relevant material in this chapter. In principle, this chapter can also be skipped and parts of it should be read whenever needed: for instance, Section 1.5 of this chapter will be needed only starting from Section 5.8, when one needs the theory of integrability of Pfaffian fibrations.

Historically, the formalism of jets arose to provide a coordinate-free formulation of PDEs on manifolds. Such a process quickly revealed the notion of Cartan form (and distribution) on the jet bundles J^kP and its central role to the entire geometric theory. The various ways of understanding these objects gave rise to different schools and approaches to the subject, e.g. depending on whether (and how) one works with vector fields or differential forms; see, among others, the monographs [62, 77, 84, 9, 91, 93].

After recalling some of the basic notions, we change a bit the formalism. The main message is that what is needed for the theory to work is not the jet bundles J^kP but just a fibration R together with the induced Cartan form, satisfying the appropriate conditions: in our language, a *Pfaffian fibration*. Of course, there are points at which the jet bundles are still important, but often they are just "noise" in the background, giving rise to unnecessarily complicated formulae. While this point may be, in principle, rather obvious to the specialists, we find it useful to spell it out in detail, taking care of the subtleties that arise along the way. We hope that, in this way, various techniques and notions that are often presented in a rather pragmatic way, via "down to earth" (but complicated) local formulae, become more transparent to people with a more geometric background.

Most of this chapter is based on the preprint [19], to which we refer for the proofs of the results in Sections 1.3-1.6.

1.1 Jets and PDEs

A Partial Differential Equation (shortly, a PDE) of order k in the function $y : \mathbb{R}^n \to \mathbb{R}^m$ is an equation of the form

$$F\left(x^i, y^\alpha, y^\alpha_J\right) = 0, \qquad y^\alpha_J := y^\alpha_{j^1 \dots j^s} = \frac{\partial^s y^\alpha}{\partial x^{j_1} \cdots \partial x^{j_s}},$$

for all multi-indices $J = (j_1 \dots j_s)$ of length $s \leq k$, where F is a vector-valued function, the Latin indices run from 1 to n and the Greek indices run from 1 to m.

However, in order to describe a conceptual theory of PDEs on manifolds, the language of jets will be very well suited, since it allows one to see the PDE as a submanifold of the k-jet bundle given by the zero locus of F. Jet spaces were introduced by Ehresmann in [31]; see [62, 84] as modern references.

More precisely, the k-jet of y at $x \in \mathbb{R}^n$ encodes all the partial derivatives of y at x up to order k: this means that two such functions have the same k-jet at x if they have the same Taylor polynomial of degree k at x. This defines an equivalence relation $\sim^k_x$ on the space of smooth maps $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, whose induced equivalence class (the k-jet of y at x) is denoted by $j^k_x y$. Such an element of this quotient has coordinates induced by the value of any representative $y : \mathbb{R}^n \to \mathbb{R}^m$ and its derivatives at x, i.e.

$$(x^i, y^\alpha, y^\alpha_J)$$

with indices as above.

The discussion can be generalised to maps between manifolds M and N or, more generally, to sections of a fibration (i.e. a surjective submersion) $\pi : P \to M$; smooth functions $f : M \to N$ arise from sections of the trivial fibration $\mathrm{pr}_1 : M \times N \to M$. A further generalisation to jets of submanifolds is possible (see e.g. Section 7.1 of [62]) but will not be needed here.

Definition 1.1.1. Given a fibration $\pi : P \to M$ (i.e. a smooth surjective submersion), the **set of its k-jets at** $x \in M$ is the quotient of the local sections defined around x w.r.t. the equivalence relation $\sigma \sim^k_x \rho$:

$$J^k_x \pi := \Gamma_{\mathrm{loc},x}(\pi) / \sim^k_x .$$

We denote the equivalence classes by $j^k_x \sigma = [\sigma]^k_x$. The set of all k-jets of π is

$$J^k \pi := \coprod_{x \in M} J^k_x \pi = \{ j^k_x \sigma \mid \sigma \in \Gamma_{\mathrm{loc}}(\pi), x \in \mathrm{dom}(\sigma) \}.$$

Sometimes one also uses the notation $J^k P$ instead of $J^k \pi$. ♦

The set $J^k \pi$ has a natural structure of smooth manifold, with a system of coordinates $(x^i, y^\alpha, y^\alpha_J)$ induced from the base coordinates x^i on M and the fibre coordinates y^α on P. In particular, the coordinates of type $y^\alpha_{j_1 \dots j_s}$ are symmetric in the j-indices, since they will represent the "higher derivatives" (of order s). Furthermore, each $J^k\pi$ defines a fibration over M, with projection map

$$\pi^k : J^k\pi \to M, \quad j^k_x\sigma \mapsto x,$$

or simply π when there is no ambiguity. When $\pi : P \to M$ is a vector bundle over M, then so is $J^k\pi$.

Remark 1.1.2. In the case $k = 0$, since $\sigma \sim^0_x \rho \Leftrightarrow \sigma(x) = \rho(x)$, the 0-jet $j^0_x\sigma$ can be identified with $\sigma(x) \in P$, hence $J^0\pi$ with P and π^0 with π. By convention, we set $J^{-1}\pi := M$ and $\pi^{-1} := \mathrm{id}_M$.

In the case $k = 1$, a jet $j^1_x\sigma$ is completely encoded in $\sigma(x) \in P$ and the differential $d_x\sigma : T_xM \to T_{\sigma(x)}P$. Actually, since σ is a section of π, this differential is completely encoded in its image

$$H_{\sigma(x)} := \mathrm{Im}(d_x\sigma) \subseteq T_{\sigma(x)}P.$$

Indeed, $d_x\sigma$ will be the inverse of $d\pi|_H$. Of course, H is not an arbitrary subspace: it is a complement in $T_{\sigma(x)}P$ of the subspace $T^\pi_{\sigma(x)}P$, where $T^\pi P$ is the bundle of π-vertical vectors:

$$T^\pi P := \ker(d\pi) \subseteq TP.$$

Such a complement is also called a **horizontal subspace** for π. Therefore, one has

$$J^1\pi \cong \{(p, H_p) \mid p \in P, H_p \subseteq T_pP \text{ horizontal}\}.$$

Note also the relationship with the notion of Ehresmann connection on $\pi : P \to M$ which, by definition, means a subbundle $H \subseteq TP$ complementary to $T^\pi P$. We see that such connections are basically the same thing as sections of the canonical projection $J^1\pi \to P$. $\Diamond$

The various jet bundles are related to each other by the obvious projections

$$\cdots \to J^2\pi \to J^1\pi \to J^0\pi = P,$$

all denoted by pr; when we want to be more precise, we use the notation π^k_l or

$$\mathrm{pr}^k_l : J^k\pi \to J^l\pi, \quad j^k\sigma \mapsto j^l\sigma \quad (k \geq l).$$

Locally these maps correspond always to the projection to the first group of coordinates, i.e. $\mathrm{pr}^k_l : (x^i, y^\alpha, y^\alpha_{j_1}, \ldots, y^\alpha_{j_1\ldots j_k}) \mapsto (x^i, y^\alpha, y^\alpha_{j_1}, \ldots, y^\alpha_{j_1\ldots j_l})$. All these maps are surjective submersions, and

$$\mathrm{pr} = \mathrm{pr}^k_{k-1} : J^k\pi \to J^{k-1}\pi$$

has a canonical structure of affine bundle modelled on the vector bundle

$$\pi^* S^k(T^*M) \otimes \mathrm{pr}^* T^\pi P,$$

where $T^\pi P$ is the bundle of π-vertical vectors (Remark 1.1.2). In particular, there is a canonical short exact sequence of vector bundles over $J^k\pi$:

$$0 \to \pi^* S^k(T^*M) \otimes \mathrm{pr}^* T^\pi P \to T^\pi J^k\pi \xrightarrow{d\,\mathrm{pr}} \mathrm{pr}^*(T^\pi J^{k-1}\pi) \to 0.$$

1.1.1 PDEs and the Cartan form

Having at hand the language of jets, one can formalise the notion of PDE (see e.g. Section 7 of [38]).

Definition 1.1.3. A **PDE of order** k on $\pi : P \to M$ is a connected submanifold

$$R \subseteq J^k\pi,$$

which fibres over M. A **(local) solution of** R is any (local) section σ of π with the property that

$$j^k_x\sigma \in R \quad \forall x \in \mathrm{dom}(\sigma). \qquad \blacklozenge$$

Typically, a PDE is also asked to satisfy some mild regularity conditions. While one could develop most of the theory with no further assumptions, these conditions simplify the exposition and avoid unnecessary technicalities. Accordingly, in the rest of the thesis we will follow Section 1.4 of [100] and require that, if $R \subseteq J^k\pi$ is a PDE, $\pi^k_{k-1}(R) \subseteq J^{k-1}\pi$ is a submanifold as well, and the projections $R \to \pi^k_{k-1}(R)$ and $\pi^k_{k-1}(R) \to \pi(R) \subseteq M$ are submersions.

The set of solutions of R, denoted by $\mathrm{Sol}(R)$, can be identified with the set of sections ξ of $R \to M$ which are **holonomic**, i.e. of the form $\xi = j^k\sigma$, for σ a section of π. An extremely useful tool to handle holonomicity is the Cartan form:

Definition 1.1.4. The **Cartan form** of $J^k\pi$ is the 1-form with values in the pullback (via the projection $\mathrm{pr} : J^k\pi \to J^{k-1}\pi$) of the vector bundle over $J^{k-1}\pi$ of vectors tangent to the fibres of $J^{k-1}\pi \to M$,

$$\omega \in \Omega^1(J^k\pi, \mathrm{pr}^* T^\pi J^{k-1}\pi)$$

or, more precisely,

$$\omega \in \Omega^1(J^k\pi, (\mathrm{pr}^k_{k-1})^* T^{\pi^{k-1}} J^{k-1}\pi),$$

which at $p = j^k_x\sigma \in J^k\pi$ is defined by

$$\omega_p := d_p \,\mathrm{pr}^k_{k-1} - d_x j^{k-1}\sigma \circ d_p\pi^k. \qquad \blacklozenge$$

The formula may be more transparent if one pictures the maps in a diagram:

$$\begin{tikzcd} T_pJ^k\pi \arrow[rr, "d_p\,\mathrm{pr}^k_{k-1}"] \arrow[dr, "d_p\pi^k"'] & & T_{p'}J^{k-1}\pi \\ & T_xM \arrow[ur, "d_xj^{k-1}\sigma"'] & \end{tikzcd} \qquad (p' = j^{k-1}_x\sigma).$$

For instance, for $k = 1$, at $p = j^1_x\sigma$, ω_p evaluated on $v \in T_pJ^1\pi$ is given by

$$\omega_p(v) := d\mathrm{pr}(v) - d_x\sigma(d_p\pi(v)) \in T^\pi_{\sigma(x)}P.$$

In local coordinates, the Cartan form of $J^k\pi$ has the local expression

$$\omega = \omega^\alpha_J \otimes \frac{\partial}{\partial y^\alpha_J}, \qquad \omega^\alpha_J := dy^\alpha_J - y^\alpha_{J_i} dx^i$$

where we sum over all multi-indices J of length $0, \ldots, k-1$, and where J_i denotes the multi-index $(j_1 \ldots j_s i)$, if $J = (j_1 \ldots j_s)$. Here is the main property of ω:

Proposition 1.1.5. *A section $\xi \in \Gamma(J^k\pi)$ is holonomic if and only if $\xi^*\omega = 0$.* 1

Conceptually, this means that we can characterise the solutions of $R \subseteq J^k\pi$ only in terms of R viewed as a bundle over M (and not as a subbundle of $J^k\pi$), together with the restriction of ω to R:

$$\mathrm{Sol}(R) \simeq \Gamma(R, \omega) := \{\xi \in \Gamma(R) \mid \xi^*\omega = 0\}.$$

In other words, for the study of PDEs, the only relevant data is a fibration $R \to M$ endowed with an appropriate 1-form: this will be our starting point for the definition 1.3.1 of Pfaffian fibrations (which forgets the ambient jet spaces).

One sometimes uses, instead of ω, its kernel

$$\mathcal{C} := \ker \omega \subseteq TJ^k\pi,$$

also called the **Cartan distribution** of $J^k\pi$. Proposition 1.1.5 can be recast by saying that a section $\xi \in \Gamma(J^k\pi)$ is holonomic if and only if it is tangent to $\mathcal{C}$, i.e.

$$d_x\xi(T_xM) \subseteq \mathcal{C}_{\xi(x)} \quad \forall x \in M.$$

To be precise, we should write ω^k and $\mathcal{C}_k$, since we can define a different Cartan form and distribution for each jet bundle; however, they are all related since $(\mathrm{pr}^k_l)^*\omega^l = \mathrm{pr}^{k-1}_{l-1} \circ \omega^k$, so we will omit this index k in most of the cases. Note also that, in local coordinates, the Cartan distribution of $J^k\pi$ is generated by the following vector fields

$$\begin{cases} D_i := \dfrac{\partial}{\partial x^i} + \displaystyle\sum_{|J|=0}^{k-1} y^\alpha_{J_i} \frac{\partial}{\partial y^\alpha_J}, & i = 1, \ldots, \dim(M), \\ Y^J_\beta := \dfrac{\partial}{\partial y^\beta_J} & |J| = k, \quad \beta = 1, \ldots, \dim(P) - \dim(M). \end{cases}$$

It is easy to show that the distribution $\mathcal{C}$ is (maximally) not involutive: while

$$[D_i, D_j] = 0, \quad \left[Y^J_\beta, Y^L_\gamma\right] = 0,$$

the mixed brackets are never in $\Gamma(\mathcal{C})$:

$$\left[Y^J_\beta, D_i\right] = \frac{\partial}{\partial y^\beta_{j_1 \ldots j_{k-1}}}.$$

1.1.2 Prolongations of PDEs

The theory of prolongations of a PDE is a powerful tool to study solutions for a PDE. The literature on this topic is very rich and dates back several decades: we mention [40,41,77,9,91,93] and we will briefly recall here some of these notions.

A prolongation of a k-order PDE R on $\pi : P \to M$ (Definition 1.1.3) can be thought of as the $(k+1)$-order PDE on π of first order differential consequences of R, with the fundamental property of having the same space of solutions. The first naïve guess to define the prolongation of R would be simply $J^1R = \{j^1_x\sigma \mid \sigma \in \Gamma(R)\}$. However, one immediately sees that J^1R fails to be a PDE of order $k+1$ on π, since J^1R is by construction a subset of $J^1(J^kP)$, while $J^{k+1}P \subseteq J^1(J^kP)$ is in general a strict inclusion. The **prolongation** $R^{(1)}$ of R is defined as

$$R^{(1)} := J^1R \cap J^{k+1}P \subseteq J^{k+1}P.$$

In general, $R^{(1)}$ may fail to be a subbundle of $J^{k+1}P$ or even to be smooth. If $R^{(1)}$ happens to be "nice enough" - in the sense that it is smooth and the projection $R^{(1)} \to R$ is a surjective submersion - then R is said to be **integrable up to order** 1. Continuing inductively one defines

$$R^{(l+1)} := \left(R^{(l)}\right)^{(1)},$$

one talks about integrability up to order k and, when this holds for all k, we say that R is **formally integrable**. In this case we obtain a tower of bundles over M

$$\cdots \to R^{(2)} \to R^{(1)} \to R,$$

each of them endowed with the restriction of the Cartan form at every order, and all the projections being surjective submersions.

The study of formal integrability of a PDE is a very useful tool in order to find its solutions. This can be best seen in the analytic case, where formal integrability becomes a sufficient condition for **integrability**, i.e. finding local solutions at every point.

Proposition 1.1.6 (Theorem 9.1 of [38]). *If $R \subseteq J^k\pi$ is an analytic formally integrable PDE, then for every $p \in R^{(l)} \subseteq J^{k+l}\pi$ over $x \in M$ there is an analytic local solution σ of R, defined on a neighbourhood of x, such that $j^{k+l}_x\sigma = p$.*

In particular, for every $p \in R$ there passes a local (analytic) solution.

However, in the smooth category this is not always true, since there are formally integrable PDEs admitting no solution: see the famous Lewy counterexample [65]. Indeed, Proposition 1.1.6 is essentially a reformulation, in the setting of PDEs on jet bundles, of the classical Cartan-Kähler theorem, which holds only in the analytic category.

To understand better the structure of the prolongations and the notion of formal integrability, one arrives at the notion of tableau, to which we dedicate the next section. They are certain linear spaces that provide the framework to handle the intricate linear algebra behind PDEs, and they also provide cohomological criteria for the integrability of PDEs.

1.2 Tableaux and Spencer cohomology

As mentioned in the previous section, prolongation is a standard and powerful technique for studying PDEs. Although such a process is, in nature, quite intricate and non-linear, it is remarkable (and probably one of the reasons for its usefulness) that much of it is controlled by linear algebra; the reason is that the prolongation is always affine in the new extra coordinates. In this section we give an overview on this linear counterpart (to be used later on, also for prolongations of G-structures); see as references [86, 39, 13].

Definition 1.2.1. Let V, W be vector spaces. A **tableau** on (V, W) is a linear subspace

$$\mathfrak{g} \subseteq \mathrm{Hom}(V, W).$$

The **first prolongation of** $\mathfrak{g}$ is the vector space

$$\mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(V, \mathfrak{g})$$

consisting of linear maps $\xi : V \to \mathfrak{g}$ satisfying

$$\xi(v)(u) = \xi(u)(v) \quad \forall u, v \in V.$$

Recursively, we define the $k^{\mathbf{th}}$**-prolongation** of $\mathfrak{g}$

$$\mathfrak{g}^{(k)} \subseteq \mathrm{Hom}(V, \mathfrak{g}^{(k-1)})$$

as the prolongation of the tableau $\mathfrak{g}^{(k-1)} \subseteq \mathrm{Hom}(V, \mathfrak{g}^{(k-2)})$, with the conventions $\mathfrak{g}^{(0)} := \mathfrak{g}$ and $\mathfrak{g}^{(-1)} := W$. ♦

Note that, even if interested only in the case $V = W$, higher prolongations require us to consider also the case $V \neq W$.

Remark 1.2.2. Let $\mathfrak{g}$ be a tableau on (V, W); since

$$\mathfrak{g}^{(k)} \subseteq \mathrm{Hom}(V, \mathfrak{g}^{(k-1)}) \subseteq \mathrm{Hom}(V, \mathrm{Hom}(V, \mathfrak{g}^{(k-2)})) \subseteq \ldots,$$

the elements $t \in \mathfrak{g}^{(k)}$ can be interpreted as symmetric k-multilinear maps

$$t : V \times \ldots \times V \to W$$

such that, for every fixed $v_1, \ldots, v_k \in V$, the linear map

$$t(\cdot, v_1, \ldots, v_k) \in \mathrm{Hom}(V, W)$$

belongs to $\mathfrak{g}$. This provides a direct definition of the prolongation $\mathfrak{g}^{(k)}$ as a subspace of $S^k V^* \otimes W$. ♢

Definition 1.2.3. The **Spencer complex** of a tableau $\mathfrak{g} \subseteq \mathrm{Hom}(V, W)$ is the bigraded vector space $C^{\bullet,\bullet}(\mathfrak{g})$ defined by

$$C^{l,m}(\mathfrak{g}) := \mathrm{Hom}(\Lambda^m V, \mathfrak{g}^{(l)}),$$

endowed with the differentials

$$\delta = \delta^{l,m} : C^{l,m}(\mathfrak{g}) \to C^{l-1,m+1}(\mathfrak{g}),$$

$$\delta(\xi)(u_0, \ldots, u_m) := \sum_{i=0}^{m} (-1)^i \xi(u_0, \ldots, \hat{u}_i, \ldots, u_m)(u_i).$$

One also defines the **Spencer cohomology** spaces of $\mathfrak{g}$

$$H^{l,m}(\mathfrak{g}) := \frac{\ker\left(\delta : C^{l,m}(\mathfrak{g}) \to C^{l-1,m+1}(\mathfrak{g})\right)}{\mathrm{Im}\left(\delta : C^{l+1,m-1}(\mathfrak{g}) \to C^{l,m}(\mathfrak{g})\right)}. \qquad \blacklozenge$$

In particular, recalling the conventions that $\mathfrak{g}^{(0)} := \mathfrak{g}$, $\mathfrak{g}^{(-1)} := W$ and $\mathfrak{g}^{(k)} := 0$ for $k \leq -2$, one sets

$$H^{-1,m}(\mathfrak{g}) := \frac{\mathrm{Hom}(\Lambda^m V, W)}{\delta\left(\mathrm{Hom}(\Lambda^{m-1} V, \mathfrak{g})\right)}.$$

Let us mention a few useful properties of the Spencer cohomology. It is clear that the first prolongation of $\mathfrak{g}$ coincides with the kernel $\ker(\delta)$ of the antisymmetrisation map δ

$$\delta = \delta^{0,1} : \mathrm{Hom}(V, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^2 V, W),$$

and similarly, for higher prolongations, $\mathfrak{g}^{(k)} = \ker(\delta^{k-1,1})$. In other words (see also Lemma 6.2 of [38]):

$$H^{l,0}(\mathfrak{g}) = H^{l,1}(\mathfrak{g}) = 0 \quad \text{for all } l.$$

Given $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{g}^{(1)}$, one checks easily the following relation between their Spencer complexes

$$(C^{l,m}(\mathfrak{g}), \delta^{l,m}) = (C^{l-1,m}(\mathfrak{h}), \delta^{l-1,m}).$$

As a consequence, for any fixed m, one obtains the chain of equalities

$$H^{l,m}(\mathfrak{g}) = H^{l-1,m}(\mathfrak{g}^{(1)}) = \ldots = H^{1,m}(\mathfrak{g}^{(l-1)}) = H^{0,m}(\mathfrak{g}^{(l)}).$$

If follows that, if $\mathfrak{g}$ is of **finite type**, i.e. $\mathfrak{g}^{(i)} = 0$ for all $i \geq k$, then

$$H^{l,m}(\mathfrak{g}) = 0 \quad \text{for all } l \geq k, m \geq 0.$$

Definition 1.2.4. A tableau $\mathfrak{g} \subseteq \mathrm{Hom}(V, W)$ is said to be **involutive** if

$$H^{l,m}(\mathfrak{g}) = 0 \quad \text{for all } l \geq 0, m \geq 0,$$

and k**-acyclic** if the condition above holds for all $l \geq 0$ and all $m \leq k$. ♦

One shows easily that, for the prolongation $\mathfrak{g}^{(1)}$ to be involutive, it is sufficient (but not necessary) that $\mathfrak{g}$ is involutive. A deep result by Singer and Sternberg (based on the Cartan-Kuranishi theorem)says that, starting with an arbitrary $\mathfrak{g}$, after a certain number of prolongations involutivity can always be achieved:

Proposition 1.2.5 (section 4.6 of [86])**.** *Given a tableau* $\mathfrak{g} \subseteq \mathrm{Hom}(V, W)$*, there exists a* $k \in \mathbb{N}$ *such that* $\mathfrak{g}^{(k)}$ *is involutive.*

Later on, in the theory of Pfaffian fibrations, we will need a small variant of the Spencer complex, namely its analogous version for a **generalised tableau**, in which the inclusion $\mathfrak{g} \hookrightarrow \mathrm{Hom}(V, W)$ is replaced by a linear map

$$\partial : \mathfrak{g} \to \mathrm{Hom}(V, W).$$

Definition 1.2.6. The **first prolongation of $\mathfrak{g}$ (with respect to ∂)** is

$$\mathfrak{g}^{(1)}(\partial) := \{\eta \in \mathrm{Hom}(V, \mathfrak{g}) \mid \partial(\eta(u))(v) = \partial(\eta(v))(u)\ \forall u, v \in V\}. \qquad ♦$$

We can regard $\mathfrak{g}^{(1)}(\partial)$ as a (classical) tableau on $(V, \mathfrak{g})$ and prolong it repeatedly, giving rise to the higher prolongations

$$\mathfrak{g}^{(i)}(\partial) \subseteq \mathrm{Hom}(V, \mathfrak{g}^{(i-1)}(\partial)).$$

In the same way one adapts also the Spencer sequence to this more general situation, giving rise to the **∂-Spencer cohomology of $\mathfrak{g}$**.

Another small variation arises when dealing with vector bundles E, F over M instead of vector spaces: all the notions discussed above extend naturally, or can be applied fibrewise. In particular, a **tableau bundle on** (E, F) is a bundle over M

$$\mathfrak{g} \subseteq \mathrm{Hom}(E, F)$$

consisting of linear subspaces $\mathfrak{g}_x \subseteq \mathrm{Hom}(E_x, F_x)$ for $x \in M$; we call it *smooth* if it is a smooth subbundle of $\mathrm{Hom}(E, F)$. We have to allow non-smooth bundles because, even if we start with a smooth tableau bundle $\mathfrak{g}$, its prolongations $\mathfrak{g}^{(i)}$ may fail to be smooth. One of the roles of the acyclicity condition (Definition 1.2.4) is to ensure the smoothness of the prolongations:

Lemma 1.2.7 (Lemma 6.5 of [38])**.** *Let* $\mathfrak{g} \subseteq \mathrm{Hom}(E, F)$ *be a tableau bundle of constant rank over a connected manifold* M*. If*

- $\mathfrak{g}$ *is 2-acyclic,*

- $\mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(E, \mathfrak{g})$ *is of constant rank,*

then the prolongation $\mathfrak{g}^{(i)} \subseteq \mathrm{Hom}(E, \mathfrak{g}^{(i-1)})$ *is also of constant rank for all* $i \geq 0$.

It is a classical result that the kernel of a vector bundle morphism is a vector subbundle if and only if it is of constant rank; then the previous statement gives a criterion for all tableau bundles $\mathfrak{g}^{(i)}$ to be smooth. Lemma 1.2.7 above also holds when dealing with a smooth tableau bundle defined by a vector bundle map $\partial : \mathfrak{g} \to \mathrm{Hom}(E, F)$ over M; in that case we are considering of course the prolongations $\mathfrak{g}^{(i)}(\partial)$ w.r.t. ∂.

1.2.1 Symbol space of PDEs

As anticipated in the previous section, we recall now the canonical tableau bundle associated to any PDE, which controls its formal integrability.

Let $R \subseteq J^k\pi$ be a PDE on the fibration $\pi : P \to M$. The **symbol space** of R is the vector subbundle $\mathfrak{g} \subseteq TR$ defined as

$$\mathfrak{g} := TR \cap \ker(d\,\mathrm{pr}^k_{k-1}).$$

Equivalently, using the Cartan form $\omega \in \Omega^1(J^k\pi, E)$ (Definition 1.1.4),

$$\mathfrak{g} := T^\pi R \cap \ker(\omega).$$

In order to interpret $\mathfrak{g}$ as a tableau bundle, recall from Section 1.1 that

$$\ker(d\,\mathrm{pr}^k_{k-1}) \cong \pi^*(S^k(T^*M)) \otimes \mathrm{pr}^*(T^\pi P).$$

Using the standard inclusion

$$S^k(T^*M) \subseteq T^*M \otimes S^{k-1}(T^*M),$$

we obtain

$$\ker(d\,\mathrm{pr}^k_{k-1}) \subseteq \mathrm{Hom}(\pi^*TM, \mathrm{pr}^* \ker(d\,\mathrm{pr}^{k-1}_{k-2})).$$

It follows that

$$\mathfrak{g} \subseteq \mathrm{Hom}(\pi^*TM, E),$$

where $E = \mathrm{pr}^*(T^\pi J^{k-1}\pi)$ is the coefficient of ω. This tableau bundle structure provides a sufficient criterion for formal integrability of PDEs in terms of the prolongations and the Spencer cohomology of $\mathfrak{g}$.

Proposition 1.2.8 (Goldschmidt formal integrability criterion). *Let R be a PDE on $\pi : P \to M$ such that*

- *the symbol space $\mathfrak{g}$ is 2-acyclic,*
- *$R^{(1)} \to R$ is surjective and the prolongation $\mathfrak{g}^{(1)}$ is of constant rank.*

Then R is formally integrable.

We refer to Theorem 8.1 in [38] for the original result and Theorem 1.5.8 in [100] for a modern proof, where Lemma 1.2.7 constitutes a crucial step.

1.3 Pfaffian fibrations

As promised, we now change a bit the formalism of PDEs explaining that, for a large part of theory to work, one can give up on jets provided we retain the Cartan form and its properties.

1

Definition 1.3.1. A **Pfaffian fibration** $(R, \theta) \to M$ consists of a fibration $\pi : R \to M$ together with a pointwise surjective form $\theta \in \Omega^1(R, \mathcal{N})$ with coefficients in some vector bundle $\mathcal{N} \to R$, such that

- θ is *π-transversal*, i.e. $\ker(\theta)$ is transversal to the π-fibres:
$$\ker(\theta) + T^{\pi}R = TR.$$
- θ is *π-involutive*, i.e. the following distribution is involutive
$$\mathfrak{g}(\theta) := T^{\pi}R \cap \ker \theta.$$

We say that θ is a **Pfaffian form** and we call $\mathcal{N}$ its **coefficient bundle**. ♦

The involutive distribution $\mathfrak{g}(\theta) \subseteq TR$ is called the **symbol space** of θ; from the π-transversality of θ it follows that it has constant rank.

As anticipated, the k-jet space $\pi : J^k\pi \to M$ of a fibration $\pi : P \to M$, endowed with its canonical Cartan form ω (Definition 1.1.4), is a Pfaffian fibration. More generally, any PDE $R \subseteq J^k\pi$ (with the assumptions required after Definition 1.1.3) is a Pfaffian fibration, together with the restriction of ω.

Remark 1.3.2. (Pfaffian distributions) We can look at surjective π-transversal 1-forms from the equivalent point of view of distributions transversal to the π-fibres (or **π-transversal distributions**). In particular, starting with a π-transversal distribution
$$\mathcal{C} \subseteq TR,$$
one defines the *symbol space* of $\mathcal{C}$
$$\mathfrak{g}(\mathcal{C}) := T^{\pi}R \cap \mathcal{C}$$
and the normal bundle
$$\mathcal{N}_{\mathcal{C}} := TR/\mathcal{C} \cong T^{\pi}R/\mathfrak{g}(\mathcal{C})$$
If, moreover $\mathfrak{g}(\mathcal{C})$ is involutive, we call $\mathcal{C}$ a *Pfaffian distribution.* We can then produce the 1-form $\theta_{\mathcal{C}} \in \Omega^1(R, \mathcal{N}_{\mathcal{C}})$ (and say that $\theta_{\mathcal{C}}$ is induced by $\mathcal{C}$) given by the projection $TR \to \mathcal{N}_{\mathcal{C}}$: by construction $\theta_{\mathcal{C}}$ satisfies $\ker(\theta_{\mathcal{C}}) = \mathcal{C}$, is surjective, π-transversal, and its symbol space coincides with that of $\mathcal{C}$.

Viceversa, if $\mathcal{C}_{\theta} := \ker(\theta)$ is already the kernel of a surjective π-transversal 1-form $\theta \in \Omega^1(R, \mathcal{N})$, then its normal bundle becomes isomorphic to the coefficient bundle $\mathcal{N}$ via the map
$$\mathcal{N}_{\mathcal{C}} \to \mathcal{N}, \quad [u] \mapsto \theta(u).$$
Under this isomorphism, θ can be trivially written as the projection map $TR \to \mathcal{N}_{\mathcal{C}}$. Clearly, $\mathcal{C}_{\theta}$ is π-transversal and its symbol space coincides with that of θ. ♢

Lemma 1.3.3. *For any fibration $R \to M$, the previous construction gives a 1-1 correspondence:*

$$\left\{ \begin{array}{c} \textit{Pfaffian distributions} \\ \mathcal{C} \subseteq TR \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{(equivalence classes) of Pfaffian forms} \\ \theta \in \Omega^1(R, \mathcal{N}) \end{array} \right\}.$$

where two forms θ_1, θ_2 are equivalent if there exists a vector bundle isomorphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ such that $\phi(\theta_1(v)) = \theta_2(v)$ $\forall v \in TR$.

In the following we will switch freely between these two viewpoints; in particular, we will also talk about *Pfaffian fibrations* $(R, \mathcal{C})$ when $\mathcal{C}$ is a Pfaffian distribution.

Remark 1.3.4 (**Pfaffian systems).** Another formalism that can be used to study the geometry of PDEs, in some sense dual to jets, is that of EDS (Exterior Differential Systems). We refer to [13] for an introduction on EDSs; recall here that a *Pfaffian system* on R is an EDS

$$\mathcal{I} \subseteq \Omega^*(R),$$

generated as an exterior differential ideal in degree one, together with some transversal (or independence) conditions. Our Pfaffian fibrations fit in this viewpoint as well: they define a Pfaffian system and the involutivity condition in the definition is equivalent to the *linearity* of the Pfaffian system in the sense of [13, Definition 2.1]. ♢

1.3.1 Integrability of Pfaffian fibrations

A (local) solution of a PDE (i.e. a holonomic section in the jet bundle language, an "integral manifold" in the EDS language) corresponds to a (local) section of the Pfaffian fibration which pullbacks the Pfaffian form to zero:

Definition 1.3.5. Given a Pfaffian fibration $\pi : (R, \theta) \to M$, a **holonomic** (local) section of (R, θ) is any (local) section σ of R with the property that $\sigma^*\theta = 0$. The set of holonomic sections is denoted by $\Gamma(R, \theta)$, and that of local ones by $\Gamma_{\text{loc}}(R, \theta)$. ♦

One of the main questions for Pfaffian fibrations is the integrability from the PDE point of view:

Definition 1.3.6. A Pfaffian fibration $\pi : (R, \theta) \to M$ is said to be **integrable** if for every point $p \in R$ there is a local holonomic section $\sigma \in \Gamma_{\text{loc}}(R, \theta)$ passing through p, i.e. such that $\sigma(\pi(p)) = p$. ♦

Remark 1.3.7. Of course the notion of holonomic section makes sense for any 1-form θ on a fibration $R \to M$, without having to be *a priori* π-transversal. However, integrability implies π-transversality of θ, which is therefore *a posteriori* a meaningful condition to ask in the definition. This can be more easily seen using $\mathcal{C} = \ker\theta$: if for any p there is a local section $\sigma : M \to R$ passing through p which is tangent to $\mathcal{C}$, then

$$T_xM = d(\pi \circ \sigma)(T_xM) = d\pi(d\sigma(T_xM)) \subseteq d\pi(\mathcal{C}_p),$$

where $x = \pi(p)$. This means that $d\pi$ is surjective when restricted to $\mathcal{C}$, i.e. $\mathcal{C}$ is π-transversal. ♢

A natural notion that comes into play when studying integrability is that of **integral element**, which is directly inspired from the homonymous notion in the theory of PDEs. Intuitively, an integral element of $(R, \mathcal{C})$ is a linear subspace $V \subseteq T_pR$, $p \in R$, which is a "good" candidate to be the tangent space of a holonomic (local) section σ that passes through p. Suppose that V is indeed tangent to σ, i.e. $V = d_p\sigma(T_{\pi(p)}M)$: this immediately implies that V is a complement of $T^\pi_p R$ in T_pR. Due to the holonomicity of σ, one obtains further that

$$V \subseteq \mathcal{C}_p, \quad \text{and} \quad [u, v]_p \in V,$$

for any $u = d\sigma(X), v = d\sigma(Y)$, with $X, Y \in \mathfrak{X}(M)$. In order to handle the last condition we introduce the following object:

Definition 1.3.8. The **curvature map** of a Pfaffian distribution $\mathcal{C} \subseteq TR$ is the $C^\infty(R)$-bilinear map

$$\kappa_{\mathcal{C}} : \Lambda^2\mathcal{C} \to \mathcal{N}_{\mathcal{C}}$$

which is defined at the level of sections by $(U, V) \mapsto [U, V] \bmod \mathcal{C}$. The Leibniz identity of the Lie bracket implies that $\kappa_{\mathcal{C}}$ is well defined as a bundle map. ♦

Alternatively, if $\mathcal{C}_\theta = \ker\theta$, the curvature map is denoted κ_θ and is

$$\kappa_\theta : \mathcal{C} \times \mathcal{C} \to \mathcal{N}, \quad (U, V) \mapsto \theta([U, V]).$$

Definition 1.3.9. Given a Pfaffian fibration $\pi : (R, \mathcal{C}) \to M$, a **partial integral element** of $(R, \mathcal{C})$ is any linear subspace $V \subseteq T_pR$ satisfying

$$V \subseteq \mathcal{C}_p, \qquad T_pR = V \oplus T^\pi_p R.$$

It is called an **integral element** if, furthermore, $\kappa_{\mathcal{C}}(V, V) = 0$. ♦

1.4 Prolongations

The prolongation of a Pfaffian fibration $\pi : (R, \mathcal{C}) \to M$ may be thought of as the space of first order differential consequences of the Pfaffian fibration, in analogy with the classical notion of prolongation of a PDE (Section 1.1.2); as such, the prolongation consists of all the integral elements of $(R, \mathcal{C})$ (Definition 1.3.9). Those can be reinterpreted as the images of all linear splittings $\zeta : T_{\pi(p)}M \to T_pR$ of $d_p\pi$ (think of ζ as the differential $d_{\pi(p)}\sigma$ for some $\sigma \in \Gamma_{\mathrm{loc}}(R)$), such that

$$\mathrm{Im}(\zeta) \subseteq \mathcal{C}_p, \quad \zeta^*(\kappa_{\mathcal{C}}) = 0.$$

The *partial* prolongation of $(R, \mathcal{C})$ takes care of the first condition.

Definition 1.4.1. The **partial prolongation of a Pfaffian fibration** $\pi : (R, \mathcal{C}) \to M$, denoted by $J^1_{\mathcal{C}}R$, is the set of all its partial integral elements (Definition 1.3.9). In other words, it is the subset of J^1R which, modulo the standard identification

$$J^1R = \{(p, \zeta) \mid p \in R, \zeta : T_{\pi(p)}M \to T_pR \text{ linear, } d\pi \circ \zeta = \mathrm{id}\}$$

(see Remark 1.1.2), is defined by

$$J^1_{\mathcal{C}}R := \{(p, \zeta) \in J^1R \mid \zeta(T_{\pi(p)}M) \subseteq \mathcal{C}_p\}.$$ ♦

Proposition 1.4.2. *The partial prolongation $J^1_{\mathcal{C}}R$ of a Pfaffian fibration $(R, \mathcal{C})$ is a smooth manifold and the projection $J^1_{\mathcal{C}}R \to M$ is a smooth surjective submersion. Moreover, $J^1_{\mathcal{C}}R$ is a Pfaffian fibration over M, together with the restriction of the Cartan distribution $\mathcal{C}$ on J^1R,*

$$\mathcal{C}^{(1)} := \mathcal{C} \cap TJ^1_{\mathcal{C}}R,$$

Remark 1.4.3. The projection $\mathrm{pr} : J^1_{\mathcal{C}}R \to R$ induces a map between holonomic sections

$$\Gamma(J^1_{\mathcal{C}}R, \mathcal{C}^{(1)}) \to \Gamma(R, \mathcal{C}), \quad \xi \mapsto \mathrm{pr} \circ \xi.$$

In fact, this map defines a 1-1 correspondence with inverse given by $\sigma \mapsto j^1\sigma$. In principle, the first jet of σ is a section of J^1R tangent to the Cartan distribution $\mathcal{C}$ (see Proposition 1.1.5), but because σ is holonomic, $d_x\sigma(T_xM)$ is a subset of $\mathcal{C}_{\sigma(x)}$ for all $x \in \mathrm{dom}(\sigma)$, i.e. $j^1\sigma$ is actually a section of the partial prolongation $J^1_{\mathcal{C}}R$, tangent to $\mathcal{C}^{(1)}$. ♢

Recall from Definition 1.3.9 that an integral element of $(R, \mathcal{C})$ is a partial integral element on which the curvature $\kappa_{\mathcal{C}}$ (Definition 1.3.8) vanishes. By the discussion above, the prolongation of a Pfaffian fibration $(R, \mathcal{C})$, defined as the set of integral elements of $(R, \mathcal{C})$, can be expressed in terms of the partial prolongation $J^1_{\mathcal{C}}R$ as follows.

Definition 1.4.4. The **prolongation of a Pfaffian fibration** $\pi : (R, \mathcal{C}) \to M$, denoted by $\mathrm{Prol}(R, \mathcal{C})$, is the set of all its integral elements (Definition 1.3.9). In other words, it is the subset of $J^1_{\mathcal{C}}R$ given by

$$\mathrm{Prol}(R, \mathcal{C}) := \{(p, \zeta) \in J^1_{\mathcal{C}}R \mid \zeta^*(\kappa_{\mathcal{C}}) = 0\},$$

where $\kappa_{\mathcal{C}}$ was defined in 1.3.8 and

$$\zeta^*(\kappa_{\mathcal{C}})(u, v) := \kappa_{\mathcal{C}}(\zeta(u), \zeta(v)) \quad \forall u, v \in T_{\pi(p)}M.$$ ♦

We point out that, in our paper [19], this was called *classical* prolongation, to distinguish it from other kinds of prolongations discussed there (see also Remark 1.4.8 below).

The study of its smoothness and underlying structure is a bit more subtle than that of the partial prolongation (for which the proof of Proposition 1.4.2 is quite straightforward). First, it is useful to remark that the prolongation of $(R, \mathcal{C})$ is the zero-set of

$$\widetilde{\kappa}_{\mathcal{C}} : J^1_{\mathcal{C}}R \to \mathrm{Hom}(\pi^*\Lambda^2 TM, \mathcal{N}_{\mathcal{C}}), \quad (p, \zeta) \mapsto (\zeta^*\kappa_{\mathcal{C}})_p.$$

A simple computation reveals that $\widetilde{\kappa}_{\mathcal{C}}$ is an affine map and that the underlying vector bundle morphism is precisely the Spencer differential of the symbol space $\mathfrak{g}(\mathcal{C})$ (defined in Remark 1.3.2).

To make sense of this claim, we provide $\mathfrak{g}(\mathcal{C})$ with the following structure of generalised tableau bundle (see Section 1.2):

$$\partial_{\mathcal{C}} : \mathfrak{g}(\mathcal{C}) \to \mathrm{Hom}(\pi^*TM, \mathcal{N}_{\mathcal{C}}), \quad \partial_{\mathcal{C}}(v)(Y) = \kappa_{\mathcal{C}}(v, \bar{Y}),$$

with $\bar{Y}$ any vector tangent to $\mathcal{C}$ that projects to Y, i.e. $d\pi(\bar{Y}) = Y$. As one can check, $\partial_{\mathcal{C}}$ is well-defined because $\mathfrak{g}(\mathcal{C})$ is involutive; the map $\partial_{\mathcal{C}}$ is also called the **symbol map** of the Pfaffian fibration $(R, \mathcal{C})$.

Then, we denote by $\delta_{\mathcal{C}}$ the Spencer differential of the generalised tableau bundle $(\mathfrak{g}(\mathcal{C}), \partial_{\mathcal{C}})$:

$$\delta_{\mathcal{C}} := \delta^{0,1} : \mathrm{Hom}(\pi^*TM, \mathfrak{g}(\mathcal{C})) \to \mathrm{Hom}(\pi^*(\Lambda^2 TM), \mathcal{N}_{\mathcal{C}})$$

$$\delta_{\mathcal{C}}(\eta_p)(X, Y) = \partial_{\mathcal{C}}(\eta_p(X))(Y) - \partial_{\mathcal{C}}(\eta_p(Y))(X).$$

We deduce that $\mathrm{Prol}(R, \mathcal{C})$ is a smooth affine subbundle of J^1R if and only if:

1. $\delta_{\mathcal{C}}$ has constant rank.

2. $\mathrm{pr} : \mathrm{Prol}(R, \mathcal{C}) \to R$ is surjective.

Related to the first point, we remark that the kernel of $\delta_{\mathcal{C}}$ is precisely the prolongation of the symbol space $\mathfrak{g}(\mathcal{C})$ w.r.t. the symbol map $\partial_{\mathcal{C}}$ (Definition 1.2.6):

$$\mathfrak{g}(\mathcal{C})^{(1)} \subseteq \mathrm{Hom}(\pi^*TM, \mathfrak{g}(\mathcal{C})).$$

Of course, $\delta_{\mathcal{C}}$ has constant rank if and only if $\mathfrak{g}(\mathcal{C})^{(1)}$ is of constant rank. Now, related to the second point, the previous discussion also implies that $\widetilde{\kappa}_{\mathcal{C}}$ induces the following map, called the **torsion of** $(R, \mathcal{C})$:

$$T = T(R) : R \to \mathrm{Hom}(\pi^*(\Lambda^2 TM), \mathcal{N}_{\mathcal{C}})/\mathrm{Im}(\delta_{\mathcal{C}}), \quad p \mapsto [(\zeta^*\kappa_{\mathcal{C}})_p].$$

It is now a simple exercise to check that the zero-set of T is precisely the image of $\mathrm{pr} : \mathrm{Prol}(R, \mathcal{C}) \to R$. In conclusion, we have the following fundamental result:

Theorem 1.4.5. *For any Pfaffian fibration $\pi : (R, \mathcal{C}) \to M$, the following are equivalent:*

1. *The prolongation* $\mathrm{Prol}(R, \mathcal{C})$ *is a smooth affine subbundle of* J^1R*.*

2. *The prolongation* $\mathfrak{g}(\mathcal{C})^{(1)}$ *of* $\mathfrak{g}(\mathcal{C})$ *is of constant rank, and* $T = 0$*.*

Moreover, in this case:

- *the vector bundle underlying the affine bundle* $\mathrm{pr} : \mathrm{Prol}(R, \mathcal{C}) \to R$ *is precisely* $\mathfrak{g}(\mathcal{C})^{(1)} \to R$*.*

- *the restriction of the Cartan distribution* $\mathcal{C}$ *of* J^1R *to* $\mathrm{Prol}(R, \mathcal{C})$*,*
$$\mathcal{C}^{(1)} := \mathcal{C} \cap T\mathrm{Prol}(R, \mathcal{C}),$$
turns $\pi^{(1)} : (\mathrm{Prol}(R, \mathcal{C}), \mathcal{C}^{(1)}) \to M$ *into a Pfaffian fibration over* M*.*

- *the symbol space of* $\mathrm{Prol}(R, \mathcal{C})$ *is* $\mathrm{pr}^* \mathfrak{g}(\mathcal{C})^{(1)} \subseteq \mathrm{Hom}((\pi^{(1)})^*TM, \mathrm{pr}^* \mathfrak{g}(\mathcal{C}))$*.*

- *the Pfaffian form* $\theta^{(1)} \in \Omega^1(\mathrm{Prol}(R, \mathcal{C}), \mathrm{pr}^* T^\pi R)$ *is related to the Pfaffian form* $\theta \in \Omega^1(R, \mathcal{N}_\mathcal{C})$ *by the projections* $\mathrm{pr} : \mathrm{Prol}(R, \mathcal{C}) \to R$ *and* $l : T^\pi R \to \mathcal{N}_\mathcal{C}$*:*
$$l \circ \theta^{(1)} = \mathrm{pr}^* \theta.$$

Remark 1.4.6. The same observation as 1.4.3 goes here: there is 1-1 correspondence between holonomic sections

$$\Gamma(\mathrm{Prol}(R, \mathcal{C}), \mathcal{C}^{(1)}) \to \Gamma(R, \mathcal{C}), \quad \xi \mapsto \mathrm{pr} \circ \xi,$$

with inverse $\sigma \mapsto j^1\sigma$. In Remark 1.4.3 we saw that $j^1\sigma$ takes values in the partial prolongation; here we observe that, as $[d\sigma(X), d\sigma(Y)]_{\sigma(x)}$ coincides with $d\sigma([X, Y]_x)$ for any $X, Y \in \mathfrak{X}(M)$ and $x \in \mathrm{dom}(\sigma)$, $j^1\sigma$ lies in $\mathrm{Prol}(R, \mathcal{C})$. ◊

Again, the motivating and inspiring example of prolongation comes from the classical definition of prolongation of a PDE $R \subseteq J^kP$ (Section 1.1.2), from which it takes the name.

Proposition 1.4.7. *Let* $R \subseteq J^kP$ *be a PDE such that the restriction of the Cartan distribution* $\mathcal{C} = \mathcal{C} \cap TR$ *is of constant rank, so that* $(R, \mathcal{C})$ *is a Pfaffian fibration. Then,*

$$\mathrm{Prol}(R, \mathcal{C}) = R^{(1)} := J^1R \cap J^{k+1}P, \qquad \mathfrak{g}(\mathcal{C})^{(1)} \cong \mathfrak{g}^{(1)},$$

where $\mathfrak{g}$ *is the symbol space of the PDE* R *(Section 1.2.1). Moreover, if* R *is integrable up to order* 1*, then* $\mathrm{pr} : R^{(1)} \to R$ *is a smooth affine subbundle of* J^1R *modelled on* $\mathfrak{g}^{(1)}$*.*

Remark 1.4.8 (Abstract Pfaffian prolongations)**.** Given two Pfaffian fibrations $(R', \mathcal{C}')$ and $(R, \mathcal{C})$ over M, it is natural to ask that a smooth map $\phi : R' \to R$ "preserves" the Pfaffian structures in the appropriate sense, i.e. induces a map on the set of holonomic sections: this can be formalised by introducing the abstract notion of *Pfaffian morphism.*

Inspired from the projection $(\mathrm{Prol}(R, \mathcal{C}), \mathcal{C}^{(1)}) \to (R, \mathcal{C})$, one can also define morphisms with specific requirements which extract, in a sense, all the fundamental properties of the prolongations of a PDE, in the same way that the conditions of a Pfaffian fibration extract the fundamental properties of the solutions of a PDE. We refer to sections 4.1 and 4.2 of [19] for further details. ◊

1.5 Integrability of Pfaffian fibrations

One can think that when we prolong a Pfaffian fibration $(R, \mathcal{C})$ over M we are trying to determine if an element $p \in R$ comes from a section of $R \to M$ which is "holonomic up to order 1"; if we prolong again then we are looking for sections which are "holonomic up to order 2", etc. If we can repeat this process indefinitely, we find a formal holonomic section of the Pfaffian fibration i.e. a Taylor series of a potential holonomic section of $(R, \mathcal{C})$.

Let us be more specific. To simplify the notation, denote by

$$R^{(1)} := \mathrm{Prol}(R, \mathcal{C})$$

the prolongation of $(R, \mathcal{C})$ from Definition 1.4.4. Under the conditions of Theorem 1.4.5, such prolongation is in turn a smooth Pfaffian fibration $(R^{(1)}, \mathcal{C}^{(1)})$ over M. Denote by $R^{(2)}$ the prolongation of $R^{(1)}$; if we apply again Theorem 1.4.5, we find conditions under which also $(R^{(2)}, \mathcal{C}^{(2)})$ is a Pfaffian fibration over M. When this process can be carried out up to "infinity" we say that $(R, \mathcal{C})$ is *formally integrable.* The goal of this section is to formalise this procedure and describing precisely the obstructions to formal integrability.

1.5.1 Integrability up to finite order

We begin with the precise definition of the properties we want to consider:

Definition 1.5.1. A Pfaffian fibration $(R, \mathcal{C})$ is called **integrable up to order** $k \geq 1$ when, for all $i = 1, \dots, k$, the prolongations

$$R^{(i)} := \mathrm{Prol}(R^{(i-1)}, \mathcal{C}^{(i-1)}) \subseteq J^1 R^{(i-1)} \qquad (\text{with } (R^{(0)}, \mathcal{C}^{(0)}) := (R, \mathcal{C}))$$

are smooth submanifolds, and the projections $R^{(i)} \to R^{(i-1)}$ are surjective submersions. ♦

In particular, if $(R, \mathcal{C})$ is integrable up to order k, it follows from Theorem 1.4.5 that each $R^{(i)}$ is a Pfaffian fibration over M, when endowed with the distribution $\mathcal{C}^{(i)} := (\mathcal{C}^{(i-1)})^{(1)}$. We call $(R^{(i)}, \mathcal{C}^{(i)})$ the i^{th} **prolongation** of the Pfaffian fibration $(R, \mathcal{C})$, for $i \leq k$.

Remark 1.5.2. This definition has some immediate consequences:

- If $R \subseteq J^l P$ is a PDE, the notion of integrability up to order k in the sense of Pfaffian fibrations coincides with the notion of integrability up to order k in the sense of PDEs (see Section 1.1.2).
- Let $(R, \mathcal{C})$ be a Pfaffian fibration integrable up to order k. Then, for every integers $i, l \leq k$ with $i + l \leq k$,
  1. The Pfaffian fibration $(R^{(i)}, \mathcal{C}^{(i)})$ is integrable up to order l, and its l^{th}-prolongation $(R^{(i)})^{(l)}$ coincide with the $(i+l)^{th}$-prolongation $R^{(i+l)}$ of $(R, \mathcal{C})$.
  2. The holonomic sections of $(R, \mathcal{C})$ are in bijections with the holonomic sections of $(R^{(i)}, \mathcal{C}^{(i)})$.

The equivalence with the definition of integrability for PDEs follows directly from Proposition 1.4.7. Properties 2 is also immediate from the definition and from Remark 1.4.6. For the first property, note that $R^{(i)} \subseteq J^i R$ is a PDE and prolongations of Pfaffian fibrations and PDEs coincide. Our statement become then precisely [38, Theorem 7.2]. $\Diamond$

We describe now the main obstructions for integrability. The first step, which takes care of $\mathrm{pr} : R^{(1)} \to R$, was discussed in Theorem 1.4.5. One needs:

(c1) the projection $\mathrm{pr} : R^{(1)} \to R$ to be surjective (which, in turn, was shown to be equivalent to the torsion T to vanish).

(c2) the prolongation $\mathfrak{g}^{(1)}$ of the symbol space $\mathfrak{g} = \mathfrak{g}(\mathcal{C})$ to be of constant rank (here $\mathfrak{g}^{(1)}$ is given by Definition 1.2.6, applied to the symbol map $\partial_{\mathcal{C}} : \mathfrak{g} \to \mathrm{Hom}(\pi^* TM, \mathcal{N}_{\mathcal{C}})$).

Under these conditions $R^{(1)}$ becomes an affine bundle over R modelled on $\mathfrak{g}^{(1)}$ and itself a smooth Pfaffian fibration (over M). Moving one step upwards, we would now like to unravel these conditions (c1) and (c2) when applied to the prolongation of $R^{(1)}$, $\mathrm{pr} : R^{(2)} \to R^{(1)}$; and then to continue this analysis inductively. First of all, the (higher) prolongations that are relevant in condition (c2) will be precisely the ones from Section 1.2:

$$\mathfrak{g}^{(i)} = \ker(\delta^{i-1,1}), \quad i > 1,$$

with $\delta^{i-1,1}$ as in Definition 1.2.3. These can also be rewritten using the following inductive lemma:

Lemma 1.5.3 (Lemma 6.3 of [38])**.** *If a Pfaffian fibration $(R, \mathcal{C})$ is integrable up to order $k \geq 1$, then we have the following canonical isomorphisms of vector bundles over $R^{(i)}$, $1 \leq i \leq k$*

$$\mathrm{pr}^* \mathfrak{g}(\mathcal{C})^{(i+1)} \simeq \mathrm{pr}^* \mathfrak{g}(\mathcal{C}^{(1)})^{(i)} \simeq \ldots \simeq \mathrm{pr}^* \mathfrak{g}(\mathcal{C}^{(i-1)})^{(2)} \simeq \mathfrak{g}(\mathcal{C}^{(i)})^{(1)}.$$

Moreover $\mathrm{pr} : R^{(i)} \to R^{(i-1)}$ *is an affine bundle modelled on the vector bundle* $\mathrm{pr}^*\mathfrak{g}(\mathcal{C})^{(i)}$ *over* $R^{(i-1)}$.

We now move to the condition (c1). For a Pfaffian fibration $(R, \mathcal{C})$ integrable up to order k, the discussion following the Definition 1.4.4 tells us that the prolongation of $(R^{(k)}, \mathcal{C}^{(k)})$ is the kernel of

$$\widetilde{\kappa}_{\mathcal{C}}^{(k)} : J^1_{\mathcal{C}^{(k)}} R^{(k)} \to \mathrm{Hom}(\pi^*\Lambda^2 TM, \mathrm{pr}^*T^\pi R^{(k-1)})$$

$$j^1_x\alpha \mapsto ((d_x\alpha)^* c_{\mathcal{C}^{(k)}})_x = (c_{\mathcal{C}^{(k)}})_x(d_x\alpha(\cdot), d_x\alpha(\cdot)).$$

In the last Hom-space we have used the identification of the normal bundle $\mathcal{N}_{\mathcal{C}^{(k)}}$ with $\mathrm{pr}^*T^\pi R^{(k-1)}$ (via the differential $d\mathrm{pr}$), Also, $\widetilde{\kappa}_{\mathcal{C}}^{(k)}$ is an affine map of affine bundles over $R^{(k)}$, where $J^1_{\mathcal{C}^{(k)}} R^{(k)} \to R^{(k)}$ is modelled on $\mathrm{Hom}(\pi^*TM, \mathfrak{g}(\mathcal{C}^{(k)}))$, with

$$\mathfrak{g}(\mathcal{C}^{(k)}) = \mathrm{pr}^*\mathfrak{g}(\mathcal{C}^{(k-1)})^{(1)} \cong \mathrm{pr}^*\mathfrak{g}^{(k)},$$

where the first equality is by Theorem 1.4.5, and the second by Lemma 1.5.3.

A computation reveals that the underlying vector bundle morphism of $\widetilde{\kappa}_{\mathcal{C}}^{(k)}$ is the pullback by pr of the Spencer map $\delta^{k,1}$. Thus, $R^{(k+1)} := \mathrm{Prol}(R^{(k)}, \mathcal{C}^{(k)})$ is a smooth affine subbundle of $J^1_{\mathcal{C}^{(k)}} R^{(k)} \to R^{(k)}$ if and only if

- $\delta^{k,1}$ has constant rank, i.e. $\ker(\delta^{k,1}) = \mathfrak{g}^{(k+1)}$ has constant rank,
- $R^{(k+1)} \to R^{(k)}$ is surjective.

Related to the last point, this discussion also implies that $\widetilde{\kappa}_{\mathcal{C}}^{(k)}$ descends to the following map:

Definition 1.5.4. Let $(R, \mathcal{C})$ be a Pfaffian fibration integrable up to order $k \geq 1$. The **torsion of order** $k+1$ of $(R, \mathcal{C})$ is defined to be the torsion T of $(R^{(k)}, \mathcal{C}^{(k)})$, i.e. the map

$$T^{k+1} := T(R^{(k)}) : R^{(k)} \to \frac{\mathrm{Hom}(\pi^*\Lambda^2 TM, \mathrm{pr}^*T^\pi R^{(k-1)})}{\delta(\mathrm{Hom}(\pi^*TM, \mathrm{pr}^*\mathfrak{g}^{(k)}))}, \quad p \mapsto [(d_x\sigma)^* c_{\mathcal{C}^{(k)}}]$$

where $j^1_x\sigma$ is any element of the partial prolongation $J^1_{\mathcal{C}^{(k)}} R^{(k)}$ s.t. $\sigma(x) = p$. By definition we set $R = R^{(0)}$ and $T^1 = T$. ♦

From the general discussion of the prolongation, we know already that the zero-set of T^k is precisely the image of $R^{(k+1)} \to R^{(k)}$. Hence, from Theorem 1.4.5 we obtain:

Proposition 1.5.5. *Let* $(R, \mathcal{C})$ *be a Pfaffian fibration integrable up to order* k. *Then* $(R, \mathcal{C})$ *is integrable up to order* $k+1$ *if and only if*

- *the torsion* T^{k+1} *vanishes,*

- *the prolongation $\mathfrak{g}^{(k+1)}$ is smooth.*

Moreover, the prolongation

$$\mathrm{pr} : (R^{(k+1)}, \mathcal{C}^{(k+1)}) \to (R^{(k)}, \mathcal{C}^{(k)})$$

has symbol $\mathfrak{g}(\mathcal{C}^{(k+1)}) = \mathrm{pr}^* \mathfrak{g}^{(k+1)}$, and it is an affine bundle over $R^{(k)}$ modelled on $\mathrm{pr}^* \mathfrak{g}^{(k+1)}$.

It is remarkable that T^k takes values in the Spencer cohomology of the symbol space:

Proposition 1.5.6 (Proposition 5.6 of [19])**.** *Let $(R, \mathcal{C})$ be a Pfaffian fibration integrable up to order $k \geq 1$. Then its torsion T^{k+1} (Definition 1.5.4) takes values in the Spencer cohomology group of the tableau bundle $\mathfrak{g} = \mathfrak{g}(\mathcal{C})$ (Definition 1.2.3)*

$$H^{k-1,2}(\mathfrak{g}) = \frac{\ker(\delta : \mathrm{Hom}(\pi^* \Lambda^2 TM, \mathrm{pr}^* \mathfrak{g}^{(k-1)}) \to \mathrm{Hom}(\pi^* \Lambda^3 TM, \mathrm{pr}^* \mathfrak{g}^{(k-2)}))}{\mathrm{Im}(\delta : \mathrm{Hom}(\pi^* TM, \mathrm{pr}^* \mathfrak{g}^{(k)}) \to \mathrm{Hom}(\pi^* \Lambda^2 TM, \mathrm{pr}^* \mathfrak{g}^{(k-1)}))}$$

where we set $\mathfrak{g}^{(-1)} := \mathcal{N}_{\mathcal{C}}$.

1.5.2 Formal integrability

Definition 1.5.7. A Pfaffian fibration is called **formally integrable** when it is integrable up to any order. ◆

When a Pfaffian fibration $(R, \mathcal{C})$ is a PDE, it follows from Corollary 1.5.2 that the definition of formal integrability coincides with the homonymous one, introduced in Section 1.1.2. In particular, formal integrability is not always a sufficient condition for integrability. However, as for PDEs, the situation is nicer in the analytic setting, where can use Proposition 1.1.6 to prove the following result:

Theorem 1.5.8 (Existence of analytic local holonomic sections). *If $(R, \mathcal{C})$ is an analytic formally integrable Pfaffian fibration, then for every $p \in R^{(k)} \subseteq J^k R$ over $x \in M$ there is an analytic local holonomic section β of $(R, \mathcal{C})$, defined in a neighbourhood of x and such that $j^k_x \beta = p$. In particular, $(R, \mathcal{C})$ is integrable (Definition 1.3.6).*

We look now for sufficient conditions for formal integrability. An immediate one follows from the fact that, if some prolongation $\mathfrak{g}^{(i)}$ of the symbol space has rank 0, then the Spencer cohomology group $H^{i,2}(\mathfrak{g})$ vanishes; in particular, by Proposition 1.5.6, the torsion T^{i+1} is zero.

Proposition 1.5.9. *Let $(R, \mathcal{C})$ be a Pfaffian fibration such that the l^{th}-prolongation $\mathfrak{g}^{(l)}$ of its symbol space $\mathfrak{g}$ vanishes. If $(R, \mathcal{C})$ is integrable up to order k and $l < k$, then it is formally integrable. Moreover, $\mathrm{pr} : R^{(j)} \to R^{(j-1)}$ is a diffeomorphism and $\mathcal{C}^{(j)} = \mathrm{pr}^* \mathcal{C}^{(j-1)}$ for all $j \geq l$.*

This proposition follows also as a corollary from a straightforward generalisation of the cohomological integrability criterion of Goldschmidt (Proposition 1.2.8):

Theorem 1.5.10. *Let $(R, \mathcal{C})$ be a Pfaffian fibration such that*

- *The symbol space $\mathfrak{g}$ is 2-acyclic, i.e. $H^{l,2}(\mathfrak{g}) = 0 \ \forall l \geq 0$,*
- *$\mathfrak{g}^{(1)}$ is smooth and $R^{(1)} \to R$ is surjective.*

Then R is formally integrable.

1.6 Linear Pfaffian fibrations

In this section we specialise the notion of Pfaffian fibration to the linear setting. Let $\pi : A \to M$ be a vector bundle; we will make use of the fibrewise addition

$$a : A \times_M A \to A, \quad (v, w) \mapsto v + w,$$

and of the zero section $\mathbf{0}(x) = (x, 0)$. The tangent vector bundle $TA \to TM$ has as structure maps the differentials of the structure maps of A; in particular the fibrewise addition is given by the differential da.

- A differential form $\theta \in \Omega^1(A, \mathcal{N})$ is called **linear** if $a^*\theta = \mathrm{pr}_1^*\theta + \mathrm{pr}_2^*\theta$, where $\mathrm{pr}_1, \mathrm{pr}_2 : A \times_M A \to A$ denote the canonical projections.
- A distribution $\mathcal{C} \subseteq TA$ is called **linear** if it is a vector subbundle of TA (over the same base TM).

Remark 1.6.1. Recall that in the previous sections we associated to any Pfaffian fibration $(R, \mathcal{C})$ a few bundles over R, such as the normal bundle $\mathcal{N}_\mathcal{C} = TR/\mathcal{C}$ or the symbol space $\mathfrak{g}(\mathcal{C}) = \mathcal{C} \cap T^\pi R$. In the linear case, when instead of R we have a vector bundle $\pi : A \to M$, these objects descend on M.

In general, if a vector bundle $\mathcal{A} \to A$ comes from M, i.e. $\mathcal{A} \cong \pi^*(\mathcal{A}_M)$ for some vector bundle $\mathcal{A}_M \to M$, then $\mathcal{A}_M$ is unique and given by

$$\mathcal{A}_M = \mathcal{A}|_M,$$

where we view $M \hookrightarrow A$ as an embedded submanifold using the zero section of A. In particular, in our situation,

$$\mathcal{N}_\mathcal{C} \cong \pi^* E, \qquad \mathfrak{g}(\mathcal{C}) \cong \pi^*(\mathfrak{g}_M(\mathcal{C})),$$

where

$$E = \mathcal{N}_\mathcal{C}|_M, \qquad \mathfrak{g}_M(\mathcal{C}) = \mathfrak{g}(\mathcal{C})|_M.$$

These properties will be proved later in a more general setting, namely that of Lie groupoids endowed with a multiplicative 1-form or distribution (see Section 3.4). $\Diamond$

Here is the analogue to Remark 1.3.2:

Remark 1.6.2 (Equivalence between linear forms and distributions). Any pointwise surjective linear form $\theta \in \Omega^1(A, \pi^*E)$ induces a distribution $\mathcal{C} = \ker(\theta) \subseteq TA$ which is clearly linear too. Conversely, any linear distribution $\mathcal{C}$ on A arises as $\ker(\theta_{\mathcal{C}})$, for $\theta_{\mathcal{C}} \in \Omega^1(A, \mathcal{N}_{\mathcal{C}})$ the pointwise surjective linear form defined by the canonical projection $TA \to \mathcal{N}_{\mathcal{C}}$; here, by Remark 1.6.1, we identify $\mathcal{N}_{\mathcal{C}} \cong \pi^*E$.

As in Lemma 1.3.3, for any vector bundle $A \to M$, this construction defines a 1-1 correspondence between linear distributions and pointwise surjective linear forms (up to isomorphism of the coefficient bundle E). $\lozenge$

We can now define the linear analogue of a Pfaffian fibration:

Definition 1.6.3. A **linear Pfaffian fibration** is a vector bundle $A \to M$, together with a pointwise surjective linear form θ (or, equivalently, a linear distribution $\mathcal{C} \subseteq TA$). $\blacklozenge$

It can be easily seen that vertical vector fields constant along the fibre of π commute. Writing any vector field tangent to the symbol space $\mathfrak{g}(\mathcal{C})$ as a linear combination of such vectors tangent to $\mathfrak{g}(\mathcal{C})$ and constant along the fibres, it follows that $\mathfrak{g}(\mathcal{C})$ is involutive. Together with Remarks 1.6.1 and 1.6.2, this implies:

Lemma 1.6.4. *If (A, θ) is a linear Pfaffian fibration, then it is a Pfaffian fibration in the sense of Definition 1.3.1.*

As in the non-linear case, the prototypical example of linear Pfaffian fibration is the k-jet vector bundle $\pi : J^kA \to M$ together with its Cartan form ω. The coefficient bundle of ω is $J^{k-1}A$ thanks to the canonical identification $\mathrm{pr}^*T^\pi(J^{k-1}A) \cong \pi^*J^{k-1}A$, with $\mathrm{pr} : J^kA \to J^{k-1}A$.

1.6.1 Infinitesimal linear Pfaffian fibration: relative connections

Linear forms and linear distributions can be encoded by a generalised version of linear connections, called *relative connections*.

Definition 1.6.5. A **connection** on a vector bundle $\pi : A \to M$, **relative** to a pointwise surjective vector bundle map $l : A \to E$ over M, is an $\mathbb{R}$-linear map

$$D : \Gamma(A) \to \Omega^1(M, E),$$

satisfying, for any section $s \in \Gamma(A)$ and function $f \in C^\infty(M)$, the Leibniz-type identity

$$D(fs)(X) = fD(s)(X) + L_X(f)l(s), \quad \forall X \in \mathfrak{X}(M).$$

We also say that (D, l) is a **relative connection** and we call l its **symbol map**. $\blacklozenge$

Proposition 1.6.6. *For any vector bundle $\pi : A \to M$ there is a 1-1 correspondence*

$$\left\{ \textit{Linear 1-forms on } A \right\} \overset{\sim}{\longleftrightarrow} \left\{ \textit{Relative connections on } A \right\}$$

which associates to each linear form $\theta \in \Omega^1(A, \pi^*E)$ the connection

$$D^\theta : \Gamma(A) \to \Omega^1(M, E), \quad s \mapsto s^*\theta$$

relative to the vector bundle map

$$l^\theta : A \to E, \quad l^\theta_{\pi(\alpha)}(v) := \theta_\alpha(v),$$

where we use the canonical identification $A_{\pi(\alpha)} \cong T^\pi_\alpha A$.

In the classical theory of linear PDEs on a vector bundle $F \to M$, Spencer used the linear differential operator $D_{clas} : \Gamma(J^k F) \to \Omega^1(M, J^{k-1}F)$ (see e.g. Section 1.3 of [87]), defined uniquely by the property that a section s of $J^k F$ is holonomic if and only if $D_{clas}(s) = 0$. We call D_{clas} the **classical Spencer operator**.

Accordingly, one can use the (easier) operator D_{clas} to encode solutions of a linear PDE on F, in the same way that one uses the Cartan form θ to encode solutions of any (possibly non-linear) PDE, by means of Proposition 1.1.5.

In other words, the Cartan form of a linear PDE $A \subseteq J^k F$ is fully encoded by the classical Spencer operator. Interpreting A as a linear Pfaffian fibration (A, θ), we recover the classical Spencer operator precisely as the relative connection D^θ associated to θ.

Chapter 2

G-structures

The aim of this chapter is to give an overview on G-structures. This will be used as a source of inspiration for our approach to almost Γ-structures, which we will develop in the next chapters. The literature is very rich with monographs on the topic: see for instance [8, 57, 89, 6].

In the first part of the chapter we recall the basic definitions and main examples. In the second part we review the two fundamental steps of the integrability problem:

- studying necessary and sufficient conditions for formal integrability,
- investigating whether formal integrability implies integrability.

The first issue has been extensively studied, and can be approached by means of the theory of prolongations of geometric structures. A few results are also available on the second (harder) step.

Most of the material presented in sections 2.4, 2.5 and 2.6 is based on the original works by Guillemin, Singer, Sternberg, Kobayashi, Albert and Molino in the 60's, such as [3, 43, 58, 86, 89]. However, it is hard to find a structured exposition in the literature, since each author adopts different conventions and assumes different backgrounds.

Our original contribution in this chapter has been therefore to carefully compare the different approaches to prolongations and integrability of G-structures, and to prove their equivalence: these seem to be well known results, but, for some of them, we could not find the arguments spelled out anywhere.

2.1 G-structures

The goal of this section is to review the classical approach to geometric structures on a manifold M by means of reductions and lifts of the structure group of the frame bundle of M. Recall first that a morphism of Lie groups

$$\phi : H \to G$$

allows one to promote any principal H-bundle over M to a principal G-bundle over M:

$$\phi_* : Bun_H(M) \to Bun_G(M), \quad Q \mapsto (Q \times G)/H,$$

where the quotient of $Q \times G$ is with respect to the left diagonal action of H given by $h \cdot (q, g) = (q \cdot h^{-1}, \phi(h)g)$; such quotient is endowed with the obvious G-action $[q, g] \cdot g' = [q, gg']$. There is a canonical map of bundles over P,

$$\Phi_\phi : Q \to \phi_*(Q), \quad q \mapsto [q, e],$$

which is ϕ-equivariant in the sense that

$$\Phi_\phi(q \cdot h) = \Phi_\phi(q) \cdot \phi(h) \quad \forall q \in Q, h \in H.$$

Note that Φ_ϕ is injective or surjective if ϕ is. In particular, if ϕ is an inclusion of a Lie subgroup

$$i : H \hookrightarrow G,$$

then $P := i_*Q$ contains Q as a subbundle, hence it can be thought of as "an extension of Q".

Definition 2.1.1. Let $\phi : H \to G$ a Lie group morphism and $P \to M$ a principal G-bundle. A **reduction of** P **to** H (with respect to ϕ) consists of a principal H-bundle $Q \to M$ and an isomorphism of principal G-bundles

$$P \cong \phi_*(Q). \qquad \blacklozenge$$

Of special interest is the case when ϕ is an inclusion $i : H \hookrightarrow G$ of a Lie subgroup. By the discussion above, a reduction of P to H is the same thing as a subspace $Q \subseteq P$ which is H-invariant and, with this action, it is a principal H-bundle over M.

Recall also the 1-1 correspondence between vector bundles of rank n and principal $GL(n, \mathbb{R})$-bundles, which associates to a vector bundle $E \to M$ its frame bundle:

$$Fr(E) = \{(x, e) \mid x \in M, e \in Fr(E_x)\}.$$

Here $Fr(E_x)$ denotes the set of all frames (ordered bases) of the vector space E_x or, equivalently, linear isomorphisms between $\mathbb{R}^n$ and E_x. This last interpretation reveals an obvious action of $GL(n, \mathbb{R})$ on $Fr(E)$, so that $Fr(E)$ becomes a principal $GL(n, \mathbb{R})$-bundle together with $\pi : (x, e) \mapsto x$,

We will be particularly interested in the frame bundle of $E = TM$, for which we use the notation

$$Fr(M) := Fr(TM).$$

Definition 2.1.2. Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup. A **G-structure on a manifold** M^n is a reduction of the structure group of $Fr(M)$ to G. ♦

Hence, a G-structure on M is just a G-invariant submanifold

$$P \subseteq Fr(M)$$

such that $\pi_{|_P} : P \to M$ is a principal G-bundle.

2.1.1 Integrability of a G-structure

The intuition behind the rather abstract approach to "geometric structures" is that what we are describing is not the actual geometric structure, but the frames of the tangent bundle that are "adapted" to the actual structure (think e.g. of orthonormal frames in the presence of a metric, and see the examples below). In other words, P should be thought of as the collection of "adapted frames".

With a similar spirit, a chart $\chi : U \to \chi(U) \subseteq \mathbb{R}^n$ of M will be called **adapted to** P if all the frames that it induces are adapted, i.e.

$$\left(\frac{\partial}{\partial \chi^1}(x), \ldots, \frac{\partial}{\partial \chi^n}(x) \right) \in Fr(T_x M),$$

belong to P for all $x \in U$.

Definition 2.1.3. A G-structure P on M is said to be **integrable** if, around each point of M, one can find a chart that is adapted to P. ♦

Integrability can also be understood as a certain "local triviality" of the G-structure. To make this precise, first note that, given any Lie subgroup $G \subseteq GL(n, \mathbb{R})$, the manifold $\mathbb{R}^n$ carries a canonical G-structure:

$$P_{\text{can}} := \mathbb{R}^n \times G \subseteq Fr(\mathbb{R}^n),$$

called **the flat G-structure** on $\mathbb{R}^n$. We also need the notion of isomorphism:

Definition 2.1.4. An **isomorphism** (or **equivalence**) between two G-structures $P \to M$ and $P' \to N$ is a diffeomorphism $\varphi : M \to N$ with the property that its lift to the frame bundles,

$$Fr(\varphi) : Fr(M) \to Fr(N), \quad (x, e_1, \ldots, e_n) \mapsto (\varphi(x), d_x\varphi(e_1), \ldots, d_x\varphi(e_n)),$$

takes P to P'. When $M = N$ and $P = P'$, we say that φ is an **automorphism** (or **symmetry**) of P; we denote by $\mathrm{Aut}_G(P)$ the resulting group of automorphisms. ♦

With these, the following characterisation of integrability is immediate:

Lemma 2.1.5. *A G-structure P on M is integrable if and only if, locally, it is isomorphic to the flat G-structure on $\mathbb{R}^n$: around $x \in M$ there exists an open neighbourhood U such that $P|_U$ is isomorphic (as a G-structure) to P_{can}.*

2.1.2 Examples

The G-structures associated to various Lie groups G correspond to well-known geometric structures; the following table contains some of them.

Group G	**G-structure**	**Integrability condition**
$\{e\} \subseteq GL(n, \mathbb{R})$	**Parallelism** on M^n (global frame of M)	$c_{ij}^k = 0$, with c_{ij}^k structure constant of the global frame ϕ
$GL^+(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$	**Orientation** on M^n	None
$SL(n, \mathbb{R}) \subseteq GL(n, \mathbb{R})$	**Volume form** on M^n	None
$O(n) \subseteq GL(n, \mathbb{R})$	**Riemannian metric** g on M^n	the Levi-Civita connection ∇_g is flat (**Flat Riemannian structure**)
$Sp(n) = Sp(2n, \mathbb{R}) \subseteq GL(2n, \mathbb{R})$ containing matrices A such that $A^T \Omega A = \Omega$, for Ω standard symplectic matrix	**Almost symplectic structure** on M^{2n} (non-degenerate 2-form $\omega \in \Omega^2(M)$)	$d\omega = 0$ (**Symplectic structure**)
$GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R})$ via $A + iB \mapsto \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$	**Almost complex structure** on M^{2n} ($J \in \mathrm{End}(TM)$ with $J^2 = -id_{TM}$)	The Nijenhuis tensor N_J vanishes (**Complex structure**)

$GL(k, n-k) \subseteq GL(n, \mathbb{R})$ containing block matrices $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ with $A \in GL(n-k, \mathbb{R})$ and $C \in GL(k, \mathbb{R})$	**Rank k distribution** on M^n (rank k subbundle $\mathcal{F} \subseteq TM$)	$\mathcal{F}$ is involutive (k-dimensional **foliation**)

Note that the integrability conditions mentioned in the last three examples are just reformulations of major classical theorems (Darboux, Newlander-Niremberg and Frobenius). They prescribe, under the integrability conditions, the existence of special charts compatible with the geometric structures.

2

Example 2.1.6. Many G-structures arise from tensors satisfying specific conditions. More precisely, consider the natural $GL(n, \mathbb{R})$-action on the space of tensors on $\mathbb{R}^n$,

$$W = T^k_l(\mathbb{R}^n) = \bigotimes^k(\mathbb{R}^n) \otimes \bigotimes^l(\mathbb{R}^n)^*,$$

so that any tensor τ on M can be interpreted as a $GL(n, \mathbb{R})$-equivariant map $\tau : Fr(M) \to W$. If the image $\tau(Fr(M)) \subseteq W$ is contained in a single orbit of the $GL(n, \mathbb{R})$-action, consider a point $y_0 \in \tau(Fr(M))$ and denote by G the isotropy group of the $GL(n, \mathbb{R})$-action on W. Then $P := \tau^{-1}(y_0)$ is a G-structure on M. This is part of a general construction from section II.3 of [3].

Moreover, a G-structure defined by a tensor τ is integrable if and only around each point of M one can find a chart where the coefficients of τ are constant. $\Diamond$

Example 2.1.7. Let $G \leq H \leq GL(n, \mathbb{R})$ be Lie subgroups, and denote the inclusion by $i : G \hookrightarrow H$. As we recalled at the beginning of Section 2.1, there is an induced map at the level of principal bundles

$$i_* : Bun_G(M) \to Bun_H(M), \quad Q \mapsto i_*(Q) = (Q \times H)/G.$$

When Q is actually a G-structure (hence $Q \subseteq Fr(M)$), it follows that also $i_*(Q)$ sits naturally inside $Fr(M)$ and, therefore, defines an H-structure. Indeed, the map $i_*(Q) \hookrightarrow Fr(M)$, $[q, g] \mapsto q \cdot g$ identifies $i_*(Q)$ with the H-saturation of Q

$$i_*(Q) \cong Q \cdot H \subseteq Fr(M).$$

In this way i_* is promoted to a map at the level of geometric structures

$$i_* : Struct_G(M) \to Struct_H(M).$$

For instance, since $Sp(n) \subseteq SL(2n, \mathbb{R}) \subseteq GL^+(2n, \mathbb{R})$, we recover the known fact that an almost symplectic structure induces a volume form, which in turn induces an orientation.

Concerning integrability, it should be clear that i_* takes integrable G-structures to integrable H-structures; of course, $i_*(Q)$ may be integrable even if Q is not integrable.

The converse problem of deciding whether a G-structure is induced from an H-structure is trickier and it relies on the topology of the Lie groups and of M. For instance, if G is also a deformation retract of H, corollary 12.6 of [88] can be (recast to modern language and) invoked to produce an inverse to i_*. Here are some examples: $O(n)$ is a deformation retract of $GL(n, \mathbb{R})$ (and, indeed, any manifold admits a Riemannian metric), $SL(n, \mathbb{R})$ is a deformation retract of $GL^+(n, \mathbb{R})$ (and, indeed, any orientation is induced by some volume form) and $SO(n)$ is a deformation retract of $SL(n, \mathbb{R})$ (and, indeed, any volume form can be obtained from an oriented orthonormal frame). Note that the first and third examples are part of a general class: any connected Lie group H deformation retracts on its maximal compact subgroup K. $\lozenge$

Remark 2.1.8. In Example 2.1.7 we discussed when a G-structure is induced from an H-structure, for $G \leq H \leq GL(n, \mathbb{R})$. The most interesting instance is when H is the entire $GL(n, \mathbb{R})$: then we are asking ourselves the more fundamental question of existence of a G-structure on a given manifold M. The answer can go either way (e.g. every manifold admits a Riemannian metric, whereas not every manifold admits an almost symplectic or an almost complex structure) and there are several results in the literature showing that the existence of such structures is obstructed topologically, by various characteristic classes.

An even more interesting (but also more difficult) question is the existence of an integrable G-structure on a given manifold M. In principle, after solving the previous *topological* problem and having found a G-structure, the question is whether it is (or it can be changed into) an integrable one: this is a *differential* problem that motivates the rest of this chapter. The key point will be to define inductively certain tensors T^k_{intr}, called intrinsic torsions, whose vanishing will be necessary conditions for integrability. In particular, a G-structure such that $T^{k-1}_{intr} = 0$ will be called **k-integrable**; a G-structure such that all the intrinsic torsions vanish will be called **formally integrable**. $\lozenge$

2.1.3 Linear G-structures

Observe that a Riemannian metric on M is a collection of inner products on the tangent spaces T_xM, varying smoothly w.r.t. $x \in M$. Similarly, an almost complex structure on M is a collection of complex structures on the tangent spaces T_xM, etc. To have a similar description for any G-structure, one needs the linear version of this concept.

Definition 2.1.9. Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup; a **linear G-structure** on the vector space V^n is a G-invariant subset $S \subseteq Fr(V)$ satisfying the following property: if $\phi, \phi' \in S$, then the matrix $A(\phi, \phi')$ associated to the change of frames is in G. $\blacklozenge$

With this, it is clear that G-structures can be interpreted as collections of linear G-structures P_x on each T_xM, varying smoothly w.r.t. $x \in M$. The notion of isomorphism between linear G-structures (i.e. the linear version of Definition 2.1.4) should be clear; in particular, for any G-structure S on a vector space V, one has the associated Lie group $GL(V,S)$ of symmetries, and its Lie algebra $\mathfrak{gl}(V,S)$:

$$GL(V,S) \subseteq GL(V), \qquad \mathfrak{gl}(V,S) \subseteq \mathfrak{gl}(V).$$

Note that $GL(V,S)$ is isomorphic to G, but not canonically. In particular, if one applies this construction fibrewise, for a G-structure P on a manifold M, one obtains two non-trivial bundles over M: a bundle of Lie groups, and a bundle of Lie algebras. We will be especially interested in the second:

Definition 2.1.10. The **bundle of infinitesimal automorphisms** of a G-structure $P \to M$ is the vector bundle

$$\mathfrak{g}_P = \mathfrak{gl}(TM, P) \subseteq \mathrm{End}(TM),$$

whose fibre at $x \in M$ is the Lie algebra of infinitesimal automorphisms of the linear G-structure P_x:

$$(\mathfrak{g}_P)_x = \mathfrak{gl}(T_xM, P_x) \subseteq \mathfrak{gl}(T_xM). \qquad \blacklozenge$$

Remark 2.1.11. Definition 2.1.10 above can also be understood via the standard construction of the associated vector bundle, i.e. the process of forming a vector bundle $P[V] \to M$ out of a principal G-bundle $P \to M$ by attaching (linear) fibres given by a representation V of G. Here $P[V] := (P \times V)/G$ is the quotient w.r.t. the diagonal G-action, and the (fibrewise) linear structure making it into a vector bundle is induced from the one on V.

For instance, when $G \subseteq GL(n,\mathbb{R})$ (so that $V = \mathbb{R}^n$ is the obvious representation of G) and P is a G-structure on M, one has

$$P[\mathbb{R}^n] \xrightarrow{\sim} TM, \quad [p,v] \mapsto p(v)$$

and, similarly,

$$P[\mathfrak{gl}(n,\mathbb{R})] \cong \mathrm{End}(TM).$$

With this, $\mathfrak{g}_P$ can be obtained by attaching to P the fibre $\mathfrak{g} = \mathrm{Lie}(G)$ (a representation of G w.r.t. the adjoint action)

$$\mathfrak{g}_P \cong P[\mathfrak{g}],$$

and its inclusion into $\mathrm{End}(TM)$ comes from the inclusion $\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$. $\lozenge$

2.1.4 Special types

Many properties of a G-structure depends on properties of the Lie group $G \subseteq GL(n, \mathbb{R})$ and its Lie algebra. Accordingly, here are some special types of Lie subalgebras $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ which will be relevant in the following. For some of them we are interpreting $\mathfrak{g}$ as a tableau

$$\mathfrak{g} \subseteq \mathrm{Hom}(V, W), \quad \text{with } V = W = \mathbb{R}^n,$$

so that the discussion from Section 1.2 applies (e.g. one can talk about its prolongations $\mathfrak{g}^{(k)}$).

Definition 2.1.12. A Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ is called

- of **finite type** $k \geq 1$ when $\mathfrak{g}^{(k)} = 0$ but $\mathfrak{g}^{(k-1)} \neq 0$.
- of **infinite type** if none of its prolongations vanishes.
- **elliptic** if it contains no linear maps of rank 1.
- **involutive** if it is involutive as in Definition 1.2.4 with $V = W = \mathbb{R}^n$.

