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Koszul duality for Lie algebroids

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ABSTRACT

This paper studies the role of dg-Lie algebroids in derived deformation theory. More precisely, we provide an equivalence between the homotopy theories of formal moduli problems and dg-Lie algebroids over a commutative dg-algebra of characteristic zero. At the level of linear objects, we show that the category of representations of a dg-Lie algebroid is an extension of the category of quasi-coherent sheaves on the corresponding formal moduli problem. We describe this extension geometrically in terms of pro-coherent sheaves.

1. Introduction

Lie algebroids appear throughout algebraic and differential geometry as natural objects describing infinitesimal geometric structures, like foliations and actions of Lie algebras [28,20]. In this paper, we study the relation between Lie algebroids and deformation theory, which can informally be described as follows: suppose that k has characteristic zero and that

$$x\colon \mathrm{Spec}(A) \longrightarrow \mathcal{M}$$

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is a map from an affine (derived) scheme to a moduli space over k. Then a formal neighborhood of $\mathcal{M}$ around x is controlled by a Lie algebroid over Spec(A). This Lie algebroid is given by the vector fields on Spec(A) that are (derived) tangent to the fibers of x.

When Spec(A) is a point, this recovers the well-known relation between deformation problems and dg-Lie algebras, which originates in the work of Kodaira and Spencer [13] and which has been put forward as a key principle in deformation theory by Deligne and Drinfeld. The classification of deformation problems by dg-Lie algebras has been extensively studied and applied [6,14,21], and has been given a precise mathematical formulation in the work of Hinich [9], Pridham [26] and Lurie [17]. An important idea in these works is to describe the formal neighborhood of a moduli space in terms of derived geometry, using its functor of points. More precisely, one can describe a formal neighborhood of $\mathcal{M}$ around x by a functor

$$\mathcal{M}^{\wedge}\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ B \longmapsto \mathcal{M}(B) \times_{\mathcal{M}(A)} \{x\} \tag{1.1}$$

sending a *derived infinitesimal thickening* Spec(A) $\longrightarrow$ Spec(B) to the space of extensions of x to a map $\tilde{x}\colon \mathrm{Spec}(B) \longrightarrow \mathcal{M}$. Such derived infinitesimal thickenings are dual to maps $B \longrightarrow A$ in the ∞-category of *connective commutative k-algebras*, i.e. the ∞-categorical localization of the category of (homologically) nonnegatively graded commutative dg-k-algebras at the quasi-isomorphisms.

For any reasonable moduli space $\mathcal{M}$, the functor $\mathcal{M}^{\wedge}$ satisfies a derived version of the Schlessinger conditions, which encodes the usual obstruction theory for existence and uniqueness of deformations [30]. The works mentioned above provide an equivalence between the homotopy theory of such functors satisfying the Schlessinger conditions and the homotopy theory of dg-Lie algebras, at least when A is a field (see also [8] for an extension to dg-Lie algebras over more general dg-algebras A).

The purpose of this paper is to provide a similar identification of the homotopy theory of dg-Lie algebroids with the homotopy theory of *formal moduli problems*, in the sense of [5,17]:

Definition 1.2. Let A be a connective commutative k-algebra, where k has characteristic zero. The ∞-category $\mathrm{CAlg}_k^{\mathrm{sm}}/A$ of *small extensions* of A is the smallest subcategory of the ∞-category of connective commutative k-algebras over A that contains A and that is closed under square zero extensions by $A[n]$ with $n \geq 0$.

A *formal moduli problem* on A is a functor

$$X\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$$

satisfying the following two conditions:

(a) $X(A) \simeq *$ is contractible.

(b) X preserves any pullback diagram of small extensions of A of the form

$$\begin{array}{ccc} B_\eta & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle (\mathrm{id},0)} \\ B & \underset{\eta}{\longrightarrow} & A \oplus A[n+1]. \end{array}$$

Such a pullback square realizes B_η as a square zero extension of B by $A[n]$ (for $n \geq 0$).

Geometrically, a formal moduli problem can be thought of as a map of stacks $x\colon \mathrm{Spec}(A) \longrightarrow X$ that realizes X as an infinitesimal thickening of $\mathrm{Spec}(A)$. We show that such a stack determines a Lie algebroid $T_{A/X}$, which can be thought of as the fiberwise vector fields on $\mathrm{Spec}(A)$ over X. This construction is part of an equivalence, with inverse sending a Lie algebroid $\mathfrak{g}$ to the 'quotient' of $\mathrm{Spec}(A)$ by the infinitesimal $\mathfrak{g}$-action:

Theorem 1.3. *Suppose that A is eventually coconnective. Then there is an equivalence of ∞-categories*

$$\mathrm{MC}\colon \mathrm{LieAlgd}_A \rightleftarrows \mathrm{FMP}_A \colon T_{A/}$$

between the ∞-category of Lie algebroids over A and the ∞-category of formal moduli problems under A.

The ∞-category of Lie algebroids has an explicit description in terms of homological algebra, as the localization of the category of dg-Lie algebroids over A at the quasi-isomorphisms. One can therefore think of the above result as a rectification result, showing that any formal moduli problem admits a rigid description in terms of chain complexes endowed with algebraic structure. In this sense, Theorem 1.3 provides a complement to the recent work of Gaitsgory and Rozenblyum [5], where Lie algebroids are defined and studied purely in terms of formal moduli problems.

Remark 1.4. Theorem 1.3 relates formal moduli problems and Lie algebroids on an affine derived scheme. Using descent methods, one can extend this result to derived Deligne-Mumford stacks. Recent work by Calaque and Grivaux [3] sketches extensions of Theorem 1.3 to more general kinds of stacks.

The proof of Theorem 1.3 follows the lines of [17], and relies on a version of Koszul duality: the small extensions of A are Koszul dual to certain free Lie algebroids over A, by means of the functor sending a dg-Lie algebroid to its cohomology (as studied already in [28]). To make efficient use of this result, it is useful to study Koszul duality at the level of *linear* objects as well. More precisely, given a quasi-coherent sheaf $\mathcal{F}$ on X, one expects the restriction $x^*\mathcal{F}$ along $x\colon \mathrm{Spec}(A) \longrightarrow X$ to carry a natural representation of the Lie algebroid $T_{A/X}$. Indeed, we prove the following result:

Theorem 1.5. *Let A be eventually coconnective and let X be a formal moduli problem on A with associated Lie algebroid $T_{A/X}$. Then there is a fully faithful, symmetric monoidal left adjoint functor*

$$\Psi_X\colon \mathrm{QC}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}$$

which induces an equivalence on connective objects.

As an application of this result, we give a simple (folklore) chain-level description of the formal moduli problem describing the deformations of a connective commutative A-algebra (or an algebra over some other operad, cf. Remark 6.18):

Proposition 1.6. *Let A be eventually coconnective and let R be a connective commutative A-algebra. Then the formal moduli problem*

$$\mathrm{Def}_R\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\;\; B \longmapsto \mathrm{CAlg}\big(\mathrm{Mod}_B\big) \times_{\mathrm{CAlg}(\mathrm{Mod}_A)} \{R\}$$

is classified by the dg-Lie algebroid of natural derivations of the map $A \longrightarrow R$. The Lie bracket is given by the commutator bracket and the anchor sends a natural derivation to its component at A.

It is well-known (cf. [29,12]) that there are convergence issues that prevent the functor Ψ_X from being an equivalence. For example, for any map $f\colon X \longrightarrow Y$ of formal moduli problems, the restriction functor

$$f^!\colon \mathrm{Rep}_{T_{A/Y}} \longrightarrow \mathrm{Rep}_{T_{A/X}}$$

preserves all limits, while the restriction functor $f^*\colon \mathrm{QC}(Y) \longrightarrow \mathrm{QC}(X)$ does not. For this reason (among many others), a more refined geometric theory of sheaves has been introduced in [4], in which quasi-coherent sheaves are replaced by *ind-coherent sheaves*. For our purposes, it seems more convenient to work with the dual notion of *pro-coherent sheaves*, which are often equivalent to ind-coherent sheaves by Serre duality (see [4, Section 9] or Remarks 7.7 and 7.11). For instance, this is the case for derived schemes which are locally almost of finite type over a field (this condition is imposed on most objects in the book [5]). The ∞-category of pro-coherent sheaves on an eventually coconnective algebra A is an extension

$$\mathrm{Mod}_A = \mathrm{QC}(A) \hookrightarrow \mathrm{QC}^!(A) = \mathrm{Ind}\big(\mathrm{Coh}_A^{\mathrm{op}}\big)$$

of the usual ∞-category of quasi-coherent sheaves, i.e. A-modules (see Section 7.1). This extension exhibits much better behavior with respect to deformation theory. For instance, we will relate pro-coherent sheaves to (quasi-coherent) Lie algebroid representations:

Theorem 1.7. *Let A be eventually coconnective and coherent, and let X be a formal moduli problem on A with associated Lie algebroid $T_{A/X}$. Then there is fully faithful embedding*

$$\mathrm{Rep}_{T_{A/X}} \hookrightarrow \mathrm{QC}^!(X)$$

whose essential image consists of the pro-coherent sheaves on X whose restriction to A is quasi-coherent.

The work of Gaitsgory and Rozenblyum [5] provides an extensive study of derived algebraic geometry in terms of ind-coherent sheaves, or equivalently (by Serre duality) in terms of pro-coherent sheaves. In particular, they develop a theory of pro-coherent Lie algebroids and their representations, which is formulated entirely in terms of formal moduli problems and their categories of pro-coherent sheaves. More precisely, they consider a slightly larger ∞-category of small extensions of A, which also contains square zero extensions of A by connective *coherent* A-modules. A Lie algebroid is then defined to be a functor

$$X\colon \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$$

that satisfies the Schlessinger conditions, analogous to conditions (a) and (b) of Definition 1.2. We will refer to such a functor as a *pro-coherent formal moduli problem*, since it has a tangent space functor

$$T_X\colon \mathrm{Coh}_A^{\geq 0} \longrightarrow \mathcal{S};\ \ E \longmapsto X\big(A \oplus E\big)$$

that defines a pro-coherent sheaf on A. Similarly, a representation of a Lie algebroid is essentially defined to be a pro-coherent sheaf on such a pro-coherent formal moduli problem [5, Vol. II, Ch. 8, §4].

Remark 1.8. One can think of pro-coherent formal moduli problems as refinements of formal moduli problems (as in Definition 1.2), which are adapted to 'almost finite type' situations (as considered in [5]). Such situations appear naturally in deformation theory, because small extensions and Postnikov stages of finitely presented algebras are typically only almost of finite type.

In terms of linear algebra, the usefulness of pro-coherent formal moduli problems can be seen as follows. Suppose that A is eventually coconnective and almost of finite type, and let $f\colon \mathrm{Spec}(A) \longrightarrow \mathcal{M}$ be a map to a derived Artin stack which is locally finitely presented. The *formal completion* of $\mathcal{M}$ at $\mathrm{Spec}(A)$ is then controlled by the cotangent complex $L_{A/\mathcal{M}}$, while the associated formal moduli problem (1.1) is only controlled by its A-linear dual $T_{A/\mathcal{M}}$. In general, one cannot retrieve $L_{A/\mathcal{M}}$ from its linear dual, unless it is perfect, for instance. This puts some further constraints on the map f (and hence on A), such as being finitely presentation.

In contrast, one can recover the cotangent complex $L_{A/\mathcal{M}}$ from its pro-coherent dual: indeed, $L_{A/\mathcal{M}}$ is a locally almost finitely presented A-module (Definition 7.3), which implies that $L_{A/\mathcal{M}} \simeq L^{\vee\vee}_{A/\mathcal{M}}$ in the ∞-category of pro-coherent sheaves (this follows from Corollary 8.9). Accordingly, the formal completion of $\mathcal{M}$ at $\mathrm{Spec}(A)$ is completely determined by a pro-coherent formal moduli problem [5, Vol. II, Ch. 5, §1.4]. Because of this, pro-coherent formal moduli problems appear for instance in the study of D-modules, i.e. ind-coherent sheaves on de Rham stacks (in other words, on formal completions of points) [5, Vol. II, Ch. 4].

In certain (somewhat restricted) situations, we show that the rectification results from Theorem 1.3 and Theorem 1.7 can also be applied to pro-coherent formal moduli problems. In particular, this allows one to study formal completions of derived stacks (locally almost of finite type) algebraically in terms of dg-Lie algebroids. Our results make use of a simple point-set model for Lie algebroids in the pro-coherent setting, based on a certain 'tame' model structure on dg-A-modules [1,25]. Unfortunately, the resulting tame homotopy theory of dg-Lie algebroids is only well behaved when A satisfies some rather strict technical assumptions at the chain level (see Warning 8.14). As an important example, these technical conditions are met when k is Noetherian and eventually coconnective and when

$$\mathrm{Spec}(A) = \mathbb{A}^n_k \times_{\mathbb{A}^m_k} \{0\}.$$

In particular, for $A = k$ we obtain a simple point-set model for the ∞-category of pro-coherent Lie algebras over any coherent, eventually coconnective A.

In the cases where we have a good point-set model for pro-coherent Lie algebroids over A, we show (Theorem 8.2) that there is an equivalence of ∞-categories

$$T_{A/}\colon \mathrm{FMP}^!_A \xrightarrow{\sim} \mathrm{LieAlgd}^!_A$$

between pro-coherent formal moduli problems and pro-coherent Lie algebroids over A. Theorem 1.7 can then be refined (Theorem 8.3) to an equivalence

$$\Psi_X\colon \mathrm{QC}^!(X) \xrightarrow{\sim} \mathrm{Rep}^!_{T_{A/X}}$$

between pro-coherent sheaves on X and pro-coherent $T_{A/X}$-representations.

Outline. The paper is outlined as follows. In Section 2, we recall the basic homotopy theory of dg-Lie algebroids over a commutative dg-algebra A, in which the weak equivalences are the quasi-isomorphisms. Our main result (Theorem 1.3) is proven in Section 5, based on results about Lie algebroid cohomology that are discussed in Section 3 and 4. Section 6 is devoted to a proof of Theorem 1.5.

In Section 7, we discuss the theory of pro-coherent sheaves on formal moduli problems and prove Theorem 1.7. Finally, Section 8 describes the homotopy theories of pro-coherent sheaves and pro-coherent Lie algebroids in model categorical terms. Using these, we establish extensions of Theorem 1.3 and Theorem 1.7 to the pro-coherent setting.

Conventions. Throughout, let $\mathbb{Q} \subseteq k$ be a fixed connective commutative dg-algebra of characteristic zero and let A be a connective cdga over k. All differential-graded objects are homologically graded, so that *connective* objects are concentrated in non-negative degrees. Homology and homotopy groups are both denoted by π_i. Given a chain complex V, we denote its suspension and cone by $V[1]$ and $V[0,1]$.

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2. Recollections on DG-Lie algebroids

In this section we recall the homotopy theory of dg-Lie algebroids over a commutative dg-algebra, based on the discussion in [23].

2.1. DG-Lie algebroids

Recall that the tangent module of a commutative dg-k-algebra A is the dg-A-module of k-linear derivations of A

$$T_A = \mathrm{Der}_k(A, A).$$

The commutator bracket endows this complex with the structure of a dg-Lie-algebra over k.

Definition 2.1. A *dg-Lie algebroid* $\mathfrak{g}$ over A (relative to k) is an unbounded dg-A-module $\mathfrak{g}$, equipped with a k-linear dg-Lie algebra structure and an *anchor map* $\rho\colon \mathfrak{g} \longrightarrow T_A$ such that

(1) ρ is both a map of dg-A-modules and dg-Lie algebras.