A Lie subgroup $G \subseteq GL(n, \mathbb{R})$ is said to be of finite type, elliptic, etc., if its Lie algebra is. ♦

Using Remark 1.2.2 one proves the following:

Proposition 2.1.13 (Proposition 1.4 of [58])**.** *If $\mathfrak{g}$ is of finite type, then it is elliptic.*

Example 2.1.14 (Lie algebras of finite type or elliptic)**.** By a direct computation, one sees that the first prolongation of the Lie algebras $\mathfrak{o}(n)$, $\mathfrak{o}(p, q)$ and $\mathfrak{u}(n)$ are zero, thus they are of finite type 1. The Lie algebra $\mathfrak{co}(n)$ of the conformal group $CO(n)$ (Example 2.2.4) is instead of type 2 (see Example 2.6 in [58]). More generally, Guillemin proved (Section 5 of [43]) that the compact subgroups of $GL(n, \mathbb{R})$ are of finite type.

Note also that, if $\mathfrak{g}$ is of finite type k, its prolongations $\mathfrak{g}^{(i)}$ is of finite type $k - i$. For this to make sense, one should interpret $\mathfrak{g}^{(i)}$ as a Lie subalgebra of $\mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}^{(i-1)})$ (see later paragraph 2.5.2).

According to Proposition 2.1.13, all finite type Lie algebras are elliptic; an example of elliptic Lie algebra of infinite type is $\mathfrak{gl}(n, \mathbb{C}) \subseteq \mathfrak{gl}(2n, \mathbb{R})$: it is infinite because any prolongation $\mathfrak{g}^{(k)}$ contains all the symmetric complex $(k+1)$-linear maps $t : \mathbb{C}^n \times \ldots \times \mathbb{C}^n \to \mathbb{C}^n$. ♢

Example 2.1.15 (Lie algebras of infinite type)**.** The following Lie algebras are not elliptic, hence also of infinite type: $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{sp}(n)$, $\mathfrak{gl}(p, q)$, $\mathfrak{gl}(p, \mathbb{R}) \times \mathfrak{gl}(q, \mathbb{R})$. If $\mathfrak{g}$ contains a Lie subalgebra of infinite type, $\mathfrak{g}$ is of infinite type too. ♢

Example 2.1.16 (Involutive Lie algebras)**.** Thanks to an alternative characterisation of involutivity due to Serre (Definition 3.1 and Proposition 4.6 of [86]), one checks easily that $GL(n,\mathbb{R})$, $GL(n,\mathbb{C})$, $SL(n,\mathbb{R})$, $Sp(n)$ and $GL(p,n-p)$ are involutive. More generally, Guillemin states (Section 6D of [43]) that, if a group is made up of invertible matrices commuting with a fixed matrix, its Lie algebra is of infinite type.

In general, all non-zero involutive algebras are of infinite type; an example that is of infinite type but not involutive is the conformal symplectic group $CSp(n)$ (see Example 2.2.4 later). ♢

Remark 2.1.17. The group of symmetries $\mathrm{Aut}_G(P)$ of a G-structure $P \to M$ (Definition 2.1.4) can be in general quite large and wild; here are some simple cases when it is known that $\mathrm{Aut}_G(P)$ is actually a finite-dimensional Lie group (see [58, 80] for a summary):

- $G = \{e\}$ (originally proved in [55]).
- G is of finite type (follows from Proposition 2.5.8).
- G is elliptic and M is compact (Theorem A of [76]). ♢

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2.2 More examples

Many geometric structures arise as combinations of the basic G-structures that we have previously mentioned in Section 2.1.2, and can often be described as G-structures as well. Here we list some of them, with the remark that the first example is the only one which will be used in the rest of the thesis.

Example 2.2.1. An **almost symplectic foliation** on M^{2k+r} is given by a rank r distribution $\mathcal{F} \subseteq TM$ together with a leaf-wise non-degenerate 2-form $\omega \in \Gamma(\Lambda^2\mathcal{F}^*)$. Such structure corresponds to a G_{SF}-structure for the subgroup $G_{SF} \subseteq GL(2k+r,\mathbb{R})$ made up of block matrices

$$\begin{bmatrix} A_{2k\times 2k} & B_{r\times 2k} \\ 0_{2k\times r} & C_{r\times r} \end{bmatrix},$$

where $A \in Sp(k)$, B is a real matrix and $C \in GL^+(r,\mathbb{R})$. An almost symplectic foliation $(\mathcal{F},\omega)$ is integrable if and only if it is a **symplectic foliation**, i.e. if $\mathcal{F}$ is involutive and ω is closed along $\mathcal{F}$.

When $r = 1$, many authors consider also the special case when the distribution is given by the kernel of a 1-form and the 2-form is globally defined. More precisely, an **almost cosymplectic structure** on M^{2k+1} is given by a pair $(\alpha,\beta) \in \Omega^1(M) \times \Omega^2(M)$ such that $\alpha \wedge \beta^k \neq 0$. This is a G-structure for the group

$$G_{CS} := \begin{bmatrix} A_{2k\times 2k} & B_{1\times 2k} \\ 0_{2k\times 1} & 1 \end{bmatrix}.$$

and it is integrable if and only if it is a **cosymplectic structure**, i.e. $d\alpha = 0, d\beta = 0$.

Still for $r = 1$, another possible variation consists in allowing ω to have coefficients in the line bundle $TM/\xi \to M$; this is called a **conformal almost symplectic foliation**, and is a G-structure for the group

$$G_{CSF} := \begin{bmatrix} A_{2k\times 2k} & B_{1\times 2k} \\ 0_{2k\times 1} & \mathbb{R}^{\times} \end{bmatrix}.$$

It is integrable if and only if it is a **conformal symplectic foliation**, i.e. $\mathcal{F}$ is involutive and ω is closed along $\mathcal{F}$. Note that, when TM/ξ is trivial, we recover the notion of almost symplectic foliation defined above. ♢

Example 2.2.2. An **almost hermitian structure** (J, g) on M^{2n} is given by an almost complex structure J and a Riemannian metric g such that $g(J\cdot, J\cdot) = g(\cdot,\cdot)$, called **hermitian metric**, with the compatibility condition that $\omega_J(\cdot,\cdot) := g(\cdot, J\cdot)$ defines an almost symplectic structure, called **hermitian 2-form**.

Equivalently, an almost hermitian structure (J, ω) is given by an almost complex structure J and an almost symplectic structure ω such that $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$, called **Kähler form**, with the compatibility condition that $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a Riemannian metric, caller **Kähler metric**.

Almost hermitian structures encode the data of three compatible structures (Riemannian, almost symplectic and almost complex), and are therefore G-structures for the unitary group

$$U(n) = O(2n, \mathbb{R}) \cap Sp(n, \mathbb{R}) \cap GL(n, \mathbb{C}) \subseteq GL(2n, \mathbb{R}).$$

One can consider three special cases of almost hermitian structures, corresponding to the integrability conditions for each of the three structures defining them:

- An **hermitian structure** is an almost hermitian structure where J is integrable (i.e. $N_J = 0$).
- An **almost Kähler structure** is an almost hermitian structure where ω is integrable (i.e. $d\omega = 0$).
- A **flat almost hermitian structure** is an almost hermitian structure where g is integrable (i.e. the Levi-Civita connection ∇_g is flat).

A $U(n)$-structure is 1-integrable, and it is called **Kähler structure**, if the first two conditions above hold. It is integrable, and it is called **flat Kähler structure**, if all the three conditions above hold. ♢

Example 2.2.3. A **pseudo-Riemannian structure** on M^n is a non-degenerate (but not necessarily positive definite) metric tensor $g \in \Gamma(S^2T^*M)$, of a certain signature type (p, q), with $p + q = n$. It can be described as an $O(p, q)$-structure, where $O(p,q)$ denotes the group of linear transformations of $\mathbb{R}^n$ preserving the bilinear form $(\cdot,\cdot)_{p,q}$ on $\mathbb{R}^n$

$$(x,y)_{p,q} := \sum_{i=1}^{p} x_i y_i - \sum_{j=p+1}^{n} x_j y_j$$

or, equivalently, the group of invertible n-matrices A such that $A^t I_{p,q} A = I_{p,q}$, where

$$I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

As in the case of positive signature, a pseudo-Riemannian structure is always 1-integrable, and it is integrable if and only if its unique compatible torsion-free connection has zero curvature. $\Diamond$

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Example 2.2.4. A **conformal (Riemannian) structure** on M^n is conformal equivalence class of Riemannian metric tensors $[g]$, i.e. a class of metrics all proportional by a positive factor; it corresponds to a G-structure for the conformal group

$$CO(n) := \{A \in GL(n,\mathbb{R}) \mid A^t A = cI \text{ for some } c \in \mathbb{R}^+\}.$$

Such a structure is integrable if and only if it is conformally flat, i.e. some metric in the equivalence class is locally conformally equivalent to the flat Euclidean one. For $n \geq 4$, a $CO(n)$-structure is always 1-integrable, while the second intrinsic torsion coincides with the Weyl curvature tensor. Its vanishing is equivalent to the conformal flatness, hence a $CO(n)$-structure is integrable if and only if it is 2-integrable.

With similar arguments, this point of view can be generalised also to conformal pseudo-Riemannian structures, using the group $CO(p,q)$, conformal symplectic structures, using the group $CSp(n)$, etc. $\Diamond$

Example 2.2.5. The **Spin group** $\mathrm{Spin}(n) = \mathrm{Spin}(\mathbb{R}^n)$ is the 2-fold covering of the special orthogonal group, i.e. it fits in the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\rho} SO(n) \to 0.$$

When $n > 2$, it can be shown that $\mathrm{Spin}(n)$ is simply connected, hence it coincides with the universal cover of $SO(n)$.

Let M^n be a manifold admitting an $SO(n)$-structure $S \to M$. A **Spin structure** on M is a reduction of the structure group of the special orthogonal frame bundle S to $\mathrm{Spin}(n)$ w.r.t. the Lie group homomorphism ρ; since ρ is surjective, this reduction is often called a lift of the structure group. More explicitly, it consists of a principal $\mathrm{Spin}(n)$-bundle $P \to M$, together with a bundle map $\Phi : P \to S$ such that

$$\Phi(p \cdot g) = \Phi(p) \cdot \rho(g) \quad \forall p \in P, g \in \mathrm{Spin}(n).$$

It can be shown that the existence of a $\mathrm{Spin}(n)$-structure is bound to the topology of M, namely is equivalent to the vanishing of the second Stiefel-Withney class $w_2(M)$. $\lozenge$

Example 2.2.6. In the following table we recap various **quaternionic structures**, i.e. G-structures on a manifold M^{4n} for a subgroup $G \subseteq GL(4n, \mathbb{R})$, with the following common property: each tangent space T_xM becomes an $\mathbb{H}$-module. We refer to [5] for a more extensive overview.

Name and group	**Definition**	**1-integrability condition**
Almost hypercomplex structure, $GL(n, \mathbb{H})$	$H = (I, J, K)$ anticommuting almost complex structures, satisfying $IJ = K$	Two of the three a.c.s. are integrable (*hypercomplex structure*)
Almost unimodular hypercomplex structure, $SL(n, \mathbb{H})$	A.h.s. H with a volume form	H is 1-integrable (*unimodular hypercomplex structure*)
Almost hypercomplex hermitian structure, Sp_n	A.h.s. H with a metric g hermitian w.r.t. I, J and K	H is 1-integrable and g is compatible with I, J and K (*hyperKähler structure*)
Almost quaternionic structure, $Sp_1 \cdot GL(n, \mathbb{H})$	Rank 3 subbundle $Q \subseteq \mathrm{End}(TM)$ generated locally by a.h.s.	Its generators are integrable as a.c.s. (*quaternionic structure*)
Almost unimodular quaternionic structure, $Sp_1 \cdot SL(n, \mathbb{H})$	A.q.s. Q with a volume form	Q is 1-integrable (*unimodular quaternionic structure*)
Almost hypercomplex quaternionic structure, $Sp_1 \cdot Sp_n$	A.q.s. Q with a metric g hermitian w.r.t. the generators of Q	Q is 1-integrable and g is compatible with all elements of $S(Q)$ (*quaternionic Kähler structure*)

In the last four examples, Sp_n denotes the **compact symplectic group**

$$Sp_n := Sp(2n, \mathbb{C}) \cap U(2n),$$

and by $\cdot$ we mean the following product of groups

$$G \cdot H := (G \times H)/\mathbb{Z}_2.$$

In the last example, $S(Q) \subseteq Q$ denotes the "sphere" of almost complex structures $aI + bJ + cK$, for all $(a, b, c) \in \mathbb{R}^3$ such that $a^2 + b^2 + c^2 = 1$. $\lozenge$

We list here some other relevant geometric structures that can described by G-structures.

Name	Definition	Group	Integrability
Real almost product structure on M^{p+q} ([60, 72, 96])	Splitting of real bundles $TM = P^p \oplus Q^q$.	$GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$	P and Q involutive.
Complex almost product structure on $M^{2(p+q)}$ ([60, 72, 96])	Splitting of complex bundles $TM \otimes \mathbb{C} = P^p \oplus Q^q$	$GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$	P and Q involutive
Almost CR structure on M^n ([60, 72])	Distribution $H^{2k} \subseteq TM$ and endomorphism $J : H \to H$ s.t. $J^2 = -id_H$.	$G \subseteq GL(2k, n-2k)$, with elements of the upper-left block in $GL(k,\mathbb{C})$	H involutive and $N_J = 0$.
Product metric on $K^k \times L^l$ ([6])	Metric $g = \kappa \times \lambda$, for κ and λ metrics on K and L	$O(k) \times O(l)$	κ and λ flat
Almost tangent structure on M^{2n} ([11, 10, 27])	Endomorphism $F : M \to M$ with $F^2 = 0$	$T(n) \subseteq GL(n,n)$, set of block matrices $\begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$, with $A \in GL(n,\mathbb{R})$.	$N_F = 0$ (Nijenhuis tensor)
Almost Finsler structure on M^n ([64])	$F : TM \to [0, +\infty]$, smooth on $TM - \{0\}$, with F_x asymmetric norm $\forall x \in M$	$F(n) \subseteq T(n)$, defined by the conditions $A \in O(n)$ and B symmetric	Gives a Finsler metric under other conditions
G_2-structure on M^7 ([12, 51, 52, 14, 53])	$\phi \in \Omega^3(M)$ locally modelled on the associative 3-form φ (defined by the cross product of $\mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$)	$G_2 = Aut(\mathbb{O})$, or $G_2 \subseteq \mathrm{Diff}(\mathbb{R}^7)$ subgroup preserving φ	1-integrable $\Leftrightarrow$ $d\phi = 0$ and $d *_{g_\phi} \phi = 0$, for g_ϕ induced metric

2.3 Abstract G-structures

A G-structure on a manifold M is not only a principal G-bundle P, but also requires having fixed:

- an inclusion of G in $GL(n,\mathbb{R})$, where n is the dimension of M,
- an inclusion of P in $Fr(M)$ as a principal G-subbundle.

This is not always ideal. The main point of this section is to explain/recall that P carries a canonical 1-form θ_P which encodes the information above, besides P

being a principal G-bundle. One should note the (non-accidental!) similarity with the Cartan form (Section 1.1.1), capturing the main properties of jet bundles and PDEs and leading to the abstract notion of Pfaffian fibration.

Definition 2.3.1. The **canonical form** of the frame bundle $\tau_M : Fr(M) \to M$ (also known as fundamental, tautological, or soldering form) is the 1-form

$$\theta_{\mathrm{can}} \in \Omega^1(Fr(M), \mathbb{R}^n)$$

defined for every $(x, \xi) \in Fr(M)$ by

$$(\theta_{\mathrm{can}})_{(x,\xi)} = \xi^{-1} \circ d_{(x,\xi)}\tau_M,$$

where we view ξ as a linear isomorphism $\xi : \mathbb{R}^n \to T_xM$. The **canonical form** θ_P **of a** G**-structure** P is the restriction

$$\theta_P := \theta_{\mathrm{can}}|_P \in \Omega^1(P, \mathbb{R}^n). \qquad \blacklozenge$$

One can easily check that θ_P is pointwise surjective and has kernel equal to $\ker(\tau_M|_P)$. Moreover, it is G-equivariant with respect to the action of G on TP induced by the action on P and the right linear action of G on $\mathbb{R}^n$ coming from the inclusion $G \subseteq GL(n, \mathbb{R})$, i.e. $x \cdot g := g^{-1}x$. And here is the promised result:

Theorem 2.3.2. *Let $P \xrightarrow{\pi} M$ be a principal G-bundle and $\theta \in \Omega^1(P, \mathbb{R}^n)$ such that θ is pointwise surjective, and $\ker(\theta) = T^\pi P$. Then there exist*

- *a unique Lie group morphism $i : G \to GL(n, \mathbb{R})$ such that θ becomes G-equivariant*
- *a unique morphism of principal bundles $j : P \to Fr(M)$ such that $\theta = j^*(\theta_{\mathrm{can}})$.*

Moreover, if i is injective, then j is an injective immersion.

Given the importance of such result for this thesis, and the fact that a (complete) proof is hard to find in the literature, we provide here the details.

Proof. For every $g \in G$, we use the surjectivity of θ_p to define the linear map

$$i_g : \mathbb{R}^n = \theta_p(T_pP) \to \theta_{p\cdot g}(T_{p\cdot g}P) = \mathbb{R}^n$$

$$\theta_p(v) \mapsto \theta_{p\cdot g}(d_pR_g(v)).$$

This is well defined because, if $\theta_p(v) = \theta_p(w)$, then $v - w \in \ker(\theta_p) = \ker(d_p\pi)$, and (since π is G-invariant)

$$d_{p\cdot g}\pi(d_pR_g(v - w)) = d_p(\pi \circ R_g)(v - w) = d_p\pi(v - w) = 0.$$

Hence $d_pR_g(v - w) \in \ker(d_{p\cdot g}\pi) = \ker(\theta_{p\cdot g})$, i.e. $\theta_{p\cdot g}(d_pR_g(v)) = \theta_{p\cdot g}(d_pR_g(w))$.

It is immediate to see that $(i_g)^{-1} = i_{g^{-1}}$, so that $i_g \in GL(n, \mathbb{R})$, and $i_{gh} = i_h i_g$, so that

$$i : G \to GL(n, \mathbb{R}), \quad g \mapsto i_g^{-1}$$

is a (Lie) group morphism, i.e. a representation of G on $\mathbb{R}^n$. Moreover, the form θ becomes by construction G-equivariant w.r.t. to such a representation:

$$((R_g)^*\theta)_p(v) = \theta_{p \cdot g}(d_p R_g(v)) = i_g(\theta_p(v)) = (g^{-1} \cdot \theta)_p(v).$$

If there was another Lie group morphism $\widetilde{i} : G \to GL(n, \mathbb{R})$ such that θ becomes G-equivariant, then for every $g \in G$, $v \in T_pP$,

$$i_g(\theta_p(v)) = ((R_g)^*\theta)_p(v) = \widetilde{i}_g(\theta_p(v)).$$

By the surjectivity of θ_p, we conclude that $i_g = \widetilde{i}_g$ for every $g \in G$, hence i is unique.

For the second part, we define the morphism

$$j : P \to Fr(M), \quad p \mapsto (\pi(p), \xi_p),$$

$$\xi_p : \mathbb{R}^n = \theta_p(T_pP) \to T_{\pi(p)}M, \quad \theta_p(v) \mapsto d_p\pi(v).$$

The linear map ξ_p is well given because, if $\theta_p(v) = \theta_p(v')$, then $v - v' \in \ker(\theta_p) = \ker(d_p\pi)$, i.e. $d_p\pi(v) = d_p\pi(v')$. With the same argument, using the fact that π is a submersion (hence $d_p\pi(T_pP) = T_{\pi(p)}M$) we can see that the map

$$T_{\pi(p)}M \to \mathbb{R}^n, \quad d_p\pi(v) \mapsto \theta_p(v)$$

is well defined. It is clearly an inverse of ξ_p, hence ξ_p is a linear isomorphism.

Moreover, for every $p \in P$, $v \in T_pP$ we have

$$(j^*\theta_{\mathrm{can}})_p(v) = (\theta_{\mathrm{can}})_{(\pi(p),\xi_p)}(d_pj(v)) =$$

$$= (\xi_p)^{-1}(d_{(\pi(p),\xi_p)}\tau_M(d_pj(v))) = (\xi_p)^{-1}(d_p\pi(v)) = \theta_p(v).$$

We show now that $\xi_{p \cdot g} = g \cdot \xi_p$. For every $y \in \mathbb{R}^n$ we have to compare

$$\xi_{p \cdot g}(y) = \xi_{p \cdot g}(\theta_{p \cdot g}(v)) = d_{p \cdot g}\pi(v) = d_{p \cdot g}(\pi \circ R_{g^{-1}})(v) = d_p\pi(d_{p \cdot g}(R_{g^{-1}}(v))),$$

$$g \cdot \xi_p(y) = \xi_p(g \cdot y) = \xi_p(\theta_p(w)) = d_p\pi(w).$$

But, thanks to the G-invariance of θ, we have

$$\theta_p(w) = g \cdot y = g \cdot \theta_{p \cdot g}(v) = g \cdot (g^{-1} \cdot \theta_p(d_{p \cdot g}R_{g^{-1}}(v))) = \theta_p(d_{p \cdot g}R_{g^{-1}}(v)).$$

Therefore, with the usual trick, $w - d_{p \cdot g}R_{g^{-1}}(v) \in \ker(\theta_p) = \ker(d_p\pi)$ and we conclude that $\xi_{p \cdot g}(y) = g \cdot \xi_p(y)$.

This implies that the map $j : P \to Fr(M)$ is G-equivariant, hence is a morphism of principal bundles:

$$j(p \cdot g) = (\pi(p \cdot g), \xi_{p \cdot g}) = (\pi(p), g \cdot \xi_p) = (\pi(p), \xi_p) \cdot g = j(p) \cdot g.$$

Suppose there was another morphism of principal bundles $\widetilde{j} : P \to Fr(M)$; then $\widetilde{j}(p)$ must be of the form $(\pi(p), \widetilde{\xi}_p)$ for some linear isomorphism $\widetilde{\xi}_p : \mathbb{R}^n \to T_{\pi(p)}M$. If $\theta = \widetilde{j}^*(\theta_{\text{can}})$, then the computations above imply that, for every $v \in T_pP$,

$$(\xi_p)^{-1}(d_p\pi(v)) = \theta_p(v) = (\widetilde{\xi}_p)^{-1}.$$

Since π is a submersion, we conclude that $\xi_p = \widetilde{\xi}_p$ for every $p \in P$, hence j is unique.

Assume now that i is injective. In order to prove the injectivity of j, assume $j(p) = j(q)$: then $\pi(p) = \pi(q)$, i.e. $q = p \cdot g$ for some $g \in G$ (the G-action is transitive on the π-fibre). Moreover, the fact that $\xi_p = \xi_q$ means that, for every $y \in \mathbb{R}^n$,

$$\xi_p(y) = \xi_q(y) = \xi_{p \cdot g}(y) = g \cdot \xi_p(y) = \xi_p(g \cdot y).$$

Since ξ_p is an isomorphism, $y = g \cdot y$, and since i is injective (the action is faithful) we conclude that $g = e$, i.e. $p = q$.

A standard result in Lie theory states that any Lie group morphism is of constant rank; if i is injective, then it is also an immersion. On the other hand, another known fact is that any morphism $P \to Q$ of principal bundles over the same manifold is an immersion if and only if the corresponding Lie group morphism $G \to H$ is an immersion (i.e. an injection); accordingly, also j is an immersion. Q.E.D.

The previous proposition motivates the following:

Definition 2.3.3. Given $G \subseteq GL(n, \mathbb{R})$, an **abstract G-structure** $\pi : (P, \theta) \to M$ on a manifold M consists of a principal G-bundle $\pi : P \to M$ and a G-invariant 1-form $\theta \in \Omega^1(P, \mathbb{R}^n)$ which is pointwise surjective and satisfies $\ker(\theta) = T^\pi P$. ♦

Note that, although this is not explicitly required, it follows that M must be n-dimensional. The (rather small) difference with classical G-structures is that, in general, such an abstract G-structure can only be immersed in $Fr(M)$. Of course, one can go one step further and allow arbitrary group homomorphism $i : G \to GL(n, \mathbb{R})$ (instead of G-inclusions) and then we can talk about abstract G-structures w.r.t. i; this generality allows to treat examples such as 2.2.5.

2.3.1 Symmetries and canonical form

Here is another fundamental result that illustrates the fact that the tautological form θ of a G-structure contains most of its information.

Proposition 2.3.4. *Let $G \subseteq GL(n,\mathbb{R})$ be a connected Lie subgroup and $P \xrightarrow{\pi} M$ a G-structure, with tautological form θ. Then there is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \text{symmetries of } P \\ \text{(Definition 2.1.4)} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{diffeomorphisms } \phi : P \to P \\ \text{satisfying } \phi^*\theta = \theta \end{array} \right\}$$

explicitly given by $\varphi \mapsto Fr(\varphi)$.

Again, we include the proof for completeness and for later references.

Proof. Let φ be a symmetry of P and $\phi = Fr(\varphi)$; then, for every $v \in T_{(x,\xi)}P$,

$$(\phi^*\theta)_{(x,\xi)}(v) = \theta_{\phi(x,\xi)}(d_{(x,\xi)}\phi(v)) = \theta_{(\varphi(x), d_x\varphi\circ\xi)}(d_{(x,\xi)}\phi(v)) =$$

$$= (d_x\varphi \circ \xi)^{-1}(d_{\phi(x,\xi)}\pi \circ d_{(x,\xi)}\phi(v)) = \xi^{-1}(d_{\phi(x)}(\varphi^{-1}) \circ d_{\phi(x,\xi)}\pi \circ d_{(x,\xi)}\phi(v)) =$$

$$= \xi^{-1}(d_{(x,\xi)}(\varphi^{-1} \circ \pi \circ \phi)(v)) = \xi^{-1}(d_{(x,\xi)}\pi(v)) = \theta_{(x,\xi)}(v),$$

hence $\phi^*\theta = \theta$.

Conversely, let $\phi \in \mathrm{Diff}(P)$ such that $\phi^*\theta = \theta$; then

$$\theta(v) = (\phi^*\theta)(v) = \theta(d\phi(v)).$$

Moreover, any $v \in TP$ is vertical if and only if $\theta(v) = 0$, since $\ker(\theta) = T^\pi P$. These two facts imply that v is vertical if and only if $d\phi(v)$ is vertical; in particular

$$d\phi(\ker(d\pi)) \subseteq \ker(d\pi).$$

By a well known result on foliations, ϕ sends leaves of $\ker(d\pi)$ to leaves of $\ker(d\pi)$. Since $\ker(d_p\pi) = T_p(\pi^{-1}(x))$, for $x = \pi(p)$, the leaves of $\ker(d\pi)$ are (the connected components of) the π-fibres $\{\pi^{-1}(x)\}_{x\in M}$, which are connected by hypothesis; hence ϕ sends fibres to fibres.

This means that we can define a map $\varphi : M \to M$ such that

$$\phi_{|\pi^{-1}(x)} : \pi^{-1}(x) \to \pi^{-1}(\varphi(x)).$$

Since ϕ is a diffeomorphism of P, φ is also a diffeomorphism of M and the two maps commute with π:

$$(\pi \circ \phi)(p) = \pi(\phi(p)) = \varphi(\pi(p)) = (\varphi \circ \pi)(p).$$

We prove now that ϕ is G-equivariant, i.e. $\phi(p \cdot g) = \phi(p) \cdot g$ for every $g \in G$. Indeed, thanks to the previous property we have that $\phi(p)$ and $\phi(p \cdot g)$ belong to the same π-fibre, hence there exists an $h \in G$ such that

$$\phi(p \cdot g) = \phi(p) \cdot h.$$

Therefore, we have only to prove that $g = h$.

In order to do so, observe first that, by the surjectivity of θ_p, any $\alpha \in \mathbb{R}^n$ can be written as $\theta_p(X_p)$, for some vector field $X \in \mathfrak{X}(P)$. Using the G-equivariance of θ, the hypothesis $\phi^*\theta = \theta$ and the definition of θ, we obtain

$$g^{-1}\alpha = g^{-1} \cdot \theta_p(X_p) = \theta_{p\cdot g}(d_p R_g(X_p)) = \theta_{\phi(p\cdot g)}(d_p(\phi \circ R_g)(X_p)) =$$

$$= (\phi(p \cdot g))^{-1}(d_p(\pi \circ \phi \circ R_g)(X_p)) = (\phi(p \cdot g))^{-1}(d_p(\pi \circ R_h \circ \phi)(X_p)) =$$
$$= \theta_{\phi(p)\cdot h}(d_p(R_h \circ \phi)(X_p)) = h^{-1} \cdot \theta_{\phi(p)}(d_p \phi(X_p)) = h^{-1} \cdot \theta_p(X_p) = h^{-1}\alpha.$$

Since the arguments holds for every α, we conclude that $g = h$.

It is left to prove that $\phi = Fr(\varphi)$. Consider $\widetilde{\phi} = Fr(\varphi)^{-1} \circ \phi$, which satisfies the following two properties:

$$\widetilde{\phi}^*\theta = \phi^*(Fr(\phi)^{-1}(\theta)) = \phi^*\theta = \theta;$$

$$(\pi \circ \widetilde{\phi})(p) = \pi(Fr(\varphi)^{-1}(\phi(p))) = \pi(d_{\pi(\phi(p))}\varphi^{-1}(\phi(p))) =$$
$$= \varphi^{-1}(\pi(\phi(p))) = \varphi^{-1}(\varphi(\pi(p))) = \pi(p).$$

This means, by the same argument as before, that $\widetilde{\phi}$ covers id_M, hence sends every fibre $\pi^{-1}(x)$ to itself. Moreover, $\widetilde{\phi}$ is an equivariant morphism on the principal G-bundle P, hence it acts on each fibre by right multiplication by some element of G. In other words,

$$\widetilde{\phi}(p) = p \cdot \widetilde{g}(\pi(p)) \qquad \forall p \in P,$$

for some smooth map $\widetilde{g} : M \to G$; however, the $\widetilde{\phi}$-invariance of θ implies that

$$\theta = \widetilde{\phi}^*\theta = (R_{\widetilde{g}})^*\theta = \widetilde{g}^{-1} \cdot \theta$$

hence the function $\widetilde{g}$ must be identically equal to the identity $e \in G$, and therefore $\widetilde{\phi}$ is the (product by the) identity; this concludes the proof that $\phi = Fr(\varphi)$. Q.E.D.

Remark 2.3.5. In the same spirit outlined at the beginning of this chapter, the previous result should be compared with the classical Lie-Bäcklund theorem for PDEs (see e.g. Section 4.4.5 of [62]); actually, it can be seen as an improved version.

More precisely, a G-structure $P \subseteq Fr(M)$ can be seen as a PDE of order 1 on $M \times \mathbb{R}^n \to M$, and its tautological form is the restriction of the Cartan form of $J^1(M \times \mathbb{R}^n)$. In general, given a PDE $P \subseteq J^k(\pi)$, the Lie-Bäcklund theorem characterises the diffeomorphism $\phi : P \to P$ preserving the Cartan form as the k-jet lifts of diffeomorphisms $\psi : J^0\pi \to J^0\pi$ (assuming π has rank at least 2). When π is the projection $M \times \mathbb{R}^n \to M$, $k = 1$ and P is a G-structure, this gives us a diffeomorphism $\psi : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ which lifts to ϕ. On the other hand, Proposition 2.3.4 is stronger since it states that ϕ comes from a diffeomorphism $\phi : M \to M$ on the base, i.e. $\psi = \phi \times id_{\mathbb{R}^n}$. $\Diamond$

With Proposition 2.3.4 in mind, symmetries of abstract G-structures are defined in the obvious way:

Definition 2.3.6. The **automorphism group** of an abstract G-structure (P, θ) is the group
$$\mathrm{Aut}(P, \theta) := \{\phi \in \mathrm{Diff}(P) \mid \phi^*\theta = \theta\}. \qquad \blacklozenge$$

Note that we do not ask the element of $\phi \in \mathrm{Aut}(P, \theta)$ to cover a diffeomorphism of M: this follows as a consequence, with the same argument used in the proof of Proposition 2.3.4.

Remark 2.3.7. Proposition 2.3.4 can be easily generalised to two distinct G-structures, $P \to M$ and $Q \to N$: a (local) diffeomorphism $\phi : P \to Q$ between them satisfies $\phi^*\theta_Q = \theta_P$ if and only if $\phi = Fr(\varphi)$, for some (local) diffeomorphism $\varphi : M \to N$. $\lozenge$

2.4 First-order obstructions to integrability

In this section we will recall a fundamental obstruction to the integrability of a G-structure, its (intrinsic) torsion. We follow closely Chapter VII of [89], to which we refer also for the missing details.

2.4.1 The (intrinsic) torsion of a G-structure

Let $\pi : P \to M$ be a G-structure and choose for every $p \in P$ an horizontal space $H_p \subseteq T_pP$, i.e. a complement to $\ker(d_p\pi) = \ker(\theta_p)$ (see also Remark 1.1.2 for notations). In particular, since $\theta_{p|H_p} : H_p \to \mathbb{R}^n$ is an isomorphism, for every $v \in \mathbb{R}^n$ there exists a unique $\xi^p_v \in H_p$ such that $\theta_p(\xi^p_v) = v$. We define the **torsion of P with respect to H_p** as the element
$$c_{H_p} \in \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n), \quad c_{H_p}(v, w) := (d\theta)_p(\xi^p_v, \xi^p_w).$$
We now interpret the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ as a tableau (see Section 1.2)
$$\mathfrak{g} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n).$$
and we make use of the Spencer differential (Definition 1.2.3)
$$\delta : \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n), \quad \delta(\xi)(u, v) = \xi(u)(v) - \xi(v)(u).$$

It follows from a simple computation that c_{H_p}, in the quotient by $\mathrm{Im}(\delta)$, does not depend on the choice of H_p, hence defines a class in the Spencer cohomology of $\mathfrak{g} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$:
$$H^{-1,2}(\mathfrak{g}) := \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)}{\delta\left(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})\right)}.$$

Definition 2.4.1. Let P be a G-structure; the map

$$T : P \to H^{-1,2}(\mathfrak{g}), \quad p \mapsto [c_{H_p}]$$

is called the **(1^{st}-order) torsion** of P. ♦

One can actually go a bit further: since G acts on $\mathbb{R}^n$ by matrix multiplication, there is an induced G-action on $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$, and $\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}))$ becomes a G-invariant subspace. Accordingly, the Spencer cohomology $H^{-1,2}(\mathfrak{g})$ is naturally a representation of G,

$$H^{-1,2}(\mathfrak{g}) \in \mathrm{Rep}(G)$$

and, therefore, as in Remark 2.1.11, we can consider the vector bundle over M associated to P and $H^{-1,2}(\mathfrak{g})$. This is precisely the Spencer cohomology of the tableau *bundle* $\mathfrak{g}_P \subseteq \mathrm{Hom}(TM, TM)$ of infinitesimal automorphisms of P (Definition 2.1.10):

$$\mathcal{H}^{-1,2}(\mathfrak{g}_P) = P[H^{-1,2}(\mathfrak{g})].$$

In particular, given the discussion from Remark 2.1.11, we see that the fibres of $\mathcal{H}^{-1,2}(\mathfrak{g}_P)$ are precisely the Spencer cohomology of the tableaux $\mathfrak{g}_x = (\mathfrak{g}_P)_x \subseteq \mathrm{Hom}(T_xM, T_xM)$

$$\mathcal{H}^{-1,2}(\mathfrak{g}_P)_x = H^{-1,2}(\mathfrak{g}_x) = \frac{\mathrm{Hom}(\Lambda^2 T_xM, T_xM)}{\delta\left(\mathrm{Hom}(T_xM, \mathfrak{g}_x)\right)} .$$

The main reason to have this discussion is the simple remark that the torsion (Definition 2.4.1) is actually G-invariant and, therefore, it descends M and can be reinterpreted as a section of $\mathcal{H}^{-1,2}(\mathfrak{g}_P)$.

Definition 2.4.2. The **intrinsic ($1^{\mathbf{st}}$ order) torsion** of a G-structure P is its torsion T, reinterpreted as a section of $\mathcal{H}^{-1,2}(\mathfrak{g}_P)$, and denoted

$$T_{intr} = T_{intr}(P) \in \Gamma(M, \mathcal{H}^{-1,2}(\mathfrak{g}_P)) = \frac{\Omega^2(M, TM)}{\delta\left(\Omega^1(M, \mathfrak{g}_P)\right)}.$$ ♦

The terminology "torsion" will be clear later in this section, when we discuss compatible connections. The adjective *intrinsic* refers to the fact that $T_{intr}(P)$ is a section of a bundle over M (and not over P). In view of the next paragraph 2.4.2, it refers also to $T_{intr}(P)$ being independent from the choice of a specific connection.

Example 2.4.3. For any Lie group G, the torsion of the canonical flat G-structure on $\mathbb{R}^n$ vanishes (simple computation).

For $G = O(n)$, the torsion of *any* $O(n)$-structure vanishes; that is because the Spencer cohomology of $\mathfrak{o}(n)$ vanishes. This can be seen using that $\ker(\delta) = \mathfrak{g}^{(1)} = 0$, i.e. δ is injective and then the dimension counting

$$\dim(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{o}(n))) = n \dim(\mathfrak{o}(n)) = n^2(n-1)/2 = \dim(\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n))$$

implies that δ is also surjective.

Another interesting case is when $G = \{e\}$ and we consider the $\{e\}$-structure on a Lie group H obtained by fixing a basis $e_1, \ldots, e_n$ of $\mathfrak{h}$ and then right-translating it to form a frame $\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}$. In this case $\mathfrak{g} = 0$, hence the torsion take values in $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$; it is not surprising that this function is actually constant, and it is encoded in the structure constants c_{ij}^k of the Lie algebra $\mathfrak{h}$ with respect to the chosen basis. Because of this example, some authors call the torsion of a G-structure its *structure function* and denoted it by c. $\diamondsuit$

Given the naturality of the frame construction, it is not surprising that the torsion is preserved under isomorphisms of G-structure:

Proposition 2.4.4 (Theorem 2.1 of [86])**.** *Let $\phi : M \to N$ be an isomorphism between the G-structures $P \to M$ and $Q \to N$. Then the 1^{st}-order torsion (Definition 2.4.1) satisfies*

$$T(P) = T(Q) \circ Fr(\phi)_{|P}.$$

In particular, the (intrinsic) torsion of any integrable G-structure vanishes.

Note however that the vanishing of the torsion of a G-structure is not a sufficient condition for its integrability: for instance, a non-flat metric is a non-integrable $O(n)$-structure, even if it has zero torsion (as discussed in Example 2.4.3). Accordingly, one uses the following terminology:

Definition 2.4.5. A G-structure is said to be **integrable up to order 1** (or 1-integrable) if its torsion T, or, equivalently, its intrinsic torsion T_{intr}, vanishes. $\blacklozenge$

Note again the (non-accidental) similarity with Definition 1.5.1 and Proposition 1.5.5 in the setting of Pfaffian fibrations.

2.4.2 Compatible connections and intrinsic torsions

In this section, which is not used in the rest of the thesis, we are going to recall the interpretation of the intrinsic torsion using compatible connections. This is extremely useful for actual computations and also provides new insights.

Definition 2.4.6. Let $P \subseteq Fr(M)$ be a G-structure. A linear connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on M is **compatible with** P if the associated principal connection $\omega \in \Omega^1(Fr(M), \mathfrak{gl}(n, \mathbb{R}))$ on $Fr(M)$ restricted on P takes values in $\mathfrak{g} = \mathrm{Lie}(G)$, i.e.

$$\omega_{|P} \in \Omega^1(P, \mathfrak{g}). \qquad \blacklozenge$$

Compatible connections always exist. Moreover, while the space of connections on M is an affine space modelled on the vector space $\Omega^1(M, \mathrm{End}(TM))$, the space of connections compatible with a given G-structure P turns out to be an affine subspace modelled on

$$\Omega^1(M, \mathfrak{g}_P) \subseteq \Omega^1(M, \mathrm{End}(TM)),$$

where $\mathfrak{g}_P \subseteq \mathrm{End}(TM)$ is the bundle of infinitesimal automorphisms of P (Definition 2.1.10):

Lemma 2.4.7. *Let ∇ be a connection on M compatible with a G-structure P. Any other P-compatible connection arises as ∇^ξ:*

$$\nabla^\xi_X(Y) := \nabla_X(Y) + \xi(X)(Y) \qquad \forall X, Y \in \mathfrak{X}(M),$$

for some $\xi \in \Omega^1(M, \mathfrak{g}_P)$.

Given a linear connection ∇ on M, we can view its torsion as a 2-form

$$T_\nabla \in \Omega^2(M, TM), \quad T_\nabla(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y].$$

Again, a simple computation implies the following:

Lemma 2.4.8. *Let ∇ be a connection on M compatible with a G-structure P and $\xi \in \Omega^1(M, \mathfrak{g}_P)$. Then*

$$T_{\nabla^\xi} - T_\nabla = \delta(\xi)$$

where δ is the Spencer differential

$$\delta : \Omega^1(M, \mathfrak{g}_P) \to \Omega^2(M, TM), \quad \delta(\xi)(X, Y) = \xi(X)(Y) - \xi(Y)(X).$$

In particular, passing to the quotient, the class

$$[T_\nabla] \in \frac{\Omega^2(M, TM)}{\delta(\Omega^1(M, \mathfrak{g}_P))} = \Gamma(M, \mathcal{H}^{-1,2}(\mathfrak{g}_P))$$

does not depend on the choice of the compatible connection ∇.

Of course, $[T_\nabla]$ will be precisely the intrinsic torsion from Definition 2.4.2

$$T_{intr}(P) \in \Gamma(M, \mathcal{H}^{-1,2}(\mathfrak{g}_P)).$$

To show this, it is enough to view the torsion of a connection ∇ compatible with P as the exterior covariant derivative D_∇ of the tautological form θ of P, and to apply the Koszul formula. We refer to Section VII.5 of [89] for further details.

Corollary 2.4.9. *A G-structure P on M is* **1-integrable** *(Definition 2.4.5) if and only if M admits a linear torsion-free connection compatible with P.*

Example 2.4.10. By unraveling the definition of compatible connection, one can compute explicitly the intrinsic torsions of the several G-structures described in the table from Section 2.1; see Section 4.2 of [22] for more details.

For instance, for $Sp(n)$-structures encoded in almost symplectic forms $\omega \in \Omega^2(M)$, the intrinsic torsion can naturally be identified with the differential $d\omega$. Similarly for $GL(n, \mathbb{C})$, when we deal with almost complex structures J and their intrinsic torsion becomes N_J, or $GL(k, n-k)$, when we deal with distributions $\mathcal{F}$ and the intrinsic torsion can be identified with a tensor controlling the involutivity of $\mathcal{F}$. Note that, in all these cases, 1-integrability implies integrability.

On the other hand, we have already mentioned that the torsion of any $O(n)$-structure vanishes (Example 2.4.3), hence any $O(n)$-structure is 1-integrable, without having to be integrable. The previous corollary can be interpreted as saying that any Riemannian manifold admits a linear torsion-free metric connection, i.e. the existence of the Levi-Civita connection. $\Diamond$

Remark 2.4.11 (uniqueness of compatible torsion-free connections)**.** If a Lie group $G \subseteq GL(n, \mathbb{R})$ is of finite type 1 (Definition 2.1.12), then any G-structure P admits at most one torsion-free compatible connection on it. This follows from the fact that the space of compatible torsion-free connections on P is an affine space modelled on $\mathfrak{g}_P^{(1)}$.

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For instance, since $\mathfrak{o}^{(1)} = 0$ (Example 2.1.14), we recover the fact that the Levi-Civita connection of a Riemannian manifold (interpreted as a compatible torsion-free as in Example 2.4.10) is unique. $\Diamond$

Remark 2.4.12 (1-integrability and holonomy)**.** The holonomy of a connection is a powerful tool to study reductions of principal bundles. We mention here, as a side remark, a couple of applications to G-structures, and refer to chapters 2-3 of [52] for proofs and details; this will not be used later in this thesis.

- Let ∇ be a connection on M^n and $G \subseteq GL(n, \mathbb{R})$ a Lie subgroup. Then M admits a G-structure P compatible with ∇ if and only if the holonomy group of ∇ is contained in G.
- Let M^n be a manifold and $G \subseteq GL(n, \mathbb{R})$ a Lie subgroup. Then M admits a 1-integrable G-structure P if and only if there is a compatible torsion-free connection ∇ on M with holonomy in G.

This point of view can be very useful since there are several classifications of the holonomy groups of connections (e.g. the one due to Berger [7]). $\Diamond$

2.5 Higher-order obstructions to integrability

The goal of this section is to describe further necessary conditions for integrability of a G-structure P. In order to do so, one needs first to produce another first-order geometric structure sitting over P, drawing ideas from the concept of prolongations of PDEs.

2.5.1 The prolongation of a G-structures

We start by describing what will turn out to be the structure algebra (and group) of the first prolongation of a G-structure: the prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g} = \mathrm{Lie}(G)$

interpreted as a tableau (Definition 1.2.1). However, while

$$\mathfrak{g} \subseteq \mathfrak{gl}(\mathbb{R}^n)$$

as a Lie subalgebra, one would like to have a similar representation of $\mathfrak{g}^{(1)}$, instead of the inclusion $\mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$ that comes from the general theory from Section 1.2. This can be realised using the inclusion

$$\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}) \hookrightarrow \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g}), \quad A \mapsto \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix},$$

so that one obtains

$$\mathfrak{g}^{(1)} \subseteq \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g})$$

(actually an abelian subalgebra).

If $G \subseteq GL(n, \mathbb{R})$ is a Lie group with Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$, we can define the **first prolongation** of G as the abelian Lie group

$$G' = G^{(1)} := \exp(\mathfrak{g}^{(1)}).$$

From the previous discussion we see that G' can be identified with the subgroup of $GL(\mathbb{R}^n \oplus \mathfrak{g})$ consisting of all linear maps of type

$$\phi_T : \mathbb{R}^n \oplus \mathfrak{g} \to \mathbb{R}^n \oplus \mathfrak{g}, \quad (v, \rho) \mapsto (v, \rho + T(v))$$

with $T \in \mathfrak{g}^{(1)}$. Since $\phi_T \circ \phi_{T'} = \phi_{T+T'}$, G' is indeed a subgroup.

To define the first prolongation of a G-structure $P \to M$ one has, unfortunately, to make first an auxiliary choice: a G-invariant complement C of $\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}))$ in $\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$. Since this choice turns out not to be so important, we will not emphasise it too much.

Given the interpretation of horizontal subspaces as 1-jets (Remark 1.1.2), the torsion of P (Definition 2.4.1) can also be seen as a map

$$\widetilde{T} : J^1P \to \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n), \quad (p, H_p) \mapsto c_{H_p}.$$

Definition 2.5.1. The **first prolongation of a G-structure** $P \to M$ (w.r.t. the complement C) is

$$P^{(1)} := \{\xi \in J^1P \mid \widetilde{T}(\xi) \in C\}. \qquad \blacklozenge$$

Observe that, in general, one has the following inclusion

$$J^1Fr(M) \longrightarrow Fr(Fr(M)), \quad (p, H_p) \mapsto (p, \phi_p)$$

where the frame

$$\phi_p : \mathbb{R}^n \times \mathfrak{gl}(n, \mathbb{R}) \to T_pFr(M)$$

is defined as $(\theta_{p|H_p})^{-1} \oplus a_p$, i.e.

$$\phi_p(v,\rho) = (\theta_{p|H_p})^{-1}(v) + a_p(\rho)$$

and $a_p : \mathfrak{gl}(n,\mathbb{R}) \to T_pFr(M)$ denotes the infinitesimal action of $\mathfrak{gl}(n,\mathbb{R})$ on $Fr(M)$.

Similarly, for any G-structure $P \subseteq Fr(M)$, one has the inclusion

$$J^1P \hookrightarrow Fr(P), \quad (p, H_p) \mapsto (p, \phi_p),$$

with the frame of P

$$\phi_p : \mathbb{R}^n \times \mathfrak{g} \to T_pP$$

described as above. Moreover

- $Fr(P)$ is a principal bundle over P, with structure group $GL(n+\dim(G)) \cong GL(\mathbb{R}^n \oplus \mathfrak{g})$,
- J^1P is a principal subbundle of $Fr(P)$, with structure group $\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$.

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The following fundamental proposition becomes then an easy computation.

Proposition 2.5.2. *The first prolongation $P^{(1)}$ of a G-structure P on M is a G'-structure on P.*

One can also see that a different choice of the complement C gives a different subbundle of J^1P translated by an element of $\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$.

Example 2.5.3. For the flat G-structure on $\mathbb{R}^n$, one has $T = 0$ (Example 2.4.3), hence $P^{(1)} = J^1P$. Moreover, since P is the trivial bundle $\mathbb{R}^n \times G$, J^1P is trivial as well, so $P^{(1)}$ is the flat G'-structure $P \times G'$ over P.

If P is an $O(n)$-structure, then C must be the zero subspace: this follows from the fact that $\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{o}(n))) = \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$ (Example 2.4.3). Moreover, since $T = 0$, for every $p \in P$ there is a unique horizontal space H_p such that $c_{H_p} = 0$, hence $P^{(1)}$ is in bijection with P. This is in accord with the fact that $P^{(1)}$ is a $O(n)^{(1)} = \{e\}$-structure over P: the prolongation of a metric carries exactly the same information of the metric. $\lozenge$

2.5.2 The higher order prolongations of a G-structure

The next step to define the higher order torsions of a G-structure is to iterate its prolongations; however, there are a few subtleties one should take care of.

Indeed, recall from Paragraph 2.5.1 that the first prolongation of $\mathfrak{g}$,

$$\mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}),$$

can be viewed as a tableau

$$\widetilde{\mathfrak{g}}^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}),$$

via the inclusion

$$\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}), \quad A \mapsto \widetilde{A} := \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}.$$

Since $\mathfrak{g}^{(1)}$ and $\widetilde{\mathfrak{g}}^{(1)}$ are the same vector space with different tableau structures, when talking about the second prolongation we have in principle two possibilities:

$$\mathfrak{g}^{(2)} := (\mathfrak{g}^{(1)})^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}), \qquad \widetilde{\mathfrak{g}}^{(2)} := (\widetilde{\mathfrak{g}}^{(1)})^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \widetilde{\mathfrak{g}}^{(1)}).$$

However, $\mathfrak{g}^{(2)}$ is related to $\widetilde{\mathfrak{g}}^{(2)}$ by the inclusion of tableaux

$$\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \widetilde{\mathfrak{g}}^{(1)}), \quad B \to \widetilde{B},$$

$$\widetilde{B}(u,\alpha)(v,\beta) := \widetilde{B(u)}(v,\beta) = (0, B(u)(v)).$$

More precisely, if $B \in \mathfrak{g}^{(2)}$, i.e.

$$B(u)(v) = B(v)(u) \quad \forall u, v \in \mathbb{R}^n,$$

then $\widetilde{B} \in \widetilde{\mathfrak{g}}^{(2)}$, i.e. for every $(u,\alpha), (v,\beta) \in \mathbb{R}^n \times \mathfrak{g}$,

$$\widetilde{B}(u,\alpha)(v,\beta) = (0, B(u)(v)) = (0, B(v)(u)) = \widetilde{B}(v,\beta)(u,\alpha).$$

This means that the second prolongation $\mathfrak{g}^{(2)}$ coincides with $\widetilde{\mathfrak{g}}^{(2)}$ modulo the inclusion above, and that it can be interpreted as an (abelian) Lie subalgebra

$$\mathfrak{g}^{(2)} \subseteq \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)})$$

via the inclusion

$$\mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathfrak{g}^{(1)}) \hookrightarrow \mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)}, \mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)}), \quad B \mapsto \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}.$$

With the same arguments, one shows that all higher prolongations of a Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$ can be interpreted uniquely as subspaces

$$\mathfrak{g}^{(k)} \subseteq \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g}^k),$$

where $\mathfrak{g}^k$ denotes the vector space $\mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \ldots \oplus \mathfrak{g}^{(k-1)}$ (we set $\mathfrak{g}^1 := \mathfrak{g}$ and $\mathfrak{g}^0 := \{0\}$). Note that all the prolongations with $k \geq 1$ are actually abelian Lie subalgebras.

The same goes for the k^{th}-prolongation of G:

$$G^{(k)} := (G^{(k-1)})' = \exp(\mathfrak{g}^{(k)}) \subseteq GL(\mathbb{R}^n \oplus \mathfrak{g}^k).$$

Definition 2.5.4. The **k^{th}-order prolongation** of a G-structure P is defined inductively as

$$P^{(k)} := (P^{(k-1)})^{(1)} \subseteq J^1 P^{(k-1)},$$

We also set $P^{(0)} := P$. ♦

By iterating Proposition 2.5.2, it follows that

Proposition 2.5.5. *The k^{th}-prolongation $P^{(k)}$ of a G-structure P over M is a $G^{(k)}$-structure over $P^{(k-1)}$.*

2.5.3 The higher order (intrinsic) torsions of a G-structure

We arrive finally to the definition of higher torsions of a G-structure P. In principle, those are naturally induced by applying the construction from Definition 2.4.1 to the prolongations $P^{(k)}$. However, the resulting map $T(P^{(k)})$ contains unnecessary information, so we are going to reduce it to the actual obstruction to integrability.

Recall that $T(P^{(k)})$ takes values in the Spencer cohomology of the tableau $\mathfrak{g}^{(k)} \subseteq \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g}^k)$, i.e.

$$\frac{\mathrm{Hom}(\Lambda^2(\mathbb{R}^n \oplus \mathfrak{g}^k), \mathbb{R}^n \oplus \mathfrak{g}^k)}{\delta(\mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}^k, \mathfrak{g}^{(k)}))}.$$

One can decompose such a space into the direct sum of smaller spaces, the most important of which is the Spencer cohomology of $\mathfrak{g}^{(k)}$, this time viewed as a tableau $\mathfrak{g}^{(k)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(k-1)})$:

$$H^{-1,2}(\mathfrak{g}^{(k)}) = \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g}^{(k-1)})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(k)}))}.$$

Let us sketch this decomposition in the case $k = 1$; the other cases follow inductively (see Sections 3.3-3.5 of [90] for more details). Splitting the domain of the numerator, it is clear that

$$\mathrm{Hom}(\Lambda^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g}) = \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n \oplus \mathfrak{g}) \oplus \mathrm{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}) \oplus \mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}),$$

where $\phi = \phi_1 \oplus \phi_2 \oplus \phi_3$; here $\phi_1(x, y) := \phi((x, 0), (y, 0))$, and similarly for ϕ_2 and ϕ_3. Taking into account the denominator, we write

$$\frac{\mathrm{Hom}(\Lambda^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathfrak{g}^{(1)}))} =$$

$$= \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n \oplus \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}))} \oplus \frac{\mathrm{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n \oplus \mathfrak{g}^1, \mathfrak{g}^{(1)}))} \oplus \frac{\mathrm{Hom}(\Lambda^2\mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g})}{\delta(\mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^{(1)}))}.$$

This is well defined by interpreting, for instance, $\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}))$ as $\delta(\mathrm{Hom}(\mathbb{R}^n \oplus 0, \mathfrak{g}^{(1)}))$, and similarly for the other two terms. Last, we have the component-wise decomposition

$$\frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n \oplus \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}))} = \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}))} \oplus \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)}{\delta(\mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^{(1)}))}, \quad [(\phi_1, \phi_2)] \mapsto [\phi_1] \oplus [\phi_2],$$

where we finally obtain the space we wanted:

$$H^{-1,2}(\mathfrak{g}^{(1)}) = \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(1)}))} \quad \text{for } \mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}).$$

Definition 2.5.6. The $(k+1)^{th}$**-order torsion** of a G-structure P is the component of the torsion $T(P^{(k)})$ of its prolongation $P^{(k)}$ which takes values in the Spencer cohomology $H^{-1,2}(\mathfrak{g}^{(k)})$ of the tableau $\mathfrak{g}^{(k)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(k-1)})$:

$$T^{(k+1)} = T^{(k+1)}(P) : P^{(k)} \to \frac{\mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g}^{(k-1)})}{\delta(\mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(k)}))}. \qquad \blacklozenge$$

Definition 2.4.5 is therefore extended as follows:

Definition 2.5.7. A G-structure P is called k**-integrable** if its torsions $T^{(1)}$, $T^{(2)}$, ..., $T^{(k)}$ vanish. $\blacklozenge$

If P is k-integrable, then the torsion $T^{(k+1)}(P)$ takes values in

$$H^{k-1,2}(\mathfrak{g}) = \frac{\ker(\delta : \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g}^{(k-1)}) \to \mathrm{Hom}(\Lambda^3\mathbb{R}^n, \mathfrak{g}^{(k-2)}))}{\mathrm{Im}(\delta : \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g}^{(k)}) \to \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{g}^{(k-1)}))}.$$

We refer to Sections 2.11-2.12 of [86] for the proof and a more extensive discussion.

In order to understand the k-integrability as an obstruction to integrability, we recall this result by Singer and Sternberg.

Proposition 2.5.8 (Theorem 2.3 of [86])**.** *Two G-structures are isomorphic (as G-structures) if and only if their first prolongations are isomorphic (as $G^{(1)}$-structures!).*

In particular, symmetries of a G-structure (Definition 2.1.4) are in bijective correspondence with symmetries of its prolongations.

In view of Lemma 2.1.5, we have the following immediate but important corollary.

Corollary 2.5.9. *Any integrable G-structure is k-integrable (Definition 2.5.7) for every k.*

Remark 2.5.10. Another consequence of Proposition 2.5.8 is that, for a group G of finite type (Definition 2.1.12), the integrability problem of a G-structure is reduced in a finite number of steps to an integrability problem of an $\{e\}$-structure. This approach has been widely studied in the literature and led to interesting ways to classify geometric structures: see for instance the ongoing work by Fernandes and Struchiner [33, 34, 35], which brought Lie groupoids and Lie algebroids into the picture. $\lozenge$

Assume now that P is a k-integrable G-structure; we generalise the discussion of Section 2.4.1 to the higher torsions

$$T^{k+1}(P) : P^{(k)} \to H^{k-1,2}(\mathfrak{g}).$$

First, one checks that $H^{k-1,2}(\mathfrak{g})$ is a representation of the Lie group $G^k := G \times G' \times \ldots \times G^{(k)}$, and $P^{(k)}$ is a principal G^k-bundle over M. Moreover, the Spencer cohomology $\mathcal{H}^{k-1,2}(\mathfrak{g}_P)$ of the tableau bundle $\mathfrak{g}_P \subseteq \mathrm{Hom}(TM, TM)$ (Definition 2.1.10) can be written using as the associated vector bundle

$$\mathcal{H}^{k-1,2}(\mathfrak{g}_P) = P^{(k)}[H^{k-1,2}(\mathfrak{g})].$$

Accordingly, $T^{k+1}(P)$ descends to M as a section of $\mathcal{H}^{k-1,2}(\mathfrak{g}_P)$ and we can give the higher order analogue of Definition 2.4.2.

Definition 2.5.11. Let P be a k-integrable G-structure. The **intrinsic** $(k+1)^{\mathbf{th}}$**-order torsion** of P is the $(k+1)^{\text{th}}$-order torsion $T^{k+1}(P)$ reinterpreted as a section of $\mathcal{H}^{k-1,2}(\mathfrak{g}_P)$ as discussed above:

$$T^{(k+1)}_{intr}(P) \in \Gamma(M, \mathcal{H}^{k-1,2}(\mathfrak{g}_P)) = \frac{\ker\left(\delta : \Omega^2(M, \mathfrak{g}_P^{(k-1)}) \to \Omega^3(M, \mathfrak{g}_P^{(k-2)})\right)}{\mathrm{Im}\left(\delta : \Omega^1(M, \mathfrak{g}_P^{(k)}) \to \Omega^2(M, \mathfrak{g}_P^{(k-1)})\right)}.$$

We set $T^{(1)}_{intr} = T_{intr}$, recovering Definition 2.4.2. ♦

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Corollary 2.5.12. *A G-structure P is k-integrable (Definition 2.5.7) if and only if*

$$T^{(i)}_{intr}(P) = 0 \quad \forall i = 1, \dots, k.$$

2.5.4 Compatible connections and second order intrinsic torsion

In Section 2.4.2 we have translated the (1^{st}-order) intrinsic torsion into a more explicit object: the torsion of any compatible connection. In this section, which again is not used in the rest of the thesis, we describe the intrinsic torsion $T^{(2)}_{intr}$ in a more concrete way as well.

Let ∇ be a connection on the manifold M and consider its curvature

$$K_\nabla \in \Omega^2(M, \mathrm{End}(TM)), \quad K_\nabla(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

If ∇ is compatible with a G-structure $P \to M$ (Definition 2.4.6), then K_∇ turns out to be an element of $\Omega^2(M, \mathfrak{g}_P)$.

Recall also the Bianchi identity, which links the torsion and the curvature of a connection:

Lemma 2.5.13. *Let ∇ be a connection on M; then*

$$d_\nabla(T_\nabla) = \delta(K_\nabla),$$

where d_∇ is the de Rham differential associated to ∇

$$d_\nabla : \Omega^2(M, TM) \to \Omega^3(M, TM)$$

and δ is the Spencer differential of the tableau bundle $\mathrm{End}(TM)$

$$\delta : \Omega^2(M, \mathrm{End}(TM)) \to \Omega^3(M, TM),$$
$$\delta(\xi)(X,Y,Z) = \xi(X,Y)(Z) + \xi(Y,Z)(X) + \xi(Z,X)(Y).$$

Consider now a G-structure P which is 1-integrable, i.e. $T_{intr} = 0$: by Corollary 2.4.9, there exists a torsion-free connection ∇ on M compatible with P; by Lemma 2.5.13, the curvature K_∇ is in the kernel of the Spencer differential of the tableau bundle $\mathfrak{g}_P \subseteq \mathrm{End}(TM)$

$$\delta : \Omega^2(M, \mathfrak{g}_P) \to \Omega^3(M, TM).$$

On the other hand, similar to Lemma 2.4.8, but with longer computations, one gets the following result.

Lemma 2.5.14 (Lemma 4.43 of [22]). *Let P be a G-structure on M, ∇ a compatible torsion-free connection, $\xi \in \Omega^1(M, \mathfrak{g}_P)$ and ∇^ξ the connection described in Lemma 2.4.7. Then the difference between the curvatures of ∇^ξ and ∇ is in the image of the Spencer differential of $\mathfrak{g}_P$*

$$\delta : \Omega^1(M, \mathfrak{g}_P^{(1)}) \to \Omega^2(M, \mathfrak{g}_P), \quad \delta(\xi)(X, Y) = \xi(X)(Y) - \xi(Y)(X).$$

In conclusion, we can give the following definition.

Definition 2.5.15. Let P be a 1-integrable G-structure on M. The **intrinsic curvature** of P is the class

$$[K_\nabla] \in \frac{\ker(\delta : \Omega^2(M, \mathfrak{g}_P) \to \Omega^3(M, TM))}{\mathrm{Im}(\delta : \Omega^1(M, \mathfrak{g}_P^{(1)}) \to \Omega^2(M, \mathfrak{g}_P))} = \Gamma(M, \mathcal{H}^{0,2}(\mathfrak{g}_P)). \qquad \blacklozenge$$

Similarly to the first intrinsic torsion T_{intr}, one can show that the intrinsic curvature $[K_\nabla]$ coincides with the second order intrinsic torsion $T^{(2)}_{intr}$ from Definition 2.5.11. We refer again to Section VII.5 of [89] for further details: here the crucial step is to view the curvature K_∇ of a connection ∇ as the exterior covariant derivative D_∇ of the connection 1-form $\Omega \in \Omega^1(P, \mathfrak{g})$ associated to ∇.

Example 2.5.16. Recall from Example 2.4.10 that any $O(n)$-structure, defining a Riemannian metric g, is 1-integrable. On the other hand, in view of the characterisation discussed above, the second intrinsic torsion coincides with the curvature of the Levi-Civita connection of g. $\lozenge$

2.5.5 Formal integrability

We conclude this section by studying the main necessary (but not sufficient) condition for integrability:

Definition 2.5.17. A G-structure is **formally integrable** if it is k-integrable (Definition 2.5.7) for every k. $\blacklozenge$

Note that formal integrability involves in principle an infinite amount of conditions. However, for involutive Lie algebras $\mathfrak{g}$ (Definition 2.1.12), all the Spencer cohomology spaces $H^{k,2}(\mathfrak{g})$ vanish, for $k \geq 0$, hence all intrinsic torsions $T^{(k)}_{intr}$ are zero, except possibly the first one $T^{(1)}_{intr} = T_{intr}$, which takes values in $\mathcal{H}^{-1,2}(\mathfrak{g})$.

Since every tableau $\mathfrak{g}$ becomes involutive after a finite number of prolongations (Proposition 1.2.5), and $H^{k,2}(\mathfrak{g}^{(i)}) = H^{k+i,2}(\mathfrak{g})$ (Section 1.2), we conclude that every G-structure has only a finite number of non-zero intrinsic torsions.

Example 2.5.18. An $O(n)$-structure is formally integrable if and only if it is 2-integrable: this follows from the fact that $\mathfrak{o}(n)$ is of finite type 1, hence all the higher intrinsic torsions $T^{(k)}_{intr}$ are zero for $k \geq 3$. Moreover, in this situation, formal integrability is sufficient for integrability, as the 2-integrability condition corresponds to the flatness of the Riemannian metric (see Example 2.5.16).

For the G-structures associated to almost symplectic structures, almost complex structures and distributions we have a different situation, but similar conclusions. In this case, their structure groups G are involutive (Example 2.1.16); accordingly, all their intrinsic torsions $T^{(k)}_{intr}$ vanish for $k \geq 2$, i.e. these G-structures are formally integrable if and only if they are 1-integrable. In this case, three powerful theorems show that formal integrability implies integrability: respectively, the Darboux theorem, the Newlander-Niremberg theorem and the Frobenius theorem. $\lozenge$

2

We have now understood that the formal integrability problem is of algebraic-cohomological nature, and is solved via a sequence of progressive obstructions. On the other hand, the integrability problem is essentially different and has no general solutions. We list here some fundamental cases where the formal integrability of a G-structure P is a sufficient condition for its integrability (see Section 5.6 of [86] for a wider overview):

- $\mathfrak{g}$ is of finite type (Proposition 5.1 in [43] and Theorem 3.3 in [89])
- $\mathfrak{g}$ (with its representation on $\mathbb{R}^n$) is irreducible (Theorem 5.4 of [86])
- $\mathfrak{g}$ is elliptic and P is real analytic (Theorem 7.1 in [69])
- $\mathfrak{g}$ is of infinite type and P is real analytic (Theorem 3.3 of [86])

Note that, in the proofs of all the results above, a crucial step is of "analytic nature", requiring e.g. the Frobenius theorem or the Cartan-Kähler theorem (see [13]).

We remark also that both Pollack [79] and Buttin-Molino [17] published a paper claiming that every formally integrable G-structure is integrable, essentially by showing that the previous cases exhaust all the possible ones. Their result is controversial: Goldschmidt [42] claims there is a hole in the arguments, while Sternberg [89] mentions it as a conjecture, and in appendix of the second edition avoids explicitly to give a judgement on the matter. As far as we know, no further paper have been written confirming or confuting their proof.

It is worth noticing that we are not aware of an actual example of a G-structure which is formally integrable but not integrable (otherwise we could immediately confute the previous conjecture). Often such a counterexample is quoted from a paper [45] by Guillemin and Sternberg, where they use the Lewy counterexample [65] to build two G-structures on $\mathbb{R}^5$ which are formally equivalent but not equivalent. However, since neither of the G-structures are integrable, this is not the counterexample we were looking for. Nevertheless, it does provide a counterexample for the integrability problem of the more general class of geometric structures we study in this thesis: almost Γ-structures (see Example 5.2.14 later).

2.6 Higher-order G-structures

The geometric structures we have considered so far were of first order. To describe higher order geometric structures, we need to introduce the higher order versions of the frame bundle and the general linear group.

Definition 2.6.1. The space of **k-jets of locally defined diffeomorphisms** between the manifolds M and N is denoted by

$$J^k(M,N) := \{j^k_x\phi \mid \phi \in \mathrm{Diff}_{\mathrm{loc}}(M,N), x \in \mathrm{dom}(\phi)\},$$

where $\mathrm{Diff}_{\mathrm{loc}}(M,N)$ is the set of locally defined diffeomorphisms between open sets of M and N. ♦

The space $J^k(M,N)$ sits inside the fibre bundle $J^k(pr) \to M$ of jets of sections of $pr : M \times N \to M$. Since being a diffeomorphism is an open condition, $J^k(M,N)$ is a smooth subbundle. Note also that some authors use, respectively, $J^k(M,N)$ and $J^k_{inv}(M,N)$ to denote what we called $J^k(pr)$ and $J^k(M,N)$.

Definition 2.6.2. The **k^{th}-order frame manifold** of M^n is the fibre of the bundle $J^k(\mathbb{R}^n,M)$ at $0 \in \mathbb{R}^n$:

$$Fr^k(M) := J^k_0(\mathbb{R}^n,M).$$

The **$k^{\mathbf{th}}$-order general linear group** is the group of jets at 0 of locally defined diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$:

$$GL^k(n,\mathbb{R}) := \{j^k_0\phi \in J^k_0(\mathbb{R}^n,\mathbb{R}^n) \mid \phi(0) = 0\}. \qquad ♦$$

It is easy to see that $Fr^k(M)$ is a principal $GL^k(n,\mathbb{R})$-bundle over M.

Remark 2.6.3. For $k = 1$, the Lie group isomorphism

$$GL^1(\mathbb{R}^n) \to GL(n,\mathbb{R}), \quad j^1_0\phi \mapsto d_0\phi$$

together with Remark 1.1.2 gives an isomorphism of principal bundles

$$Fr^1(M) \cong Fr(M). \qquad \Diamond$$

To generalise the previous remark, we describe first the lift of a diffeomorphism $\phi : M \to N$ to the k-frames:

$$Fr^k(\phi) : Fr^k(M) \to Fr^k(N), \quad j^k_0\psi \mapsto j^k_0(\phi \circ \psi).$$

One sees that $Fr^k(\phi)$ is a morphism of principal $GL^k(\mathbb{R}^n)$-bundles. For $k = 1$ we recover Definition 2.1.4, and we set $Fr^0(\phi) := \phi$.

Consider now the k^{th}**-order tangent bundle** of M^n

$$T^kM := J^k_0(\mathbb{R}, M),$$

which is a vector bundle of rank kn over M. Indeed, to define a vector space structure on each fibre, it is enough to pick a chart φ of M and use the bijection

$$T^kM \to (\mathbb{R}^n)^k, \quad j^k_0 f \mapsto \left(\frac{d}{dt}\Big|_{t=0} (\varphi \circ f)(t), ..., \frac{d^k}{dt^k}\Big|_{t=0} (\varphi \circ f)(t) \right).$$

Moreover, there is an injection of principal bundles over M

$$Fr^k(M) \hookrightarrow Fr(T^kM),$$

described by the fibrewise injection at any $x \in M$

$$(Fr^k(M))_x = (J^k_0(\mathbb{R}^n, M))_x \hookrightarrow Fr(T^k_xM), \quad j^k_0\phi \mapsto Fr^k(\phi),$$

where we interpreted the frames of the vector space T^k_xM as linear maps

$$\mathbb{R}^{kn} = (J^k_0(\mathbb{R}, \mathbb{R}^n))_0 \to (J^k_0(\mathbb{R}, M))_x.$$

Note that, in the formulae above, by $Fr(T^kM)$ we mean the frame bundle of the *vector bundle* $T^kM \to M$, and not that of the manifold T^kM.

Moreover, the group $GL^k(\mathbb{R}^n)$ can be viewed as a subgroup of $GL(kn, \mathbb{R})$. For $n = 1$, this inclusion can be given explicitly by

$$j^k_0\phi \mapsto \begin{bmatrix} \phi'(0) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \phi^{(k)}(0) \end{bmatrix}$$

For $n > 1$, the element $\phi^{(i)}(0)$ along the diagonal are replaced by block matrices of partial derivatives; see [78, 83, 2] for more details.

2.6.1 Intermezzo: general linear groups of higher orders

Consider the tower of Lie groups (and Lie group homomorphisms)

$$\ldots \xrightarrow{\pi^4_3} GL^3(\mathbb{R}^n) \xrightarrow{\pi^3_2} GL^2(\mathbb{R}^n) \xrightarrow{\pi^2_1} GL^1(\mathbb{R}^n) \cong GL(n, \mathbb{R}),$$

and the corresponding tower of Lie algebras (and Lie algebra homomorphisms)

$$\dots \xrightarrow{\pi_3^4} \mathfrak{gl}^3(\mathbb{R}^n) \xrightarrow{\pi_2^3} \mathfrak{gl}^2(\mathbb{R}^n) \xrightarrow{\pi_1^2} \mathfrak{gl}^1(\mathbb{R}^n) \cong \mathfrak{gl}(n, \mathbb{R}).$$

For any subgroup $G^k \subseteq GL^k(\mathbb{R}^n)$, and its associated Lie algebra $\mathfrak{g}^k = \mathrm{Lie}(G^k)$, we will consider the symbol spaces

$$\dot{G}^k := \{g \in G^k \mid \pi^k_{k-1}(g) = j_0^{k-1} id\}, \quad \dot{\mathfrak{g}}^k := \mathrm{Lie}(\dot{G}^k)$$

Any subgroup $G^k \subseteq GL^k(\mathbb{R}^n)$ projects down to $G^i := \pi^k_i(G^k) \subseteq GL^i(\mathbb{R}^n)$:

$$G^k \to G^{k-1} \to \dots \to G^1.$$

In order to extend the tower $\{G^i\}_i$ from above, one would like a subgroup of $GL^{k+1}(\mathbb{R}^n)$ that projects down to G^k. There is an obvious choice, which is also the largest possible: the preimage $\widetilde{G}^k$ of G^k via the projection

$$\pi^{k+1}_k : GL^{k+1}(\mathbb{R}^n) \to GL^k(\mathbb{R}^n).$$

On the other hand, we have the definition of prolongation:

Definition 2.6.4. The **prolongation** of a Lie subgroup $G^k \subseteq GL^k(\mathbb{R}^n)$ is

$$(G^k)^{(1)} := \{j_0^{k+1} f \in GL^{k+1}(\mathbb{R}^n) \mid j_0^k f \in G^k, dFr^k(f)(\mathbb{R}^n \oplus \dot{\mathfrak{g}}^k) = \mathbb{R}^n \oplus \dot{\mathfrak{g}}^k\}. \quad \blacklozenge$$

The prolongation $(G^k)^{(1)}$ is a (non-abelian) Lie subgroup of $GL^{k+1}(\mathbb{R}^n)$ projecting on G^k. This implies that the prolongation $(G^{k-1})^{(1)}$ of the projection G^{k-1} of G^k coincides with the projection to order k of the prolongation $(G^k)^{(1)}$.

For a better understanding of $(G^k)^{(1)}$, consider $k = 1$, i.e. start with a subgroup $G \subseteq GL(n, \mathbb{R})$ (Remark 2.6.3); then its prolongation is

$$G^{(1)} = \{j_0^2 f \in GL^2(\mathbb{R}^n) \mid j_0^1 f \in G, d_{j_0^1 id} Fr(f)(\mathbb{R}^n \oplus \mathfrak{g}) = \mathbb{R}^n \oplus \mathfrak{g}\} \subseteq GL^2(\mathbb{R}^n).$$

Proposition 2.6.5. *Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup. The prolongation $G^{(1)}$ (Definition 2.6.4) is isomorphic to the product $G \times G'$, where G' is the abelian prolongation of G (Section 2.5.1), with the following "double semi-direct" group structure*

$$(g_1, T_1) \cdot (g_2, T_2) := (g_1 g_2, g_1 T_2 + T_1 g_2).$$

Proof. We show first that $G^{(1)}$ is isomorphic to the subgroup

$$H = \{\phi^g_T : (v, \rho) \mapsto (gv, g\rho + T(v)) \mid g \in G, T \in \mathfrak{g}^{(1)}\} \subseteq GL(\mathbb{R}^n \oplus \mathfrak{g})$$

by the following map: every element $j_0^2 f \in G^{(1)}$ is sent to the restriction of $d_{j_0^1 id} Fr(f)$ to $\mathbb{R}^n \oplus \mathfrak{g}$. This assignment does not depend on the choice of f, since all elements in the second jet share the same first and second derivatives at 0. Moreover, it is a group morphism since

$$j_0^2 f_1 \circ j_0^2 f_2 = j_0^2 (f_1 \circ f_2) \mapsto dFr(f_1 \circ f_2) = dFr(f_1) \circ dFr(f_2).$$

Now we check that $d_{j_0^1 id} Fr(f)$ takes the form ϕ^g_T, for

- $g = j_0^1 f \in G$ (since $j_0^2 f$ is in $(\pi_1^2)^{-1}(G)$)
- $T = d_0^2 f \in \mathfrak{g}^{(1)} \subseteq \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$ (since $T(v) = d_0^2 f(v, \cdot) = d_v(d_0 f(\cdot)) \in \mathfrak{g}$ and the hessian $d_0^2 f$ is bilinear and symmetric)

Take $v = \gamma'(0) \in \mathbb{R}^n \oplus \mathfrak{g} = T_{j_0^1 id}(\mathbb{R}^n \times G)$, for some curve $\gamma(t) = j_0^1 f_t \in \mathbb{R}^n \times G \cong Fr(\mathbb{R}^n)$; then one writes $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $\gamma_1(t) = f_t(0) \in \mathbb{R}^n$ and $\gamma_2(t) = d_0 f_t \in G$ and compute

$$d_{j_0^1 id} Fr(f)(v) = \left.\frac{d}{dt}\right|_{t=0} Fr(f)\gamma(t) = \left.\frac{d}{dt}\right|_{t=0} j_0^1(f \circ f_t) =$$

$$\left(\left.\frac{d}{dt}\right|_{t=0} (f \circ f_t(0)), \left.\frac{d}{dt}\right|_{t=0} d_0(f \circ f_t) \right) = \left(\left.\frac{d}{dt}\right|_{t=0} f(\gamma_1(t)), \left.\frac{d}{dt}\right|_{t=0} d_{\gamma_1(t)} f(\gamma_2(t)(\cdot)) \right) =$$

$$= \left(d_0 f(v_1), \left.\frac{d}{dt}\right|_{t=0} \left.\frac{d}{d\tau}\right|_{\tau=0} f(\bar\gamma(t,\tau)) \right) = \left(g v_1, \left.\frac{d}{dt}\right|_{t=0} d_{\bar\gamma(0)} f \cdot \bar\gamma'(0) \right) =$$

$$= \left(g v_1, \left.\frac{d}{dt}\right|_{t=0} d_{\gamma_1(t)} f \cdot \gamma_2(t)(\cdot) \right) = \left(g v_1, d^2_{\gamma_1(0)} f(\gamma_1'(0), \gamma_2(0)(\cdot)) + d_{\gamma_1(0)} f \gamma_2'(0) \right) =$$

$$= \left(g v_1, d_0^2 f(v_1, e(\cdot)) + d_0 f v_2 \right) = \left(g v_1, d_0^2 f(v_1, \cdot) + g v_2 \right) = (g v_1, T v_1 + g v_2)\,.$$

In the third line, for each t, $\bar\gamma(\tau)$ is a curve starting in $\gamma_1(t)$ with speed $\gamma_2(t)$ at 0.

It is only left to check that

$$H \to G' \times G, \quad \phi_T^g \mapsto (g, T)$$

is a group isomorphism, i.e.

$$\phi_{T_1}^{g_1} \circ \phi_{T_2}^{g_2} = \phi_{g_1 T_2 + T_1 g_2}^{g_1 g_2} \mapsto (g_1 g_2, g_1 T_2 + T_1 g_2) = (g_1, T_1) \cdot (g_2, T_2).$$

This follows from an easy computation:

$$\phi_{T_1}^{g_1} \circ \phi_{T_2}^{g_2}(v, \rho) = \phi_{T_1}^{g_1}(g_2 v, g_2 \rho + T_2(v)) =$$

$$= (g_1 g_2 v, g_1 g_2 \rho + g_1 T_2(v) + T_1 g_2(v)) = \phi_{g_1 T_2 + T_1 g_2}^{g_1 g_2}(v, \rho).$$

Q.E.D.

Corollary 2.6.6. *For any Lie subgroup $G \subseteq GL(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}$,*

$$\mathrm{Lie}(G^{(1)}) \cong \mathfrak{g} \oplus \mathfrak{g}^{(1)}.$$

More generally, Guillemin proved in section 3 of [43] that, given a Lie subgroup $G \subseteq GL(n, \mathbb{R})$, the Lie algebra of its prolongation $G^{(k)}$ can be described in terms of the Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R})$:

$$\mathrm{Lie}(G^{(k)}) = \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \ldots \oplus \mathfrak{g}^{(k)}.$$

2.6.2 Higher order G-structures

Consider the tower of principal bundles over M

$$\dots \xrightarrow{\pi_3^4} Fr^3(M) \xrightarrow{\pi_2^3} Fr^2(M) \xrightarrow{\pi_1^2} Fr^1(M) \cong Fr(M),$$

While first-order G-structures were subbundle of $Fr(M)$, a G^k**-structure of order** k **on** M is a reduction of the structure group of $Fr^k(M)$ to a Lie subgroup $G^k \subseteq GL^k(\mathbb{R}^n)$.

Any G^k-structure $P^k \subseteq Fr^k(M)$ projects down to $P^i = \pi_i^k(P^k) \subseteq Fr^i(M)$:

$$P^k \to P^{k-1} \to \dots \to P^1,$$

and each P^i is a G^i-structure, for the projection $G^i = \pi_i^k(G^k) \subseteq GL^i(\mathbb{R}^n)$.

In order to extend the tower $\{P^i\}_i$ from above, one would like a subbundle of $Fr^{k+1}(M)$ that projects down to P^k. There is an obvious choice, which is also the largest possible: the preimage $\widetilde{P}^k$ of P^k via the projection

$$\pi_k^{k+1} : Fr^{k+1}(M) \to Fr^k(M).$$

On the other hand, we have the definition of prolongation:

Definition 2.6.7. The 1^{st} **prolongation** of a G^k-structure $P^k \subseteq Fr^k(M)$ is

$$(P^k)^{(1)} := \{z = j_0^{k+1}\phi \in Fr^{k+1}(M) \mid w := \pi_k^{k+1}(z) \in P^k, d_{(0,e)}Fr^k(\phi)(\mathbb{R}^n \oplus \dot{\mathfrak{g}}^k) = T_w P^k\}.$$

♦

We will sometimes refer to $(P^k)^{(1)}$ as Molino-Albert prolongation (they introduced it in [3]). However, while it can always be defined, $(P^k)^{(1)}$ may not be a principal bundle over M. To fix this, we consider a further hypothesis:

Definition 2.6.8. A G^k-structure P^k is k**-integrable** if over each $x \in M$ there exists a $j_0^{k+1}\phi \in Fr^{k+1}(M)$, with $w = j_0^k\phi \in P^k$ and such that

$$d_{(0,e)}Fr^k(\phi)(\mathbb{R}^n \oplus \dot{\mathfrak{g}}^k) = T_w P^k.$$ ♦

Under the hypothesis of k-integrability, the prolongation $(P^k)^{(1)}$ is a principal $(G^k)^{(1)}$-subbundle of $Fr^{k+1}(M)$ projecting on P^k. It follows that the projection P^{k-1} of P^k is $(k-1)$-integrable, and its prolongation $(P^{k-1})^{(1)}$ coincides with the projection to order k of the prolongation $(P^k)^{(1)}$.

Definition 2.6.9. A G^k-structure $P^k \subseteq Fr^k(M)$ is $(k+1)$**-integrable** if it is k-integrable and its first prolongation $(P^k)^{(1)}$ is $(k+1)$-integrable.

The **second prolongation** of a $(k+1)$-integrable G^k-structure P^k is

$$(P^k)^{(2)} := \left((P^k)^{(1)}\right)^{(1)}.$$

In an analogous way one defines $(k+l)$-integrability and l^{th}-prolongation; P^k is called **formally integrable** if it is $(k+l)$-integrable for every $l \geq 0$. ♦

Example 2.6.10. For any Lie subgroup $G^k \subseteq GL^k(\mathbb{R}^n)$, the **flat G^k-structure** on $\mathbb{R}^n$ is

$$P^k_{\text{can}} = \mathbb{R}^n \times G^k \subseteq Fr^k(\mathbb{R}^n). \qquad \lozenge$$

By a direct computation, it is formally integrable and its l^{th}-prolongation is $\mathbb{R}^n \times (G^k)^{(l)}$. In the case $k = l = 1$, this agrees with Example 2.5.3: if $P = \mathbb{R}^n \times G$ is the flat G-structure on $\mathbb{R}^n$, then

$$P^{(1)} = J^1 P \cong P \times G' = \mathbb{R}^n \times G \times G' \cong \mathbb{R}^n \times G^{(1)}.$$

Accordingly, $P^{(1)}$ is both the flat $G^{(1)}$-structure over $\mathbb{R}^n$ and the flat G'-structure over P.

Definition 2.6.11. A G^k-structure $P^k \subseteq Fr^k(M)$ is **integrable** if it is locally isomorphic to the flat G^k-structure P^k_{can} by the lift of a chart on M. ♦

2

Of course, as one expects, if P^k is integrable, then it is formally integrable.

2.6.3 Equivalence between integrabilities of 1^{st}-order G-structures

In Definition 2.6.8 we recalled the notion of k-integrability for a G-structure of order k. If $k = 1$, one expects to recover the standard definition 2.4.5 of 1-integrability for G-structures of order 1. Similarly, the higher integrability of a G-structure of order 1 according to Definition 2.6.9 should coincide with Definition 2.5.7.

These equivalences seem to be well known, but not written anywhere, so we thought it was useful to spell out the details.

In order to do so, we have first to revise a standard explicit way to interpret the vanishing of the intrinsic torsions.

Recall from Lemma 2.1.5 that the integrability of a G-structure P amounts to the local equivalence between P and the canonical flat G-structure $\mathbb{R}^n \times G$, i.e. to the existence at any point of a locally defined diffeomorphism $\phi \in \text{Diff}_{\text{loc}}(\mathbb{R}^n, M)$ such that the submanifolds $Fr(\phi)(\mathbb{R}^n \times G)$ and P coincide. This condition can be "approximated" by asking that $Fr(\phi)(\mathbb{R}^n \times G)$ and P are "close enough", motivating the following definition, originally due to Guillemin (Section 4 of [43]):

Definition 2.6.12. A G-structure P on M is called **uniformly k-flat** if, for every $x \in M$, there exists $\phi \in \text{Diff}_{\text{loc}}(\mathbb{R}^n, M)$ with $\phi(0) = x$ such that

- $Fr(\phi)$ sends $(0, e) \in \mathbb{R}^n \times G$ to $j^1_0\phi \in P$
- The submanifold $Fr(\phi)(\mathbb{R}^n \times G) \subseteq Fr(M)$ has order of contact k with $P \subseteq Fr(M)$ at $j^1_0\phi$.

Recall that two submanifolds $Y_1, Y_2 \subseteq X^n$ of dimension l have order of contact k at $p \in X$ if there are coordinates $(y^i)_{i=1}^n$ around p to express Y_1 as the zero locus of $(y^i)_{i=1}^l$ and Y_2 as the zero locus of some smooth functions $(f_i)_{i=1}^l$ whose derivatives up to order k vanish at p. One also says that Y_1 and Y_2 have the same k-jet at p, and write $j^k_p Y_1 = j^k_p Y_2$. ♦

Note that, in order to state Definition 2.6.12, we do not need the entire ϕ, but it is enough to know its derivative up to order $k+1$, i.e. to know the jet $j^{k+1}_0 \phi \in Fr^{k+1}(M)$.

Proposition 2.6.13 ([43], corollary to theorem 4.1). *Given a G-structure $P \to M$, the following properties are equivalent:*

- *P is k-integrable, i.e. the intrinsic torsions $T^{(i)}_{intr}$ (Definition 2.5.11) vanish for all $i \leq k$.*
- *P is uniformly k-flat (Definition 2.6.12).*

We can now prove the desired equivalence.

Proposition 2.6.14. *Let $G \subseteq GL(n,\mathbb{R})$ a Lie subgroup and P be a first-order G-structure on M. Then the Molino definition 2.6.9 of k-integrability coincides with the standard definition 2.6.9 of k-integrability.*

Proof. As a consequence of Proposition 2.6.13, it is enough to prove the eqivalence with the Guillemin definition 2.6.12 of uniform k-flatness.

Let us start with $k=1$. As we already observed, both definitions begin with the existence of an element $z = j^2_0\phi \in \widetilde{P}^2$ over every $x \in M$. Since $\mathbb{R}^n \oplus \dot{\mathfrak{g}}^1 = \mathbb{R}^n \oplus \mathfrak{g} = T_{(0,e)}(\mathbb{R}^n \times G)$, we can compute its image through the differential of $Fr(\phi)$ as

$$d_{(0,e)}Fr(\phi)(\mathbb{R}^n \oplus \mathfrak{g}) = d_{(0,e)}Fr(\phi)(T_{(0,e)}(\mathbb{R}^n \times G)) = T_{j^1_0\phi}(Fr(\phi)(\mathbb{R}^n \times G)).$$

The condition in Molino's definition is that such tangent space is equal to $T_w P = T_{j^1_0\phi}P$, i.e. that $Fr(\phi)(\mathbb{R}^n \times G)$ and P have "order of contact 1" at $j^1_0\phi$, which is the condition in Guillemin's definition.

Let us consider now the case $k=2$; for $k \geq 2$ we proceed inductively in the same way. A G-structure $P \subseteq Fr(M)$ is Molino-2-integrable if it is 1-integrable and its first prolongation $P^{(1)}$ is 1-integrable; the first condition is trivially satisfied by a 2-integrable G-structure according to Guillemin (uniform 2-flatness implies uniform 1-flatness), hence we have only to show the equivalence between the second parts.

First, we compute

$$d_{(0,e)}Fr^2(\phi)(\mathbb{R}^n \oplus \dot{\mathfrak{g}}^2) = d_{(0,e)}Fr^2(\phi)(\mathbb{R}^n \oplus \mathfrak{g}^{(1)}) =$$

$$= d_{(0,e)}Fr^2(\phi)(T_{(0,e)}(\mathbb{R}^n \times G^{(1)})) = T_{j_0^2\phi}(Fr^2(\phi)(\mathbb{R}^n \times G^{(1)})).$$

On the other side, we have

$$Fr^2(\phi)(\mathbb{R}^n \times G^{(1)}) = \{Fr(\phi)(j_0^2 f) \mid j_0^2 f \in G^{(1)}\} =$$

$$= \{j_0^2(\phi \circ f) \mid j_0^1 f \in G, j_0^2 f(\mathbb{R}^n \oplus \mathfrak{g}) = \mathbb{R}^n \oplus \mathfrak{g}\} =$$

$$= \{z = j_0^2(\phi \circ f) \in \widetilde{Fr(\phi)(\mathbb{R}^n \times G)}^{2} \mid d_{(0,e)}Fr^2(f)(\mathbb{R}^n \oplus \mathfrak{g}) = \mathbb{R}^n \oplus \mathfrak{g}\},$$

and

$$\widetilde{z}^{-1}(T_w(Fr(\phi)(\mathbb{R}^n \times G))) = d_{(0,e)}Fr((\phi \circ f)^{-1})(T_w Fr(\phi)(\mathbb{R}^n \times G)) =$$

$$= T_{(0,e)}(Fr(\phi \circ f)^{-1} Fr(\phi)(\mathbb{R}^n \times G)) = T_{(0,e)}(Fr(f^{-1})(\mathbb{R}^n \times G)) =$$

$$= d_{(0,e)}Fr(f^{-1})(\mathbb{R}^n \oplus \mathfrak{g}),$$

which, put together, yield

$$Fr^2(\phi)(\mathbb{R}^n \times G^{(1)}) = (Fr(\phi)(\mathbb{R}^n \times G))^{(1)}.$$

In conclusion, since the tangent spaces of $Fr^2(\phi)(\mathbb{R}^n \times G^{(1)})$ and $(Fr(\phi)(\mathbb{R}^n \times G))^{(1)}$ coincide, we invoke Lemma 2.6.15 to see that $Fr(\phi)(\mathbb{R}^n \times G)$ and P have contact of order 2 if and only if their prolongation have contact of order 1.

Then we conclude that $d_{(0,e)}Fr^2(\phi)(\mathbb{R}^n \oplus \dot{\mathfrak{g}}^2)$ is equal to $T_{j_0^2\phi}P^{(1)}$ if and only if $Fr(\phi)(\mathbb{R}^n \times G)$ and P have contact of order 2 at $j_0^2\phi$, i.e. $P^{(1)}$ is 2-integrable. Q.E.D.

Lemma 2.6.15. *Two G-structures $P, Q \subseteq Fr(M)$ have order of contact 2 at $w = j_0^1\phi \in P \cap Q$ if and only if their prolongations $P^{(1)}, Q^{(1)} \subseteq Fr^2(M)$ have order of contact 1 at $z = j_0^2\phi \in P^{(1)} \cap Q^{(1)}$.*

Proof. Since contact of order 2 implies contact of order 1, assume $T_wP = T_wQ$; then

$$T_z\widetilde{P}^2 = T_z(\pi_1^2)^{-1}(P) = d_z\pi_1^2(T_wP) = d_z\pi_1^2(T_wQ) = T_z(\pi_1^2)^{-1}(P) = T_z\widetilde{Q}^2.$$

In particular, the tangent space of $P^{(1)} \subseteq \widetilde{P}^2$ coincides with that of $Q^{(1)}$, since they are both subsets of the same space imposing the same condition $\widetilde{z}(\mathbb{R}^n \oplus \mathfrak{g}) = T_wP = T_wQ$; then the prolongations $P^{(1)}$ and $Q^{(1)}$ have order of contact 1.

Viceversa, assuming that such prolongations have the same tangent space at z, we see first that their projection P and Q have order of contact 1, and then we write the above argument backwards in order to obtain the conclusion. Q.E.D.

2.6.4 Canonical form of higher order G-structures

As for $Fr(M)$, the higher order frame bundle $Fr^k(M)$ (Definition 2.6.2) carries a canonical 1-form; we use as references [56] and [59, Section IV.16].

Definition 2.6.16. The **canonical (or tautological) form of the k^{th}-frame bundle** of M is the 1-form

$$\theta^k \in \Omega^1(Fr^k(M), \mathbb{R}^n \oplus \mathfrak{gl}^{k-1}(\mathbb{R}^n))$$

defined for every $u = j^k_0\phi \in Fr^k(M)$ as

$$\theta^k_u(v) := \widetilde{u}^{-1}(d_u\pi^k_{k-1}(v))$$

where $\widetilde{u} = d_{(0,e)}Fr^{k-1}(\phi)$, and we identify its domain with the coefficients of θ:

$$\mathbb{R}^n \oplus \mathfrak{gl}^{k-1}(\mathbb{R}^n) \cong T_{(0,e)}(\mathbb{R}^n \times GL^{k-1}(\mathbb{R}^n)) \cong T_{(0,e)}Fr^{k-1}(\mathbb{R}^n).$$

Let $G^k \subseteq GL^k(\mathbb{R}^n)$ be a Lie subgroup which projects to G^{k-1}, and let $\mathfrak{g}^{k-1} = \mathrm{Lie}(G^{k-1})$. The **canonical form of a G^k-structure** $P \subseteq Fr^k(M)$ is the restriction of θ^k to P, namely

$$\theta^k_P \in \Omega^1(P, \mathbb{R}^n \oplus \mathfrak{g}^{k-1}). \qquad \blacklozenge$$

Note that $\widetilde{u}$ does not depend on the entire $\phi : \mathbb{R}^n \to M$, but only on $u = j^k_0\phi$. In particular, when $k = 1$,

$$\widetilde{u} = d_0(Fr^0(\phi)) = d_0\phi,$$

therefore θ^1 coincides with the canonical form θ on $Fr(M)$ from Definition 2.3.1.

Remark 2.6.17. A G^k-structure P^k is k-integrable (Definition 2.6.8) if over each $x \in M$ there exists $z \in Fr^{k+1}(M)$ with $w = \pi^{k+1}_k(z) \in P^k$ and such that

$$\theta^{k+1}_z(T_z\widetilde{P}^{k+1}) = \mathbb{R}^n \oplus \dot{\mathfrak{g}}^k.$$

This follows from

$$(\theta^{k+1})_z(T_z\widetilde{P}^{k+1}) = \widetilde{z}^{-1}(d_z\pi^k_{k-1}(T_w((\pi^{k+1}_k)^{-1}(P^k)))) = \widetilde{z}^{-1}(T_wP^k).$$

Indeed, an element in $T_z\widetilde{P}^{k+1}$ is the speed of a curve on $\widetilde{P}^{k+1} := (\pi^{k+1}_k)^{-1}(P^k)$, on which we apply the differential of the surjective map π^k_{k-1}, resulting in the speed of a curve on P^k, i.e. an element of T_zP^k.

Similarly, the prolongation of P^k can be written using θ:

$$(P^k)^{(1)} = \{z \in \widetilde{P}^{k+1} \mid \theta^{k+1}_z(T_z\widetilde{P}^{k+1}) = \mathbb{R}^n \oplus \dot{\mathfrak{g}}^k\} \subseteq Fr^{k+1}(M). \qquad \lozenge$$

The tautological form θ^k of a G^k-structure P^k is pointwise surjective, G^k-invariant and satisfies $\ker(\theta^k) = \ker(d\pi^k_{k-1} : TP^k \to TP^{k-1})$. Moreover, the projection

$$\pi^k_{k-1} : P^k \to P^{k-1}$$

is a principal $\dot{G}^k$-bundle, since the symbol space $\dot{G}^k$ of $G^k \subseteq GL^k(\mathbb{R}^n)$ induces the short exact sequence

$$0 \to \dot{G}^k \hookrightarrow G^k \to G^{k-1} \to 0.$$

It follows that (P^k, θ^k) is an abstract G^k-structure (Definition 2.3.3), hence by Theorem 2.3.2 we have

$$P^k \hookrightarrow Fr(P^{k-1}).$$

By iteration, we see that

$$P^k \subseteq (Fr)^k(M).$$

2

2.6.5 Example of higher order G-structures

The vast majority of known G-structures are of first order: we list here some second order G-structures of relevance in geometry.

Example 2.6.18. Consider the Lie group $PGL(\mathbb{R}^n) := GL(n+1,\mathbb{R})/\mathbb{R}^\times$ and its natural action on the projective space $\mathbb{P}^n := (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times$; we denote by $PGL_0(\mathbb{R}^n)$ its isotropy group at the point $0 := [e_n] \in \mathbb{P}^n$, where $(e_0, \dots, e_n)$ denotes the standard basis of $\mathbb{R}^{n+1}$. A **projective structure** on a manifold M^n is a $PGL_0(\mathbb{R}^n)$-structure of second order (see [58], chapter IV, and [3], chapter V).

To make sense of this definition, recall that the group $PGL(\mathbb{R}^n)$ is the group of projective transformations $\mathbb{P}^n \to \mathbb{P}^n$, hence $PGL_0(\mathbb{R}^{n+1})$ consists of the projective transformations $\mathbb{P}^n \to \mathbb{P}^n$ which fix 0. Since $\mathbb{P}^n$ is an n-dimensional manifold, we can use local coordinates of $\mathbb{P}^n$ around 0 in order to view each element $f \in PGL_0(\mathbb{R}^n)$ as a locally defined diffeomorphism between opens of $\mathbb{R}^n$. Under this identification, one checks that a 2-jet of f at 0 is an element of $GL^2(\mathbb{R}^n)$, i.e. $PGL_0(\mathbb{R}^n) \subseteq GL^2(\mathbb{R}^n)$.

For instance, once we realise the projective space as an homogeneous space $PGL(\mathbb{R}^n)/PGL_0(\mathbb{R}^n)$, there is a natural projective structure on $M^n = \mathbb{P}^n$ given by the principal $PGL_0(\mathbb{R}^n)$-bundle $PGL(\mathbb{R}^n) \to \mathbb{P}^n$. Here we view $PGL(\mathbb{R}^n) \subseteq Fr^2(\mathbb{P}^n)$ with a similar argument as above, i.e. identifying the 2-jets of elements of $PGL(\mathbb{R}^n)$ at 0 with the 2-frames of $\mathbb{P}^n$, and defining an injective map $PGL(\mathbb{R}^n) \to Fr^2(\mathbb{P}^n)$. ♢

Example 2.6.19. Consider the group of conformal transformations $O(n+1,1)$ (see Example 2.2.3) and its natural action on the n-dimensional **Möbius space**

$$S^n := \{x \in \mathbb{P}^{n+1} \mid x^t S x = 0\} \qquad \text{where } S = \begin{bmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

We denote by $O_0(n+1,1)$ its isotropy group at the point $0 := [e_{n+1}] \in \mathbb{P}^{n+1}$, called also conformal parabolic subgroup. A **conformal structure** on a manifold M^n is a $O_0(n+1,1)$-structure of second order (see [58], chapter IV).

As in Example 2.6.18, to make sense of this definition, one can see that $O_0(n+1,1)$ consists of the conformal transformations $f : S^n \to S^n$ which fix 0. Since S^n is an n-dimensional manifold, diffeomorphic to the n-sphere in $\mathbb{R}^{n+1}$, using local coordinates around 0 we identify the 2-jets of elements $f \in O_0(n+1,1)$ with elements of $GL^2(\mathbb{R}^{n+2})$.

For instance, there is a natural conformal structure on the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ given by the principal $O_0(n+1,1)$-bundle $O(n+1,1) \to S^n$, where we view $O(n+1,1) \subseteq Fr^2(S^n)$ with a similar argument as above.

Last, let us show that conformal structures in the sense defined here are in correspondence with conformal structures in the sense of Example 2.2.4.

If $P \subseteq Fr^2(M)$ is an $O_0(n+1,1)$-structure, we can quotient it by the subgroup of $O_0(n+1,1)$ made up of matrices with $a = 1$ and $A = I$; we obtain a $CO(n)$-structure of first order on M, i.e. a conformal structure from Example 2.2.4. Conversely, a $CO(n)$-structure $P \subseteq Fr^1(M)$ can be always prolonged to a $O_0(n+1,1)$-structure of order 2. Indeed, P is 1-integrable, as discussed in Example 2.2.4. By Proposition 2.6.14, it is also 1-integrable according to Definition 2.6.8; then its prolongation $P^{(1)} \subseteq Fr^2(M)$ is a $CO^{(1)}$-structure of order 2 on M, and one can check that $CO^{(1)} = O_0(n+1,1)$. $\lozenge$

Example 2.6.20. A **torsion-free Ehresmann connection on** $Fr(M)$ is a G-structure of order 2 on M for the group $GL^1(\mathbb{R}^n)$, seen as a subgroup of $GL^2(\mathbb{R}^n)$ (see [3], chapter V, [86], chapter 2).

First, an Ehresmann connection on $Fr(M)$ is equivalent to a global section of $J^1Fr(M)$; this follows from a general equivalence between Ehresmann connections on fibre bundles $E \to M$ and global sections of J^1E, already mentioned in Remark 1.1.2.

With similar arguments, a torsion-free connection is equivalent to a global section σ of $Fr^2(M) \subseteq J^1Fr(M)$. This comes intuitively from the fact that the torsion measures precisely how "non-symmetric" are the coordinates of $J^1Fr(M)$ corresponding to second-order derivatives. Having torsion equal to zero means precisely that all such coordinates are symmetric, hence they restrict to the coordinates of $Fr^2(M)$.

Last, the image $\sigma(M) \subseteq Fr^2(M)$ becomes a $GL^1(\mathbb{R}^n)$-structure on M. The integrability of such G-structure is equivalent to the connection being flat: indeed, the local model is precisely the flat connection on $\mathbb{R}^n$ defined by $\nabla_X(\frac{\partial}{\partial x^i}) = 0$ for all $X \in \mathfrak{X}(M)$, and we know that a connection ∇ is flat and torsion-free if and only if the manifold admits local coordinates for which the restrictions ∇_U to each chart is the standard flat connection.

This example, first due to Kobayashi [56], was generalised by Kolář [59, Section IV.17] to higher orders: a torsion-free connection on $Fr^k(M)$ is a G-structure of order $k+1$ for the group $GL^k(\mathbb{R}^n)$, seen as a subgroup of $GL^{k+1}_0(\mathbb{R}^n)$. $\lozenge$

Chapter 3

Lie groupoids and multiplicative forms

In this chapter we review some fundamental building blocks of this thesis: Lie groupoids, Lie algebroids and Morita equivalences. Several proofs will be only sketched or omitted, since they are classical results easily found in the literature; see e.g. [67, 73, 68, 23].

An important topic in differential geometry concerns the adaptation of various geometric structures on Lie groupoids. For instance, symplectic, Poisson and contact structures on Lie groupoids have been prominently studied in the literature [21, 54, 66, 71, 98], as well as their infinitesimal counterparts on Lie algebroids (see e.g. [15]). The relevant objects here are differential forms "compatible" with the groupoid multiplication - also called *multiplicative*.

While multiplicative forms with trivial coefficients have become an established tool in this area (see [61] for a recent survey), multiplicative forms with non-trivial coefficients (and their infinitesimal counterparts) have been investigated for the first time only very recently, e.g. applied to the relevant example of Jacobi structures ([24]). Accordingly, besides reviewing the standard theory of multiplicative forms, we will prove several new results for multiplicative forms with coefficients, concerning the interplay between their regularity, surjectivity, transversality and invariance by the right and left multiplication.

3.1 Lie groupoids

A **groupoid** $\mathcal{G}$ over M is a small category where all the arrows are invertible; M denotes the set of objects and $\mathcal{G}$ the set of arrows. The five structure maps are the following:

- the source $s : \mathcal{G} \to M$ (which associates to every arrow $g : x \to y$ its starting object $s(g) = x$)

- the target $t : \mathcal{G} \to M$ (which associates to every arrow $g : x \to y$ its ending object $t(g) = y$)
- the multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}, \quad (g, h) \mapsto gh$ (which associated to a pair of composable arrows $(g, h) \in \mathcal{G}^{(2)} := \mathcal{G} \,{}_s\!\times_t \mathcal{G}$ their composition gh)
- the unit $u : M \to \mathcal{G}, \quad x \mapsto 1_x$ (hence $1_{t(g)} \cdot g = g$ and $g \cdot 1_{s(g)} = g$)
- the inversion $i : \mathcal{G} \to \mathcal{G}, \quad g \mapsto g^{-1}$ (hence $g \cdot g^{-1} = 1_{t(g)}$, $g^{-1} \cdot g = 1_{s(g)}$)

Such a groupoid will be denote by $\mathcal{G} \rightrightarrows M$, or simply $\mathcal{G}$.

A **topological groupoid** is a groupoid where $\mathcal{G}$ and M are topological spaces, the structure maps are continuous and s and t are open maps. A **Lie groupoid** is a groupoid where $\mathcal{G}$ and M are smooth manifolds, s and t are smooth submersions, and m, u and i are smooth maps (this makes sense because, with the previous assumptions, $\mathcal{G}^{(2)}$ becomes a submanifold). In order to include some natural examples (see Example 4.1.9 later), we do not impose on $\mathcal{G}$ the standard regularity conditions of smooth manifolds, i.e. being Hausdorff or second countable.

A **subgroupoid** $\mathcal{H} \rightrightarrows N$ of $\mathcal{G} \rightrightarrows M$ is a subcategory ($\mathcal{H} \subseteq \mathcal{G}, N \subseteq M$) of $\mathcal{G} \rightrightarrows M$. A **Lie subgroupoid** of a Lie groupoid is a subgroupoid with the extra requirement that $\mathcal{H} \subseteq \mathcal{G}$ is an immersed submanifold.

The **orbit** of $\mathcal{G}$ through the point $x \in M$, denoted by $\mathcal{O}(x)$ or $\mathcal{G}x$, is the set of all the points of M which can be joined to x through an arrow in $\mathcal{G}$, i.e. $\mathcal{O}(x) = t(s^{-1}(x))$. The orbits form a partition of M; if $\mathcal{G}$ is a Lie groupoid, then all orbits are immersed submanifolds. Last, the collection of the orbits, called the **orbit space** and denoted by $M/\mathcal{G}$, becomes a topological space with the quotient topology induced by M, though usually it is not smooth nor Hausdorff.

The **isotropy group** at the point $x \in M$ is the set $\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$ of arrows starting and ending in x; the composition between them is always defined and $\mathcal{G}_x$ is a group. If x and y sit in the same orbit, then any arrow $g \in \mathcal{G}$ from x to y induces an isomorphism $\mathcal{G}_x \to \mathcal{G}_y, h \mapsto ghg^{-1}$. If $\mathcal{G}$ is, respectively, a topological or Lie groupoid, then its isotropy groups are, respectively, topological or Lie groups. Last, for every $x \in M$, $\mathcal{G}_x$ acts freely and properly on $s^{-1}(x)$, and $t : s^{-1}(x) \to \mathcal{O}(x)$ is a principal $\mathcal{G}_x$-bundle.

A **groupoid morphism** between two groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is a functor between them, explicitly given by two maps $(F : \mathcal{G} \to \mathcal{H}, f : M \to N)$. In a similar way, one defines morphisms between topological and Lie groupoids. Note that a (topological or Lie) groupoid morphism $(F : \mathcal{G} \to \mathcal{H}, f : M \to N)$ induces for any point $x \in M$ a (topological or Lie) group morphism $F_x : \mathcal{G}_x \to \mathcal{H}_{f(x)}$ between the isotropy groups.

Here are some interesting special classes of groupoids: a Lie groupoid $\mathcal{G}$ over M is said to be:

- **transitive** if any two points $x, y \in M$ are related by at least one arrow of $\mathcal{G}$ (equivalently, if $\mathcal{G}$ has only one orbit)
- **proper** if the map $(s,t) : \mathcal{G} \to M \times M$ is proper. As a consequence, the isotropy Lie groups $\mathcal{G}_x = (s,t)^{-1}(x,x)$ are compact, the orbits $\mathcal{O}_x = (s,t)(s^{-1}(x))$ are closed and the orbit space $M/\mathcal{G}$ is Hausdorff.
- **étale** if the source map $s : \mathcal{G} \to M$ is a local diffeomorphism.

For étale groupoids, arrows $g \in \mathcal{G}$ are closely related to germs of diffeomorphisms defined from neighbourhoods of $x = s(g)$ to $y = t(g)$. More precisely, any g defines

$$\Phi_g := \mathrm{germ}_x \left(t \circ (s_{|U_g})^{-1} \right) : (M,x) \to (M,y),$$

where U_g is an open neighbourhood of g on which s becomes a diffeomorphism. One says that $\mathcal{G}$ is **effective** if it is étale and the map $g \mapsto \Phi_g$ is injective.

3

Definition 3.1.1. A **local bisection** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a smooth map $\sigma : U \subseteq M \to \mathcal{G}$ defined on an open subset U of M such that $s \circ \sigma = id_U$ and $t \circ \sigma : U \to t(\sigma(U))$ is a diffeomorphism. If $U = M$, we say that σ is a **(global) bisection**. We denote by $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ the sets of local bisections, and by $\mathrm{Bis}(\mathcal{G})$ the set of global ones. ♦

By construction, any bisection σ induces a diffeomorphism on the base,

$$\Phi_\sigma := t \circ \sigma : M \to M.$$

The resulting map into the diffeomorphism group of M,

$$\Phi : \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M), \quad \sigma \mapsto \Phi_\sigma,$$

can be made into a group homomorphism by defining the "composition" of two bisections $\sigma_i : M \to \mathcal{G}$ by

$$\sigma_2\sigma_1 : M \to \mathcal{G}, \quad x \mapsto \sigma_1(\Phi_{\sigma_2}(x))\sigma_2(x).$$

Note the the unit map $u : M \to \mathcal{G}$ becomes the unit element of $\mathrm{Bis}(\mathcal{G})$ and the inverse of σ is

$$\sigma^{-1} : M \to \mathcal{G}, \quad x \mapsto i(\sigma(\Phi_\sigma^{-1}(x))).$$

Of course, the previous construction makes sense for local bisections as well, giving rise to a map into the set of locally defined diffeomorphisms of M

$$\Phi_{\mathrm{loc}} : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}) \to \mathrm{Diff}_{\mathrm{loc}}(M).$$

which preserves the partial composition. With this, one obtains the following characterisation of effectiveness for étale groupoids $\mathcal{G}$ (hinting to what effectiveness may mean when $\mathcal{G}$ is not étale):

$$\mathcal{G} \quad \text{is effective} \quad \iff \quad \Phi_{\text{loc}} \quad \text{is injective.}$$

3.1.1 Examples

We now illustrate the definitions by several examples.

Example 3.1.2. A **Lie group** G is a Lie groupoid $G \rightrightarrows \{*\}$ over a point. It is trivially transitive, isomorphic to its unique isotropy group and with $\mathrm{Bis}_{\text{loc}}(G) = \mathrm{Bis}(G) = G$; it is proper if and only if G is compact, it is étale if and only if G is discrete, and it is effective only when G is trivial. $\lozenge$

Example 3.1.3. The **unit groupoid** of the manifold M is the groupoid $M \rightrightarrows M$ with $s = t = id_M$; this forces the multiplication to be trivial, i.e. $xx = x = 1_x$. It is not transitive, each singleton $\{x\}$ is an orbit, and all the isotropy groups are trivial; the only local bisections are the identities id_U of each open $U \subseteq M$; it is étale, effective and proper. $\lozenge$

Example 3.1.4. The **pair groupoid** of the manifold M is the product

$$M \times M \rightrightarrows M.$$

It has a unique arrow (x, y) between y and x, the multiplication is $(x, y)(y, z) = (x, z)$, the unit $1_x = (x, x)$ and the inverse $(x, y)^{-1} = (y, x)$. It is transitive and has trivial isotropy groups; it is proper but not étale. Note that, in this case, (local) bisections correspond to (locally defined) diffeomorphisms of M, i.e. Φ and its local version are bijections. $\lozenge$

Example 3.1.5. If $\mu : P \to M$ is a surjective submersion, its **fibred pair groupoid** (or submersion groupoid) is the subgroupoid of the pair groupoid of P defined by the fibred product

$$P \times_\mu P = \{(p, q) \in P \times P \mid \mu(p) = \mu(q)\} \rightrightarrows P.$$

Unlike the pair groupoid, in general it is not transitive, its orbits are the fibres of μ, but it has still trivial isotropy groups. When M is a point, we recover the pair groupoid of P. $\lozenge$

Example 3.1.6. If G is a Lie group acting (say, from the left) on a manifold M, its **action groupoid** is the Cartesian product

$$G \times M \rightrightarrows M,$$

where a pair (g, x) is viewed as an arrow from x to $g \cdot x$; the multiplication is $(h, g \cdot x)(g, x) = (hg, x)$, the unit $1_x = (e_G, x)$ and the inverse $(g, x)^{-1} = (g^{-1}, g \cdot x)$.

Both its orbits and its isotropy groups coincide with the orbits and the isotropy groups of the G-action, whereas $\mathrm{Bis}_{\mathrm{loc}}(G \times M) = \mathcal{C}^\infty_{\mathrm{loc}}(M, G)$, i.e. locally defined smooth functions on M with values in G. Moreover, many properties of the groupoid can be recovered from those of the group action:

- $G \times M$ is transitive/proper $\Leftrightarrow$ the G-action is transitive/proper,
- $G \times M$ is étale $\Leftrightarrow$ the group G is discrete,
- $G \times M$ is effective if the G-action is free and G is discrete (but not conversely).

$\Diamond$

Example 3.1.7. Given the vector bundle $E \to M$, its **general linear groupoid**

$$GL(E) \rightrightarrows M$$

is the set of the triples (x, ϕ_{xy}, y), where $\phi_{xy} : E_x \to E_y$ is a linear isomorphism. The source and target of (x, ϕ_{xy}, y) are, respectively, x and y, whereas the multiplication is given by $(y, \phi_{yz}, z)(x, \phi_{xy}, y) = (x, \phi_{yz}\phi_{xy}, z)$, the unit by $1_x = (x, id_{E_x}, x)$ and the inversion by $(x, \phi_{xy}, y)^{-1} = (y, \phi_{xy}^{-1}, x)$. It is transitive, with isotropy groups all isomorphic to $GL(k, \mathbb{R})$, where $k = \mathrm{rank}(E)$; in particular, if $E = M \times \mathbb{R}^k$ is trivial, then $GL(E)$ coincides with $M \times GL(k, \mathbb{R})$. $\Diamond$

3

Example 3.1.8. Let $\mathcal{G}$ be a Lie groupoid over M; its k^{th}**-jet groupoid** is the set of k-jets of its local bisections

$$J^k\mathcal{G} = \{ j^k_x\sigma \mid \sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}), x \in \mathrm{dom}(\sigma) \subseteq M \}.$$

Indeed, $J^k\mathcal{G}$ has a groupoid structure over M induced by the product in $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$:

$$j^k_{t(\rho(x))}\sigma \cdot j^k_x\rho = j^k_x(\sigma \cdot \rho).$$

On the other hand, the jet space $J^k(s)$ of $s : \mathcal{G} \to M$ (Definition 1.1.1) provides $J^k\mathcal{G} \subseteq J^k(s)$ with a topological and smooth structure which makes it into an open dense submanifold. Note also that the orbits of $J^k\mathcal{G}$ are the same as those of $\mathcal{G}$; it follows that $J^k\mathcal{G}$ is transitive if and only if $\mathcal{G}$ is. $\Diamond$

Example 3.1.9. The **pullback** of a groupoid $\mathcal{G} \rightrightarrows M$ w.r.t. a surjective submersion $\mu : P \to M$ is the fibred product

$$\mu^*(\mathcal{G}) = P \,{}_\mu\!\times_t \mathcal{G} \,{}_s\!\times_\mu P \rightrightarrows P,$$

whose arrows are triples (p, g, q), with source q and target p; the multiplication is $(r, h, p)(p, g, q) = (r, hg, q)$, the unit $(p, 1_{\mu(p)}, p)$ and the inverse $(p, g, q)^{-1} = (q, g^{-1}, p)$. There is a canonical groupoid homomorphism

$$\mu^*(\mathcal{G}) \to \mathcal{G}, \quad (p, g, q) \mapsto g,$$

which allows one to compare the pullback with the original groupoid. For instance, the isotropy group $\mu^*(\mathcal{G})_p$ is isomorphic to $\mathcal{G}_{\mu(p)}$ and the orbits of $\mu^*(\mathcal{G})$ are the μ-preimages of the orbits of $\mathcal{G}$; it follows that $\mu^*(\mathcal{G})$ is transitive if and only if $\mathcal{G}$ is so. $\Diamond$

Example 3.1.10. If $\mu : P \to M$ is a surjective submersion, the pullback of the unit groupoid $M \rightrightarrows M$ is the fibred pair groupoid $P \times_\mu P$ of Example 3.1.5.

Another special case is the **Čech groupoid** $M_{\mathcal{U}}$ of a (countable) open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of a manifold M. Denoting by $\mu : P = \coprod_{i \in I} U_i \to M$ the obvious submersion, then $M_{\mathcal{U}}$ is the μ-pullback of the unit groupoid $M \rightrightarrows M$, i.e.

$$M_{\mathcal{U}} = \{(i, x, j) \mid x \in U_i \cap U_j\} \rightrightarrows P. \qquad \lozenge$$

Example 3.1.11. Given a principal G-bundle $P \xrightarrow{\pi} M$, its **gauge groupoid** $\mathrm{Gauge}(P) = P \otimes_G P$ is the quotient of the pair groupoid of P with respect to the diagonal action of G, i.e.

$$(P \times P)/G \rightrightarrows P/G \cong M.$$

The arrows $[p, q]$ have source $[q]$ and target $[p]$, the multiplication is $[p, q][q, r] = [p, r]$, the unit $1_{[p]} = [p, p]$ and the inverse $[p, q]^{-1} = [q, p]$. Its isotropy groups are all isomorphic to G, and it is proper if and only if G is compact.

It inherits the transitivity from the pair groupoid. Actually, gauge groupoids exhaust all transitive Lie groupoids: given a transitive Lie groupoid $\mathcal{G} \rightrightarrows M$, fixing a point $x \in M$, the s-fibre $P = s^{-1}(x)$ is a principal bundle over M with structure group the isotropy group $G = \mathcal{G}_x$, and

$$\mathrm{Gauge}(P) \to \mathcal{G}, \quad [g, h] \mapsto g \cdot h^{-1}$$

becomes an isomorphism. This induces a bijective correspondence

$$\left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{transitive Lie groupoids over } M \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{principal bundles over } M \end{array} \right\}.$$

An interesting example is that of the general linear groupoid $GL(E)$ of $E \to M$, which corresponds to the frame bundle $Fr(E) \to M$ (a principal $GL(k, \mathbb{R})$-bundle, with $k = \mathrm{rank}(E)$).

Returning to the general discussion of the gauge groupoid $\mathcal{G} = \mathrm{Gauge}(P)$ of a principal G-bundle P, its bisections are related to the automorphisms of P as a principal G-bundle. Indeed, any such automorphism $\Phi : P \to P$ gives rise to the bisection

$$\sigma[p] = [\Phi(p), p];$$

it is not difficult to see now that this gives a bijective correspondence:

$$\left\{ \begin{array}{c} \text{(local) bisections} \\ \text{of the gauge groupoid } \mathcal{G} \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(local) automorphisms} \\ \text{of the principal } G\text{-bundle } P \end{array} \right\}. \qquad \lozenge$$

3.1.2 Actions and representations

We recall the notion of action of Lie groupoids. With the case of Lie groups in mind, the main novelty is that groupoids $\mathcal{G} \rightrightarrows M$ act on maps $\mu : P \to M$ rather than just on manifolds P. Or, if one wants to emphasise P, one first has to specify a smooth map $\mu : P \to M$, and then talk about actions of $\mathcal{G}$ on P along μ.

The action is given by specifying how an arbitrary arrow $g : x \to y$ of $\mathcal{G}$ acts along the fibres of μ, by maps

$$\mu^{-1}(x) \to \mu^{-1}(y), \quad p \mapsto g \cdot p.$$

To formalise this we consider the fibred product

$$\mathcal{G} \ {}_s\!\times_\mu P := \{(g, p) \in \mathcal{G} \times P \mid s(g) = \mu(p)\}.$$

Definition 3.1.12. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a smooth map $\mu : P \to M$, a (left) **action of the Lie groupoid** $\mathcal{G}$ on P along μ is a smooth map

$$m_P : \mathcal{G} \ {}_s\!\times_\mu P \to P, \quad (g, p) \mapsto g \cdot p$$

such that $t(g) = \mu(g \cdot p)$ and the usual action properties are satisfied: $1_{\mu(p)} \cdot p = p$ and $(gh) \cdot p = g \cdot (h \cdot p)$ for all the $p \in P, g, h \in \mathcal{G}$ for which these operations make sense. One says also that P is a $\mathcal{G}$**-space**, and the map μ is sometimes called **moment map** of the action. 3

Right actions are defined similarly, but swapping t and s. ♦

It is clear that an action of $\mathcal{G}$ on μ restricts to an action of the isotropy group $\mathcal{G}_x$ on the fibre $\mu^{-1}(x)$.

Example 3.1.13. Every Lie groupoid $\mathcal{G}$ acts on itself; indeed, the groupoid multiplication $\mathcal{G} \ {}_s\!\times_t \mathcal{G} \to \mathcal{G}$ can be seen both as a left action (with moment map t) and as a right action (with moment map s). ♢

As for group actions (Example 3.1.6), any groupoid action gives rise to an action groupoid with space of arrows the fibred product mentioned above

$$\mathcal{G} \ {}_s\!\times_\mu P \rightrightarrows P.$$

The transitivity/properness of the action is equivalent, by definition, to the transitivity/properness of the action groupoid. Similarly, the orbits and the isotropy groups of the action are the ones of the action groupoid, and so is the orbit space $P/\mathcal{G}$. The action is called **free** if

$$g \cdot p = p \Rightarrow g = 1_{\mu(p)} \quad \forall (g, p) \in \mathcal{G} \ {}_s\!\times_\mu P.$$

Definition 3.1.14. A **representation** of a groupoid $\mathcal{G}$ is a vector bundle $E \xrightarrow{\pi} M$ together with a (left) action of $\mathcal{G}$ along π which is fibrewise linear, in the sense that, for an arrow $g : x \to y$ of $\mathcal{G}$, the resulting bijection $g : E_x \to E_y$ is linear. Equivalently, it is a Lie groupoid map $\mathcal{G} \to GL(E)$.

If $\mathcal{H}$ is another groupoid over M, a $(\mathcal{G}, \mathcal{H})$**-vector bundle** is a representation $E \to M$ of $\mathcal{G}$ together with a fibrewise linear *right* action of $\mathcal{H}$ on E, commuting with the $\mathcal{G}$-action. ♦

We denote by $\mathrm{Rep}(\mathcal{G})$ the set of equivalence classes (in the obvious sense) of representations of $\mathcal{G}$.

Example 3.1.15. For a Lie group G interpreted as a Lie groupoid $G \rightrightarrows \{*\}$, we recover the usual notion of representation. In general, the fibre at $x \in M$ of any representation of $\mathcal{G} \rightrightarrows M$ becomes a representation of the isotropy $\mathcal{G}_x$, giving rise to a canonical map

$$\mathrm{ev}_x : \mathrm{Rep}(\mathcal{G}) \to \mathrm{Rep}(\mathcal{G}_x), \quad E \to E_x.$$

For transitive groupoids, the situation becomes even more familiar when described from the perspective of principal bundles. Indeed, if $\mathcal{G} = \mathrm{Gauge}(P)$, for a principal G-bundle $\pi : P \to M$ (see the 1-1 correspondence discussed in Example 3.1.11), then the evaluation ev_x is a bijection, with inverse given by the associated bundle (recalled in Remark 2.1.11):

$$\mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Rep}\,(\mathcal{G})\,, \quad V \mapsto P[V] := (P \times V)/G.$$

Here is how representations look like in a few other simple classes of groupoids:

- For the unit groupoid $M \rightrightarrows M$ we are looking at arbitrary vector bundles over M.
- For the pair groupoid $M \times M \rightrightarrows M$ we are looking at vector bundles over M together with a trivialisation.
- For the action groupoid $G \times M \rightrightarrows M$ we are looking at G-equivariant vector bundles over M. ♢

3.1.3 Principal bundles

Finally, we recall the notion of principal bundles w.r.t. a Lie groupoid $\mathcal{G} \rightrightarrows M$. As in the case of Lie groups, they correspond to right actions which are proper and free; by the usual arguments, such an action of $\mathcal{G}$ on $\mu : P \to M$ gives rise to a smooth quotient $P/\mathcal{G}$, making the canonical projection $P \to P/\mathcal{G}$ into a submersion. Or, if one wants to emphasise the base $B = P/\mathcal{G}$ of the bundle:

Definition 3.1.16. A (right) **principal $\mathcal{G}$-bundle** over B is a (right) $\mathcal{G}$-space P w.r.t. some map $\mu : P \to M$, together with a $\mathcal{G}$-invariant surjective submersion $\pi : P \to B$, such that the map

$$P\ {}_t\!\times_\mu \mathcal{G} \to P \times_\pi P, \quad (p, g) \mapsto (p \cdot g, p)$$

is a diffeomorphism. By $\mathcal{G}$-invariance we mean $\pi(p \cdot g) = \pi(p)$ for all $(p, g) \in P\ {}_t\!\times_\mu \mathcal{G}$. Left principal bundles are defined similarly. ♦

We represent the situation above by the following diagram:

$$\begin{tikzcd} & P \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \curvearrowleft & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ B & & & M \end{tikzcd}$$

We will often further assume (without saying it) that μ is a surjective submersion.

Example 3.1.17. One can proceed with a discussion similar to that from Example 3.1.15. The most interesting aspect is that for transitive Lie groupoids $\mathcal{G}$ over M, choosing $x \in M$, one obtains a bijective correspondence

$$\left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{principal } \mathcal{G}\text{-bundles over } B \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{principal } \mathcal{G}_x\text{-bundles over } B \end{array} \right\},$$

which sends P to its fibre $P_x := \mu^{-1}(x)$. ♢

The **gauge groupoid** $\mathrm{Gauge}(P) = P \otimes_{\mathcal{G}} P$ (Example 3.1.11) can be defined also for a principal $\mathcal{G}$-bundle $\pi : P \to B$, as long as the moment map $\mu : P \to M$ is a surjective submersion: it is the quotient

$$(P \times_\mu P)/\mathcal{G} \rightrightarrows P/\mathcal{G} \cong B$$

of the fibred pair groupoid $P \times_\mu P$ (Example 3.1.5) by the diagonal $\mathcal{G}$-action.

The same holds for the vector bundle associated to a principal bundle and a representation (similarly to Example 3.1.15):

Proposition 3.1.18. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\pi : P \to B$ be a (left) principal $\mathcal{G}$-bundle with moment map $\mu : P \to M$. There is a map*

$$\Phi : \mathrm{Rep}(\mathcal{G}) \to \mathrm{Rep}(\mathrm{Gauge}(P)), \quad E \mapsto P[E] := (P \times_M E)/\mathcal{G},$$

uniquely characterised by the fact that, for any representation E,

$$\mu^* E \to \pi^* \Phi(E), \quad (p, e) \mapsto (p, [p, e])$$

is an isomorphism of $(\mathcal{G}, \mathrm{Gauge}(P))$-vector bundles over P (see Definition 3.1.14). Here $\mu^* E$ and $\pi^* \Phi(E)$ are viewed as $(\mathcal{G}, \mathrm{Gauge}(P))$-vector bundles by considering the following $\mathcal{G}$-actions

$$\mathcal{G} \times_M \mu^* E \to \mu^* E, \quad g \cdot (p, e) = (g \cdot p, g \cdot e),$$

$$\mathcal{G} \times_M \pi^* F \to \pi^* F, \quad g \cdot (p, f) = (g \cdot p, f),$$

and similar actions of $\mathrm{Gauge}(P)$ from the right.

Furthermore, when $\mathcal{G}$ is transitive, the correspondence Φ is bijective, as a consequence of Examples 3.1.15 and 3.1.17.

In particular, by combining the previous proposition with Example 3.1.17, one obtains the following useful result:

Lemma 3.1.19. *Let $\mathcal{G}$ be a transitive Lie groupoid on M, $E \to M$ a representation of $\mathcal{G}$ and P a principal $\mathcal{G}$-bundle over B with moment map $\mu : P \to M$. For any $x \in M$, the associated bundle*

$$P[E] := (P \times_M E)/\mathcal{G}$$

is isomorphic to the associated bundle

$$P_x[E_x] := (P_x \times E_x)/\mathcal{G}_x,$$

where $E_x \in \mathrm{Rep}(\mathcal{G}_x)$ is the fibre of E over x and the principal $\mathcal{G}_x$-bundle $P_x \to B$ is the μ-fibre of P over x.

For Lie groups G, principal G-bundles are locally trivial and, therefore, can be described using transition functions. Moving to Lie groupoids, the triviality is more subtle: this is due to the fact that the base of $\mathcal{G}$ is a manifold M and no longer just a point. Accordingly, one has a trivial $\mathcal{G}$-bundle over B for any smooth map $f : B \to M$:

Definition 3.1.20. The **trivial $\mathcal{G}$-bundle over** B associated to a smooth map $f : B \to M$ is the fibred product

$$P_f^{triv} := \{(x, g) \in B \times \mathcal{G} \mid f(x) = t(g)\}$$

endowed with the obvious $\mathcal{G}$-action $(x, g) \cdot h = (x, gh)$. ♦

Given a principal $\mathcal{G}$-bundle $\pi : P \to B$ as above, choosing a local section $\sigma : U \to P$ for every open $U \subseteq B$, one obtains a smooth map

$$f_\sigma := \mu \circ \sigma : U \to M$$

and an isomorphism between $P|_U$ and the corresponding trivial bundle

$$P|_U \xrightarrow{\cong} P_{f_\sigma}^{triv}, \quad \sigma(x)g \leftarrow\!\shortmid (x, g).$$

Looking at what happens on overlaps (when changing σ) one arrives at the notion of cocycle.

Definition 3.1.21. A **$\mathcal{G}$-cocycle** on B consists of:

- a collection of maps $f_\alpha : U_\alpha \to M$ with $\{U_\alpha\}_\alpha$ a (countable) open cover of B,
- a collection of maps $\phi_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{G}$ on the intersections $U_{\alpha\beta} := U_\alpha \cap U_\beta$,

such that:

- for every $x \in U_{\alpha\beta}$, $\phi_{\alpha\beta}(x) \in \mathcal{G}$ has source $f_\beta(x)$ and target $f_\alpha(x)$,
- on triple intersections $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, the cocycle identity holds:

$$\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x) \quad \forall x \in U_{\alpha\beta\gamma}. \qquad \blacklozenge$$

Note that, in this case, one also has $\phi_{\alpha\alpha}(x) = 1_{f_\alpha(x)}$ and $\phi_{\alpha\beta}(x)^{-1} = \phi_{\beta\alpha}(x)$. Equivalently, one can also use the Čech groupoid $B_\mathcal{U}$ associated to $\mathcal{U} := \{U_\alpha\}_\alpha$ (Example 3.1.10) to pack together the previous data into a groupoid morphism

$$\phi : B_\mathcal{U} \to \mathcal{G}, \quad (\alpha, x, \beta) \mapsto \phi_{\alpha\beta}(x)$$

covering $(x, \alpha) \mapsto f_\alpha(x)$.

As for Lie groups, one says that two cocycles $\phi_{\alpha\beta}$ and $\phi'_{\alpha\beta}$ (on different open covers) are equivalent if there is a common refinement where the two cocycles are cohomologous, i.e. $\phi'_{\alpha\beta}(x) = F_\alpha(x)\phi_{\alpha\beta}(x)F_\beta(x)$ for some functions $F_\alpha : U_\alpha \to \mathcal{G}$. Then one obtains:

Proposition 3.1.22 (Proposition I.3.1 of [75]). *For every Lie groupoid $\mathcal{G} \rightrightarrows M$ and any manifold B one has a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{(isomorphism classes of)} \\ \textit{principal } \mathcal{G}\textit{-bundles over } B \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{(equivalence classes of)} \\ \mathcal{G}\textit{-cocycles on } B \end{array} \right\}.$$

Explicitly, the principal bundle corresponding to a $\mathcal{G}$-cocycle $\{(f_\alpha, \phi_{\alpha\beta})\}_{\alpha\beta}$ is

$$P = \coprod_\alpha P^{triv}_{f_\alpha} / \sim$$

where two elements $(x, g) \in P^{triv}_{f_\alpha}$ and $(x', g') \in P^{triv}_{f_\beta}$ are equivalent if and only if $x' = x \in U_{\alpha\beta}$ and $g' = \phi_{\alpha\beta}(x)g$. The projection to B is $\pi : [x, g] \mapsto x$, the moment map is $\mu : [x, g] \mapsto f_\alpha(x)$ and the action is $[x, g] \cdot h = [x, gh]$.

3.2 Lie algebroids

We now pass to the infinitesimal counterpart of Lie groupoids: a **Lie algebroid** over a manifold M is a vector bundle $A \to M$ together with a vector bundle map $\rho : A \to TM$ (called **anchor**) and a Lie algebra structure $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$ satisfying the Leibniz rule

$$[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad \forall X, Y \in \Gamma(A), f \in \mathcal{C}^\infty(M).$$

It follows from the definition that ρ induces a Lie algebra homomorphism

$$\rho : \Gamma(A) \to \mathfrak{X}(M)$$

where the space of vector fields $\mathfrak{X}(M) := \Gamma(TM)$ is endowed with the usual Lie bracket.

Similarly to the Lie algebra of a Lie group, one can talk about the **Lie algebroid** $A = \mathrm{Lie}(\mathcal{G})$ **of a Lie groupoid** $\mathcal{G} \rightrightarrows M$. As vector bundle, it is

$$A := u^* \ker(ds) = \coprod_{x \in M} T_{1_x} s^{-1}(x) \subseteq T\mathcal{G}.$$

The anchor map at $x \in M$ is given by

$$\rho_x = d_{1_x}(t_{|s^{-1}(x)}) : A_x \to T_x M.$$

To describe the Lie bracket on $\Gamma(A)$ we will make use of right translations. More precisely, the right translation by an element $g \in \mathcal{G}$ defines a diffeomorphism between s-fibres:

$$R_g : s^{-1}(t(g)) \to s^{-1}(s(g)), \quad h \mapsto hg;$$

the differential of R_g at $1_{t(g)}$, denoted still by R_g, is then a linear isomorphism between the vertical s-spaces

$$R_g : T^s_h \mathcal{G} \to T^s_{hg} \mathcal{G}.$$

An s-vertical vector field $X \in \mathfrak{X}^s(\mathcal{G}) := \Gamma(T^s\mathcal{G})$ is called **right-invariant** if

$$R_g(X_h) = X_{hg} \quad \forall (h, g) \in \mathcal{G}^{(2)}.$$

One can easily see that right-invariant vector fields form a Lie subalgebra

$$\mathfrak{X}^s_{inv}(\mathcal{G}) \subseteq \Gamma(T^s\mathcal{G})$$

and that there is an isomorphism

$$\Gamma(A) \to \mathfrak{X}^s_{inv}(\mathcal{G}), \quad \alpha \mapsto \alpha^R, \qquad \text{with } (\alpha^R)_g := R_g(\alpha_{t(g)}).$$

Therefore, one obtains the desired Lie bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$, uniquely characterised by

$$([\alpha, \beta]_A)^R = [\alpha^R, \beta^R] \quad \forall \alpha, \beta \in \Gamma(A).$$

The (local) **flow** of a (local) section $\alpha \in \Gamma(A)$ is the 1-parameter family of (local) bisections

$$\phi^\varepsilon_\alpha : M \to \mathcal{G}, \quad x \mapsto \phi^\varepsilon_{\alpha^R}(1_x)$$

defined using the flow of the right-invariant vector field α^R; by construction, it sends a point $x \in M$ into an arrow in $\mathcal{G}$ from x to $\phi^\varepsilon_{\rho(\alpha)}(x)$.

Remark 3.2.1. Not all Lie algebroids arise from a Lie groupoid; those which do are called **integrable**, and in that case there is always a unique s-simply connected Lie groupoid which integrates the algebroid (as in the analogous result for Lie groups, known as Lie I theorem). See [23] for several results on the integrability of Lie algebroids to Lie groupoids and further references. ♢

Returning to a general Lie algebroid $A \to M$, its **isotropy Lie algebra** at the point $x \in M$ is the kernel of the anchor map at x:

$$\mathfrak{g}_x(A) := \ker(\rho_x).$$

When $A = \mathrm{Lie}(\mathcal{G})$ comes from a Lie groupoid $\mathcal{G}$, then $\mathfrak{g}_x(A) = \mathrm{Lie}(\mathcal{G}_x)$ is the Lie algebra of the isotropy group $\mathcal{G}_x$.

An **orbit** of a Lie algebroid $A \to M$ is any connected immersed submanifold $\mathcal{O} \subseteq M$ satisfying

$$T_x\mathcal{O} = \rho_x(A_x) \subseteq T_xM$$

for all $x \in \mathcal{O}$, and maximal with these properties. The existence of such orbits through each point, and the fact that they form a partition of M, is not trivial, and requires for instance the Stefan-Sussmann theorem for singular foliations. The most transparent case is when the anchor has constant rank; then the image of the anchor $\rho(A) \subseteq TM$ defines a smooth distribution which is involutive; by Frobenius theorem, $\rho(A)$ is a foliation on M, and then the orbits of A are the resulting leaves. In general, if $A = \mathrm{Lie}(\mathcal{G})$ comes from a Lie groupoid, the orbits of A coincide with the connected components of the orbits of $\mathcal{G}$.

Some interesting special classes of algebroids are the ones that are **regular**, in the sense that the anchor has constant rank or, even more restrictively, the ones that are **transitive**, in the sense that the anchor is surjective.

3

3.2.1 Examples

Example 3.2.2. A **Lie algebra** $\mathfrak{g}$ is a Lie algebroid over a point, with trivial anchor and its own bracket on $\Gamma(\mathfrak{g}) = \mathfrak{g}$. If $\mathfrak{g} = \mathrm{Lie}(G)$, it is the Lie algebroid of the Lie groupoid $G \rightrightarrows \{*\}$ (Example 3.1.2). ♢

Example 3.2.3. The **tangent bundle** $A = TM$ of a manifold M is a Lie algebroid with anchor $\rho = id_{TM}$ and Lie bracket on $\Gamma(A) = \mathfrak{X}(M)$ given by standard bracket of vector fields. It is characterised by the fact that it is transitive and all the isotropy Lie algebras are trivial; it is the Lie algebroid of the pair groupoid $M \times M \rightrightarrows M$ (Example 3.1.4). ♢

Example 3.2.4. If $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ is an (infinitesimal) action of a Lie algebra $\mathfrak{g}$ on a manifold M, its **action algebroid** is the trivial vector bundle

$$\mathfrak{g} \times M \to M$$

with anchor $\rho : \mathfrak{g} \times M \to TM$ given by the action map ρ, and Lie bracket on $\mathcal{C}^\infty(M, \mathfrak{g})$ uniquely determined by the Leibniz identity and the following condition on constant sections $c_u \in C^\infty(M, \mathfrak{g})$ (corresponding to $u \in \mathfrak{g}$):

$$[c_u, c_v] = c_{[u,v]_\mathfrak{g}}.$$

When $\mathfrak{g}$ is the Lie algebra of a Lie group G and the Lie algebra action is the infinitesimal counterpart of a G-action on M, it is not surprising (and it is not difficult to see) that $\mathfrak{g} \times M$ is the Lie algebroid of the action groupoid $G \times M \rightrightarrows M$ (Example 3.1.6). $\diamondsuit$

Example 3.2.5. Let $\pi : A \to M$ be a Lie algebroid, with anchor and bracket denoted ρ_A and $[\cdot, \cdot]_A$, respectively. Its k^{th}**-jet algebroid** is the space of k-jets of its sections

$$J^k A := \{j^k_x \alpha \mid \alpha \in \Gamma(A), x \in \mathrm{dom}(\alpha)\}.$$

Its structure of Lie algebroid over M is induced by that of A: the bracket is uniquely determined by

$$[j^k \alpha, j^k \beta] := j^k [\alpha, \beta]_A,$$

and the anchor is

$$\rho : J^k A \to TM, \quad j^k_x \alpha \mapsto \rho_A(\alpha(x)).$$

If A integrates to a Lie groupoid $\mathcal{G}$, one sees easily that the Lie algebroid $J^k A$ coincides with the Lie algebroid of the Lie groupoid $J^k \mathcal{G}$ (Example 3.1.8). $\diamondsuit$

Example 3.2.6. Given a principal G-bundle $\pi : P \to M$, the **Atiyah algebroid** of P is the quotient of the tangent bundle $TP \to P$ by the G-action

$$A(P) := TP/G \to P/G = M.$$

It has anchor $[d\pi] : TP/G \to TM$ and Lie bracket arising from the standard Lie bracket of vector fields on P, via the canonical identification with G-equivariant vector fields

$$\Gamma(A(P)) \cong \mathfrak{X}^G(P).$$

This algebroid is transitive and all its isotropy Lie algebras are isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$. Again, it is not surprising (and it is not difficult to see) that $A(P)$ is the Lie algebroid of the gauge groupoid $\mathrm{Gauge}(P)$ (Example 3.1.11). $\diamondsuit$

Example 3.2.7. If (M, π) is a Poisson manifold, the **cotangent Lie algebroid** of M is the vector bundle $T^*M \to M$ with anchor

$$\pi^\# : T^*M \to TM, \quad \alpha \mapsto \pi(\alpha, \cdot)$$

and Lie bracket

$$[\alpha, \beta] := \mathcal{L}_{\pi^\#(\alpha)} \beta - \mathcal{L}_{\pi^\#(\beta)} \alpha - d\langle \alpha, \pi^\#(\beta) \rangle,$$

where we denote by $\langle \cdot, \cdot \rangle$ the pairing between cotangent and tangent bundle. When T^*M is integrable, its s-simply connected integration $\Sigma \rightrightarrows M$ can be given a structure of symplectic groupoid (see Example 3.4.8 later). $\diamondsuit$

3.2.2 Actions and representations

Next, we discuss the infinitesimal counterpart of groupoid actions.

Definition 3.2.8. Let $A \to M$ be a Lie algebroid and $\mu : P \to M$ a smooth map. A **Lie algebroid action** of A on P w.r.t. μ is a Lie algebra morphism

$$a : \Gamma(A) \to \mathfrak{X}(P)$$

such that, for every $p \in P$, $X \in \Gamma(A)$, $f \in \mathcal{C}^\infty(M)$,

$$d_p\mu(a(X)_p) = \rho_{\mu(p)}(X_{\mu(p)}), \qquad a(f \cdot X) = (f \circ \mu) \cdot a(X). \qquad \blacklozenge$$

For a Lie algebra $\mathfrak{g} \to \{*\}$, the conditions above are trivially satisfied, and one recovers the usual notion of Lie algebra action on a manifold.

Example 3.2.9. The main source of Lie algebroid actions is represented by Lie groupoid actions. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and m_P be a left $\mathcal{G}$-action on $\mu : P \to M$ (Definition 3.1.12); its induced **infinitesimal (left) action** is the Lie algebroid action of $A = \mathrm{Lie}(\mathcal{G})$ on P defined by:

$$a : \Gamma(A) \to \mathfrak{X}(P), \quad a(\alpha)_p := d_{1_{\mu(p)}} m_P(\cdot, p)(\alpha_{\mu(p)}).$$

It is easy to check that, if the $\mathcal{G}$-action is free, the induced A-action is injective.

We recall also an alternative way to compute the differential of a groupoid action m_P (which will be handy in several computations in the following chapters): for every $\alpha \in \ker(d_g s)$, $\beta \in \ker(d_p\mu)$,

$$d_g m_P(\cdot, p)(\alpha) = d_{(g,p)} m_P(\alpha, 0), \qquad d_p m_P(g, \cdot)(\beta) = d_{(g,p)} m_P(0, \beta).$$

In particular, the first equation gives an equivalent definition of infinitesimal actions:

$$a(\alpha)_p = d_{(1_{\mu(p)}, p)} m_P(\alpha_{\mu(p)}, 0).$$

If m_P is a right $\mathcal{G}$-action, the infinitesimal action is defined analogously as

$$a(\alpha)_p := d_{1_{\mu(p)}}(m_P(p, \cdot) \circ i)(\alpha_{\mu(p)}).$$

Recall that an action of the Lie algebra $\mathfrak{g}$ on M induces the action algebroid $\mu^*\mathfrak{g} = \mathfrak{g} \times M$ from Example 3.2.4; similarly, if we have a Lie algebroid action of A on P, the vector bundle $\mu^*A \to P$ can be provided with a Lie algebroid stucture such that $a : \mu^*A \to TP$ is a morphism between Lie algebroids over P (Proposition 4.2 of [68]). When the action comes from a Lie groupoid $\mathcal{G}$, then μ^*A is clearly the Lie algebroid of the action groupoid $\mathcal{G}\ {}_s\times_\mu P$. $\lozenge$

The notion of representation of a Lie groupoid (Definition 3.1.14) has also an infinitesimal counterpart.

Definition 3.2.10. Given a Lie algebroid A over M, an **A-connection** on a vector bundle $E \to M$ is a map

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$$

which is $\mathcal{C}^\infty(M)$-linear in the first argument, $\mathbb{R}$-linear in the second, and satisfies the Leibniz rule

$$\nabla_\alpha(f\sigma) = f\nabla_\alpha(\sigma) + \mathcal{L}_{\rho(\alpha)}(f)\sigma \quad \forall \alpha \in \Gamma(A), \sigma \in \Gamma(E), f \in \mathcal{C}^\infty(M),$$

where $\rho : A \to TM$ is the anchor of A. The **curvature tensor** of ∇ is

$$R_\nabla \in \mathrm{Hom}(\Lambda^2 A, \mathrm{Hom}(E, E)), \quad R_\nabla(\alpha, \beta) := \nabla_{[\alpha,\beta]} - [\nabla_\alpha, \nabla_\beta],$$

and ∇ is called **flat** if $R_\nabla = 0$. A **representation** of A is a vector bundle $E \to M$ together with a flat A-connection ∇. ♦

Lemma 3.2.11. *Let $E \to M$ be a representation of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and $A = \mathrm{Lie}(\mathcal{G})$ its Lie algebroid. Then there is an induced representation of A on E*

$$\nabla_\alpha \sigma(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\phi^\varepsilon_\alpha(x))^{-1} \cdot \sigma(t(\phi^\varepsilon_\alpha(x)))$$

where $\alpha \in \Gamma(A), \sigma \in \Gamma(E), x \in M$ and $\cdot$ denotes the $\mathcal{G}$-action on E.

Example 3.2.12. If A is the tangent algebroid TM from Example 3.2.3, we recover the standard notion of connection on a vector bundle on M, and the standard notion of flatness.

For a Lie algebra $\mathfrak{g}$ interpreted as a Lie algebroid (Example 3.2.2), an A-connection ∇ is just a linear map $\mathfrak{g} \to \mathfrak{gl}(V)$ and the flatness condition makes it into a Lie algebra map, recovering the usual notion of representation.

In full generality, for any vector bundle $E \to M$ one can define a Lie algebroid $\mathfrak{gl}(E) \to M$ such that representations of any Lie algebroid $A \to M$ on E correspond to Lie algebroid morphisms $A \to \mathfrak{gl}(E)$. Clearly, if E comes from a representation of a Lie groupoid $\mathcal{G}$ via Lemma 3.2.11, then the Lie algebroid morphism is the differential of the corresponding Lie groupoid morphism $\mathcal{G} \to GL(E)$ (Definition 3.1.14). ♢

3.2.3 Basic forms on principal bundles

The infinitesimal counterpart of a Lie groupoid action allows us to describe a special class of differential forms on principal $\mathcal{G}$-bundles $\pi : P \to B$: the basic ones. First, let us recall the following characterisation of the **π-vertical vector fields** on P.

Proposition 3.2.13. *For any principal $\mathcal{G}$-bundle $P \xrightarrow{\pi} B$, the π-vertical bundle of P coincides with the image of the infinitesimal action $a : A \to TP$ (Example 3.2.9),*

$$\ker(d\pi) = \mathrm{Im}(a).$$

This yields a short exact sequence of vector bundles

$$0 \to \mathrm{Im}(a) \hookrightarrow TP \xrightarrow{d\pi} TB \to 0.$$

Dual to π-vertical vector fields, for any fibration $\pi : P \to B$ one can talk about **π-horizontal (or semi-basic) differential forms** ω on P, possibly with coefficients; these forms are characterised by the condition

$$\iota_X \omega = 0 \quad \forall X \in \mathfrak{X}^\pi(P).$$

For principal bundles, given the previous proposition, the horizontality of ω can be rephrased as

$$\iota_{a(\alpha)} \omega = 0 \quad \forall \alpha \in \Gamma(A).$$

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Definition 3.2.14. A **basic form** on a (left) principal $\mathcal{G}$-bundle $\pi : P \to B$ with coefficients in a representation $E \to M$ of $\mathcal{G}$ is a horizontal form $\theta \in \Omega^k(P, \pi^* E)$ which is also $\mathcal{G}$-equivariant, i.e.

$$\theta_{g \cdot p}([d_{(p,g)} m_P(\tilde{X})]) = g \cdot \theta_p([X]) \quad \forall (g,p) \in \mathcal{G}\ {}_s\!\times_\mu P, X \in T_p P.$$

where $\tilde{X}$ is any vector in $T_{(g,p)}(\mathcal{G} \times P)$ such that $dpr_2(\tilde{X}) = X$.

To make sense of the formula above, recall (e.g. from Section 8.8 of [100]) that any horizontal form θ can be interpreted as a form on the normal bundle of any orbit $\mathcal{O} \subseteq P$ of the action groupoid $\mathcal{G} \times P$:

$$\theta : N\mathcal{O} \to (\mu^* E)_{|\mathcal{O}}, \quad [X] \mapsto \theta_p(X). \qquad \blacklozenge$$

There is also another characterisation of basic forms, which will be useful in the later chapters.

Lemma 3.2.15 (Proposition 8.8.3 of [100]). *A form $\theta \in \Omega^k(P, \mu^* E)$ is $\mathcal{G}$-basic if and only if*

$$g \cdot (pr_2^* \theta)_{(g,p)} = (m^* \theta)_{(g,p)} \quad \forall (g,p) \in \mathcal{G}\ {}_s\!\times_\mu P.$$

Recall that, for principal G-bundles, the space of basic forms with coefficients in $V \in \mathrm{Rep}(G)$ is canonically isomorphic to the space of forms on the base B with coefficients in the associated vector bundle $P[V] = (P \times V)/G$ (see Remark 2.1.11):

$$\Omega^k(B, P[V]) \xrightarrow{\cong} \Omega^k_{bas}(P, V) \quad \omega \mapsto \pi^* \omega.$$

Lemma 3.2.15 is used to generalise such a result in the setting of $\mathcal{G}$-bundles:

Proposition 3.2.16. *Let $\pi : P \to B$ be a principal $\mathcal{G}$-bundle, $E \in \mathrm{Rep}(\mathcal{G})$ and $P[E]$ the associated vector bundle over B from Proposition 3.1.18. Then the pull-back from B to P induces an isomorphism*

$$\Omega^k(B, P[E]) \xrightarrow{\cong} \Omega^k_{bas}(P, \pi^* E), \quad \omega \mapsto \pi^*\omega.$$

The proof in the case $k = 1$ can be found in Section 8.8 of [100], but it can be directly generalised to forms of any degree.

3.3 Morita equivalences

The obvious notion of isomorphism between Lie groupoids is too strict; a more useful notion is that of Morita equivalence.

Definition 3.3.1. A **Morita equivalence** between the Lie groupoids $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ is a manifold P together with two surjective submersions $\mu_1 : P \to M_1$ and $\mu_2 : P \to M_2$

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll, "\mu_1"'] \arrow[drr, "\mu_2"] & \curvearrowleft & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2
\end{tikzcd}$$

and two actions, a left action of $\mathcal{G}_1$ along μ_1 and a right action of $\mathcal{G}_2$ along μ_2, such that P is a principal bi-bundle. This means that

- μ_2 is a left principal $\mathcal{G}_1$-bundle,
- μ_1 is a right principal $\mathcal{G}_2$-bundle,
- the two actions commute, i.e. $g_1 \cdot (p \cdot g_2) = (g_1 \cdot p) \cdot g_2$ for all $g_i \in \mathcal{G}_i$ and $p \in P$ for which the compositions are defined. ⧫

It is well-known (see e.g. Proposition 5.12 of [73] and the remark thereafter) that Morita equivalence is an equivalence relation. For reflexivity, see Example 3.1.13; for transitivity, given two Morita equivalences

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll, "\mu_1"'] \arrow[drr, "\mu_2"] & \curvearrowleft & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & Q \arrow[dll, "\tau_1"'] \arrow[drr, "\tau_2"] & \curvearrowleft & \mathcal{G}_3 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2 & & & & M_3
\end{tikzcd}$$

the product bundle $(P \times_{M_2} Q)/\mathcal{G}_2$ becomes a Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_3$.

Lemma 3.3.2. *Any Morita equivalence P between $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ induces:*

- *A canonical homeomorphism $M_1/\mathcal{G}_1 \to M_2/\mathcal{G}_2$ between the orbit spaces, associating to an orbit $\mathcal{O}_1 \subseteq M_1$ the orbit $\mathcal{O}_2 \subseteq M_2$ uniquely determined by $\mu_1^{-1}(\mathcal{O}_1) = \mu_2^{-1}(\mathcal{O}_2)$.*

- *For any $p \in P$, an isomorphism between the isotropy groups $(\mathcal{G}_1)_{\mu_1(p)}$ and $(\mathcal{G}_2)_{\mu_2(p)}$.*

Example 3.3.3. An isomorphism (F, f) between two Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ can be turned into a Morita equivalence by considering the trivial principal $\mathcal{G}_2$-bundle P_f^{triv} (Definition 3.1.20) and using F to make it into a principal $\mathcal{G}_1$-bundle on M_2. $\Diamond$

Example 3.3.4. Let $f : N \to M$ be a submersion. Every Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to the pullback groupoid $f^*\mathcal{G} \rightrightarrows N$ (Example 3.1.9). As bibundle P we take again the trivial principal $\mathcal{G}$-bundle P_f^{triv} (Definition 3.1.20), endowed with the obvious actions. For instance, we get the following consequences:

- The fibred pair groupoid $N \times_f N$ (Example 3.1.5) is Morita equivalent to the unit groupoid of M (Example 3.1.3).

- If $\mathcal{U}$ and $\mathcal{V}$ are two open covers of M, their associated Čech groupoids (Example 3.1.10) are Morita equivalent (being Morita equivalent to the same unit groupoid $M \rightrightarrows M$). $\Diamond$

Example 3.3.5. Here are other immediate examples of Morita equivalences.

- Two Lie groups are Morita equivalent (as Lie groupoids) if and only if they are isomorphic (as Lie groups).

- Two unit groupoids $M \rightrightarrows M$ and $N \rightrightarrows N$ are Morita equivalent if and only if M and N are diffeomorphic.

- Every pair groupoid $M \times M \rightrightarrows M$ is Morita equivalent to the trivial group $\{e\} \rightrightarrows \{*\}$. Indeed, $M \times M$ acts trivially on $P = M$ along id_M by $(x, y) \cdot y = x$, so M is both a $(M \times M)$- and an $\{e\}$-principal bundle.

- Any transitive Lie groupoid $\mathcal{G} \rightrightarrows M$ is Morita equivalent to its isotropy group $\mathcal{G}_x$ at any point $x \in M$. Such Morita equivalence is given explicitly by $P = s^{-1}(x)$, which is both a principal $\mathcal{G}$-bundle over $\{*\}$ and a principal $\mathcal{G}_x$-bundle over M (see Example 3.1.11). $\Diamond$

Example 3.3.6. Let $P \to N$ be a left principal $\mathcal{G}$-bundle, whose moment map is a surjective submersion; then $\mathcal{G}$ is Morita equivalent to the gauge groupoid $P \otimes_{\mathcal{G}} P$ (Example 3.1.11). Indeed, $P \otimes_{\mathcal{G}} P$ acts on $P \to N$ from the right in a principal way as

$$p \cdot [p, q] = q,$$

and P becomes a principal bi-bundle, since this right action is compatible with the previous left $\mathcal{G}$-action:

$$(g \cdot p) \cdot [p, q] = (g \cdot p) \cdot [g \cdot p, g \cdot q] = g \cdot q = g \cdot (p \cdot [p, q]). \qquad \Diamond$$

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The following proposition tells us that any principal $\mathcal{G}$-bundle P can be uniquely "complemented" to a Morita equivalence via its gauge groupoid.

Proposition 3.3.7. *Let P be a Morita equivalence between $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$. Then we have the isomorphisms of Lie groupoids*

$$\mathcal{G}_2 \cong \mathrm{Gauge}_{\mathcal{G}_1}(P), \qquad \mathcal{G}_1 \cong \mathrm{Gauge}_{\mathcal{G}_2}(P).$$

Proof. We describe explicitly an isomorphism $P \otimes_{\mathcal{G}_1} P \to \mathcal{G}_2$, associating to each equivalence class $[(p,q)] \in P \otimes_{\mathcal{G}_1} P$ the unique element $g(p,q) \in \mathcal{G}_2$ such that $q = g(p,q) \cdot p$; the other isomorphism is analogous, swapping 1 and 2.

This is possible because, for a representative $(p,q) \in P \times_{\mu_1} P$, we have $\mu_1(p) = \mu_1(q)$, i.e. p and q are in the same μ_1-fibre; μ_1 being a principal $\mathcal{G}_2$-bundle, $\mathcal{G}_2$ acts freely and transitively on each μ_1-fibre, therefore such $g(p,q)$ is unique. Moreover, since μ_1 is $\mathcal{G}_2$-invariant, for every other representative $(p',q') \in P \times_{\mu_1} P$, p' and q' are in the same fibre of p and q, hence are linked by the same $g(p,q) \in \mathcal{G}_2$. Q.E.D.

3.3.1 Morita morphisms

Lie groupoids, together with Lie groupoid morphisms, form a category. The notion of Morita morphism extends the morphisms of this category in such a way that

- Lie groupoid (iso)morphisms become a special case of Morita (iso)morphisms,
- Morita isomorphisms are Morita equivalences.

Definition 3.3.8. A **Morita morphism** $P : \mathcal{G}_1 \rightsquigarrow \mathcal{G}_2$ from a Lie groupoid $\mathcal{G}_1 \rightrightarrows M_1$ to a Lie groupoid $\mathcal{G}_2 \rightrightarrows M_2$ (or between $\mathcal{G}_1$ and $\mathcal{G}_2$) is given by a principal $\mathcal{G}_2$-bundle $P \xrightarrow{\mu_1} M_1$, with $\mathcal{G}_2$-action along $P \xrightarrow{\mu_2} M_2$, together with a $\mathcal{G}_1$-action on P along μ_1, such that the two actions commute. ♦

In other words, a Morita morphism is a Morita equivalence without the condition that μ_2 is a principal bundle:

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll, "\mu_1"'] \arrow[drr, "\mu_2"] & \curvearrowleft & \mathcal{G}_2 \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2
\end{tikzcd}$$

A composition of two Morita morphisms is well-defined: given $P : \mathcal{G}_1 \rightsquigarrow \mathcal{G}_2$ and $Q : \mathcal{G}_2 \rightsquigarrow \mathcal{G}_3$, $(P \times_{M_2} Q)/\mathcal{G}_2$ is a Morita morphism between $\mathcal{G}_1$ and $\mathcal{G}_3$. Moreover, any groupoid $\mathcal{G}$ defines a Morita identity morphism, which allows us to talk about Morita inverses and Morita isomorphisms, and to prove the following result:

Proposition 3.3.9 (Corollary II.1.7 of [75])**.** *A Morita morphism P is an isomorphism (in the Morita category) if and only if P is a Morita equivalence.*

Example 3.3.10. Let (F, f) be a morphism between two Lie groupoids $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$. Then there is a Morita morphism given by P_f^{triv} (Definition 3.1.20): it is a principal $\mathcal{G}_2$-bundle over M_1, whereas $\mathcal{G}_1$ acts on P_f^{triv} by $g \cdot (x, h) = (x, F(g)h)$. In particular, an isomorphism (F, f) gives rise to the Morita equivalence from Example 3.3.3. ♢

Example 3.3.11. Here are a couple of other examples:

- Morita morphisms between a unit groupoid $M \rightrightarrows M$ and a Lie groupoid $\mathcal{G} \rightrightarrows X$ are the same thing as principal $\mathcal{G}$-bundles over M.
- Morita morphisms between $\mathcal{G} \rightrightarrows M$ and $GL(k, \mathbb{R}) \rightrightarrows \{*\}$ are the same thing as representations of rank k of $\mathcal{G}$ (Definition 3.1.14). ♢

Remark 3.3.12 (Clarifications on the terminology)**.** There are other definitions of equivalences between Lie groupoids in the literature, expressed in the language of category theory, and an abundance of names to denote them. In the rest of this thesis we are not going to use them, but we briefly recall them in this remark to help the readers who are more familiar with them, as well as to avoid misunderstandings with the terminology used in Definition 3.3.8.

In Section 4.2 of [28] the term *Morita map* refers to a Lie groupoid morphism which is fully faithful and essentially surjective (as a functor between categories), with further requirements to take smoothness into account. In Section I.2 of [75], this is called an *essential equivalence*, while Morita morphisms are called *Hilsum-Skandalis maps*. On the other hand, in Section 5.4 of [73] essential equivalences are rebaptised *weak equivalences*, to avoid confusions with the notion of *strong equivalence*, which requires the two Lie groupoids to be equivalent as categories and the two functors defining the equivalence to be Lie groupoid morphisms.

The precise relation between Morita maps (in the sense of [28]) and Morita morphisms (in the sense of Definition 3.3.8) can be found e.g. in Section 4.6 of [28] and goes as follows. A *generalised map* between two Lie groupoids $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ consists of another Lie groupoid $\mathcal{H} \rightrightarrows N$, together with a Morita map $\mathcal{H} \to \mathcal{G}_1$ and a Lie groupoid morphism $\mathcal{H} \to \mathcal{G}_2$. There is a 1-1 correspondence between Morita morphisms P from $\mathcal{G}_1$ to $\mathcal{G}_2$ and generalised maps, given by $\mathcal{H} := \mathcal{G}_1 \times_{M_1} P \times_{M_2} \mathcal{G}_2 \rightrightarrows P$, and $P := (\mathcal{G}_1 \times_{M_1} N \times_{M_2} \mathcal{G}_2)/\mathcal{H}$. In particular, two Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are Morita equivalent if and only if there exist two Morita maps from a third groupoid $\mathcal{H}$. ♢

3.4 Multiplicative forms

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $f \in \mathcal{C}^\infty(\mathcal{G})$. One can ask that the function

$$f : (\mathcal{G}, m) \to (\mathbb{R}, +)$$

is also a Lie groupoid morphism between $\mathcal{G}$ and the additive group $(\mathbb{R}, +)$, i.e.

$$f(gh) = f(g) + f(h) \quad \forall (g, h) \in \mathcal{G}^{(2)}.$$

Denoting by pr_i the projections on each component of $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, the formula above is equivalent to $m^* f = \mathrm{pr}_1^* f + \mathrm{pr}_2^* f$ and motivates the following definition:

Definition 3.4.1. Let $\mathcal{G}$ be a Lie groupoid; a differential form $\omega \in \Omega^k(\mathcal{G})$ is called **multiplicative** if

$$m^*\omega = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\omega,$$

where $m : \mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is the multiplication of $\mathcal{G}$ and $\mathrm{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$ are the projections on the i^{th}-component. ♦

If $E \to M$ is a representation of $\mathcal{G}$ (Definition 3.1.14), there is a notion of multiplicativity for forms with coefficients in the pullback bundle t^*E.

Definition 3.4.2. Let $\mathcal{G}$ be a Lie groupoid and E a representation; a differential form $\omega \in \Omega^k(\mathcal{G}, t^*E)$ is called **multiplicative** if

$$m^*\omega = \mathrm{pr}_1^*\,\omega + g \cdot \mathrm{pr}_2^*\,\omega.$$

More precisely, the condition is:

$$(m^*\omega)_{(g,h)} = (\mathrm{pr}_1^*\,\omega)_{(g,h)} + g \cdot (\mathrm{pr}_2^*\,\omega)_{(g,h)} \quad \forall (g,h) \in \mathcal{G}^{(2)}.$$ ♦

3.4.1 Examples

Example 3.4.3. If $\phi : \mathcal{H} \to \mathcal{G}$ is a Lie groupoid morphism, E a representation of $\mathcal{G}$ and $\omega \in \Omega^k(\mathcal{G}, t^*E)$ a multiplicative form, then the pullback $\phi^*\omega \in \Omega^k(\mathcal{H}, \phi^*(t^*E))$ is a multiplicative form. ♢

Example 3.4.4. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\alpha \in \Omega^k(M)$ a differential form on its base space. Then the differential form

$$\delta\alpha := s^*\alpha - t^*\alpha \in \Omega^k(\mathcal{G})$$

is multiplicative. This can be generalised to differential forms with coefficients, though the formulae become more cumbersome: if $E \to M$ is a representation of $\mathcal{G}$ and $\alpha \in \Omega^k(M, E)$ a differential form with coefficients in E, then the differential form

$$\delta\alpha \in \Omega^k(\mathcal{G}, t^*E), \qquad (\delta\alpha)_g := g \cdot s^*\alpha - t^*\alpha \quad \forall g \in \mathcal{G}$$

is multiplicative. A multiplicative form of this type is also called **cohomologically trivial**. ♢

Example 3.4.5. Let $\mathcal{G}$ be a Lie groupoid and $J^k\mathcal{G}$ its k-jet groupoid (Example 3.1.8); its Cartan form (Definition 1.1.4)

$$\omega \in \Omega^1(J^k\mathcal{G}, T^s J^{k-1}\mathcal{G}) = \Omega^1(J^k\mathcal{G}, t^*J^{k-1}A)$$

is multiplicative.

To explain this claim, we observe that $J^k\mathcal{G}$ has a natural representation on the jet algebroid $J^{k-1}A$ (Example 3.2.5), called **adjoint representation**, since it resembles the homonymous representation of a Lie group on its Lie algebra. More precisely, an element $j^k_x\sigma \in J^k\mathcal{G}$ acts on $\alpha_x = \frac{d}{d\varepsilon}\big|_{\varepsilon=0}\gamma(\varepsilon) \in J^{k-1}_x A = \mathrm{Lie}(J^{k-1}\mathcal{G})_x$ as

$$j^k\sigma \cdot \alpha_x := j^k\sigma \cdot \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} j^{k-1}_x\sigma \cdot \gamma(\varepsilon) \cdot (j^{k-1}_x\sigma)^{-1}.$$

Since the right translation induces a diffeomorphism between the s-vertical bundle of any Lie groupoid and the t-pullback of its algebroid, we obtain

$$T^s J^{k-1}\mathcal{G} \xrightarrow{\cong} t^* \mathrm{Lie}(J^{k-1}\mathcal{G}) = t^* J^{k-1}A.$$

Accordingly, ω takes values in a representation of $J^k\mathcal{G}$, and one can prove directly its multiplicativity (see Proposition 2.4.3 of [100]). ♢

Example 3.4.6. Let G be a Lie group, V a representation of G and $\mathfrak{g} = \mathrm{Lie}(G)$ endowed with the adjoint action of G. There is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Multiplicative 1-forms} \\ \omega \in \Omega^1(G, V) \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} G\text{-equivariant linear maps} \\ l : \mathfrak{g} \to V \end{array} \right\}$$

Explicitly, l is ω evaluated at the unit $e \in G$. Moreover, multiplicative V-valued k-forms on G are zero for $k \geq 2$. These claims require Lemma 3.4.11 and will be proved after it. ♢

Example 3.4.7. Let $\pi : (A, \theta) \to M$ be a linear Pfaffian fibration (Definition 1.6.3); the form $\theta \in \Omega^1(A, \pi^*E)$ is multiplicative.

Indeed, any vector bundle $A \to M$ is a bundle of abelian Lie groups, hence a Lie groupoid, where $m = a$ is the fibrewise addition and both s and t coincide with the projection to the base manifold. A representation of such a Lie groupoid is just a vector bundle $E \to M$ such that each fibre E_x is a representation of the abelian group A_x, for all $x \in M$. The linearity of θ was defined precisely as its multiplicativity. ♢

Example 3.4.8. A **symplectic groupoid** $(\mathcal{G}, \omega)$ is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a symplectic form $\omega \in \Omega^2(\mathcal{G})$ such that the graph of the multiplication map m

$$Gr(m) \subseteq (\mathcal{G} \times \mathcal{G} \times \mathcal{G}, \omega \oplus \omega \oplus -\omega)$$

is a Lagrangian submanifold. One can easily see that it is equivalent to ω being multiplicative. This has been one of the first instances of multiplicative forms naturally arising from geometric contexts [21, 98, 66]. ♢

Example 3.4.9. A **contact groupoid** $(\mathcal{G}, \theta, r)$ is a Lie groupoid $\mathcal{G} \rightrightarrows M$ endowed with a contact form $\theta \in \Omega^1(\mathcal{G})$ and a function $r \in \mathcal{C}^\infty(\mathcal{G})$ such that

$$m^*\theta = \mathrm{pr}_1^*\,\theta + \mathrm{pr}_1^*(e^r) \cdot \mathrm{pr}_2^*\,\theta.$$

One can see that this condition is equivalent to the multiplicativity of θ with respect to the following non-trivial representation of $\mathcal{G}$ on the trivial line bundle $L_r = M \times \mathbb{R} \to M$:

$$(L_r)_{s(g)} \xrightarrow{g\cdot} (L_r)_{t(g)}, \quad x \mapsto e^{r(g)}x.$$

This has been observed first in [54] and further understood in [26]. $\Diamond$

3.4.2 Properties

We focus on the case of 1-forms and we discuss the first consequences of the multiplicativity condition. According to the philosophy explained in Remark 1.3.2, a 1-form ω on $\mathcal{G}$ can be thought of as dual to

$$H := \ker(\omega) \subseteq T\mathcal{G}.$$

Of course, for this to be a smooth object (vector subbundle of $T\mathcal{G}$), one needs the assumption that θ is of constant rank.

Lemma 3.4.10. *Let $\omega \in \Omega^1(\mathcal{G}, t^*E)$ be a multiplicative form; then*

$$u^*\omega = 0, \qquad (i^*\omega)_g = -g^{-1} \cdot \omega_g \quad \forall g \in \mathcal{G}.$$

In particular, $H = \ker(\omega)$ satisfies for all $x \in M$, $g \in \mathcal{G}$,

$$du(T_xM) \subseteq H_{1_x}, \qquad di(H_g) = H_{i(g)}.$$

Proof. Consider the smooth map $id_{\mathcal{G}} \times i : \mathcal{G} \to \mathcal{G}_2, \quad g \mapsto (g, g^{-1})$, which satisfies

$$(m \circ id_{\mathcal{G}} \times i)(g) = m(g, g^{-1}) = 1_{t(g)} = (u \circ t)(g).$$

Using the multiplicativity condition of ω we get

$$t^*u^*\omega = (u \circ t)^*\omega = (m \circ id_{\mathcal{G}} \times i)^*\omega = (id_{\mathcal{G}} \times i)^*(m^*\omega) =$$

$$= (id_{\mathcal{G}} \times i)^*((\mathrm{pr}_1)^*\omega + g \cdot (\mathrm{pr}_2)^*\omega) = (id_{\mathcal{G}} \times i)^*(\mathrm{pr}_1)^*\omega + g \cdot (id_{\mathcal{G}} \times i)^*(\mathrm{pr}_2)^*\omega =$$

$$= (\mathrm{pr}_1 \circ id_{\mathcal{G}} \times i)^*\omega + g \cdot (\mathrm{pr}_2 \circ id_{\mathcal{G}} \times i)^*\omega = id_{\mathcal{G}}^*\omega + g \cdot i^*\omega = \omega + g \cdot i^*\omega.$$

However, on the other hand,

$$u^*\omega = (id_M)^*u^*\omega = (t \circ u)^*u^*\omega = u^*t^*u^*\omega = u^*(\omega + g \cdot i^*\omega) =$$

$$= u^*\omega + g \cdot u^*i^*\omega = u^*\omega + g \cdot (i \circ u)^*\omega = u^*\omega + g \cdot u^*\omega.$$

This proves that $u^*\omega = 0$. Plugging this result in the first equation we get

$$0 = t^*(0) = t^*(u^*\omega) = \omega + g \cdot i^*\omega,$$

which concludes the first part of the proof.

For the second part, note that, for every $v \in T_x M$,

$$\omega_{1_x}(d_x u(v)) = (u^*\omega)_x(v) = 0,$$

hence $d_x u(v) \in H_{1_x}$. We conclude that $\mathrm{Im}(d_x u) \subseteq H_{1_x}$ for all $x \in M$.