(2) the failure of the Lie bracket to be A-bilinear is governed by the Leibniz rule

$$[X, a \cdot Y] = (-1)^{Xa} a[X, Y] + \rho(X)(a) \cdot Y.$$

Let $\mathrm{LieAlgd}^{\mathrm{dg}}{}_A$ be the category of dg-Lie algebroids over A, with maps given by A-linear maps over T_A that preserve the Lie bracket.

Example 2.2. Any dg-A-module E gives rise to an *Atiyah dg-Lie algebroid* $\mathrm{At}(E)$ over A, which can be described as follows: a degree n element of $\mathrm{At}(E)$ is a tuple (v, ∇_v) consisting of a derivation $v\colon A \longrightarrow A$ (of degree n), together with a k-linear map $\nabla_v\colon E \longrightarrow E$ (of degree n) such that

$$\nabla_v(a \cdot e) = v(a) \cdot e + (-1)^{|a| \cdot n} a \cdot \nabla_v(e)$$

for all $a \in A$ and $e \in E$. This becomes a dg-A-module under pointwise multiplication and a dg-Lie algebra under the commutator bracket. The anchor map is the obvious projection $\mathrm{At}(E) \longrightarrow T_A$ sending (v, ∇_v) to v.

Example 2.3. Similarly, suppose that $E \in \mathrm{Mod}_A^{\mathrm{dg}}$ has the structure of an algebra over a k-linear dg-operad $\mathcal{P}$. Then there is a sub dg-Lie algebroid $\mathrm{At}_{\mathcal{P}}(E) \subseteq \mathrm{At}(E)$ consisting of the tuples (v, ∇_v) where ∇_v is a $\mathcal{P}$-algebra derivation.

The following is the main result of [23]:

Proposition 2.4. *The category of dg-Lie algebroids over A carries the* projective *semi-model structure, in which a map is a weak equivalence (fibration) if and only if it is a quasi-isomorphism (degreewise surjective). Furthermore, the forgetful functor to the projective model structure on dg-A-modules*

$$\mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$$

is a right Quillen functor that preserves all sifted homotopy colimits.

Remark 2.5. Let $0 \longrightarrow \mathfrak{g} \longrightarrow T_A$ be a fibrant-cofibrant replacement of the initial dg-Lie algebroid over A. Then there is a Quillen equivalence

$$\mathrm{LieAlgd}_A^{\mathrm{dg}} \rightleftarrows \mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}.$$

Using the simplicial structure from [32] and the fact that every object in $\mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}$ is fibrant, one finds that $\mathfrak{g}/\mathrm{LieAlgd}_A^{\mathrm{dg}}$ is a genuine (combinatorial) model category.

Definition 2.6. Let A be a cofibrant connective cdga over k. We define the *∞-category of Lie algebroids over A* to be the ∞-categorical localization

$$\mathrm{LieAlgd}_A := \mathrm{LieAlgd}_A^{\mathrm{dg}}\Big[\{\text{quasi-iso}\}^{-1}\Big].$$

This is a locally presentable ∞-category since $\mathrm{LieAlgd}_A^{\mathrm{dg}}$ is Quillen equivalent to a combinatorial model category (Remark 2.5).

Remark 2.7. The condition that A is cofibrant over k guarantees that the tangent module T_A has the correct homotopy type.

Definition 2.8. We will say that a dg-Lie algebroid $\mathfrak{g}$ is *A-cofibrant* when it is cofibrant as a dg-A-module. Every cofibrant dg-Lie algebroid is A-cofibrant. Conversely, if $\mathfrak{g}$ is A-cofibrant, then it has an explicit cofibrant replacement $Q(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g}$, described as follows [23, Section 5]: without differential, $Q(\mathfrak{g})$ is freely generated by the A-linear map

$$\Big(\mathrm{Sym}_A^{\geq 1}\mathfrak{g}[1]\Big)[-1] \xrightarrow{\ \pi\ } \mathfrak{g} \xrightarrow{\ \rho\ } T_A$$

where π is the obvious projection. As a dg-Lie algebra, $Q(\mathfrak{g})$ is a quotient of the A-linear extension of the usual operadic bar-cobar resolution $\Omega B\mathfrak{g}$ of the dg-Lie algebra underlying $\mathfrak{g}$ [15, Theorem 11.3.6] (also denoted $\mathcal{LC}(\mathfrak{g})$ by Quillen [27, Appendix B]). This determines the differential.

2.2. Representations

Recall that a *representation* of a dg-Lie algebroid $\mathfrak{g}$ is a dg-A-module E, together with a Lie algebra representation $\nabla\colon \mathfrak{g}\otimes_k E \longrightarrow E$ such that (without Koszul signs)

$$\nabla_{a\cdot X}(e) = a\cdot\nabla_X(e) \qquad\qquad \nabla_X(a\cdot e) = a\cdot\nabla_X(e) + \rho(X)(a)\cdot e$$

for all $a\in A, X\in\mathfrak{g}$ and $e\in E$. Equivalently, a $\mathfrak{g}$-representation on E is a map

$$\mathfrak{g} \longrightarrow \mathrm{At}(E)$$

to the Atiyah Lie algebroid of E (Example 2.2). Such representations can be organized into a symmetric monoidal category $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$, with tensor product given by $E\otimes_A F$, endowed with the $\mathfrak{g}$-representation

$$\nabla_X(e\otimes f) = \nabla_X(e)\otimes f + e\otimes\nabla_X(f).$$

The internal hom is given by $\mathrm{Hom}_A(E,F)$, endowed with the $\mathfrak{g}$-representation by conjugation.

Example 2.9. Let $\mathcal{P}$ be a dg-operad. Then a $\mathcal{P}$-algebra in $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$ is simply a $\mathcal{P}$-algebra in $\mathrm{Mod}_A^{\mathrm{dg}}$, equipped with a $\mathfrak{g}$-representation acting by $\mathcal{P}$-algebra derivations. Equivalently, such a $\mathcal{P}$-algebra structure on a $\mathfrak{g}$-representation E is determined by a map of dg-Lie algebroids $\mathfrak{g} \longrightarrow \mathrm{At}_{\mathcal{P}}(E)$ (Example 2.3).

One can identify $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$ with the category of left modules over the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ [28, Section 2]. In particular, it carries the projective model structure, where a map is a weak equivalence (fibration) if it is a quasi-isomorphism (surjection). We define the ∞-*category of* $\mathfrak{g}$-*representations* to be the associated ∞-category

$$\mathrm{Rep}_{\mathfrak{g}} := \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}\Big[\{\text{quasi-iso}\}^{-1}\Big].$$

Lemma 2.10. *Any map between dg-Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ induces a Quillen adjunction between the projective model structures*

$$f_*\colon \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \rightleftarrows \mathrm{Rep}_{\mathfrak{h}}^{\mathrm{dg}}\colon f^!$$

where $f^!$ restricts a representation along f. When $\mathfrak{g}$ and $\mathfrak{h}$ are A-cofibrant and f is a weak equivalence, this Quillen adjunction is a Quillen equivalence.

Proof. The map f induces a map of universal enveloping algebras $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$, which gives rise to the Quillen pair $(f_*, f^!)$. The left adjoint f_* is given by the functor $E \mapsto \mathcal{U}(\mathfrak{h}) \otimes_{\mathcal{U}(\mathfrak{g})} E$.

For the second part, recall that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has a natural PBW filtration, obtained by declaring generators from A to have weight 0 and generators from $\mathfrak{g}$ to have weight 1 [28, Section 3]. The associated graded is a graded cdga and comes equipped with a surjective map of graded cdgas

$$\mathrm{Sym}_A(\mathfrak{g}) \longrightarrow \mathrm{gr}\big(\mathcal{U}(\mathfrak{g})\big).$$

When $\mathfrak{g}$ is projective as a graded A-module, this map (or rather, the underlying map of graded graded-commutative algebras) is an isomorphism by the PBW theorem of [28], which applies verbatim in the graded setting.

Now suppose that $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is a weak equivalence between A-cofibrant dg-Lie algebroids. The map $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ is compatible with the PBW filtrations and is given by $\mathrm{Sym}_A(f)\colon \mathrm{Sym}_A(\mathfrak{g}) \longrightarrow \mathrm{Sym}_A(\mathfrak{h})$ on the associated graded. Since $\mathfrak{g}$ and $\mathfrak{h}$ are cofibrant as dg-A-modules, $\mathrm{Sym}_A(f)$ is a weak equivalence. By induction along the PBW filtration, this implies that $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ is a weak equivalence, so that $(f_*, f^!)$ is a Quillen equivalence. □

Lemma 2.11. *Suppose that $\mathfrak{g}$ is A-cofibrant. Then the category $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$ also carries a combinatorial model structure where a map is a weak equivalence (cofibration) if the underlying map of dg-A-modules is a weak equivalence (cofibration).*

Proof. We can apply the criterion from [16, Proposition A.2.6.8], using that there are sets of generating (trivial) cofibrations by [16, Lemma A.3.3.3]. To see that a map with the right lifting property against the cofibrations is a weak equivalence, note that it is in particular a trivial fibration of dg-A-modules. Indeed, $\mathcal{U}(\mathfrak{g})$ is cofibrant as a dg-A-module (by the PBW theorem), so that any projective cofibration of $\mathfrak{g}$-representations is also a cofibration of dg-A-modules. □

Remark 2.12. The '*A-injective*' model structure from Lemma 2.11 has the following properties:

(1) The identity functor is a Quillen equivalence from the projective to the A-injective model structure (both have the same weak equivalences). In particular, the ∞-category $\mathrm{Rep}_{\mathfrak{g}}$ can also be studied using the A-injective model structure.

(2) For any map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ between A-cofibrant dg-Lie algebroids, the functor $f^!$ is the left adjoint in a Quillen pair between the A-injective model structures

$$f^!\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{h}} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\colon f_!.$$

The right Quillen functor $f_!$ sends E to the coinduction $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}\big(\mathcal{U}(\mathfrak{h}), E\big)$. This is a Quillen equivalence when f is a Quillen equivalence, because the (left derived) functor of ∞-categories $f^!\colon \mathrm{Rep}_{\mathfrak{h}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ agrees with the right derived functor of Lemma 2.10.

(3) The A-injective model structure manifestly defines a monoidal model structure on $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$, and each $f^!$ is a monoidal left Quillen functor. It follows that $\mathrm{Rep}_{\mathfrak{g}}$ is a closed symmetric monoidal ∞-category [19, Example 4.1.7.6, Lemma 4.1.8.8]. In fact, when $\mathfrak{g}$ is A-cofibrant, one can show that the projective model structure satisfies the pushout-product axiom as well: this uses that the tensor product of free representations $\mathcal{U}(\mathfrak{g}) \otimes_A \mathcal{U}(\mathfrak{g})$ is isomorphic to the free representation generated by $\mathcal{U}(\mathfrak{g})$. However, the unit A is not cofibrant in the projective model structure.

In particular, the properties from Remark 2.12 imply that a map between A-cofibrant dg-Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ induces a symmetric monoidal functor $f^!\colon \mathrm{Rep}_{\mathfrak{h}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ between presentable (closed) symmetric monoidal ∞-categories. Since the ∞-category of Lie algebroids can be obtained from the category of A-cofibrant dg-Lie algebroids by inverting the quasi-isomorphisms, we obtain a functor

$$\mathrm{Rep}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}}/\mathrm{Mod}_A := \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})/\mathrm{Mod}_A$$

to the ∞-category of presentable (closed) symmetric monoidal ∞-categories over Mod_A. This functor sends $\mathfrak{g}$ to the forgetful functor $\mathrm{Rep}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}_A$ (or equivalently, the functor restricting along the map of Lie algebroids $0 \longrightarrow \mathfrak{g}$). For a more precise description of this functor in terms of fibrations, see Section 6.2.

Lemma 2.13. *The functor* Rep *preserves all limits.*

Proof. We can forget about the symmetric monoidal structures, since the forgetful functor $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ preserves limits and detects equivalences. For any map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, the restriction functor $f^!$ is both a left and a right adjoint. It therefore suffices to show that the functor $\mathrm{Rep}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{R}}/\mathrm{Mod}_A$ preserves limits.

To see this, consider $\mathcal{U}(\mathfrak{g})$ as a unital associative algebra in the category of dg-A-*bimodules* (over $\mathbb{Q}$), using the map of algebras $A \longrightarrow \mathcal{U}(\mathfrak{g})$. The category of dg-$A$-bimodules carries a monoidal model structure, whose weak equivalences and fibrations are transferred along the forgetful functor $\mathrm{BiMod}_A^{\mathrm{dg}} \longrightarrow \mathrm{LMod}_A^{\mathrm{dg}}$. Since we are working in characteristic zero, there exists a transferred model structure on the category of associative algebras in $\mathrm{BiMod}_A^{\mathrm{dg}}$ as well. By [19, Theorem 4.1.8.4], this is a model for the ∞-category of algebras in the monoidal ∞-category BiMod_A of A-bimodules (in [19, Theorem 4.1.8.4], the monoid axiom and symmetry of the monoidal structure are only used for the existence of a model structure on algebras).

The functor Rep now factors over the ∞-category of algebras in A-bimodules

$$\mathrm{LieAlgd}_A^{\mathrm{op}} \xrightarrow{\ \mathcal{U}\ } \mathrm{Alg}\big(\mathrm{BiMod}_A\big)^{\mathrm{op}} \xrightarrow{\ \mathrm{LMod}\ } \mathrm{Pr}^{\mathrm{R}}/\mathrm{Mod}_A.$$

The functor LMod preserves all limits by [19, Theorem 4.8.5.11] (see the remarks just above Corollary 4.8.5.13 in [19]). It remains to verify that $\mathcal{U}\colon \mathrm{LieAlgd}_A \longrightarrow \mathrm{Alg}\big(\mathrm{BiMod}_A\big)$ preserves colimits. At the chain level, $\mathcal{U}$ admits a right adjoint, sending an algebra B to the dg-Lie algebroid of tuples $(b \in B, v \in T_A)$ such that $[b, a] = v(a)$ in B. Because it is not clear to us that this adjoint descends to the level of ∞-categories, we will verify by hand that $\mathcal{U}$ preserves (homotopy) colimits. First, observe that $\mathcal{U}$ preserves sifted colimits, since the composite

$$\mathrm{LieAlgd}_A \xrightarrow{\ \mathcal{U}\ } \mathrm{Alg}\big(\mathrm{BiMod}_A\big) \xrightarrow{\ \mathrm{forget}\ } \mathrm{BiMod}_A$$

preserves sifted colimits by [23, Theorem 4.22]. To see that $\mathcal{U}\colon \mathrm{LieAlgd}_A \longrightarrow \mathrm{Alg}\big(\mathrm{BiMod}_A\big)$ preserves finite coproducts, note that $\mathrm{LieAlgd}_A$ is generated under sifted colimits by free Lie algebroids $F(V)$ on A-modules V, equipped with the zero map $0\colon V \longrightarrow T_A$ (cf. [23, Corollary 3.8]). It therefore suffices to verify that the composite

$$\mathrm{Mod}_A \xrightarrow{\ F\ } \mathrm{LieAlgd}_A \xrightarrow{\ \mathcal{U}\ } \mathrm{Alg}\big(\mathrm{BiMod}_A\big)$$

preserves finite coproducts. Unraveling the definitions, one sees that for any cofibrant dg-A-module V, the dg-algebra $\mathcal{U}(F(V))$ is naturally equivalent to the A-linear tensor algebra $T_A(V)$. In other words, the above functor is naturally equivalent to the functor

$$\mathrm{Mod}_A \xrightarrow{\ \Delta\ } \mathrm{BiMod}_A \xrightarrow{\ \mathrm{Free}\ } \mathrm{Alg}\big(\mathrm{BiMod}_A\big)$$

sending an A-module to the free algebra on V, considered as a symmetric A-bimodule. This functor clearly preserves colimits. □