Last, for every $v \in H_g$ we have

$$\omega_{g^{-1}}(d_g i(v)) = (i^*\omega)_g(v) = -g^{-1} \cdot \omega_g(v) = 0,$$

hence $d_g i(v) \in H_{g^{-1}}$. We conclude that $d_g i(H_g) \subseteq H_{g^{-1}}$ for every $g \in \mathcal{G}$; since di is a diffeomorphism, the equality holds. Q.E.D.

Lemma 3.4.11. *Let $\mathcal{G}$ be a Lie groupoid, E a representation and $\omega \in \Omega^k(\mathcal{G}, t^*E)$ a multiplicative form. Then, for every $g \in \mathcal{G}$ from x to y:*

- $(L_g)^*(\omega_{|t^{-1}(y)}) = g \cdot \omega_{|t^{-1}(x)}$,
- $(R_g)^*(\omega_{|s^{-1}(x)}) = \omega_{|s^{-1}(y)}$.

Proof. Applying the second formula from Example 3.2.9 to the groupoid multiplication m, we see that, for $(g, h) \in \mathcal{G}_2$ and $Y \in T_h(t^{-1}(s(g)))$,

$$d_h L_g(Y) = d_h m(g, \cdot)(Y) = d_{(g,h)} m_{|T(\{g\} \times t^{-1}(s(g)))}(0, Y).$$

Therefore, using the multiplicativity of ω, we obtain

$$((L_g)^*\omega)_h(Y) = \omega_{g \cdot h}(d_h L_g(Y)) =$$

$$= \omega_{m(g,h)}(d_{(g,h)} m_{|T(\{g\} \times t^{-1}(s(g)))}(0, Y)) = \cancel{\omega_g(0)} + g \cdot \omega_h(Y).$$

Similarly, the first formula from Example 3.2.9 yields, for $(h, g) \in \mathcal{G}^{(2)}$ and $X \in T_g(s^{-1}(t(g)))$,

$$d_h R_g(X) = d_h m(\cdot, g)(X) = d_{(h,g)} m_{|T(s^{-1}(t(g)) \times \{g\})}(X, 0),$$

and with the same argument we conclude that

$$((R_g)^*\omega)_h(X) = \omega_h(X) + \cancel{h \cdot \omega_g(0)}.$$ Q.E.D.

Proof of Example 3.4.6. Assume ω is multiplicative and define the linear map

$$l := \omega_e : T_e G = \mathfrak{g} \to V.$$

We can apply Lemma 3.4.11 to any $u \in T_e G$ (since it lies both in the s- and the t-fibre of G) to prove the G-equivariance of l:

$$l(Ad_g(u)) = \omega_e(d_e(L_g R_{g^{-1}})(u)) = ((L_g R_{g^{-1}})^*\omega)_e(u) =$$

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$$= (R_{g^{-1}})^*(g \cdot \omega)_e(u) = g \cdot \omega_e(u) = g \cdot l(u).$$

Viceversa, given any G-equivariant linear map $l : \mathfrak{g} \to V$, we define a form $\omega \in \Omega^1(G, V)$ by imposing

$$\omega_g(X) := l(d_g R_{g^{-1}}(X)).$$

Its multiplicativity follows from the fact that $d_{(e,e)}m(u, v) = u + v$: for every $X \in T_gG, Y \in T_hG$ we have

$$(m^*\omega)_{(g,h)}(X, Y) = \omega_{gh}(d_{(g,h)}m(X, Y)) = l(d_{gh}R_{h^{-1}g^{-1}}d_{(g,h)}m(X, Y)) =$$

$$= l(d_{(g,h)}(m \circ (R_{g^{-1}}, L_g R_{g^{-1}} R_{h^{-1}}))(X, Y)) =$$

$$= l(d_{(e,e)}m(d_g R_{g^{-1}}(X), d_h(L_g R_{g^{-1}} R_{h^{-1}})(Y))) =$$

$$= l(d_g R_{g^{-1}}(X) + d_h(L_g R_{g^{-1}} R_{h^{-1}})(Y)) = l(d_g R_{g^{-1}}(X)) + l(Ad_g(d_h R_{h^{-1}}(Y))) =$$

$$= \omega_g(X) + g \cdot \omega_h(Y) = ((\mathrm{pr}_1)^*\omega)_{(g,h)}(X, Y) + g \cdot ((\mathrm{pr}_2)^*\omega)_{(g,h)}(X, Y).$$

Last, the correspondence is bijective since $d_e R_{e^{-1}} = id_{\mathfrak{g}}$ and $\omega_e(d_g R_{g^{-1}}(X)) = \omega_g(X)$ (using again Lemma 3.4.11).

On the other hand, if $\omega \in \Omega^k(G, V)$ is multiplicative, using the same arguments above, we can define $l = \omega_e : \Lambda^k\mathfrak{g} \to V$ and prove that, for all $X_1, \dots, X_k \in T_gG, Y_1, \dots, Y_k \in T_hG$,

$$l(d_g R_{g^{-1}}(X_1) + Ad_g(d_h R_{h^{-1}}(Y_1)), \dots, d_g R_{g^{-1}}(X_k) + Ad_g(d_h R_{h^{-1}}(Y_k))) =$$

$$= l(d_g R_{g^{-1}}(X_1), \dots, d_g R_{g^{-1}}(X_k)) + l(Ad_g(d_h R_{h^{-1}}(Y_1)), \dots, l(Ad_g(d_h R_{h^{-1}}(Y_k)))).$$

But, since l is skew and multilinear, this means that the mixed terms of the kind

$$l(d_g R_{g^{-1}}(X_1), \dots, Ad_g(d_h R_{h^{-1}}(Y_k)))$$

must all vanish; this implies $l = 0$, hence $\omega = 0$. Q.E.D.

The following statement is of chief importance for our further results and is new even for trivial coefficients.

Proposition 3.4.12. *Any multiplicative form $\omega \in \Omega^1(\mathcal{G}, t^*E)$ on a Lie groupoid $\mathcal{G}$ (with coefficient in a representation E) is automatically s-transversal at the units (Definition 1.3.1).*

Furthermore, if $\ker(\omega)_{|M}$ is of constant rank, the following properties are equivalent:

1. *ω has constant rank,*
2. *ω is s-transversal,*
3. *ω is t-transversal.*

Proof. For $v \in T_{1_x}\mathcal{G}$ we consider $v' = d_x u(d_{1_x} s(v))$ and we write

$$v = v' + (v - v').$$

From Lemma 3.4.10, $v' \in H_{1_x}$; this proves the first statement, since $v - v' \in \ker(d_{1_x} s)$:

$$d_{1_x} s(v - v') = d_{1_x} s(v) - d_{1_x}(s \circ u \circ s)(v) = d_{1_x} s(v) - d_{1_x}(id_M \circ s)(v) = 0.$$

For the second part, let us denote by $\cap_g$ the intersection

$$\cap_g := H_g \cap \ker(d_g s), \quad \mathfrak{g} \in \mathcal{G}.$$

Observe that, for $g = 1_x$, the dimension of $\cap_{1_x}$ is constant:

$$\dim(\cap_{1_x}) = \dim(H_{1_x}) + \dim(\ker(d_{1_x} s)) - \dim(T_{1_x}\mathcal{G}) = \dim(H_{1_x}) - \dim(M).$$

On the other hand, using Lemma 3.4.11, one sees that

$$d_g R_{h|\cap_g} : \cap_g \to \cap_h \quad \forall (g,h) \in \mathcal{G}^{(2)}$$

is an isomorphism. Then the dimension of $\cap_g$ is constant:

$$\dim(\cap_{g_1}) = \dim(\cap_{1_{s(g_1)}}) = \dim(\cap_{1_{s(g_2)}}) = \dim(\cap_{g_2}) \quad \forall g_1, g_2 \in \mathcal{G}.$$

Condition 2 amounts to

$$\dim(H_g) + \dim(\ker(d_g s)) - \dim(\cap_g) = \dim(\mathcal{G}) \quad \forall g \in \mathcal{G}.$$

Since $\dim(\ker(d_g s))$ and $\dim(\cap_g)$ are constant for all $g \in \mathcal{G}$, then $\dim(\ker(\omega_g))$ is constant, so condition 1 holds. Viceversa, if ω has constant rank, then

$$\dim(H_g) + \dim(\ker(d_g s)) - \dim(\cap_g) = \text{const} \quad \forall g \in \mathcal{G}.$$

In particular, const is the value taken at $g = 1_x$, which is equal to $\dim(\mathcal{G})$; then condition 2 holds.

Last, we show that conditions 2 and 3 are equivalent. Assume that ω is s-transversal: then, for every $g \in \mathcal{G}$, $d_g s$ is surjective when restricted to H_g. However, since $t = s \circ i$,

$$d_{g^{-1}} t_{|H_{g^{-1}}} = d_g s_{|H_g} \circ d_{g^{-1}} i_{|H_{g^{-1}}},$$

where we used Lemma 3.4.10 to see that $d_g i$ sends $H_{g^{-1}}$ into H_g.

Being a composition of surjective maps, $d_{g^{-1}} t_{|H_{g^{-1}}}$ is surjective as well for every $g \in \mathcal{G}$, hence ω is t-transversal. The converse is proved analogously. Q.E.D.

This proposition has two important consequences.

Lemma 3.4.13. *Let $\mathcal{G}$ be a Lie groupoid, E a representation and $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative 1-form with constant rank. Then there is a vector subbundle $E(\omega) \subseteq E$ with rank equal to* $\mathrm{rank}(\omega)$*, such that*

- *$E(\omega)$ is a representation of $\mathcal{G}$,*
- *$\omega \in \Omega^1(\mathcal{G}, t^*E(\omega))$ is pointwise surjective.*

Proof. We define $E(\omega) \to M$ as the subbundle of E with fibres

$$E(\omega)_x = \mathrm{Im}(\omega_{1_x}) \subseteq E_x \quad \forall x \in M.$$

Then $E(\omega)$ is automatically a smooth vector bundle, since it is the image of the vector bundle map

$$\omega_{|TM} : TM \to E,$$

and ω is of constant rank.

Let us prove that $\omega \in \Omega^1(\mathcal{G}, t^*E(\omega))$ is pointwise surjective, i.e. that

$$\omega_g : T_g\mathcal{G} \to E(\omega)_{t(g)}$$

is surjective for any $g \in \mathcal{G}$; this amounts to $\mathrm{Im}(\omega_g) = \mathrm{Im}(\omega_{1_x})$, where $x = t(g)$. By Proposition 3.4.12 we have

$$\mathrm{Im}(\omega_g) = \omega_g(T_g\mathcal{G}) = \omega_g(\ker(d_g s) + \ker(\omega_g)) = \omega_g(\ker(d_g s)).$$

Consider $v = \omega_g(X_g) \in \mathrm{Im}(\omega_g)$, for some $X_g \in \ker(d_g s) \subseteq T_g\mathcal{G}$; by Lemma 3.4.11 we have

$$v = \omega_g(X_g) = ((R_{g^{-1}})^*\omega)_g(X_g) = \omega_{1_x}(d_g R_h(X_g)).$$

We conclude that $v \in \mathrm{Im}(\omega_{1_x})$; this proves that $\mathrm{Im}(\omega_g) \subseteq \mathrm{Im}(\omega_{1_x})$. The same argument shows also the converse, so in conclusion ω is pointwise surjective.

Last, we prove that $E(\omega)$ is a representation of $\mathcal{G}$, i.e. that, if $g \in \mathcal{G}$ and $v \in E(\omega)_{s(g)}$, then $g \cdot v \in E_{t(g)}$ belongs to $E(\omega)_{t(g)}$. Pick an element $h \in \mathcal{G}$ such that $s(g) = t(h)$; then v can be written as

$$v = \omega_h(Y_h) \in E(\omega)_{t(h)} = E(\omega)_{s(g)},$$

for $Y_h \in \ker(d_h s)$. Using again Lemma 3.4.11 we conclude that

$$g \cdot v = ((L_g)^*\omega)_h(Y_h) = \omega_{gh}(d_h L_g(Y_h)) \in \mathrm{Im}(\omega_{gh}) = E(\omega)_{t(gh)} = E(\omega)_{t(g)}.$$

Q.E.D.

We will soon interpret $s : (\mathcal{G}, \omega) \to M$ as a Pfaffian fibration, in the sense of Definition 1.3.1. Accordingly, we will be interested in

$$\mathfrak{g}(\omega) := \ker(ds) \cap \ker(\omega).$$

Lemma 3.4.14. *Let $\mathcal{G}$ be a Lie groupoid, E a representation and $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form with constant rank. Then*

1. *$\mathfrak{g}(\omega)$ is a smooth vector bundle over $\mathcal{G}$*

2. *$\mathfrak{g}(\omega)$ is a pullback by t of a vector bundle over M. Equivalently,*

$$\mathfrak{g}(\omega) \cong t^*\mathfrak{g}_M(\omega),$$

where $\mathfrak{g}_M(\omega) = \mathfrak{g}(\omega)_{|M}$.

Proof. For the first claim, it is enough to use Proposition 3.4.12: for every $g \in \mathcal{G}$,

$$\dim(\mathfrak{g}(\omega)_g) = \dim(\ker(d_g s) \cap \ker(\omega_g)) =$$

$$= \dim(\ker(d_g s)) + \dim(\ker(\omega_g)) - \dim(T_g\mathcal{G}) = \mathrm{const}\,.$$

For the second claim, recall from the definition of Lie algebroid that

$$T^s\mathcal{G} \cong t^*A, \quad X_g \mapsto d_g R_{g^{-1}}(X_g) = dm(X_g, 0_{g^{-1}}).$$

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Thus, if X_g belongs also to $\ker(\omega)$, then its right-translated does it too (by Lemma 3.4.11), and the conclusion follows. Q.E.D.

3.4.3 Multiplicative distributions

In chapter 1 (see e.g. Lemma 1.3.3) we have extensively used the dualism between surjective 1-forms and hyperplane distributions; here we discuss the multiplicative version of such dualism.

Definition 3.4.15. A regular distribution $H \subseteq T\mathcal{G}$ on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is called **multiplicative** if it is also a subgroupoid of the tangent groupoid $T\mathcal{G} \rightrightarrows TM$ (over the entire TM). More explicitly, a **multiplicative distribution** on $\mathcal{G}$ is a regular distribution $H \subseteq T\mathcal{G}$ such that:

1. H is closed under dm, i.e. $dm(H, H) \subseteq H$

2. H is closed under di, i.e. $di(H) \subseteq H$

3. $du(TM) \subseteq H$

The vector bundle

$$\mathfrak{g}_M(H) := (\ker(ds) \cap H)_{|M}$$

is called the **symbol space** of H. Note that

$$\mathfrak{g}_M(H) \subseteq \ker(ds)_{|M} = A. \qquad \blacklozenge$$

Proposition 3.4.16. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $E \to M$ a representation. There is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{Multiplicative} \\ \textit{distributions } H \subseteq T\mathcal{G} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{(Isomorphism classes of) representations} \\ E \in \mathrm{Rep}(\mathcal{G}) \textit{ and pointwise surjective} \\ \textit{multiplicative forms } \theta \in \Omega^1(\mathcal{G}, t^*E) \end{array} \right\}.$$

It is immediate to check, using Lemma 3.4.10, that a multiplicative form θ with values in E induces the multiplicative distribution $H_\theta = \ker(\theta)$ such that $T\mathcal{G}/H \cong t^*E$. For the converse we refer to Proposition 6.1.14 of [81]; in particular, E is the vector bundle $A/\mathfrak{g}_M(H) \to M$.

Note also that, by Proposition 3.4.12, multiplicative distributions are s-transversal, i.e. $H + \ker(ds) = T\mathcal{G}$.

Example 3.4.17. The Cartan distribution $\mathcal{C}_\omega = \ker(\omega) \subseteq J^k\mathcal{G}$ on a jet groupoid $J^k\mathcal{G}$ is a multiplicative distribution, since the Cartan form ω is multiplicative (Example 3.4.5) and pointwise surjective. ♢

Example 3.4.18. A linear distribution $H \subseteq TA$ on a vector bundle $A \to M$, viewed as a bundle of abelian Lie groups, is multiplicative, in the same spirit that a linear 1-form on a vector bundle is multiplicative (Example 3.4.7)

More precisely, the duality between multiplicative distributions and multiplicative 1-forms on Lie groupoids from Proposition 3.4.16 restricts to the duality between linear distributions and linear 1-forms on vector bundles from Remark 1.6.2. The properties on linear distributions of Remark 1.6.1 follow then immediately from such duality and Lemma 3.4.14. ♢

Remark 3.4.19. Although we will not use it in this thesis, we remark that Definition 3.4.15 can be relaxed by asking $H \subseteq T\mathcal{G}$ to be a subgroupoid, not necessarily over the entire TM. This has been introduced in [49, 50] and applied to foliations. ♢

3.4.4 Infinitesimal multiplicative forms

Multiplicative forms often arise from geometric structures on Lie groupoids; in this framework, an important feature concerns the possibility to understand these structures from an infinitesimal point of view, as objects on a Lie algebroid.

We recall here the approach using Spencer operators from [25, 81]. An equivalent way to study multiplicative forms from the infinitesimal point of view (also known as *infinitesimally multiplicative forms*, or IM-forms) has been carried out in [15], and later generalised to multiplicative tensors in [16] and to multiplicative forms with values in VB-groupoids in [29]. A different approach to these topics, which avoids the global groupoid picture and makes use of forms on graded manifolds, is provided in [94].

Definition 3.4.20. Let A be a Lie algebroid over M and (E, ∇) a representation of A (Definition 3.2.10). A **Spencer operator** (D, l) on A with values in E consists of a vector bundle map $l : A \to E$ and an $\mathbb{R}$-linear map

$$D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E)$$

such that, for every $X \in \mathfrak{X}(M), \alpha, \alpha' \in \Gamma(A), f \in \mathcal{C}^\infty(M)$,

$$\begin{cases} D_X(f\alpha) = fD_X(\alpha) + L_X(f)l(\alpha) \\ D_{\rho(\alpha')}(\alpha) = \nabla_\alpha(l(\alpha')) - l[\alpha, \alpha'] \\ D([\alpha, \alpha']) = \tilde{\nabla}_\alpha(D(\alpha')) - \tilde{\nabla}_{\alpha'}(D(\alpha)) \end{cases}$$

In the last equation we interpreted D as a map $\Gamma(A) \to \Omega^1(M, E)$ and we used the Lie derivative operator $\tilde{\nabla}_\alpha$ induced by the representation ∇ of A:

$$\tilde{\nabla}_\alpha : \Omega^1(M, E) \to \Omega^1(M, E)$$

$$(\tilde{\nabla}_\alpha \omega)(X) := \nabla_\alpha(\omega(X)) - \omega([\rho(\alpha), X]).$$ ♦

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Example 3.4.21 (Relative connections). Any vector bundle A can be thought as a Lie algebroid with zero bracket and zero anchor. Then, a Spencer operator (D, l) on A with values in a representation E coincides with a relative connection D on A with symbol map $l : A \to E$ (Definition 1.6.5).

From the other perspective, a Spencer operator on a Lie algebroid A is a relative connection on A, which is furthermore compatible with the Lie bracket and the anchor of A. ♢

As promised, the infinitesimal counterpart of multiplicative forms are Spencer operators.

Proposition 3.4.22 (Theorem 4.3.1 of [81])**.** *Let $\mathcal{G}$ be a Lie groupoid on M, E a representation and $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative 1-form. Then there is a Spencer operator (D^ω, l^ω) on the Lie algebroid $A = \mathrm{Lie}(\mathcal{G})$ with values in the induced representation E of A (Lemma 3.2.11). Explicitly, it is defined for every $x \in M$, $\alpha \in \Gamma(A)$, $X \in \mathfrak{X}(M)$ by*

$$l^\omega(\alpha)(x) := u^*(i_\alpha \omega)(x) = \omega_{1_x}(\alpha(x)),$$

$$D^\omega_X(\alpha)(x) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\phi^\varepsilon_\alpha(x))^{-1} \cdot \omega_{\phi^\varepsilon_\alpha(x)}(d_x\phi^\varepsilon_\alpha(X_x)).$$

Moreover, the correspondence between such multiplicative forms and their Spencer operators is

- *injective if $\mathcal{G}$ is s-connected,*
- *bijective if $\mathcal{G}$ is s-simply connected.*

Example 3.4.23. Consider a linear form θ on a vector bundle, interpreted as a multiplicative form on a Lie groupoid (Example 3.4.7). Then the relative connection (D^θ, l^θ), induced by θ according to Proposition 1.6.6, coincides with the Spencer operator (D^θ, l^θ), induced by θ according to Proposition 3.4.22. $\Diamond$

Remark 3.4.24. In the rest of the thesis we will not use the explicit formula of D^ω from Proposition 3.4.22. However, we mention that it can be written more compactly (as well as more similarly to the equation for l^ω) as

$$D^\omega(\alpha) = u^*(\mathcal{L}_\alpha \omega),$$

where $\mathcal{L}$ is defined via a natural connection on t^*E; see Remark 10.5 of [95] for further details. $\Diamond$

For cohomologically trivial multiplicative forms (Example 3.4.4), the induced Spencer operator can be described in simpler terms.

Proposition 3.4.25. *Let $\mathcal{G}$ be a Lie groupoid, $E \to M$ a representation, $\theta \in \Omega^1(M, E)$, and consider the multiplicative form $\tilde{\theta} = g \cdot s^*\theta - t^*\theta \in \Omega^1(\mathcal{G}, t^*E)$. Then the Spencer operator $(D^{\tilde{\theta}}, l^{\tilde{\theta}})$ from Proposition 3.4.22 can be described as follows: for every $X \in \mathfrak{X}(M), \alpha \in \Gamma(A)$,*

$$l^{\tilde{\theta}}(\alpha) = -\theta(\rho_A(\alpha)),$$

$$D_X^{\tilde{\theta}}(\alpha) = -(\tilde{\nabla}_\alpha \theta)(X) = -\nabla_\alpha(\theta(X)) + \theta([\rho_A(\alpha), X]).$$

Both equalities follow by direct computations; for instance, for the first one:

$$(l^{\tilde{\theta}}\alpha)(x) = \tilde{\theta}_{1_x}(\alpha(x)) = 1_x \cdot (s^*\theta)_{1_x}(\alpha(x)) - (t^*\theta)_{1_x}(\alpha(x)) =$$

$$= \theta_{s(1_x)}(\cancel{d_{1_x} s(\alpha(x))}) - \theta_{t(1_x)}(d_{1_x} t(\alpha(x))) = -\theta_x(\rho_A(\alpha(x))).$$

Another natural way to obtain new Spencer operators is via pullbacks. Again, the next statement follows by straightforward computations.

Proposition 3.4.26. *Let $A \to M$ be a Lie algebroid acting on a submersion $\mu : P \to M$ (Definition 3.2.8), E a representation of A, and denote by $\mu^*E \to P$ the pullback representation of the action algebroid $\mu^*A \to P$ (Example 3.2.9).*

For every Spencer operator (D, l) on A with values in E, there is an induced Spencer operator (μ^*D, μ^*l) on μ^*A with values in μ^*E, uniquely defined by its action on sections of the kind $\mu^*\alpha = \alpha \circ \mu \in \Gamma(\mu^*A)$, for $\alpha \in \Gamma(A)$:

$$(\mu^*l)(\mu^*\alpha) := \mu^*(l(\alpha))$$

$$(\mu^*D)_{\tilde{X}}(\mu^*\alpha) := \mu^*(D_X(\alpha))$$

where $\tilde{X} \in \mathfrak{X}(P)$ is a μ-projectable lift of $X \in \mathfrak{X}(M)$.

We conclude by listing a number of interesting objects associated to any Spencer operator, which will be useful in the context of prolongations of Pfaffian groupoids.

Definition 3.4.27. Let (D, l) be a Spencer operator on a Lie algebroid $A \to M$ with values in a representation E. We define the following objects:

- The differential associated to D
$$d_D : \Omega^1(M, A) \to \Omega^2(M, E)$$
$$d_D\beta(X, Y) := D_X(\beta(Y)) - D_Y(\beta(X)) - l(\beta[X, Y]).$$
One can actually define such a Kozsul-type differential d_D for forms of any degree; in particular, on 0-forms we set
$$d_D : \Omega^0(M, A) \to \Omega^1(M, E), \quad \beta \mapsto D_\bullet \beta.$$

- The antisymmetric $\mathcal{C}^\infty(M)$-bilinear bracket on A
$$\{\cdot, \cdot\}_D : \Gamma(A) \times \Gamma(A) \to \Gamma(E)$$
$$\{\alpha, \beta\}_D := D_{\rho(\alpha)}(\beta) - D_{\rho(\beta)}(\alpha) - l([\alpha, \beta]_A).$$

- The induced antisymmetric $\mathcal{C}^\infty(M)$-linear bracket on $\Omega^1(M, A)$
$$\{\cdot, \cdot\}_D : \Omega^1(M, A) \times \Omega^1(M, A) \to \Omega^2(M, E)$$
$$\{\eta_1, \eta_2\}_D(X, Y) := \{\eta_1(X), \eta_2(Y)\}_D - \{\eta_1(Y), \eta_1(X)\}_D.$$

- The **Maurer-Cartan operator**
$$MC : \Omega^1(M, A) \to \Omega^2(M, E),$$
$$MC(\eta) = d_D\eta - \frac{1}{2}\{\eta, \eta\}_D.$$
More explicitly, for every $X, Y \in \mathfrak{X}(M)$,
$$MC(\eta)(X, Y) := l\Big([\eta(X), \eta(Y)] - \eta([X, Y])\Big) +$$
$$+ D_{X - \rho(\eta(X))}(\eta(Y)) - D_{Y - \rho(\eta(Y))}(\eta(X)).$$ ♦

Chapter 4

Lie pseudogroups

In this chapter we will recall the notion of pseudogroup of diffeomorphisms, originally introduced in the pioneering works by Lie and Cartan; the first modern exposition is due to Ehresmann [32], and a more recent one is [100].

It will be of capital importance to distinguish the class of *Lie* pseudogroups inside that of pseudogroups. On the one hand, many common examples satisfy this extra condition. On the other hand, the notion of "almost structure" (which we will discuss in the next chapter) makes sense only for Lie pseudogroups.

For a more conceptual framework for handling Lie pseudogroups, we also present the notion of Pfaffian groupoid: it has been introduced in [81] as the "multiplicative version" of a Pfaffian fibration (originally called a *Pfaffian bundle*). Here we slightly weaken its definition in order to include further natural examples. 4

As a particular case of the integrability of Pfaffian fibrations discussed in chapter 1, we will then study the integrability of Pfaffian groupoids, and then we focus on those arising from a Lie pseudogroup.

Last, we mention how the framework can be made even more general by the introduction of generalised pseudogroups, made up by bisections of a Lie groupoid, instead of diffeomorphisms of a manifold.

4.1 Pseudogroups

Recall from Definition 2.6.1 that $\mathrm{Diff}_{\mathrm{loc}}(X) := \mathrm{Diff}_{\mathrm{loc}}(X, X)$ denotes the set of all diffeomorphisms between open sets of a manifold X.

Definition 4.1.1. A **pseudogroup on a manifold** X is a subset $\Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$ satisfying the following conditions:

1. $id_X \in \Gamma$
2. $\phi \circ \psi \in \Gamma$ for all $\phi, \psi \in \Gamma$ such that $\mathrm{dom}(\phi) \subseteq \mathrm{Im}(\psi)$

3. $\phi^{-1} \in \Gamma$ for all $\phi \in \Gamma$

4. $\phi_{|U} \in \Gamma$ for all $\phi \in \Gamma$ and $U \subseteq \mathrm{dom}(\phi)$ open subset

5. $\phi \in \Gamma$ for all $\phi \in \mathrm{Diff}_{\mathrm{loc}}(X)$ such that there exists $\{U_i\}_i$ open cover of $\mathrm{dom}(\phi)$ with $\phi_{|U_i} \in \Gamma\ \forall i$ ♦

One refers to axioms 1-2-3 as "group-like" and axioms 4-5 as "sheaf-like".

4.1.1 Examples

Example 4.1.2. Given a subset $\Gamma_0 \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$, the **pseudogroup generated by** Γ_0, denoted by $\Gamma = \langle \Gamma_0 \rangle$, is the smallest pseudogroup containing Γ_0. Its elements are all the possible compositions, inversions, restrictions and glueings of elements in $\Gamma_0 \cup \{id_X\}$.

In particular, any subgroup of $\mathrm{Diff}(X)$ generates a pseudogroup on X, by taking all the possible restrictions of its elements. The two extreme cases of subgroups $\{id_X\}$ and $\mathrm{Diff}(X)$ give rise to the trivial pseudogroup $\{id_U \mid U \subseteq X \text{ open}\}$ and to the entire $\mathrm{Diff}_{\mathrm{loc}}(X)$. ♢

Example 4.1.3. The **pseudogroup of analytic diffeomorphisms** is the subset $\Gamma^\omega \subseteq \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ made up of the analytic maps. ♢

Example 4.1.4. Let (X, g) be a Riemannian manifold; the **pseudogroup of local isometries** of (X, g) is the set Γ_g of locally defined diffeomorphisms ϕ such that $\phi^* g = g$. More generally, given any G-structure P on X, the set of its local symmetries (Definition 2.1.4) is a pseudogroup. ♢

Example 4.1.5. Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup. The **pseudogroup associated to** G, denoted by Γ_G, is the set of all locally defined diffeomorphisms ϕ of $\mathbb{R}^n$ whose Jacobian $d_x\phi$ at any point $x \in \mathrm{dom}(\phi)$ is in G.

Actually Γ_G can be seen as the pseudogroup Γ of local symmetries of the canonical G-structure $P_{\mathrm{can}} = \mathbb{R}^n \times G$ (Example 4.1.4), since each element $\phi \in \Gamma$ preserves P_{can}, i.e.

$$Fr(\phi)(x, g) = (\phi(x), d_x\phi \cdot g) \in P_{\mathrm{can}} \quad \forall (x, g) \in P_{\mathrm{can}},$$

hence $\phi \in \Gamma$ if and only if $d_x\phi \in G$ for all $x \in \mathrm{dom}(\phi)$, i.e. $\phi \in \Gamma_G$.

In many examples, this fact is even more transparent by the use of coordinates. For instance, to show that the pseudogroup $\Gamma_{g_{\mathrm{can}}}$ of local isometries of $(\mathbb{R}^n, g_{\mathrm{can}})$ coincides with $\Gamma_{O(n)}$, consider an element $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ which changes the standard coordinates (x^i) into other coordinates (X^i). The condition that ϕ preserves the standard metric $g_{\mathrm{can}} = \delta_{ij} dx^i \otimes dx^j$ is equivalent to

$$\delta_{ij} dX^i \otimes dX^j = \delta_{ij} \frac{\partial X^i}{\partial x^l} \frac{\partial X^j}{\partial x^k} dx^l \otimes dx^k = \delta_{ij} \delta^i_l \delta^j_k dx^l \otimes dx^k = \delta_{lk} dx^l \otimes dx^k,$$

which is in turn equivalent to

$$\frac{\partial X^i}{\partial x^l}\frac{\partial X^j}{\partial x^k} = \delta^i_l \delta^j_k.$$

This means that the Jacobian matrix $d_x\phi = \left(\frac{\partial X^i}{\partial x^j}(x)\right)_{ij}$ is in $O(n)$ for every point $x \in \mathbb{R}^n$, i.e. $\phi \in \Gamma_{O(n)}$. $\Diamond$

Example 4.1.6. Here is an example of a pseudogroup Γ not of type Γ_G but still defined by conditions on the Jacobians: Γ is the set of locally defined diffeomorphisms ϕ of $\mathbb{R}^n$ whose Jacobian $d_x\phi$, viewed as a matrix in $GL(n,\mathbb{R})$, has locally constant determinant w.r.t. every point x. Since the constants may be different in different connected components, Γ cannot be described as a Γ_G for some G.

Equivalently, Γ contains the elements $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ such that, on each connected component of $\mathrm{dom}(\phi)$,

$$\phi^*(dx^1 \wedge \ldots \wedge dx^n) = c_\phi dx^1 \wedge \ldots \wedge dx^n$$

for some constant $c_\phi \in \mathbb{R} - \{0\}$. It follows that Γ contains $\Gamma_{SL(n,\mathbb{R})}$. $\Diamond$

Example 4.1.7. Let $G \subseteq GL^k(\mathbb{R}^n)$ be a Lie subgroup of the k^{th}-order general linear group $GL^k(\mathbb{R}^n)$ (Definition 2.6.2). The **pseudogroup associated to** G, denoted by Γ_G, is the set of locally defined diffeomorphisms ϕ of $\mathbb{R}^n$ such that $j^k_0(\tilde{\phi}_x) \in G$ for every point $x \in \mathrm{dom}(\phi)$, where $\tilde{\phi}_x : z \mapsto \phi(z+x) - \phi(x)$ defines a diffeomorphism sending 0 to 0. In the case $k = 1$ we recover Example 4.1.5. $\Diamond$

Example 4.1.8. If Γ is a pseudogroup on X and $U \subseteq X$ an open set, the **restriction of the pseudogroup** Γ to U, denoted by $\Gamma_{|U}$, is the pseudogroup consisting of the elements of Γ with domain and image inside U.

Given a diffeomorphism $f : Y \to X$, then

$$f^*\Gamma := \{f^{-1} \circ \phi \circ f_{|f^{-1}(\mathrm{dom}\,\phi)} \mid \phi \in \Gamma\}$$

is a pseudogroup on Y, called the **pullback pseudogroup**. $\Diamond$

4.1.2 Pseudogroups as (étale) groupoids

We recall a standard way to describe pseudogroups using Lie groupoids, originally due to Haefliger (section I.6 of [46]; see also Example 5.23 of [73] for a modern reference). The key idea is to encode a pseudogroup by the germs of its elements.

Example 4.1.9. For any pseudogroup Γ on X, the set of its germs

$$\mathrm{Germ}(\Gamma) := \{\mathrm{germ}_x(\phi) \mid \phi \in \Gamma, x \in \mathrm{dom}(\phi)\}$$

is an étale effective Lie groupoid on X, called the **germ groupoid of** Γ. Algebraically, $\mathrm{Germ}(\Gamma)$ is a groupoid over X, with multiplication

$$\mathrm{germ}_{\phi(x)}(\psi) \cdot \mathrm{germ}_x(\phi) = \mathrm{germ}_x(\psi \circ \phi).$$

The sheaf topology on $\mathrm{Germ}(\Gamma)$ is the (possibly non-Hausdorff and non-second countable) topology generated by subsets of the kind

$$\hat{U}_{\phi_0} := \{\mathrm{germ}_x(\phi_0) \mid x \in U\} \subseteq \mathrm{Germ}(\Gamma),$$

for every $\phi_0 \in \Gamma$ and $U = \mathrm{dom}(\phi_0) \subseteq X$. Moreover, $\mathrm{Germ}(\Gamma)$ is a finite-dimensional smooth manifold modelled on X, where the charts are given by

$$\chi_{\phi_0} : U \to \hat{U}_{\phi_0}, \quad x \mapsto \mathrm{germ}_x(\phi_0).$$

for each $\phi_0 \in \Gamma$.

To prove that $\mathrm{Germ}(\Gamma)$ is étale, note that, for any $\phi \in \Gamma$, the map

$$b_\phi : \mathrm{dom}(\phi) \to \mathrm{Germ}(\Gamma), \quad x \mapsto \mathrm{germ}_x(\phi)$$

is a local bisection of $\mathrm{Germ}(\Gamma)$. Possibly restricting its domain, b_ϕ is a diffeomorphism onto its image, which inverts locally the source map $s : \mathrm{Germ}(\Gamma) \to X, \mathrm{germ}_x(\phi) \mapsto x$, hence s is a local diffeomorphism.

Last, for the effectiveness, consider two local bisections b_1 and b_2 of $\mathrm{Germ}(\Gamma)$: by the étale property, they can be written, up to restriction, as $b_i = b_{\phi_i}$ for some $\phi_1, \phi_2 \in \Gamma$. If $t \circ b_1 = t \circ b_2$ on the intersection U of their domains, for every $x \in U$ we have $\phi_1(x) = \phi_2(x)$, hence $b_1 = b_2$:

$$b_1(x) = \mathrm{germ}_x(\phi_1) = \mathrm{germ}_x(\phi_2) = b_2(x) \quad \forall x \in U. \qquad \lozenge$$

Conversely, any étale Lie groupoid induces a pseudogroup on its base. The description is slightly neater in the effective case:

Example 4.1.10. Let $\mathcal{G} \rightrightarrows X$ be an étale effective Lie groupoid; the **pseudogroup associated to** $\mathcal{G}$ is

$$\Gamma(\mathcal{G}) := \{t \circ b \mid b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})\} \subseteq \mathrm{Diff}_{\mathrm{loc}}(X).$$

Using the composition of bisections (Definition 3.1.1), one sees that $\Gamma(\mathcal{G})$ satisfies the group-like properties of a pseudogroup. The first sheaf-like axiom is immediate: given $t \circ b : U \to X$ and an open $V \subseteq U$, then $(t \circ b)_{|V} = t \circ b_{|V} \in \Gamma(\mathcal{G})$ because $b_{|V}$ is still a local bisection.

For the second sheaf-like axiom, effectiveness is used. Consider an element $\phi \in \mathrm{Diff}_{\mathrm{loc}}(X)$ and an open cover $\{U_i\}_i$ of $\mathrm{dom}(\phi)$ such that $\phi_{|U_i} = t \circ b_i \in \Gamma(\mathcal{G})$, for some bisections $b_i \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$; we define a map $b : \mathrm{dom}(\phi) \to \mathcal{G}$ as $b(x) = b_i(x)$ if $x \in U_i$. If $x \in U_i \cap U_j$, then $b_i(x) = b_j(x)$, i.e. b is well defined: indeed,

$$t \circ (b_i)_{|U_i \cap U_j} = (t \circ b_i)_{|U_i \cap U_j} = (t \circ b_j)_{|U_i \cap U_j} = t \circ (b_j)_{|U_i \cap U_j}$$

and the effectiveness of $\mathcal{G}$ implies $(b_i)_{|U_i \cap U_j} = (b_j)_{|U_i \cap U_j}$. Moreover, b is a section of s and $t \circ b = \phi \in \Gamma(\mathcal{G})$, hence $b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ and $\Gamma(\mathcal{G})$ is a pseudogroup. $\lozenge$

The previous two constructions yields the following correspondence:

Proposition 4.1.11. *Given the manifold X, there is a bijective correspondence*

$$\left\{\textit{pseudogroups on } X\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\textit{(isomorphism classes of)}\\ \textit{étale effective Lie groupoids over } X\end{array}\right\}$$

$$\Gamma \mapsto \mathrm{Germ}(\Gamma), \qquad\qquad \Gamma(\mathcal{G}) \leftarrow\!\shortmid \mathcal{G}.$$

4.1.3 Jet groupoids of pseudogroups

Besides the germ groupoid of a pseudogroup, one is also interested in the groupoids of its jets. Recall from Definition 2.6.1 that $J^k(X,X)$ denotes the manifold of k-jets of diffeomorphisms between open sets of X.

Definition 4.1.12. The k^{th}**-jet groupoid of a pseudogroup** Γ on the manifold X is the set

$$J^k\Gamma := \{j^k_x\phi \in J^k(X,X) \mid \phi \in \Gamma, x \in \mathrm{dom}(\phi)\}$$

with the groupoid multiplication

$$j^k_{\phi(x)}\psi \cdot j^k_x\phi = j^k_x(\psi \circ \phi),$$

where each $j^k_x\phi \in J^k\Gamma$ is an arrow from x to $\phi(x)$, and has inverse $j^k_{\phi^{-1}(x)}\phi^{-1}$. ♦

It is clear that an element $\mathrm{germ}_x(\phi) \in \mathrm{Germ}(\Gamma)$ induces an element $j^k_x\phi \in J^k\Gamma$ for every k; accordingly, a pseudogroup Γ on X induces a tower of groupoids over X

$$\mathrm{Germ}(\Gamma) \to J^\infty\Gamma \to \ldots \to J^{k+1}\Gamma \to J^k\Gamma \to \ldots \to J^2\Gamma \to J^1\Gamma \to J^0\Gamma \subseteq X \times X,$$

where all the arrows are surjective. However, the data of all the jets of the elements of Γ are not enough to encode its germs, i.e. we cannot in general reconstruct the entire Γ starting from $J^k\Gamma$. We review now the fundamental class of pseudogroups where this happens.

Definition 4.1.13. A pseudogroup Γ over X is of **order** k if it can be reconstructed from its k-jets, i.e. for any $\phi \in \mathrm{Diff}_{\mathrm{loc}}(X)$,

$$j^k_x\phi \in J^k\Gamma \quad \forall x \in \mathrm{dom}(\phi) \Rightarrow \phi \in \Gamma. \qquad ♦$$

Example 4.1.14. The pseudogroup $\Gamma = \mathrm{Diff}_{\mathrm{loc}}(X)$ is of order 0. The groupoid $J^k\Gamma = J^k(X,X)$ is also called the **groupoid of invertible k^{th}-order jets**, and it coincides with the jet groupoid $J^k\mathcal{G}$ (Example 3.1.8) for the pair groupoid $\mathcal{G} = X \times X$ on X (Example 3.1.4).

Note that, for $X = \mathbb{R}^n$, the isotropy group of $J^k(X,X)$ at 0 coincides with the general linear group $GL^k(\mathbb{R}^n)$ (Definition 2.6.2): this gives a Lie groupoid isomorphism

$$J^k(X,X) \to \mathbb{R}^n \times \mathbb{R}^n \times GL^k(\mathbb{R}^n), \quad j^k_x\phi \mapsto (x, \phi(x), j^k_0\tilde{\phi}_x),$$

where $\tilde{\phi}_x(z) = \phi(z+x) - \phi(x)$. In particular, for $k = 1$, using Remark 2.6.3, we obtain the Lie groupoid isomorphism

$$J^1(\mathbb{R}^n, \mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}^n \times GL(n, \mathbb{R}), \quad j^1_x\phi \mapsto (x, \phi(x), d_x\phi). \qquad \Diamond$$

Example 4.1.15. Let $G \subseteq GL(n,\mathbb{R})$ be a Lie subgroup and Γ_G the associated pseudogroup (Example 4.1.5): then the jet groupoid $J^1\Gamma_G$ is isomorphic to $\mathbb{R}^n \times \mathbb{R}^n \times G$. This can be seen for instance by restricting the isomorphism between $J^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\mathbb{R}^n \times \mathbb{R}^n \times GL(n,\mathbb{R})$ from Example 4.1.14.

By the same isomorphism, it follows that Γ_G is of order 1: if, for every $x \in \mathrm{dom}(\phi)$, $j^1_x\phi \in J^1\Gamma_G$, then $d_x\phi \in G$, which is precisely the condition so that $\phi \in \Gamma_G$.

Last, $J^1\Gamma_G$ is Morita equivalent to G, in the sense of Definition 3.3.1; this follows from the fact that $\mathbb{R}^n \times \mathbb{R}^n \times G$ is the gauge groupoid $\mathrm{Gauge}(P)$ of the trivial principal G-bundle $P = \mathbb{R}^n \times G$; by Example 3.3.6, it is Morita equivalent to G. $\Diamond$

Example 4.1.16. Let $G \subseteq GL^k(\mathbb{R}^n)$ be a Lie subgroup and Γ_G the associated pseudogroup (Example 4.1.7); then Γ_G is of order k, by arguments similar to the previous example.

Moreover, there is a Morita morphism from the jet groupoid $J^k\Gamma_G$ to G. This follows from Example 3.3.10 applied to the groupoid map

$$J^k\Gamma_G \to \mathbb{R}^n \times \mathbb{R}^n \times G, \quad j^k_x\phi \mapsto (x, \phi(x), j^k_0(\tilde{\phi}_x)).$$

Since $\mathbb{R}^n \times \mathbb{R}^n \times G$ is Morita equivalent to G, by composition we have a Morita morphism from $J^k\Gamma_G$ to G. $\Diamond$

Example 4.1.17. The pseudogroup Γ from Example 4.1.6 (locally defined diffeomorphisms of $\mathbb{R}^n$ with locally constant Jacobian determinant) is of order 2. Indeed, in the case $n = 1$, the determinant of the Jacobian of a diffeomorphism ψ is its first derivative ψ'; if $\psi \in \Gamma$ we ask ψ' to be constant, i.e. $\psi'' = 0$. Given $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$, if $j^2_x\phi \in J^2\Gamma$ for every x, then for each x there is a $\psi_x \in \Gamma$ having the same first and second derivative of ϕ at x. This gives no information on $\phi'(x)$ (meaning that $J^1\Gamma = J^1(\mathbb{R}^n, \mathbb{R}^n)$); however, $\phi''(x) = \psi''_x(x) = 0$ for every x, hence $\phi'(x) = \mathit{const}$ (on each connected component of its domain), i.e. $\phi \in \Gamma$. When $n \geq 2$ the proof is identical but more cumbersome to write, using first and second derivatives. $\Diamond$

Example 4.1.18. The pseudogroup Γ^ω (Example 4.1.3) does not have an order. Indeed, the Taylor polynomial of order k of a smooth function is an analytic representative of its k-jet, i.e. $J^k\Gamma^\omega = J^k(\mathbb{R}^n, \mathbb{R}^n)$. On the other hand, any $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $j^k_x\phi \in J^k\Gamma^\omega$ for every k, even if ϕ is not in Γ^ω. $\Diamond$

4.1.4 Transitive pseudogroups

Definition 4.1.19. Given a pseudogroup $\Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$, the **orbits** of Γ are the equivalence classes w.r.t. the following equivalence relation on X:

$$x \sim y \Leftrightarrow \exists\ \phi \in \Gamma \text{ such that } x \in \mathrm{dom}(\phi), y = \phi(x).$$

The pseudogroup Γ is called **transitive** if it has only one orbit, i.e. for every $x, y \in X$, there exists $\phi \in \Gamma$ sending x to y. ♦

Example 4.1.20. If a pseudogroup Γ on $\mathbb{R}^n$ contains all translations, then it is transitive. For instance, Γ^ω (Example 4.1.3) and Γ_G (Example 4.1.5) are transitive, since translations are analytic and have Jacobian equal to the identity.

A pseudogroup Γ is transitive if and only if $J^k\Gamma$ is transitive for any k, if and only if $\mathrm{Germ}(\Gamma)$ is transitive. This shows also that the correspondence from Proposition 4.1.11, can be restricted to a correspondence between transitive étale effective Lie groupoids and transitive pseudogroups. ♢

Here is another important object associated to a pseudogroup.

Definition 4.1.21. The **isotropy group of order k at** $x_0 \in X$ of a pseudogroup Γ is the isotropy group $(J^k\Gamma)_{x_0}$ of its jet groupoid $J^k\Gamma$. For $k = 1$, we talk about the **linear isotropy group** at x_0:

$$(J^1\Gamma)_{x_0} \subseteq GL(T_{x_0}X).$$ ♦

4

If Γ is transitive, its isotropy groups at any two points are isomorphic, so one talks just of the isotropy group of Γ of order k. For instance, let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup; using Example 4.1.15, the linear isotropy group of Γ_G (Example 4.1.5) is

$$(J^1\Gamma_G)_0 \cong (\mathbb{R}^n \times \mathbb{R}^n \times G)_0 = G.$$

This gives a useful characterisation of the transitive pseudogroups of the type Γ_G.

Lemma 4.1.22. *Let Γ be a transitive pseudogroup of order 1 on $\mathbb{R}^n$ with linear isotropy group G. Then the following are equivalent:*

- $\Gamma = \Gamma_G$,
- Γ *contains the translations.*

Proof. It is clear that Γ_G contains translations. Conversely, let $\phi \in \Gamma$ and $x \in \mathrm{dom}(\phi)$; since the translation φ_x (sending 0 to x) belongs to Γ and has differential $d_y\varphi_x = id_{\mathbb{R}^n}$ at any $y \in \mathbb{R}^n$, we have

$$d_x\phi = d_0(\varphi_x) \circ d_x(\varphi_x^{-1} \circ \phi \circ \varphi_x) \circ d_x\varphi_x^{-1} = d_x(\varphi_x^{-1} \circ \phi \circ \varphi_x).$$

However, $\varphi_x^{-1} \circ \phi \circ \varphi_x$ sends x to x, hence $j^1_x\phi$ belongs to the isotropy group $(J^1\Gamma)_x = G$; this shows that $\Gamma \subseteq \Gamma_G$. On the other hand, if $\phi \in \Gamma_G$, then for every $x \in \mathrm{dom}(\phi)$, $j^1_x\phi \in G = (J^1\Gamma)_x \subseteq J^1_x\Gamma$; using the order 1 hypothesis, ϕ must be inside Γ. Q.E.D.

Without the order 1 hypothesis, Lemma 4.1.22 does not hold: for instance, the pseudogroup Γ^ω (Definition 4.1.3) contains translations and has isotropy group $GL(n,\mathbb{R})$, but does not coincide with $\Gamma_{GL(n,\mathbb{R})} = J^k(\mathbb{R}^n,\mathbb{R}^n)$.

Returning to the pseudogroup Γ_G, its higher order isotropy groups can be computed by means of prolongations of $G \subseteq GL(n,\mathbb{R})$ introduced in chapter 2.

Lemma 4.1.23. *Let $G \subseteq GL(n,\mathbb{R})$ be a Lie subgroup, Γ_G its associated pseudogroup (Example 4.1.5) and $J^2\Gamma_G \rightrightarrows \mathbb{R}^n$ its 2-jet groupoid (Definition 4.1.12). Then the isotropy group of $J^2\Gamma_G$ at 0 coincides with the prolongation 2.6.4 of G, i.e.*

$$(J^2\Gamma_G)_0 = G^{(1)}.$$

Proof. We prove that the identity map defines a Lie group diffeomorphism

$$id : (J^2\Gamma_G)_0 \to G^{(1)}.$$

Take an element $j^2_0\phi \in (J^2\Gamma_G)_0$; we want to show that it belongs to

$$G^{(1)} = \{j^2_0\phi \in GL^2(\mathbb{R}^n) \mid j^1_0\phi \in G, dFr(\phi)(\mathbb{R}^n \oplus \mathfrak{g}) = \mathbb{R}^n \oplus \mathfrak{g}\}.$$

The first condition is immediate by definition of Γ_G. For the second, we write explicitly the differential

$$d_{j^1_0 id}Fr(\phi): \quad v = \gamma'(0) \mapsto dFr(\phi)(v) = \left.\frac{d}{dt}\right|_{t=0} Fr(\phi\circ\gamma)(t) = \left.\frac{d}{dt}\right|_{t=0} j^1_0(\phi\circ f)(t),$$

where $\gamma(t) = j^1_0 f(t)$ is a curve in $Fr(\mathbb{R}^n) \cong \mathbb{R}^n \times GL(n,\mathbb{R})$ starting at $\gamma(0) = j^1_0 f(0) = j^1_0 id$ (here we use the same symbol for the time parameter $0 \in \mathbb{R}$ and the vector $0 \in \mathbb{R}^n$).

If v is in $\mathbb{R}^n \oplus \mathfrak{g}$, then $f(t)$ will be in G for all t; as a consequence,

$$j^1_0(\phi \circ f)(t) = j^1_0\phi \circ j^1_0 f(t)$$

is the composition of two elements in G, hence is in G itself, yielding $dFr(\phi)(v) \in \mathbb{R}^n \oplus \mathfrak{g}$.

Viceversa, take $j^2_0 f \in G^{(1)}$: it follows that $j^1_0 f \in G$ (and $f(0) = 0$), so we have just to prove that $j^1_x f$ is in G for all other $x \in \mathrm{dom}(f)$ in order to conclude that $f \in \Gamma_G$; then $j^2_0 f \in (J^2\Gamma_G)_0$.

We use the other hypothesis on f, i.e. that the linear isomorphism

$$d_{j^1_0 id}Fr(f) : \mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}^n \oplus \mathfrak{gl}(n,\mathbb{R})$$

preserves $\mathbb{R}^n \oplus \mathfrak{g}$. This is actually equivalent to the fact that $Fr(f)$ can be restricted to a local diffeomorphism $\mathbb{R}^n \times G \to \mathbb{R}^n \times G$. In turn, this means that $Fr(f)$ sends $j^1_x\phi \cong (\phi(x), d_x\phi)$ to

$$j^1_x(f\circ\phi) \cong (f(\phi(x)), d_{\phi(x)}f \circ d_x\phi) \in \mathbb{R}^n \times G,$$

i.e. that $d_{\phi(x)} f \circ d_x\phi \in G \; \forall x \in \mathrm{dom}(\phi)$.

Last, since $d_x\phi$ is already in G, the previous fact is equivalent to asking $d_{\phi(x)} f \in G$ for all $\phi(x) \in \mathrm{Im}(\phi) = \mathrm{dom}(f)$, and this is precisely the condition $f \in \Gamma_G$. Q.E.D.

Remark 4.1.24. The above result can be rewritten as

$$(J^2\Gamma_G)_0 = ((J^1\Gamma_G)_0)^{(1)}.$$

Using similar computations, for any Γ pseudogroup on $\mathbb{R}^n$ (even not transitive), one sees that

$$(J^k\Gamma)_{x_0} \cong ((J^{k-1}\Gamma)_{x_0})^{(1)} \quad \forall x_0 \in \mathbb{R}^n. \qquad \diamondsuit$$

4.2 Lie pseudogroups

The jet groupoids $J^k\Gamma$ of a pseudogroup Γ are (topological) subgroupoids of the Lie groupoids $J^k(X, X)$, but may fail to be Lie subgroupoids. Since smoothness is fundamental for many results, we will restrict our attention to the class of pseudogroups which safisfy it.

Definition 4.2.1. A pseudogroup Γ over X is called a **Lie pseudogroup** if it has finite order (Definition 4.1.13) and, for every $k \in \mathbb{N}$,

- the groupoid $J^k\Gamma$ is a smooth submanifold of $J^k(X, X)$,
- the groupoid morphism $\pi^k_{k-1} : J^k\Gamma \to J^{k-1}\Gamma$ is a smooth submersion. ♦

It follows from the definition that $J^k\Gamma$ is a Lie subgroupoid. In particular, the tower of groupoids from Section 4.1.3 4

$$\mathrm{Germ}(\Gamma) \to J^\infty\Gamma \to \ldots \to J^{k+1}\Gamma \to J^k\Gamma \to \ldots \to J^2\Gamma \to J^1\Gamma \to J^0\Gamma \subseteq X \times X$$

becomes a tower of *Lie* groupoids, where all the arrows but the left-most are submersions (since the germ topology and smooth structure is very different from the jet one).

Most pseudogroups arising from applications, e.g. Γ_G (Example 4.1.5), are Lie pseudogroups. For an explicit (and less natural) example of pseudogroup which is not Lie, see Example 3.3.4 of [100].

Remark 4.2.2 (Lie pseudogroups as PDEs)**.** Historically, Lie pseudogroups arised from the study of symmetries of PDEs. Note also that, if Γ is a Lie pseudogroup on X, then every jet groupoid $J^k\Gamma$ is a PDE of order k on $\mathrm{pr} : X \times X \to X$.

We remark that the precise regularity conditions a pseudogroup should satisfy in order to be called *Lie* are not standard in the literature, and depends on which problems one is interested to tackle. For instance, one could weaken Definition 4.2.1 by asking that, for a pseudogroup Γ of order k, only $J^k\Gamma$ and $J^{k-1}\Gamma$ are smooth and $\pi^k_{k-1} : J^k\Gamma \to J^{k-1}\Gamma$ is a submersion. This is the minimal regularity condition that makes sure $J^k\Gamma$ is a PDE (see also Remark 3.3.3 of [100]). $\diamondsuit$

If Γ is a Lie pseudogroup, the isotropy groups $(J^k\Gamma)_{x_0}$ become Lie groups.

Definition 4.2.3. Let Γ be a Lie pseudogroup over X. The **isotropy algebra of order** k of Γ at $x_0 \in X$ is the Lie algebra of its isotropy group $(J^k\Gamma)_{x_0}$. If $k = 1$, we call it the **linear isotropy algebra** at x_0. ♦

For a Lie pseudogroup Γ the projection $\pi^k_{k-1} : J^k\Gamma \to J^{k-1}\Gamma$ is a submersion, hence one can consider the **symbol space** of $J^k\Gamma$:

$$\mathfrak{g}^k := \ker(d\pi^k_{k-1} : TJ^k\Gamma_{|X} \to TJ^{k-1}\Gamma_{|X}) \to X.$$

It is a vector bundle on X, whose pullback on $J^{k-1}\Gamma$ models the affine bundle $J^k\Gamma \to J^{k-1}\Gamma$; this follows from the fact that $\mathfrak{g}^k$ coincides with the symbol space of $J^k\Gamma$, viewed as a PDE (Remark 4.2.2), and from the general theory of PDEs (Section 1.2.1).

For $k = 1$, the fibre of $\mathfrak{g}^1$ over $x \in X$ is the linear isotropy algebra at x:

$$\mathfrak{g}^1_x = \mathrm{Lie}(J^1\Gamma)_x.$$

On the other hand, for $k \geq 1$ we have only an inclusion:

$$\mathfrak{g}^k_x \subseteq \mathrm{Lie}(J^k\Gamma)_x.$$

4.2.1 Special kinds of Lie pseudogroups

In view of Definition 4.2.1, one can describe some special classes of Lie pseudogroups using the theory of Lie groupoids. They will be used in Section 5.2 when discussing solutions to the integrability problem for almost Γ-structures.

Definition 4.2.4. A Lie pseudogroup Γ of order 1 over X is called **irreducible** if the following canonical representation of the Lie groupoid $J^1\Gamma$ on the vector bundle TX is irreducible (i.e. it has no proper subrepresentations):

$$j^1_x\phi \cdot v := d_x\phi(v), \quad \text{for all } j^1_x\phi \in J^1\Gamma, v \in T_xX.$$ ♦

If Γ is transitive, it is enough to ask that the induced representation of the linear isotropy group $(J^1\Gamma)_{x_0}$ on $T_{x_0}X$ (for some $x_0 \in X$) is irreducible.

Definition 4.2.5. A Lie pseudogroup Γ over X is called of **finite type** k if the symbol space $\mathfrak{g}^k$ vanishes, or, equivalently, if $J^k\Gamma = J^{k-1}\Gamma$ (since $J^k\Gamma \to J^{k-1}\Gamma$ is an affine bundle modelled on $\mathfrak{g}^k$). ♦

Another fundamental class of Lie pseudogroups are the elliptic ones.

Definition 4.2.6. Let Γ be a transitive Lie pseudogroup of order 1 over X. Then Γ is called **elliptic** when its linear isotropy algebra is elliptic, i.e. it does not contain linear maps of order 1 (Definition 2.1.12). ♦

Example 4.2.7. Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup and consider the transitive Lie pseudogroup Γ_G (Example 4.1.5). Then Γ_G is finite type, irreducible or elliptic if and only if G is so. $\Diamond$

We recall now that the ellipticity of a Lie pseudogroup is equivalent to the system of linear PDEs defining the Γ-vector fields being elliptic. We follow the original works by Guillemin, Kumpera, Spencer and Malgrange [44, 63, 69] and reformulate their results.

Definition 4.2.8. Let Γ be a pseudogroup on X; a vector field on X is called a **Γ-vector field** when its flow is made up of diffeomorphisms in Γ. ♦

Note that Γ-vector fields form a Lie subalgebra (Proposition 3.3 of [44]).

Proposition 4.2.9 (theorem 15.1 of [44])**.** *Let Γ be a transitive Lie pseudogroup of order 1 over X. There exists a linear differential operator D_Γ of order 1 on the vector bundle TX such that*

- $\ker(D_\Gamma)$ *coincides with the set of Γ-vector fields (see Definition 4.2.8),*
- *the pseudogroup Γ is elliptic if and only if the operator D_Γ is elliptic.*

A fundamental consequence of Proposition 4.2.9 is the following:

Proposition 4.2.10. *Let X be an analytic manifold, and $\Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$ a transitive elliptic Lie pseudogroup generated by analytic diffeomorphisms (see Example 4.1.2). Then*

- *every Γ-vector field is analytic,*
- *every element of Γ is analytic.*

For the Lie pseudogroup $\Gamma = \Gamma_G$, the operator D_Γ becomes more concrete, and Proposition 4.2.9 is much easier to prove, by means of linear algebra. Furthermore, there is a more powerful version of Proposition 4.2.10:

Proposition 4.2.11 ([3, Corollary 14.2 and discussion thereafter])**.** *A Lie subgroup $G \subseteq GL(n, \mathbb{R})$ is elliptic (as in Definition 2.1.12) if and only if every element of Γ_G is real analytic.*

Example 4.2.12. Since $GL(n, \mathbb{C})$ is an elliptic group, the elements of $\Gamma_{GL(n,\mathbb{C})}$ are real analytic. By Example 4.1.5, we find therefore a more elegant (though more sophisticated) way to prove a classical result: that the bi-holomorphic maps of $\mathbb{C}^n$ are real analytic. Indeed, this follows also from the (non-trivial) fact that solutions of the Cauchy-Riemann equations are real analytic. $\Diamond$

4.3 The abstract framework for Lie pseudogroups: Pfaffian groupoids

Definition 4.3.1. A **Pfaffian groupoid** $(\mathcal{G}, \omega)$ over M consists of

- a Lie groupoid $\mathcal{G} \rightrightarrows M$
- a representation $E \to M$ of $\mathcal{G}$
- a differential form $\omega \in \Omega^1(\mathcal{G}, t^*E)$

such that

1. ω is multiplicative (Definition 3.4.2)
2. ω is of constant rank
3. ω is s-involutive (as in Definition 1.3.1).

The subbundle

$$\mathfrak{g} := (\ker(\omega) \cap \ker(ds))_{|M} \subseteq \mathrm{Lie}(\mathcal{G})$$

is called the **symbol space** of $(\mathcal{G}, \omega)$; by the s-involutivity, it is a Lie subalgebroid of $\mathrm{Lie}(\mathcal{G})$:

$$\mathfrak{g} \subseteq \ker(ds)_{|M} = \mathrm{Lie}(\mathcal{G}).$$

♦

By Lemma 3.4.13, we could replace the representation E with $E(\omega)$ so that $\omega \in \Omega^1(\mathcal{G}, t^*E(\omega))$ is a pointwise surjective form. This suggests to define the following class of Pfaffian groupoids.

Definition 4.3.2. A Pfaffian groupoid $(\mathcal{G}, \omega)$ is called **full** if the form ω is pointwise surjective. ♦

Remark 4.3.3 (Pfaffian groupoids and Pfaffian fibrations)**.** A full Pfaffian groupoid is the same thing as a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a representation E and a multiplicative form $\theta \in \Omega^1(\mathcal{G}, t^*E)$ such that $s : (\mathcal{G}, \theta) \to M$ is a Pfaffian fibration (Definition 1.3.1): this is an immediate consequence of Proposition 3.4.12.

In other words, a full Pfaffian groupoid amounts, by Proposition 3.4.16, to a multiplicative distribution $\mathcal{C} \subseteq T\mathcal{G}$ such that $\mathcal{C} \cap \ker(ds)$ is involutive. This was also the original definition of a Pfaffian groupoid ([81, Definition 6.1.4]); we have decided to weaken it in order to allow some natural examples where the form is not surjective (see 4.3.8 and 4.3.9), as well as to single out which properties depend on the surjectivity of the form and which are common to all Pfaffian groupoids.

In view of this discussion, we will consider the symbol space of $(\mathcal{G}, \omega)$ viewed as a Pfaffian fibration (Definition 1.3.1):

$$\mathfrak{g}(\omega) := \ker(\omega) \cap \ker(ds).$$

From the second part of Lemma 3.4.14 we have the following relations with the symbol space $\mathfrak{g}$ of $\mathcal{G}$ viewed as a Pfaffian groupoid:

$$\mathfrak{g} = \mathfrak{g}(\omega)_{|M}, \qquad \mathfrak{g}(\omega) \cong t^*\mathfrak{g}.$$

Moreover, $\mathfrak{g}(\omega)$ is s-involutive as per Definition 1.3.1 if and only if $\mathfrak{g} \subseteq \mathrm{Lie}(G)$ is a Lie subalgebroid. $\lozenge$

Recall from Proposition 3.4.12 that condition 2 of Definition 4.3.1 can be replaced with ω being either s-transversal or t-transversal. The choice of considering the source or the target is conventional also in condition 3: with a proof similar to that of Proposition 3.4.12, one can check that the differential of the inversion i induces an isomorphism

$$\ker(ds) \cap \ker(\omega) \xrightarrow{\cong} \ker(dt) \cap \ker(\omega),$$

and that ω is s-involutive if and only if it is t-involutive.

Here is a second important class of Pfaffian groupoids.

Definition 4.3.4. A Pfaffian groupoid $(\mathcal{G}, \omega)$ is called **Lie-Pfaffian**, or of Lie type, if it satisfies the additional condition

$$\ker(\omega) \cap \ker(dt) = \ker(\omega) \cap \ker(ds). \qquad \blacklozenge$$

Remark 4.3.5. For a Pfaffian groupoid $(\mathcal{G}, \omega)$, an important consequence of being Lie-Pfaffian is that TM becomes a representation of $\mathcal{G}$. More precisely, any $g \in \mathcal{G}$ defines a linear isomorphism

$$\lambda_g : T_{s(g)}M \to T_{t(g)}M, \quad X \mapsto dt(\tilde{X}),$$

where $\tilde{X}$ is any element in $\ker(\omega_g)$ such that $d_g s(\tilde{X}) = X$. If $\tilde{X}'$ is another such vector, then the Lie-Pfaffian property yields

$$\tilde{X} - \tilde{X}' \in \ker(\omega) \cap \ker(ds) = \ker(\omega) \cap \ker(dt),$$

so that $dt(\tilde{X}) = dt(\tilde{X}')$, i.e. λ is well defined. $\lozenge$

A Pfaffian groupoid $(\mathcal{G}, \omega)$ can be described infinitesimally by its Lie algebroid A together with the Spencer operator (D, l) induced by the multiplicative form ω (Proposition 3.4.22). In particular, the map

$$l : A \to E, \quad l_x(v) := \omega_{1_x}(v)$$

defines a morphism between vector bundles over M, with kernel equal to the symbol space $\mathfrak{g}$ (Definition 4.3.1). It can be used to detect the two properties we have introduced above (fullness and Lie type).

4

Proposition 4.3.6. *Let $(\mathcal{G},\omega)$ be a Pfaffian groupoid over M, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$ taking values in a representation E of $\mathcal{G}$; denote by A the Lie algebroid of $\mathcal{G}$ and let $l : A \to E$ be the vector bundle map described above.*

1. *$(\mathcal{G},\omega)$ is full $\Leftrightarrow$ l is surjective*
2. *$(\mathcal{G},\omega)$ is Lie-Pfaffian $\Leftrightarrow$ $\ker(l) \subseteq \ker(\rho)$.*
3. *If $(\mathcal{G},\omega)$ is both full and Lie-Pfaffian, then E admits a unique structure of Lie algebroid such that $l : A \to E$ is a Lie algebroid morphism.*

Proof. By the s-transversality of ω (Proposition 3.4.12), we have for every $g \in \mathcal{G}$

$$\omega_g(T_g\mathcal{G}) = \omega_g(\ker(\omega_g) + \ker(d_g s)) = \omega_g(\ker(d_g s)).$$

In particular this holds for $g = 1_x$; then, if ω is pointwise surjective, ω_{1_x} remains surjective when restricted to $A_x = \ker(d_{1_x} s)$, i.e. l is surjective as well.

The converse follows from the multiplicativity of ω together with the formulae of Example 3.2.9:

$$\omega_g(a_g(\alpha)) = l_{t(g)}(\alpha), \quad g \in \mathcal{G}, \alpha \in A_{t(g)},$$

where a denotes the infinitesimal action of the left action of $\mathcal{G}$ on itself (Example 3.1.13).

For the second part, assume that $\mathcal{G}$ is Lie-Pfaffian; if $v \in \mathfrak{g}_x = \ker(d_{1_x} s) \cap \ker(\omega_{1_x}) = \ker(d_{1_x} t) \cap \ker(\omega_{1_x})$, then $\rho_x(v) = d_{1_x} t(v) = 0$, so $\ker(l) = \mathfrak{g} \subseteq \ker(\rho)$.

Conversely, we show that the subspaces $\ker(d_g s) \cap \ker(\omega_g)$ and $\ker(d_g t) \cap \ker(\omega_g)$ of $T_g\mathcal{G}$ coincides for every $g \in \mathcal{G}$; actually, by Lemma 3.4.11, it is enough to prove it only in the case $g = 1_x$. From the hypothesis $\mathfrak{g}_x \subseteq \ker(\rho_x) = \ker(d_{1_x} t)$ we get the inclusion

$$\ker(d_{1_x} s) \cap \ker(\omega_{1_x}) \subseteq \ker(d_{1_x} t) \cap \ker(\omega_{1_x});$$

since we already know that these two spaces have the same dimensions, they must coincide.

For the third claim, the anchor map of E is induced by ρ, and is well defined because of condition 1 and 2:

$$\rho_E : E \to TM, \quad l(\alpha) \mapsto \rho(\alpha).$$

Similarly, the bracket on E is induced by that on A:

$$[l(\alpha), l(\beta)]_E := l([\alpha,\beta]_A), \quad \forall \alpha, \beta \in \Gamma(A).$$

In order to see that $[\cdot,\cdot]_E$ is well defined, we consider $\alpha, \alpha' \in A$ such that $l(\alpha) = l(\alpha')$ and prove

$$l([\alpha,\beta]_A) = l([\alpha',\beta]_A).$$

Since l is part of a Spencer operator, it is enough to use condition 2 of this proposition and the second equation of Definition 3.4.20:

$$l([\alpha - \alpha', \beta]_A) = D_{\cancel{\rho_A(\alpha-\alpha')}}(\beta) - \nabla_\beta \cancel{l(\alpha - \alpha')} = 0.$$ Q.E.D.

Remark 4.3.7. In the original definition of Pfaffian groupoid, the name *Lie-Pfaffian* comes from the fact that the representation E becomes a Lie algebroid. However, in our definition, this requires the fullness property to hold.

It would therefore be appropriate to impose in Definition 4.3.4 the condition that E has a Lie algebroid structure such that the (possibly non-surjective) map $l : A \to E$ becomes a Lie algebroid morphism.

However, for the theorems we prove and the examples we consider in this thesis, we will need such a condition only for Pfaffian groupoids which are full (for which it is automatic), so we decided to omit it from the general definition of Lie-Pfaffian. $\diamondsuit$

4.3.1 Examples

Example 4.3.8. Any Lie group G, endowed with any representation V and multiplicative 1-form $\omega \in \Omega^1(G, V)$, is a Lie-Pfaffian groupoid.

Let us prove this. Since the source $s : G \to \{*\}$ is a trivial map, we have $\ker(ds) = TG$; then ω satisfies $\ker(ds)+\ker(\omega) = TG$, hence is of constant rank by Proposition 3.4.12. Then we check that ω is s-involutive: since $\ker(ds)\cap\ker(\omega) = \ker(\omega)$, it is enough to show the involutivity of the kernel of ω. In order to do so, recall that ω is encoded by the G-equivariant map $l = \omega_e : \mathfrak{g} \to V$ (Example 3.4.6); then

$$\ker(\omega) = \coprod_{g\in G} \ker(\omega_g) = \coprod_{g\in G} d_e R_g(\ker(l)).$$

Since l is G-equivariant, it is a morphism between G-representations (respectively, the adjoint representation of G on $\mathfrak{g}$ and the representation V); then l is also a morphism between the associated $\mathfrak{g}$-representations, i.e.

$$l(\alpha \cdot v) = \alpha \cdot l(v) \quad \forall \alpha \in \mathfrak{g}, v \in \mathfrak{g}.$$

However, the adjoint representation of $\mathfrak{g}$ on itself coincides with the Lie bracket, hence we have

$$l([\alpha, v]) = \alpha \cdot l(v) \quad \forall \alpha \in \mathfrak{g}, v \in \mathfrak{g}.$$

From this formula it follows that $\ker(l) \subseteq \mathfrak{g}$ is a Lie ideal; *a fortiori*, it is also a Lie subalgebra. Using the fact that any right-invariant distribution $D \subseteq TG$ is involutive if and only if $D_e \subseteq \mathfrak{g}$ is a Lie subalgebra, we conclude that $\ker(\omega)$ is involutive, so (G, ω) is a Pfaffian groupoid. Last, the Lie type condition is trivial since the source and the target map coincide.

Every Lie group admits the following two "extreme" Lie-Pfaffian structures:

- for an arbitrary representation V and $l = 0$, we get the zero form $\omega = 0 \in \Omega^1(G, V)$; then $(G, 0)$ is a non-full Lie-Pfaffian groupoid with symbol space equal to its Lie algebra $\mathfrak{g}$.

- for the adjoint representation $V = \mathfrak{g}$ and $l = id_{\mathfrak{g}}$, we get the Maurer-Cartan form $\omega_{MC} \in \Omega^1(G, \mathfrak{g})$; then (G, ω_{MC}) is a full Lie-Pfaffian groupoid with zero symbol space. $\lozenge$

Example 4.3.9. Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid over M; its isotropy group $(\mathcal{G}_x, \omega_{|\mathcal{G}_x})$ at any point $x \in M$ is a Pfaffian groupoid, since the multiplicativity condition restricts to $\mathcal{G}_x$. Note that, by Example 4.3.8, $\mathcal{G}_x$ is always of Lie type, even if $\mathcal{G}$ is not. On the other hand, even if $\mathcal{G}$ is full, $\mathcal{G}_x$ is not necessarily full. $\lozenge$

Example 4.3.10. Let $\mathcal{G}$ be a Lie groupoid; then its k^{th}-jet groupoid $J^k\mathcal{G}$ (Example 3.1.8), together with its Cartan form ω (Definition 1.1.4), is a full Lie-Pfaffian groupoid.

Indeed, recall from Example 3.4.5 that $\omega \in \Omega^1(J^k\mathcal{G}, t^*J^{k-1}A)$ is multiplicative. On the other hand, one sees that $J^k\mathcal{G} \subseteq J^k(s)$ is a PDE on the source map s, hence a Pfaffian fibration; by Remark 4.3.3, this shows that $J^k\mathcal{G}$ is a full Pfaffian groupoid.

The Lie-Pfaffian condition can be proved directly using the definition of the Cartan form ω (see Proposition 6.1.18 of [81]), or, more elegantly, by noticing that $l = \pi^k_{k-1} : J^kA \to J^{k-1}A$ is a Lie algebroid morphism and checking the equivalent condition $\ker(l) \subseteq \ker(\rho)$ of Proposition 4.3.6. $\lozenge$

Example 4.3.11. Let Γ be a Lie pseudogroup on X (Definition 4.2.1); then its jet groupoid $J^k\Gamma$ (Definition 4.1.12), together with its Cartan form ω, is a full Lie-Pfaffian groupoid.

To prove this, recall that $J^k\Gamma$ is a PDE on $\mathrm{pr} : X \times X \to X$ (Remark 4.2.2), hence $s : J^k\Gamma \to X$ is a Pfaffian fibration. Moreover, $J^k\Gamma$ sits inside the full Lie-Pfaffian groupoid $J^k\mathcal{G}$, for $\mathcal{G} = X \times X$ (Example 4.3.10). Accordingly, one has only to check that the Cartan form ω remains multiplicative when restricted to $J^k\Gamma$ and to prove the Lie-Pfaffian condition: these are straightforward computations, for which we refer to Proposition 3.4.1 of [100].

Knowing that $(J^k\Gamma, \omega)$ is a Pfaffian groupoid, we can consider its symbol space $\mathfrak{g}(\omega)$; as one expects, it coincides with the symbol space $\mathfrak{g}^k = \ker(d\pi^k_{k-1})_{|X}$ of $J^k\Gamma$ viewed as a PDE (see also Section 4.2).

For instance, let us consider the Pfaffian groupoid $J^1\Gamma_G$, for Γ_G the Lie pseudogroup associated to a Lie subgroup $G \subseteq GL(n, \mathbb{R})$ (Definition 4.1.5); its symbol space is the trivial vector bundle on $\mathbb{R}^n$ with fibre $\mathfrak{g}$. This follows immediately using the fact that $J^1\Gamma_G \cong \mathbb{R}^n \times \mathbb{R}^n \times G$ (Example 4.1.15) and that, for every $j^1_x id_{\mathbb{R}^n} \in J^1\Gamma_G$,

$$d_{j^1_x id_{\mathbb{R}^n}} \pi^1_0 : T_x\mathbb{R}^n \oplus T_x\mathbb{R}^n \oplus \mathfrak{g} \to T_x\mathbb{R}^n \otimes T_x\mathbb{R}^n$$

is the projection on the first two components. $\lozenge$

Example 4.3.12. Let $(\mathcal{G}, \omega)$ be a full Pfaffian groupoid over M and $\mu : P \to M$ a surjective submersion; the pullback groupoid (Definition 3.1.9)

$$\mu^*\mathcal{G} = P \times_M \mathcal{G} \times_M P \rightrightarrows P$$

has a natural structure of full Pfaffian groupoid.

To see this, consider the groupoid morphism

$$\tau : \mu^*\mathcal{G} \to \mathcal{G}, \quad (p, g, q) \mapsto g;$$

if $E \in \mathrm{Rep}(\mathcal{G})$ is the coefficient of ω, then $\mu^* E$ is a representation of $\mu^*\mathcal{G}$ and

$$\tau^*\omega \in \Omega^1(\mu^*\mathcal{G}, \tau^*(t^*E))$$

is a pointwise surjective multiplicative form (Example 3.4.3). Moreover, since τ is a surjective submersion and $s : (\mathcal{G}, \omega) \to M$ is a Pfaffian fibration (Remark 4.3.3), then $s : (\mu^*\mathcal{G}, \tau^*\omega) \to P$ is a Pfaffian fibration as well, hence $(\mu^*\mathcal{G}, \tau^*\omega)$ is a full Pfaffian groupoid. ♢

Definition 4.3.13. Let $\mathcal{G}$ be a Lie groupoids acting on $\mu : (P, \beta) \to M$ and $\alpha \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form of constant rank. Then $(\mathcal{G}, \alpha)$ is a Pfaffian groupoid if and only if the action groupoid $(\mathcal{G}\ {}_s\!\times_\mu P, \mathrm{pr}_1^*\,\alpha)$ is Pfaffian. Indeed, we only have to prove that α is s-involutive if and only if $(\mathrm{pr}_1)^*\alpha$ is pr_2-involutive, and this follows from

$$\ker(\mathrm{pr}_1^*\,\alpha) \cap \ker(d\,\mathrm{pr}_2) = \{(X, v) \in T\mathcal{G}\ {}_{ds}\!\times_{d\mu} TP \mid \alpha(X) = 0, v = 0\} =$$
$$= (\ker(\alpha) \cap \ker(ds)) \times \{0\}. \qquad ♦$$

Note that $\mathcal{G}$ being Lie-Pfaffian is independent from $\mathcal{G}\ {}_s\!\times_\mu P$ being Lie-Pfaffian. For instance, the Pfaffian group $(G, 0)$ from Example 4.3.8 is automatically of Lie type; however, if G acts on P, the Pfaffian groupoid $(G \times P, 0)$ is not always Lie-Pfaffian.

On the other hand, in the general setting above, one can consider also the space $\mathcal{G}\ {}_s\!\times_\mu P$ as a Lie groupoid over M (by composing the structure maps with μ). Then $(\mathcal{G}, \alpha)$ is a Pfaffian groupoid if and only if $(\mathcal{G}\ {}_s\!\times_\mu P, \mathrm{pr}_1^*\,\alpha)$ is a Pfaffian groupoid (over M); again, this follows from

$$\ker(\mathrm{pr}_1^*\,\alpha) \cap \ker(d(\mu \circ \mathrm{pr}_2)) = (\ker(\alpha) \cap \ker(ds)) \times \ker(d\mu).$$

Moreover, one can check directly that the first Pfaffian groupoid is of Lie type if and only if the other is.

Example 4.3.14. A linear Pfaffian fibration (F, θ) (Example 1.6.3) is a full Lie-Pfaffian groupoid. Indeed, the form θ is multiplicative (Example 3.4.7), it is Pfaffian (Lemma 1.6.4) and the Lie type condition is automatic because source and target coincide. ♢

4.4 Integrability of Pfaffian groupoids

In this section we are going to apply and further investigate the results on the integrability of Pfaffian fibrations from Sections 1.4 and 1.5 in the Lie-theoretic setting. In particular, taking advantage of the entire structure of a Pfaffian groupoid, the prolongation of a Pfaffian groupoid in the sense of Definition 1.4.4 becomes a Pfaffian groupoid as well. The results discussed in this section will be very useful to develop the theory of integrability for principal Pfaffian bundles in Section 5.8.

First of all, one needs to specialise the notion of holonomic section of a Pfaffian fibration to the realm of Pfaffian groupoids.

Definition 4.4.1. A **holonomic bisection of a Pfaffian groupoid** $(\mathcal{G}, \omega)$ is a bisection $\sigma \in \mathrm{Bis}(\mathcal{G})$ (Definition 3.1.1) such that $\sigma^*\omega = 0$.

The set of holonomic bisections is denoted by $\mathrm{Bis}(\mathcal{G}, \omega)$ and that of local holonomic bisections by $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$.

A Pfaffian groupoid $(\mathcal{G}, \omega) \rightrightarrows M$ is **integrable** if and only if for every $g \in \mathcal{G}$ there is a local holonomic bisection $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ such that $\sigma(x) = g$, for $x = s(g) \in M$. ⧫

It is clear that, if $\sigma_1, \sigma_2 \in \mathrm{Bis}(\mathcal{G})$ are holonomic, so is the product $\sigma_1 \cdot \sigma_2$, i.e. $\mathrm{Bis}(\mathcal{G}, \omega)$ is a group. Note also that, if a full Pfaffian groupoid is integrable, its underlining Pfaffian fibration $s : (\mathcal{G}, \omega) \to X$ (see Remark 4.3.3) is integrable as well (as in Definition 1.3.6).

Example 4.4.2. Every jet groupoid $J^k\Gamma$ of a Lie pseudogroup Γ (Example 4.3.11) is integrable. To see this, it is enough to notice that its holonomic bisections are in 1-1 correspondence with the elements of Γ:

$$\Gamma \leftrightarrow \mathrm{Bis}_{\mathrm{loc}}(J^k\Gamma, \omega), \quad \phi \mapsto (j^k\phi : x \mapsto j^k_x\phi).$$

Then, for every $j^k_x\phi \in J^k\Gamma$, the bisection $j^k\phi$ is holonomic and sends x to $j^k_x\phi$. ◊

In the following paragraphs, in order to study the integrability conditions for a Pfaffian groupoid, we need to endow its symbol space with a structure of *generalised tableau bundle* (Section 1.2).

Definition 4.4.3. Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$. Its **associated tableau bundle** is given by its symbol space $\mathfrak{g}$ (Definition 4.3.1) together with the symbol map

$$\mathfrak{g} \xrightarrow{j} \mathrm{Hom}(TM, E), \quad \beta \mapsto D^\omega_\bullet(\beta),$$

where

$$D^\omega : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E)$$

is the Spencer operator on $A = \mathrm{Lie}(\mathcal{G})$ associated to the multiplicative form ω (Proposition 3.4.22). The Pfaffian groupoid is called **standard** when j is injective. The **Spencer cohomology associated to** $(\mathcal{G}, \omega)$ is defined as the Spencer cohomology of the associated tableau bundle (Definition 1.2.3 and 1.2.6). ⧫

Example 4.4.4. Given any Lie groupoid $\mathcal{G}$, the jet groupoid $J^k\mathcal{G}$, with the Pfaffian structure from Example 4.3.10, is standard.

Indeed, the Cartan form ω takes values in the adjoint representation $E = J^{k-1}A$, and every element $\alpha \in \mathfrak{g}_x \subseteq T_{1_x}\mathcal{G}$ can be seen as a map

$$\widetilde{\alpha} \in \mathrm{Hom}(T_xM, E_x), \quad v \mapsto (\kappa_\omega)_x(d_x\sigma(v), \alpha),$$

where κ_ω is the curvature of ω from Definition 1.3.8, and σ is any holonomic bisection of $(\mathcal{G}, \omega)$. Since Cartan distribution is maximally non-integrable, the map κ_ω is non-degenerate; then it follows that the map $\alpha \mapsto \widetilde{\alpha}$ is injective and $J^k\mathcal{G}$ is standard. ♢

4.4.1 The prolongation of a Pfaffian groupoid

From now on, in this section all Pfaffian groupoids are going to be full. This allows us to switch freely between surjective (Pfaffian) multiplicative 1-forms ω and regular (Pfaffian) multiplicative distributions $\mathcal{C} = \ker(\omega)$, as we did in chapter 1.

Consider the underlying Pfaffian fibration $s : (\mathcal{G}, \omega) \to M$ (Remark 4.3.3) and apply to it the results from Section 1.4. For instance, we recall from Definition 1.4.1 the notion of **partial prolongation of a Pfaffian groupoid** $(\mathcal{G}, \mathcal{C}) \rightrightarrows M$:

$$J^1_{\mathcal{C}}\mathcal{G} := \{(g, \zeta) \in J^1\mathcal{G} \mid \zeta(T_{s(g)}M) \subseteq \mathcal{C}_g\}.$$

What is new in the Pfaffian groupoid setting is that $J^1_{\mathcal{C}}\mathcal{G}$ is also a subgroupoid of $J^1\mathcal{G}$: the proof is a straightforward computation (Lemma 6.2.19 of [81]), based on the fact that $J^1_{\mathcal{C}}\mathcal{G}$ can be viewed as the kernel of

$$e : J^1\mathcal{G} \to s^*\,\mathrm{Hom}(TM, E), \quad (g, \zeta) \mapsto g \cdot \omega_g(\zeta(\cdot)),$$

and on the multiplicativity of ω. Then Proposition 1.4.2 becomes

Proposition 4.4.5. *The partial prolongation $J^1_{\mathcal{C}}\mathcal{G}$ of a Pfaffian groupoid $(\mathcal{G}, \mathcal{C})$, when endowed with the restriction of the Cartan distribution $\mathcal{C}_{Cartan}$ of $J^1\mathcal{G}$*

$$\mathcal{C}^{(1)} := \mathcal{C}_{Cartan} \cap TJ^1_{\mathcal{C}}\mathcal{G},$$

is a Lie-Pfaffian groupoid over M.

Note that $J^1_{\mathcal{C}}\mathcal{G}$ is Lie-Pfaffian (Definition 4.3.4) even if $\mathcal{G}$ is not of Lie type: this follows from the fact that $J^1\mathcal{G}$ is always of Lie type.