3. Lie algebroid cohomology

The purpose of this section is to prove the following:

Proposition 3.1. *Let A be a cofibrant commutative dg-k-algebra. There is an adjunction of ∞-categories*

$$C^*\colon \mathrm{LieAlgd}_A \rightleftarrows \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}} : \mathfrak{D} \tag{3.2}$$

between the ∞-category of Lie algebroids over A and the ∞-category of unbounded commutative k-algebras over A (i.e. cdgas up to quasi-isomorphism). The right adjoint sends $B \longrightarrow A$ to the dual in Mod_A of the map between cotangent complexes $L_A \longrightarrow L_{A/B}$ (over k).

On A-cofibrant dg-Lie algebroids, the left adjoint in (3.2) is given by the *Chevalley-Eilenberg complex* (with trivial coefficients). Recall that for any representation E of a dg-Lie algebroid $\mathfrak{g}$, this complex (with coefficients in E) is given by the graded vector space

$$C^*(\mathfrak{g}, E) := \mathrm{Hom}_A\Big(\mathrm{Sym}_A\mathfrak{g}[1], E\Big)$$

with the usual Chevalley-Eilenberg (or de Rham) differential (without Koszul signs)

$$\begin{aligned}(\partial\alpha)(X_1, \ldots, X_n) = \partial_E\big(\alpha(X_1, \ldots, X_n)\big) - \sum_i \alpha(X_1, \ldots, \partial X_i, \ldots, X_n)\\ + \sum_i \nabla_{X_i}\Big(\alpha(X_1, \ldots, X_n)\Big) - \sum_{i<j} \alpha\Big([X_i, X_j], X_1, \ldots, X_n\Big).\end{aligned}$$

There is a natural k-linear augmentation map $C^*(\mathfrak{g}, E) \longrightarrow E$, evaluating at the unit element of $\mathrm{Sym}_A\big(\mathfrak{g}[1]\big)$.

Remark 3.3. The shuffle product of forms defines a lax symmetric monoidal structure on $C^*(\mathfrak{g}, -)$, which is compatible with the augmentation in the sense that there is a commuting square

$$\begin{array}{ccc} C^*(\mathfrak{g}, E) \otimes_k C^*(\mathfrak{g}, F) & \xrightarrow{\times} & C^*(\mathfrak{g}, E \otimes_A F) \\ \downarrow & & \downarrow \\ E \otimes_k F & \longrightarrow & E \otimes_A F. \end{array}$$

In particular, taking coefficients with values in the commutative algebra A, one obtains a functor

$$C^*\colon \mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \Big(\mathrm{CAlg}_k^{\mathrm{dg}}/A\Big)^{\mathrm{op}};\ \mathfrak{g} \longmapsto C^*(\mathfrak{g}) := C^*(\mathfrak{g}, A).$$

For every dg-Lie algebroid $\mathfrak{g}$, the Chevalley-Eilenberg complex yields a lax symmetric monoidal functor $C^*(\mathfrak{g}, -)\colon \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_{C^*(\mathfrak{g})}^{\mathrm{dg}}$.

The proof of Proposition 3.1 is given at the very end of this section and uses some formal properties of the Chevalley-Eilenberg complex. To understand these properties, it will be useful to first give a slightly more model-categorical characterization of the Chevalley-Eilenberg complex.

3.1. The cotangent complex of a Lie algebroid

Consider the right Quillen functor

$$\mathfrak{g} \oplus (-)\colon \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{dg}}/\mathfrak{g}$$

sending a $\mathfrak{g}$-representation E to the square zero extension of $\mathfrak{g}$ by E. If we denote the value of the left adjoint functor on $\mathfrak{g}$ itself by $\Upsilon_{\mathfrak{g}}$, then the other values of the left adjoint are given by

$$\Big(f\colon \mathfrak{h} \longrightarrow \mathfrak{g}\Big) \longmapsto f_*\Upsilon_{\mathfrak{h}} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \Upsilon_{\mathfrak{h}}. \tag{3.4}$$

Definition 3.5. Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. The *cotangent complex* $L_{\mathfrak{g}}$ of $\mathfrak{g}$ is the value of the left derived functor of (3.4) on the identity map of $\mathfrak{g}$. In other words, it is the universal $\mathfrak{g}$-representation classified by

$$\mathrm{Map}_{\mathcal{U}(\mathfrak{g})}(L_{\mathfrak{g}}, E) \simeq \mathrm{Map}_{/\mathfrak{g}}\big(\mathfrak{g}, \mathfrak{g} \oplus E\big).$$

Example 3.6. Let $\mathfrak{g} = F(V)$ be the free dg-Lie algebroid generated by an A-linear map $V \longrightarrow T_A$ whose domain is cofibrant. Then $\mathfrak{g}$ is a cofibrant dg-Lie algebroid and for any $\mathfrak{g}$-representation E, there is a natural bijection between sections $\mathfrak{g} \longrightarrow \mathfrak{g} \oplus E$ and A-linear maps $V \longrightarrow E$. It follows that $L_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g}) \otimes_A V$ is the free $\mathfrak{g}$-representation generated by the dg-A-module V.

More generally, one can use the cofibrant replacement $Q(\mathfrak{g})$ of Definition 2.8 to compute the cotangent complex:

Proposition 3.7. *Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. Then the cotangent complex $L_\mathfrak{g}$ can be modeled by the cofibrant left $\mathcal{U}(\mathfrak{g})$-module*

$$L_\mathfrak{g} = \mathcal{U}(\mathfrak{g}) \otimes_A \left(\mathrm{Sym}_A^{\geq 1} \mathfrak{g}[1]\right)[-1] \tag{3.8}$$

with differential given (modulo Koszul signs) by

$$\begin{aligned} \partial(u \otimes X_1 \dots X_n) = (\partial u) \otimes X_1 \dots X_n + \sum_i u \otimes X_1 \dots \partial(X_i) \dots X_n \\ \overset{(n>1)}{+} \sum_i u \cdot X_k \otimes X_1 \dots X_n + \sum_{i<j} u \otimes [X_i, X_j] X_1 \dots X_n. \end{aligned} \tag{3.9}$$

The first term in the second line only applies when $n > 1$.