Similarly, from Definition 1.4.4 we have the **prolongation of a Pfaffian groupoid** $(\mathcal{G}, \mathcal{C}) \rightrightarrows M$:

$$\mathrm{Prol}(\mathcal{G}, \mathcal{C}) := \{(g, \zeta) \in J^1_{\mathcal{C}}\mathcal{G} \mid (\zeta^*\kappa_{\mathcal{C}})_g = 0\}.$$

As for the partial prolongation, one checks that $\mathrm{Prol}(\mathcal{G},\mathcal{C})$ is a subgroupoid by viewing it as the kernel of

$$\widetilde{\kappa}_{\mathcal{C}} : J^1_{\mathcal{C}}\mathcal{G} \to s^*\mathrm{Hom}(\Lambda^2 TM, E), \quad (g,\zeta) \mapsto g^{-1}\cdot \zeta^*\kappa_{\mathcal{C}},$$

and by showing that $\widetilde{\kappa}_{\mathcal{C}}$ is a cocycle:

$$\widetilde{\kappa}_{\mathcal{C}}(\xi_h\cdot\eta_g) = g^{-1}\cdot\widetilde{\kappa}_{\mathcal{C}}(\xi_h) + \widetilde{\kappa}_{\mathcal{C}}(\eta_g) \qquad \forall(\xi_h,\eta_g)\in TJ^1_{\mathcal{C}}\mathcal{G}\ {}_{ds}\times_{dt} TJ^1_{\mathcal{C}}\mathcal{G}.$$

This can be proved directly, using the multiplicativity of ω (see Lemma 6.2.30 of [81]), or as a consequence of Proposition 5.8.9 in the case $P = \mathcal{G}$.

In order for $\mathrm{Prol}(\mathcal{G},\mathcal{C})$ to be a Pfaffian groupoid, consider the prolongation

$$\mathfrak{g}(\mathcal{C})^{(1)} \subseteq \mathrm{Hom}(TM, \mathfrak{g}(\mathcal{C})).$$

of the tableau bundle $\mathfrak{g}(\mathcal{C})$ (Definition 4.4.3), as well as the **torsion of** $(\mathcal{G},\mathcal{C})$ (Definition 1.5.4):

$$T : \mathcal{G} \to s^*\,\mathrm{Hom}(\Lambda^2 TM, E)/\mathrm{Im}(\delta), \quad g \mapsto [(\zeta^*\kappa_{\mathcal{C}})_g].$$

Then Theorem 1.4.5 becomes

Theorem 4.4.6. *For any Pfaffian groupoid $(\mathcal{G},\mathcal{C}) \rightrightarrows M$, the following are equivalent:*

1. *The prolongation* $\mathrm{Prol}(\mathcal{G},\mathcal{C})$ *is a smooth affine subbundle of* $J^1\mathcal{G}$.
2. *The prolongation* $\mathfrak{g}(\mathcal{C})^{(1)}$ *of* $\mathfrak{g}(\mathcal{C})$ *is of constant rank, and* $T = 0$.

Moreover, in this case:

- *the vector bundle underlying the affine bundle* $\mathrm{Prol}(\mathcal{G},\mathcal{C}) \to \mathcal{G}$ *is* $\mathrm{pr}^*\,\mathfrak{g}(\mathcal{C})^{(1)} \to \mathcal{G}$.
- *the restriction of the Cartan distribution* $\mathcal{C}_{Cartan}$ *of* $J^1\mathcal{G}$ *to* $\mathrm{Prol}(\mathcal{G},\mathcal{C})$

  $$\mathcal{C}^{(1)} := \mathcal{C}_{Cartan} \cap T\mathrm{Prol}(\mathcal{G},\mathcal{C}),$$

  turns $(\mathrm{Prol}(\mathcal{G},\mathcal{C}), \mathcal{C}^{(1)})$ *into a Pfaffian groupoid over M.*
- *the symbol space of* $\mathrm{Prol}(\mathcal{G},\mathcal{C})$ *is* $\mathfrak{g}(\mathcal{C})^{(1)} \subseteq \mathrm{Hom}(TM,\mathfrak{g}(\mathcal{C}))$.
- *the Pfaffian form $\omega^{(1)} \in \Omega^1(\mathrm{Prol}(\mathcal{G},\mathcal{C}), t^*A)$ is related to the Pfaffian form $\omega \in \Omega^1(\mathcal{G}, t^*E)$ by the projection* $\mathrm{pr} : \mathrm{Prol}(\mathcal{G},\mathcal{C}) \to \mathcal{G}$ *and the map $l : A \to E$ from Proposition 4.3.6:*

  $$l \circ \omega^{(1)} = pr^*\omega.$$

As in Remark 1.4.6, there is a 1-1 correspondence between holonomic bisections (Definition 4.4.1):

$$\mathrm{Bis}(\mathrm{Prol}(\mathcal{G},\mathcal{C}), \mathcal{C}^{(1)}) \to \mathrm{Bis}(\mathcal{G},\mathcal{C}), \quad \xi \mapsto \mathrm{pr}\circ\xi.$$

Example 4.4.7. As the motivation and main example for the prologation of Pfaffian fibrations came from PDEs on jet bundles, here one should have in mind what happens for jet bundles of Lie groupoids. Accordingly, the analogue to Proposition 1.4.7 is Proposition 6.2.33 of [81]: for a Lie groupoid $\mathcal{G} \rightrightarrows M$ and its jet groupoid $J^k\mathcal{G}$ (Example 3.1.8), one has

$$\mathrm{Prol}(J^k\mathcal{G}, \mathcal{C}) = J^{k+1}\mathcal{G}. \qquad \diamondsuit$$

Remark 4.4.8 (Maurer-Cartan equation for Pfaffian groupoids)**.** Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid over M, and assume that its prolongation $\mathcal{G}^{(1)} := \mathrm{Prol}(\mathcal{G}, \omega)$ satisfies the conditions in Theorem 4.4.6. Then the Cartan form $\omega^{(1)} \in \Omega^1(\mathcal{G}^{(1)}, t^*A)$ satisfies the Maurer-Cartan equation (see Theorem 6.2.17 of [81] for the proof), i.e.

$$MC(\omega^{(1)}) = 0.$$

To make sense of this claim, one has to consider the pullback (t^*D, t^*l) of the Spencer operator (D, l) of ω (Proposition 3.4.26) via the target map $t : \mathcal{G}^{(1)} \to M$. It is a Spencer operator on the pullback Lie algebroid t^*A with values in t^*E, so we can consider the associated Maurer-Cartan operator from Definition 3.4.27:

$$MC : \Omega^1(\mathcal{G}^{(1)}, t^*A) \to \Omega^2(\mathcal{G}^{(1)}, t^*E), \quad \eta \mapsto d_D\eta - \frac{1}{2}\{\eta, \eta\}_D. \qquad \diamondsuit$$

4.4.2 Integrability of a Pfaffian groupoid

Definition 4.4.9. A Pfaffian groupoid $(\mathcal{G}, \mathcal{C})$ over M is called **integrable up to order** $k \geq 1$ or **formally integrable** when its underlying Pfaffian fibration $s : (\mathcal{G}, \mathcal{C}) \to M$ is so (Definitions 1.5.1 and 1.5.7). ♦

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The obstructions to formal integrability are therefore those introduced for Pfaffian fibrations. More precisely, for a Pfaffian groupoid $(\mathcal{G}, \mathcal{C})$ integrable up to order $k \geq 1$, we can consider the **torsion of order** $k+1$ from Definition 1.5.4. By Proposition 1.5.6, it is a map

$$T^{k+1} : \mathcal{G}^{(k)} \to H^{k-1,2}(\mathfrak{g}) = \frac{\ker(\delta : \mathrm{Hom}(\Lambda^2 TM, \mathfrak{g}^{(k-1)}) \to \mathrm{Hom}(\Lambda^3 TM, \mathfrak{g}^{(k-2)}))}{\mathrm{Im}(\delta : \mathrm{Hom}(TM, \mathfrak{g}^{(k)}) \to \mathrm{Hom}(\Lambda^2 TM, \mathfrak{g}^{(k-1)}))},$$

where $\mathfrak{g}$ is the symbol space of $(\mathcal{G}, \omega)$ with the tableau structure from Definition 4.4.3. Note that here the notations are simpler than in the Pfaffian fibration case, since all the relevant vector bundles are over M. Accordingly, Proposition 1.5.5 becomes:

Proposition 4.4.10. *Let $(\mathcal{G}, \mathcal{C})$ be a Pfaffian groupoid with symbol space $\mathfrak{g}$ and integrable up to order k. Then $(\mathcal{G}, \mathcal{C})$ is integrable up to order $k+1$ if and only if*

- *the intrinsic torsion T^{k+1} vanishes,*
- *the prolongation $\mathfrak{g}^{(k+1)}$ is smooth.*

Moreover, the prolongation

$$\mathrm{pr} : (\mathcal{G}^{(k+1)}, \mathcal{C}^{(k+1)}) \to (\mathcal{G}^{(k)}, \mathcal{C}^{(k)})$$

has symbol $\mathfrak{g}(\mathcal{C}^{(k+1)}) = \mathfrak{g}^{(k+1)}$*, and it is an affine bundle over* $\mathcal{G}^{(k)}$ *modelled on* $\mathrm{pr}^*\mathfrak{g}^{(k+1)}$*.*

4.4.3 Example: integrability of $J^k\Gamma$

We study now the prolongations of the Pfaffian groupoid $J^k\Gamma$ (Example 4.3.10), for a Lie pseudogrop Γ. First of all, we recall from Section 4.2 that

$$J^{k+1}\Gamma \to J^k\Gamma$$

is an affine bundle modelled on the (pullback of the) symbol space

$$\mathfrak{g}^{k+1} := \ker(d\pi^{k+1}_k : TJ^k\Gamma \to TJ^{k-1}\Gamma)_{|X} \subseteq \mathrm{Hom}(TX, \mathfrak{g}^k).$$

On the other hand, by Theorem 4.4.6,

$$(J^k\Gamma)^{(1)} \to J^k\Gamma$$

is an affine bundle modelled on the (pullback of the) prolongation $(\mathfrak{g}^k)^{(1)}$ of the symbol space $\mathfrak{g}^k \subseteq \mathrm{Hom}(TX, \mathfrak{g}^{k-1})$ of $J^k\Gamma$.

The relation between these spaces is proved in Proposition 3.5.1 of [100]

$$J^{k+1}\Gamma \subseteq (J^k\Gamma)^{(1)}, \qquad \mathfrak{g}^{k+1} \subseteq (\mathfrak{g}^k)^{(1)}.$$

In the ideal situations, the inclusions above become equalities; if this is the case, then $J^k\Gamma$ is trivially integrable up to order 1. Actually, in section 3.5 of [100] it is conjectured that for any Lie pseudogroup Γ there exists a $k_0 \in \mathbb{N}$ such that $(J^i\Gamma)^{(1)} = J^{i+1}\Gamma$ for $i \geq k_0$, i.e. the ideal situation happens after a finite number of prolongations. Note that this is equivalent to asking that $\mathfrak{g}^{i+1} = (\mathfrak{g}^i)^{(1)}$ for $i \geq k_0$, which is purely an algebraic condition on tableau bundles.

For tableaux, i.e. when the base manifold is a point, this is already known to be true (see section 4.6 of [86]), hence for every $x \in X$ there exists a k_0 such that $\mathfrak{g}^{i+1}_x = (\mathfrak{g}^i_x)^{(1)}$ for $i \geq k_0$. Strong evidence that the conjecture is true in its full generality (i.e. such a k_0 can be chosen in a uniform way) comes from the relevant examples where this happens (see Example 6.1.4 later).

However, even if $J^{k+1}\Gamma \subset (J^k\Gamma)^{(1)}$, the Pfaffian groupoid $J^k\Gamma$ is always integrable up to order 1 by Theorem 4.4.6, assuming the mild condition that $(\mathfrak{g}^k)^{(1)}$ is of constant rank. Indeed, $J^k\Gamma$ is a PDE (Remark 4.2.2) and its Pfaffian prolongation is $(J^k\Gamma)^{(1)} := J^1(J^k\Gamma) \cap J^{k+1}(X \times X)$ (Proposition 1.4.7), whose projection on $J^k\Gamma$ is clearly surjective.

Remark 4.4.11. Consider a Lie pseudogroup Γ of order 1 satisfying the ideal situation discussed above with $k_0 = 1$, i.e. $(J^1\Gamma)^{(k)} = J^{k+1}\Gamma$ for all k. Then Γ is of finite type (Definition 4.2.5) if and only if the symbol space $\mathfrak{g}^1$ of $J^1\Gamma$, viewed as a generalised tableau bundle (Definition 4.4.3), is of finite type. This follows immediately from the fact that $(\mathfrak{g}^1)^{(k)} = \mathfrak{g}^{k+1}$ for every k.

In particular, in the transitive case, Γ is of finite type if and only if its linear isotropy algebra is of finite type; this follows from the fact that the fibres of $\mathfrak{g}^1$ coincide with the linear isotropy Lie algebras of Γ. $\lozenge$

4.5 A more general framework: generalised pseudogroups

The simple remark that locally defined diffeomorphisms of X are local bisections of the pair groupoid $X \times X \to X$ (Definition 3.1.4) allows one to introduce a more general notion of pseudogroups, given by bisections of Lie groupoids.

Definition 4.5.1. A **generalised pseudogroup** Γ on a Lie groupoid $\mathcal{G} \rightrightarrows X$ is a subset $\Gamma \subseteq \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ such that the following conditions hold:

1. $u \in \Gamma$
2. $\sigma_2 \cdot \sigma_1 \in \Gamma$ for all $\sigma_1, \sigma_2 \in \Gamma$ such that $\mathrm{Im}(t \circ \sigma_2) \subseteq \mathrm{dom}(\sigma_1)$
3. $\sigma^{-1} \in \Gamma$ for all $\sigma \in \Gamma$
4. $\sigma_{|U} \in \Gamma$ for all $\sigma \in \Gamma$ and $U \subseteq \mathrm{dom}(\sigma)$ open subset
5. $\sigma \in \Gamma$ for all $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ such that there exists an open cover $\{U_i\}_i$ of $\mathrm{dom}(\sigma)$ with $\sigma_{|U_i} \in \Gamma\ \forall i$

A generalised pseudogroup Γ induces the *classical* pseudogroup

$$\Gamma_{cl} := \{t \circ \sigma \mid \sigma \in \Gamma\} \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$$

called the **shadow** of Γ. To be precise, Γ_{cl} satisfy only axioms 1-4 of Definition 4.1.1; the last axiom holds if $\mathcal{G}$ is effective (by an argument similar to the one in Example 4.1.10). $\blacklozenge$

The most relevant example of a generalised pseudogroup is the set $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$ of holonomic bisections of a Pfaffian groupoid $(\mathcal{G}, \omega)$ (Definition 4.4.1). In particular, consider $\mathcal{G} = J^k\Gamma$ the jet groupoid of a Lie pseudogroup Γ, interpreted as a Pfaffian groupoid (Example 4.3.11). Then the shadow of $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$ coincides with Γ.

Here are some other examples of generalised pseudogroups $\Gamma \subseteq \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ and their shadows:

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- For $\mathcal{G}$ a pair groupoid, Γ coincides with its shadow, hence it is a classical pseudogroup.
- For $\mathcal{G}$ a Lie group, Γ is a Lie subgroup of $\mathcal{G}$, and its shadow contains only one element.
- For any $\mathcal{G}$, the two extreme cases of generalised pseudogroups, and their respective shadows, are

$$\Gamma = \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}), \qquad \Gamma_{cl} = (\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}))_{cl} = \Gamma(\mathcal{G}) \quad \text{(Example 4.1.10)},$$

$$\Gamma = \{u_{|U} \mid U \subseteq X \text{ open}\}, \qquad \Gamma_{cl} = \{id_U \mid U \subseteq X \text{ open}\}.$$

Most of the definitions and the constructions for classical pseudogroups can be extended to generalised pseudogroups. For instance, one can talk about transitivity, define the jet groupoids $J^k\Gamma \subseteq J^k\mathcal{G}$ and the étale groupoid $\mathrm{Germ}(\Gamma) \rightrightarrows X$, etc. We will not pursue such generality, but we mention that these notions can be of use also in the framework of (almost) Γ-structures, to which the next chapter is devoted (see Example 5.1.14, where we will sketch a possible application).

Chapter 5

Almost Γ-structures

This chapter constitutes the core of the thesis. We start by reviewing the definition of Γ-structure and enrich it with new examples, applications and alternative points of view.

The entire motivation of this thesis is to develop a framework to study the "almost version" of Γ-structures. For instance, symplectic, complex or contact structures fit in the framework of Γ-structures.

The most prominent example of Γ-structures is given by integrable G-structures; however, there are also other interesting geometric objects which do not fit in this picture, and which make therefore the theory of Γ-structures interesting to study on its own.

With the same spirit, it is natural to look for a description of non-integrable G-structures in terms of Γ. Starting with almost symplectic structures, almost complex structures, etc., we arrive to the general notion of "almost Γ-structure" for every *Lie* pseudogroup Γ, possibly non-transitive.

To study (or even make sense of) almost Γ-structures, we introduce the concept of principal Pfaffian bundle. This terminology is inspired by the analogy between Pfaffian groupoids and symplectic groupoids (viewed as Lie groupoids endowed with a multiplicative structure). 5

An instance of this analogy comes from Hamiltonian actions (and their infinitesimal version in terms of the "moment map condition"), which inspired our notion of Pfaffian actions (and their infinitesimal version in terms of Spencer operators). Similarly, we develop the theory of Pfaffian Morita equivalence between Pfaffian groupoid, inspired by symplectic Morita equivalence between symplectic groupoids.

We stress that the introduction of principal Pfaffian bundles does not constitute simply a technical tool to use in a proof, nor it is a generalisation for the sake of generalising: it is precisely the "right" notion to understand the non-integrable data of a geometric structure. Accordingly, we will introduce the Cartan form of an almost Γ-structure, which is the analogue of the tautological form of a G-

structure. This allows us to talk about abstract Γ-structures and their symmetries, or to interpret them as Pfaffian Morita morphisms.

Last, we recast the results on prolongations and integrability of Pfaffian fibrations (which we proved in [19] and recapped in the first chapter) to the setting of principal Pfaffian bundles. These notions, which seems quite abstract at this stage, will be applied in the next chapter to the integrability problem for almost Γ-structures.

5.1 Γ-structures

The formalism of (integrable) G-structures interprets geometric structures on a manifold M as the frames of TM adapted to the structure. The formalism of Γ-structures changes this point of view by interpreting geometric structures as atlases of M, where the changes of coordinates are elements of a pseudogroup.

Definition 5.1.1. Let Γ be a pseudogroup on a manifold X and M a manifold with $\dim(M) = \dim(X)$. A **Γ-atlas (modelled on X)** on M is a collection $\mathcal{A} = \{(U_i, \chi_i)\}_{i \in I}$ consisting of

- open subsets $U_i \subseteq M$,
- diffeomorphisms $\chi_i : U_i \to \chi_i(U_i) \subseteq X$,

such that

- $\{U_i\}_{i \in I}$ is an open cover of M,
- $\chi_i \circ \chi_j^{-1} \in \Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(X) \quad \forall i, j \in I$.

Two Γ-atlases on M are **equivalent** if their union is still a Γ-atlas on M. A **Γ-structure** on M is an equivalence class of Γ-atlases. ⧫

If $X = \mathbb{R}^n$ and $\Gamma = \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$, a Γ-atlas $\mathcal{A}$ is an ordinary smooth atlas on M, and a Γ-structure is an ordinary smooth structure on M. With this in mind, when picking a representative for a general Γ-structure, we can always take the maximal Γ-atlas containing it, i.e. the union of all the possible Γ-atlases compatible with the given one. In the following we will often use interchangeably the words Γ-structure and maximal Γ-atlas.

Moreover, in several situations it will be more practical to write a Γ-atlas as a collection of diffeomorphisms $\chi : U \subseteq X \to \chi(U) \subseteq M$, i.e. inverses of charts; it will be clear from the context which convention we are applying.

Example 5.1.2. Let Γ be a pseudogroup on X and $\mathcal{A}$ a Γ-atlas on M; if Γ is transitive, then $\mathcal{A}$ can be replaced by an atlas modelled on $\mathbb{R}^n$.

Let us sketch this procedure. First, fix an element $x_0 \in X$ and an open neighbourhood $U_0 \subseteq X$ of x_0; then the Γ-atlas $\mathcal{A}$ induces a $\Gamma_{|U_0}$-atlas $\widetilde{\mathcal{A}}$ on M, where $\Gamma_{|U_0}$ is the restriction of Γ to U_0 (Example 4.1.8). The elements of $\widetilde{\mathcal{A}}$ are the diffeomorphisms $\bar{\chi}_x$ defined as follows. For every element χ of $\mathcal{A}$ and any point $x \in \mathrm{dom}(\chi) \subseteq X$, consider a diffeomorphism ϕ which sends x_0 to x; by the transitivity of Γ, there is always such a ϕ in Γ. Then $\bar{\chi}_x := \chi \circ \phi_x$, with the appropriate restrictions, is by construction a diffeomorphism from a neighbourhood of x_0 in U_0 to a neighbourhood of $\chi(x_0)$ in M.

Second, consider any diffeomorphism f between a neighbourhood of 0 in $\mathbb{R}^n$ and a neighbourhood of x_0 in U_0; composing the elements $\bar{\chi}_x$ of $\widetilde{\mathcal{A}}$ with f, with the appropriate restrictions, we obtain an $f^*\Gamma_{|U_0}$-atlas modelled on $\mathbb{R}^n$, for $f^*\Gamma_{|U_0}$ the pullback pseudogroup from Example 4.1.8. ◊

Example 5.1.3. If $f : M_1 \to M_2$ is a diffeomorphism and $\mathcal{A}$ a Γ-atlas on M_1, the set

$$f_*(\mathcal{A}) := \{(U, f \circ \chi) \mid (U, \chi) \in \mathcal{A}\}$$

defines a Γ-atlas on M_2. ◊

5.1.1 A first class of examples: integrable G-structures

When $\Gamma = \Gamma_G$ (Example 4.1.5), we recover integrable G-structures.

Proposition 5.1.4. *Given a Lie subgroup $G \subseteq GL(n, \mathbb{R})$ and a manifold M^n, there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{(isomorphism classes of)} \\ \textit{integrable } G\textit{-structures on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \Gamma_G\textit{-structures on } M \right\}.$$

Proof. Let $\mathcal{A}$ be the maximal Γ_G-atlas representing a Γ_G-structure on M; in order to define a G-structure, consider the set of all first order jets of the charts at all points v in $\mathbb{R}^n$, i.e.

$$J^1\mathcal{A} := \{j^1_v\chi \mid (U, \chi) \in \mathcal{A}, v \in U\} \subseteq J^1(\mathbb{R}^n, M).$$

Recall that the jet bundle of a trivial bundle is a (trivial) fibre bundle too; accordingly, $J^1\mathcal{A}$ inherits a structure of fibre bundle over $\mathbb{R}^n$, with the projection map $j^1_v\chi \mapsto v$. Using Remark 2.6.3, we consider the fibre of $J^1\mathcal{A}$ at the point $0 \in \mathbb{R}^n$

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$$J^1_0\mathcal{A} \subseteq J^1_0(\mathbb{R}^n, M) \cong Fr(M).$$

The principal $GL(n, \mathbb{R})$-action on $Fr(M)$ restricts to a principal G-action on $J^1_0\mathcal{A}$

$$j^1_0\chi \cdot g = j^1_0(\chi \circ g)$$

where we interpret g as a diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$, so that $\chi_i \circ g$ is still a chart of $\mathcal{A}$. Then the projection

$$\pi : J^1_0\mathcal{A} \to M, \quad j^1_0\chi \mapsto \chi(0)$$

makes $J^1_0\mathcal{A}$ into a principal G-bundle, sitting inside $Fr(M)$, and thus it defines a G-structure.

Last, to check the integrability of $J^1_0\mathcal{A}$, it is enough to notice that every point $x \in M$ is the image of $0 \in \mathbb{R}^n$ through some chart χ, since a Γ-atlas over $\mathbb{R}^n$ is, in particular, a smooth atlas. It follows that the corresponding element $j^1_0\chi \in J^1_0\mathcal{A}$ induces a frame of TM at $\chi(0) = x$, namely $\left(\frac{\partial}{\partial\chi^1}(x), \dots, \frac{\partial}{\partial\chi^n}(x)\right)$, which is in the fibre $(J^1_0\mathcal{A})_x$.

Viceversa, if $P \subseteq Fr(M)$ is an integrable G-structure, we obtain a Γ_G-atlas by taking a maximal atlas of M and considering only the charts (U, χ_U) which give rise to adapted frames $\left(\frac{\partial}{\partial\chi^1_U}(x), \dots, \frac{\partial}{\partial\chi^n_U}(x)\right) \in P_x$ for some $x \in U$.

Since the G-action of a principal bundle is transitive on the fibres, for any $x \in U \cap V$ there exists an element $g \in G$ such that

$$\left(\frac{\partial}{\partial\chi^1_V}(x), \dots, \frac{\partial}{\partial\chi^n_V}(x)\right) = \left(\frac{\partial}{\partial\chi^1_U}(x), \dots, \frac{\partial}{\partial\chi^n_U}(x)\right) g$$

hence $\mathrm{Jac}_x(\chi_U^{-1} \circ \chi_V) = g \in G$ for all $x \in U$. This means that such charts form an atlas of M with changes of coordinates in Γ_G. Q.E.D.

Example 5.1.5. If $\Gamma_1 \subseteq \Gamma_2$, then a Γ_1-structure $\mathcal{A}_1$ on M induces a Γ_2-structure $\mathcal{A}_2$ on M.

For instance, the first part of Example 2.1.7 can be recast as follows: if $H \leq G \leq GL(n, \mathbb{R})$, then a Γ_H-structure induces a Γ_G-structure. On the other hand, if H is also a deformation retract of G, then the second part of Example 2.1.7 guarantees that any Γ_G-structure induces a Γ_H-structure. $\lozenge$

Example 5.1.6. Every pseudogroup Γ on a manifold X defines a Γ-structure $\mathcal{A}^\Gamma_{mod}$ on X itself, called **the model Γ-structure**. Indeed, it is enough to take as charts all the elements of Γ. They form a maximal Γ-atlas because the pseudogroup preserves composition, inversion and restrictions (see section I.1 of [79]). For instance, the model Γ_G-structure corresponds, under Proposition 5.1.4, to the flat G-structure $\mathbb{R}^n \times G$. $\lozenge$

5.1.2 A second class of examples: (G, X)-structures

Another interesting example of Γ-structures are the manifolds that, roughly speaking, look like another manifold X with a given symmetry group G.

These structures have been introduced in 1977 by Thurston [92], who formalised and gave a partial solution to the following problem (posed in 1936 by Ehresmann [30], and inspired by the famous Erlangen program of Klein): which manifolds can be modelled on a given homogeneous space? Besides Thurston's work, useful surveys on the topic are [37, 99].

Definition 5.1.7. Consider a manifold X and a finite-dimensional Lie group $G \subseteq \mathrm{Diff}(X)$ with the following property: if $g, h \in G$ and $U \subseteq X$ is any open, then

$$g_{|U} = h_{|U} \Rightarrow g = h.$$

A (G, X)**-structure** on a manifold M consists of an atlas on M, modeled on X, whose changes of coordinates are restrictions of elements of G. The manifold M is also called a (G, X)-manifold. ♦

It is clear that (G, X)-structures are particular cases of Γ-structures: one can define the pseudogroup $\Gamma_{(G,X)} \subseteq \mathrm{Diff}_{\mathrm{loc}}(X)$ generated by all the possible restrictions of elements of G, and see that a (G, X)-structure is the same thing as a $\Gamma_{(G,X)}$-atlas.

Remark 5.1.8. A **Klein geometry** is a homogeneous space $M = G/H$, where H is a Lie subgroup of G. This is the modern formalisation (see [85]) of the original ideas of Klein; one can see that, under some assumptions, (G, X)-structures can be seen as Klein geometries.

Indeed, Thurston proved (sections 3.4-3.5 of [92]) that "nice" (G, X)-structures (up to diffeomorphisms) on a manifold M are in 1-1 correspondence with "nice" discrete subgroups (up to conjugacy) $H \subseteq G$ such that $M = G/H$. This means that we can see a (G, X)-structure as a manifold modelled on a homogeneous space. ♢

Example 5.1.9. An **affine structure** on M is a (G, X)-manifold for the group $G = \mathrm{Aff}(\mathbb{R}^n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ of affine transformations of $X = \mathbb{R}^n$, i.e. G consists of the diffeomorphisms of $\mathbb{R}^n$ of the form $x \mapsto Ax + v$, for $A \in GL(n, \mathbb{R})$. Note that $\Gamma_{(\mathrm{Aff}(\mathbb{R}^n),\mathbb{R}^n)}$ is a pseudogroup of order 2 and is contained in the pseudogroup Γ from Example 4.1.6.

On the other hand, a well known result by Auslander and Markus states that an affine structure is equivalent to an affine connection on M (i.e. a linear connection on TM) which is flat and torsion-free; this is in turn an integrable second-order G-structure (see Example 2.6.20).

Because of this alternative point of view, there are several topological obstructions in the literature concerning the existence of affine structures on a manifold. This is in accord with the fact that not all $(\mathrm{Aff}(\mathbb{R}^n), \mathbb{R}^n)$-structures are "nice" in the sense of Remark 5.1.8 (such niceness is indeed a topological condition), otherwise any manifold would be a homogeneous space. ♢

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Example 5.1.10. Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup and $\widetilde{G} \subseteq \mathrm{Diff}(\mathbb{R}^n)$ the group of diffeomorphisms whose Jacobian is at any point a constant element of G. Then a $(\widetilde{G}, \mathbb{R}^n)$-structure on a manifold M is a special kind of integrable G-structure on M, since the pseudogroup $\Gamma_{(\widetilde{G},\mathbb{R}^n)}$ is contained in Γ_G (using Example 5.1.5 and Proposition 5.1.4).

For instance, for $G = O(n)$, one has the group $\widetilde{G} = E(n) := O(n) \ltimes \mathbb{R}^n \subseteq \mathrm{Diff}(\mathbb{R}^n)$ of Euclidean isometries of $\mathbb{R}^n$ with the Euclidean metric. An $(E(n), \mathbb{R}^n)$-manifold, also called **Euclidean manifold**, or affine flat Riemannian manifold, is a flat Riemannian manifold (i.e. an integrable $O(n)$-structure) such that the changes of coordinates are affine. Since $E(n) \subseteq \mathrm{Aff}(\mathbb{R}^n)$, any Euclidean manifold possesses an affine structure from Example 5.1.9.

With the same principle, one can describe the following (G, X)-manifolds:

- **Minkowski manifolds** (or affine flat Lorentzian manifolds), where $G = O(n-1,1) \ltimes \mathbb{R}^n$ is the group of isometries of the Minkowski space $X = \mathbb{R}^n$
- **Elliptic (or spherical) manifolds**, where $G = O(n+1)$ is the group of isometries of the sphere $X = S^n$
- **Hyperbolic manifolds**, where G is the group of isometries of the hyperbolic space $X = H^n$. ♢

5.1.3 Other examples of Γ-structures

We list now some other relevant examples of Γ-structures which are not G- or (G, X)-structures for any Lie group G.

Example 5.1.11. A **contact structure** on a manifold M^{2k+1} is given by a rank $2k$ distribution $\xi \subseteq TM$ locally given as $\xi = \ker(\alpha)$, for some 1-forms $\alpha \in \Omega^1(M)$ satisfying $\alpha \wedge (d\alpha)^k \neq 0$, called *contact forms*.

The relevant pseudogroup is Γ_{cont}, the set of local contactomorphisms of the standard contact structure ξ_{can} on $\mathbb{R}^{2k+1}$. Here, if $\mathbb{R}^{2k+1}$ has coordinates $(x^1, \dots, x^k, y^1, \dots, y^k, z)$, then ξ_{can} is given by the kernel of the (global) form

$$\theta_{\mathrm{can}} := dz + \sum_{i=1}^{k} y^i dx^i,$$

and $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{2k+1})$ is a local contactomorphism if it preserves ξ_{can}. A Darboux-like theorem prescribes the local normal form of any contact structure ξ, i.e. the local equivalence with ξ_{can} (see e.g. Theorem 2.5.1 of [36]); this means that contact structures on M are the same thing of Γ_{cont}-structures.

A contact structure ξ is said to be **co-orientable** if the form α defining ξ is global, or, equivalently, if the line bundle TM/ξ is trivial. Such data is also equivalent to giving a positive conformal class $[\alpha]$ of contact forms on M, where α and α' are in relation if there is a positive function f such that $\alpha' = f\alpha$. We denote by Γ^c_{cont} the pseudogroup of diffeomorphisms of $\mathbb{R}^{2k+1}$ which preserves not only the distribution ξ_{can} but also the positive conformal class of θ_{can}; then a coorientable contact structure on M is the same thing as a Γ^c_{cont}-structure.

Last, one can also consider the pseudogroup Γ^s_{cont} of local *strict* contactomorphisms, i.e. diffeomorphisms $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{2k+1})$ preserving the form θ_{can}, not only its conformal class; then a Γ^s_{cont}-structure on M is the same thing as a **contact form** (or **strict contact structure**).

Other "contact-like" geometric structures, such as Engel structures and even-contact structures (see e.g. Chapter 6 of [74]), can be reformulated as well as Γ-structures, for Γ the pseudogroup of automorphisms of their local models. $\Diamond$

Example 5.1.12. Consider the pseudogroup $\Gamma^\omega \subseteq \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ of local analytic diffeomorphisms of $\mathbb{R}^n$ (see Example 4.1.3): a Γ^ω-structure is an **analytic structure on** M.

Note that, since a complex structure induces an analytic atlas, any $\Gamma_{GL(n,\mathbb{C})}$-structure on M^{2n} induces a Γ^ω-structure. This could also be seen directly from the fact that the elements of $\Gamma_{GL(n,\mathbb{C})}$ are analytic, i.e. $\Gamma_{GL(n,\mathbb{C})} \subseteq \Gamma^\omega$ (Example 4.2.12). $\Diamond$

Example 5.1.13. Let Γ be the pseudogroup of locally defined diffeomorphisms preserving the standard volume form of $\mathbb{R}^n$ up to a constant (Example 4.1.6). A Γ-structure on M is a **flat conformal volume form**, i.e. an equivalence class of volume forms $[\Omega]$, with

$$\Omega_1 \sim \Omega_2 \Leftrightarrow \exists c \in \mathbb{R} - \{0\} \text{ s.t. } \Omega_1 = c\Omega_2.$$

This is not just a conformal class, since we exclude the possibility that the two volume forms are related by a non-constant (never-vanishing) function. Since $\Gamma_{SL(n)} \subseteq \Gamma$, any volume form on M induces a Γ-structure. $\Diamond$

Example 5.1.14. It is possible to extend the notion of Γ-structure also to *generalised* pseudogroups (Definition 4.5.1). We are not going to investigate it in this thesis, but we will sketch how this extension allows us to include more examples into the framework of Γ-structures.

For instance, a **Poisson structure** is not a Γ-structure in the ordinary sense, since there is no "standard Poisson model" to define Γ, besides in the regular case (see also [20]). However, there are several classes of Poisson structures with milder regularity conditions, such as **log-symplectic structures**, which admit multiple local models and can be described by means of specific Lie algebroids. By considering an integrating Lie groupoid $\mathcal{G}$, one could define a generalised pseudogroup $\Gamma \subseteq \mathrm{Bis}(\mathcal{G})$ of "symmetries" of the local models, which is in general not transitive, and which recasts the specific kind of Poisson structure as a Γ-structure. $\Diamond$

5.1.4 Γ-structures as étale principal bundles

We are going now to reinterpret Γ-structures as principal bundles with étale moment maps. The following proof has the same spirit as the one of Proposition 4.1.11, where the Haefliger groupoid $\mathrm{Germ}(\Gamma) \rightrightarrows M$ was used in order to establish a bijection between pseudogroups and étale effective groupoids. The original proof is scattered in sections 4-5-6 of chapter III of [46] and is presented using old notations; we have rewritten both the statement and the proof in more explicit and modern terms.

Proposition 5.1.15. *Let Γ be a pseudogroup on X and M a manifold of the same dimension; then there is a bijective correspondence*

$$\left\{\Gamma\text{-}\mathit{structures\ on\ } M\right\} \stackrel{\sim}{\longleftrightarrow} \left\{\begin{array}{c}\mathit{(isomorphism\ classes\ of)}\\ \mathit{principal}\ \mathrm{Germ}(\Gamma)\text{-}\mathit{bundles\ over\ } M\\ \mathit{with\ étale\ moment\ maps}\end{array}\right\}.$$

Proof. Let $\mathcal{A}$ be a Γ-atlas representing a Γ-structure on M. For any two charts χ_α on U_α and χ_β on $U_\beta \subseteq X$, we define

$$\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Germ}(\Gamma), \quad x \mapsto \mathrm{germ}_{\chi_\beta(x)}(\chi_\alpha \circ \chi_\beta^{-1}).$$

Since $\phi_{\alpha\beta}$ satisfies the cocycle identity, the collection $\{(\chi_\alpha, \phi_{\alpha\beta})\}_{\alpha,\beta}$ is a Germ(Γ)-cocycle on M (Definition 3.1.21). By Proposition 3.1.22, we can glue such a cocycle in a principal Germ(Γ)-bundle over M:

$$\mathrm{Germ}(\mathcal{A}) := \bigcup_{\chi \in \mathcal{A}} \{\mathrm{germ}_x(\chi) \mid x \in \mathrm{dom}(\chi)\}.$$

Note that the isomorphism class of Germ($\mathcal{A}$) does not depend on the chosen representative $\mathcal{A}$: different Γ-atlases yield cohomologous cocycles, which glue to isomorphic principal bundles.

More explicitly, the topological and smooth structure on Germ($\mathcal{A}$) is defined as we did for Germ(Γ). A basis for the sheaf topology is given by

$$\hat{U} := \{\mathrm{germ}_x(\chi) \mid x \in U\} \subseteq \mathrm{Germ}(\mathcal{A}), \qquad \text{for } (U, \chi) \in \mathcal{A}$$

and Germ($\mathcal{A}$) is a manifold modelled on X, with charts

$$\xi : U \to \hat{U}, \quad x \mapsto \mathrm{germ}_x(\chi).$$

The moment map

$$\mu : \mathrm{Germ}(\mathcal{A}) \to X, \quad \mathrm{germ}_x(\chi) \mapsto x.$$

is étale since locally it can be inverted by smooth maps of the kind $x \mapsto \mathrm{germ}_x(\chi)$, for any chart χ defined around x. The Lie groupoid action of Germ(Γ) on Germ($\mathcal{A}$) along μ is

$$(\mathrm{germ}_x(\phi), \mathrm{germ}_x(\chi)) \mapsto \mathrm{germ}_{\phi(x)}(\chi \circ \phi^{-1})$$

and then the surjective submersion

$$\pi : \mathrm{Germ}(\mathcal{A}) \to M, \quad \mathrm{germ}_x(\chi) \mapsto \chi(x)$$

defines a principal Germ(Γ)-bundle over M.

Conversely, given a principal Germ(Γ)-bundle $\pi : P \to M$ along an étale map $\mu : P \to X$, consider a Germ(Γ)-cocycle $\{(f_\alpha, \phi_{\alpha\beta})\}_{\alpha,\beta}$ on an open cover

$\{U_\alpha \subseteq M\}_\alpha$ (Definition 3.1.21) which defines P. Since $\mathrm{Germ}(\Gamma)$ is étale, it follows from dimension-counting that π is étale too; since μ is étale as well, the maps $f_\alpha : U_\alpha \to X$ are of constant rank, so they are diffeomorphisms on their images.

Moreover, $\{f_\alpha\}_\alpha$ defines a Γ-atlas on M modelled on X: to see this, consider $\phi_{\alpha\beta}(x) \in \mathrm{Germ}(\Gamma)$, for some $x \in U_{\alpha\beta}$. From the definition of cocycle, there is a diffeomorphism $\varphi_{\alpha\beta} \in \Gamma$ such that

$$\phi_{\alpha\beta}(x) = \mathrm{germ}_{f_\beta(x)}(\varphi_{\alpha\beta}) \in \mathrm{Germ}(\Gamma).$$

Moreover, $\varphi_{\alpha\beta}$ sends elements of the form $f_\beta(x)$ to elements of the form $f_\alpha(x)$ for all x in an open subset of $U_{\alpha\beta}$; this concludes the proof since, restricting f_α and f_β,

$$f_\alpha \circ f_\beta^{-1} = \varphi_{\alpha\beta} \in \Gamma.$$ Q.E.D.

Recall that, for transitive pseudogroups Γ, the groupoid $\mathrm{Germ}(\Gamma)$ is transitive, so we can talk about its isotropy group. Using the correspondences from Proposition 5.1.15 and Example 3.1.17, one checks the following:

Corollary 5.1.16. *Let Γ be a transitive pseudogroup on X, and Γ_0 the isotropy group of* $\mathrm{Germ}(\Gamma)$*. For any manifold M of the same dimension of X, there is a bijective correspondence*

$$\left\{ \Gamma\text{-structures on } M \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{principal } \Gamma_0\text{-bundles over } M \end{array} \right\}.$$

5.1.5 Morphisms and symmetries of Γ-structures

Definition 5.1.17. Let X, M_1 and M_2 be manifolds of the same dimension, and consider a pseudogroup Γ on X and two Γ-structures on M_1 and M_2, represented by the maximal atlases $\mathcal{A}_1$ and $\mathcal{A}_2$.

An **isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$** is a diffeomorphism $f : M_1 \to M_2$ such that, using the notation of Example 5.1.3, $f_*(\mathcal{A}_1) = \mathcal{A}_2$. If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$, we call f a **symmetry** of $\mathcal{A}$. We denote the pseudogroup of local symmetries of a Γ-structure $\mathcal{A}$ by $\mathrm{Aut}_\Gamma(\mathcal{A})$. ♦

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Note that a diffeomorphism $f : M_1 \to M_2$ is an isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ if and only if its lift to the germ bundles

$$\hat{f} : \mathrm{Germ}(\mathrm{Diff}_{\mathrm{loc}}(X, M_1)) \to \mathrm{Germ}(\mathrm{Diff}_{\mathrm{loc}}(X, M_2)), \quad \mathrm{germ}_x \phi \mapsto \mathrm{germ}_x(f \circ \phi)$$

takes $\mathrm{Germ}(\mathcal{A}_1)$ to $\mathrm{Germ}(\mathcal{A}_2)$. This characterisation is more similar to the one of isomorphism between G-structures (Definition 2.1.4) and will be useful in the context of almost Γ-structures.

Example 5.1.18. A local isomorphism between two Γ_G-structures $\mathcal{A}_i$ on M_i is the same thing as a local isomorphism between the corresponding integrable G-structures $P_i = J^1_0\mathcal{A}_i$ on M_i (Definition 2.1.4).

The pseudogroup of symmetries of the model Γ-structure $\mathcal{A}^{\Gamma}_{mod}$ on M (Example 5.1.6) is Γ. This is analogous to the fact that the group of symmetries of a linear G-structure is isomorphic to G. ♢

5.2 Almost Γ-structures

We have seen in Proposition 5.1.4 that flat Riemannian metrics, symplectic structures, or in general any integrable G-structure, can be described as a Γ-structure for a suitable (Lie) pseudogroup Γ_G. It is natural to look for a description of the non-integrable versions of such objects, i.e. Riemannian metrics, almost symplectic structures, etc. in terms of Γ_G and not of G, as well as to understand the integrability condition via Γ_G. This is the starting point to define a notion of "almost Γ-structure" for any *Lie* pseudogroup, not necessarily of the form Γ_G, and possibly not even transitive.

Remark 5.2.1. The concept of "almost Γ-structure" has been introduced for the first time by Guillemin and Sternberg in [44] for transitive Lie pseudogroups Γ on X of order k. Denoting by G its isotropy Lie group of order k (Definition 4.1.21), they define an almost Γ-structure on M of order k as a principal G-subbundle on M

$$P^k_{x_0} \subseteq J^k_{x_0}(X, M).$$

In [4], Albert and Molino gave a slightly different definition, but still only for transitive Lie pseudogroups and making use of principal bundles with the isotropy as structure group.

On the other hand, our definition will not require any condition on the Lie pseudogroup; this is because we rely on principal groupoid bundles, a language that was not fully developed until the past two decades, and thus was not available to the previous authors, who had to rely on (isotropy) groups and restrict themselves to transitive pseudogroups. ♢

Let Γ be a Lie pseudogroup on X (Definition 4.2.1) and M a manifold of the same dimension of X. Then the Lie groupoid $J^k\Gamma$ acts on the jet space $J^k(X, M)$ (Definition 2.6.1), as in the diagram

$$\begin{tikzcd} & (J^k(X,M),\theta) \arrow[dl, "\pi"'] \arrow[dr, "\mu"] \arrow[r, phantom, "\curvearrowleft"] & (J^k\Gamma,\omega) \arrow[d, shift left] \arrow[d, shift right] \\ M & & X \end{tikzcd}$$

where $\mu : j^k_x\chi \mapsto x$ and the action is explicitly given by

$$j^k_{\phi(x)}\chi \cdot j^k_x\phi = j^k_x(\chi \circ \phi).$$

Moreover, the projection $\pi : j^k_x\chi \mapsto \chi(x)$ becomes a principal $J^k\Gamma$-bundle over M.

Definition 5.2.2. Given a Γ-atlas $\mathcal{A}$ on M, the **$k^{\mathbf{th}}$-order data of the Γ-structure** $[\mathcal{A}]$ is the set

$$J^k\mathcal{A} := \{j^k_x\chi \mid (U,\chi) \in \mathcal{A}, x \in U\} \subseteq J^k(X,M). \qquad \blacklozenge$$

The restriction of the $J^k\Gamma$-action described above on $J^k\mathcal{A}$ makes the projection

$$\pi^k : J^k\mathcal{A} \to M, \quad j^k_x\chi \mapsto \chi(x)$$

into a principal $J^k\Gamma$-subbundle of $J^k(X,M)$.

Definition 5.2.3. Let Γ be a Lie pseudogroup on X. A **$k^{\mathbf{th}}$-order almost Γ-structure** on M is a principal $J^k\Gamma$-subbundle $P \subseteq J^k(X,M)$. Such a P is called **integrable** when $P = J^k\mathcal{A}$ for some Γ-structure $\mathcal{A}$. ◆

Later in the thesis, we often write only "almost Γ-structure" when $k = 1$.

5.2.1 A first class of examples: G-structures

Recall from Proposition 5.1.4 that there is a bijective correspondence between Γ_G-structures and integrable G-structures.

Proposition 5.2.4. *Let Γ be a transitive Lie pseudogroup over $\mathbb{R}^n$ of order k, and denote by $G = (J^k\Gamma)_0 \subseteq GL^k(\mathbb{R}^n)$ its isotropy Lie group (Definition 4.1.21). Given a manifold M^n, there is a bijective correspondence*

$$\left\{ k^{th}\text{-order almost } \Gamma\text{-structures on } M \right\} \xleftrightarrow{\sim} \left\{ k^{th}\text{-order } G\text{-structures on } M \right\}.$$

Moreover, if Γ contains the translations, the correspondence restricts to

$$\left\{ \begin{array}{c} \text{integrable } k^{th}\text{-order} \\ \text{almost } \Gamma\text{-structures on } M \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{integrable } k^{th}\text{-order} \\ G\text{-structures on } M \end{array} \right\}.$$

Proof. Using the fact that $J^k\Gamma$ is a transitive Lie groupoid with isotropy G, the first correspondence follows directly from the one between principal $J^k\Gamma$-bundles and principal G-bundles over M (Example 3.1.17), together with the fact that $J^k_0(\mathbb{R}^n, M) = Fr^k(M)$ (Remark 2.6.3).

More explicitly, given the principal $J^k\Gamma$-bundle $P \subseteq J^k(\mathbb{R}^n, M)$, the fibre P_0 on $0 \in \mathbb{R}^n$ is naturally a principal $(J^k\Gamma)_0$-bundle on M, sitting inside $Fr^k(M)$. Viceversa, given a G-structure $P_0 \subseteq Fr^k(M)$, we recover $P = (s^{-1}(0) \times P_0)/G$ as a principal $J^k\Gamma$-bundle over M and check directly that is it isomorphic to a subbundle of $J^k(\mathbb{R}^n, M)$.

For the second part, consider an integrable almost Γ-structure P, i.e. $P = J^k\mathcal{A}$ for a Γ-atlas $\mathcal{A}$ on M; we show that, for every $x \in M$, there is a principal bundle isomorphism $Fr(\phi^{-1})$ between the restrictions of the G^k-structure P_0 and of the trivial G^k-structure $\mathbb{R}^n \times G^k$ to neighbourhoods of $x \in M$ and $\phi^{-1}(x) \in \mathbb{R}^n$. First, pick a chart $\phi \in \mathcal{A}$ whose inverse $\phi^{-1} : U \subseteq M \to V \subseteq \mathbb{R}^n$ sends x to y. The lift of such a diffeomorphism to the k-frames is the following isomorphism of principal $GL^k(\mathbb{R}^n)$-bundles

$$Fr^k(\phi^{-1}) : Fr^k(U) \subseteq Fr^k(M) \to Fr^k(V) \cong V \times GL^k(\mathbb{R}^n)$$

$$j^k_0 f \mapsto j^k_0(\phi^{-1} \circ f) \cong (x_0 = \phi^{-1}(f(0)), j^k_0(\varphi^{-1}_{x_0} \circ \phi^{-1} \circ f)),$$

where $\varphi_{x_0} \in \Gamma$ denotes the translation sending 0 to x_0. We restrict $Fr^k(\phi^{-1})$ further to $(P_0)_{|U}$ and check that it remains an isomorphism between principal G^k-bundles, i.e. that the image is in $V \times G^k$. If $j^k_0 f \in (P_0)_{|U}$, then f must be a chart of the Γ-atlas $\mathcal{A}$, thus the composition $\phi^{-1} \circ f$ is a change of coordinates in such atlas, i.e. a locally defined diffeomorphism which belongs to Γ. Since $\varphi_{x_0} \in \Gamma$ sends 0 to x_0, then $\varphi^{-1}_{x_0} \circ \phi^{-1} \circ f$ sends 0 to

$$\varphi^{-1}_{x_0}(\phi^{-1} \circ f)(0) = \varphi^{-1}_{x_0}(x_0) = 0,$$

so that the composition $j^k_0(\varphi^{-1}_{x_0} \circ \phi^{-1} \circ f)$ is in $(J^k\Gamma)_0 = G^k$, as we wanted. This proves that P_0 is integrable, i.e. locally isomorphic to the flat G^k-structure.

Conversely, assume that P_0 is integrable and define $\mathcal{A}$ as the set of all the diffeomorphisms $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n, M)$ between restrictions of P_0 and $\mathbb{R}^n \times G^k$ (from Definition 2.6.11): we will show that $P = J^k\mathcal{A}$.

Since $j^k_0 id_{\mathbb{R}^n}$ always belongs to G^k, we have for every $\phi \in \mathcal{A}$

$$j^k_0\phi = Fr^k(\phi)(j^k_0 id_{\mathbb{R}^n}) = (Fr^k(\phi^{-1}))^{-1}(j^k_0 id_{\mathbb{R}^n}) \in P_0.$$

Similarly, for $x \neq 0$, we consider again the translation $\varphi_x \in \Gamma$ (which sends 0 to x) and see that

$$j^k_x\phi = j^k_0(\phi \circ \varphi_x) \circ j^k_x\varphi^{-1}_x = j^k_0(\phi \circ \varphi_x) \cdot j^k_x\varphi^{-1}_x$$

is the action on an element of $P_0 \subseteq P$ (with the previous argument) by an element of $J^k\Gamma$, i.e. it stays in P; in conclusion, $J^k\mathcal{A} \subseteq P$.

On the other hand, take $p \in P_0$; since $Fr^k(\phi)$ is a diffeomorphism (for $\phi \in \mathcal{A}$), it is in particular surjective, so there is $j^k_0\psi \in G^k \subseteq J^k\Gamma$ such that $p = Fr^k(\phi)(j^k_0\psi) = j^k_0(\phi \circ \psi)$. Then $p \in J^k\mathcal{A}$ since ϕ is in $\mathcal{A}$ and $\mathcal{A}$ is closed w.r.t. composition with elements of Γ, so we have $P_0 \subseteq J^k\mathcal{A}$.

Similarly, if $p = j^k_x f \in P$ with $x \neq 0$, we write it as $j^k_0(f \circ \varphi_x) \circ j^k_x\varphi^{-1}_x$, where the first term is in $P_0 \subseteq J^k\mathcal{A}$ (by the previous part) and the second in $J^k\Gamma$, hence the composition is again in $J^k\mathcal{A}$; in conclusion, $P \subseteq J^k\mathcal{A}$.

Last, to show that $\mathcal{A}$ is a Γ-atlas: take $\phi_1, \phi_2 \in \mathcal{A}$ and consider the k-jet of the change of coordinates $\phi^{-1}_1 \circ \phi_2 \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$ at a point $x \in \mathbb{R}^n$:

$$j^k_x(\phi^{-1}_1 \circ \phi_2) = j^k_0(\phi^{-1}_1 \circ \phi_2 \circ \varphi_x) \circ j^k_x\varphi^{-1}_x = Fr^k(\phi^{-1}_1)(j^k_0(\phi_2 \circ \varphi_x)) \circ j^k_x\varphi^{-1}_x.$$

However, $j^k_x\phi_2 \in J^k\mathcal{A} = P$ and $j^k_0\varphi_x \in J^k\Gamma$, so $j^k_0(\phi_2 \circ \varphi_x) \in P$ as well (actually in P_0), its image by $Fr^k(\phi_1^{-1})$ is in $G^k \subseteq J^k\Gamma$, hence also the composition with $j^k_x\varphi_x^{-1} \in J^k\Gamma$ is in $J^k\Gamma$. In conclusion, $j^k_x(\phi_1^{-1} \circ \phi_2) \in J^k_x\Gamma$ for every x in its domain, thus $\phi_1^{-1} \circ \phi_2 \in \Gamma$ (Γ being a pseudogroup of order k). Q.E.D.

Example 5.2.5. Let $G \subseteq GL(n,\mathbb{R})$ be a Lie subgroup; since the pseudogroup Γ_G contains the translations, we can apply Proposition 5.2.4. This gives a bijective correspondence which extends that of Proposition 5.1.4:

$$\left\{ 1^{\text{st}}\text{-order almost } \Gamma_G\text{-structures on } M \right\} \xleftrightarrow{\sim} \left\{ G\text{-structures on } M \right\}.$$

Similarly, for higher order Lie subgroups $G \subseteq GL^k(n,\mathbb{R})$ and the pseudogroups $J^k\Gamma_G$ from Example 4.1.16, one has

$$\left\{ k^{\text{th}}\text{-order almost } \Gamma_G\text{-structures on } M \right\} \xleftrightarrow{\sim} \left\{ k^{\text{th}}\text{-order } G\text{-structures on } M \right\},$$

since the isotropy of $J^k\Gamma_G$ is G and $J^k_0(\mathbb{R}^n, M) = Fr^k(M)$. $\Diamond$

5.2.2 Other examples of almost Γ-structures

Example 5.2.6. Let Γ^c_{cont} be the pseudogroup of coorientable contactomorphisms from Example 5.1.11; an almost Γ^c_{cont}-structure of order 1 on M^{2k+1} is the same thing as an **almost contact structure**, i.e. a codimension 1 distribution $\xi \subseteq TM$ together with a 2-form $\beta \in \Omega^2(\xi)$ such that β_x is a linear symplectic form on each hyperplane $\xi_x \subseteq T_xM$.

Let us prove this. First, Γ^c_{cont} is transitive, so we can apply the first part of Proposition 5.2.4. Then, one shows that the linear isotropy group of Γ_{cont} is the group G_{CSF} from Example 2.2.1 (see Lemma 5.5.13 later); it follows that almost Γ_{cont}-structures are the same thing as G_{SF}-structures, i.e. almost symplectic foliations of codimension 1. It is well known, and one can easily see, that these are precisely hyperplane distributions $\xi \subseteq TM$ together with a nondegenerate 2-form $\beta \in \Omega^2(\xi)$, i.e. such that β_x is nondegenerate for every $x \in M$, hence they are almost contact structures. 5

It is very important to notice that, while almost symplectic foliations and almost coorientable contact structures are the same thing, their integrable counterparts (symplectic foliations and coorientable contact structures) are not: in the first case the defining distribution has to be involutive, while in the second maximally non-integrable. A first reason for the two integrability conditions to be radically different comes from Proposition 5.2.4: the second part does not hold since Γ^c_{cont} does not contain translations. This phenomenon will be better interpreted in Sections 5.5.3 and 6.3.1 using Pfaffian Morita equivalences. $\Diamond$

Example 5.2.7. Here are two small and obvious variations of the previous example. Consider the pseudogroup Γ^s_{cont} of *strict* contactomorphisms from Example

5.1.11; then an almost Γ^s_{cont}-structure of order 1 is the same thing of an **almost contact form**, i.e. a pair $(\alpha, \beta) \in \Omega^1(M) \times \Omega^2(M)$ such that $\alpha \wedge \beta^k \neq 0$.

Moreover, almost contact forms coincide with the almost cosymplectic structures described in Example 2.2.1, which are G_{CS}-structures. With this spirit, the same comments as in Example 5.2.6 apply. In particular, there are two different integrability conditions on (α, β): imposing $\beta = d\alpha$ gives rise to a Γ^s_{cont}-structure, i.e. a contact form (which in turn induces a contact structure), while $d\alpha = 0, d\beta = 0$ gives rise to an integrable G_{CS}-structure, i.e. a **cosymplectic structure** (which in turn induces a symplectic foliation).

Similarly, if Γ_{cont} is the pseudogroup of all contactomorphisms of $\mathbb{R}^{2k+1}$ from Example 5.1.11, then an almost Γ_{cont}-structure on M is the same thing as an **almost contact structure**, i.e. a codimension 1 distribution $\xi \subseteq TM$ together with a 2-form $\beta \in \Omega^2(\xi, TM/\xi)$ such that β_x is a linear symplectic form on each hyperplane $\xi_x \subseteq T_xM$. In turn, this coincides with the notion of an almost conformal symplectic foliation from Example 2.2.1, which are G_{CSF}. ◊

Example 5.2.8. An almost Γ^ω-structure P of any order k, for Γ^ω the pseudogroup of analytic diffeomorphisms of $\mathbb{R}^n$ (Example 4.1.3), is a smooth structure.

Indeed, recall from Example 4.1.18 that $J^k\Gamma^\omega = J^k(\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n))$; then P coincides with an almost $\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^n)$-structure, i.e. a smooth structure. Moreover, P is always integrable since a well-known result by Whitney guarantees that every smooth manifold admits an analytic atlas underlying the smooth structure, i.e. a Γ^ω-atlas (Example 5.1.12).

This example shows that the hypotheses of Proposition 5.2.4 are sufficient but not necessary: the statement holds for Γ^ω, which contains translations, but it is not of order 1. Indeed, the isotropy group of Γ^ω is

$$G^k = (J^k\Gamma^\omega)_0 = GL^k(\mathbb{R}^n),$$

so a G^k-structure is just a smooth structure, which is always integrable. ◊

Example 5.2.9. Let Γ be a Lie pseudogroup on X; the **model almost Γ-structure of order** k on X is the subbundle

$$P^\Gamma_{mod} := J^k\Gamma \subseteq J^k(X, X).$$

It is immediate to check that P^Γ_{mod} is the k^{th}-order data of the model Γ-structure $\mathcal{A}^\Gamma_{mod}$ (Example 5.1.6), i.e. it is always integrable. ◊

5.2.3 Morphisms and symmetries

Definition 5.2.10. Let Γ be a pseudogroup on X; an **isomorphism between two almost Γ-structures** $P_1 \subseteq J^k(X, M_1)$ and $P_2 \subseteq J^k(X, M_2)$ is a diffeomorphism $f : M_1 \to M_2$ such that its lift $\hat{f}$ to the k-jet bundles

$$\hat{f} : J^k(X, M_1) \to J^k(X, M_2), \quad j^k_x\phi \mapsto j^k_x(f \circ \phi)$$

takes P_1 to P_2. If $M_1 = M_2$ and $P_1 = P_2$, we call f a **symmetry**. We denote the pseudogroup of local symmetries of an almost Γ-structure P by $\mathrm{Aut}_\Gamma(P)$. ♦

If $f : M_1 \to M_2$ is an isomorphism of Γ-structures (Definition 5.1.17), then it is also an isomorphism between their k^{th}-order data.

Example 5.2.11. For any Lie pseudogroup Γ on X, the pseudogroup of symmetries of P^Γ_{mod} (Definition 5.2.9) is $\mathrm{Aut}_\Gamma(P^\Gamma_{mod}) = \Gamma$.

For a Lie subgroup $G \subseteq GL(n, \mathbb{R})$, the pseudogroup of symmetries of an almost Γ_G-structure coincides with the pseudogroup of symmetries of the associated G-structure (Example 5.2.5). ♢

Remark 5.2.12. An almost Γ-structure P is integrable if and only if every element $p \in P$ has a representative $j^k_x f$ for a local isomorphism $f : X \to M$ between the model almost Γ-structure on X (Definition 5.2.9) and P. For $\Gamma = \Gamma_G$, we recover Lemma 2.1.5 for G-structures. ♢

5.2.4 Integrability of almost Γ-structures

Recall from Proposition 5.1.15 that, for any pseudogroup Γ, a Γ-atlas $\mathcal{A}$ is fully encoded in the principal $\mathrm{Germ}(\Gamma)$-bundle $\mathrm{Germ}(\mathcal{A})$. When Γ is a *Lie* pseudogroup, we can also consider the k^{th}-order data $J^k\mathcal{A}$ from Definition 5.2.2; then $\mathcal{A}$ induces the following tower of principal bundles over M:

$$\mathrm{Germ}(\mathcal{A}) \to J^\infty\mathcal{A} \to \ldots \to J^{k+1}\mathcal{A} \to J^k\mathcal{A} \to \ldots \to J^2\mathcal{A} \to J^1\mathcal{A},$$

whose structure groupoids fit in the tower of Lie groupoids

$$\mathrm{Germ}(\Gamma) \to J^\infty\Gamma \to \ldots \to J^{k+1}\Gamma \to J^k\Gamma \to \ldots \to J^2\Gamma \to J^1\Gamma.$$

An almost Γ-structure (of any order) is then integrable if it comes from something at the top level of the tower. Moreover, any k^{th}-order almost Γ-structure $P \subseteq J^k(X, M)$ induces a l^{th}-order almost Γ-structure $Q = \pi^k_l(P) \subseteq J^l(X, M)$ for all $l \leq k$. The converse (i.e. climbing the tower a finite amount of steps) is not always possible and motivates the following definition.

Definition 5.2.13. Let Γ be a Lie pseudogroup on X and $Q \subseteq J^l(X, M)$ an l^{th}-order almost Γ-structure on M. Then Q is called

- **integrable up to order** k (or k-integrable) if there is a $(k + l)^{th}$-order structure $P = Q^{(k)} \subseteq J^{k+l}(X, M)$ such that the projection $\pi^{k+l}_l : P \to Q$ is a surjective submersion.
- **formally integrable** if it is integrable up to any order. ♦

We can compare this language with the analogous one for G-structures of higher order and k-integrability; with this spirit, in chapter 6 we will develop a theory of obstructions for formal integrability. On the other hand, for the following special cases of transitive pseudogroups Γ (see Section 4.2.1), the formal integrability of an almost Γ-structure P is a sufficient condition for its integrability:

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- Γ is of finite type (Proposition 5.1 in [43] and Theorem 3.3 in [89])
- Γ is irreducible (Theorem 5.4 of [86])
- Γ is elliptic and P is real analytic (Theorem 7.1 in [69])
- Γ is of infinite type and P is real analytic (Theorem 3.3 of [86])

These results either come directly from the corresponding ones for G-structures (see Section 2.5.5) or have been proved in the framework of almost Γ-structures using the original definition for transitive Lie pseudogroups (Remark 5.2.1).

Example 5.2.14. There is a counterexample of a formally integrable almost Γ-structure which is not integrable. This is based on a counterexample by Guillemin and Sternberg [45], who built two G-structures P_1 and P_2 on $\mathbb{R}^5$ and studied their equivalence problem. This turned out to be equivalent to a famous smooth differential equation which Lewy proved [65] to be formally solvable, but without actual smooth (or even differentiable) solutions. Pollack [79] pointed out that P_1 can be reinterpreted as a Γ-structure, for Γ the pseudogroup of symmetries of P_2, and that by construction P_1 is formally integrable but not integrable. $\lozenge$

5.3 The abstract framework for almost Γ-structures: principal Pfaffian bundles

An almost Γ-structure $P \subseteq J^k(X, M)$ carries a richer structure than just that of principal $J^k\Gamma$-bundle: both P and $J^k\Gamma$ are endowed with the restriction of the respective Cartan forms of the jet bundles $J^k(X, M)$ and $J^k(X, X)$, which satisfy various natural compatibility conditions and, furthermore, encode the fundamental information about P.

In this chapter, we describe an abstract way of thinking of an almost Γ-structure, forgetting the ambient jet bundle and extracting the main properties of the two Cartan forms mentioned above. This is the same philosophy we adopted in chapter 1 (going from PDEs to Pfaffian fibrations) and in chapter 2 (going from G-structures to abstract G-structures).

In order to do so, we are going first to develop the theory of compatible actions and their infinitesimal counterpart in the framework of Pfaffian groupoids; actually, we start with something slightly more general, in order to separate the properties arising from the multiplicativity and those arising from the Pfaffian conditions.

5.3.1 Multiplicative actions

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid acting (on the left) on the manifold P along the map $\mu : P \to M$ (Definition 3.1.12); denote by m_P the action map, defined on the fibred product

$$\mathcal{G}\ {}_s\!\times_\mu P := \{(g,p) \in \mathcal{G} \times P \mid s(g) = \mu(p)\}.$$

Moreover, let E be a representation of $\mathcal{G}$, $\alpha \in \Omega^k(\mathcal{G}, t^*E)$ a multiplicative form (Definition 3.4.2) and $\beta \in \Omega^k(P, \mu^*E)$ a differential form; we represent this setting in the following diagram:

$$\begin{tikzcd}
(\mathcal{G}, \alpha) \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \arrow[r, phantom, "\curvearrowright"] & (P, \beta) \arrow[dl, "\mu"] \\
M &
\end{tikzcd}$$

Definition 5.3.1. The $\mathcal{G}$-action on P is called **multiplicative** (with respect to α and β) if

$$(m_P^*\beta)_{(g,p)} = (\mathrm{pr}_1^*\,\alpha)_{(g,p)} + g \cdot (\mathrm{pr}_2^*\,\beta)_{(g,p)} \quad \forall (g,p) \in \mathcal{G}\ {}_s\!\times_\mu P.$$

As for multiplicative forms, we will often use the sloppy notation

$$m_P^*\beta = \mathrm{pr}_1^*\,\alpha + g \cdot \mathrm{pr}_2^*\,\beta.$$

Multiplicative right actions are defined analogously, with the condition

$$g \cdot (m_P^*\beta)_{(p,g)} = (\mathrm{pr}_1^*\,\beta)_{(p,g)} + (\mathrm{pr}_2^*\,\alpha)_{(p,g)} \quad \forall (p,g) \in P\ {}_\mu\!\times_t \mathcal{G}. \qquad \blacklozenge$$

Remark 5.3.2. The formula defining a multiplicative action can be written as an equality between multiplicative forms on the action groupoid $\mathcal{G}\ {}_s\!\times_\mu P \rightrightarrows P$ with coefficients in the vector bundle $(\mu \circ m_P)^*E = (t \circ \mathrm{pr}_1)^*E$:

$$m_P^*\beta - g \cdot \mathrm{pr}_2^*\,\beta = \mathrm{pr}_1^*\,\alpha.$$

Indeed, the target and source of the action groupoid $\mathcal{G}\ {}_s\!\times_\mu P$ coincide with the action map m_P and the projection pr_2; then the left-hand side is a multiplicative form by Example 3.4.4.

Similarly, the right-hand side is the pullback of the multiplicative form α by the Lie groupoid morphism $\mathrm{pr}_1 : \mathcal{G}\ {}_s\!\times_\mu P \to \mathcal{G}$, hence is multiplicative by Example 3.4.3. $\lozenge$

Example 5.3.3. Let $\mathcal{G}$ be a Lie groupoid acting on $\mu : P \to M$, E a representation of $\mathcal{G}$ and $\alpha \in \Omega^k(M, E)$. Then the $\mathcal{G}$-action m_P is multiplicative with respect to the cohomologically trivial multiplicative form $\omega = \delta\alpha \in \Omega^k(\mathcal{G}, t^*E)$ (Example 3.4.4) and the differential form $\theta = -\mu^*\alpha \in \Omega^k(P, \mu^*E)$:

$$m_P^*\theta - \mathrm{pr}_1^*\,\omega - g \cdot \mathrm{pr}_2^*\,\theta =$$

$$= \cancel{-(\mu \circ m_P)^*\alpha} - \cancel{g \cdot (s \circ \mathrm{pr}_1)^*\alpha} + \cancel{(t \circ \mathrm{pr}_1)^*\alpha} + \cancel{g \cdot (\mu^* \circ \mathrm{pr}_2)^*\alpha} = 0. \qquad \lozenge$$

5.3.2 Infinitesimally multiplicative action

We are going to prove an equivalent characterisation for a Lie groupoid action to be multiplicative (Definition 5.3.1) using its infinitesimal data, i.e. the Lie algebroid and the Spencer operator (Definition 3.4.20) associated to the multiplicative form. This is analogous to the moment map condition for Hamiltonian actions - see Example 5.3.6 below.

Since in the following we will apply these results to Pfaffian groupoids, we will focus on the case of 1-forms; nevertheless, many results could be proved in greater generality, using multiplicative forms of any degree and a suitable generalisation of Proposition 3.4.22 to Spencer operators of higher order (see e.g. Sections 2.2-2.3 of [25]).

Proposition 5.3.4. *Let $\mathcal{G}$ be a Lie groupoid, $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form with coefficients in a representation E of $\mathcal{G}$ and consider the associated Spencer operator (D, l) from Proposition 3.4.22.*

Assume moreover that $\mathcal{G}$ acts on $\mu : P \to M$, denote its infinitesimal action by $a : \Gamma(A) \to \mathfrak{X}(P)$ (Definition 3.2.9), and let $\theta \in \Omega^1(P, \mu^*E)$.

1. *If the action of $(\mathcal{G}, \omega)$ on (P, θ) is multiplicative, then*

$$\begin{cases} \theta(a(\alpha)) = l(\alpha) \\ \nabla'_\beta(\theta(Y)) = \theta[\rho'(\beta), Y] + D'_Y(\beta) \end{cases}$$

   for all $\alpha \in \Gamma(A), \beta \in \Gamma(\mu^*A), Y \in \mathfrak{X}(P)$. Here ρ', ∇' and D' are, respectively, the anchor, the induced representation and Spencer operator of the pullback algebroid μ^*A (Example 3.2.9).

2. *If $\mathcal{G}$ is also s-connected, the converse holds.*

Proof. For the first claim, consider the following differential forms in $\Omega^1(\mathcal{G} \times_M P, (t \circ \mathrm{pr}_1)^*E)$:

$$\Theta = g \cdot (\mathrm{pr}_2)^*\theta - m_P^*(\theta), \quad \Omega = (\mathrm{pr}_1)^*(-\omega).$$

By definition, the $\mathcal{G}$-action is multiplicative if and only if $\Theta = \Omega$.

According to Remark 5.3.2, both Θ and Ω are multiplicative, hence they induce the Spencer operators (D^Θ, l^Θ) and (D^Ω, l^Ω) (Proposition 3.4.22).

In order to compute the Spencer operator of Θ, consider the action groupoid $\mathcal{G} \times_M P \rightrightarrows P$ and its algebroid $\mathrm{Lie}(\mathcal{G} \times_M P) = \mu^*A$, whose anchor is the differential of m_P, i.e. the infinitesimal action a (Example 3.2.9). Applying Proposition 3.4.25, we have for all $\alpha \in \Gamma(A)$, $p \in P$, $x = \mu(p)$, and every $Y \in \mathfrak{X}(P)$

$$l^\Theta(\mu^*\alpha)(p) = -\theta_p(\rho_{\mu^*A}(\mu^*\alpha)(p)) = -\theta_p(a_p(\alpha_x)),$$

$$D^\Theta_Y(\mu^*\alpha) = -(\tilde{\nabla}_{\mu^*\alpha}\theta)(Y) = -\nabla'_{\mu^*\alpha}(\theta(Y)) + \theta([\rho'(\mu^*\alpha), Y]).$$

Similarly, for the Spencer operator of Ω, we consider the Lie groupoid morphism $\mathrm{pr}_1 : \mathcal{G} \times_M P \to \mathcal{G}$ over $\mu : P \to M$. By Proposition 3.4.26, for every $\alpha \in \Gamma(A)$, $p \in P$, $x = \mu(p)$ and every $Y \in \mathfrak{X}(P)$,

$$l^{\Omega}(\mu^*\alpha)(p) = (\mu^* l^{-\omega})_p(\alpha) = l_x^{-\omega}(\alpha_x) = -\omega_{1_x}(\alpha_x),$$

$$D_Y^{\Omega}(\tilde{\alpha}) = (\mu^* D^{-\omega})_Y(\tilde{\alpha}) = -D'_Y(\mu^*\alpha)$$

Since $(D^{\Theta}, l^{\Theta}) = (D^{\Omega}, l^{\Omega})$, we conclude by comparing the previous four expressions replacing $\mu^*\alpha$ with any section β of the pullback algebroid $\mu^* A$.

It is interesting to notice that the first equation follows also directly from the multiplicativity of the $(\mathcal{G}, \omega)$-action m_P on (P, θ), together with the formulae from Example 3.2.9:

$$\theta_p(a_p(\alpha_x)) = \theta_p(d_{(1_x,p)} m_P(\alpha_x, 0)) = (m_P^*\theta)_{(1_x,p)}(\alpha_x, 0) = \omega_{1_x}(\alpha_x) + \cancel{\theta_p(0)} = l_x(\alpha_x).$$

For the second part, it is enough to use the hypothesis of s-connectedness of $\mathcal{G}$ in the second part of Proposition 3.4.22. Then we can conclude that the action is multiplicative if and only if $(D^{\Theta}, l^{\Theta}) = (D^{\Omega}, l^{\Omega})$. Q.E.D.

Example 5.3.5. Let G be a connected Lie group acting on a manifold P and V a representation of G. Given $\omega \in \Omega^1(G, V)$ multiplicative and $\theta \in \Omega^1(P, V)$, then the (G, ω)-action on (P, θ) is multiplicative if and only if

$$\begin{cases} \theta \circ a = l \\ \theta \text{ is } G\text{-invariant} \end{cases}$$

where $a : \mathfrak{g} \to TP$ is the infinitesimal action and $l : \mathfrak{g} \to V$ is the G-equivariant linear map defining ω (Example 3.4.6).

This follows from Proposition 5.3.4; the first condition is identical, while the second reduces to

$$\nabla_\alpha(\theta(Y)) = \theta([a(\alpha), Y]),$$

since any Spencer operator on the Lie algebroid $\mathfrak{g} \to \{*\}$ is trivial. In turn, this is equivalent to θ being infinitesimally equivariant and therefore, since G is connected, also G-equivariant.

As in the proof of Proposition 5.3.4, note that the G-equivariance follows directly from the multiplicativity without G being necessarily connected:

$$\theta_{g\cdot p}(g \cdot X) = \theta_{g\cdot p}(d_{(g,p)} m_P(0, X)) = (m_P^*\theta)_{(g,p)}(0, X) = \cancel{\omega_g(0)} + g \cdot \theta_p(\theta) \quad \Diamond$$

Example 5.3.6. Let $(\mathcal{G}, \omega) \rightrightarrows M$ be a symplectic groupoid (Example 3.4.8) acting from the left on a symplectic manifold (P, θ) along the map $\mu : P \to M$; the action is called **Hamiltonian** if

$$m_P^*\theta = \mathrm{pr}_1^*\,\omega + \mathrm{pr}_2^*\,\theta.$$

This coincides with Definition 5.3.1 in the particular case when the coefficients are trivial. In fact, the theory of Hamiltonian actions constitutes an important source of inspiration for this entire chapter, and motivates some terminology.

We recall also the well known infinitesimal condition for a $(\mathcal{G}, \omega)$-action to be Hamiltonian; it is the so-called moment map condition:

$$\iota_{a(\alpha)}\theta = \mu^*(I(\alpha)) \quad \forall \alpha \in \Gamma(A),$$

where A is the Lie algebroid of $\mathcal{G}$, $a : \Gamma(A) \to \mathfrak{X}(P)$ the associated infinitesimal action (Definition 3.2.9), and $I : A \to T^*M$ is the isomorphism between A and the cotangent algebroid T^*M (Example 3.2.7). Such an infinitesimal condition is the analogue of the first equation of Proposition 5.3.4, which can be rewritten as

$$\iota_{a(\alpha)}\theta = \mu^*(\omega_{|M}(\alpha)) \quad \forall \alpha \in \Gamma(A). \qquad \diamondsuit$$

5.3.3 Multiplicative actions on principal bundles

When a multiplicative action of a Lie groupoid $\mathcal{G}$ on P is also principal, the corresponding gauge groupoid $\mathrm{Gauge}(P)$ (Definition 3.1.11) carries a multiplicative form and its action on P is multiplicative as well.

Proposition 5.3.7. *Let $\mathcal{G}$ be a Lie groupoid and $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form with coefficient in a representation E of $\mathcal{G}$. Let also P be a left principal $\mathcal{G}$-bundle over X, whose moment map μ is a submersion, and $\theta \in \Omega^1(P, \mu^*E)$ a differential form such that the $\mathcal{G}$-action is multiplicative.*

Then $\mathrm{Gauge}(P)$ carries a unique multiplicative form $\hat{\omega}$ such that

$$\tau^*\hat{\omega} = \mathrm{pr}_1^*\,\theta - \mathrm{pr}_2^*\,\theta,$$

for τ the projection from $P \times_\mu P$. Moreover, the action of $(\mathrm{Gauge}(P), \hat{\omega})$ on (P, θ) is multiplicative.

Proof. Let us represent on the following diagram the spaces and the maps we are going to use.

$$\begin{tikzcd}
 & & (P \times_\mu P, \tilde{\theta}) \arrow[dl, shift right, "\tilde{s}"'] \arrow[dl, shift left, "\tilde{t}"] \arrow[dr, "\tau"] \arrow[ddll, bend left, "\tilde{\mu}"] & \\
(\mathcal{G}, \omega) \arrow[urr, bend left, "\tilde{m}_P"] \arrow[r, bend left, "m_P"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & (P, \theta) \arrow[dl, "\mu"] \arrow[drr, "\pi"] & & ((P \times_\mu P)/\mathcal{G}, \hat{\omega}) \arrow[ll, bend right, "\hat{m}_P"'] \arrow[d, shift right, "\hat{s}"'] \arrow[d, shift left, "\hat{t}"] \\
M & & & P/\mathcal{G}
\end{tikzcd}$$

The proof will be carried out in four steps:

1. The cohomologically trivial form $\tilde{\theta} = \tilde{s}^*\theta - \tilde{t}^*\theta \in \Omega^1(P \times_\mu P, \tilde{\mu}^*E)$ is basic.
2. There is a unique form $\hat{\omega} \in \Omega^k(\mathrm{Gauge}(P), P[E])$ such that $\tau^*\hat{\omega} = \tilde{\theta}$.
3. $\hat{\omega}$ is multiplicative.
4. The action of Gauge(P) on P is multiplicative.

For the first, we denote by pr the projections from $\mathcal{G}\ {}_s\!\times_\mu P$ on the first and second component, and by $\tilde{\mathrm{pr}}$ the projections from $\mathcal{G}\ {}_s\!\times_{\tilde{\mu}} (P \times_\mu P)$ to either one of the three components or two of them.

Note that the vector bundle $\mu^*E \to P$ is a trivial representation of the groupoid $P \times_\mu P$: then both $\tilde{s}^*\theta$ and $\tilde{t}^*\theta$ belong to the same fibre and the $g\cdot$ of Definition 3.4.2 becomes redundant, so we omit it. Using the multiplicativity of m_P we find

$$(\tilde{m}_P)^*\tilde{\theta} = (\tilde{m}_P)^*(\tilde{s}^*\theta) - (\tilde{m}_P)^*(\tilde{t}^*\theta) = (\tilde{s} \circ (\tilde{m}_P))^*\theta - (\tilde{t} \circ (\tilde{m}_P))^*\theta =$$

$$\begin{aligned}
&= (m_P \circ \tilde{\mathrm{pr}}_{13})^*\theta - (m_P \circ \tilde{\mathrm{pr}}_{12})^*\theta = \tilde{\mathrm{pr}}_{13}^*(m_P^*\theta) - \tilde{\mathrm{pr}}_{12}^*(m_P^*\theta) = \\
&= \tilde{\mathrm{pr}}_{13}^*(\mathrm{pr}_1^*\,\omega) + \tilde{\mathrm{pr}}_{13}^*(g \cdot \mathrm{pr}_2^*\,\theta) - \tilde{\mathrm{pr}}_{12}^*(\mathrm{pr}_1^*\,\omega) - \tilde{\mathrm{pr}}_{12}^*(g \cdot \mathrm{pr}_2^*\,\theta) = \\
&= \cancel{\tilde{\mathrm{pr}}_1^*\omega} + g \cdot \tilde{\mathrm{pr}}_3^*\theta - \cancel{\tilde{\mathrm{pr}}_1^*\omega} - g \cdot \tilde{\mathrm{pr}}_2^*\theta = \\
&= g \cdot ((\tilde{s} \circ \tilde{\mathrm{pr}}_{23})^*\theta - (\tilde{t} \circ \tilde{\mathrm{pr}}_{23})^*\theta) = g \cdot \tilde{\mathrm{pr}}_2^*(\tilde{s}^*\theta - \tilde{t}^*\theta) = g \cdot \tilde{\mathrm{pr}}_2^*\tilde{\theta}.
\end{aligned}$$

By Lemma 3.2.15 we conclude that $\tilde{\theta}$ is basic.

The second part follows from Proposition 3.2.16: since

$$\Omega^1(\mathrm{Gauge}(P), P[E]) \to \Omega^1_{bas}(P \times_\mu P, \tilde{\mu}^*E), \quad \omega \mapsto \tau^*\omega$$

is an isomorphism, and $\tilde{\theta} \in \Omega^1_{bas}(P \times_\mu P, \tilde{\mu}^*E)$, then $\hat{\omega}$ exists and is unique.

For the third part, denote by $\bar{m}$, $\bar{\mathrm{pr}}_1$ and $\bar{\mathrm{pr}}_2$ the maps

$$(P \times_\mu P)\ {}_{\tilde{t}}\!\times_{\tilde{s}} (P \times_\mu P) \to P \times_\mu P$$

corresponding to the multiplication of the groupoid $P \times_\mu P$ and to the projections of $(P \times_\mu P)\ {}_{\tilde{t}}\!\times_{\tilde{s}} (P \times_\mu P)$ on the first and second component, and by $[\bar{m}]$ and $[\bar{\mathrm{pr}}_i]$ the projections of those maps to the quotient $(P \times_\mu P)/\mathcal{G}$. With the usual arguments we get

$$(\tau \times \tau)^*([\bar{m}]^*\hat{\omega}) = ([\bar{m}] \circ (\tau \times \tau))^*\hat{\omega} = (\tau \circ \bar{m})^*\hat{\omega} = \bar{m}^*(\tau^*\hat{\omega}) = \bar{m}^*\tilde{\theta} =$$

$$\begin{aligned}
&= \bar{\mathrm{pr}}_1^*\tilde{\theta} + \bar{\mathrm{pr}}_2^*\tilde{\theta} = \bar{\mathrm{pr}}_1^*(\tau^*\hat{\omega}) + \bar{\mathrm{pr}}_2^*(\tau^*\hat{\omega}) = (\tau \circ \bar{\mathrm{pr}}_1)^*\hat{\omega} + (\tau \circ \bar{\mathrm{pr}}_2)^*\hat{\omega} = \\
&= ([\bar{\mathrm{pr}}_1] \circ (\tau \times \tau))^*\hat{\omega} + ([\bar{\mathrm{pr}}_2] \circ (\tau \times \tau))^*\hat{\omega} = (\tau \times \tau)^*([\bar{\mathrm{pr}}_1]^*\hat{\omega} + [\bar{\mathrm{pr}}_2]^*\hat{\omega}).
\end{aligned}$$

By the injectivity of the pullback we get the multiplicativity of $\hat{\omega}$.

For the fourth claim, we see first that the action of the fibred pair groupoid $P \times_\mu P$ on P

$$\hat{m}_P : P\ {}_{id}\times_{\tilde{t}} (P \times_\mu P) \to P, \quad (p,(p,q)) \mapsto q$$

is multiplicative with respect to θ and $\tilde{\theta}$:

$$(\hat{\mathrm{pr}}_1)^*\theta + (\hat{\mathrm{pr}}_2)^*\tilde{\theta} = \cancel{(\hat{\mathrm{pr}}_1)^*\theta} + (\tilde{s} \circ \hat{\mathrm{pr}}_2)^*\theta - \cancel{(\tilde{t} \circ \hat{\mathrm{pr}}_2)^*\theta} = (\hat{m}_P)^*\theta.$$

Then, when passing to the action $[\hat{m}_P]$ from the quotient $(P \times_\mu P)/\mathcal{G}$, the multiplicativity condition is preserved:

$$(id_P,\tau)^*([\hat{m}]^*\theta) = ([\hat{m}] \circ (id_P,\tau))^*\theta = \hat{m}^*\theta = \hat{\mathrm{pr}}_1^*\theta - \hat{\mathrm{pr}}_2^*\tilde{\theta} =$$

$$= ([\hat{\mathrm{pr}}_1] \circ (id_P,\tau))^*\theta + (\tau \circ \hat{\mathrm{pr}}_2)^*\hat{\omega} = ([\hat{\mathrm{pr}}_1] \circ (id_P,\tau))^*\theta + ([\hat{\mathrm{pr}}_2] \circ (id_P,\tau))^*\hat{\omega} =$$

$$= (id_P,\tau)^*([\hat{\mathrm{pr}}_1]^*\theta + [\hat{\mathrm{pr}}_2]^*\hat{\omega}).$$

Again, by the injectivity of the pullback we get the multiplicativity of the Gauge(P)-action on P. Q.E.D.

5.3.4 Pfaffian actions

We focus now on multiplicative 1-forms which are part of a Pfaffian groupoid structure, specialising the definition of multiplicative actions to that of "Pfaffian actions".

Definition 5.3.8. Let $(\mathcal{G},\omega)$ be a Pfaffian groupoid over M, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$ for E a representation of $\mathcal{G}$. Moreover, let $\mathcal{G}$ act on a fibration $\mu : P \to M$, and let $\theta \in \Omega^1(P, \mu^*E)$. The $(\mathcal{G},\omega)$-action on (P,θ) is called **Pfaffian** if

- The action is multiplicative w.r.t. ω and θ (Definition 5.3.1),
- $\mathfrak{g}_\mu(\theta) := \ker(\theta) \cap \ker(d\mu) \subseteq TP$ is an involutive distribution. ♦

The second condition is related to $\mu : (P,\theta) \to M$ being a Pfaffian fibration, in the sense of Definition 1.3.1. However, we do not require that θ is μ-transversal: this would exclude some natural examples where this property does not hold. The reason we ask the μ-involutivity condition on θ will be clear soon: it has to do with making the Pfaffian structure "invariant", in a suitable sense, under Morita equivalence. It is also clear, at this point, why we changed the term *Pfaffian bundle* from [81] to *Pfaffian fibration*: we wanted to avoid the counterintuitive statement that a principal Pfaffian bundle is not necessarily a Pfaffian bundle.

Example 5.3.9. Let (G,ω) be a Pfaffian group (Example 4.3.8), with $\omega \in \Omega^1(G,V)$, and assume that G acts multiplicatively on a manifold P with respect to some form $\theta \in \Omega^1(P, V)$. By Example 5.3.5, this is equivalent to θ being G-equivariant and

$$\theta \circ a = l,$$

where $a : \mathfrak{g} \to TP$ is the infinitesimal action and $l : \mathfrak{g} \to V$ is the G-equivariant linear map defining ω (Example 3.4.6).

In general, the (G, ω)-action on (P, θ) is not necessarily Pfaffian, since

$$\ker(\theta) \cap \ker(d\mu) = \ker(\theta) \subseteq TP$$

may not be involutive. A sufficient condition for this to happen is the transitivity of the G-action: it is well known that such condition is equivalent to $\mathrm{Im}(a) = TP$, so that, using $\theta \circ a = l$,

$$\ker(\theta) = a(\ker(l)),$$

which is an involutive distribution.

However, soon we will move to the case when P is a principal G-bundle, where the G-action cannot be transitive. In such cases, the involutivity of $\ker(\theta)$ has to be imposed, together with the condition $\theta \circ a = l$ and the fact that θ is G-equivariant. For instance, using the two Pfaffian structures on G examined in Example 4.3.8 (corresponding, respectively, to $l = 0$ and $l = id_{\mathfrak{g}}$) we have the following:

- For $\omega = 0$, the (G, ω)-action on (P, θ) is Pfaffian if and only if $\theta(a(v)) = 0$ and $\ker(\theta)$ is involutive. In particular, θ is a basic form.
- For $\omega = \omega_{MC}$, the (G, ω)-action on (P, θ) is Pfaffian if and only if $\theta(a(v)) = v$ and $\ker(\theta)$ is involutive. In particular, θ is a flat connection on P. $\Diamond$

Example 5.3.10. The action of a Pfaffian groupoid $(\mathcal{G}, \omega)$ on itself by left/right multiplication (Example 3.1.13) is Pfaffian because Definition 5.3.1 coincides exactly with Definition 3.4.2 and ω is both s-involutive and t-involutive.

Similarly, for any $x \in M$, the left action of $(\mathcal{G}, \omega)$ on the fibre $(s^{-1}(x), \omega_{|_{s^{-1}(x)}})$, and the right action of the Pfaffian isotropy group $(\mathcal{G}_x, \omega_{|\mathcal{G}_x})$ (Example 4.3.9) on $(s^{-1}(x), \omega_{|s^{-1}(x)})$ are Pfaffian as well. $\Diamond$

We discuss now the fundamental example that motivates this section: the $J^k\Gamma$-action of an almost Γ-structure is Pfaffian with respect to the Cartan forms.

Proposition 5.3.11. *Let Γ be a Lie pseudogroup on X and P be an almost Γ-structure over M (Definition 5.2.3). Then the (principal) action of the Pfaffian groupoid $J^k\Gamma$ (Example 4.3.11) on P is Pfaffian with respect to the Cartan form ω of $J^k(X, X)$ (which is multiplicative by Example 3.4.5) and to the Cartan form θ of $J^k(X, M)$.*

Proof. We consider $k = 1$ for simplicity; the general case can be obtained by similar computations, or by identifying $J^k\Gamma \subseteq J^1(J^{k-1}\Gamma)$ and $P \subseteq J^1(J^{k-1}(X, M))$ inside the space of 1-jets.

Since the Cartan form of $J^1\Gamma$ has coefficients in t^*E, for $E = TX$, the coefficients of the Cartan form $\theta \in \Omega^1(P, \pi^*TM)$ have to be μ^*E. Accordingly, we are going to use the following alternative version of the Cartan form $\theta \in \Omega^1(P, \mu^*E)$:

$$\theta_{j^1_x\phi}(v) := d(\phi^{-1} \circ \pi^1_0)(v) - d\pi^1(v).$$

Let us check the multiplicativity of the action, for any $p = j^1_{f(x)}\phi \in P$ and $g = j^1_x f \in J^k\Gamma$. On the one hand, using $\pi \circ m_P = \pi \circ \mathrm{pr}_1$ and $\mu \circ m_P = s \circ \mathrm{pr}_2$,

$$g \cdot (m_P^*\theta)_{(p,g)}(v,\alpha) = df \circ d(\phi \circ f)^{-1} \circ d\pi \circ dm_P(v,\alpha) - df \circ d\mu \circ dm_P(v,\alpha) =$$

$$= d\phi^{-1} \circ d\pi \circ d\,\mathrm{pr}_1(v,\alpha) - df \circ ds \circ d\,\mathrm{pr}_2(v,\alpha)$$

On the other hand, since $\mu \circ \mathrm{pr}_1 = t \circ \mathrm{pr}_2$,

$$(\mathrm{pr}_1^*\,\theta)_{(p,g)}(v,\alpha) + (\mathrm{pr}_2^*\,\omega)_{(p,g)}(v,\alpha) = \theta_p(d\,\mathrm{pr}_1(v,\alpha)) + \omega_g(d\,\mathrm{pr}_2(v,\alpha)) =$$

$$= d\phi^{-1} \circ d\pi \circ d\,\mathrm{pr}_2(v,\alpha) - \cancel{d\mu \circ d\,\mathrm{pr}_1(v,\alpha)} + \cancel{dt \circ d\,\mathrm{pr}_2(v,\alpha)} - df \circ ds \circ d\,\mathrm{pr}_2(v,\alpha).$$

Last, the action is Pfaffian since

$$\ker(\theta) \cap \ker(d\mu) = \ker(d\pi^1_0) \subseteq TP,$$

which is an involutive distribution. Q.E.D.

In the case $X = M$ and $P = \mathcal{G}$ we recover the proof that the Cartan form of $J^k\Gamma$ is multiplicative (see e.g. Proposition 2.4.3 of [100]).

Remark 5.3.12. The slight modification of the Cartan form is just a technicality to make sure that the coefficients match; if the action was from the left, we would have not had such a problem. Moreover, it is worth noticing that we have only applied to the standard Cartan form the isomorphism $\Phi : \pi^*TM \to \mu^*TX$, that we are going to prove in Lemma 5.3.18. ♢

5.3.5 Principal Pfaffian bundles

Definition 5.3.13. Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid over X, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$ a multiplicative form with coefficients in a representation E of $\mathcal{G}$. A (left) **principal Pfaffian bundle** over M is a principal $\mathcal{G}$-bundle P together with a differential form $\theta \in \Omega^1(P, \mu^*E)$, as in the diagram

$$\begin{tikzcd}
(\mathcal{G},\omega) \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & (P,\theta) \arrow[dll, "\mu"'] \arrow[dr, "\pi"] & \\
X & & & M
\end{tikzcd}$$

such that the $(\mathcal{G}, \omega)$-action on (P, θ) is Pfaffian (Definition 5.3.8). ♦

We emphasise that a principal Pfaffian bundle is not necessarily a Pfaffian fibration (in the sense of Definition 1.3.1), neither over M nor over X. However, the following lemma shows that $\pi : (P, \theta) \to M$ is always π-involutive; using this, we will prove a characterisation for π to be a Pfaffian fibration.

Lemma 5.3.14. *Let $\pi : (P, \theta) \to M$ be a principal Pfaffian $\mathcal{G}$-bundle. Then the μ-pullback of the symbol space $\mathfrak{g}$ (Definition 4.3.1) of the Pfaffian groupoid $(\mathcal{G}, \omega)$ is isomorphic to the space $\mathfrak{g}_\pi(\theta) := \ker(d\pi) \cap \ker(\theta)$:*

$$\mathfrak{g}_{\mu(p)} \cong \mathfrak{g}_\pi(\theta)_p \quad \forall p \in P.$$

In particular, $\mathfrak{g}_\pi(\theta) \subseteq TP$ is a regular involutive distribution on P.

Proof. Let A be the Lie algebroid of $\mathcal{G}$; the isomorphism will be induced by the infinitesimal action $a : \mu^* A \to TP$ (Definition 3.2.9). Using Proposition 3.2.13 and the fact that infinitesimal free actions are injective, we see that

$$a_p : A_{\mu(p)} \to \mathrm{Im}(a_p) = \ker(d_p \pi).$$

is an isomorphism. Therefore, we have only to show that a_p sends $\mathfrak{g}_{\mu(p)} = A_{\mu(p)} \cap \ker(\omega_{1_{\mu(p)}})$ to $\mathfrak{g}_\pi(\theta)_p = \ker(d_p\pi) \cap \ker(\theta_p)$. Consider $\alpha \in \mathfrak{g}_{\mu(p)}$; since the action is multiplicative, by Proposition 5.3.4

$$\theta_p(a_p(\alpha)) = \omega_{1_{\mu(p)}}(\alpha) = 0,$$

therefore $a_p(\alpha) \in \mathfrak{g}_\pi(\theta)_p$. Conversely, if $a_p(\alpha) \in \mathfrak{g}_\pi(\theta)_p$, for some $\alpha \in A_{\mu(p)}$, then $\alpha \in \ker(\omega_{1_{\mu(p)}})$, hence $\alpha \in \mathfrak{g}_{\mu(p)}$.

Last, the infinitesimal action $a_p = d_{1_{\mu(p)}} m_P(p, \cdot)$ preserves the Lie brackets: since the symbol space $\mathfrak{g}$ of $\mathcal{G}$ is involutive, then $\mathfrak{g}_\pi(\theta)$ is involutive too. Q.E.D.

And here is the promised characterisation:

Theorem 5.3.15. *Let (P, θ) be a principal Pfaffian $\mathcal{G}$-bundle over M, and assume that θ is pointwise surjective; then* 5

$$\pi : (P, \theta) \to M$$

is a Pfaffian fibration (Definition 1.3.1) if and only if the Pfaffian groupoid $\mathcal{G}$ is full (Definition 4.3.2).

Proof. The proof is based on dimension counting. Since θ is surjective,

$$\dim(P) = \dim(\ker(\theta_p)) + \dim(\mathrm{Im}(\theta_p)) = \dim(\ker(\theta_p)) + \mathrm{rank}(E), \quad p \in P.$$

Similarly, the rank of $\mathfrak{g}_\pi(\theta)$ (see Lemma 5.3.14) can be computed either by linear algebra or via the isomorphism with the symbol space $\mathfrak{g}$ of $\mathcal{G}$:

$$\dim(\mathfrak{g}_\pi(\theta)_p) = \dim(\ker(d_p\pi)) + \dim(\ker(\theta_p)) - \dim(\ker(d_p\pi) + \ker(\theta_p)), \quad p \in P$$

$$\dim(\mathfrak{g}_\pi(\theta)_p) = \dim(\mathfrak{g}_x) = \dim(\ker(d_{1_x}s)) - \dim(\mathrm{Im}(\omega_{1_x})), \quad x = \pi(p).$$

Last, by Proposition 3.2.13, we remark that

$$\dim(\ker(d_{1_{\mu(p)}}s)) = \dim(A_{\mu(p)}) = \cancel{\dim(\ker(a_p))} + \dim(\mathrm{Im}(a_p)) = \dim(\ker(d_p\pi)).$$

Putting everything together, we obtain

$$\dim(P) - \dim(\ker(d_p\pi) + \ker(\theta_p)) = \mathrm{rank}(E) - \dim(\mathrm{Im}(\omega_{1_{\mu(p)}})).$$

The vanishing of the right-hand side is equivalent to $(\mathcal{G}, \omega)$ being full, while the vanishing of the left-hand side is equivalent to θ being π-transversal. By Lemma 5.3.14, this is equivalent to $\pi : (P, \theta) \to M$ being a Pfaffian fibration. Q.E.D.

Definition 5.3.16. Let P be a principal Pfaffian bundle (Definition 5.3.13), as in the following diagram

$$\begin{tikzcd}
(\mathcal{G},\omega) \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & (P,\theta) \arrow[dll, "\mu"'] \arrow[dr, "\pi"] & \\
X & & & M
\end{tikzcd}$$

Then P is called **Lie-Pfaffian bundle** when, in addition, the involutive distribution (see Lemma 5.3.14)

$$\mathfrak{g}_\pi(\theta) := \ker(d\pi) \cap \ker(\theta) \subseteq TP$$

coincides with the involutive distribution (see Definition 5.3.8)

$$\mathfrak{g}_\mu(\theta) := \ker(d\mu) \cap \ker(\theta) \subseteq TP.$$

In this case, $\mathfrak{g}_\pi(\theta) = \mathfrak{g}_\mu(\theta)$ is called **symbol space** of P and denoted by $\mathfrak{g}(P)$ or $\mathfrak{g}(\theta)$. ♦

This Lie-Pfaffian condition will be understood better when dealing with Morita equivalences (see Example 5.4.7). It is trivially satisfied, for instance, when π and μ are the source and target of a Lie-Pfaffian groupoid; however, the condition $\mathfrak{g}_\pi(\theta) = \mathfrak{g}_\mu(\theta)$ is stronger than just requiring $(\mathcal{G}, \omega)$ to be Lie-Pfaffian, as the following theorem shows.

Theorem 5.3.17. *Let $\pi : (P, \theta) \to M$ be a principal Pfaffian $(\mathcal{G}, \omega)$-bundle (Definition 5.3.13); the Pfaffian groupoid $\mathcal{G}$ is Lie-Pfaffian (Definition 4.3.4) if and only if*

$$\ker(d\pi) \cap \ker(\theta) \subseteq \ker(d\mu) \cap \ker(\theta).$$

In particular, if (P, θ) is Lie-Pfaffian, then so is $(\mathcal{G}, \omega)$.

Proof. Consider $v \in \ker(d_p\pi) \cap \ker(\theta_p)$; by Proposition 3.2.13, $v = a_p(\alpha)$ for some $\alpha \in A_{\mu(p)}$, and by Proposition 5.3.4,

$$\omega_{1_{\mu(p)}}(\alpha) = \theta_p(a_p(\alpha)) = \theta_p(v) = 0.$$

Then $\alpha \in \ker(\omega_{1_{\mu(p)}}) \cap A_{\mu(p)}$, i.e. α is in the symbol space $\mathfrak{g}$ of $(\mathcal{G}, \omega)$. Using the Lie-Pfaffian hypothesis, $\alpha \in \ker(\rho)$ by Proposition 4.3.6, hence

$$d_p\mu(v) = d_p(a_p(\alpha)) = d_p\mu(d_{1_{\mu(p)}} m_P(\cdot, p))(\alpha) = d_{1_{\mu(p)}} t(\alpha) = \rho_{\mu(p)}(\alpha) = 0.$$

Conversely, consider $\alpha \in \mathfrak{g}_x = \ker(d_{1_x} s) \cap \ker(\omega_{1_x})$; using backwards the arguments above one has, for $p \in \mu^{-1}(x)$,

$$a_p(\alpha) \in \ker(d_p\pi) \cap \ker(\theta_p).$$

Then $a_p(\alpha) \in \ker(d_p\mu)$, hence

$$\rho_x(\alpha) = d_{1_x} t(\alpha) = d_p\mu(d_{1_x} m_P(\cdot, p)(\alpha)) = d_p\mu(a_p(\alpha)) = 0.$$

This means that $\mathfrak{g} \subseteq \ker(\rho)$, hence $(\mathcal{G}, \omega)$ is of Lie type by Proposition 4.3.6. Q.E.D.

Here is another similarity between Lie-Pfaffian groupoids and principal bundles. Recall that a Lie-Pfaffian groupoid over X has a representation on TX (see Remark 4.3.5), which can be interpreted as an isomorphism $s^*TX \to t^*TX$ of vector bundles over $\mathcal{G}$. For a principal Lie-Pfaffian bundle we can prove the following generalisation, which will be useful later.

Lemma 5.3.18. *Let $(\mathcal{G}, \omega) \rightrightarrows X$ be a full Pfaffian groupoid, $\pi : (P, \theta) \to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M (Definition 5.3.16) with moment map $\mu : P \to X$, and assume moreover* $\dim(M) = \dim(X)$.

Then there is a canonical isomorphism of vector bundles over P

$$\Phi : \pi^*TM \to \mu^*TX.$$

Proof. Since $\mathcal{G}$ is full, then $\pi : (P, \theta) \to M$ becomes a Pfaffian fibration (Theorem 5.3.15): in particular, $d\pi_{|\ker(\theta)} : \ker(\theta) \to TM$ is pointwise surjective.

Accordingly, we define Φ fibrewise for all $p \in P$:

$$\Phi_p : T_yM \to T_xX, \quad u \mapsto \Phi_p(u) = d_p\mu(\bar{u}), \qquad y = \pi(p), x = \mu(p),$$

where $\bar{u}$ is an element of $\ker(\theta_p)$ such that $d_p\pi(\bar{u}) = u$.

First of all, consider $\bar{u}$ and $\bar{\bar{u}}$ in $\ker(\theta_p)$, both projecting on u through $d_p\pi$. This implies that $d_p\pi(\bar{\bar{u}} - \bar{u}) = u - u = 0$, hence, since the principal bundle is Lie-Pfaffian,

$$\bar{\bar{u}} - \bar{u} \in \ker(d_p\pi) \cap \ker(\theta_p) = \ker(d_p\mu) \cap \ker(\theta_p).$$

This means that $d_p\mu(\bar{\bar{u}}) = d_p\mu(\bar{u})$, i.e. $\Phi_p(u)$ is well defined.

Moreover, if $u \in \ker(\Phi_y)$, then $d_p\mu(\bar{u}) = 0$; with the same argument,

$$\bar{u} \in \ker(d_p\mu) \cap \ker(\theta_p) = \ker(d_p\pi) \cap \ker(\theta_p),$$

hence $u = d_p\pi(\bar{u}) = 0$.

We conclude that each Φ_p is an injective linear map between vector spaces of the same dimension, hence a linear isomorphism; since μ and π are smooth, $\Phi : \pi^*TM \to \mu^*TX$ is smooth as well, hence is an isomorphism of vector bundles.

Q.E.D.

As anticipated, when $X = M$ and $P = \mathcal{G}$ is a full Lie-Pfaffian groupoid interpreted as a principal Lie-Pfaffian $\mathcal{G}$-bundle (see also Example 5.4.7 later), we recover as particular case the action of $\mathcal{G}$ on TM described in Remark 4.3.5.

5.4 Pfaffian Morita equivalence

In this section we use our theory of principal Pfaffian bundles to adapt the notion of Morita equivalence to Pfaffian groupoids.

Definition 5.4.1. Let $(\mathcal{G}_1, \omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$ be two Pfaffian groupoids, with ω_i taking values in the representations E_i of $\mathcal{G}_i$. A **Pfaffian Morita equivalence** between them is given by

- A Morita equivalence (P, μ_1, μ_2) between the Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$
- A $(\mathcal{G}_1\text{-}\mathcal{G}_2)$-vector bundle $E \to P$, i.e. a vector bundle on which both groupoids act, and such that the two actions are compatible
- A pointwise surjective differential form $\theta \in \Omega^1(P, E)$
- The choice of two isomorphisms $E \cong \mu_i^* E_i$ such that both $(\mathcal{G}_i, \omega_i)$-actions on (P, θ) are Pfaffian (using the two isomorphisms to interpret θ as a form with the suitable coefficients in both cases).

This can be represented in a diagram

$$\begin{tikzcd}
(\mathcal{G}_1,\omega_1) \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & (P,\theta) \arrow[dll, "\mu_1"'] \arrow[drr, "\mu_2"] & \curvearrowleft & (\mathcal{G}_2,\omega_2) \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & & & M_2
\end{tikzcd}$$

Note that both $\mu_i^* E_i$ are $(\mathcal{G}_1\text{-}\mathcal{G}_2)$-vector bundles over P: one action is induced by the $\mathcal{G}_i$-action on E, the other is trivial. Accordingly, when we say "isomorphisms $E \cong \mu_i^* E_i$" we mean actual isomorphisms as $(\mathcal{G}_1\text{-}\mathcal{G}_2)$-vector bundles, not just as vector bundles. This is the reason that we ask E to have *a priori* a $(\mathcal{G}_1\text{-}\mathcal{G}_2)$-vector bundle structure. ♦

Proposition 5.4.2. *The composition of Pfaffian Morita equivalences is a Pfaffian Morita equivalence.*

Proof. Consider a Pfaffian Morita equivalence (P, θ_1) between two Pfaffian groupoids $(\mathcal{G}_1, \omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$, and another Pfaffian Morita equivalence (Q, θ_2) between $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$ and a third Pfaffian groupoid $(\mathcal{G}_3, \omega_3) \rightrightarrows M_3$. We know from Section 3.3 that the associated Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_3$ is given by the fibred product

$$R := (P \times_{M_2} Q)/\mathcal{G}_2.$$

We have to check that R is a Pfaffian Morita equivalence too, by defining a suitable differential form $\bar{\theta}$ on R. With an argument similar to that of Proposition 5.3.7, we start by defining

$$\theta \in \Omega^1(P \times_{M_2} Q, \mathrm{pr}^* E_2), \qquad \theta_{(p,q)}(v, w) := (\theta_1)_p(v) + (\theta_2)_q(w).$$

Note that $P \times_{M_2} Q$ can be viewed as a principal left $\mathcal{G}_2$-bundle over R, with action

$$g \cdot (p, q) := (p \cdot g^{-1}, g \cdot q).$$

Accordingly, we prove now that θ is basic by checking the equivalent condition from Lemma 3.2.15:

$$\begin{aligned}
(m^*\theta)_{(g,p,q)}(\alpha, v, w) &= \theta_{(p\cdot g^{-1}, g\cdot q)}(dm_P(v, di(\alpha)), dm_Q(\alpha, w)) = \\
&= (\theta_1)_{p\cdot g^{-1}}(dm_P(v, di(\alpha))) + (\theta_2)_{(g,q)}(dm_Q(\alpha, w)) = \\
&= g \cdot (pr_1^*\theta_1)_{(p,g^{-1})}(v, di(\alpha)) + g \cdot (pr_2^*\omega)_{(p,g^{-1})}(v, di(\alpha)) + \\
&\quad + g \cdot (pr_2^*\theta_2)_{(g,q)}(\alpha, w) + (pr_1^*\omega)_{(g,q)}(\alpha, w) = \\
&= g \cdot (\theta_1)_p(v) + g \cdot \omega_{g^{-1}}(di(\alpha)) + g \cdot (\theta_2)_q(w) + \omega_g(\alpha) = \\
&= g \cdot \theta_{(p,q)}(v, w) - \omega_g(\alpha) + \omega_g(\alpha) = g \cdot \theta_{(p,q)}(v, w),
\end{aligned}$$

where in the third line we used the multiplicativity of the $\mathcal{G}_2$-actions on P and Q, and in the last line we applied Lemma 3.4.10. From Proposition 3.2.16, θ descends to the following form on the base R:

$$\bar{\theta} \in \Omega^1(R, (P \times_{M_2} Q)[E_2]), \quad \bar{\theta}_{[p,q]}([v, w]) := (\theta_1)_p(v) + (\theta_2)_q(w).$$

With similar computations, one checks that both the induced $\mathcal{G}_1$- and $\mathcal{G}_3$-actions on R are Pfaffian with respect to $\bar{\theta}$. Q.E.D.

Recall from Example 3.3.6 that any principal bundle whose moment map is a surjective submersion can always be "completed" to a Morita equivalence with its gauge groupoid. Analogously, we show that any principal *Pfaffian* bundle can be completed to a *Pfaffian* Morita equivalence with its gauge groupoid.

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Proposition 5.4.3. *In the setting of Proposition 5.3.7, assume that (P, θ) be a principal Pfaffian $(\mathcal{G}, \omega)$-bundle. Then (P, θ) defines a Pfaffian Morita equivalence with* $(\mathrm{Gauge}(P), \hat{\omega})$.

Proof. Since the moment map μ is a submersion, we know from Example 3.3.6 that P defines a Morita equivalence between $\mathcal{G}$ and $\mathrm{Gauge}(P)$, and from Proposition 5.3.7 that there is a multiplicative form $\hat{\omega}$ on $\mathrm{Gauge}(P)$ such that the action of $(\mathrm{Gauge}(P), \hat{\omega})$ on (P, θ) is multiplicative. Accordingly, we have only to show that $\hat{\omega}$ is $\hat{s}$-involutive and that the $(\mathrm{Gauge}(P), \hat{\omega})$-action on (P, θ) is Pfaffian.

Observe that the first part of Lemma 5.3.14 depends only on the multiplicativity of the action, so it can be applied to the multiplicative $(\mathcal{G}, \hat{\omega})$-action on $(P, \theta) \to M$. Denoting by $\mathfrak{g} = \ker(d\hat{s}) \cap \ker(\hat{\omega})$ the symbol space of $(\mathrm{Gauge}(P), \hat{\omega})$, we get

$$\pi^* \mathfrak{g} \cong \mathfrak{g}_\mu(\theta).$$

Since the $(\mathcal{G}, \omega)$-action on (P, θ) is Pfaffian, the right-hand side is an involutive distribution, hence so is $\mathfrak{g}$, i.e. $\hat{\omega}$ is $\hat{s}$-involutive.

On the other hand, applying the same Lemma 5.3.14 to (P, θ), this time viewed as a principal Pfaffian $(\mathcal{G}, \omega)$-bundle over X, one gets that $\mathfrak{g}_\pi(\theta)$ is an involutive distribution, hence the $(\mathrm{Gauge}(P), \hat{\omega})$-action on (P, θ) is Pfaffian. Q.E.D.

The proof above shows the crucial importance of the second condition in Definition 5.3.8: if $\ker(d\mu) \cap \ker(\theta)$ was not an involutive distribution, $\hat{\omega}$ might fail to be $\hat{s}$-involutive.

However, the other properties that a Pfaffian groupoid might satisfy (being full and/or of Lie type) are not necessarily preserved by a Pfaffian Morita equivalence (see Section 5.4.1 for counterexamples). In order to find the precise conditions for this to happen, we can use Theorems 5.3.15 and 5.3.17, since any Pfaffian Morita equivalence (P, θ) is a principal Pfaffian bundle with respect to both actions.

Theorem 5.4.4. *Let (P, θ) be a Pfaffian Morita equivalence between two Pfaffian groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$. Using the same notations as in Definition 5.4.1:*

- *The distributions $\mathfrak{g}_{\mu_i}(\theta) = \ker(\theta) \cap \ker(d\mu_i) \subseteq TP$ are involutive.*
- *$\mathcal{G}_1$ is a full Pfaffian groupoid if and only if $\mu_2 : (P, \theta) \to M_2$ is a Pfaffian fibration; similarly for $\mathcal{G}_2$.*
- *$\mathcal{G}_1$ is a Lie-Pfaffian groupoid if and only if $\ker(d\mu_2) \cap \ker(\theta) \subseteq \ker(d\mu_1) \cap \ker(\theta)$; similarly for $\mathcal{G}_2$.*

It follows immediately, using Lemma 5.3.14, that

Corollary 5.4.5. *Let (P, θ) be a Pfaffian Morita equivalence between $\mathcal{G}_1$ and $\mathcal{G}_2$. Then (P, θ) is Lie-Pfaffian, i.e.*

$$\ker(\theta) \cap \ker(d\mu_1) = \ker(\theta) \cap \ker(d\mu_2),$$

if and only if both Pfaffian groupoids $(\mathcal{G}_1, \omega_1)$ *and* $(\mathcal{G}_2, \omega_2)$ *are of Lie type.*

Moreover, the symbol space $\mathfrak{g}(\theta)$ *of* (P, θ) *(Definition 5.3.16) is isomorphic to the pullbacks* $\mu_i^* \mathfrak{g}_i$ *of the symbol spaces* $\mathfrak{g}_i$ *(Definition 4.3.1) of* $(\mathcal{G}_i, \omega_i)$*.*

In the "nicest case" when both Pfaffian groupoids are full and of Lie type, a Pfaffian Morita equivalence preserves further properties.

Theorem 5.4.6. *Let* $(\mathcal{G}_1, \omega_1) \rightrightarrows M_1$ *and* $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$ *be full Lie-Pfaffian groupoids, with* ω_i *taking values in* E_i*, and assume they are Pfaffian Morita equivalent.*

Then their dimensions, the dimensions of their bases, and the ranks of the representations E_i *coincide. Furthermore, the fibres of the Spencer cohomologies associated to* $(\mathcal{G}_1, \omega_1)$ *and* $(\mathcal{G}_2, \omega_2)$ *(Definition 4.4.3) are isomorphic.*

Proof. The surjectivity of $\omega_i \in \Omega^1(\mathcal{G}_i, t_i^* E_i)$ yields, for $x \in M_i$,

$$\dim(T_{1_x}\mathcal{G}_i) = \dim(\ker((\omega_i)_{1_x})) + \dim(\mathrm{Im}((\omega_i)_{1_x})) = \dim(\ker((\omega_i)_{1_x})) + \mathrm{rank}(E_i).$$

Therefore, the symbol space $\mathfrak{g}_i$ of $\mathcal{G}_i$ has rank

$$\begin{aligned}\mathrm{rank}(\mathfrak{g}_i) &= \dim(\ker((\omega_i)_{1_x})) + \dim(\ker(d_{1_x} s_i)) - \dim(\ker((\omega_i)_{1_x}) + \ker(d_{1_x} s_i)) = \\ &= \cancel{\dim(T_{1_x}\mathcal{G})} - \mathrm{rank}(E_i) + \dim(\ker(d_{1_x} s_i)) - \cancel{\dim(T_{1_x}\mathcal{G}_i)} = \dim(\ker(d_{1_x} s_i)) - \mathrm{rank}(E_i).\end{aligned}$$

Let P be a Pfaffian Morita equivalence, as in Definition 5.4.1; then the ranks of the representations E_1 and E_2 coincide because $\mu_1^* E_1$ and $\mu_2^* E_2$ are both isomorphic to E. Moreover, Corollary 5.4.5 gives an isomorphism $\mu_1^* \mathfrak{g}_1 \cong \mu_2^* \mathfrak{g}_2$ between vector bundles on P; then the fibres of the symbol spaces $\mathfrak{g}_i$ are isomorphic, hence the s-fibres of $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same dimensions. As remarked in the proof of Theorem 5.3.15, it follows that

$$\dim(\ker(d_p \mu_2)) = \dim(\ker(d_{1_{\pi(p)}} s_1)) = \dim(\ker(d_{1_{\mu(p)}} s_2)) = \dim(\ker(d_p \mu_1)).$$

This means that the fibres of P, with respect to either principal bundle structure, have the same dimensions, hence also the base manifolds has the same dimension; it follows also that $\dim(\mathcal{G}_1) = \dim(\mathcal{G}_2)$.

Last, using the isomorphism $\Phi : \mu_1^* TM_1 \to \mu_2^* TM_2$ from Lemma 5.3.18 and the fact that $\mu^* E_1 \cong \mu^* E_2$, we have an isomorphism between $\mu_1^* \operatorname{Hom}(TM_1, E_1)$ and $\mu_2^* \operatorname{Hom}(TM_2, E_2)$; then the following diagrams commute by unraveling Definition 4.4.3 of the symbol maps j_1 and j_2:

$$\begin{tikzcd}
\mu_1^* \mathfrak{g}_1 \arrow[r, "j_1"] \arrow[d, "\cong"'] & \mu_1^* \operatorname{Hom}(TM_1, E_1) \arrow[d, "\cong"] \\
\mu_2^* \mathfrak{g}_2 \arrow[r, "j_2"'] & \mu_2^* \operatorname{Hom}(TM_2, E_2)
\end{tikzcd}$$

This means that the fibres of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic not only as vector spaces, but also as tableaux; then the fibres of their Spencer cohomology

$$(H^{\bullet,\bullet}(\mathfrak{g}_1))_x = H^{\bullet,\bullet}((\mathfrak{g}_1)_x), \quad (H^{\bullet,\bullet}(\mathfrak{g}_2))_y = H^{\bullet,\bullet}((\mathfrak{g}_2)_y), \quad x \in M_1, y \in M_2$$

are isomorphic as well. Q.E.D.

5.4.1 Examples of principal Pfaffian bundles and Pfaffian Morita equivalences

Example 5.4.7. Consider a full Pfaffian groupoid $(\mathcal{G}, \omega)$; using Example 3.1.13, $P = \mathcal{G}$ is a Morita equivalence between two copies of itself. Since both $\mathcal{G}$-actions are Pfaffian (Example 5.3.10) and ω is surjective, (P, ω) is a Pfaffian Morita equivalence.

Note that $\mathcal{G}$ is Lie-Pfaffian as a Pfaffian groupoid (Definition 4.3.4) if and only if it is Lie-Pfaffian as a principal Pfaffian bundle (Definition 5.3.16). Moreover, even if $\mathcal{G}$ is not Lie-Pfaffian, the conclusion of Theorem 5.4.6 holds trivially, showing that its hypotheses were only sufficient conditions. $\diamondsuit$

Example 5.4.8. A transitive full Pfaffian groupoid $(\mathcal{G}, \omega)$ is Pfaffian Morita equivalent to its isotropy group $\mathcal{G}_x$, seen as a Lie-Pfaffian group (Example 4.3.9). This follows immediately from Examples 3.3.5 and 5.3.10, by taking $P = s^{-1}(x)$ and $\theta = \omega_{|P}$, which is surjective since ω is s-transversal.

Moreover, this example confirms that being Lie-Pfaffian or being full is not necessarily Pfaffian Morita invariant. For instance, we know that $\mathcal{G}_x$ is Lie-Pfaffian, and, equivalently,

$$\mathfrak{g}_{s_{|P}}(\theta) = \ker(ds) \cap \ker(\omega) \supseteq \ker(dt) \cap \ker(ds) \cap \ker(\omega) = \mathfrak{g}_{t_{|P}}(\theta),$$

but the converse inclusion is not necessarily true, i.e. $(\mathcal{G}, \omega)$ may not be Lie-Pfaffian. Even more, if $\mathfrak{g}_{s_{|P}}(\theta) = \mathfrak{g}_{t_{|P}}(\theta)$, then $\mathcal{G}$ is Lie-Pfaffian (and full) but this prevents $\mathcal{G}_x$ to be full. The reason is that $\ker(ds) \cap \ker(\omega) \subseteq \ker(dt)$ implies that $d_g t_{|\ker(\theta)}$ is never surjective, i.e. $t : (P, \theta) \to M$ cannot be a Pfaffian fibration. $\diamondsuit$

Example 5.4.9. Given a Lie pseudogroup Γ on X, any almost Γ-structure $P \subseteq J^k(X, M)$ of order k (Definition 5.2.3) is a principal Lie-Pfaffian $J^k\Gamma$-bundle over M together with the restriction θ of the Cartan form, with $J^k\Gamma$ the Pfaffian groupoid from Example 4.3.11.

Indeed, $(P, \theta) \to M$ is a principal Pfaffian bundle since the (principal) action of $(J^k\Gamma, \omega)$ on (P, θ) is Pfaffian (Proposition 5.3.11). Let us prove that the Lie-type condition holds, using the notations of Definition 5.3.16.

Recall first that μ is induced from the projection $\pi^k : J^k(X, M) \to X$; then, for any $v \in \ker(\theta) \cap \ker(d\mu)$,

$$\theta_{j^k_x \phi}(v) = d\pi^k_{k-1}(v) - d(j^{k-1}\phi) \circ \cancel{d\pi^k(v)} = 0,$$

so that $d\pi^k_{k-1}(v) = 0$. Recalling also that π is induced from the projection $\pi^k_0 : J^k(X, M) \to J^0(X, M) = X \times M$, we have

$$d\pi(v) = d\,\mathrm{pr}_2 \circ d\pi^k_0(v) = d\,\mathrm{pr}_2 \circ d\pi^{k-1}_0 \circ \cancel{d\pi^k_{k-1}(v)} = 0,$$

which proves that $v \in \ker(\theta) \cap \ker(d\pi)$. Since the Pfaffian groupoid $J^k\Gamma$ is of Lie type, the converse inclusion $\ker(\theta) \cap \ker(d\pi) \subseteq \ker(\theta) \cap \ker(d\mu)$ holds as well by applying Theorem 5.3.17. $\diamondsuit$

Example 5.4.10. Let G be a Lie group and $\pi : (P, \theta) \to M$ be an abstract G-structure (Definition 2.3.3), with tautological form $\theta \in \Omega^1(P, \mathbb{R}^n)$. If $\omega \in \Omega^1(G, V)$ is a multiplicative form (with coefficients in any representation V), then (P, θ) is a principal Lie-Pfaffian (G, ω)-bundle if and only if $\omega = 0$. Moreover, in such a situation, $(\mathrm{Gauge}(P), \hat{\omega})$ is a full Lie-Pfaffian groupoid.

Let us show the first claim. The form θ satisfies by definition

$$\ker(\theta) = \ker(d\pi) = \mathrm{Im}(a),$$

so $\ker(\theta)$ is an involutive distribution. In order for the action to be Pfaffian, according to Example 5.3.9, the infinitesimal Lie algebra action $a : \mathfrak{g} \to \mathfrak{X}(P)$ should satisfy the condition

$$\theta(a(v)) = l(v) \quad \forall v \in \mathfrak{g}.$$

However, the left-hand side is always zero, therefore we must ask $l = 0$, for $l : \mathfrak{g} \to V$ the G-equivariant map defining ω (see Example 3.4.6), i.e. $\omega = 0$. It is then immediate to see that $\ker(\pi) \cap \ker(\theta) = \ker(\theta)$, hence the principal Pfaffian bundle $\pi : (P, \theta) \to M$ is Lie-Pfaffian (Definition 5.3.16).

For the second claim, note that θ is surjective by definition, hence Proposition 5.4.3 implies that (P, θ) defines a Pfaffian Morita equivalence between $(G, \omega = 0)$ and $(\mathrm{Gauge}(P), \hat{\omega})$. In particular, by Corollary 5.4.5, both Pfaffian groupoids are of Lie type: we already knew it for G, but not yet for its gauge groupoid. Similarly, applying Theorem 5.3.15, we see that $(\mathrm{Gauge}(P), \hat{\omega})$ is full, since $(P, \theta) \to \{*\}$ is trivially a Pfaffian fibration.

Note also that $(P, \theta) \xrightarrow{\pi} M$ is not a Pfaffian fibration since $(G, 0)$ is not full. Actually, π is a Pfaffian fibration if and only if the distribution

$$\ker(d\pi) + \ker(\theta) = \ker(d\pi) = \ker(\theta)$$

coincides with the entire TP; this happens only when $d\pi(v) = 0$ for every $v \in TP$, i.e. in the trivial case $M = \{*\}$. $\diamondsuit$

Example 5.4.11. The previous example becomes even more transparent in the particular case when $G \subseteq GL(n, \mathbb{R})$ is a Lie subgroup and $P_{\mathrm{can}} = \mathbb{R}^n \times G$ is the canonical G-structure on $\mathbb{R}^n$: then $(P_{\mathrm{can}}, \theta)$ becomes a Morita equivalence between the non-full Lie-Pfaffian groupoid $(G, 0)$ and the full Pfaffian groupoid $(J^1\Gamma_G, \omega)$ (Example 4.3.11). Indeed, the groupoid $\mathrm{Gauge}(P)$ becomes $J^1\Gamma_G \rightrightarrows \mathbb{R}^n$ (using the isomorphism of Example 4.1.15), the tautological form $\theta \in \Omega^1(P, \mathbb{R}^n)$ can be written explicitly as $\theta_{(x,g)}(v, \alpha) = v$, and the multiplicative form $\hat{\omega}$ coincides with the Cartan form ω. $\diamondsuit$

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Example 5.4.12. Example 5.4.11 shows the necessity of the fullness hypothesis in Theorem 5.4.6 for having isomorphic Spencer cohomologies.

Indeed, the symbol space of $(G, 0)$ is its Lie algebra $\mathfrak{g}$ (Example 4.3.8) with the trivial tableau structure

$$\mathfrak{g} \to \mathrm{Hom}(T\{*\}, V) = 0, \quad v \mapsto 0.$$

On the other hand, $(J^1\Gamma_G, \omega)$ has symbol space $\mathbb{R}^n \times \mathfrak{g} \to \mathbb{R}^n$ (Example 4.3.11), with a non-trivial tableau structure

$$\mathbb{R}^n \times \mathfrak{g} \to \mathrm{Hom}(T\mathbb{R}^n, T\mathbb{R}^n).$$

In conclusion, the fibres of the symbol spaces are isomorphic as vector spaces but not as tableaux, hence their Spencer cohomologies are different. ◊

Example 5.4.13. Examples 5.4.11 and 5.4.10 may be combined as follows: given an abstract G-structure (P, θ) on M, there is a Pfaffian Morita equivalence between the Pfaffian groupoids $J^1\Gamma_G \rightrightarrows \mathbb{R}^n$ and $\mathrm{Gauge}(P) \rightrightarrows M$ given by the bibundle

$$(P_{\mathrm{can}} \times P)/G \cong \mathbb{R}^n \times P.$$

One sees immediately that the dimensions of the spaces involved coincide: this is in accord with Theorem 5.4.6, since both groupoids are full and of Lie type. ◊

Example 5.4.14. Let $(\mathcal{G}, \omega)$ be a full Pfaffian groupoid over M, $\mu : P \to M$ a surjective submersion and consider the corresponding pullback Pfaffian groupoid $(\mu^*\mathcal{G}, \tau^*\omega)$ over P (Example 4.3.12). Then the Morita equivalence between $\mathcal{G}$ and $\mu^*\mathcal{G}$ from Example 3.3.4 is actually a Pfaffian Morita equivalence.

To see this, it is enough to endow the bibundle

$$P_\mu^{triv} = P \ {}_\mu\times_t \mathcal{G}$$

with the pointwise surjective differential form $\theta = (pr_2)^*\omega$: one checks easily that the actions of $\mathcal{G}$ and $\mu^*\mathcal{G}$ on (P_μ^{triv}, θ) are multiplicative. ◊

5.4.2 Pfaffian Morita morphisms

Definition 5.4.15. Let $(\mathcal{G}_1, \omega_1) \rightrightarrows M_1$ and $(\mathcal{G}_2, \omega_2) \rightrightarrows M_2$ two Pfaffian groupoids, with ω_i taking values in the representations E_i of $\mathcal{G}_i$. A **Pfaffian Morita morphism** between them is given by

- A Morita morphism (P, μ_1, μ_2) from the Lie groupoid $\mathcal{G}_1$ to the Lie groupoid $\mathcal{G}_2$ (Definition 3.3.8)
- A $(\mathcal{G}_1$-$\mathcal{G}_2)$-vector bundle $E \to P$, i.e. a vector bundle on which both groupoids act, and such that the two actions are compatible
- A pointwise surjective differential form $\theta \in \Omega^1(P, E)$
- A surjective map $\phi : \mu_1^* E_1 \to E$ and an isomorphism $E \cong \mu_2^* E_2$ of $(\mathcal{G}_1, \mathcal{G}_2)$-vector bundles such that both $(\mathcal{G}_i, \omega_i)$-actions on (P, θ) are Pfaffian (using the two maps to see θ as a form with the suitable coefficients in both cases).

In other words, a Pfaffian Morita morphism is the same thing as a Pfaffian Morita equivalence (Definition 5.4.1), without the condition that μ_2 is a principal bundle and that ϕ is injective. This can be represented in a diagram

$$\begin{tikzcd}
(\mathcal{G}_1,\omega_1) \arrow[r, phantom, "\curvearrowright"] \arrow[d, shift left] \arrow[d, shift right] & (P,\theta) \arrow[dl, "\mu_1"] \arrow[dr, "\mu_2"'] & (\mathcal{G}_2,\omega_2) \arrow[l, phantom, "\curvearrowleft"] \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2
\end{tikzcd}$$

but we will often use the shorter notation $(P,\theta) : (\mathcal{G}_1,\omega_1) \rightsquigarrow (\mathcal{G}_2,\omega_2)$.

A **Lie-Pfaffian Morita morphism** is a Pfaffian Morita morphism where the principal bundle is Lie-Pfaffian, i.e. such that

$$\ker(\theta) \cap \ker(d\mu_1) = \ker(\theta) \cap \ker(d\mu_2). \qquad \blacklozenge$$

Example 5.4.16. Any principal Pfaffian $(\mathcal{G},\omega)$-bundle (P,θ) over M can be viewed as a Pfaffian Morita morphism from M to $\mathcal{G}$, endowing the unit groupoid $M \rightrightarrows M$ with the zero form. Moreover, (P,θ) is Lie-Pfaffian as a Pfaffian Morita morphism if and only if it is Lie-Pfaffian as a principal Pfaffian bundle (Definition 5.3.16). $\lozenge$

Example 5.4.17. Let $(\mathcal{G},\omega)$ be a full (Lie-)Pfaffian groupoid integrable up to order 1 (Definition 4.4.9). Then there is a (Lie-)Pfaffian Morita morphism from the prolongation $(\mathcal{G}^{(1)},\omega^{(1)})$ to $(\mathcal{G},\omega)$

$$\begin{tikzcd}
(\mathcal{G}^{(1)},\omega^{(1)}) \arrow[r, phantom, "\curvearrowright"] \arrow[d, shift left] \arrow[d, shift right] & (\mathcal{G},\omega) \arrow[dl, "t"] \arrow[dr, "s"'] & (\mathcal{G},\omega) \arrow[l, phantom, "\curvearrowleft"] \arrow[d, shift left] \arrow[d, shift right] \\
M & & M
\end{tikzcd}$$

with the obvious $\mathcal{G}^{(1)}$-action on induced by the right action of $\mathcal{G}$ on itself. Denoting by E_1 the representation of $\mathcal{G}$ where ω takes values, we set $E := t^*E_1$ (which is trivially isomorphic to s^*E_1). Last, since $\omega^{(1)}$ takes values in the adjoint representation $E_2 := A$ of $\mathcal{G}^{(1)}$ (Example 3.4.5), the surjective map

$$\phi : t^*A \to t^*E$$

is induced from the surjective map $l : A \to E$ from Proposition 4.3.6. $\lozenge$

As for Pfaffian Morita equivalences, one shows that

Proposition 5.4.18. *The composition of two (Lie-)Pfaffian Morita morphisms is a (Lie-)Pfaffian Morita morphism.*

Proof. The proof is essentially the same of Proposition 5.4.2; the only new part is to show that the Lie-Pfaffian property is preserved, i.e. that, given the Lie-Pfaffian Morita morphisms (P_1,θ_1) and (P_2,θ_2),

$$\begin{tikzcd}
(\mathcal{G}_1,\omega_1) \arrow[r, phantom, "\curvearrowright"] \arrow[d, shift left] \arrow[d, shift right] & (P_1,\theta_1) \arrow[dl, "\pi_1"] \arrow[dr, "\tau_1"'] & (\mathcal{G}_2,\omega_2) \arrow[l, phantom, "\curvearrowleft"] \arrow[r, phantom, "\curvearrowright"] \arrow[d, shift left] \arrow[d, shift right] & (P_2,\theta_2) \arrow[dl, "\pi_2"] \arrow[dr, "\tau_2"'] & (\mathcal{G}_3,\omega_3) \arrow[l, phantom, "\curvearrowleft"] \arrow[d, shift left] \arrow[d, shift right] \\
M_1 & & M_2 & & M_3
\end{tikzcd}$$

the following Pfaffian Morita morphism is Lie-Pfaffian

$$\begin{tikzcd}
(\mathcal{G}_1,\omega_1) \arrow[r, bend left] \arrow[d, shift left] \arrow[d, shift right] & (R,\theta) \arrow[dl, "\mu_1"] \arrow[dr, "\mu_2"'] & (\mathcal{G}_3,\omega_3) \arrow[l, bend right] \arrow[d, shift left] \arrow[d, shift right] \\
M & & M_2
\end{tikzcd}$$

$$R := (P_1 \ {}_{\tau_1}\!\times_{\pi_2} P_2)/\mathcal{G}_2, \qquad \mu_1[p_1,p_2] := \pi_1(p_1), \qquad \mu_2[p_1,p_2] := \tau_2(p_2),$$
$$[p_1,p_2]\cdot g_3 := [p_1,p_2\cdot g_3], \qquad \theta_{[p_1,p_2]}([v,w]) := (\theta_1)_{p_1}(v) + (\theta_2)_{p_2}(w).$$

First, since P_2 is Lie-Pfaffian, $\mathcal{G}_3$ is a Lie-Pfaffian groupoid by Theorem 5.3.17. Therefore, applying again Theorem 5.3.17 but to R, it is enough to prove that

$$\ker(\theta)\cap\ker(d\mu_1)\subseteq\ker(\theta)\cap\ker(d\mu_2).$$

Take $[v,w]\in\ker(\theta)\cap\ker(d\mu_1)$; by Lemma 5.3.14, $[v,w]=a_R(\alpha)$ for some $\alpha\in\ker(\omega_3)\cap\ker(ds_2)$, where a_R is the infinitesimal action of $\mathcal{G}_3$ on R. Since such an action is defined using the $\mathcal{G}_3$-action on P_2, one sees easily that

$$w = a_{P_2}(\alpha),$$

where a_{P_2} is the infinitesimal action of $\mathcal{G}_3$ on P_2. It follows by Proposition 5.3.4 that

$$\theta_2(w)=\theta_2(a_{P_2}(\alpha))=\omega_3(\alpha)=0.$$

Now we use the hypothesis that $[v,w]\in\ker(\theta)$: by definition of θ,

$$0=\theta([v,w])=\theta_1(v)+\cancel{\theta_2(w)}=\theta_1(v),$$

so that $v\in\ker(\theta_1)$. Using the other hypothesis $[v,w]\in\ker(d\mu_1)$,

$$0=d\mu_1([v,w])=d\pi_1(v).$$

In conclusion, $v\in\ker(\theta_1)\cap\ker(d\pi_1)$; since (P_1,θ_1) is Lie-Pfaffian, $v\in\ker(\theta_1)\cap\ker(d\tau_1)$, hence

$$0=d\tau_1(v)=d\pi_2(w),$$

where we used that fact that $(v,w)\in TP_1 \ {}_{d\tau_1}\!\times_{d\pi_2} TP_2$. This means that $w\in\ker(d\pi_2)\cap\ker(\theta_2)=\ker(d\tau_2)\cap\ker(\theta_2)$ (since P_2 is Lie-Pfaffian), hence

$$d\mu_2[v,w]=d\tau_2(w)=0,$$

which conclude our proof. Q.E.D.

The same remarks of Section 3.3.1 apply here: any Pfaffian groupoid $(\mathcal{G},\omega)$ can be interpreted as a Pfaffian Morita identity morphism. This allows us to talk about Pfaffian Morita inverses and show the following fundamental result:

Proposition 5.4.19. *An invertible Pfaffian Morita morphism, i.e. a Pfaffian Morita isomorphism, is the same thing of a Pfaffian Morita equivalence.*

Moreover, a Pfaffian Morita equivalence between Lie-Pfaffian groupoids is an invertible Lie-Pfaffian morphism.

Given Proposition 3.3.9, one has only to check that the composition of any invertible Pfaffian Morita morphism with its inverse is the Pfaffian Morita identity morphism. The second claim follows immediately from Theorem 5.3.17.

Using Example 5.4.16, one has the following important application:

Corollary 5.4.20. *Let $(\mathcal{G}_i, \omega_i)$ be two Pfaffian groupoids over X_i which are Pfaffian Morita equivalent, and let M be a manifold. Then there is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{Principal Pfaffian} \\ (\mathcal{G}_1, \omega_1)\textit{-bundle on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Principal Pfaffian} \\ (\mathcal{G}_2, \omega_2)\textit{-bundle on } M \end{array} \right\}$$

$$(R_1, \theta_1) \quad \longmapsto \quad (R_2, \theta_2)$$

If both $\mathcal{G}_i$ are Lie-Pfaffian, then

$$(R_1, \theta_1) \textit{ is Lie-Pfaffian } \Leftrightarrow (R_2, \theta_2) \textit{ is Lie-Pfaffian.}$$

5.5 Abstract almost Γ-structures

As we did in Section 2.3 for G-structures, we note here that an almost Γ-structure P is a principal bundle with a given inclusion of P in $J^1(X, M)$. We will prove that such information can be encoded in the canonical 1-form on P induced by the Cartan form of $J^1(X, M)$.

Definition 5.5.1. Let Γ be a Lie pseudogroup on X and $P \subseteq J^1(X, M)$ an almost Γ-structure on M (Definition 5.2.3). The the **Cartan form of** P is the restriction θ of the Cartan form θ_{can} of $J^1(X, M)$ (Definition 1.1.4) to P. ♦

The Cartan form θ plays for almost Γ-structures the same role that the tautological form (Definition 2.3.1) played for G-structures. For instance, since any almost Γ-structure is a principal Lie-Pfaffian bundle (Example 5.4.9), we have an analogue of the immersion theorem 2.3.2.

Theorem 5.5.2. *Assume* $\dim(X) = \dim(M)$*, let Γ be a Lie pseudogroup on X and consider the associated Pfaffian groupoid $J^1\Gamma$ (Example 4.3.11), whose Pfaffian form $\omega \in \Omega^1(J^1\Gamma, s^*E)$ takes values in the adjoint representation of $J^1\Gamma$ (Example 3.4.5), i.e.*

$$E := \mathrm{Lie}(J^0\Gamma) \subseteq \mathrm{Lie}(X \times X) = TX.$$

Moreover, let $\pi : (P, \theta) \to M$ be a principal Lie-Pfaffian $J^1\Gamma$-bundle (Definition 5.3.16), with $\theta \in \Omega^1(P, \mu^*E)$. Then there exists a morphism of principal bundles

$$j : P \to J^1(X, M),$$

such that j is an immersion and

$$j^*\theta_{\mathrm{can}} = \bar{\Phi} \circ \theta,$$

where $\theta_{\mathrm{can}} \in \Omega^1(J^1(X,M), \mathrm{pr}^* TM)$ is the Cartan form of $J^1(X,M)$ and $\bar{\Phi}$ is minus the isomorphism from Lemma 5.3.18 between μ^*TX and π^*TM.

Proof. Let us define, for every $p \in P$,

$$j(p) := (p, \Phi_p^{-1} : T_{\mu(p)}X \xrightarrow{\cong} T_{\pi(p)}M) \in J^1(X,M),$$

where we interpret $J^1(X,M)$ as in Remark 1.1.2.

To show that j is a morphism of principal $J^1\Gamma$-bundles, i.e.

$$j(p \cdot g) = (p \cdot g, \Phi_{p\cdot g}^{-1}) = (p, \Phi_p^{-1}) \cdot g = j(p) \cdot g \quad \forall (p,g) \in P\ {}_{\mu}\!\times_t J^1\Gamma,$$

we need $\Phi_{p\cdot g}^{-1} = \Phi_p^{-1} \cdot g$. Indeed, for any $g = j^1_x \Phi^{-1} \in J^1\Gamma$ and $v \in \ker(\theta_{p\cdot g})$, one has

$$\begin{aligned}(\Phi_p^{-1} \cdot g)(d_{p\cdot g}\mu(v)) = \Phi_p^{-1}(d_x\Phi^{-1} \circ d_{p\cdot g}\mu(v)) = \Phi_p^{-1}(d_p\mu(\bar{v})) = \\ = d_p\pi(\bar{v}) = d_{p\cdot g}\pi(v) = \Phi_{p\cdot g}^{-1}(d_{p\cdot g}\mu(v)),\end{aligned}$$

where $\bar{v} = d_{p\cdot g} m_P(\cdot, g^{-1})(v) \in \ker(\theta_p)$. It follows by the standard theory of principal bundles that j is an immersion (see the proof of Theorem 2.3.2).

We prove now the relation between θ and the Cartan form of $J^1(X,M)$. By definition of θ_{can}, for every $v \in T_pP$

$$(j^*\theta_{\mathrm{can}})_p(v) = (\theta_{\mathrm{can}})_{j(p)}(dj(v)) = d\pi^1_0(dj(v)) - \Phi_p^{-1}(d\pi^1(dj(v))) = d\pi(v) - \Phi_p^{-1}(d\mu(v)).$$

Recall that, since $(J^1\Gamma, \omega)$ is full, then $(P,\theta) \to M$ is a Pfaffian fibration (Theorem 5.3.15). Accordingly, we can split v as

$$v = v_1 + v_2, \qquad \text{for } v_1 \in \ker(\theta_p) \text{ and } v_2 \in \ker(d_p\pi),$$

and show that $(j^*\theta_{\mathrm{can}})_p = -\Phi_p^{-1} \circ \theta_p$ on each term. For the first one, we use the definition of Φ^{-1} to get

$$(j^*\theta_{\mathrm{can}})_p(v_1) = d_p\pi(v_1) - \Phi_p^{-1}(d\mu(v_1)) = d_p\pi(v_1) - d_p\pi(v_1) = 0 = -\Phi^{-1}(\theta_p(v_1)).$$

For the second term, recall that $\ker(d_p\pi)$ is the image of the infinitesimal action $a_p : A_{\mu(p)} \to T_pP$ (Proposition 3.2.13); then $v_2 = a_p(\alpha)$ for some $\alpha \in A_{\mu(p)}$, where $A \subseteq TX \times TX$ is the Lie algebroid of $J^1\Gamma$, and

$$d_p\mu(v_2) = d_p\mu(a_p(\alpha)) = d_{1_{\mu(p)}}(\mu \circ m_P(p,\cdot) \circ i)(\alpha) = d_{1_{\mu(p)}} t(\alpha) = \rho_A(\alpha).$$

Since $(J^1\Gamma, \omega)$ is Lie-Pfaffian, the representation $E = TX$ of $J^1\Gamma$ is a Lie algebroid over X (Proposition 4.3.6), with anchor $\rho_E = id_{TX}$. Since the anchor of A coincides with $l : A \to E$, by Proposition 5.3.4,

$$\rho_A(\alpha) = l(\alpha) = \omega_{1_{\mu(p)}}(\alpha) = \theta_p(a_p(\alpha)) = \theta_p(v_2).$$

In conclusion,

$$(j^*\theta_{\mathrm{can}})_p(v_2) = \cancel{d\pi(v_2)} - \Phi_p^{-1}(d\mu(v_2)) = -\Phi_p^{-1}(\theta_p(v_2)) = \bar{\Phi}_p(\theta_p(v)),$$

so that $(j^*\theta_{\mathrm{can}})(v) = \bar{\Phi}(\theta(v))$ holds for every $v \in TP$. Q.E.D.

Under the hypotheses of Theorem 5.5.2, and if j is an embedding, then $j(P)$ is an almost Γ-structure (of order 1) on M. One can prove a similar result starting with a principal $J^k\Gamma$-bundle. This motivates the following definition, analogous to 2.3.3.

Definition 5.5.3. Let Γ be a Lie pseudogroup over X^n; an **abstract almost Γ-structure of order** k on M^n is a principal Lie-Pfaffian $J^k\Gamma$-bundle

$$\begin{tikzcd} & (P,\theta) \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & (J^k\Gamma,\omega) \arrow[l, bend right] \arrow[d, shift left] \arrow[d, shift right] \\ M & & X \end{tikzcd}$$

♦

Proposition 5.5.4. *Let Γ be a transitive Lie pseudogroup over $\mathbb{R}^n$ and denote by $G = J^k_x\Gamma$ its isotropy Lie group of order k at any point $x \in \mathbb{R}^n$. Then there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{abstract } k^{th}\textit{-order} \\ \textit{almost } \Gamma\textit{-structures on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{abstract } k^{th}\textit{-order} \\ G\textit{-structures on } M \end{array} \right\}.$$

Moreover, if Γ contains the translations, such a correspondence restricts to

$$\left\{ \begin{array}{c} \textit{integrable abstract } k^{th}\textit{-order} \\ \textit{almost } \Gamma\textit{-structures on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{integrable abstract } k^{th}\textit{-order} \\ G\textit{-structures on } M \end{array} \right\}.$$

Proof. The proof goes as in Proposition 5.2.4, which associates to a principal $J^k\Gamma$-bundle $\pi : P \to M$ (with moment map $\mu : P \to \mathbb{R}^n$) the fibre $P_x := \mu^{-1}(x)$, which is by construction a $(J^k\Gamma)_x$-bundle over M.

The extra piece of information is given by the Lie-Pfaffian structure of P, i.e. by the form $\theta \in \Omega^1(P, \mu^*E)$; we have to check that P_x is an abstract G-structure on M (Definition 2.3.3) together with the restriction

$$\theta_{|P_x} \in \Omega^1(P_x, T_x\mathbb{R}^n).$$

Indeed, $\theta_{|P_x}$ is surjective (since θ is so) and G-equivariant (it follows easily from the multiplicativity of the $J^1\Gamma$-action on P). Moreover, for $k = 1$, one obtains

$$\ker(\theta_{|P_x}) = \ker(d\pi_{|P_x})$$

by combinining Proposition 5.5.2 with Lemma 5.5.5.

For $k > 1$, one can generalise those results or prove directly the equivalent equality $\ker(\theta) \cap \ker(d\mu) = \ker(d\pi) \cap \ker(d\mu)$. The inclusion $\subseteq$ holds since (P, θ) is Lie-Pfaffian (Definition 5.3.16), whereas the inclusion $\supseteq$ follows from Propositions 3.2.13, 4.3.6 and 5.3.4, as in the proof of Proposition 5.5.2. Q.E.D.

Lemma 5.5.5. *Let Γ be a transitive pseudogroup over $\mathbb{R}^n$ with linear isotropy group $G \subseteq GL(n, \mathbb{R})$ (Definition 4.1.21), $P \subseteq J^1(\mathbb{R}^n, M)$ an almost Γ-structure on M^n and θ its Cartan form (Definition 5.5.1).*

Then, for any $x \in \mathbb{R}^n$, the fibre $P_x := \mu^{-1}(x)$ is a G-structure and the restriction $\theta_{|P_x}$ coincides with the tautological form $\theta_{tau} \in \Omega^1(P_x, \mathbb{R}^n)$, up to an isomorphism between the coefficients.

Proof. By Proposition 5.2.4, $P_x \subseteq J^1_x(\mathbb{R}^n, M)$ is a G-structure and the restriction of the Cartan form $\theta \in \Omega^1(J^1(\mathbb{R}^n, M), T^{\pi^0} J^0(\mathbb{R}^n, M))$ on the fibre $J^1_x(\mathbb{R}^n, M)$ is an element $\theta_{|P_x} \in \Omega^1(P_x, T^{\pi^0}_x J^0(\mathbb{R}^n, M))$. Since $\mu : P \to \mathbb{R}^n$ is the restriction of the projection $\pi^1 : J^1(\mathbb{R}^n, M) \to \mathbb{R}^n$, then for any $v \in P_x$,

$$\theta_{j^1_x \phi}(v) = d\pi^1_0(v) - d\phi \circ \cancel{d\pi^1(v)} = d\pi^1_0(v) = d_x\phi((\theta_{tau})_{j^1_x \phi}(v)). \qquad \text{Q.E.D.}$$

5.5.1 Symmetries

The Cartan form of an almost Γ-structure (Definition 5.5.1) can be used to characterise its symmetries (Definition 5.2.10), as we did in Proposition 2.3.4. As in Remark 2.3.5, this is the natural analogue of the Lie-Bäcklund theorem.

Proposition 5.5.6. *Let Γ be a Lie pseudogroup on X, $P \subseteq J^1(X, M)$ an almost Γ-structure on M and θ its Cartan form. For any local automorphism $(\phi, \varphi) \in \mathrm{Aut}_{\mathrm{loc}}(P)$ of the principal bundle $P \to M$, define the smooth function $\widetilde{g} : P \to \mathcal{G}$ by $\widetilde{g} = j^1_{\varphi(f(x))}\varphi$; then (ϕ, φ) satisfies $\phi^*\theta = \widetilde{g} \cdot \theta$ if and only if $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ is a local symmetry of P (Definition 5.2.10) and $\phi \in \mathrm{Diff}_{\mathrm{loc}}(P)$ is its lift to the 1-jets, i.e.*

$$\phi(j^1_x f) = j^1_x(\varphi \circ f).$$

Proof. Let $\phi \in \mathrm{Diff}_{\mathrm{loc}}(P)$ be the lift of a symmetry $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ of P, and $p = j^1_x f \in P$; then

$$(\phi^*\theta)_p(v) = \theta_{j^1_x(\varphi \circ f)}(d\phi(v)) =$$

$$= d\pi^1_0(d\phi(v)) - d(\varphi \circ f) d\pi^1(d\phi(v)) = d(\varphi \circ \pi^1_0)(v) - d\varphi \circ d(f \circ \pi^1)(v) =$$

$$= j^1_{\varphi(f(x))}\varphi \cdot d\pi^1_0(v) - j^1_{\varphi(f(x))}\varphi \cdot df(d\pi^1(v)) = \widetilde{g}(p) \cdot \theta_{j^1_x f}(v) = (\widetilde{g} \cdot \theta)_p(v).$$

The converse goes on the same lines of the proof of Proposition 2.3.4. Q.E.D.

This motivates the following definition:

Definition 5.5.7. The pseudogroup of local symmetries of an (abstract) almost Γ-structure $\pi : (P, \theta) \to M$ of order k is

$$\Gamma_\pi(P, \theta) := \{(\phi, \varphi) \in \mathrm{Aut}_{\mathrm{loc}}(P) \mid \phi^*\theta = \widetilde{g} \cdot \theta \text{ for } \widetilde{g} : P \to J^k\Gamma \text{ described above } \}.$$

♦

Since $J^k\Gamma$ is Lie-Pfaffian and holonomically effective (Example 5.6.3), Proposition 5.6.4 gives the isomorphism

$$\Phi : \mathrm{Bis}_{\mathrm{loc}}(J^k\Gamma, \omega) \to \Gamma_\pi(P, \theta).$$

Recall from Example 4.4.2 that the holonomic bisections of $(J^k\Gamma, \omega)$ are in 1-1 correspondence with the elements of Γ: then

Corollary 5.5.8. *The pseudogroup of local symmetries of an (abstract) almost Γ-structure of order k coincides with the pseudogroup Γ.*

If $\Gamma = \Gamma_G$, we recover the fact that the pseudogroup of symmetries of a G-structure is identified with Γ_G (see Example 4.1.5).

5.5.2 Morita invariance of almost Γ-structures

It is clear from Example 3.3.11 that an almost Γ-structure P can be interpreted as a Morita morphism between M and $J^k\Gamma$. Taking into account the Cartan form θ of P (Definition 5.5.1), one sees that (P, θ) becomes a Lie-Pfaffian Morita morphism (Definition 5.4.15) to the Pfaffian groupoid $J^k\Gamma$, where M is the unit groupoid $M \rightrightarrows M$ endowed with the zero form. This point of view becomes particularly useful when dealing with abstract structures: the following result follows from the definitions.

Proposition 5.5.9. *Let Γ be a Lie pseudogroup on X and M a manifold of the same dimension of X; for every k, there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Abstract order } k \textit{ almost} \\ \Gamma\textit{-structures on } M \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Lie-Pfaffian Morita morphisms from} \\ \textit{the unit groupoid } M \textit{ to } J^k\Gamma \end{array} \right\}.$$

5

With this correspondence one deduces right away that an abstract almost Γ-structure P of order k induces an abstract almost Γ-structure Q of order $h < k$; it is enough to compose (applying Proposition 5.4.18) P with the Pfaffian Morita morphism from $J^k\Gamma$ to $J^h\Gamma$ defined by the Lie groupoid morphism $J^k\Gamma \to J^h\Gamma$ (see Example 3.3.10).

Similarly, the Morita morphism from $J^k\Gamma_G$ to G of Example 4.1.16 is actually a Pfaffian Morita morphism, and yields the following

Corollary 5.5.10. *Let M^n be a manifold and G a Lie subgroup of the general linear group $GL^k(\mathbb{R}^n)$ (Definition 2.6.2); there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Abstract order } k \textit{ almost} \\ \Gamma_G\textit{-structures on } M \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Pfaffian Morita morphisms from} \\ \textit{the unit groupoid } M \textit{ to } G \end{array} \right\}.$$

And here is the fundamental consequence of Proposition 5.5.9: two Pfaffian Morita equivalent (jet groupoids of) pseudogroups induce the same almost geometric structures:

Corollary 5.5.11. *Let Γ_1, Γ_2 be Lie pseudogroups over X_1 and X_2 such that the Lie-Pfaffian groupoids $J^k\Gamma_1$ and $J^k\Gamma_2$ are Pfaffian Morita equivalent; then there is a bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Abstract order } k \textit{ almost} \\ \Gamma_1\textit{-structures on } M \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{Abstract order } k \textit{ almost} \\ \Gamma_2\textit{-structures on } M \end{array} \right\}.$$

This follows immediately by applying Corollary 5.4.20.

5.5.3 Motivating example: contact structures

In this section we reinterpret the fact that the first-order data of coorientable contact structures and symplectic foliations are the same; and similarly for contact forms and cosymplectic structures, and for contact structures and conformal symplectic foliations.

This is actually an immediate consequence of Proposition 5.5.12 below together with Corollary 5.5.11. Here the relevant pseudogroups are, on the one hand, Γ^c_{cont}, Γ^s_{cont} and Γ_{cont} from Example 5.1.11, and on the other hand $\Gamma_{SF} := \Gamma_{G_{SF}}$, $\Gamma_{CS} := \Gamma_{G_{CS}}$ and $\Gamma_{CSF} := \Gamma_{G_{CSF}}$, with the Lie groups G_{SF}, G_{CS} and G_{CSF} from Example 2.2.1.

Proposition 5.5.12. *The Pfaffian groupoids $J^1\Gamma^c_{cont}$ and $J^1\Gamma_{SF}$ are Pfaffian Morita equivalent; similarly for $J^1\Gamma^s_{cont}$ and $J^1\Gamma_{CS}$, and for $J^1\Gamma_{cont}$ and $J^1\Gamma_{CSF}$.*

Proof. Recall that a transitive Pfaffian groupoid is Pfaffian Morita equivalent to its isotropy group in any point (Example 5.4.8); since both Γ_{cont} and Γ_{SF} are transitive, and Lemma 5.5.13 yields $(J^1\Gamma^c_{cont})_0 = (J^1\Gamma_{SF})_0$, by composition of Pfaffian Morita equivalences (Proposition 5.4.2) we get that the Pfaffian groupoids $J^1\Gamma^c_{cont}$ and $J^1\Gamma_{SF}$ are Pfaffian Morita equivalent. The other two claims are analogous. Q.E.D.

Lemma 5.5.13. *The isotropy groups at 0 of the jet groupoids $J^1\Gamma^c_{cont}$ and $J^1\Gamma_{SF}$ coincide; similarly for $J^1\Gamma^s_{cont}$ and $J^1\Gamma_{CS}$, and for $J^1\Gamma_{cont}$ and $J^1\Gamma_{CSF}$.*

Proof. Let us begin with the second claim. A generic element of $(J^1\Gamma^s_{cont})_0$ is of the form $j^1_0\phi$, for $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ such that $\phi(0) = 0$ and $\phi^*\theta_{\mathrm{can}} = \theta_{\mathrm{can}}$, with $\theta_{\mathrm{can}} := x^i dy^i + dt$ the canonical contact form on $\mathbb{R}^{2n+1}$, as explained in Example 5.1.11. Since $j^1_x\phi$ is equivalent to $(x, \phi(x), d_x\phi)$ (via Remark 1.1.2), it is enough to compute the matrix $d_0\phi$.

If ϕ changes coordinates from (x^i, y^i, t) to (X^i, Y^i, T), then we have to impose $X^i dY^i + dT = x^i dy^i + dt$ as well as $dX^i \wedge dY^i = dx^i \wedge dy^i$ (if ϕ preserves θ, it preserves also its differential); after computing the differentials of the second variables using the first ones (similarly to Example 4.1.5), this is tantamount to

$$\begin{cases} X^i \frac{\partial Y^i}{\partial x^j} + \frac{\partial T}{\partial x^j} = 0 \ \ \forall j \\ X^i \frac{\partial Y^i}{\partial y^j} + \frac{\partial T}{\partial y^j} - x^j = 0 \ \ \forall j \\ X^i \frac{\partial Y^i}{\partial t} + \frac{\partial T}{\partial t} - 1 = 0 \end{cases}$$

$$\begin{cases} \frac{\partial X^i}{\partial x^j}\frac{\partial Y^i}{\partial y^k} - \frac{\partial X^i}{\partial y^k}\frac{\partial Y^i}{\partial x^j} = \delta^k_j \ \ \forall j,k \\ \frac{\partial X^i}{\partial x^j}\frac{\partial Y^i}{\partial t} - \frac{\partial X^i}{\partial t}\frac{\partial Y^i}{\partial x^j} = 0 \ \ \forall j \\ \frac{\partial X^i}{\partial y^j}\frac{\partial Y^i}{\partial t} - \frac{\partial X^i}{\partial t}\frac{\partial Y^i}{\partial y^j} = 0 \ \ \forall j \end{cases}$$

However, imposing the conditions $X^i(0,\dots,0) = Y^i(0,\dots,0) = T(0,\dots,0) = 0$ and computing everything at the origin, the first system becomes

$$\begin{cases} \frac{\partial T}{\partial x^j}(0,\dots,0) = 0 \ \ \forall j \\ \frac{\partial T}{\partial y^j}(0,\dots,0) = 0 \ \ \forall j \\ \frac{\partial T}{\partial t}(0,\dots,0) = 1 \end{cases}$$

In conclusion, the Jacobian $d_0\phi$ is a matrix of the form

$$\begin{bmatrix} \frac{\partial X^i}{\partial x^j}(0,\dots,0) & \frac{\partial X^i}{\partial y^j}(0,\dots,0) & \frac{\partial X^i}{\partial t}(0,\dots,0) \\ \frac{\partial Y^i}{\partial x^j}(0,\dots,0) & \frac{\partial Y^i}{\partial y^j}(0,\dots,0) & \frac{\partial Y^i}{\partial t}(0,\dots,0) \\ \frac{\partial T}{\partial x^j}(0,\dots,0) & \frac{\partial T}{\partial y^j}(0,\dots,0) & \frac{\partial T}{\partial t}(0,\dots,0) \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix}$$

where the upper-left square matrix is in $Sp(n) \subseteq GL(2n,\mathbb{R})$ and the determinants of $\begin{pmatrix} A & C \\ D & F \end{pmatrix}$ and $\begin{pmatrix} B & C \\ E & F \end{pmatrix}$ are zero.

On the other hand, a generic element of $(J^1\Gamma_{CS})_0$ is of the form $j^1_0\phi$, with $\phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ such that $\phi(0) = 0$, $\phi^*\omega_{\mathrm{can}} = \omega_{\mathrm{can}}$ (where $\omega_{\mathrm{can}} := dx^i \wedge dy^i$ is the canonical symplectic form on $\mathbb{R}^{2n}$) and $\phi^*\alpha = \alpha$ (where $\alpha = dt$ is the 1-form generating the codimension 1 foliation on $\mathbb{R}^{2n+1}$, given by the leaves $t = t_0$). Since $j^1_x\phi$ is equivalent to $(x, \phi(x), d_x\phi)$, also in this case it is enough to compute the matrix $d_0\phi$ and check that it coincides with the one found for Γ_{cont}. 5

If ϕ changes coordinates from (x^i, y^i, t) to (X^i, Y^i, T), then we have to impose $dX^i \wedge dY^i = dx^i \wedge dy^i$ as well as $dT = dt$; after computing the differentials of the second variables using the first ones, this is tantamount to

$$\begin{cases} \frac{\partial X^i}{\partial x^j}\frac{\partial Y^i}{\partial y^k} - \frac{\partial X^i}{\partial y^k}\frac{\partial Y^i}{\partial x^j} = \delta^k_j \ \ \forall j,k \\ \frac{\partial X^i}{\partial x^j}\frac{\partial Y^i}{\partial t} - \frac{\partial X^i}{\partial t}\frac{\partial Y^i}{\partial x^j} = 0 \ \ \forall j \\ \frac{\partial X^i}{\partial y^j}\frac{\partial Y^i}{\partial t} - \frac{\partial X^i}{\partial t}\frac{\partial Y^i}{\partial y^j} = 0 \ \ \forall j \end{cases}$$

$$\begin{cases} \frac{\partial T}{\partial x^j}(0,\dots,0) = 0 \ \ \forall j \\ \frac{\partial T}{\partial y^j}(0,\dots,0) = 0 \ \ \forall j \\ \frac{\partial T}{\partial t}(0,\dots,0) = 1 \end{cases}$$

where everything is computed at the origin.

In conclusion, also the Jacobian $d_0\phi$ is a matrix of the form

$$\begin{bmatrix} \frac{\partial X^i}{\partial x^j}(0,\dots,0) & \frac{\partial X^i}{\partial y^j}(0,\dots,0) & \frac{\partial X^i}{\partial t}(0,\dots,0) \\ \frac{\partial Y^i}{\partial x^j}(0,\dots,0) & \frac{\partial Y^i}{\partial y^j}(0,\dots,0) & \frac{\partial Y^i}{\partial t}(0,\dots,0) \\ \frac{\partial T}{\partial x^j}(0,\dots,0) & \frac{\partial T}{\partial y^j}(0,\dots,0) & \frac{\partial T}{\partial t}(0,\dots,0) \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix}$$

where the upper-left square matrix is in $Sp(n) \subseteq GL(2n, \mathbb{R})$ and the determinants of $\begin{pmatrix} A & C \\ D & F \end{pmatrix}$ and $\begin{pmatrix} B & C \\ E & F \end{pmatrix}$ are zero.

The same arguments can be carried out in the case of diffeomorphisms preserving not the contact form θ_{can} but only its positive conformal class, or even only the whole contact distribution, and of diffeomorphisms preserving not the generating 1-form α but only its positive conformal class, or even only the whole symplectic foliation. This means that, at the beginning, we impose either $\phi^*\theta_{\text{can}} = f\theta_{\text{can}}$ or $\phi^*\alpha = f\alpha$, for a function $f \in \mathcal{C}^\infty(M)$ (either positive or non-zero); then f and its derivatives appear in the computations, but, when imposing $\phi(0) = 0$, almost all new terms are killed, obtaining the same equations except for

$$\frac{\partial T}{\partial t}(0,\dots,0) = f(0,\dots,0).$$

According to f being a positive or a non-zero function, then we obtain the same kind of matrix except for number in the bottom-right corner, which would be any element of either $GL^+(1,\mathbb{R}) = \mathbb{R}_{>0}$ or $GL(1,\mathbb{R}) = \mathbb{R}^\times$, not necessarily 1. Q.E.D.

One can also replicate the computations of Lemma 5.5.13 for higher orders: given a k-jet $j^k_0\phi$, for ϕ in Γ_{CS} or Γ^s_{cont}, one has to describe the higher order derivatives of X, Y and T, by taking the differential consequences of the formulae above (i.e. applying a numer of times the operators $\frac{\partial}{\partial x^j}$, $\frac{\partial}{\partial y^j}$ and $\frac{\partial}{\partial t}$) and computing them in $0 \in \mathbb{R}^{2n+1}$.

It is easy to see that, already at the second order, all the second derivatives of T for elements of Γ_{CS}, i.e. $\frac{\partial^2 T}{\partial x^j \partial x^k}$, $\frac{\partial^2 T}{\partial x^j \partial y^k}$, etc. must vanish identically, whereas this does not hold for elements of Γ^s_{cont}.

This means that the isotropy groups at 0 of Γ^s_{cont} and Γ_{CS} of order ≥ 2 are not isomorphic, hence $J^k\Gamma^s_{cont}$ and $J^k\Gamma_{CS}$ are Morita equivalent only for $k = 1$. Of course, this is just a reflection of the fact that the higher order data of contact structures and of symplectic foliations do not coincide.

5.6 An application of principal Pfaffian bundles: realising generalised pseudogroups as classical ones

Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid: the set of its local holonomic bisections (Definition 4.4.1)

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega) \subseteq \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$$

is a generalised pseudogroup on $\mathcal{G}$ (Definition 4.5.1). Here we show that a principal Pfaffian $(\mathcal{G}, \omega)$-bundle (P, θ) allows one to represent such generalised pseudogroup as a classical one, namely:

Definition 5.6.1. Let $\pi : P \to M$ be a surjective submersion and $\theta \in \Omega^1(P, \pi^* F)$ a differential form with coefficients in a vector bundle $F \to M$. The **pseudogroup of local symmetries** of (P, θ) is the set

$$\Gamma_\pi(P, \theta) \subseteq \mathrm{Diff}_{\mathrm{loc}}(P)$$

of diffeomorphisms ϕ satisfying

$$\pi \circ \phi = \pi, \qquad \phi^* \theta = \theta. \qquad \blacklozenge$$

Proposition 5.6.2. *Let $(\mathcal{G}, \omega) \rightrightarrows X$ be a Pfaffian groupoid and (P, θ) a principal Pfaffian $(\mathcal{G}, \omega)$-bundle over M. Then the action of (bisections of) $\mathcal{G}$ on P induces an injective anti-homomorphism*

$$\Phi : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega) \to \Gamma_\pi(P, \theta).$$

Here we interpret the coefficients $\mu^* E$ of θ as $\mu^* F$, for $F \to M$ the vector bundle associated to E by Proposition 3.1.18, so that $\Gamma_\pi(P, \theta)$ is well defined.

Proof. The proof is divided in three steps:

1. Define an anti-homomorphism $\Phi : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}) \to \mathrm{Diff}_{\mathrm{loc}}(P)$ for any $\mathcal{G}$-action on $\mu : P \to X$.

2. Prove the injectivity of Φ, using the freeness of the $\mathcal{G}$-action and the surjectivity of the moment map μ.

3. Since (P, θ) is a principal Pfaffian bundle, show that Φ restricts to

$$\Phi : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega) \to \Gamma_\pi(P, \theta).$$

For every local bisection $\sigma : U \subseteq X \to \mathcal{G}$ we define the following diffeomorphism

$$\Phi(\sigma) = \phi_\sigma : \mu^{-1}(U) \to \mathrm{Im}(\phi_\sigma) = \mu^{-1}(t \circ \sigma(U)) \subseteq P,$$

5

$$\phi_\sigma(p) := \sigma(\mu(p)) \cdot p = m_P \circ (\sigma \circ \mu, id_P)(p).$$

Note that ϕ_σ is well defined because $s(\sigma(\mu(p))) = \mu(p)$, hence we can use the groupoid action for every p such that $\mu(p) \in U$.

Moreover, Φ is a pseudogroup anti-homomorphism, i.e. $\Phi(\sigma_2 \cdot \sigma_1) = \Phi(\sigma_1) \circ \Phi(\sigma_2)$, because (we use $\cdot$ indistinguishably for the product in $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ and the $\mathcal{G}$-action)

$$\phi_{\sigma_2 \cdot \sigma_1}(p) = (\sigma_2 \cdot \sigma_1)(\mu(p)) \circ p = (\sigma_1((t \circ \sigma_2)(\mu(p)))\sigma_2(\mu(p))) \cdot p =$$

$$= \sigma_1(t(\sigma_2(\mu(p)))) \cdot (\sigma_2(\mu(p)) \cdot p) = \sigma_1(\mu(\sigma_2(\mu(p)) \cdot p)) \cdot (\sigma_2(\mu(p)) \cdot p) =$$

$$= \sigma_1(\mu(\phi_{\sigma_2}(p))) \cdot (\phi_{\sigma_2}(p)) = \phi_{\sigma_1}(\phi_{\sigma_2}(p)) = \phi_{\sigma_1} \circ \phi_{\sigma_2}(p).$$

For the second part, assume $\phi_{\sigma_1} = \phi_{\sigma_2}$; then we have, for every p in their domains,

$$\sigma_1(\mu(p)) \cdot p = \sigma_2(\mu(p)) \cdot p.$$

Since the action is free, $\sigma_1(\mu(p)) = \sigma_2(\mu(p))$, and the surjectivity of μ implies $\sigma_1 = \sigma_2$, therefore Φ is injective.

For the third part, take a holonomic bisection $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$, i.e. $\sigma^*\omega = 0$; the multiplicativity of the action yields

$$(\phi_\sigma)^*\theta_p = (m_P \circ (\sigma \circ \mu, id_P))^*\theta_p = (\sigma \circ \mu, id_P)^*((m_P)^*\theta_{(\sigma(\mu(p)),p)}) =$$

$$= (\sigma \circ \mu, id_P)^*((\mathrm{pr}_1)^*\omega + \sigma(\mu(p)) \cdot (\mathrm{pr}_2)^*\theta)_{(\sigma(\mu(p)),p)} =$$

$$= (\mathrm{pr}_1 \circ (\sigma \circ \mu, id_P))^*\omega_{\sigma(\mu(p))} + \sigma(\mu(p)) \cdot (\mathrm{pr}_2 \circ (\sigma \circ \mu, id_P))^*\theta_p =$$

$$= (\sigma \circ \mu)^*\omega_{\sigma(\mu(p))} + \sigma(\mu(p)) \cdot (id_P)^*\theta_p = \mu^*(\cancel{\sigma^*\omega})_p + \sigma(\mu(p)) \cdot \theta_p = \sigma(\mu(p)) \cdot \theta_p,$$

where $g = \sigma(\mu(p)) \in \mathcal{G}$ sends the $s(\sigma(\mu(p))) = \mu(p)$-fibre to the $t(\sigma(\mu(p))) = \mu(\phi_\sigma(p))$-fibre of the representation $E \to X$.

By the $\mathcal{G}$-invariance of π, we conclude that ϕ_σ is in $\Gamma_\pi(P, \theta)$ since

$$(\pi \circ \phi_\sigma)(p) = \pi(\sigma(\mu(p)) \cdot p) = \pi(p). \qquad \text{Q.E.D.}$$

We provide now a general condition on $(\mathcal{G}, \omega)$ for the injective morphism $\Phi : \mathrm{Bis}(\mathcal{G}, \omega) \to \Gamma_\pi(P, \theta)$ to be also surjective. We say that a Pfaffian groupoid $(\mathcal{G}, \omega) \rightrightarrows X$ is **holonomically effective** when, for $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$, $t \circ \sigma = id$ implies $\sigma(x) = 1_x$ for all $x \in \mathrm{dom}(\sigma)$; equivalently, when the following map is injective

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega) \to \mathrm{Diff}_{\mathrm{loc}}(X), \quad \sigma \mapsto t \circ \sigma.$$

Example 5.6.3. Any effective Pfaffian groupoid (see Section 3.1) is in particular holonomically effective. A fundamental example of holonomically effective groupoid which is not effective is given by the jet groupoid $J^k\Gamma$ of a pseudogroup Γ (Example 4.3.11). In this case, the map

$$\mathrm{Bis}_{\mathrm{loc}}(J^k\Gamma, \omega) \to \Gamma \subseteq \mathrm{Diff}_{\mathrm{loc}}(M), \quad \sigma \mapsto t \circ \sigma$$

is even bijective (see Example 4.4.2). ◊

Proposition 5.6.4. *Let $(\mathcal{G}, \omega)$ be a holonomically effective Lie-Pfaffian groupoid over X and $\pi : (P, \theta) \to M$ be a principal Pfaffian $\mathcal{G}$-bundle, and assume that the moment map μ has connected fibres. Then the injective morphism from Proposition 5.6.2*

$$\Phi : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega) \to \Gamma_\pi(P, \theta)$$

is also surjective.

Proof. For every $\phi \in \Gamma_\pi(P, \theta)$, we see that p and $\phi(p)$ are in the same π-fibre, since $\pi(\phi(p)) = \pi(p)$; this means that, π being a principal bundle, $\phi(p) = \bar{g}(p) \cdot p$, for some element $\bar{g}(p) \in \mathcal{G}$ depending on p. All the elements of $\Gamma_\pi(P, \theta)$ can be expressed in this form, so Φ is surjective if all such $\bar{g}$ can be written as $\sigma \circ \mu$ for some holonomic bisection σ.