Proof. Since $\mathfrak{g}$ is A-cofibrant, it suffices to find a $\mathfrak{g}$-representation corepresenting the functor

$$\mathrm{Rep}_\mathfrak{g}^{\mathrm{dg}} \longrightarrow \mathrm{Set}; \ \ E \longmapsto \mathrm{Hom}_{/\mathfrak{g}}(Q(\mathfrak{g}), E).$$

By [23, Corollary 6.16], there is a natural bijection between maps $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E$ over $\mathfrak{g}$ and 0-cycles in the kernel of the augmentation map $C^*(\mathfrak{g}, E[1]) \longrightarrow E[1]$. On the other hand, unwinding the definition of the complex $L_\mathfrak{g}$ in (3.8), one sees that there is a short exact sequence

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}\big(L_\mathfrak{g}, E\big) \longrightarrow C^*\big(\mathfrak{g}, E[1]\big) \longrightarrow E[1]. \tag{3.10}$$

Passing to 0-cycles, one finds that the complex $L_\mathfrak{g}$ given in (3.8) indeed represents maps $Q(\mathfrak{g}) \longrightarrow \mathfrak{g} \oplus E$ over $\mathfrak{g}$. □

Remark 3.11. The cotangent complex (3.8) comes with a $\mathcal{U}(\mathfrak{g})$-linear map

$$L_\mathfrak{g} = \mathcal{U}(\mathfrak{g}) \otimes_A \left(\mathrm{Sym}_A^{\geq 1} \mathfrak{g}[1]\right)[-1] \longrightarrow \mathcal{U}(\mathfrak{g})$$

sending $u \otimes X_1 \dots X_n$ to zero when $n > 1$ and to $u \cdot X_1$ when $n = 1$. The *Koszul complex* $K(\mathfrak{g})$ of $\mathfrak{g}$ is the mapping cone of this map. It fits onto a cofiber sequence

$$L_\mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow K(\mathfrak{g}).$$

Unraveling the definitions, $K(\mathfrak{g})$ can be identified with $\mathcal{U}(\mathfrak{g}) \otimes_A \mathrm{Sym}_A\big(\mathfrak{g}[1]\big)$, with differential given by formula (3.9), but where the term in the second line is also included when $n = 1$. The Chevalley-Eilenberg complex $C^*(\mathfrak{g}, E)$ can be identified with $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(K(\mathfrak{g}), E)$, so that the above cofiber sequence induces (a shift of) the fiber sequence (3.10) on mapping complexes.

Remark 3.12. The composite map

$$L_{\mathfrak{g}} = \mathcal{U}(\mathfrak{g}) \otimes_A \left(\mathrm{Sym}_A^{\geq 1} \mathfrak{g}[1]\right)[-1] \longrightarrow \mathcal{U}(\mathfrak{g}) \xrightarrow{u \mapsto u \cdot 1} A$$

is equal to zero, so that there is a $\mathcal{U}(\mathfrak{g})$-linear map $K(\mathfrak{g}) \longrightarrow A$. When $\mathfrak{g}$ is A-cofibrant, this map is a weak equivalence. Indeed, the PBW filtration on $\mathcal{U}(\mathfrak{g})$ (see the proof of Lemma 2.10) and the filtration on $\mathrm{Sym}_A(\mathfrak{g}[1])$ by polynomial degree determine a total filtration on the Koszul complex $\mathcal{K}(\mathfrak{g})$. The map on the associated graded is the obvious projection

$$\mathrm{Sym}_A(\mathfrak{g}[0,1]) = \mathrm{Sym}_A(\mathfrak{g}) \otimes_A \mathrm{Sym}_A(\mathfrak{g}[1]) \longrightarrow A$$

from the symmetric algebra on the cone $\mathfrak{g}[0,1]$, which is a weak equivalence.

In other words, $C^*(\mathfrak{g}, E)$ is a model for the derived mapping space $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(A, E)$. The lax symmetric monoidal structure on $C^*(\mathfrak{g}, -)$ arises from the fact that A is a cocommutative coalgebra in $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}}$.

3.2. The free case

When $\mathfrak{g} = F(V)$ is the free dg-Lie algebroid generated by a cofibrant dg-A-module over T_A, we now have two different (but weakly equivalent) descriptions of the cotangent complex $L_{\mathfrak{g}}$: Example 3.6 simply evaluates the left Quillen functor (3.4) on $\mathfrak{g}$ itself, while Proposition 3.7 computes the value of (3.4) on the 'cobar' resolution $Q(\mathfrak{g})$. Of course, the first description is significantly smaller than the second. To compare the two, note that there is a canonical section of the map $Q(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g}$, induced by the canonical inclusion

$$V \hookrightarrow \mathfrak{g} \longrightarrow \left(\mathrm{Sym}_A \mathfrak{g}[1]\right)[-1] \subseteq Q(\mathfrak{g}).$$

Applying (3.4) to this section produces a weak equivalence between the two models for the cotangent complex $L_{\mathfrak{g}}$, given by the $\mathcal{U}(\mathfrak{g})$-linear extension of the above inclusion

$$\mathcal{U}(\mathfrak{g}) \otimes_A V \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}) \otimes_A \left(\mathrm{Sym}_A^{\geq 1} \mathfrak{g}[1]\right)[-1].$$

Restriction along this map induces a weak equivalence to a significantly smaller complex

$$\kappa\colon C^*(\mathfrak{g}, A) \xrightarrow{\sim} A^V = A \oplus_{\rho^\vee} \mathrm{Hom}_A(V[1], A);\;\; \alpha \longmapsto \left(\alpha(1), \alpha|_{V[1]}\right).$$

The codomain A^V has the natural structure of a commutative dg-algebra, the *square zero extension* of A by $V[1]^\vee = \mathrm{Hom}_A(V[1], A)$ classified by the map

$$\rho^\vee\colon \Omega_A \longrightarrow V^\vee;\ d_{\mathrm{dR}}a \longmapsto \big(v \mapsto \rho(v)(a)\big).$$

In other words, A^V fits into a (homotopy) pullback square of cdgas

$$\begin{array}{ccc} A^V = A \oplus_{\rho^\vee} V[1]^\vee & \longrightarrow & A \oplus V[0,1]^\vee \\ \downarrow & & \downarrow \\ A & \xrightarrow[\rho^\vee]{} & A \oplus V^\vee. \end{array} \tag{3.13}$$

Using this pullback square, one can easily check that the functor

$$A^{(-)}\colon \mathrm{Mod}^{\mathrm{dg}}_A/T_A \longrightarrow \big(\mathrm{CAlg}^{\mathrm{dg}}_k/A\big)^{\mathrm{op}}; V \longmapsto A^V = A \oplus_{\rho^\vee} V[1]^\vee.$$

is a left Quillen functor from the model category of dg-A-modules to the model category of cdgas over A. Its right adjoint sends $B \longrightarrow A$ to the mapping fiber of the map

$$T_A = \mathrm{Der}_k(A, A) \longrightarrow \mathrm{Der}_k(B, A),$$

together with its natural projection to T_A. We may therefore summarize the previous discussion by the following result:

Corollary 3.14. *There is a natural transformation to a right Quillen functor*

$$\mathrm{Mod}^{\mathrm{dg}}_A/T_A \underset{A^{(-)}}{\overset{C^*\circ F}{\rightrightarrows}} \big(\mathrm{CAlg}^{\mathrm{dg}}_k/A\big)^{\mathrm{op}} \qquad (\kappa\colon C^*\circ F \Rightarrow A^{(-)})$$

which is a quasi-isomorphism when restricted to cofibrant dg-A-modules over T_A.

Let us finally turn to the proof of Proposition 3.1:

Proof (of Proposition 3.1). Consider the functor

$$\mathrm{LieAlgd}^{\mathrm{dg}}_A \xrightarrow{C^*} \big(\mathrm{CAlg}^{\mathrm{dg}}_k/A\big)^{\mathrm{op}} \xrightarrow{\ker} \big(\mathrm{Mod}^{\mathrm{dg}}_k\big)^{\mathrm{op}}$$

sending a dg-Lie algebroid to the kernel of the map $C^*(\mathfrak{g}) \longrightarrow A$. By Proposition 3.7, this functor can be identified with the composite

$$\mathrm{LieAlgd}^{\mathrm{dg}}_A \xrightarrow{Q} \mathrm{LieAlgd}^{\mathrm{dg}}_A \xrightarrow{\Upsilon} \mathrm{Rep}^{\mathrm{dg}}_{T_A} \xrightarrow{\mathrm{Hom}_{\mathcal{U}(T_A)}(-,A)} \big(\mathrm{Mod}^{\mathrm{dg}}_k\big)^{\mathrm{op}}.$$

The functor Q sends an A-cofibrant dg-Lie algebroid to its cofibrant replacement, the functor Υ is the functor (3.4) sending $\mathfrak{g}$ to $\mathcal{U}(T_A) \otimes_{\mathcal{U}(\mathfrak{g})} \Upsilon_{\mathfrak{g}}$ and the last functor takes the maps from a T_A-representation to A. Since the last two functors are left Quillen functors, it follows that C^* preserves weak equivalences between A-cofibrant dg-Lie algebroids. We therefore obtain functors of ∞-categories

$$\mathrm{LieAlgd}_A \xrightarrow{C^*} \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}} \xrightarrow{\ker} \big(\mathrm{Mod}_k\big)^{\mathrm{op}}.$$

The composition of these two functors preserves all colimits. Since the functor $\ker\colon \mathrm{CAlg}_k/A \longrightarrow \mathrm{Mod}_k$ detects equivalences and preserves all limits, it follows that C^* preserves all colimits. By the adjoint functor theorem, it follows that C^* admits a right adjoint $\mathfrak{D}\colon \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}} \longrightarrow \mathrm{LieAlgd}_A$.