To show that this happens, we consider any section $\rho \in \Gamma(\mu)$ and prove that the function $\sigma = \bar{g} \circ \rho : X \to \mathcal{G}$ is a holonomic bisection. It is clearly a section of s since for all $x \in X$

$$s(\sigma(x)) = s(\bar{g}(\rho(x))) = \mu(\rho(x)) = x.$$

Moreover, σ is holonomic, as $\sigma^*\omega = \rho^*(\bar{g}^*\omega)$ and $\bar{g}^*\omega = 0$:

$$g \cdot \theta = \phi^*\theta = (m_P \circ (\bar{g}, id))^*\theta = (\bar{g}, id)^*(pr_1^*\omega + g \cdot pr_2^*\theta) = \bar{g}^*\omega + g \cdot \theta.$$

We check now that σ is a bisection. In general, the map $d_p\bar{g} : T_pP \to T_{\bar{g}(p)}\mathcal{G}$ takes values in $\ker(\omega_{\bar{g}(p)})$, since $\omega(d\bar{g}(v)) = (\bar{g}^*\omega)(v) = 0$. However, when we restrict $\bar{g}$ to the fibre $\mu^{-1}(x)$, for $x = \mu(p)$, the commutativity of the diagram reads

$$d_{\bar{g}(p)}s(d_p(\bar{g}_{|\mu^{-1}(x)}(v))) = d_p(s \circ \bar{g}_{|\mu^{-1}(x)})(v) = d_p(\mu_{|\mu^{-1}(x)})(v) = (d_p\mu)_{|\ker(d_p\mu)}(v) = 0.$$

In conclusion, using the Lie-Pfaffian hypothesis,

$$d_p(\bar{g}_{|\mu^{-1}(x)})(v) \in \ker(d_{\bar{g}(p)}s) \cap \ker(\omega_{\bar{g}(p)}) = \ker(d_{\bar{g}(p)}t) \cap \ker(\omega_{\bar{g}(p)}).$$

This means that

$$(dt \circ d\bar{g}_{|\mu^{-1}(x)})_p = d_p(t \circ \bar{g}_{|\mu^{-1}(x)}) = 0,$$

hence, using the assumption that the μ-fibres are connected, $t \circ \bar{g}_{|\mu^{-1}(x)}$ is constant for all elements in $\mu^{-1}(x)$, i.e. equal to an element of X depending only on x.

If we call such element $\xi(x)$, we have defined a map $\xi : X \to X$ such that

$$t(\sigma(x)) = t(\bar{g}(\rho(x))) = \xi(x),$$

since $\rho(x) \in \mu^{-1}(x)$. Moreover, ξ is invertible: if $\bar{g}^{-1} : p \mapsto \bar{g}(p)^{-1}$, then $t(\bar{g}^{-1} \circ \rho(p)) = \xi^{-1}(x)$. Being also clearly smooth, ξ is a diffeomorphism, which proves that σ is a holonomic bisection of $(\mathcal{G}, \omega)$.

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Last, σ does not depend on the choice of ρ. For a different $\rho' \in \Gamma(\mu)$, $\rho'(x)$ and $\rho(x)$ belong to the same μ-fibre, since $\mu(\rho'(x)) = x = \mu(\rho(x))$. As a consequence, since $t \circ \bar{g}$ is constant on the μ-fibres,

$$t \circ (\bar{g} \circ \rho') = t \circ (\bar{g} \circ \rho).$$

This means that the holonomic sections $\bar{g} \circ \rho'$ and $\bar{g} \circ \rho$ coincide because $\mathcal{G}$ was holonomically effective, so $\sigma = \bar{g} \circ \rho$ is independent from ρ.

It is then enough to choose any number of smooth local sections of μ whose domains cover the entire $M = \mu(P)$ (one smooth global section would do, but it does not necessarily exist) to define a global σ which satisfies (with possibly different sections ρ depending on the points)

$$\sigma \circ \mu = \bar{g} \circ \rho \circ \mu = \bar{g}.$$

as we asked at the beginning. Q.E.D.

In easier situations, e.g. when dealing with G-structures, we have a more explicit version of Proposition 5.6.4:

Proposition 5.6.5. *Let (G, ω) be a Pfaffian group, $\mu : (P, \theta) \to M$ a principal Pfaffian G-bundle, and consider the corresponding Pfaffian gauge groupoid ($\mathcal{G} = \mathrm{Gauge}(P), \hat{\omega}$) from Proposition 5.4.3 and the trivial principal Pfaffian $\mathcal{G}$-bundle $\pi : P \to \{*\}$. Then the injective morphism $\Phi : \mathrm{Bis}(\mathcal{G}, \omega) \to \Gamma_\pi(P, \theta)$ from Proposition 5.6.2 is also surjective.*

Proof. First of all, the morphism

$$\Phi' : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}) \to \mathrm{Diff}_{\mathrm{loc}}(P), \quad \sigma \mapsto \phi_\sigma$$

has an explicit inverse

$$\Psi' : \mathrm{Diff}_{\mathrm{loc}}(P) \to \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}), \quad \phi \mapsto \sigma_\phi,$$

given by the correspondence between automorphisms of a principal bundle and bisections of its gauge groupoid (see Example 3.1.11):

$$\sigma_\phi([p]) := [\phi(p), p].$$

Indeed, we check that $\Psi' = \Phi'^{-1}$: for every $p \in P$,

$$\Phi'(\Psi'(\phi))(p) = \phi_{\sigma_\phi}(p) = \sigma_\phi(\mu(p)) \cdot p = \sigma_\phi([p]) \cdot p = [\phi(p), p] \cdot p = \phi(p),$$

$$\Psi'(\Phi'(\sigma))([p]) = \sigma_{\phi_\sigma}([p]) = [\phi_\sigma(p), p] = [\sigma([p]) \cdot p, p] = \sigma([p]).$$

In order to produce an explicit inverse to Φ, we claim that Ψ' restricts to a morphism

$$\Psi : \Gamma_\pi(P, \theta) \to \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega).$$

To see this, one uses the multiplicativity of the $\mathcal{G}$-action m_P, together with Proposition 5.3.7:

$$\mu^*((\sigma_\phi)^*\omega) = \phi^*\theta - g \cdot \theta,$$

so that $\phi \in \Gamma(P, \theta)$ if and only if $\sigma_\phi \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$ (the condition $\pi \circ \phi = \pi$ is automatic). Then Ψ is an inverse of Φ, i.e. Φ is an isomorphism. Q.E.D.

Example 5.6.6. Let $\mu : P \to M$ be a G-structure and $\mathcal{G}$ its gauge groupoid, viewed as a subgroupoid

$$\mathcal{G} = \mathrm{Gauge}(P) \subseteq J^1(M, M)$$

and endowed with the Cartan form ω. The generalised pseudogroup $(\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega))$ is identified with the classical pseudogroup of local symmetries of P.

To see this, we observe first that the local holonomic bisections of $\mathcal{G}$ are identified with the jets of the symmetries of P, i.e. with elements of the kind

$$\sigma = j^1\varphi, \quad \varphi \in \mathrm{Aut}_G(P).$$

In particular, for each such σ, one has

$$\phi_{j^1\varphi}(p) = j^1\varphi(\mu(p)) \cdot p = j^1_{\mu(p)}\varphi \cdot p = Fr(\varphi)(p),$$

Our statement follows by applying Proposition 5.6.5 to P, viewed as a principal Pfaffian $\mathcal{G}$-bundle as in Example 5.4.10, so that the isomorphism Φ becomes

$$\Phi : (\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)) \to \Gamma_\pi(P, \theta), \quad j^1\varphi \mapsto Fr(\phi). \qquad \diamondsuit$$

5.7 Another application of principal Pfaffian bundles: Cartan bundles

In Section 5.1.2 we have briefly mentioned that (G, X)-structures have as particular cases a nice class of Klein geometries and one of integrable G-structures.

On the other hand, Klein geometries are particular instances of the so-called Cartan geometries, while integrable G-structures are of course particular G-structures. In this section, after recalling the notions above, we introduce a more general concept, that of **Cartan bundle**, which encompasses both Cartan geometries and G-structures. At the end, we show that Cartan bundles are closely related to principal Pfaffian bundles; we are currently investigating the advantages of this approach in the forthcoming paper [1]. This section is based on the preprint [18], to which we refer for more details.

Definition 5.7.1. A **Klein geometry** is a pair of a Lie group G and a Lie subgroup $H \subseteq G$ such that the quotient manifold G/H is connected and H contains no proper subgroups which are normal in G.

A **Klein pair** is a pair of a Lie algebra $\mathfrak{g}$ and a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\mathfrak{h}$ contains no proper ideal of $\mathfrak{g}$. ♦

Any Klein geometry induces a Klein pair, which, in turn, can be considered as the infinitesimal counterpart of the geometry. In the following, we will fix a Klein pair $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{h} = \mathrm{Lie}(H)$ and such that the adjoint representation $Ad : H \to GL(\mathfrak{h})$ can be extended to some representation $H \to GL(\mathfrak{g})$. This will be the "local model" for a Cartan geometry. As reference, we use chapters 4 and 5 of [85].

Definition 5.7.2 (Definition 3.1 of [85, chapter 5])**.** A **Cartan geometry** (P, ω) modelled on $(\mathfrak{g}, \mathfrak{h})$ is a principal H-bundle $P \to M$ together with a form $\omega \in \Omega^1(P, \mathfrak{g})$, such that

- ω is a pointwise isomorphism
- θ is H-equivariant, i.e. $(R_h)^*\omega = h^{-1} \cdot \omega$ for every $h \in H$
- $\omega(X^R) = X$ for every $X \in \mathfrak{h}$, with $X^R \in \mathfrak{X}(H)$ the right-invariant vector field associated to X, interpreted as $X^R \in \mathfrak{X}(P)$ via bundle trivialisations.

♦

There is a nice local description of Cartan geometries in terms of the so-called "Cartan gauges", which play the same role of charts in a smooth atlas.

Definition 5.7.3 (Definitions 1.2-1.3 of [85, chapter 5])**.** A **Cartan gauge** on M is a pair (U, θ_U), where $U \subseteq M$ is an open set, $\theta_U \in \Omega^1(U, \mathfrak{g})$ and

$$\mathrm{pr} \circ (\theta_U)_u : T_u U \to \mathfrak{g}/\mathfrak{h}$$

is an isomorphism for every $u \in U$, where $\mathrm{pr} : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the projection.

Two Cartan gauges $(U_\alpha, \theta_\alpha)$ and (U_β, θ_β) are **compatible, or gauge-equivalent**, if there exists a function $k : U_\alpha \cap U_\beta \to H$ such that,

$$(\theta_\beta)_x = k(x)^{-1} \cdot (\theta_\alpha)_x + (k^*\omega_H)_x \qquad \forall x \in U_\alpha \cap U_\beta,$$

where ω_H is the Maurer-Cartan form of H.

A **Cartan atlas** $\mathcal{A} = \{(U_\alpha, \theta_\alpha)\}_\alpha$ is a collection of compatible Cartan gauges, with $\{U_\alpha\}_\alpha$ open cover of M. ♦

It is proved in sections 5.2-5.3 of [85] that there is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{(isomorphism classes of)} \\ \text{Cartan geometries} \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(equivalence classes of)} \\ \text{Cartan atlases on } M \end{array} \right\}.$$

It associates to any Cartan atlas $\{(U_\alpha, \theta_\alpha)\}_\alpha$ the Cartan geometry (P, ω) defined by

$$P := \coprod_\alpha (U_\alpha \times H)/\sim, \quad \omega_{[u,h]}([v, y]) := Ad(h^{-1})\theta_u(v) + (\omega_{MC})_h(y).$$

Cartan geometries and G-structures can now be put together:

Definition 5.7.4. A **Cartan bundle** (P, θ) is a principal H-bundle $P \xrightarrow{\pi} M$, for H a Lie group, together with a representation $V \in \mathrm{Rep}(H)$ and a differential form $\theta \in \Omega^1(P, V)$ such that

- $\ker(\theta) \subseteq T^\pi P$ and it is an involutive distribution
- θ is H-equivariant, i.e.
$$(R_h)^* \theta_p(v) = h^{-1} \cdot \theta_p(v)$$
for all $p \in P, h \in H, v \in T_p P$. ♦

As anticipated, this general definition has the following two extreme cases, when $\ker(\theta)$ is the largest or the smallest possible distribution (in both situations, involutivity comes for free).

Example 5.7.5. An abstract H-structure (P, θ) on M^n (Definition 2.3.3) is a Cartan bundle with $V = \mathbb{R}^n$ the representation of H induced by the immersion $H \to GL(n, \mathbb{R})$ (see Theorem 2.3.2). In particular, the form $\theta \in \Omega^1(P, \mathbb{R}^n)$ satisfies $\ker(\theta) = T^\pi P$. ♢

Example 5.7.6. A Cartan geometry (P, θ) (Definition 5.7.2) modelled on the Klein pair $(\mathfrak{g}, \mathfrak{h})$ is a Cartan bundle with $V = \mathfrak{g}$ the representation of H which extends the adjoint representation of H. In particular, the form $\theta \in \Omega^1(P, \mathfrak{g})$ satisfies $\ker(\theta) = 0$. ♢

Proposition 5.7.7. *For any manifold M there is a 1-1 correspondence*

$$\left\{ \begin{array}{c} \textit{(isomorphism classes of)} \\ \textit{transitive Lie-Pfaffian groupoids on } M \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \textit{(isomorphism classes of)} \\ \textit{Cartan bundles on } M \end{array} \right\}.$$

Proof. Recall from Example 3.1.11 that transitive Lie groupoids with isotropy H are in 1-1 correspondence with principal H-bundles: it is then enough to show the correspondence between their differential forms.

Let $(\mathcal{G}, \omega)$ be a full transitive Lie-Pfaffian groupoid, with ω taking values in $E \in \mathrm{Rep}(\mathcal{G})$, and fix any $x \in M$; the principal H-bundle $P := s^{-1}(x) \xrightarrow{t} M$ is a Cartan bundle with the representation $V := E_x \in \mathrm{Rep}(\mathcal{G}_x)$ and the differential form
$$\theta \in \Omega^1(P, V), \qquad \theta_g(v) := g^{-1} \cdot \omega_g(v).$$

Indeed, from Lemma 3.4.11 (based on the multiplicativity of ω), it follows that θ is H-equivariant:
$$((R_h)^*\theta)_g(v) = \theta_{gh}(dR_h(v)) = (gh)^{-1} \cdot \omega_{gh}(dR_h(v)) =$$
$$= h^{-1} g^{-1} \cdot ((R_h)^* \omega)_g(v) = h^{-1} \cdot g^{-1} \cdot \omega_g(v) = h^{-1} \cdot \theta_g(v).$$

Moreover, $\ker(\theta)$ can be computed as
$$\ker(\theta) = \ker(\omega) \cap TP = \ker(\omega_P) = \ker(\omega) \cap \ker(ds).$$

Since ω is s-involutive, $\ker(\theta)$ is involutive, and since $(\mathcal{G},\omega)$ is of Lie type, $\ker(\theta)$ is contained in $\ker(dt)$.

Conversely, consider a Cartan bundle (P,θ); the gauge groupoid $\mathcal{G} := (P \times P)/H$ carries the representation $E := P[V]$ and the following differential form $\omega \in \Omega^1(\mathcal{G}, t^*E)$:

$$\omega_{[p,q]}([v,w]) = \theta_p(v) - [p,q]^{-1} \cdot \theta_q(w).$$

From the H-equivariance of θ, it follows that ω is well defined, by the same arguments used in Proposition 5.4.2; moreover, ω is multiplicative by costruction.

Last, it follows from the definition of ω that

$$\ker(ds) \cap \ker(\omega) = [\ker(\theta), \mathrm{Im}(a)];$$

since $\ker(\theta)$ is involutive, ω is s-involutive. Analogously,

$$\ker(dt) \cap \ker(\omega) = [\mathrm{Im}(a), \ker(\theta)],$$

hence $(\mathcal{G},\omega)$ is of Lie type. Q.E.D.

Corollary 5.7.8. *Let (P,θ) be a Cartan bundle; then (P,θ) is a principal Pfaffian bundle, with the action of the corresponding Pfaffian groupoid from Proposition 5.7.7.*

Proof. We have only to check that the action of Gauge(P) on P is Pfaffian with respect to θ and ω. On the one hand,

$$\ker(d\pi) = \ker(\omega),$$

which is involutive by definition of Cartan bundle. On the other hand, the action is multiplicative since

$$((m_P)^*\theta)_{([p,q],q)}([v,w],w) = \theta_p(dm_P([v,w],w)) =$$

$$= \theta_p(v) = \omega_{[p,q]}([v,w]) + [p,q]^{-1} \cdot \theta_q(w).$$

Q.E.D.

5.8 Integrability of principal Pfaffian bundles

We conclude this chapter by recasting the results on integrability of Pfaffian bundles (Section 1.5) in the framework of principal Pfaffian bundles, making use of the integrability of Pfaffian groupoids (Section 4.4). The purpose of this discussion will be to tackle the formal integrability problem for almost Γ-structures in the next chapter.

We start by proving two fundamental theorems, which describe how the principal Pfaffian structure behaves with respect to projections and prolongations.

Theorem 5.8.1. *Let $(\mathcal{G},\omega)$ be a full Pfaffian groupoid over X, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$, and integrable up to order 1 (Definition 4.4.9). Let also $(\widetilde{P},\widetilde{\theta})$ be a principal Pfaffian $(\mathcal{G}^{(1)},\omega^{(1)})$-bundle over M, where $\widetilde{\theta}$ has coefficients in $A = \mathrm{Lie}(\mathcal{G})$.*

Then there is a principal Pfaffian $(\mathcal{G},\omega)$-bundle (P,θ) over M, where $\theta \in \Omega^1(P,\mu^*E)$, together with a surjective submersion

$$\tau_P : (\widetilde{P},\widetilde{\theta}) \to (P,\theta)$$

such that

- *τ_P is $\mathcal{G}^{(1)}$-equivariant (with the $\mathcal{G}^{(1)}$-action on P induced by that on $\mathcal{G}$ via τ_G)*
- *τ_P preserves the forms $\widetilde{\theta}$ and θ, in the sense that*

$$\tau_P^*\theta = l \circ \widetilde{\theta},$$

If both $\widetilde{P}$ and $\mathcal{G}$ are Lie-Pfaffian (Definitions 5.3.16 and 4.3.4), then P is Lie-Pfaffian as well.

Proof. We apply Proposition 5.4.18 to the (Lie)-Pfaffian Morita morphism $\widetilde{P} : M \rightsquigarrow \mathcal{G}^{(1)}$ (Example 5.4.16) and the (Lie)-Pfaffian Morita morphism $\mathcal{G} : \mathcal{G}^{(1)} \rightsquigarrow \mathcal{G}$ (Example 5.4.17). This defines a principal (Lie)-Pfaffian $(\mathcal{G},\omega)$-bundle (P,θ), as in the following diagram:

$$\begin{tikzcd}
 & (\widetilde{P},\widetilde{\theta}) \arrow[d, dashed, "\tau_P"] \arrow[ddr, bend left, "\widetilde{\mu}"] \arrow[ddl, bend right, "\widetilde{\pi}"'] & (\mathcal{G}^{(1)},\omega^{(1)}) \arrow[l, bend right, "m_{\widetilde{P}}"'] \arrow[d, "\tau_{\mathcal{G}}"] \\
 & (P,\theta) \arrow[dl, dashed, "\pi"] \arrow[dr, dashed, "\mu"'] & (\mathcal{G},\omega) \arrow[l, dashed, bend right, "m_P"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
M & & X
\end{tikzcd}$$

Since P is constructed as $P := (\widetilde{P}\ {}_{\widetilde{\mu}}\times_t \mathcal{G})/\mathcal{G}^{(1)}$, the surjective submersion

$$\tau_P : \widetilde{P} \to P, \quad \widetilde{p} \mapsto [\widetilde{p}, 1_{\widetilde{\mu}(\widetilde{p})}],$$

makes the diagram commute. Let us check the $\mathcal{G}^{(1)}$-equivariance: for any $(\widetilde{p},\widetilde{g}) \in \widetilde{P}\ {}_{\widetilde{\mu}}\times_{t\circ\tau_G} \mathcal{G}^{(1)}$

$$\tau_P(\widetilde{p}\cdot\widetilde{g}) = [\widetilde{p}\cdot\widetilde{g}, 1_{\widetilde{\mu}(\widetilde{p}\cdot\widetilde{g})}] = [\widetilde{p}, \tau_G(\widetilde{g})] = [\widetilde{p}, 1_{\widetilde{\mu}(\widetilde{p})}]\cdot\widetilde{g} = \tau_P(\widetilde{p})\cdot\widetilde{g}.$$

Similarly, for the second condition we use the definition of θ:

$$((\tau_P)^*\theta)_{\widetilde{p}}(\alpha) = \theta_{(\widetilde{p},1_{\widetilde{\mu}(\widetilde{p})})}([\alpha,0]) = l \circ \widetilde{\theta}_{\widetilde{p}}(\alpha) \quad \forall \alpha \in T_{\widetilde{p}}\widetilde{P}.$$

Q.E.D.

5

Remark 5.8.2. A submersion between Pfaffian fibrations with the property of τ_P above is called a *normalised Pfaffian prolongation* in section 4.2 of our paper [19]. $\diamondsuit$

We have just proved that, roughly speaking, a principal Pfaffian $\mathcal{G}^{(1)}$-bundle admits a nice "projection" to a principal Pfaffian $\mathcal{G}$-bundle.

We are interested in proving a sort of "converse" of this result, i.e. that (under suitable conditions) a principal Pfaffian $\mathcal{G}$-bundle can be prolonged to a principal Pfaffian $\mathcal{G}^{(1)}$-bundle. We start by introducing the following fundamental definition.

Definition 5.8.3. Let $(\mathcal{G}, \omega) \rightrightarrows X$ be a full Pfaffian groupoid and $\pi : (P, \theta) \to M$ a principal Pfaffian $\mathcal{G}$-bundle on M, with θ surjective. Then P is called

- **integrable up to order** k if is integrable up to order k as a Pfaffian fibration $\pi : (P, \theta) \to M$ (see Theorem 5.3.15),
- **formally integrable** if it is integrable up to any order. $\blacklozenge$

Note that we do not ask any integrability conditions on the Pfaffian groupoid $(\mathcal{G}, \omega)$. This is indeed a necessary condition for the integrability of a principal Pfaffian $\mathcal{G}$-bundle (P, θ). Since it is often easier to check, it will be used to simplify the integrability problem of (P, θ).

Theorem 5.8.4. *Let $\pi : (P, \theta) \to M$ be a principal Lie-Pfaffian $(\mathcal{G}, \omega)$-bundle on M integrable up to order k. Then the Pfaffian groupoid $(\mathcal{G}, \omega) \rightrightarrows X$ is integrable up to order k.*

Proof. Since P is Lie-Pfaffian, by Corollary 5.4.5 the symbol space $\mathfrak{g}(P)$ of $(P, \theta) \to M$ is isomorphic to the symbol space $\mathfrak{g}$ of $(\mathcal{G}, \omega) \rightrightarrows X$. In particular, for any $i \leq k$, all the prolongations $\mathfrak{g}(P)^{(i)}$ are smooth (Theorem 1.5.5), therefore the prolongations $\mathfrak{g}^{(i)}$ are smooth as well.

By the same result, the integrability of P implies that the intrinsic torsions $\tau_P^{(i)}$ of the Pfaffian fibrations $P^{(i)}$ vanish; using the multiplicativity of the action, it follows from Proposition 5.8.9 discussed below that the intrinsic torsions $\tau_{\mathcal{G}}^{(i)}$ vanish as well. Then $(\mathcal{G}, \omega)$ is integrable up to order k by Proposition 4.4.10. Q.E.D.

Here is another natural but not obvious consequence of integrability of a principal Pfaffian bundle: the prolongations are principal Pfaffian bundles as well.

Theorem 5.8.5. *Let $\pi : (P, \theta) \to M$ be a principal Pfaffian $\mathcal{G}$-bundle on M integrable up to order k. Then, for every $i \leq k$, $(P^{(i)}, \theta^{(i)})$ is a principal Pfaffian $\mathcal{G}^{(i)}$-bundle over M.*

If P is Lie-Pfaffian (Definition 5.3.16), then $P^{(i)}$ is Lie-Pfaffian as well (and so are $\mathcal{G}$ and $\mathcal{G}^{(i)}$, by Theorem 5.3.17).

Proof. Since prolongations are defined recursively, it is enough to prove the theorem for $i = 1$, i.e. in the situation below:

$$\begin{tikzcd}
& (P^{(1)},\theta^{(1)}) \arrow[d, "\tau_P"] \arrow[ddl, bend right, "\pi^{(1)}"'] \arrow[ddr, "\mu^{(1)}"] & (\mathcal{G}^{(1)},\omega^{(1)}) \arrow[l, bend right, "m_P^{(1)}"'] \arrow[d, "\tau_{\mathcal{G}}"] \\
& (P,\theta) \arrow[dl, "\pi"] \arrow[dr, "\mu"'] & (\mathcal{G},\omega) \arrow[l, bend right, "m_P"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
M & & X
\end{tikzcd}$$

Given the multiplicative action $m_P : P \times_X \mathcal{G} \to P$, we define a new action of $\mathcal{G}^{(1)}$ on J^1P along $J^1P \to X, j^1_x\sigma \mapsto \mu(\sigma(x))$:

$$m_P^{(1)} : J^1P \times_X \mathcal{G}^{(1)} \to J^1P, \quad (j^1_x\sigma, j^1_y\rho) \mapsto j^1_x(\sigma \cdot \rho),$$

where $\sigma \cdot \rho \in \Gamma(P)$ is the local section of P defined by $(\sigma \cdot \rho)(x) := m_P(\sigma(x), \rho(y))$. This is well defined for all $(j^1_x\sigma, j^1_y\rho) \in J^1P \times_X \mathcal{G}^{(1)}$, since

$$\mu(\sigma(x)) = t(\rho(y))$$

implies that $(\sigma(x), \rho(y)) \in P \times_X \mathcal{G}$.

The action $m_P^{(1)}$ restricts to the partial prolongation $J^1_\theta P$. To see this, consider a composable pair $(j^1_x\sigma, j^1_x\rho)$ and assume $j^1_x\sigma \in J^1_\theta P$, i.e. $\theta_{\sigma(x)}(d_x\sigma(v)) = 0$ for $v \in T_xM$. Then, using the second formula from Example 3.2.9, we have

$$d_x(\sigma \cdot \rho)(v) = d_{\sigma(x)}(m_P(\cdot, \rho(y))) \circ d_x\sigma(v) = d_{(\sigma(x),\rho(y))}m_P(d_x\sigma(v), 0).$$

By the multiplicativity of m_P, we conclude that $j^1_x(\sigma \cdot \rho) \in J^1_\theta P$:

$$\theta_{\sigma(x)}(d_x(\sigma \cdot \rho)(v)) = ((m_P)^*\theta)_{(\sigma(x),\rho(y))}(d_x\sigma(v), 0) = \theta_{\sigma(x)}(d_x\sigma(v)) = 0.$$

Similarly, the action restricts to the prolongation $P^{(1)}$. In the situation above, if we assume further that $j^1_x\sigma \in P^{(1)}$, i.e. $\kappa_\theta(d_x\sigma(v), d_x\sigma(w)) = 0$, we can use the previous computation together with Lemma 5.8.7 (proved later) to check that also $j^1_x(\sigma \cdot \rho) \in P^{(1)}$:

$$\kappa_\theta(d_x(\sigma \cdot \rho)(v), d_x(\sigma \cdot \rho)(w)) = (m_P)^*\kappa_\theta((d_x\sigma(v), 0), (d_x\sigma(w), 0)) =$$

$$= \kappa_\theta(d_x\sigma(v), d_x\sigma(w)) = 0.$$

The $(\mathcal{G}^{(1)}, \omega^{(1)})$-action $m_P^{(1)}$ on $(P^{(1)}, \theta^{(1)})$ is multiplicative: this follows directly from the multiplicativity of m_P and the facts that $\mathrm{pr} \circ \theta^{(1)} = (\tau_P)^*\theta$ and $\mathrm{pr} \circ \omega^{(1)} = (\tau_{\mathcal{G}})^*\omega$ (Theorems 1.4.5 and 4.4.6):

$$g \cdot (m_P^{(1)})^*(\mathrm{pr} \circ \theta^{(1)}) = g \cdot (\tau_P \circ m_P^{(1)})^*\theta = (\tau_P, \tau_{\mathcal{G}})^*(g \cdot m_P^*\theta) =$$

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$$= (\tau_P, \tau_{\mathcal{G}})^*(\mathrm{pr}_1^* \theta) + (\tau_P, \tau_{\mathcal{G}})^*(\mathrm{pr}_2^* \omega) =$$
$$= (\tau_P \circ \mathrm{pr}_1)^*\theta + (\tau_{\mathcal{G}} \circ \mathrm{pr}_2)^*\omega = \mathrm{pr}_1^*(\mathrm{pr} \circ \theta^{(1)}) + \mathrm{pr}_2^*(\mathrm{pr} \circ \omega^{(1)}).$$

Rewriting the above formula at any point in $P^{(1)} \times_X \mathcal{G}^{(1)}$, one obtains Definition 5.3.1. Moreover, the surjective map $d\tau_P$ sends $\ker(\theta^{(1)}) \cap \ker(d\mu^{(1)})$ into the involutive distribution $\ker(\theta) \cap \ker(d\mu)$, hence $\ker(\theta^{(1)}) \cap \ker(d\mu^{(1)})$ is involutive as well, i.e. the multiplicative action is Pfaffian.

Using the diffeomorphism $P \times_X \mathcal{G} \to P \times_M P$, one can check that $P^{(1)} \times_X \mathcal{G}^{(1)} \to P^{(1)} \times_M P^{(1)}$ is a diffeomorphism as well; in conclusion, $(P^{(1)}, \theta^{(1)})$ is a principal Pfaffian $\mathcal{G}^{(1)}$-bundle over M.

Last, assume that P is Lie-Pfaffian and consider a tangent vector $v \in TP^{(1)}$. If $v \in \ker(\theta^{(1)}) \cap \ker(d\pi^{(1)})$, then the projection $v' = d\tau_P(v) \in TP$ belongs to $\ker(\theta) \cap \ker(d\pi)$. By the Lie-Pfaffian hypothesis, v' belongs also to $\ker(\theta) \cap \ker(d\mu)$, hence in particular

$$d\mu^{(1)}(v) = d\mu(d\tau_P(v)) = d\mu(v') = 0$$

This proves that $\ker(\theta^{(1)}) \cap \ker(d\pi^{(1)}) \subseteq \ker(\theta^{(1)}) \cap \ker(d\mu^{(1)})$; one checks analogously the other inclusion and obtains that $P^{(1)}$ is Lie-Pfaffian. Q.E.D.

5.8.1 From torsion to intrinsic torsion

The torsion T of a Pfaffian fibration $(P, \theta) \to M$ is a map defined on P. The goal of this section is to understand how, in presence of a principal Pfaffian action, T descends to a map defined on M, which we call *intrinsic* torsion. This is of course inspired by what happens with torsions of a G-structure, and will be applied in a similar spirit to almost Γ-structures.

We begin by proving the compatibility of the curvature and the torsion with the Pfaffian action.

Lemma 5.8.6. *Let (P, α) and (P, β) be Pfaffian fibrations over M, with the same coefficient bundles $\mathcal{N} \to P$ (Definition 1.3.1), $f \in \mathcal{C}^\infty(P)$ and $\phi : Q \to P$ a submersion. Then, if*

$$(P, \alpha + \beta), \qquad (P, f\alpha), \quad \text{and} \quad (Q, \phi^*\alpha)$$

are Pfaffian fibrations over M, their curvatures (Definition 1.3.8) can be computed in terms of those of (P, α) and (P, β):

$$\kappa_{\alpha+\beta}{}_{|\ker(\alpha)\cap\ker(\beta)} = (\kappa_\alpha + \kappa_\beta)_{|\ker(\alpha)\cap\ker(\beta)}, \qquad \kappa_{f\alpha} = f\kappa_\alpha, \qquad \kappa_{\phi^*\alpha} = \phi^*\kappa_\alpha.$$

Proof. For $p \in P$, $X, Y \in \ker(\alpha_p) \cap \ker(\beta_p) \subseteq \ker((\alpha + \beta)_p)$, we compute directly

$$(\kappa_{\alpha+\beta})_p(X, Y) = (\alpha+\beta)_p([\tilde{X}, \tilde{Y}]_p) = \alpha_p([\tilde{X}, \tilde{Y}]_p) + \beta_p([\tilde{X}, \tilde{Y}]_p) = (\kappa_\alpha + \kappa_\beta)_p(X, Y),$$

$$(\kappa_{f\alpha})_p(X, Y) = (f\alpha)_p([\tilde{X}, \tilde{Y}]_p) = f(p)(\kappa_\alpha)_p(X, Y) = (f\kappa_\alpha)_p(X, Y),$$

where, as per definition of the curvature κ, we choose any vector fields $\tilde{X}$ and $\tilde{Y}$ tangent to the Pfaffian distribution and such that $\tilde{X}_p = X$ and $\tilde{Y}_p = Y$.

Similarly, for $q \in Q$, $X, Y \in \ker(\phi^*\alpha)_q$ (so that $d_q\phi(X), d_q\phi(Y) \in \ker(\alpha_{\phi(q)})$), we choose vector fields $\tilde{X}$ and $\tilde{Y}$ which are ϕ-projectable and compute

$$(\kappa_{\phi^*\alpha})_q(X,Y) = (\phi^*\alpha)_q([\tilde{X},\tilde{Y}]_q) = \alpha_{\phi(q)}(d_q\phi([\tilde{X},\tilde{Y}]_q)) =$$

$$= \alpha_{\phi(q)}([\widetilde{d_q\phi(X)}, \widetilde{d_q\phi(Y)}]_{\phi(q)}) = (\kappa_\alpha)_{\phi(q)}(d_q\phi(X), d_q\phi(Y)) = (\phi^*\kappa_\alpha)_q(X,Y).$$

Q.E.D.

Lemma 5.8.7. *Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid and (P,θ) a (right) principal Pfaffian $\mathcal{G}$-bundle on M, with θ surjective.*

Then the curvature κ_θ of the Pfaffian fibration $s : (P,\theta) \to M$ (Definition 1.3.8) is multiplicative w.r.t. the Pfaffian $(\mathcal{G},\omega)$-action m_P: for every $(p,g) \in P \times_X \mathcal{G}$

$$g \cdot (m_P^*\kappa_\theta)_{(p,g)} = (\mathrm{pr}_1^*\,\kappa_\theta)_{(p,g)} + (\mathrm{pr}_2^*\,\kappa_\omega)_{(p,g)}$$

on $\ker(\mathrm{pr}_1^*\,\theta) \cap \ker(\mathrm{pr}_2^*\,\omega)$.

Proof. By definition of κ_θ, we compute, for every $X, Y \in \ker(\mathrm{pr}_1^*\,\theta) \cap \ker(\mathrm{pr}_2^*\,\omega) \subseteq T(P \times_X \mathcal{G})$,

$$g \cdot (m_P^*\kappa_\theta)_{(p,g)}(X,Y) = g \cdot (\kappa_\theta)_{m_P(p,g)}(d_{(p,g)}m_P(X), d_{(p,g)}m_P(Y)) =$$

$$= g \cdot \theta_{p\cdot g}([\widetilde{d_{(p,g)}m_P(X)}, \widetilde{d_{(p,g)}m_P(Y)}]_{p\cdot g}) = g \cdot \theta_{p\cdot g}(d_{(p,g)}m_P([\tilde{X},\tilde{Y}]_{(p,g)})) =$$

$$= g \cdot (m_P^*\theta)_{(p,g)}([\tilde{X},\tilde{Y}]_{(p,g)}) = (g \cdot m_P^*\theta)_{(p,g)}([\tilde{X},\tilde{Y}]_{(p,g)}) = (\kappa_{g\cdot m_P^*\theta})_{(p,g)}(X,Y).$$

It is enough then to use the multiplicativity of m_P and Lemma 5.8.6:

$$g \cdot (m_P^*\kappa_\theta)_{(p,g)} = (\kappa_{g\cdot m_P^*\theta})_{(p,g)} = (\kappa_{\mathrm{pr}_1^*\,\theta + \mathrm{pr}_2^*\,\omega})_{(p,g)} = (\mathrm{pr}_1^*\,\kappa_\theta)_{(p,g)} + (\mathrm{pr}_2^*\,\kappa_\omega)_{(p,g)}.$$

Q.E.D.

Remark 5.8.8. If $P = \mathcal{G}$ is the principal Pfaffian bundle with the right $(\mathcal{G},\omega)$-multiplication on itself (see Examples 3.1.13 and 5.4.7), we recover the fact that the curvature of a Pfaffian groupoid is multiplicative; this was proved directly in Lemma 6.2.31 of [81]. $\lozenge$

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Proposition 5.8.9. *Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid, (P,θ) a principal Lie-Pfaffian $\mathcal{G}$-bundle on M, with θ surjective, and assume moreover* $\dim(M) = \dim(X)$.

Then the torsion T_P of the Pfaffian fibration $(P,\theta) \to M$ (defined in Section 1.4) is compatible with the torsion $T_\mathcal{G}$ of the Pfaffian groupoid $(\mathcal{G},\omega)$ in the following sense:

$$T_P(p \cdot g) = g^{-1} \cdot T_P(p) + \Phi^* T_\mathcal{G}(g) \quad \forall (p,g) \in P\ {}_\mu\!\times_t \mathcal{G},$$

where Φ is given by Lemma 5.3.18.

Proof. Recall that T_P is defined as

$$T_P : P \to \mathrm{Hom}(\pi^*\Lambda^2 TM, \mu^* E)/\operatorname{Im}(\delta), \quad p \mapsto [(\xi^*\kappa_\theta)_p],$$

for some (p, ξ) in the partial prolongation $J^1_\theta P$ (Definition 1.4.2). Accordingly, it is enough to prove that the map

$$\widetilde{\kappa}_\theta : J^1_\theta P \to \mathrm{Hom}(\pi^*\Lambda^2 TM, \mu^* E), \quad (p, \xi) \mapsto (\xi^*\kappa_\theta)_p$$

is "multiplicative" w.r.t. the Pfaffian action in the following sense. Consider $(p, g) \in P\ {}_\mu\times_t \mathcal{G}$, $(p, \xi) \in J^1_\theta P$ and $(g, \eta) \in J^1_\omega \mathcal{G}$ such that the following linear map is well defined

$$\xi \cdot \eta : T_{\pi(p)} M \to T_{p\cdot g} P, \quad v \mapsto d_{(p,g)} m_P(\xi(v), \eta(\Phi_{\pi(p)}(v))).$$

Thanks to the multiplicativity of the action m_P, one checks that $\xi \cdot \eta$ takes values in $\ker(\theta_{p\cdot g})$, i.e. $(p \cdot g, \xi \cdot \eta) \in J^1_\theta \mathcal{G}$:

$$\theta_{p\cdot g}(\xi\cdot\eta(v)) = \theta_{p\cdot g}(d_{(p,g)} m_P(\xi(v), \eta(\Phi_{\pi(p)}(v)))) = (m_P^*\theta)_{(p,g)}(\xi(v), \eta(\Phi_{\pi(p)}(v))) =$$
$$= (g^{-1} \cdot \mathrm{pr}_1^*\theta + g^{-1} \cdot \mathrm{pr}_2^*\omega)_{(p,g)}(\xi(v), \eta(\Phi_{\pi(p)}(v))) =$$
$$= g^{-1} \cdot \theta_p(\xi(v)) + g^{-1} \cdot \omega_g(\eta(\Phi_{\pi(p)}(v))) = g^{-1} \cdot 0 + g^{-1} \cdot 0 = 0.$$

The "multiplicativity" of $\widetilde{\kappa}_\theta$ w.r.t. the action amounts then to

$$\widetilde{\kappa}_\theta(\xi \cdot \eta) = g^{-1} \cdot \widetilde{\kappa}_\theta(\xi) + \Phi^*\widetilde{\kappa}_\omega(\eta).$$

This follows from the multiplicativity of κ_θ (Lemma 5.8.7): for every $u, v \in T_{\pi(p)} M$,

$$\widetilde{\kappa}_\theta(\xi \cdot \eta)(u, v) = ((\xi \cdot \eta)^*\kappa_\theta)_{p\cdot g}(u, v) = (\kappa_\theta)_{p\cdot g}(\xi \cdot \eta(u), \xi \cdot \eta(v)) =$$
$$= (\kappa_\theta)_{p\cdot g}(d_{(p,g)} m_P(\xi_p(u), \eta_g(\Phi_{\pi(p)}(u))), d_{(p,g)} m_P(\xi_p(v), \eta_g(\Phi_{\pi(p)}(v)))) =$$
$$= (m_P^*\kappa_\theta)_{(p,g)}((\xi_p(u), \eta_g(\Phi_{\pi(p)}(u))), (\xi_p(v), \eta_g(\Phi_{\pi(p)}(v)))) =$$
$$= (g^{-1}\cdot(\mathrm{pr}_1^*\kappa_\theta)_{(p,g)} + g^{-1}\cdot(\mathrm{pr}_2^* c_\omega)_{(p,g)})((\xi_p(u), \eta_g(\Phi_{\pi(p)}(u))), (\xi_p(v), \eta_g(\Phi_{\pi(p)}(v)))) =$$
$$= g^{-1} \cdot (\kappa_\theta)_p(\xi_p(u), \xi_p(v)) + g^{-1} \cdot (c_\omega)_g(\eta_g(\Phi_{\pi(p)}(u)), \eta_g(\Phi_{\pi(p)}(v))) =$$
$$= g^{-1} \cdot \widetilde{\kappa}_\theta(\xi)(u, v) + \widetilde{\kappa}_\omega(\eta)(\Phi(u), \Phi(v)) = (g^{-1} \cdot \widetilde{\kappa}_\theta(\xi) + \Phi^*\widetilde{\kappa}_\omega(\eta))(u, v).$$

By the definition of T_P, this concludes the proof. Q.E.D.

The last ingredient to define the intrinsic torsion of a principal Pfaffian bundle is a representation of the acting Pfaffian groupoid. This is provided by the Spencer cohomology of its symbol space:

Lemma 5.8.10. *Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid, $\pi : (P,\theta) \to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M (with moment map $\mu : P \to X$), and assume moreover* $\dim(M) = \dim(X)$.

Then the Spencer cohomology of the symbol space $\mathfrak{g}(\theta)$ of P,

$$H^{-1,2}(\mathfrak{g}(\theta)) = \mathrm{Hom}(\pi^*\Lambda^2 TM, \mu^* E)/\,\mathrm{Im}(\delta) \to P,$$

is a representation of $\mathcal{G}$.

Similarly, if (P,θ) is integrable up to order k, then

$$H^{k-1,2}(\mathfrak{g}(\theta)) \in \mathrm{Rep}(\mathcal{G}^{(k)}).$$

Proof. We shall define the following left action:

$$\mathcal{G} \times H^{-1,2}(\mathfrak{g}(\theta)) \to H^{-1,2}(\mathfrak{g}(\theta)), \quad (g,[\xi]) \mapsto [g\cdot\xi].$$

First, since $\mathcal{G}$ acts on E via the representation, it also acts on $\mathrm{Hom}(\pi^*\Lambda^2 TM, \mu^*E)$: if $\xi : \Lambda^2 T_{\pi(p)}M \to E_{\mu(p)} = E_{t(g)}$, then

$$g\cdot\xi : \Lambda^2 T_{\pi(p\cdot g)}M \to E_{\mu(p\cdot g)} = E_{s(g)}, \quad (u,v) \mapsto g\cdot\xi(u,v)$$

Moreover, the $\mathcal{G}$-action on P is transported to $\mathrm{Hom}(\pi^*TM, \mathfrak{g}(\theta))$ via the differential of the action map: more precisely, an element $g \in \mathcal{G}$ acts on $\eta : T_{\pi(p)}M \to \mathfrak{g}(\theta_{p\cdot g})$ from the left as

$$g\cdot\eta : T_{\pi(p)}M \to \mathfrak{g}(\theta_p) \subseteq T_pP, \quad v \mapsto (g\cdot\eta)(v) = d_{p\cdot g}(m_P(\cdot,g^{-1}))(\eta(v)).$$

This is well defined since, for any $\eta(v) \in \mathfrak{g}(\theta_{p\cdot g}) = \ker(\theta_{p\cdot g}) \cap \ker(d_{p\cdot g}\pi)$, then

$$\theta_p((g\cdot\eta)(v)) = \theta_p(d_{(p\cdot g,g^{-1})}m_P(\eta(v),0)) = g\cdot\cancel{\theta_{p\cdot g}(\eta(v))} + g\cdot\omega_g(0) = 0$$

$$d_p\pi((g\cdot\eta)(v)) = d_{p\cdot g}(\pi\circ m_P(\cdot,g^{-1}))(\eta(v)) = d_{p\cdot g}\pi(\eta(v)) = 0.$$

Last, we check the compatibility with the quotient by the image of δ. We need to show that $[\xi]=[\xi'] \Rightarrow [g\cdot\xi] = [g\cdot\xi']$; in other words, if $\xi - \xi' = \delta_p(\eta)$, for some $\eta \in \mathrm{Hom}(\pi^*TM, \mathfrak{g}(\theta))$, then we prove

$$\delta_p(g\cdot\eta) = g\cdot\delta_{p\cdot g}(\eta).$$

This comes from the following computation, taking $u,v \in T_{\pi(p)}M = T_{\pi(p\cdot g)}M$ and the respective $\bar{u}_p,\bar{v}_p \in \ker(\theta_p)$ s.t. $d_p\pi(\bar{u}_p) = u$, $d_p\pi(\bar{v}_p) = v$ (similarly for $\bar{u}_{p\cdot g}$ and $\bar{v}_{p\cdot g}$):

$$\delta_p(g\cdot\eta)(u,v) = (\kappa_\theta)_p((g\cdot\eta)(u),\bar{v}_p) - (\kappa_\theta)_p((g\cdot\eta)(v),\bar{u}_p) =$$

$$= (\kappa_\theta)_p(d_{p\cdot g}(m_P(\cdot,g^{-1}))(\eta(u)),\bar{v}_p) - (\kappa_\theta)_p(d_{p\cdot g}(m_P(\cdot,g^{-1}))(\eta(v)),\bar{u}_p) =$$

$$= (\kappa_\theta)_p(d_{(p\cdot g,g^{-1})}m_P(\eta(u),0), d_{(p\cdot g,g^{-1})}m_P(\bar{v}_{p\cdot g},\beta))+$$

$$-(\kappa_\theta)_p(d_{(p\cdot g,g^{-1})}m_P(\eta(v),0),d_{(p\cdot g,g^{-1})}m_P(\bar{u}_{p\cdot g},\beta))=$$

$$=(m_P^*\kappa_\theta)_{(p\cdot g,g^{-1})}((\eta(u),0),(\bar{v}_{p\cdot g},\beta))-(m_P^*\kappa_\theta)_{(p\cdot g,g^{-1})}((\eta(v),0),(\bar{u}_{p\cdot g},\beta))=$$

$$=g\cdot(\mathrm{pr}_1^*\,\kappa_\theta)_{(p\cdot g,g^{-1})}((\eta(u),0),(\bar{v}_{p\cdot g},\beta))+g\cdot(\mathrm{pr}_2^*\,c_\omega)_{(p\cdot g,g^{-1})}((\eta(u),0),(\bar{v}_{p\cdot g},\beta))+$$

$$-g\cdot(\mathrm{pr}_1^*\,\kappa_\theta)_{(p\cdot g,g^{-1})}((\eta(v),0),(\bar{u}_{p\cdot g},\beta))+g\cdot(\mathrm{pr}_2^*\,c_\omega)_{(p\cdot g,g^{-1})}((\eta(v),0),(\bar{u}_{p\cdot g},\beta))=$$

$$=g\cdot(\kappa_\theta)_{p\cdot g}(\eta(u),\bar{v}_{p\cdot g})+g\cdot\cancel{(c_\omega)_{g^{-1}}(0,\beta)}-g\cdot(\kappa_\theta)_{p\cdot g}(\eta(v),\bar{u}_{p\cdot g})-g\cdot\cancel{(c_\omega)_{g^{-1}}(0,\beta)}=$$

$$=g\cdot(\kappa_\theta)_{p\cdot g}(\eta(u),\bar{v}_{p\cdot g})-g\cdot(\kappa_\theta)_{p\cdot g}(\eta(v),\bar{u}_{p\cdot g})=g\cdot\delta_{p\cdot g}(\eta)(u.v)=(g\cdot\delta_{p\cdot g}(\eta))(u,v).$$

In the third line we replaced $\bar{v}_p$ with $d_pm_P(\bar{v}_{p\cdot g},\beta)$, for some $\beta\in\ker(\omega_{g^{-1}})$ (similarly for $\bar{u}_{p\cdot g}$). This can be done because $d_{(p\cdot g,g^{-1})}m_P(\bar{v}_{p\cdot g},\beta)$ is an element of $\ker(\theta_p)$:

$$\theta_p(d_{(p\cdot g,g^{-1})}m_P(\bar{v}_{p\cdot g},\beta))=(m_P^*\theta)_{(p\cdot g,g^{-1})}(\bar{v}_{p\cdot g},\beta)=g\cdot\cancel{\theta_{p\cdot g}(\bar{v}_{p\cdot g})}+g\cdot\cancel{\omega_{g^{-1}}(\beta)}=0,$$

whose image through $d_p\pi$ is v:

$$d_p\pi(d_{(p\cdot g,g^{-1})}m_P(\bar{v}_{p\cdot g},\beta))=d_{(p\cdot g,g^{-1})}(\pi\circ m_P)(\bar{v}_{p\cdot g},\beta)=$$

$$=d_{(p\cdot g,g^{-1})}(\pi\circ\mathrm{pr}_1)(\bar{v}_{p\cdot g},\beta)=d_{p\cdot g}\pi(d_{(p\cdot g,g^{-1})}\,\mathrm{pr}_1(\bar{v}_{p\cdot g},\beta))=d_{p\cdot g}\pi(\bar{v}_{p\cdot g})=v.$$

If (P,θ) is integrable up to order k, then also $(\mathcal{G},\omega)$ is integrable up to order k (Theorem 5.8.4), so we can consider $\mathcal{G}^{(k)}$. The fact that $H^{k-1,2}(\mathfrak{g}(\theta))$ is a representation of $\mathcal{G}^{(k)}$ can be proved by a direct computation, or by applying the first part of this Lemma to the principal Pfaffian $(\mathcal{G}^{(k)},\omega^{(k)})$-bundle $(P^{(k)},\theta^{(k)})$, and using the fact that $H^{-1,2}(\mathfrak{g}(\theta)^{(k)})\cong H^{k-1,2}(\mathfrak{g}(\theta))$ (see the discussion in Section 1.2). Q.E.D.

And here is the promised result, which will play a key role in the formal integrability problem, discussed in the next section.

Proposition 5.8.11. *Let $(\mathcal{G},\omega)\rightrightarrows X$ be a full Pfaffian groupoid, $\pi:(P,\theta)\to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M with symbol space $\mathfrak{g}=\mathfrak{g}(\theta)$, and assume moreover $\dim(M)=\dim(X)$.*

If the torsion $T_\mathcal{G}$ of $\mathcal{G}$ vanishes, then the torsion T_P of P descends to a section T_{intr} of the vector bundle over M associated to P by the representation $H^{-1,2}(\mathfrak{g})$:

$$T_{intr}:M\to P[H^{-1,2}(\mathfrak{g})].$$

Similarly, if (P,θ) is integrable up to order k, and the torsion $T^{k+1}(\mathcal{G})=T(\mathcal{G}^{(k+1)})$ of $\mathcal{G}^{(k+1)}$ vanishes, then the torsion $T^{k+1}(P)=T(P^{(k)})$ of P (Definition 1.5.4) descends to a section of the associated bundle

$$T^{k+1}_{intr}:M\to P^{(k)}[H^{k-1,2}(\mathfrak{g})].$$

Proof. We use Lemma 5.8.10 to consider the associated vector bundle $P[H^{-1,2}(\mathfrak{g})]$ and define

$$T_{intr}(x) := [p, T(P)(p)],$$

for some $p \in \pi^{-1}(x)$. Let us prove that T_{intr} is well defined. Since $T_{\mathcal{G}} = 0$, then the torsion T_P of P is $\mathcal{G}$-equivariant by Lemma 5.8.9:

$$T_P(p \cdot g) = g^{-1} \cdot T_P(p).$$

The conclusion follows by

$$[p \cdot g, T(P)(p \cdot g)] = [p \cdot g, g^{-1} \cdot T(P)(p)] = [p, T(P)(p)] \quad \forall (p, g) \in P \times_X \mathcal{G}.$$

The same argument applies also for T^{k+1}_{intr}. Q.E.D.

5.8.2 Intrinsic torsion and formal integrability

This section is devoted to finding conditions under which the converse of Theorem 5.8.4 holds.

Assume that $(\mathcal{G}, \omega)$ is full and integrable up to order 1; then, by Theorem 1.5.5, a principal Lie-Pfaffian $(\mathcal{G}, \omega)$-bundle (P, θ) is integrable up to order 1 (Definition 5.8.3) if and only if

- the intrinsic torsion T of the Pfaffian fibration $(P, \theta) \to M$ (Definition 1.5.4) vanishes,
- the prolongation $\mathfrak{g}(\theta)^{(1)}$ of its symbol space $\mathfrak{g}_\pi(\theta) = \mathfrak{g}(\theta)$ (Definition 5.3.16) is smooth.

However, by Corollary 5.4.5, $\mathfrak{g}(\theta)$ is isomorphic to the symbol space $\mathfrak{g}$ of $(\mathcal{G}, \omega)$, whose prolongation $\mathfrak{g}^{(1)}$ is smooth since $(\mathcal{G}, \omega)$ is integrable (Proposition 4.4.10). In most situations, it follows that $\mathfrak{g}(\theta)^{(1)}$ is automatically smooth: this is the case if the tableau structures of $\mathfrak{g}(\theta)$ and $\mathfrak{g}$ are isomorphic, such as under the hypotheses of Theorem 5.4.6.

In the following, we will always assume that $\mathfrak{g}(\theta)^{(1)}$ is smooth. The only obstruction to the integrability of P is then the vanishing of its torsion

$$T : P \to H^{-1,2}(\mathfrak{g}(\theta)).$$

Applying Proposition 5.8.11, T can be interpreted as a section

$$T_{intr} : M \to P[H^{-1,2}(\mathfrak{g}(\theta))] =: \mathcal{H}^{-1,2}(\mathfrak{g}(\theta)),$$

and $T = 0$ if and only if $T_{intr} = 0$. We call the associated vector bundle $\mathcal{H}^{-1,2}(\mathfrak{g}(\theta)) \to M$ the **Spencer cohomology of the principal Pfaffian bundle** P, and the section T_{intr} the **intrinsic torsion** (of order 1) of P.

In conclusion, by applying Theorem 1.4.5, we have shown the following.

Theorem 5.8.12. *Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid and $\pi : (P,\theta) \to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M with θ surjective. Assume that*

- $\dim(M) = \dim(X)$
- *the prolongation $\mathfrak{g}(\theta)^{(1)}$ of its symbol space is smooth*
- *the Pfaffian groupoid $(\mathcal{G},\omega)$ is integrable up to order 1 (Definition 4.4.9)*

Then (P,θ) is integrable up to order 1 as a principal Pfaffian bundle (Definition 5.8.3) if and only if its intrinsic torsion $T_{intr} : M \to \mathcal{H}^{-1,2}(\mathfrak{g}(\theta))$ vanishes.

Moreover, when this is the case,

$$\mathfrak{g}(\omega^{(1)}) \cong \mathfrak{g}(\omega)^{(1)}, \qquad \mathfrak{g}(\theta^{(1)}) \cong \mathfrak{g}(\theta)^{(1)}.$$

One defines inductively the intrinsic torsions of higher orders.

Definition 5.8.13. Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid and $\pi : (P,\theta) \to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M. Assume moreover that $\dim(M) = \dim(X)$ and that P is integrable up to order k.

The **intrinsic torsion of order** $k+1$ of P is the section

$$T^{k+1}_{intr} \in \Gamma(M, \mathcal{H}^{k-1,2}(\mathfrak{g}(\theta)))$$

induced by Proposition 5.8.11 from the torsion $T^{k+1} : P^{(k)} \to H^{k-1,2}(\mathfrak{g})$ of the Pfaffian fibration $(P,\theta) \to M$. Here $\mathcal{H}^{k-1,2}(\mathfrak{g}(\theta))$ denotes the **Spencer cohomology of the principal Pfaffian bundle** P, i.e. the associated vector bundle

$$\mathcal{H}^{k-1,2}(\mathfrak{g}(\theta)) := P^{(k)}[H^{k-1,2}(\mathfrak{g}(\theta))] \to M. \qquad \blacklozenge$$

Theorem 5.8.14. *Let $(\mathcal{G},\omega) \rightrightarrows X$ be a full Pfaffian groupoid and $\pi : (P,\theta) \to M$ a principal Lie-Pfaffian $\mathcal{G}$-bundle on M. Assume moreover that*

- $\dim(M) = \dim(X)$
- *the prolongations $\mathfrak{g}(\theta)^{(i)}$ of its symbol space are of constant (non-zero) rank $\forall i = 1,\dots,k$*
- *the Pfaffian groupoid $(\mathcal{G},\omega)$ is integrable up to order k (Definition 4.4.9).*

Then (P,θ) is integrable up to order k as a principal Pfaffian bundle (Definition 5.8.3) if and only if the intrinsic torsions T^i_{intr} vanish for $i = 1,\dots,k$.

Moreover, when this is the case, for all $i = 1,\dots,k$,

$$\mathfrak{g}(\omega^{(i)}) \cong \mathfrak{g}(\omega^{(i-1)})^{(1)} \cong \mathfrak{g}(\omega)^{(i)}, \qquad \mathfrak{g}(\theta^{(i)}) \cong \mathfrak{g}(\theta^{(i-1)})^{(1)} \cong \mathfrak{g}(\theta)^{(i)}.$$

In particular, if $\mathcal{G}$ is formally integrable and $\mathfrak{g}(\theta)^{(i)}$ is smooth for every i, then P is formally integrable if and only if all the intrinsic torsions vanish.

Proof. Assume that all T^i_{intr} vanish; since $T^1_{intr} = 0$, we apply Theorem 5.8.12 to get integrability up to order 1.

Then, using the facts that $\mathfrak{g}(\omega^{(1)}) \cong \mathfrak{g}(\omega)^{(1)}$ and $\mathfrak{g}(\theta^{(1)}) \cong \mathfrak{g}(\theta)^{(1)}$ (Propositions 4.4.10 and 5.8.12), together with Lemma 5.3.14, we see that

$$\mathfrak{g}(\omega^{(1)})^{(1)} \cong (\mathfrak{g}(\omega)^{(1)})^{(1)} = \mathfrak{g}(\omega)^{(2)} \cong \mathfrak{g}(\theta)^{(2)} = (\mathfrak{g}(\theta)^{(1)})^{(1)} \cong \mathfrak{g}(\theta^{(1)})^{(1)}.$$

This means that the prolongation of $\mathfrak{g}(\theta^{(1)})$ is a smooth vector bundle; combining this with the vanishing of the second intrinsic torsion $T^2_{intr} = T_{intr}(P^{(1)})$, we can now apply again Theorem 5.8.12 on $(P^{(1)}, \theta^{(1)})$, obtaining the principal Pfaffian bundle $P^{(2)} = \mathrm{Prol}(P^{(1)}, \theta^{(1)})$ over M. Our hypotheses allow us to iterate this argument $k-1$ times, obtaining integrability up to order k.

Viceversa, if P is integrable up to order k, we apply Theorem 1.4.5 for every $i = 1, \dots, k$ in order to get $T(P^{(i)}) = 0$, which, together with $T(\mathcal{G}^{(i)}) = 0$, implies $T^i_{intr} = 0$. Q.E.D.

5.8.3 Maurer-Cartan equation for principal Pfaffian bundles

For any Pfaffian groupoid $(\mathcal{G}, \omega)$, the prolongation $(\mathcal{G}^{(1)}, \omega^{(1)})$ is related to $(\mathcal{G}, \omega)$ by a fundamental property: the Maurer-Cartan equation. This was discovered in [81] and was recalled in Remark 4.4.8. We now prove a similar result for prolongations of principal Pfaffian bundles.

Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid over X, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$, and consider its Lie algebroid A and the associated Spencer operator (D^ω, l^ω). Moreover, let (P, θ) be a principal Pfaffian $(\mathcal{G}, \omega)$-bundle over M, with moment map μ and $\theta \in \Omega^1(P, \mu^*E)$.

As in Remark 4.4.8, in order to state the Maurer-Cartan equation we consider the pullback (μ^*D, μ^*l) of the Spencer operator (D, l) of ω (Proposition 3.4.26) via the moment map $\mu : P^{(1)} \to X$. It is a Spencer operator on the pullback Lie algebroid μ^*A with values in μ^*E, so we can consider the associated Maurer-Cartan operator from Definition 3.4.27:

$$MC : \Omega^1(P^{(1)}, \mu^*A) \to \Omega^2(P^{(1)}, \mu^*E), \quad \eta \mapsto d_D\eta - \frac{1}{2}\{\eta, \eta\}_D.$$

The following result generalises Theorem 6.2.17 of [81] (which is recovered when $P = \mathcal{G}$, as in Example 3.1.13).

Proposition 5.8.15. *Let $(\mathcal{G}, \omega)$ be a full Pfaffian groupoid over X and (P, θ) be a principal Lie-Pfaffian $(\mathcal{G}, \omega)$ bundle over M integrable up to order 1: then*

$$MC(\theta^{(1)}) = 0.$$

Proof. Since $(P^{(1)}, \theta^{(1)}) \to M$ is a Pfaffian fibration (Theorem 5.3.15), $\theta^{(1)}$ is π-transversal, hence it is enough to prove $MC(\theta^{(1)})(X, Y) = 0$ in the following three cases of $X, Y \in TP^{(1)}$:

1. both X and Y are in $\ker(\theta^{(1)})$
2. both X and Y are in $\ker(d\pi)$
3. $X \in \ker(\theta^{(1)})$ and $Y \in \ker(d\pi)$

For the first part, since $\theta^{(1)}(X) = \theta^{(1)}(Y) = 0$, every term in $MC(\theta^{(1)})(X,Y)$ besides $l(\theta^{(1)}[X,Y])$ vanishes automatically. However, by Theorem 1.4.5,

$$l(\theta^{(1)}[X,Y]) = \theta(d\pi^1_0[X,Y]) = \theta([d\pi^1_0(X), d\pi^1_0(Y)]) = (\pi^1_0)^*\kappa_\theta(X,Y) = 0,$$

where in the last equality we use the definition of the prolongation $P^{(1)}$.

For the second part, we use the fact that $\ker(d\pi) = \mathrm{Im}(a)$, so that $X = a(\alpha), Y = a(\beta)$, for some $\alpha, \beta \in A^{(1)} = \mathrm{Lie}(\mathcal{G}^{(1)})$. By Theorems 5.8.4 and 5.8.5, $(\mathcal{G}, \omega)$ is integrable up to order 1 and $(P^{(1)}, \theta^{(1)})$ is a principal Pfaffian bundle; in particular, since the action is multiplicative, Proposition 5.3.4 yields

$$\theta^{(1)} \circ a = l^{(1)},$$

where a is the infinitesimal $\mathcal{G}^{(1)}$-action on P and $l^{(1)} : A^{(1)} \to A$ is the Lie algebroid morphism from Proposition 4.3.6. Together with the fact that $d\mu \circ a = \rho^{(1)} = \rho \circ l^{(1)}$, we obtain

$$(\mu^*D^\omega)_X(\theta^{(1)}(Y)) = (\mu^*D^\omega)_{a(\alpha)}(\omega^{(1)}(\beta)) = D^\omega_{\rho^{(1)}(\alpha)}(\omega^{(1)}(\beta)) = D^\omega_{\rho(\theta^{(1)}(Y))}(\theta^{(1)}(X)),$$

$$l(\theta^{(1)}[X,Y]) = l(l^{(1)}[\alpha,\beta]) = l[l^{(1)}(\alpha), l^{(1)}(\beta)] = l[\theta^{(1)}(X), \theta^{(1)}(Y)].$$

Putting these pieces together,

$$MC(\theta^{(1)})(a(\alpha), a(\beta)) = 0.$$

For the third part, the formula $MC(\theta^{(1)})(X,Y) = 0$ reduces to

$$(\mu^*D^\omega)_Y(\theta^{(1)}(X)) - l(\theta^{(1)}[X,Y]) = 0.$$

Since $X = a(\alpha)$ for some $\alpha \in A^{(1)}$, one can check that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi^\varepsilon_{a(\alpha)}(p) = a(\alpha)_p,$$

which allows us to describe the flow of X in terms of the flow of α:

$$\phi^\varepsilon_{a(\alpha)} : p \mapsto p \cdot i(\phi^\varepsilon_\alpha(\mu(p))) = m_P \circ (id_P, i \circ \phi^\varepsilon_\alpha \circ \mu)(p).$$

For every $p \in P^{(1)}$, denote $\tilde{p} = \phi^\varepsilon_{a(\alpha)}(p)$, so that $p = \tilde{p} \cdot \phi^{-\varepsilon}_\alpha(\mu(p))$, and compute

$$\theta^{(1)}_p([a(\alpha), Y]_p) = \theta^{(1)}_p\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d_{\tilde{p}}\phi^{-\varepsilon}_{a(\alpha)}(Y_{\tilde{p}})\right) =$$

$$= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \theta^{(1)}_p\Big(d_{\tilde{p}}(m_P \circ (id, i \circ \phi^{-\varepsilon}_\alpha \circ \mu))(Y_{\tilde{p}})\Big) =$$

$$= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi^{-\varepsilon}_\alpha(\mu(p)) \cdot \cancel{\theta^{(1)}_{\tilde{p}}(Y_{\tilde{p}})} + \phi^{-\varepsilon}_\alpha(\mu(p)) \cdot \omega^{(1)}_{\phi^{-\varepsilon}_\alpha(\mu(p))}\Big(d\phi^{-\varepsilon}_\alpha(d_{\tilde{p}}\mu(Y_{\tilde{p}}))\Big) =$$

$$= D^{\omega^{(1)}}_{d\mu(Y)}(\alpha)(p),$$

where we used the multiplicativity of the action and the definition of $D^{\omega^{(1)}}$. In conclusion, we have

$$l(\theta^{(1)}[a(\alpha), Y]) = l(D^{\omega^{(1)}}_{d\mu(Y)}(\alpha)) = (D^\omega)_{d\mu(Y)}(l^{(1)}(\alpha)) = (\mu^* D^\omega)_Y(\theta^{(1)}(X)),$$

where in the second passage we used the first part of Proposition 6.2.46 of [81].
Q.E.D.

Chapter 6

Integrability of almost Γ-structures

In this chapter we apply the machinery of principal Pfaffian bundles, developed in chapter 5, to study the integrability of almost Γ-structures. As for G-structure, we will focus mainly on structures of order 1, but everything can be done in full generality.

The first section of this chapter deals with a characterisation of formal integrability in terms of the intrinsic torsions. This is followed by some relevant examples and particular cases. For instance, for $\Gamma = \Gamma_G$, we recover the intrinsic torsions for G-structures discussed in chapter 2.

In the second part, we investigate when a formally integrable almost Γ-structure is integrable. This is a different kind of problem, which we recast as solving a PDE with symmetries. We will show how this is related to the corresponding linearised problem, and indicate which analytic techniques may be used to solve it.

In the last section, we obtain further results on the integrability of Morita equivalent geometric structures, motivated by the similarities and the differences in the integrability problem of contact structures and symplectic foliations. In particular, we show that being integrable is, in general, not Pfaffian Morita invariant.

6.1 Formal integrability of almost Γ-structures

We are going to tackle the formal integrability problem for almost Γ-structures by describing a sequence of progressive obstructions to the k-integrability. More precisely, we will prove a criterion for an almost Γ-structure of order k to be integrable up to order $k+1$; such proof relies on the theory of prolongations for principal Lie-Pfaffian bundles, which we developed in Section 5.8.

Although the theory of formal integrability of a G-structure constitutes a useful source of inspiration, one should avoid to establish easy parallelisms and mimic all the definitions. Indeed, a direct approach would be too restrictive, since it would be meaningful only for transitive pseudogroups, and would not make full use of the Lie-Pfaffian properties.

Definition 6.1.1. Let Γ be a Lie pseudogroup over X, P an almost Γ-structure of order 1 over M (Definition 5.2.3) and θ its Cartan form (Definition 5.5.1). Consider the Spencer cohomology of (P,θ) (Definition 5.8.13), viewed as a principal Lie-Pfaffian bundle with symbol space $\mathfrak{g}$:

$$\mathcal{H}^{k-1,2}(\mathfrak{g}) := P^{(k)}[H^{k-1,2}(\mathfrak{g})].$$

Then the **intrinsic torsion of order** $k+1$ of P is the intrinsic torsion

$$T^{k+1}_{intr} \in \Gamma(M, \mathcal{T}^k(P))$$

of the principal Lie-Pfaffian bundle (P,θ). ♦

Theorem 6.1.2. *Let P be an almost Γ-structure of order l together with its Cartan form θ, and assume that*

- *the first prolongation $\mathfrak{g}(\theta)^{(1)}$ of its symbol space is of constant (non-zero) rank,*
- *$(J^l\Gamma)^{(1)} = J^{l+1}\Gamma$.*

Then P is integrable up to order 1 as an almost Γ-structure (Definition 5.2.13) if and only if its first intrinsic torsion T_{intr} vanishes.

Proof. Since (P,θ) is a principal Lie-Pfaffian bundle, we can apply Theorem 5.8.12: the intrinsic torsion T_0 vanishes if and only if (P,θ) is integrable up to order 1 according to Definition 5.8.13, i.e. the prolongation $P^{(1)}$ is smooth and the projection $P^{(1)} \to P$ is a surjective submersion.

By Theorem 5.8.5, $(P^{(1)},\theta^{(1)})$ is a principal Lie-Pfaffian $(J^1\Gamma)^{(1)}$-bundle, and the immersion theorem 5.5.2 allows us to identify $P^{(1)}$ as a submanifold of $J^{k+1}(X,M)$; the last hypothesis $(J^k\Gamma)^{(1)} = J^{k+1}\Gamma$ says that $P^{(1)}$ is an almost Γ-structure of order $k+1$ projecting on P, i.e. that P is integrable up to order $k+1$ as an almost Γ-structure. Q.E.D.

The proof above can be iterated any number of times by applying Theorem 5.8.14, and obtaining the following criterion for formal integrability.

Theorem 6.1.3. *Let P be an almost Γ-structure of order l together with its Cartan form θ, and assume that*

- *the prolongations $\mathfrak{g}(\theta)^{(i)}$ of its symbol space are of constant (non-zero) rank $\forall i = 1,\dots,k$.*

- $(J^i\Gamma)^{(1)} = J^{i+1}\Gamma$ $\forall i = 1, \ldots, k$.

Then P is integrable up to order k if and only if all the intrinsic torsions T^i_{intr} vanish for $i = 1, \ldots, k$.

In particular, if $\mathfrak{g}(\theta)^{(i)} \neq 0$ is smooth and $(J^i\Gamma)^{(1)} = J^{i+1}\Gamma$ $\forall i$, P is formally integrable if and only if all the intrinsic torsions vanish.

6.1.1 Discussion on the hypotheses of the criterion

The hypotheses of Theorem 6.1.3 are very mild. In this section, we show that they are always satisfied in all the fundamental examples.

Example 6.1.4. The condition $(J^k\Gamma)^{(1)} = J^{k+1}\Gamma$ holds for the pseudogroup Γ_G.

We prove this for $k = 1$ (for simplicity). Since both $(J^1\Gamma_G)^{(1)}$ and $J^2\Gamma_G$ are transitive Pfaffian groupoids, they are Pfaffian Morita equivalent to their isotropy group (Example 5.4.8); on the other hand, such isotropy groups

$$((J^1\Gamma_G)_0)^{(1)} = G^{(1)}, \qquad (J^2\Gamma_G)_0,$$

are isomorphic by Lemma 4.1.23, so that $(J^1\Gamma_G)^{(1)}$ and $J^2\Gamma_G$ are Pfaffian Morita equivalent by composition.

Moreover, $(J^1\Gamma_G)^{(1)}$ and $J^2\Gamma_G$ are both Lie-Pfaffian, hence they have the same dimension by Theorem 5.4.6. Their s-fibres are principal bundles over $\mathbb{R}^n$ with the isotropy as structure group; since $J^2\Gamma_G \subseteq (J^1\Gamma_G)^{(1)}$ (Section 4.4.3), such s-fibres coincide, hence the entire subgroupoid $J^2\Gamma_G$ coincides with $(J^1\Gamma_G)^{(1)}$. $\lozenge$

Example 6.1.5. The condition $(J^k\Gamma)^{(1)} = J^{k+1}\Gamma$ holds for the pseudogroup Γ_{cont}. Again, we prove this for $k = 1$. We know that $J^1\Gamma_{cont}$ is Pfaffian Morita equivalent to $J^1\Gamma_G$, for $G = G_{CSF}$ (Proposition 5.5.12): both groupoids are full and Lie-Pfaffian, hence their symbol spaces and their prolongations coincide (Theorem 5.4.6). Using the equality $(J^1\Gamma_G)^{(1)} = J^2\Gamma_G$ from Example 6.1.4, we obtain the following inclusion of symbol spaces

$$\mathfrak{g}_{J^2\Gamma_{cont}} \subseteq (\mathfrak{g}_{J^1\Gamma_{cont}})^{(1)} \cong (\mathfrak{g}_{J^1\Gamma_G})^{(1)} = \mathfrak{g}_{J^2\Gamma_G}.$$

By counting the dimensions of the fibres of $\mathfrak{g}_{J^2\Gamma_{cont}}$ and $\mathfrak{g}_{J^2\Gamma_G}$, we see that the equality must hold; in particular, by the discussion from Section 4.4.3, $J^2\Gamma_{cont} = (J^1\Gamma_{cont})^{(1)}$. $\lozenge$

Example 6.1.6. The smoothness of $\mathfrak{g}^{(i)}(\omega)$ (hence of $\mathfrak{g}^{(i)}(\theta)$, by Lemma 5.3.14) holds for the pseudogroup Γ_G; moreover, $\mathfrak{g}^{(i)}(\omega)$ are of non-zero rank if and only if G is of infinite type. 6

To prove this, it is enough to recall that, by Example 4.3.11, the symbol space $\mathfrak{g}(\omega)$ of $J^1\Gamma_G$ is the trivial vector bundle $\mathbb{R}^n \times \mathrm{Lie}(G) \to \mathbb{R}^n$. Its prolongations are then trivial vector bundles as well, i.e.

$$\mathfrak{g}^{(i)}(\omega) = (\mathbb{R}^n \times \mathrm{Lie}(G))^{(i)} = \mathbb{R}^n \times \mathrm{Lie}(G)^{(i)};$$

accordingly, $\mathfrak{g}^{(i)}(\omega)$ is always smooth. Moreover, $\mathfrak{g}^{(i)}(\omega)$ are of non-zero rank if and only if $\mathrm{Lie}(G)^{(i)} \neq 0\ \forall i$, i.e. $\mathrm{Lie}(G)$ is of infinite type. $\Diamond$

Example 6.1.7. The smoothness of $\mathfrak{g}^{(i)}(\omega)$ holds for every transitive pseudogroup Γ over X; moreover, $\mathfrak{g}^{(i)}(\omega)$ are of non-zero rank if and only if the linear isotropy group is of infinite type.

Indeed, denote by G the linear isotropy group of Γ; then the transitive groupoid $J^1\Gamma$ is Pfaffian Morita equivalent to $J^1\Gamma_G$, since they are both Pfaffian Morita equivalent to $(G, 0)$ by Example 5.4.8. Both groupoids are full and of Lie type, therefore the prolongations of their symbol spaces are isomorphic by Theorem 5.4.6. It follows that the prolongations $\mathfrak{g}^{(i)}$ of the symbol space of $J^1\Gamma$ will always be smooth, and non-zero if and only if $\mathrm{Lie}(G)$ is of infinite type. $\Diamond$

6.1.2 Special cases of the criterion

Proposition 6.1.8 (Formal integrability of almost Γ-structure of finite type)**.** *Let Γ be a Lie pseudogroup of finite type k (Definition 4.2.5) and P an almost Γ-structure. Assume that P is integrable up to order k; then*

- *P is formally integrable,*
- *All its higher prolongations $P^{(k+1)}$, $P^{(k+2)}$, etc. coincide with $P^{(k)}$.*

Proof. Consider for simplicity $k = 1$; then the symbol space $\mathfrak{g}$ of the Pfaffian groupoid $J^1\Gamma$ satisfies the condition $\mathfrak{g}^{(1)} = 0$. It follows that all the jet groupoids $J^k\Gamma$ are equal to $J^1\Gamma$. Then, all the hypotheses of Theorem 6.1.2 are fulfilled except the non-zero rank of $\mathfrak{g}^{(1)}$.

However, it is possible to see that this hypothesis is needed only in the passage from $T(P) = 0$ to the smoothness of $P^{(1)}$. Indeed, our integrability criterion is based on Theorem 1.4.5; if one looks carefully in its proof (which we wrote in [19]), the condition $\mathfrak{g}^{(1)} = 0$ turns out to be equivalent to $\ker(TP^{(1)} \to TP) = 0$, i.e. to the surjective vector bundle map $TP^{(1)} \to TP$ being also fibrewise injective. It follows that the projection $P^{(1)} \to P$ is a surjective bundle map between manifolds of the same dimensions, and its fibres are connected (being affine spaces), hence $P^{(1)} = P$.

In conclusion, in this case, the prolongation $P^{(1)}$ is trivially smooth, and moreover it stabilises at P. Q.E.D.

Another important special case of Theorem 6.1.3 is inspired by the Goldschmidt criterion for formal integrability of a PDE (Proposition 1.2.8).

Proposition 6.1.9 (Goldschmidt criterion of formal integrability for Γ-structures)**.** *Let P be an almost Γ-structure of order k on M and $\mathfrak{g}$ its symbol space. Assume that*

- *$\mathfrak{g}$ is 2-acyclic, i.e. $H^{l,2}(\mathfrak{g}) = 0\ \forall l \geq 0$ (Definition 1.2.4),*

- *the prolongation $\mathfrak{g}^{(1)}$ is smooth,*
- *the intrinsic torsion T_{intr} of P vanishes.*

Then P is formally integrable.

Proof. We apply Lemma 1.2.7 to the symbol space of P, obtaining the smoothness of all prolongations $\mathfrak{g}^{(i)}$. Since each intrinsic torsion T^{k+1}_{intr} takes values in $H^{k-1,2}(\mathfrak{g})$, 2-acyclicity hypothesis means that $T^{k+1}_{intr} = 0\ \forall k \geq 1$; together with $T_{intr} = T^1_{intr} = 0$, we obtain the formal integrability. Q.E.D.

6.1.3 Formal integrability of almost Γ_G-structures

We have already proved in Examples 6.1.4 and 6.1.6 that the Lie pseudogroup $\Gamma = \Gamma_G$ satisfies the hypotheses of our formal integrability criterion. In view of the correspondence between almost Γ_G-structures and G-structures from Example 5.2.5, we recover now the intrinsic torsions discussed in Sections 2.4.1 and 2.5.3.

Proposition 6.1.10. *Let $G \subseteq GL(n, \mathbb{R})$ be a Lie subgroup and Γ_G the associated Lie pseudogroup (Example 4.1.5). If P is an almost Γ_G-structure on M, and P_0 the correspondent G-structure on M by Example 5.2.5, then the intrinsic torsions 2.4.2 and 6.1.1 coincide up to a sign.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. By Example 4.3.11, we know that the symbol space of Γ_G is the trivial bundle

$$\mathfrak{g}_{\mathbb{R}^n} := \mathbb{R}^n \times \mathfrak{g} \to \mathbb{R}^n.$$

Accordingly, the Spencer cohomology $H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n})$ of $\mathfrak{g}_{\mathbb{R}^n}$, viewed as a tableau bundle, has as fibre over 0 the Spencer cohomology $H^{-1,2}(\mathfrak{g})$, viewed as a tableau. Since $H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n})$ is a representation of $\mathcal{G}$, we apply Lemma 3.1.19 to see that the bundle $\mathcal{H}^{-1,2}(\mathfrak{g}_{\mathbb{R}^n})$ is isomorphic to $H^{-1,2}(\mathfrak{g})$.

Recall now that the intrinsic torsion of P is induced by the torsion

$$T : P \to H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n}), \quad p \mapsto [\zeta^* \kappa_\theta]$$

By the Koszul formula, one sees that

$$\kappa_\theta(v, w) = \theta([V, W]) = -d\theta(V, W) + V(\cancel{\theta(W)}) - W(\cancel{\theta(V)}) = -T(p)(v, w).$$

Accordingly, under the isomorphism $H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n}) \cong \mathbb{R}^n \times H^{-1,2}(\mathfrak{g})$, we interpret T as

$$T : P \to H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n}), \quad p \mapsto (\mu(p), -T(p))$$

where $T : P_0 \to H^{-1,2}(\mathfrak{g})$ is the torsion of the G-structure P_0.

By the transitivity of $\mathcal{G}$, in any class $[p] \in P/\mathcal{G} = M$ there is a representative $p \in P_0$; then the intrinsic torsion

$$T_{intr}(P) : M \to P[H^{-1,2}(\mathfrak{g}_{\mathbb{R}^n})], \quad [p] \mapsto [p, T(p)]$$

can be rewritten as

$$M \to P_0[H^{-1,2}(\mathfrak{g})], \quad [p] \mapsto [p, \mu(p), -T(p)] = -[p, 0, T(p)],$$

hence it coincides with $-T$. Q.E.D.

6.2 Integrability of almost Γ-structures

In this section we are going to tackle the integrability problem of an almost Γ-structure with a different approach. In principle, this could lead to stronger results (under stronger assumptions, of course), reaching actual integrability and not just the formal one as in the previous section.

As first step, we recast the integrability problem into the problem of finding enough holonomic sections.

Proposition 6.2.1. *Let Γ be a Lie pseudogroup of order 1, $P \subseteq J^1(X, M)$ a 1^{st}-order almost Γ-structure and θ its Cartan form:*

$$\begin{tikzcd}
 & (P,\theta) \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \curvearrowleft (J^1\Gamma,\omega) \arrow[d, shift left] \arrow[d, shift right] \\
M \arrow[ur, dashed, bend right, "\sigma"'] & & X \arrow[ul, dashed, bend left, "\rho"]
\end{tikzcd}$$

Then the following are equivalent:

1. *P is integrable as an almost Γ-structure (Definition 5.2.3).*
2. *for every $p \in P$ there is a local holonomic section ρ of μ passing through p.*
3. *for every $p \in P$ there is a local holonomic section σ of π passing through p.*
4. *$\pi : (P, \theta) \to M$ is integrable as a Pfaffian fibration (Definition 1.3.6).*

Proof. If P is integrable to a Γ-atlas $\mathcal{A}$, i.e. $P = J^1\mathcal{A}$, then every element $p \in P$ can be expressed as $j^1_x\phi = (j^1\phi)(x)$, with $\phi \in \mathcal{A}$ hence there is a local holonomic section $\rho = j^1\phi$ of μ such that $\rho(x) = p$.

Conversely, assume 2 and consider a local holonomic section $\rho \in \Gamma_{\mathrm{loc}}(\mu)$. Then, for any $x \in \mathrm{dom}(\rho)$, the map $d_x(\pi \circ \rho)$ restricts to a surjective map between vector spaces of the same dimensions, i.e. to a linear isomorphism. Accordingly, $\pi \circ \rho$ can be restricted to a diffeomorphism f_ρ on a sufficiently small neighbourhood of a non-critical point.

Then, the collections $\{V_\rho := \mathrm{Im}(f_\rho)\}_\rho$ determines an open cover of M because of the surjectivity of π: every $y \in M$ is $\pi(p)$ for some $p \in P$, for which there exists a ρ by hypothesis, so $y = \pi(p) = \pi(\rho(x)) = f_\rho(x) \in \mathrm{Im}(f_\rho) = V_\rho$.

Moreover, we claim that

$$\mathcal{A} := \{(V_\rho, f_\rho) \mid \rho \in \Gamma_{\mathrm{loc}}(\mu)\}$$

is a Γ-atlas on M modelled on X. We have just to check that the changes of coordinates belong to Γ. For any local holonomic sections ρ_1 and ρ_2, consider $y \in V_{\rho_1} \cap V_{\rho_2} \subseteq M$: by construction there are two points $x_i \in U_i := \mathrm{dom}(f_{\rho_i}) \subseteq X$ such that $f_{\rho_i} : U_i \to V_{\rho_i}$ sends x_i to y. It follows that the 1-jets of f_{ρ_1} and f_{ρ_2} belong to the same equivalence class of elements in P modulo the $J^1\Gamma$-action, i.e.

$$y = [j^1_{x_1} f_{\rho_1}] = [j^1_{x_2} f_{\rho_2}].$$

In particular, there is an element $j^1_{x_1}\psi \in J^1\Gamma$, with $\psi \in \Gamma$ sending x_1 to x_2, such that

$$j^1_{x_1} f_{\rho_1} = j^1_{x_2} f_{\rho_2} \cdot j^1_{x_1}\psi = j^1_{x_1}(f_{\rho_2} \circ \psi);$$

this implies therefore $j^1_{x_1}(f^{-1}_{\rho_2} \circ f_{\rho_1}) = j^1_{x_1}\psi \in J^1\Gamma$.

In conclusion, the generic change of coordinates $(f^{-1}_{\rho_2} \circ f_{\rho_1})_{|f^{-1}_{\rho_1}(V_{\rho_1} \cap V_{\rho_2})}$ satisfies

$$j^1_x(f^{-1}_{\rho_2} \circ f_{\rho_1}) \in J^1\Gamma \quad \forall x \in f^{-1}_{\rho_1}(V_{\rho_1} \cap V_{\rho_2});$$

since Γ is a pseudogroup of order 1, then $(f^{-1}_{\rho_2} \circ f_{\rho_1})_{|f^{-1}_{\rho_1}(V_{\rho_1} \cap V_{\rho_2})}$ belongs to Γ and $\mathcal{A}$ is a Γ-atlas.

Finally, we prove that $P = J^1\mathcal{A}$. One the one hand, $P \subseteq J^1\mathcal{A}$ because every $p = \rho(x)$ is by construction $j^1_x f_\rho$. Conversely, every element $j^1_x f_\rho \in J^1\mathcal{A}$ defines a $\rho(x) \in P \subseteq J^1(X, M)$; in conclusion, $P = J^1\mathcal{A}$ is integrable.

It is easy to prove that 2 and 3 are equivalent. Indeed, for any local holonomic section $\rho \in \Gamma_{\mathrm{loc}}(\pi)$ passing through some $p \in P$, we already described a diffeomorphism f_ρ. Then $\sigma := \rho \circ (f_\rho)^{-1}$ defines a local holonomic section of μ passing through p.

Last, by definition of integrability of a Pfaffian fibration, 3 and 4 coincide.

Q.E.D.

6.2.1 Twisting sections by a bisection

Proposition 6.2.1 provides an integrability criterion for almost Γ-structures: every point of P should be in the image of a local holonomic section of π. Since, given $p \in P$, it is always possible to find a section σ such that $\sigma(x_0) = p$, for some $x_0 \in M$, the actual problem is to make σ into a holonomic section. One way to look at this problem is to "twist" such a (possibly non-holonomic) section σ by a local bisection of $J^1\Gamma$ in order to produce a holonomic one.