It remains to describe this right adjoint $\mathfrak{D}$, at least at the level of the underlying A-modules. To this end, observe that the composite

$$\big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}} \xrightarrow{\mathfrak{D}} \mathrm{LieAlgd}_A \xrightarrow{U} \mathrm{Mod}_A/T_A$$

is right adjoint to the functor $C^* \circ F\colon \mathrm{Mod}_A/T_A \longrightarrow \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}}$. Corollary 3.14 provides a natural equivalence $\kappa\colon C^* \circ F \longrightarrow A^{(-)}$, so that $U \circ \mathfrak{D}$ is equivalent to the derived right adjoint to the left Quillen functor $A^{(-)}$. Because A is cofibrant over k, the discussion preceding Corollary 3.14 shows that this derived functor sends $B \longrightarrow A$ to the A-linear dual of $L_A \longrightarrow L_{A/B}$. $\square$

Remark 3.15. The functor $\mathfrak{D}$ does not admit a straightforward point-set description. However, when $B \longrightarrow A$ is a *cofibration* of commutative dg-algebras, its image under $\mathfrak{D}$ can be identified with the sub-dg-Lie algebroid $T_{A/B} := \mathrm{Der}_B(A, A) \longrightarrow T_A$ of B-linear derivations of A. To see this, note that there is a natural diagram of commutative dg-k-algebras over A

$$\begin{array}{ccccc} B & \xrightarrow{f} & C^*(T_{A/B}) & \longrightarrow & C^*(\tilde{T}_{A/B}) \\ & {\scriptstyle\phi}\searrow & \downarrow & \swarrow & \\ & & A & & \end{array}$$

where $\tilde{T}_{A/B} \longrightarrow T_{A/B}$ is a cofibrant replacement. For any $b \in B$, its image under f is the map

$$\mathrm{Sym}_A\big(T_{A/B}[1]\big) \xrightarrow{T_{A/B} \mapsto 0} A \xrightarrow{\phi(b)\cdot(-)} A.$$

The map f respects the differential because the derivations in $T_{A/B}$ are B-linear. The composition $B \longrightarrow C^*(\tilde{T}_{A/B})$ is adjoint to a map $\tilde{T}_{A/B} \longrightarrow \mathfrak{D}(B \longrightarrow A)$. At the level of the underlying A-modules, this map is simply given by the composite map

$$\tilde{T}_{A/B} \xrightarrow{\sim} T_{A/B} \longrightarrow L^\vee_{A/B}.$$

Because $B \longrightarrow A$ was a cofibration, the second map is a weak equivalence. It follows that the dg-Lie algebroid $\tilde{T}_{A/B}$ (and therefore $T_{A/B}$) is weakly equivalent to $\mathfrak{D}(B \longrightarrow A)$.

4. Koszul duality

The purpose of this section is to show that the adjunction from Proposition 3.1

$$C^* \colon \mathrm{LieAlgd}_A \rightleftarrows \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}} \colon \mathfrak{D}$$

is not too far from being an equivalence: when restricted to suitably finite-dimensional dg-Lie algebroids, the functor C^* is fully faithful.

Proposition 4.1. *Let A be a cofibrant connective commutative dg-k-algebra and let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid. Suppose $\mathfrak{g}$ satisfies the following conditions:*

(i) *As a graded A-module, $\mathfrak{g}$ is free on a set of generators x_i.*
(ii) *There are finitely many x_i in each single degree, and all generators have (homological) degree ≤ -1.*

Then the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$ can be identified with the map of dg-A-modules

$$\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$$

from $\mathfrak{g}$ into its A-linear bidual.

Corollary 4.2. *Let A be a cofibrant commutative dg-k-algebra which is eventually coconnective. Then $C^* \colon \mathrm{LieAlgd}_A \longrightarrow \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}}$ is fully faithful on all Lie algebroids that can be modeled by A-cofibrant dg-Lie algebroids satisfying conditions (i) and (ii) of Proposition 4.1.*

Proof. It suffices to verify that the map $\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$ is a quasi-isomorphism. Forgetting about the differential, we can write $\mathfrak{g} = \bigoplus_{n<0} A^{\oplus k_n}[n]$. It follows that $\mathfrak{g}^\vee$ is a cofibrant dg-A-module, so that $\mathfrak{g}^{\vee\vee}$ is a model for the derived bidual of $\mathfrak{g}$. It therefore suffices to pick a weak equivalence $A \longrightarrow \overline{A}$ to a bounded cdga and verify that the map $\mathfrak{g} \otimes_A \overline{A} \longrightarrow (\mathfrak{g} \otimes_A \overline{A})^{\vee\vee}$ is a quasi-isomorphism. Unraveling the definitions, this map can be identified with

$$\bigoplus_{n<0} \overline{A}^{\oplus k_n}[n] \longrightarrow \prod_{n<0} \overline{A}^{\oplus k_n}[n],$$

which is an isomorphism because $\overline{A}$ is bounded. $\square$

To prove Proposition 4.1, let us start by considering the map of commutative dg-algebras over A

$$c\colon C^*(\mathfrak{g}) \longrightarrow A^{\mathfrak{g}} = A \oplus_{\rho^\vee} \mathfrak{g}[1]^\vee \tag{4.3}$$

which sends $\alpha\colon \mathrm{Sym}_A\big(\mathfrak{g}[1]\big) \longrightarrow A$ to the restriction $\big(\alpha(1), \alpha\big|\mathfrak{g}[1]\big)$. Just above Corollary 3.14, we have seen that the functor $\mathfrak{g} \mapsto A^{\mathfrak{g}}$ was a left Quillen functor, whose derived right adjoint sent

$$\big(B \longrightarrow A\big) \longmapsto \big(L^\vee_{A/B} \longrightarrow L^\vee_A = T_A\big).$$

Lemma 4.4. *Let $\mathfrak{g}$ be an A-cofibrant dg-Lie algebroid and consider the A-linear map $\mathfrak{g} \longrightarrow L^\vee_{A/C^*(\mathfrak{g})}$ adjoint to (4.3). This map is equivalent to the A-linear map underlying the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$.*

Proof. Let $F\colon \mathrm{Mod}_A/T_A \leftrightarrows \mathrm{LieAlgd}_A\colon U$ be the free-forgetful adjunction. The A-linear map $U(\mathfrak{g}) \longrightarrow U\mathfrak{D}C^*(\mathfrak{g})$ in Mod_A/T_A corresponds by adjunction to the map

$$C^*(\mathfrak{g}) \longrightarrow C^*(F(\mathfrak{g}))$$

in CAlg/A. The composition of this map with the equivalence $\kappa\colon C^*(F(\mathfrak{g})) \longrightarrow A^{\mathfrak{g}}$ from Corollary 3.14 is exactly the map (4.3). This means that the maps

$$\mathfrak{g} \longrightarrow L^\vee_{A/C^*(\mathfrak{g})} \quad \text{and} \quad U(\mathfrak{g}) \longrightarrow U\mathfrak{D}C^*(\mathfrak{g})$$

are identified under the adjoint equivalence between $U\mathfrak{D}$ and the functor sending $B \longrightarrow A$ to $L^\vee_{A/B}$. $\square$