Lemma 6.2.2. *Let $P \subseteq J^1(X, M)$ be an almost Γ-structure as in the following diagram:*

$$\begin{tikzcd} & (P,\theta) \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \curvearrowleft (J^1\Gamma,\omega) \arrow[d, shift left] \arrow[d, shift right] \\ M & & X \end{tikzcd}$$

For any σ local section of π, define

$$\widetilde{\sigma} := \sigma \circ (\phi_\sigma)^{-1} \in \Gamma_{\mathrm{loc}}(\mu).$$

where $\phi_\sigma \in \mathrm{Diff}_{\mathrm{loc}}(M, X)$ is a suitable restriction of $\mu \circ \sigma$. Furthermore, let $b \in \mathrm{Bis}_{\mathrm{loc}}(J^1\Gamma)$ and $p \in P$ such that $b(\mu(p)) = 1_{\mu(p)}$, and define

$$\phi_{\sigma,b} : \mathrm{dom}(\sigma) \to P, \quad \phi_{\sigma,b}(x) := \sigma(x) \cdot (b(\phi_\sigma(x)))^{-1} = m_P \circ (\sigma, \overline{b} \circ \phi_\sigma)(x),$$

where we denote $\overline{b} := i \circ b$ (a section of the target map).

$$\begin{tikzcd} & (P,\theta) \arrow[dl, "\pi"'] \arrow[dr, "\mu"] & \curvearrowleft (J^1\Gamma,\omega) \arrow[d, shift left] \arrow[d, shift right] \\ M \arrow[ur, dashed, bend left=60, "\phi_{\sigma,b}"] \arrow[ur, dashed, bend right, "\sigma"'] \arrow[rr, dashed, bend right=40, "\phi_\sigma"'] & & X \arrow[ul, dashed, bend left, "\widetilde{\sigma}"] \arrow[u, dashed, bend right=60, "b"'] \end{tikzcd}$$

Then

- *$\phi_{\sigma,b}$ is a local section of π.*
- *If σ passes through p, i.e. $\sigma(x_0) = p$ for some $x_0 \in M$, then $\phi_{\sigma,b}$ passes through p as well.*
- *$\phi_{\sigma,b}$ is holonomic if and only if*

$$\widetilde{\sigma}^*\theta + \overline{b}^*\omega = 0.$$

Proof. The map $\phi_{\sigma,b}$ is a local section of π since σ is so and π is $J^1\Gamma$-invariant. Moreover, $\phi_{\sigma,b}$ passes through p if σ does so:

$$\phi_{\sigma,b}(x_0) = \sigma(x_0) \cdot \overline{b}(\mu(\sigma(x_0))) = p \cdot \overline{b}(\mu(p)) = p \cdot (1_{\mu(p)})^{-1} = p.$$

Last, using the multiplicativity of the $J^1\Gamma$-action m_P, we compute

$$(\phi_{\sigma,b})^*\theta = (\sigma, \overline{b} \circ \phi_\sigma)^*(m_P^*\theta) = (\sigma, \overline{b} \circ \phi_\sigma)^*(\mathrm{pr}_1^*\,\theta + \mathrm{pr}_2^*\,\omega) =$$

$$= \sigma^*\theta + (\overline{b} \circ \phi_\sigma)^*\omega = (\phi_\sigma)^*(\widetilde{\sigma}^*\theta) + (\phi_\sigma)^*\overline{b}^*\omega = (\phi_\sigma)^*(\widetilde{\sigma}^*\theta + \overline{b}^*\omega).$$

Since ϕ_σ is a diffeomorphism, the holonomicity of $\phi_{\sigma,b}$ is equivalent to $\widetilde{\sigma}^*\theta + \overline{b}^*\omega = 0$. Q.E.D.

Note that, if b is already holonomic, then $\phi_{\sigma,b}$ is holonomic if and only if σ is holonomic. Hence, to achieve integrability, we are left with finding, for every p and any σ, a suitable (non-holonomic) bisection b which makes $\phi_{\sigma,b}$ holonomic.

6.2.2 The non-linear Spencer complex

Building on the previous discussion, we are going to rephrase the integrability problem in a more refined way. From now on, by $\Omega^k_{\mathrm{loc}}(X,F)$ we mean the space of differential k-forms with coefficients in F defined on an open set of X.

Proposition 6.2.3. *Let $(\mathcal{G},\omega)$ be a full Lie-Pfaffian groupoid over X, with $\omega \in \Omega^1(\mathcal{G}, t^*E)$ and $A = \mathrm{Lie}(\mathcal{G})$. Assume moreover that $(\mathcal{G},\omega)$ is integrable up to order 1 (Definition 4.4.9), denote by $(\mathcal{G}^{(1)},\omega^{(1)})$ its prolongation and consider the map*

$$F : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \to \Omega^1_{\mathrm{loc}}(X,A), \quad b \mapsto -\overline{b}^*\omega^{(1)}$$

and the Maurer-Cartan operator from Definition 3.4.27

$$MC : \Omega^1_{\mathrm{loc}}(X,A) \to \Omega^2_{\mathrm{loc}}(X,E) \quad \eta \mapsto d_D\eta - \frac{1}{2}\{\eta,\eta\}_D.$$

Then the sequence

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \xrightarrow{F} \Omega^1_{\mathrm{loc}}(X,A) \xrightarrow{MC} \Omega^2_{\mathrm{loc}}(X,E) \tag{$*$}$$

is a "cochain complex", in the sense that $MC \circ F = 0$.

We call the sequence $(*)$ the **(non-linear) Spencer complex** of $(\mathcal{G},\omega)$.

Proof. For any $b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)})$, it is enough to compute

$$(MC \circ F)(b) = MC(-\overline{b}^*\omega^{(1)}) = -\overline{b}^*\cancel{MC(\omega^{(1)})} = 0,$$

where in the second step we used the naturality of MC with respect to any section of $t : \mathcal{G}^{(1)} \to X$, and in the third we used $MC(\omega^{(1)}) = 0$ (Remark 4.4.8). Q.E.D.

Now we prove how the exactness of the Spencer complex is related to the integrability problem. By "exactness" we mean the following: for every $\eta \in \Omega^1_{\mathrm{loc}}(X,A)$ such that $MC(\eta) = 0$, there exists a $b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)})$ such that $F(b) = \eta$.

Theorem 6.2.4. *Let Γ be a Lie pseudogroup on X and assume that the Spencer complex $(*)$ associated to the Pfaffian groupoid $J^1\Gamma$ is exact. Then, if an almost Γ-structure P over M is integrable up to order 1, it is integrable.*

Proof. For any section σ of $\pi^{(1)} : P^{(1)} \to M$, we consider $\psi = \widetilde{\sigma}^*\theta^{(1)} \in \Omega^1_{\mathrm{loc}}(X,A)$, where $\widetilde{\sigma}$ denotes the corresponding section of $\mu^{(1)} : P \to X$ from Lemma 6.2.2.

Using Proposition 5.8.15 and the same arguments of Proposition 6.2.3, we show that MC sends ψ to 0:

$$MC(\psi) = MC(\widetilde{\sigma}^*\theta^{(1)}) = \widetilde{\sigma}^*\cancel{MC(\theta^{(1)})} = 0.$$

Then, by the exactness of the sequence, $\psi = F(b)$ for some bisection $b \in \mathrm{Bis}(\mathcal{G}^{(1)})$, i.e.

$$\widetilde{\sigma}^*\theta^{(1)} = -\overline{b}^*\omega^{(1)}.$$

By Lemma 6.2.2, this implies that the section $\phi_{\sigma,b}$ of $P^{(1)} \to M$ is holonomic.

For any $\bar{p} \in P$, we consider a preimage $p \in P^{(1)}$ via the surjective map $\pi_0^1 : P^{(1)} \to P$. By the discussion above, there is a holonomic section $\phi_{\sigma,b}$ passing through p. Then $\overline{\phi_{\sigma,b}} = \pi_0^1 \circ \phi_{\sigma,b}$ is by construction a holonomic section of $\pi : P \to M$ passing through $\bar{p}$, therefore P is integrable by Proposition 6.2.1. Q.E.D.

The main problem becomes then to understand when the Spencer complex $(*)$

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \xrightarrow{F} \Omega^1_{\mathrm{loc}}(X, A) \xrightarrow{MC} \Omega^2_{\mathrm{loc}}(X, E),$$

is exact. One way to tackle it is to describe its linearisation around the unit bisection $u : X \to \mathcal{G}^{(1)}, x \mapsto 1_x$ (see next paragraph):

$$\Gamma_{\mathrm{loc}}(A^{(1)}) = \Omega^0_{\mathrm{loc}}(X, A^{(1)}) \xrightarrow{d_u F} \Omega^1_{\mathrm{loc}}(X, A) \xrightarrow{d_{F(u)} MC} \Omega^2_{\mathrm{loc}}(X, E),$$

where $A^{(1)}$ denotes the Lie algebroid of $\mathcal{G}^{(1)}$.

Ideally, if the linearised complex is exact, we would like the non-linear complex to be exact too. To make this precise, one needs to introduce some analysis in the picture; this goes beyond the scope of this thesis and will be subject of a future paper.

Our plan is to apply some recent results on PDEs with symmetries ([97, 70]), in order to produce an explicit obstruction to integrability, on the lines of Theorem 6.1.3. While formal integrability was equivalent to the vanishing of the intrinsic torsions T_i, for integrability we expect a stronger condition, involving the theory of Fréchet spaces and the notion of tameness developed by Hamilton [48] precisely for this kind of problems.

6.2.3 The linearised Spencer complex

Proposition 6.2.5. *In the setting of Proposition 6.2.3, the linearisation of the complex* $(*)$

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \xrightarrow{F} \Omega^1_{\mathrm{loc}}(X, A) \xrightarrow{MC} \Omega^2_{\mathrm{loc}}(X, E)$$

around the global bisection $u \in \mathrm{Bis}(\mathcal{G}^{(1)})$, $u : x \mapsto 1_x$, is the complex

$$\Omega^0_{\mathrm{loc}}(X, A^{(1)}) \xrightarrow{f} \Omega^1_{\mathrm{loc}}(X, A) \xrightarrow{g} \Omega^2_{\mathrm{loc}}(X, E),$$

with

$$f(\alpha) = d_{D^{\omega^{(1)}}}(\alpha), \quad g(\eta) = d_{D^\omega}(\eta)$$

Proof. First, we show that

$$T_u \operatorname{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) = \Omega^0_{\mathrm{loc}}(X, A^{(1)}),$$

where $A^{(1)} = \operatorname{Lie}(\mathcal{G}^{(1)})$. This follows from interpreting any section $\alpha \in \Omega^0_{\mathrm{loc}}(X, A^{(1)}) = \Gamma(A^{(1)} \to X)$ as the speed at $\varepsilon = 0$ of its flow (see Section 3.2)

$$\phi^\varepsilon_\alpha : x \mapsto \phi^\varepsilon_{\alpha^R}(1_x).$$

Since ϕ^ε_α is a bisection of $\mathcal{G}^{(1)}$ and $\phi^0_\alpha = u$, then α can indeed be viewed as a tangent vector of $\operatorname{Bis}(\mathcal{G}^{(1)})$ at the unit u. Conversely, any vector

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \varphi^\varepsilon \in T_u \operatorname{Bis}(G^{(1)})$$

defines the following section of $A^{(1)} \to X$:

$$x \mapsto \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \varphi^\varepsilon(x) \in A^{(1)}_x.$$

The other two objects of the complex, $\Omega^1_{\mathrm{loc}}(X, A)$ and $\Omega^2_{\mathrm{loc}}(X, E)$, are already (infinite-dimensional) vector spaces; then their tangent space (at $F(u) = 0$ and at $G(F(u)) = 0$) coincide with themselves.

Let us linearise the maps. Since

$$F : b \mapsto -\overline{b}^* \omega^{(1)},$$

we get

$$\begin{aligned} d_u F(\alpha) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(\phi^\varepsilon_\alpha) = -\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\phi^{-\varepsilon}_\alpha)^* \omega^{(1)} = \\ &= -\omega^{(1)} \left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} d\phi^{-\varepsilon}_\alpha \right) = D^{\omega^{(1)}}_\bullet(\alpha) = d_{D^{\omega^{(1)}}}(\alpha) \end{aligned}$$

where in the penultimate step we used the computations in Lemma 2.5.2 of [100].

Last, the linearisation of

$$MC : \eta \mapsto MC(\eta) = d_D \eta - \{\eta, \eta\}$$

is

$$\begin{aligned} d_0 MC(\eta) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} MC(\varepsilon\eta) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Big(d_D(\varepsilon\eta) - \{\varepsilon\eta, \varepsilon\eta\} \Big) = \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Big(\varepsilon d_D \eta - \varepsilon^2 \{\eta, \eta\} \Big) = d_D \eta. \end{aligned}$$

Q.E.D.

This linearised sequence is of course a complex since the non-linear one was. This can be also checked directly: for any $\alpha \in \Gamma(A)$, $g(f(\alpha)) = 0$ is equivalent to

$$D^\omega_V(D^{\omega^{(1)}}_W(\alpha)) - D^\omega_W(D^{\omega^{(1)}}_V(\alpha)) - l^\omega(D^{\omega^{(1)}}_{[V,W]}(\alpha)) = 0 \quad \forall V, W \in \mathfrak{X}(X),$$

and this is indeed true by the second part of Proposition 6.2.46 of [81].

6.2.4 The geometry of the non-linear Spencer complex

We study now some properties of the maps composing the non-linear Spencer complex $(*)$. First of all, let us investigate where the map F takes values.

Proposition 6.2.6. *In the setting of Proposition 6.2.3, the image of the map*

$$F : \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \to \Omega^1_{\mathrm{loc}}(X, A), \quad b \mapsto -\bar{b}^* \omega^{(1)}$$

is contained into

$$\{\beta \in \Omega^1_{\mathrm{loc}}(X, A) \mid \forall x \in X, id - \rho \circ \beta_x : T_xX \to T_xX \text{ is invertible}\}.$$

Proof. For any bisection b of $\mathcal{G}^{(1)}$, consider $\beta = \bar{b}^* \omega^{(1)} \in \Omega^1_{\mathrm{loc}}(X, A)$. Note first that, for any $W \in T\mathcal{G}^{(1)}$, $\widetilde{W} \in \ker(\omega^{(1)})$ such that $ds^{(1)}(W) = ds^{(1)}(\widetilde{W})$, the definition of $\omega^{(1)}$ yields

$$\omega^{(1)}(W) = \omega^{(1)}(W - \widetilde{W}) = d\pi^1_0(W - \widetilde{W}),$$

For any $V \in TX$, $W = d\bar{b}(V) \in T\mathcal{G}^{(1)}$, we have therefore

$$\rho(\beta(V)) = \rho(\omega^{(1)}(d\bar{b}(V))) = \rho(\omega^{(1)}(W)) = \rho(d\pi^1_0(W - \widetilde{W})) =$$

$$= dt \circ d\pi^1_0 \circ d\bar{b}(V) - dt \circ d\pi^1_0(\widetilde{W}) = d(t^{(1)} \circ \bar{b})(V) - dt^{(1)}(\widetilde{W}) = V - dt^{(1)}(\widetilde{W}).$$

Since $(\mathcal{G}^{(1)}, \omega^{(1)})$ is Lie-Pfaffian, we conclude that

$$(id_{TX} - \rho \circ \beta)(V) = dt^{(1)}(\widetilde{W}) = \lambda \circ d(s \circ \bar{b})(V),$$

where λ is the representation of $\mathcal{G}^{(1)}$ on TX described in Remark 4.3.5; here we used the fact that $\widetilde{W} \in \ker(\omega^{(1)})$ and $ds^{(1)}(\widetilde{W}) = ds^{(1)}(W) = d(s^{(1)} \circ \bar{b})(V)$.

Accordingly, since b is a bisection,

$$id_{TX} - \rho \circ \beta = \lambda \circ d(s \circ \bar{b})$$

is the composition of two isomorphisms, hence it is invertible. Q.E.D.

The space we have described in the previous proposition may seem quite arbitrary. We are going now to see that it is actually a very natural space to consider, and it is endowed with a rich algebraic structure.

Proposition 6.2.7. *In the setting of Proposition 6.2.3, the open subbundle*

$$\mathcal{L} := \{\beta_x : T_xX \to A_x \mid id_{T_xX} - \rho \circ \beta_x \text{ is invertible}\} \subseteq \mathrm{Hom}(TX, A).$$

is in bijection with the subbundle

$$\mathcal{K} := \ker(J^1\mathcal{G} \to \mathcal{G}) \subseteq J^1\mathcal{G},$$

where we recall that the kernel of a morphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ between Lie groupoids over X is the bundle of Lie groups $\ker(\Phi) \subseteq \mathcal{G}_1$ *consisting of elements $g \in \mathcal{G}_x$ such that $\Phi(g) = 1_x$.*

In particular, it follows by Proposition 6.2.6 that F takes values in $\Gamma_{\mathrm{loc}}(\mathcal{L})$.

Proof. For every $x \in X$, consider the canonical splitting

$$T_{1_x}\mathcal{G} \cong du(T_x X) \oplus A_x.$$

Accordingly, any element $(1_x, \xi) \in \mathcal{K}$ can be described by a map $\xi : T_x X \to du(T_x X) \oplus A_x$. Since ξ is a section of ds, it is of the form

$$\xi : T_x X \to du(T_x X) \oplus A_x, \quad V \mapsto (du(V), -\beta_x(V)),$$

for some $\beta_x : T_x X \to A_x$. Moreover, since $dt \circ \xi$ is an isomorphism, and

$$dt(\xi(V)) = dt(du(V), \beta_x(V)) = V - \rho(\beta_x(V)) = (id - \rho\beta_x)(V),$$

we conclude that $id_{T_x X} - \rho \circ \beta_x$ is invertible. Q.E.D.

Note that $\mathcal{K}$ is a bundle of non-abelian groups over $\mathcal{G}$, and such group structure is transported on the set of its bisections $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{K})$. Analogously, $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{K})$ has two natural actions on $\Omega^2_{\mathrm{loc}}(X, E)$. All these algebraic structures are naturally induced on $\Gamma_{\mathrm{loc}}(\mathcal{L})$, and can also be described more explicitly:

Lemma 6.2.8. *In the setting of Proposition 6.2.3, the group structure on $\Gamma_{\mathrm{loc}}(\mathcal{L}) \subseteq \Omega^1_{\mathrm{loc}}(X, A)$ induced by $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{K})$ can be given explicitly by*

$$\beta * \gamma := \beta + \gamma(id_{TX} - \rho \circ \beta) \quad \forall \beta, \gamma \in \Gamma_{\mathrm{loc}}(\mathcal{L}).$$

Similarly, the action of the group $\Gamma_{\mathrm{loc}}(\mathcal{L})$ on $\Omega^2_{\mathrm{loc}}(X, E)$, both from the left and from the right, can be described as follows: if $\beta \in \Gamma_{\mathrm{loc}}(\mathcal{L})$ and $\zeta \in \Omega^2_{\mathrm{loc}}(X, E)$, then

$$\beta \cdot \zeta := \zeta \circ (id_{TX} - \rho \circ \beta), \qquad \zeta \cdot \beta = (id_E - l \circ \beta \circ \rho_E) \circ \zeta.$$

Proof. The multiplication on $\mathcal{K}$ is induced by that of the isotropy group of $J^1\mathcal{G}$, which in turns comes from the multiplication of bisections:

$$j^1_x \sigma_1 \cdot j^1_x \sigma_2 := j^1_x(\sigma_1 \cdot \sigma_2),$$

with

$$(\sigma_1 \cdot \sigma_2)(x) := m(\sigma_2 \circ (t \circ \sigma_1), \sigma_1).$$

Under the bijection from Proposition 6.2.7, we identify $d_x\sigma_i$ with β_i, so that

$$\beta_1 * \beta_2 = d_x(\sigma_1 \cdot \sigma_2) = dm(d\sigma_2 \circ dt(d\sigma_1), d\sigma_1) =$$

$$= dm(\beta_2 \circ (id - \rho\beta_1), \beta_1) = \beta_2 \circ (id_{TX} - \rho \circ \beta_1) + \beta_1,$$

where we used the standard fact that the differential of a Lie group multiplication is the addition on the tangent space.

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It follows that this induced operation on $\mathcal{L}$ is associative and 0 is the neutral element. Moreover, with similar computations, using

$$(j^1_x\sigma)^{-1} := j^1_{\phi_\sigma(x)}(\sigma^{-1}), \quad \sigma^{-1}(y) := i(\sigma(t\circ\sigma)^{-1}(y)),$$

we obtain

$$\beta^{-1} = di(d\sigma \circ d(t\circ\sigma)^{-1}) = di(\beta(id-\rho\beta)^{-1}) = -\beta(id-\rho\beta)^{-1}.$$

The same argument apply for the second part, starting with the following actions of $\mathcal{K}$ on $\Omega^2_{\mathrm{loc}}(X,E)$:

$$j^1_x\sigma \cdot \zeta := \zeta \circ d(t\circ\sigma), \qquad \zeta \cdot j^1_x\sigma := l \circ d(\sigma\circ t)\circ d\sigma \circ \rho_E \circ \zeta. \qquad \text{Q.E.D.}$$

Lemma 6.2.9. *In the setting of Proposition 6.2.3, the map F is a group morphism, i.e.*

$$F(b_1\cdot b_2) = F(b_1) * F(b_2) \qquad \forall b_1, b_2 \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}).$$

In particular, if J denotes the inversion of the group $\Gamma(\mathcal{L})$,

$$F(b^{-1}) = J(F(b)) \qquad \forall b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}).$$

Proof. Since the product of two bisections b_1, b_2 is

$$(b_1\cdot b_2)(x) = b_2(\phi_{b_1}(x))\cdot b_1(x) = m\circ(b_2\circ\phi_{b_1}, b_1)(x),$$

we compute

$$F(b_1\cdot b_2) = -\overline{b_1\cdot b_2}^*\omega^{(1)} = -(i\circ m\circ(b_2\circ\phi_{b_1}, b_1))^*\omega^{(1)} =$$
$$= -(b_2\circ\phi_{b_1}, b_1)^* m^*(i^*\omega^{(1)}) = -h^{-1}\cdot(b_2\circ\phi_{b_1})^*(i^*\omega^{(1)}) - b_1^*(i^*\omega^{(1)}) =$$
$$= -h^{-1}\cdot\phi_{b_1}^*(i\circ b_2)^*\omega^{(1)} - (i\circ b_1)^*\omega^{(1)} = h^{-1}\cdot\phi_{b_1}^*F(b_2) + F(b_1),$$

where in the second line we use the fact that $i^*\omega^{(1)}$ is the pullback of a multiplicative form, hence is multiplicative (Example 3.4.3), and $h = b_1(x)$ (if we compute the entire expression in x). On the other hand,

$$F(b_1) * F(b_2) = F(b_1) + F(b_2) - F(b_2)\rho F(b_1),$$

so we are left to prove that

$$F(b_2)\rho F(b_1) = F(b_2) - h^{-1}\cdot\phi_{b_1}^*F(b_2).$$

Fix $x\in X$ and $V\in T_xX$; if $b_1(x) = j^1_x\sigma_x$, for some bisection $\sigma_x\in\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$, then we use the definition of the Cartan form of $J^1\mathcal{G}$ to obtain

$$F(\beta_2)\rho F(b_1)_x(V) = -\omega^{(1)}_{\overline{b_2(x)}}\circ d_x\overline{b_2}\circ d_{1_x}t\circ(-\omega^{(1)}_{\overline{b_1(x)}})\circ d_x\overline{b_1}(V) =$$

$$= \omega^{(1)}_{\overline{b_2}(x)} \circ d_x\overline{b_2} \circ d_{1_x}t\Big(d_x(i\circ\sigma_x)(V) - d_x(\sigma_x^{-1}\circ t\circ b_1)(V)\Big) =$$

$$= \omega^{(1)}_{\overline{b_2}(x)}\Big(d_x\overline{b_2}\circ d(t\circ i)\circ d_x\sigma_x(V)\Big) - \omega^{(1)}_{\overline{b_2}(x)}\Big(d_x\overline{b_2}\circ d_{1_x}t\circ d\sigma_x^{-1}\circ d_x\phi_{b_1}(V)\Big) =$$

$$= \omega^{(1)}_{\overline{b_2}(x)}\Big(d_x\overline{b_2}\circ d_x(s\circ\sigma_x)(V)\Big) - \omega^{(1)}_{\overline{b_2}(x)}\Big(d_x\overline{b_2}\circ d(t\circ\sigma_x^{-1})\circ d_x\phi_{b_1}(V)\Big) =$$

$$= \omega^{(1)}_{\overline{b_2}(x)}(d_x\overline{b_2}(V)) - h^{-1}\cdot\omega^{(1)}_{\overline{b_2}(\phi_{b_1}(x))}(d_{\phi_{b_1}(x)}\overline{b_2}\circ d_x\phi_{b_1}(V)) =$$

$$= (\overline{b_2}^*\omega^{(1)})_x(V) - h^{-1}\cdot(\overline{b_2}^*\omega^{(1)})_{\phi_{b_1}(x)}(d_x\phi_{b_1}(V)) =$$

$$= F(b_2)_x(V) - h^{-1}\cdot\phi_{b_1}^* F(b_2)_x(V). \qquad \text{Q.E.D.}$$

6.2.5 An alternative non-linear Spencer complex

In this section we will investigate a slight modification of the non-linear Spencer complex $(*)$. The overall aim will be to work with a sequence whose objects have a richer structure - namely, they are DGLAs.

We will replace the Maurer-Cartan operator MC with the following map

$$G : \Omega^1_{\text{loc}}(X,A) \to \Omega^2_{\text{loc}}(X,E), \quad \psi \mapsto d_D\psi - \frac{1}{2}l\circ[[\psi,\psi]].$$

Here d_D is still the operator described in Definition 3.4.27 and $l : A \to E$ is the Lie algebroid map described in Proposition 4.3.6. The bracket

$$[[\cdot,\cdot]] : \Omega^1_{\text{loc}}(X,A)\times\Omega^1_{\text{loc}}(X,A) \to \Omega^2_{\text{loc}}(X,A)$$

is part of a graded Lie algebra structure on $\Omega^*_{\text{loc}}(X,A)$ that will be discussed in detail elsewhere [82]. Here we just use the expression in degree 1:

$$[[\beta,\gamma]](V,W) := [\beta(V),\gamma(W)] + [\gamma(V),\beta(W)] +$$

$$+\beta\rho_E\Big(D_V(\gamma(W)) - D_W(\gamma(V))\Big) - \beta\Big([\rho(\gamma(V)),W] - [\rho(\gamma(W)),V]\Big) +$$

$$+\gamma\rho_E\Big(D_V(\beta(W)) - D_W(\beta(V))\Big) - \gamma\Big([\rho(\beta(V)),W] - [\rho(\beta(W)),V]\Big),$$

where (D,l) is the Spencer operator associated to $\omega\in\Omega^1(\mathcal{G},t^*E)$ (Proposition 3.4.22) and $\rho_E : E\to TM$ the anchor map ρ of E (Proposition 4.3.6).

We are going to show that G and MC are more similar than they look.

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Proposition 6.2.10. *In the setting of Proposition 6.2.3, for any $\beta\in\Gamma_{\text{loc}}(\mathcal{L})$,*

$$\beta\cdot G(J(\beta)) = -MC(\beta)\cdot J(\beta).$$

Here J denotes the inversion of the group $\Gamma_{\text{loc}}(\mathcal{L})$ and $\cdot$ the two $\Gamma_{\text{loc}}(\mathcal{L})$-actions on $\Omega^2_{\text{loc}}(X,E)$ (see Lemma 6.2.8).

Proof. For any $V, W \in \mathfrak{X}(X)$, denote

$$\widetilde{V} = (id - \rho \circ \beta)(V), \qquad \widetilde{W} = (id - \rho \circ \beta)(W),$$

so that $J(\beta)(\widetilde{V}) = -\beta(V)$ and $J(\beta)(\widetilde{W}) = -\beta(V)$. We can compute

$$\beta \cdot G(J(\beta))(V, W) = G(J(\beta))(\widetilde{V}, \widetilde{W}) =$$

$$= D_{\widetilde{V}}(-\beta(W)) - D_{\widetilde{W}}(-\beta(V)) - l(J(\beta)[\widetilde{V}, \widetilde{W}]) - l([[J(\beta), J(\beta)]](\widetilde{V}, \widetilde{W}))$$

We collect now the terms after $-l$ and use the definition of $[[\cdot, \cdot]]$:

$$J(\beta)[\widetilde{V}, \widetilde{W}] + [[J(\beta), J(\beta)]](\widetilde{V}, \widetilde{W}) =$$

$$= J(\beta)\Big([V, W] - \cancel{[V, \rho(\beta(W))]} - \cancel{[\rho(\beta(V)), W]} + \cancel{[\rho(\beta(V)), \rho(\beta(W))]}\Big) +$$

$$+[-\beta(V), -\beta(W)] + J(\beta)\rho_E\Big(D_{\widetilde{V}}(-\beta(W)) - D_{\widetilde{W}}(-\beta(V))\Big)+$$

$$+J(\beta)\Big(\cancel{[\rho(\beta(V)), W]} - \cancel{[\rho(\beta(V)), \rho(\beta(W))]} - \cancel{[\rho(\beta(W)), V]} + [\rho(\beta(W)), \rho(\beta(V))]\Big) =$$

$$= J(\beta)\Big([V, W] + [\rho(\beta(V)), \rho(\beta(W))] - \rho_E(D_{\widetilde{V}}(\beta(W)) - D_{\widetilde{W}}(\beta(V)))\Big) + [\beta(V), \beta(W)].$$

On the other hand, recall the definition of MC:

$$MC(\beta)(V, W) = D_{V-\rho(\beta(V))}(\beta(W)) - D_{W-\rho(\beta(W))}(\beta(V)) - l\Big(\beta([V, W]) - [\beta(V), \beta(W)]\Big).$$

Accordingly,

$$-MC(\beta) \cdot J(\beta)(V, W) = -(1 - lJ(\beta)\rho_E)MC(\beta)(V, W) =$$

$$= -D_{\widetilde{V}}(\beta(W)) + D_{\widetilde{W}}(\beta(V)) - l([\beta(V), \beta(W)]) + l(\beta([V, W]))+$$

$$+lJ(\beta)\rho_E\Big(D_{\widetilde{V}}(\beta(W)) - D_{\widetilde{W}}(\beta(V))\Big) - lJ(\beta)\rho_E l\Big(\beta([V, W]) - \underline{[\beta(V), \beta(W)]}\Big).$$

Comparing the expression above with

$$\beta \cdot G(J(\beta))(V, W) = -D_{\widetilde{V}}(\beta(W)) + D_{\widetilde{W}}(\beta(V)) - l([\beta(V), \beta(W)])+$$

$$-lJ(\beta)\Big([V, W] + \underline{[\rho(\beta(W)), \rho(\beta(V))]} - \rho_E(D_{\widetilde{V}}(\beta(W)) - D_{\widetilde{W}}(\beta(V)))\Big),$$

and using the facts that ρ is a Lie algebra morphism and $\rho_E l = \rho$, we are left to prove that

$$l(\beta([V, W])) - lJ(\beta)\rho\beta([V, W]) = -lJ(\beta)[V, W],$$

which follows immediately from $J(\beta)(\rho \circ \beta - id) = \beta$. Q.E.D.

We conclude this section by proving that, for the purpose of integrability of almost Γ-structures, nothing changes if we use MC or G.

Proposition 6.2.11. *Let $(\mathcal{G},\omega)$ be a full Lie-Pfaffian groupoid over X, integrable up to order 1, with $\omega \in \Omega^1_{\mathrm{loc}}(\mathcal{G}, t^*E)$ and $A = \mathrm{Lie}(\mathcal{G})$. Then the alternative Spencer sequence*

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)}) \xrightarrow{F} \Gamma_{\mathrm{loc}}(\mathcal{L}) \xrightarrow{G} \Omega^2_{\mathrm{loc}}(X,E) \tag{$**$}$$

is a "cochain complex", in the sense that $G \circ F = 0$.

Moreover, $(**)$ is exact if and only if the Spencer sequence $(*)$ is exact.

In view of Lemma 6.2.6, we replaced $\Omega^1(X,A)$ with $\Gamma_{\mathrm{loc}}(\mathcal{L})$ in the sequence $(*)$.

Proof. For the first part, it is enough to apply Proposition 6.2.10 to $F(b) \in \Gamma(\mathcal{L})$:

$$G(F(b)) = -F(b) \cdot MC(J(F(b))) \cdot F(b) = -F(b) \cdot \cancel{MC(F(b^{-1}))} \cdot F(b) = 0,$$

where in the second step we used Lemma 6.2.9.

For the second part, assume for instance that the sequence $(**)$ is exact and consider any $\beta \in \Gamma_{\mathrm{loc}}(\mathcal{L})$ such that $MC(\beta) = 0$. By Proposition 6.2.10 we have

$$G(J(\beta)) = -J(\beta) \cdot \cancel{MC(\beta)} \cdot J(\beta) = 0.$$

Then, from the exactness, there exists some $b \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}^{(1)})$ such that

$$J(\beta) = F(b).$$

In conclusion, by Lemma 6.2.9,

$$\beta = J(J(\beta)) = J(F(b)) = F(b^{-1}),$$

hence the sequence $(*)$ is exact. The converse is analogous. Q.E.D.

It follows that the conclusion of Theorem 6.2.4 still holds:

Corollary 6.2.12. *Let Γ be a Lie pseudogroup and assume that the complex $(**)$ associated to the Pfaffian groupoid $J^1\Gamma$ is exact. Then, if an almost Γ-structure P over M is integrable up to order 1, it is integrable.*

6.3 (Non)-invariance of integrability under Morita equivalence

The equivalent condition for the integrability of an almost Γ-structure from Proposition 6.2.1 motivates the following:

Definition 6.3.1. Let $(\mathcal{G},\omega) \rightrightarrows X$ be a Pfaffian groupoid and $\pi : (P,\theta) \to M$ a principal Pfaffian $\mathcal{G}$-bundle with moment map μ (Definition 5.3.13); then P is

- **left-integrable** if for every $p \in P$ there is a local holonomic section $\sigma \in \Gamma_{\mathrm{loc}}(\mu)$ passing through p.
- **right-integrable** if for every $p \in P$ there is a local holonomic section $\rho \in \Gamma_{\mathrm{loc}}(\pi)$ passing through p.
- **integrable** if it is both left and right integrable. ♦

Remark 6.3.2. If, like in the setting of Proposition 6.2.1, the dimensions of M and X in Definition 6.3.1 coincide, then left and right integrability are equivalent. Indeed, for any holonomic section $\sigma \in \Gamma_{\mathrm{loc}}(\pi)$ and $x \in M$, $d_x(\pi \circ \sigma)$ restricts to a surjective map between vector spaces of the same dimensions, i.e. a linear isomorphism. Accordingly, $\pi \circ \sigma$ can be restricted to a diffeomorphism on a sufficiently small neighbourhood of a non-critical point, and $\rho = \sigma \circ (\pi \circ \sigma)^{-1}$ defines a local holonomic section of μ. ♢

Example 6.3.3 (integrability as Pfaffian fibrations)**.** A principal Pfaffian $\mathcal{G}$-bundle $\pi : (P, \theta) \to M$ is right-integrable if and only if it is integrable as a Pfaffian fibration over M. Note that we do not need to ask π to be a Pfaffian fibration beforehand, e.g. by applying Theorem 5.3.15: the π-involutivity holds by Lemma 5.3.14, and by Remark 1.3.7 the integrability condition implies the π-transversality of θ. ♢

Example 6.3.4 (integrability as G-structures)**.** Let X be a point and (P, θ) an abstract G-structure, viewed as a principal Pfaffian bundle (Example 5.4.10); then it is trivially right-integrable, and it is left-integrable if and only if it is integrable as G-structure.

Indeed, one checks directly that P is left-integrable if and only if the associated almost Γ_G-structure (Proposition 5.2.5) is left-integrable, which is in turn equivalent (by Proposition 6.2.1) to its integrability as almost Γ_G-structure, i.e. to the integrability of P as a G-structure. ♢

6.3.1 Integrable Pfaffian Morita equivalences

Definition 6.3.5. A Pfaffian Morita equivalence (P, θ) between two Pfaffian groupoids $(\mathcal{G}_i, \omega_i) \rightrightarrows X_i$ is called

- **left-integrable** if and only if for every $p \in P$ there is a holonomic section $\sigma_1 \in \Gamma(\pi_1)$ passing through p and such that $\pi_2 \circ \sigma_1$ is of constant rank; in particular, P is left-integrable as principal Pfaffian $\mathcal{G}_1$-bundle.
- **right-integrable** if and only if for every $p \in P$ there is a holonomic section $\sigma_2 \in \Gamma(\pi_2)$ passing through p and such that $\pi_1 \circ \sigma_2$ is of constant rank; in particular, P is right-integrable as principal Pfaffian $\mathcal{G}_1$-bundle.
- **integrable** if it is both left- and right-integrable. ♦

With the same argument of Remark 6.3.2 we observe that, when $\dim(X_1) = \dim(X_2)$, left and right integrability are equivalent.

Not all Pfaffian Morita equivalences are integrable: here is an important counterexample.

Proposition 6.3.6. *The Pfaffian Morita equivalence between $J^1\Gamma_{cont}$ and $J^1\Gamma_{CSF}$ from Proposition 5.5.12 is not integrable.*

Proof. Recall from Proposition 5.5.12 that the Pfaffian Morita equivalence $(\bar{P}, \bar{\theta})$ between $J^1\Gamma_{cont}$ and $J^1\Gamma_{CSF}$ is given by the composition of the Pfaffian Morita equivalence between $J^1\Gamma_{cont}$ and $(J^1\Gamma_{cont})_0$ with the one between $(J^1\Gamma_{CSF})_0$ and $J^1\Gamma_{CSF}$, as shown in the following diagram:

$$\begin{tikzcd}
 & & (\bar{P}, \bar{\theta}) \arrow[dll, bend right, "\bar{\pi}_1"'] \arrow[ddrr, bend left, "\bar{\pi}_2"] & & \\
(\mathcal{G}_1, \omega^1) \arrow[urr, bend left, "\bar{m}_1"] \arrow[r, bend left, "m_1"] \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "t_1"] & (P_1, \omega^1_{|P_1}) \arrow[dl, "\pi_1"] \arrow[drr, "\mu_1"'] & (G, 0) \arrow[l, bend right, "m_1^G"'] \arrow[r, bend left, "m_2^G"] \arrow[d, shift right] \arrow[d, shift left] & (P_2, \omega^2_{|P_2}) \arrow[dl, "\mu_2"] \arrow[dr, "\pi_2"'] & (\mathcal{G}_2, \omega^2) \arrow[ull, bend right, "\bar{m}_2"'] \arrow[l, bend right, "m_2"'] \arrow[d, shift right, "s_2"'] \arrow[d, shift left, "t_2"] \\
\mathbb{R}^n & & \{0\} & & \mathbb{R}^n
\end{tikzcd}$$

where

$$\mathcal{G}_1 := J^1\Gamma_{CSF}, \qquad \mathcal{G}_2 := J^1\Gamma_{cont}, \qquad G := (\mathcal{G}_1)_0 \cong (\mathcal{G}_2)_0,$$
$$P_1 := (s_1)^{-1}(0), \qquad P_2 := (t_2)^{-1}(0), \qquad \bar{P} := (P_1 \times P_2)/G,$$
$$\pi_1(j^1_0\phi) := \phi(0), \qquad \pi_2(j^1_x\phi) := x,$$
$$\bar{\pi}_1[p_1, p_2] := \pi_1(p_1), \qquad \bar{\pi}_2[p_1, p_2] := \pi_2(p_2),$$
$$m_1(j^1_x\phi, j^1_0\psi) = j^1_0(\phi \circ \psi), \qquad m_2(j^1_y\psi, j^1_x\phi) = j^1_0(\psi \circ \phi),$$
$$\bar{m}_1(g, [p_1, p_2]) = [g \cdot p_1, p_2], \qquad \bar{m}_2([p_1, p_2], g) = [p_1, p_2 \cdot g],$$
$$\bar{\theta}_{[p_1,p_2]}([v, w]) := \omega^1_{p_1}(v) + \omega^2_{p_2}(w).$$

Assume $(P, \bar{\theta})$ is integrable: then for every point $p = [p_1, p_2] \in \bar{P}$ there is a holonomic section $\sigma : \mathbb{R}^n \to \bar{P}$ passing through p. Using the equivalence relation on $P_1 \times P_2$ defined by the principal G-action, σ can be rewritten as

$$\sigma(x) = [e_G, \bar{\sigma}(x)],$$

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where $\bar{\sigma} : \mathbb{R}^n \to P_2$ is a section of π_2. Moreover, using the definition of $\bar{\theta}$, one sees that the holonomicity of σ is equivalent to the holonomicity of $\bar{\sigma}$.

This means that, for every $p_2 = j^1_x\phi \in P_2$, there exists a holonomic section passing through p_2, i.e. something of the form

$$y \mapsto j^1_y\phi \in P_2 = (t_2)^{-1}(0),$$

for a fixed $\phi \in \Gamma_{cont}$. But this is impossible, since $t_2(j^1_y\phi) = \phi(y) = 0$ for all $y \in \mathrm{dom}(\phi)$, i.e. ϕ is not an element in Γ_{cont}. Then such a σ cannot exist, and the Pfaffian Morita equivalence is not integrable. Q.E.D.

Recall from Example 4.4.2 that both Pfaffian groupoids $J^1\Gamma_{cont}$ and $J^1\Gamma_{CSF}$ are integrable; then the previous proposition shows that the integrability of the Pfaffian groupoids is not sufficient for the integrability of the Pfaffian Morita equivalence. Nevertheless, it is a necessary condition, analogously to what happened for formal integrability (Theorem 5.8.4):

Proposition 6.3.7. *If two Pfaffian groupoids $(\mathcal{G}_i, \omega_i) \rightrightarrows X_i$ are integrable Pfaffian Morita equivalent (Definition 6.3.5), they are both integrable as Pfaffian groupoids (Definition 4.4.1).*

Proof. Assume that the Pfaffian Morita equivalence is given by a principal Pfaffian $\mathcal{G}_1$-bundle (P, θ): without loss of generality, $\mathcal{G}_2$ can be assumed to be the gauge groupoid of P (Proposition 3.3.7).

In order to show that $\mathcal{G}_2$ is integrable, for every $[p, q] \in \mathcal{G}_2$ we look for a holonomic bisection $\rho_2 \in \mathrm{Bis}(\mathcal{G}_2, \omega_2)$ whose image contains $[p, q]$; in particular, being a section of the source map, it will be $\rho_2([q]) = [p, q]$.

Pick a representative $(p, q) \in P \times_{\pi_1} P$ of the class $[p, q]$; by the left integrability of the Morita equivalence, there is a holonomic section $\sigma_1 \in \Gamma(\pi_1)$ whose image contains p (in particular, $\sigma_1(\pi_1(p)) = p$) and such that $\pi_2 \circ \sigma_1$ is a diffeomorphism. Similarly, by the right integrability, there is a holonomic section $\sigma_2 \in \Gamma(\pi_2)$, whose image contains q (in particular, $\sigma_2([q]) = q$) such that $\pi_1 \circ \sigma_2$ is a diffeomorphism.

We define then

$$\rho_2(y) := [\sigma_1((\pi_1 \circ \sigma_2)(y)), \sigma_2(y)] \in \mathcal{G}_2,$$

since both components are in the fibred product $P \times_{\pi_1} P$. Such ρ_2 is clearly smooth, satisfies $s \circ \rho_2 = id$ (since σ_2 is a section of π_2) and $t \circ \rho_2 = \pi_2 \circ \sigma_1 \in \mathrm{Diff}_{\mathrm{loc}}(X_1, X_2)$ (by the integrability of the Morita equivalence), hence it is a bisection of $\mathcal{G}_2$. Moreover it sends the element $[q]$ to

$$\rho_2([q]) = [\sigma_1(\pi_1(\sigma_2([q]))), \sigma_2([q])] = [\sigma_1(\pi_1(q)), q] = [\sigma_1(\pi_1(p)), q] = [p, q].$$

Last, ρ is holonomic because σ_1 and σ_2 are so:

$$(\rho_2)^*\omega_2 = (\tau \circ \widetilde{\rho}_2)^*\omega_2 = (\widetilde{\rho}_2)^*(\tau^*\omega_2) = (\widetilde{\rho}_2)^*(\mathrm{pr}_1^*\theta - \mathrm{pr}_2^*\theta) =$$

$$= (\sigma_1 \circ (\pi_1 \circ \sigma_2))^*\theta - (\sigma_2)^*\theta = (\pi_1 \circ \sigma_2)^*\cancel{(\sigma_1)^*\theta} - \cancel{(\sigma_2)^*\theta} = 0.$$ Q.E.D.

In particular, Proposition 6.3.7 shows that the property of being integrable, for a Pfaffian groupoid, is *integrable* Pfaffian Morita invariant.

Last, we want to understand better the geometric meaning of Proposition 6.3.6. In order to do so, we need first the following relation between the integrability of a Pfaffian Morita equivalences and of a principal Pfaffian bundle.

Proposition 6.3.8. *Let $(\mathcal{G}_i, \omega_i)$ be two Pfaffian groupoids over X_i which are integrable Pfaffian Morita equivalent. Then the 1-1 correspondence from Corollary 5.4.20*

$$\left\{ \begin{array}{c} \textit{Principal Pfaffian} \\ (\mathcal{G}_1, \omega_1)\textit{-bundle on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Principal Pfaffian} \\ (\mathcal{G}_2, \omega_2)\textit{-bundle on } M \end{array} \right\}$$

$$(R_1, \theta_1) \quad \longmapsto \quad (R_2, \theta_2)$$

restricts to

$$\left\{ \begin{array}{c} \textit{Integrable principal Pfaffian} \\ (\mathcal{G}_1, \omega_1)\textit{-bundle on } M \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Integrable principal Pfaffian} \\ (\mathcal{G}_2, \omega_2)\textit{-bundle on } M \end{array} \right\}$$

$$(R_1, \theta_1) \quad \longmapsto \quad (R_2, \theta_2)$$

Proof. We are going to prove that, if (R_1, θ_1) is integrable, then also (R_2, θ_2) is integrable. We use the notations from Proposition 5.4.18: for every $[r_1, p] \in R_2$ we have to describe a holonomic section $\rho_2 : M \to R_2$ passing through $[r_1, p]$.

By the integrability of R_1, there exists a holonomic section ρ_1 through r_1, i.e. $\rho_1(x_0) = r_1$ for some $x_0 \in M$. Similarly, by the integrability of P, there exists a holonomic section σ through p, i.e. $\sigma(y_0) = p$ for some $y_0 \in M_1$. Then, for every $x \in M$, let

$$\rho_2(x) := [\rho_1(x), \sigma(\tau_1(\rho_1(x)))];$$

one checks that ρ_2 is a well-defined section of $\pi_2 : R_2 \to M$ such that $\rho_2(x_0) = [r_1, p]$.

Last, to check that ρ_2 is holonomic, we consider the projection to the quotient

$$\mathrm{pr} : R_1 \, {}_{\tau_1}\!\times_{\mu_1} P \to R_2,$$

and use the definition of θ_2 to compute

$$(\rho_2)^*\theta_2 = (\mathrm{pr} \circ (\rho_1, \sigma \circ \tau_1 \circ \rho_1))^*\theta_2 = (\rho_1, \sigma \circ \tau_1 \circ \rho_1)^*(\mathrm{pr}^* \theta_2) =$$

$$= (\rho_1, \sigma \circ \tau_1 \circ \rho_1)^*(\mathrm{pr}_1^* \theta_1 + \mathrm{pr}_2^* \theta) = \cancel{(\rho_1)^* \theta_1} + (\tau_1 \circ \rho_1)^*(\cancel{\sigma^* \theta}) = 0. \qquad \text{Q.E.D.}$$

Remark 6.3.9. From Corollary 5.5.11 we know that a principal Lie-Pfaffian $J^1\Gamma_{cont}$-bundle, i.e. an almost contact structure, is the same thing as a principal Lie-Pfaffian $J^1\Gamma_{SF}$-bundle, i.e. an almost symplectic foliation.

We have already proved in Proposition 6.3.6 that the canonical Pfaffian Morita equivalence between $J^1\Gamma_{cont}$ and $J^1\Gamma_{SF}$ is not integrable. If there was another

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Pfaffian Morita equivalence which was integrable, then Proposition 6.3.8, would give a contradiction, i.e. that the integrability of an almost contact structure is equivalent to the integrability of an almost symplectic foliation. This means that the non-integrability of the Morita equivalence is the key property that explains the different integrability conditions for these two geometric structures. $\Diamond$

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Index

Summary for Muggles

This thesis is about almost geometric structures on smooth manifolds. In this short summary for non-mathematicians[1] we give an informal introduction to these concepts and outline the main results.

Smooth manifolds Students in high school learn how to describe 2-dimensional geometric pictures by means of algebraic equations. For instance, $x^2 + y^2 = 1$ defines a circle of radius 1; similar formulae can be used for lines, parabolae, hyperbolae, etc. The unknowns x and y define the so-called *Cartesian coordinate system* for the standard Euclidean plane (i.e. the infinite piece of paper where we imagine to draw our axes and our pictures).

Cartesian geometry can be studied also in dimension 3, where we deal with pictures in a coordinate grid given by three axes, x, y and z, which defines the standard Euclidean space. This point of view is easily generalisable to any dimension, leading to the notion of *Euclidean space of dimension* n: any point in it can be described by the coordinates $x_1, \ldots, x_n$.

A *manifold* is a "curved space" that locally looks like the Euclidean space of a certain dimension n. In other words, for any point on the manifold, we can provide n coordinates to describe a small region around it. One should think of these as "curved coordinates grids".

The adjective *smooth* refers to the absence of singularities - there are no edges, corners or any kind of sharp points. More formally, let us look at any two different points on a manifold: the coordinates describing the regions around them will be different, but they will be related by some functions, the *changes of coordinates*. A manifold is smooth when all these functions admit derivatives of any order (singularities would arise in points where we cannot derivate).

Let us look at the most common examples of smooth manifolds in low dimensions.

- Curves are smooth manifolds of dimension 1. Indeed, any arc of a curve

[1] It is important to remark that, while Muggles are born without magical powers, mathematics can be learnt by everybody! In other words, unlike in J. K. Rowling's world, here everybody is a potential wizard, but unfortunately only a few decide to pursue a (formal or informal) magic education besides learning some basic spells.

can be approximated by small line segments, which can be described by one coordinate (the length).

- Surfaces are smooth manifolds of dimension 2. Indeed, any area of a surface can be approximated by small rectangles, which can be described by two coordinates (length and width). For instance, the surface of the Earth is a 2-dimensional manifold[2] because it can be fully encoded by two well-known numbers: the latitude and the longitude. The same applies to the surface of a doughnut without sprinkles (mathematically known as a *torus*), of a maschera di carnevale cookie (a genus 2 torus), of a pretzel (a genus 3 torus), or of other smooth pastries with any number of holes.

- Our universe (from a Newtonian point of view) is a smooth manifold of dimension 3: any point in the space can be described by three global coordinates, after having fixed an origin (for instance, the Sun).

Manifolds of higher dimensions are more difficult to picture graphically for humans with a three-dimensional mindset[3] and can be described only mathematically. However, this is one of the many instances where a general definition eventually pays off the price of its abstractness. Indeed, higher dimensional manifolds appear naturally, and are very relevant, in modern physics: for instance, the space-time of general relativity is a manifold of dimension 4, and in (some version of) string theory, physicists see the universe as a manifold of dimension 11.

Transformations on smooth manifolds One can perform different kinds of transformations on manifolds: for instance, they can be moved, rotated, stretched, compressed, or even cut, pinched, etc. Of course, a random transformation may change our manifold into a very different kind of space. In geometry one is often interested in transformations, called *diffeomorphisms*, which preserve the smoothness of our manifold, as well as its "shape". This means, among other things, that the dimension is preserved; moreover, the transformation admits an "inverse transformation" satisfying the same properties.

For instance, a function transforming a circle in a square is not a diffeomorphism (it does not preserve smoothness), whereas transforming a circle into an oval is a diffeomorphism (roughly speaking, this means that there are no differences between them from the point of view of smooth manifold). More generally, translations and rotations are diffeomorphisms of any manifold, while cutting or pinching are not.

A diffeomorphism does not necessarily have to act on the whole manifold, i.e. being global. For instance, instead of rotating an entire sphere, one can rotate only

[2]For flat-earthers: this statement is still true (provided your Earth does not have a defined boundary). But you probably will not be reading this thesis in any case.

[3]Concerning this topic (n-dimensional beings who cannot conceive $(n+1)$-dimensionality), I suggest to read the beautiful novel *Flatland* by E. Abbott Abbott (1884).

the northern hemisphere. This is an example of a *locally defined diffeomorphism*, or (more imprecise but shorter) local diffeomorphism.

Any two (global) diffeomorphism can be applied consecutively, obtaining a third diffeomorphism, called the *composition* of the previous two. In particular, the composition of three or more diffeomorphisms is associative, i.e. does not require parentheses to be written. These algebraic properties can be described in a more compact way, by saying that the set $\mathrm{Diff}(M)$ of all diffeomorphisms of a manifold M has the structure of a *group*.

On the other hand, the set $\mathrm{Diff}_{\mathrm{loc}}(M)$ of local diffeomorphisms is the prototypical example of a *pseudogroup*. This is defined in the same way of a group, but without the requirement that any two elements can be composed; however, when they can, these elements satisfy the same properties of the elements of a group.

Geometric structures A geometric structure on a smooth manifold is an extra piece of information that allows one to perform specific tasks on that manifold. Let us describe a few of them.

- The most intuitive class of geometric structures are *(Riemannian) metrics*; they are tools which enable us to measure distances between points on a manifold. For the Euclidean space, the standard way of measuring lengths of straight lines is an example. On a curved space, a metric simply associates a positive number to any two points. For instance, the distance between any two points on the surface of a sphere can be computed as the length of the shortest arc of curve between them (this leads to the concept of geodesics, which is used to determine the best trajectory for planes around the Earth[4]).
- A slight variation of this notion is a *pseudo-Riemannian metric*: this arises in general relativity, where it has the purpose of measuring distances between two events in the 4-dimensional space-time universe. There can be counterintuitive consequences: unlike a Riemannian metric, here the distance between two points is not necessarily a positive number, but can also be negative (if the temporal separation between the two events is greater than the spatial separation).
- Another important geometric structure coming from physics, more precisely Hamiltonian mechanics[5], is a *symplectic structure*. It can be defined only on manifolds of dimension $2n$, which are interpreted as the phase space of a particle moving in an n-dimensional space: the first n coordinates of the manifold defines its position, the other n its velocity. The role of a symplectic structure is to describe the dynamics of the particle in terms of the energy

[4]For flat-earthers: unlike footnote 1, this is not true anymore for you. Time to start asking yourselves some questions.

[5]Hamiltonian mechanics is a reformulation of the classical Newtonian mechanics, due to R. Hamilton (1833). In particular, in this formalism, Newton's second law $F = ma$ is replaced by the so-called Hamilton equations.

of the physical system. More generally, a symplectic structure can take any function in input, and gives as output a trajectory where the function does not change; in the previous physical example, the symplectic structure tells us where the energy of the system is conserved.

- The last class of geometric structures we describe in this introduction are *(regular) foliations*. These are used to partition a manifold into "smaller" manifolds, called "leaves". The adjective regular refers to the fact that all leaves must have the same dimension. An easy example (which explains the name of this structure) is given by the three-dimensional Euclidean space, which is foliated by an infinite amount of parallel two-dimensional Euclidean planes (imagine an infinitely high stack of infinitely large sheets of papers). Another example is given by the foliation of the surface of a torus in parallel circles of the same radius (think of a slinky toy[6] where you join the two ends).

  As a counterexample, note that the parallels of the Earth[7] (circles of constant latitude, e.g. the Equator or the Tropics) do describe a partition of its surface, but not a regular foliation. The reason is that, while most leaves are 1-dimensional (since they are standard circles), the north and the south poles are covered by two circles of radius zero, which consist only of one point, and are therefore 0-dimensional manifolds. The meridians of the Earth do not describe a regular foliation either: they do not even form a partition, since through the north and the south poles there is more than one leaf.[8]

Symmetries of geometric structures Many interesting phenomena appear when studying transformations that preserve not only "the shape" of a manifold, but also an additional geometric structure. A *(local) symmetry* of a geometric structure on a manifold is precisely a (local) diffeomorphism which preserves the structure.

For instance, let us consider a Riemannian metric on a sphere. As we discussed earlier, translations and rotations are diffeomorphisms; moreover, they preserve the distances between any two points, hence they are symmetries of the metric (these are also known as *isometries*). On the other hand, if we rotate only the northern hemispere, we have a local diffeomorphism which is not an isometry (the distance between two points in two different hemisphere can change).

Similarly, stretching the sphere or the torus (entirely or partially) is a diffeomorphism, but is not an isometry. However, stretching the torus to a bigger one

[6]If you do not know it, think of the Slinky character from *Toy Story*, and remove head, tail and paws.

[7]Last comment for flat-earthers: strangely enough, what follows is also true for you. Although I doubt you talk about north and south poles.

[8]Actually, one can prove that there are no regular foliations on the 2-dimensional sphere, beside the trivial ones (where there is only one leaf - the entire sphere - or where every point is a leaf).

is a symmetry of the slinky toy foliation by parallel circles described earlier. In this case, each stretched circle deforms into another circle of bigger radius, which is still a manifold of dimension 1.

The reason we discussed the concept of symmetry is that a geometric structure on a manifold can be equivalently described via the symmetries of its local model. Let us be more precise. We said at the beginning that a manifold M is a space locally modelled on an Euclidean space of a given dimension. Assume that such an Euclidean space is endowed with a "canonical" geometric structure, and call Γ its pseudogroup of local symmetries. Then any geometric structure of the same kind on M can be described using the structure on the local model: it is enough to ask that the changes of coordinates of M are elements of Γ, i.e. they are diffeomorphisms of the Euclidean space preserving its canonical geometric structure. With this point of view, the geometric structure on M is also called a Γ-*structure*.

Almost structures in geometry As we mentioned many times, a manifold M is a curved space that locally looks like an Euclidean space of a fixed dimension n; this means that at each point of M we can consider the space (of dimension n) tangent to M, which we call *tangent space*. The key remark is that a geometric structure on M induces a linear geometric structure of the same kind on each tangent space; the adjective linear refers to the fact that the tangent spaces are not only manifolds on their own, but have other special properties.

We call the collection of these linear data an *almost* (Γ-)*structure* on M. This should be interpreted as the "shadow" of the geometric structure: if we see an object, we can imagine its shadow, but seeing a shadow may not be enough to imagine which object projects it. Stretching a bit this metaphor, we could even think of shadows as independent entities[9], so that, if we see one, we do not even know if there is an object projecting it in the first place.

Indeed, if we are given only a linear structures on each tangent space of a manifold (an almost structure), we cannot always reconstruct the entire geometric structure on the manifold we started from. Actually, we cannot even be sure that there was a structure a priori! The *integrability problem* consists precisely in understanding under which conditions (called integrability conditions) a given almost structure comes from a structure (then we call the almost structure *integrable*).

A class of examples of almost Γ-structure are *distributions* of rank k on a manifold of dimension n (for $k \leq n$): roughly speaking, they consist in a collection of k little arrows at any point on the manifold, which are compatible between them in a suitable way.

We illustrate this concept with an example, for $k = 1$ and $n = 2$. Consider the surface of a giant doughnut and imagine to put on each point a sprinkle with the shape of an arrow pointing in any direction; this defines a distribution on it.

[9]Shadows separated from their original bodies appear e.g. in *Peter Pan* and *Once Upon a Time*.

If we start from a point, we can walk on the manifold[10] following these arrows; the path we create is called an integral line. If these integral lines define smooth manifolds of the same dimension and which partition M, then we obtain a regular foliation (e.g. the one by parallel circles described earlier), i.e. the distribution is integrable.

Differential equations As we discussed earlier, the integrability problem consists in understanding when a given almost Γ-structure comes from a Γ-structure. Since the integrability conditions correspond to a differential equation of order 1, we are going now to recall the basics of this theory.

A *(partial) differential equation* (also known as PDE) is an equation where the unknowns are a function and its derivatives. It is said to be of order k if in the equation there are derivatives up to order k. These equations, studied since Newton's time, constitute the most common mathematical tool to model real world phenomena, from astrophysics to economics. However, unlike algebraic equations, for which many formulae and standard techniques are available, differential equations are, in general, extremely difficult to solve[11].

In experimental sciences and engineering, the standard approach to PDEs consists in avoiding to solve them directly, and instead approximating their solutions using powerful computers. The softwares they use are based on techniques from numerical analysis which, in turn, come from theoretical mathematics. In particular, one way to build a solution is to start by finding some function f_1 that does not necessarily solve the PDE, but does it "at order 1". This means that, if we eventually find the actual solution f, it will have the same first derivatives of f_1. After finding f_1, we are going to improve this approximation by looking for a function f_2: this is required to have not only the same first derivatives of f (and of f_1), but also the same second derivatives of f.

While computers have to stop at a certain point (and are left only with a numerical approximation of the solution), we can try to repeat this algebraic process indefinitely. If this is possible, we have built a sequence of functions that look increasingly more similar to the supposed solution; we call such a sequence a *formal solution.* If the equation we started from was the integrability condition of an almost Γ-structure, then a formal solution is a *formally integrable almost Γ-structure*. Formal integrability is a necessary condition for integrability, and is usually easier to check. Moreover, if the setting of our problem is "nice enough" (and this is what happens in most of the examples), the recursive procedure outlined above reaches a conclusion after a finite amount of steps; then any formal solution is an actual solution, and formal integrability is a sufficient condition for integrability.

[10] Assuming we are wearing suction cup shoes, or the doughnut exercises enough gravity to keep our feet anchored to it.

[11] The general solution of the Navier-Stokes equation - a fundamental law in fluid mechanics - is worth one million dollars. This is one of the seven Millennium problems, posed in 2000 by the Clay Mathematics Institute, only one of which has been ever solved up to now.

Main results of this thesis Many authors have already studied (and solved) the integrability problem for various examples of almost Γ-structures. Moreover, between the 60's and the 80's, a general approach was developed to tackle the integrability problem for a special class of pseudogroups Γ (using the established theory of "G-structures"). However, what was still missing was a complete approach which works for a general Γ-structure. Actually, there was not even a definition of almost Γ-structure in terms of a generic Γ!

In this thesis we filled this gap, i.e. we provided a general definition of almost Γ-structures, which relies on more powerful mathematical objects, namely principal groupoid bundles, which were not available in the 80's. Moreover, we give a precise description of the obstructions to the formal integrability of any almost Γ-structure, in terms of some "algebraic spaces", called Spencer cohomology groups, associated to the pseudogroup Γ. The problem of understanding when a formal solution coincides with an actual solution requires different techniques. In this case, we described a sufficient analytic condition (as opposed to the algebraic ones mentioned before), under which all formally integrable almost Γ-structures are integrable. A last problem we tackled in this thesis is to prove that "equivalent" pseudogroups induce "equivalent" almost structures, as well as to understand when integrability is preserved under this equivalence.

As a final remark, I would like to mention that the way we solved these problems required an extra layer of abstractness. More precisely, we developed the general framework of principal Pfaffian bundles, which are the abstract objects encoding the key properties of almost Γ-structures. Then, we translated the integrability problem for almost Γ-structures in the setting of principal Pfaffian bundles, solved it in this abstract framework, and retranslated back to "concrete" almost Γ-structures. This is again an example of how going more abstract can help to solve concrete problems.

Samenvatting voor Dreuzels

Dit proefschrift gaat over bijna meetkundige structuren op gladde variëteiten. In deze korte samenvatting voor niet-wiskundigen[1] geven we een informele introductie tot deze concepten en een overzicht van de belangrijkste resultaten.

Gladde variëteiten Studenten op de middelbare school leren hoe ze tweedimensionale meetkundige plaatjes kunnen beschrijven door middel van algebraïsche vergelijkingen. Bijvoorbeeld, $x^2 + y^2 = 1$ definieert een cirkel met straal 1; vergelijkbare formules kunnen worden gebruikt voor lijnen, paraboolen, hyperbolen, enz. De onbekenden x en y definiëren het zogenaamde *Cartesische coördinatenstelsel* voor het standaard Euclidische vlak (d.w.z. het denkbeeldige oneindige stuk papier waar we onze assen en plaatjes tekenen).

Cartesische meetkunde kan ook worden bestudeerd in dimensie 3, waar we plaatjes beschrijven met een coördinatenrooster gegeven door drie assen, x, y en z, die de standaard Euclidische ruimte definiëren. Dit idee is makkelijk te generaliseren naar elke dimensie, wat leidt tot het begrip van *Euclidische ruimte van dimensie n*: elk punt hierin kan worden beschreven door de coördinaten $x_1, \dots, x_n$.

Een *variëteit* is een "gekromde ruimte" die lokaal lijkt op de Euclidische ruimte van een bepaalde dimensie n. Met andere woorden, voor elk punt op de variëteit kunnen we n coördinaten kiezen om een klein gebiedje rond het punt te beschrijven. Men moet hierover nadenken als "gekromde coördinatenroosters".

Het adjectief *glad* verwijst naar de afwezigheid van singulariteiten - er zijn geen randen, hoeken of andere soorten scherpe punten. Formeler, laten we naar twee verschillende punten op een variëteit kijken: de coördinaten die de regio's eromheen beschrijven zijn anders, maar ze zullen gerelateerd zijn via bepaalde functies, de *veranderingen van coördinaten*. Een variëteit is glad wanneer al deze functies afgeleiden van elke orde hebben (singulariteiten zouden ontstaan in punten waar we niet kunnen differentieren).

Laten we kijken naar de meest voorkomende voorbeelden van gladde variëteiten in lage dimensies.

[1] Het is belangrijk op te merken dat, terwijl Dreuzels worden geboren zonder magische krachten, wiskunde door iedereen kan worden geleerd! Met andere woorden, hier is iedereen, anders dan in de wereld van J.K. Rowling, een potentiële tovenaar, maar helaas besluiten slechts enkelen een (formele of informele) magisch opleiding te volgen, afgezien van enkele basis toverspreuken.

- Krommen zijn gladde variëteiten van dimensie 1. Inderdaad, elke boog van een kromme kan worden benaderd door kleine lijnsegmenten, die kunnen worden beschreven door één coördinaat (de lengte).
- Oppervlakken zijn gladde variëteiten van dimensie 2. Inderdaad, elk gebied van een oppervlak kan dat worden benaderd door kleine rechthoeken, die kunnen worden beschreven door twee coördinaten (lengte en breedte). Het oppervlak van de aarde is bijvoorbeeld een tweedimensionale variëteit[2] omdat het volledig kan worden beschreven door twee vertrouwde getallen: de breedte- en lengtegraad. Hetzelfde geldt voor het oppervlak van een donut zonder hagelslag (wiskundig bekend als een *torus*), van een maschera di carnevale koekje (een genus 2 torus), van een pretzel (een genus 3 torus), of van andere gladde gebakjes met een aantal gaten.
- Ons universum is (vanuit een Newtoniaans opzicht) een gladde variëteit van dimensie 3: elk punt in de ruimte kan worden beschreven door drie globale coördinaten, nadat een oorsprong is vastgesteld (bijvoorbeeld de Zon).

Variëteiten van hogere dimensies zijn moeilijker grafisch in te beelden voor mensen met een driedimensionale manier van denken[3] en kunnen alleen wiskundig worden beschreven. Dit is echter een van de vele gevallen waarin een algemene definitie uiteindelijk zijn vruchten afwerpt, ten koste van meer abstractie. Hoger dimensionale variëteiten verschijnen op natuurlijke wijze, en zijn zeer relevant in de moderne natuurkunde: de ruimtetijd van de algemene relativiteitstheorie is bijvoorbeeld een variëteit van dimensie 4, en in (een versie van) de snaartheorie, zien natuurkundigen het universum als een variëteit van dimensie 11.

Transformaties op gladde variëteiten Men kan verschillende soorten transformaties op variëteiten uitvoeren: ze kunnen bijvoorbeeld worden verplaatst, gedraaid, uitgerekt, verkleind, of zelfs gesneden, geknepen, etc. Natuurlijk kan een willekeurige transformatie onze variëteit veranderen in een heel ander soort ruimte. In de meetkunde is men vaak geïnteresseerd in transformaties, *diffeomorfismen* genoemd, die de gladheid van ons variëteit behouden, evenals zijn "vorm". Dit betekent, onder andere, dat de dimensie is behouden; bovendien laat de transformatie een "inverse transformatie" toe die aan dezelfde eigenschappen voldoet.

Een functie die een cirkel in een vierkant transformeert, is bijvoorbeeld geen diffeomorfisme (het behoudt geen gladheid), terwijl het transformeren van een cirkel in een ovaal een diffeomorfisme is (grof gezegd, betekent dit dat er vanuit het gezichtspunt van gladde variëteiten geen verschillen tussen hen zijn). Meer algemeen, zijn translaties en rotaties diffeomorfismen van alle variëteiten, terwijl snijden of knijpen dat niet zijn.

[2]Voor flat-earthers: deze stelling is nog steeds waar (op voorwaarde dat jullie aarde geen gedefinieerde grens heeft). Maar waarschijnlijk zullen jullie dit proefschrift toch niet lezen.

[3]Betreffende dit onderwerp (n-dimensionale wezens die geen $(n+1)$-dimensionaliteit kunnen opvatten), raad ik aan om de prachtige roman *Flatland* van E. Abbott Abbott (1884) te lezen.

Een diffeomorfisme hoeft niet noodzakelijkerwijs op de hele variëteit te werken, d.w.z. globaal zijn. In plaats van de hele bol te draaien, kan men bijvoorbeeld alleen het noordelijk halfrond draaien. Dit is een voorbeeld van een *lokaal gedefinieerd diffeomorfisme*, of wel (onnauwkeuriger, maar korter) lokaal diffeomorfisme.

Elk tweetal (globale) diffeomorfismen kunnen opeenvolgend worden toegepast, waarbij een derde diffeomorfisme wordt verkregen, de *samenstelling* van de vorige twee genoemd. De compositie van drie of meer diffeomorfismen is in het bijzonder associatief, d.w.z. er zijn geen haakjes nodig. Deze algebraïsche eigenschappen kunnen op een compactere manier worden beschreven door te zeggen dat de verzameling $\mathrm{Diff}(M)$ van alle diffeomorfismen van een variëteit M de structuur heeft van een *groep*.

Aan de andere kant is de verzameling $\mathrm{Diff}_{\mathrm{loc}}(M)$ van lokale diffeomorfismen het prototypische voorbeeld van een *pseudogroep*. Dit wordt op dezelfde manier als een groep gedefinieerd, maar zonder de eis dat elke twee elementen kunnen worden samengesteld; wanneer dat echter wel kan, moeten deze elementen voldoen aan dezelfde eigenschappen als de elementen van een groep.

Meetkundige structuren Een meetkundige structuur op een gladde variëteit is een extra stuk informatie waarmee men specifieke taken op de variëteit kan uitvoeren. Laten we er een paar beschrijven.

- De meest intuitive klasse van meetkundige structuren zijn *(Riemanniaanse) metrieken*; dit zijn gereedschappen waarmee we afstanden tussen punten op een variëteit kunnen meten. Voor de Euclidische ruimte, is de standaardmanier om lengtes van rechte lijnen te meten een voorbeeld. Op een gekromde ruimte geeft een metriek simpelweg een positief getal aan elk paar punten. De afstand tussen twee willekeurige punten op de oppervlak van een bol kan bijvoorbeeld worden berekend als de lengte van de kortste boog tussen de punten (dit leidt tot het concept van geodeten, dat wordt gebruikt om te bepalen wat voor vliegtuigen het beste traject rond de Aarde is[4]).

- Een kleine variatie van deze notie is een *pseudo-Riemanniaanse metriek*: deze komt voort uit de algemene relativiteitstheorie, met als doel het meten van afstanden tussen twee gebeurtenissen in het 4-dimensionale ruimtetijd universum. Er kunnen tegen-intuïtieve gevolgen zijn: in tegenstelling tot bij een Riemanniaanse metriek, is hier de afstand tussen twee punten niet noodzakelijkerwijs een positief getal, maar kan ook een negatief getal zijn (als de temporele scheiding tussen de twee gebeurtenissen groter is dan de ruimtelijke scheiding).

- Een andere belangrijke meetkundige structuur uit de natuurkunde, preciser de Hamiltoniaanse mechanica[5], is een *symplectische structuur*. Het

[4]Voor flat-earthers: in tegenstelling tot voetnoot 1, geldt dit niet meer voor jullie. Tijd om jezelf wat vragen te stellen.

[5]Hamiltoniaanse mechanica is een herformulering van de klassieke Newtoniaanse mechanica,

kan alleen worden gedefinieerd op variëteiten van dimensie $2n$, die worden geïnterpreteerd als de faseruimte van een deeltje bewegend door een n-dimensionale ruimte: de eerste n coördinaten van de variëteit bepalen zijn positie, de andere n zijn snelheid. De rol van een symplectische structuur is te beschrijven de dynamiek van het deeltje in termen van de energie van het fysische systeem. In het algemeen, kan een symplectische structuur elke functie als input nemen, en geeft als output het traject waarin de functie niet verandert; in het vorige fysische voorbeeld, vertelt de symplectische structuur ons waar de energie van het systeem is behouden.

- De laatste klasse van meetkundige structuren die we in deze introductie beschrijven, zijn *(reguliere) foliaties*. Deze worden gebruikt om een variëteit in "kleinere" variëteiten te verdelen, "bladeren" genoemd. Het adjectief regulier verwijst naar het feit dat alle bladeren dezelfde dimensie moeten hebben. Een eenvoudig voorbeeld (wat de naam van deze structuur verklaart) wordt gegeven door de driedimensionale Euclidische ruimte, welke is gefolieerd door een oneindige hoeveelheid parallelle tweedimensionale Euclidische vlakken (stel je een oneindig grote stapel oneindig grote vellen papier voor). Een ander voorbeeld is de foliatie van het oppervlak van een torus in parallelle cirkels van dezelfde straal (denk aan slinky speelgoed[6] waarbij je de twee uiteinden aan elkaar plakt).

  Als tegenvoorbeeld, merk op dat de breedtecirkels van de Aarde[7] (cirkels van constante breedtegraad, b.v. de evenaar of de keerkringen) beschrijven een verdeling van haar oppervlak, maar geen reguliere foliatie. De reden is dat, terwijl de meeste bladeren 1-dimensionaal zijn (omdat ze standaardcirkels zijn), de noord- en de zuidpool worden bedekt door twee cirkels met straal nul, die slechts uit één punt bestaan, en zijn daarom 0-dimensionale variëteiten. Ook de lengtecirkels van de Aarde beschrijven geen reguliere foliatie: ze vormen zelfs geen verdeling, omdat door er de noord- en zuidpool meer dan één blad gaat[8].

Symmetrieën van meetkundige structuren Veel interessante fenomenen verschijnen wanneer we transformaties bestuderen die niet alleen "de vorm" van een variëteit behouden, maar ook een extra meetkundige structuur. Een *(lokale) symmetrie* van een meetkundige structuur op een variëteit is een (lokaal) diffeomorfisme die de structuur behoudt.

bedacht door R. Hamilton (1833). In het bijzonder wordt in dit formalisme de tweede wet van Newton, $F = ma$, vervangen door de zogenaamde Hamilton-vergelijkingen.

[6]Als je dit niet kent, denk dan aan het Slinky-personage uit *Toy Story* en verwijder kop, staart en poten.

[7]Laatste opmerking voor flat-earthers: vreemd genoeg, wat volgt is ook waar voor jullie. Hoewel ik betwijfel of jullie het over noord- en zuidpolen hebben.

[8]Men kan bewijzen dat er geen reguliere foliaties op de tweedimensionale sfeer bestaan, naast de triviale (waarbij er maar één blad is - de hele bol - of waarbij elk punt een blad is).

Laten we bijvoorbeeld een Riemanniaanse metriek op een bol bekijken. Zoals we eerder besproken hebben, zijn translaties en rotaties diffeomorfismen; bovendien behouden ze de afstanden tussen twee willekeurige punten, daarom zijn het symmetrieën van de metriek (deze worden ook wel *isometrieën* genoemd). Aan de andere kant, als we alleen het noordelijk halfrond roteren, hebben we een lokaal diffeomorfisme dat geen isometrie is (de afstand tussen twee punten op twee verschillende halfronden kan veranderen).

Op dezelfde manier is het uitrekken van de bol of de torus (geheel of gedeeltelijk) een diffeomorfisme, maar geen isometrie. Het uitrekken van de torus naar een grotere torus is echter een symmetrie van de slinkyfoliatie door parallelle cirkels zoals eerder beschreven. In dit geval, vervormt elke uitgerekte cirkel tot een andere circkel met een grotere straal, welke nog steeds een variëteit van dimensie 1 is.

De reden dat we het concept van symmetrie hebben besproken, is dat een meetkundige structuur op een variëteit equivalent kan worden beschreven via de symmetrieën van het lokale model. Laten we preciezer zijn. We zeiden in het begin dat een variëteit M een ruimte is die lokaal gemodelleerd op een Euclidische ruimte met een gegeven dimensie. Neem aan dat zo'n Euclidische ruimte een "canonieke" meetkundige structuur heeft, en noem Γ zijn pseudogroep van lokale symmetrieën. Dan kan elke meetkundige structuur van dezelfde soort op M worden beschreven met behulp van de structuur op het lokale model: het is voldoende om te eisen dat de veranderingen van coördinaten van M elementen van Γ zijn, d.w.z. het zijn diffeomorfismen van de Euclidische ruimte die zijn canonieke meetkundige structuur behouden. Vanuit dit opzichtpunt, wordt de meetkundige structuur op M ook wel een Γ-*structuur* genoemd.

Bijna structuren in de meetkunde Zoals we al vaak hebben gezegd, is een variëteit M een gekromde ruimte die lokaal lijkt op een Euclidische ruimte met een vaste dimensie n; dit betekent dat we op elk punt van M de vlakke ruimte (van dimensie n) die M raakt kunnen beschouwen, die we *raakruimte* noemen. Een meetkundige structuur op M een lineaire meetkundige structuur van dezelfde soort op elke raakruimte induceert; het adjectief lineair verwijst naar het feit dat de raakruimten niet alleen variëteiten zijn, maar ook andere speciale eigenschappen hebben.

We noemen de verzameling van deze lineaire gegevens een *bijna* (Γ-)*structuur* op M. Dit moet worden geïnterpreteerd als de "schaduw" van de meetkundige structuur: als we een object zien, kunnen we ons zijn schaduw voorstellen, maar een schaduw zien is misschien niet genoeg om ons voor te stellen welk object het werpt. Als we deze metafoor een beetje doortrekken, kunnen we zelfs over schaduwen denken als onafhankelijke entiteiten[9], zodat we, als we er één zien, niet eens weten of er een object is die het werpt.

[9]Schaduwen gescheiden van hun oorspronkelijke lichamen verschijnen b.v. in *Peter Pan* en *Once Upon a Time*.

Als we alleen een lineaire structuur op elke raakruimte van een variëteit (een bijna structuur) hebben, kunnen we niet altijd de hele meetkundige structuur op de variëteit reconstrueren. Eigenlijk kunnen we niet eens zeker weten dat er een a priori structuur was! Het *integreerbaarheidsprobleem* betreft het begrijpen van onder welke voorwaarden (integreerbaarheidscondities genoemd) een gegeven bijna structuur van een structuur komt (dan noemen we de bijna structuur *integreerbaar*).

Een klasse voorbeelden van bijna Γ-structuren zijn *distributies* van rang k op een variëteit van dimensie n (voor $k \leq n$): grof gezegd, bestaan ze uit een verzameling van k kleine pijlen op elk punt op die variëteit, die zich goed gedragen.

We illustreren dit concept met een voorbeeld voor $k = 1$ en $n = 2$. Beschouw het oppervlak van een gigantische donut en stel je voor dat je op elk punt een hagelslagje zet met de vorm van een pijl wijzend in een willekeurige richting; dit definieert een distributie op de donut. Als wij in een punt beginnen, we kunnen over de variëteit langs deze pijlen lopen[10]; het pad we creëren wordt een integrale lijn genoemd. Als deze integrale lijnen gladde variëteiten van dezelfde dimensie definiëren en die M verdelen, dan krijgen we een reguliere foliatie (b.v. de foliatie door parallelle cirkels zoals eerder beschreven), d.w.z. dat de distributie is integreerbaar.

Differentiaalvergelijkingen Zoals eerder besproken, bestaat het integreerbaarheidsprobleem uit het bepalen wanneer een gegeven bijna Γ-structuur uit een Γ-structuur voortkomt. Sinds de integreerbaarheidscondities een differentiaalvergelijking van orde 1 zijn, gaan we nu de fundamenten van deze theorie herhalen.

Een *(partiële) differentiaalvergelijking* (ook wel bekend als PDV) is een vergelijking waarbij de onbekenden een functie en haar afgeleiden zijn. Er wordt gezegd dat de vergelijking het van orde k is als er afgeleiden tot orde k voorkomen. Deze vergelijkingen, bestudeerd sinds de tijd van Newton, zijn het meest gebruikte wiskundige hulpmiddel bij het modelleren van echte wereldfenomenen, van de astrofysica tot aan de economie. In tegenstelling tot algebraïsche vergelijkingen, waarvoor veel formules en standaardtechnieken beschikbaar zijn, zijn differentiaalvergelijkingen over het algemeen extreem moeilijk om op te lossen[11].

In experimentele wetenschap en engineering bestaat de standaardaanpak van PDV's door ze niet direct op te lossen, maar in plaats daarvan hun oplossingen te benaderen met behulp van krachtige computers. De software die gebruikt wordt is gebaseerd op technieken uit de numerieke analyse, die op zijn beurt uit de theoretische wiskunde voortkomen. In het bijzonder, één manier om een oplossing te bouwen is om te beginnen met het vinden van een functie f_1 die niet noodzakelij-

[10] Stel je voor dat we zuignapschoenen dragen, of dat de donut voldoende zwaartekracht uitoefent om onze voeten te verankeren.

[11] De algemene oplossing van de Navier-Stokes-vergelijking - een fundamentele wet in de vloeistofmechanica - is een miljoen dollar waard. Dit is een van de zeven Millennium-problemen, die in 2000 gesteld zijn door het Clay Mathematics Institute, waarvan er tot nu toe slechts één is opgelost.

kerwijs de PDV hoeft op te lossen, maar dit "tot orde 1"doet. Dit betekent dat, als we uiteindelijk de echte oplossing f vinden, deze dezelfde eerste afgeleiden zal hebben als f_1. Na het vinden van f_1, gaan wij het benaderen verbeteren door te zoeken naar een functie f_2: deze moet niet alleen dezelfde eerste afgeleiden als f (en f_1) hebben, maar ook dezelfde tweede afgeleiden van f.

Terwijl computers op een bepaald punt moeten stoppen (en alleen een numerieke benadering van de oplossing geven), kunnen we proberen dit algebraïsche proces voor onbepaalde tijd te herhalen. Als dit mogelijk is, hebben we een reeks functies gebouwd die steeds beter vergelijkbaar zijn met de veronderstelde oplossing; we noemen zo'n reeks een *formele oplossing*. Als de vergelijking waar we mee begonnen was de integreerbaarheidsconditie van een bijna Γ-structuur, dan is een formele oplossing een *formeel integreerbare bijna* Γ*-structuur* genoemd. Formele integreerbaarheid is een noodzakelijke voorwaarde voor integreerbaarheid en is meestal gemakkelijker te controleren. Bovendien, als de setting van ons probleem "mooi genoeg" is (en dit is het geval in de meeste voorbeelden), komt de hierboven beschreven recursieve procedure tot een conclusie na een eindige hoeveelheid stappen. Dan is elke formele oplossing een feitelijke oplossing en formele integreerbaarheid is een voldoende voorwaarde voor integreerbaarheid.

Belangrijkste resultaten van dit proefschrift Veel auteurs hebben het integreerbaarheidsprobleem voor verschillende voorbeelden van bijna Γ-structuren bestudeerd (en opgelost). Tussen de jaren 60 en 80, werd een algemeen kader ontwikkeld om het integreerbaarheidsprobleem voor een speciale klasse pseudogroepen aan te pakken (met behulp van de gevestigde theorie van "G-structuren"). Wat echter nog ontbrak was een complete kader wat werkt voor een algemene Γ-structuur. Eigenlijk was er niet eens een definitie van bijna Γ-structuur in termen van een generieke Γ!

In dit proefschrift hebben we deze leemte opgevuld, d.w.z. we hebben een algemene definitie van bijna Γ-structuren gegeven, die krachtigere wiskundige objecten nodig heeft, namelijk de groepoïde-hoofdvezelbundels, die in de jaren 80 niet beschikbaar waren. Bovendien geven we een precieze beschrijving van de obstructies voor de formele integreerbaarheid van een bijna Γ-structuur, in termen van enkele "algebraïsche ruimtes", Spencer cohomologiegroepen genaamd, geassocieerd met de pseudogroep Γ. Het begrijpen wanneer een formele oplossing samenvalt met een werkelijke oplossing vereist anderen technieken. In dit geval hebben we een voldoende analytische conditie (in tegenstelling tot de eerder genoemde algebraïsche), waaronder alle formeel integreerbare bijna Γ-structuren integreerbaar zijn. Een laatste probleem dat is behandeld in dit proefschrift is het bewijzen dat "gelijkwaardige" pseudogroepen "gelijkwaardige" bijna structuren induceren, evenals het begrijpen wanneer integreerbaarheid onder deze gelijkwaardigheid behouden blijft.

Tot slot wil ik nog vermelden dat de manier waarop we deze problemen hebben opgelost een extra laag abstractie vereiste. Meer precies, we ontwikkelden het algemeen kader van Pfaffiaanse hoofdvezelbundels, welke de abstracte objecten zijn die de fundamentele eigenschappen van bijna Γ-structuren beschrijven. Vervolgens hebben we het probleem van integreerbaarheid voor bijna Γ-structuren vertaald in de taal van Pfaffiaanse hoofvezeldbundels, het probleem in dit abstract kader opgelost, en opnieuw vertaald naar "concrete" bijna Γ-structuren. Dit is nog een voorbeeld waarbij abstracter werken kan helpen bij het oplossen van concrete problemen.

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Curriculum vitae

Francesco Cattafi was born on the 11th of November 1991 in Biella, a small city in Piedmont, Italy. He is the son of Santi Cattafi and Antonella Turco, and he has a younger sister, Eleonora. He completed high school in 2010 at Liceo Scientifico Statale "Amedeo Avogadro".

In September 2010 he moved to Torino, where he lived for five years at Collegio Universitario Renato Einaudi and he studied mathematics at the Department "Giuseppe Peano" of Università degli studi di Torino. He obtained a bachelor degree (2013, supervised by Lorenzo Fatibene) and a master degree (2015, supervised by Marcella Palese), both *cum laude*, with theses in mathematical physics.

In September 2015 Francesco moved to the Netherlands to begin his PhD at Universiteit Utrecht under the supervision of Marius Crainic. During the years 2015-2019, besides carrying out research in differential geometry, he has been teaching assistant for a total of ten bachelor and master courses, and he has been in charge of organising the local seminars of his research group (the "Friday Fish").

He defended his thesis, entitled "*A general approach to almost structures in geometry*", on February 26th, 2020.