To use Lemma 4.4, we will have to compute the relative cotangent complex of the map $C^*(\mathfrak{g}) \longrightarrow A$. Unfortunately, $C^*(\mathfrak{g})$ has the structure of a power series algebra, which means that $C^*(\mathfrak{g})$ is not cofibrant and computing its cotangent complex requires some effort. Let us therefore introduce the following 'global' variant of the Chevalley-Eilenberg complex:

Construction 4.5. Let A be a cofibrant commutative dg-k-algebra and let $\mathfrak{g}$ be a dg-Lie algebroid over A satisfying conditions (i) and (ii) of Proposition 4.1, i.e. $\mathfrak{g} \cong \bigoplus_{n<0} A^{\oplus k_n}[n]$ as a graded A-module. Let

$$C^*_{\mathrm{poly}}(\mathfrak{g}) := \mathrm{Sym}_A\big(\mathfrak{g}[1]^\vee\big) \subseteq C^*(\mathfrak{g})$$

be the graded-subalgebra of $C^*(\mathfrak{g})$ consisting of graded A-linear maps $\mathrm{Sym}_A\big(\mathfrak{g}[1]\big) \longrightarrow A$ that vanish on some $\mathrm{Sym}_A^{\geq n}\mathfrak{g}[1]$. This graded subalgebra of $C^*(\mathfrak{g})$ is closed under the Chevalley-Eilenberg differential of $C^*(\mathfrak{g})$, which sends a function vanishing on $\mathrm{Sym}_A^{\geq n}\mathfrak{g}[1]$ to a function vanishing on $\mathrm{Sym}_A^{\geq n+1}\mathfrak{g}[1]$.

Example 4.6. Let $\mathfrak{g} = A^{\oplus n}[-1]$ be the trivial dg-Lie algebroid on n generators of degree -1. Then $C^*(\mathfrak{g})$ is isomorphic to the ring of power-series $A[[x_1, ..., x_n]]$ and the inclusion $C^*_{\mathrm{poly}}(\mathfrak{g}) \subseteq C^*(\mathfrak{g})$ is the inclusion of the polynomial algebra $A[x_1, ..., x_n] \subseteq A[[x_1, ..., x_n]]$.

Warning 4.7. Note that $C^*_{\mathrm{poly}}(-)$ does not preserve quasi-isomorphisms.

Lemma 4.8. *Let A be a cofibrant commutative dg-k-algebra and let $\mathfrak{g}$ be a dg-Lie algebroid over A such that $\mathfrak{g} \cong \bigoplus_{n<0} A^{\oplus k_n}[n]$ as a graded A-module. Then the following hold:*

(1) *$C^*_{\mathrm{poly}}(\mathfrak{g})$ is a cofibrant commutative dg-k-algebra.*
(2) *the map on Kähler differentials (relative to the base cdga k)*

$$\Omega_{C^*_{\mathrm{poly}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{g})} A \longrightarrow \Omega_A$$

can be identified with the projection map $\Omega_A \oplus \mathfrak{g}[1]^\vee \longrightarrow \Omega_A$. Here $\Omega_A \oplus \mathfrak{g}[1]^\vee$ is the mapping fiber of the dual of the anchor map, i.e. it has differential

$$\partial\big(d_{\mathrm{dR}}(a), \alpha\big) = \Big(d_{\mathrm{dR}}(\partial_A a), \partial_{\mathfrak{g}[1]^\vee}(\alpha) + \rho^\vee(d_{\mathrm{dR}} a)\Big)$$

where $\rho^\vee : \Omega_A \longrightarrow \mathfrak{g}^\vee$ is the adjoint of the anchor map $\mathfrak{g} \longrightarrow T_A$.

Proof. Since $\mathfrak{g}$ is given by $\bigoplus_{i<0} A^{\oplus k_i}[i]$ as a graded A-module, $C^*_{\mathrm{poly}}(\mathfrak{g})$ is a polynomial algebra over A, generated by the free module $\mathfrak{g}[1]^\vee$. This module has generators in degrees ≥ 0 and A is cofibrant, so that $C^*_{\mathrm{poly}}(\mathfrak{g})$ is the retract of a connective graded polynomial ring over k. This implies that $C^*_{\mathrm{poly}}(\mathfrak{g})$ is cofibrant.

For the second assertion, note that $C^*_{\mathrm{poly}}(\mathfrak{g})$ is freely generated over A by the graded A-module $\mathfrak{g}[1]^\vee$. It follows that

$$\Omega_{C^*_{\mathrm{poly}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{g})} A \cong \Omega_A \oplus \mathfrak{g}[1]^\vee$$

as a graded A-module. The map $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow A$ sends all $\mathfrak{g}[1]^\vee$ to zero, so that the induced map on Kähler differentials is just the projection $\Omega_A \oplus \mathfrak{g}[1]^\vee \longrightarrow \Omega_A$. Furthermore, the Chevalley-Eilenberg differential is given by

$$\begin{aligned} A \ni a &\mapsto \partial_A a + \rho^\vee(d_{\mathrm{dR}} a) \\ \mathfrak{g}[1]^\vee \ni \alpha &\mapsto \partial_{\mathfrak{g}[1]^\vee}\alpha \qquad \mathrm{mod}\ \big(\mathfrak{g}[1]^\vee\big)^2. \end{aligned}$$

This shows that the differential on $\Omega_A \oplus \mathfrak{g}[1]^\vee$ is given as in the lemma. □

Lemma 4.9. *Let A be a nonnegatively graded commutative $\mathbb{Q}$-algebra, let V be a finite dimensional $\mathbb{Q}$-vector space and let W be a degreewise finite-dimensional graded $\mathbb{Q}$-vector space, concentrated in degrees ≤ -1. Then there is a natural isomorphism of graded-commutative A-algebras*

$$\mathrm{Hom}_{\mathbb{Q}}\big(\mathrm{Sym}_{\mathbb{Q}} V, A\big) \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}(W^\vee) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}\big(\mathrm{Sym}_{\mathbb{Q}}(V \oplus W), A\big)$$

where $\mathrm{Sym}_{\mathbb{Q}} W^\vee$ is the graded polynomial algebra on the dual vector space of W.

Proof. Observe that there is an isomorphism of graded cocommutative coalgebras $\mathrm{Sym}_{\mathbb{Q}}(V \oplus W) \cong \mathrm{Sym}_{\mathbb{Q}} V \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}} W$. There is a natural map of graded-commutative algebras

$$\mathrm{Hom}_{\mathbb{Q}}\big(\mathrm{Sym}_{\mathbb{Q}} V, A\big) \otimes_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Sym}_{\mathbb{Q}} W, \mathbb{Q}) \xrightarrow{\ \mu\ } \mathrm{Hom}_{\mathbb{Q}}\big(\mathrm{Sym}_{\mathbb{Q}} V \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}} W, A\big).$$

sending two maps $\alpha\colon \mathrm{Sym}_{\mathbb{Q}} V \longrightarrow A$ and $\beta\colon \mathrm{Sym}_{\mathbb{Q}} W \longrightarrow \mathbb{Q}$ to $\alpha \otimes \beta$. Using that $\mathrm{Sym}_{\mathbb{Q}} W$ is free on generators of degrees ≤ -1, with finitely many generators in each degree, one can identify $\mathrm{Sym}_{\mathbb{Q}}(W^\vee) \simeq \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Sym}_{\mathbb{Q}} W, \mathbb{Q})$. Using this, it follows that μ is an isomorphism. □

Lemma 4.10. *Let A be a cofibrant commutative dg-k-algebra and let $\mathfrak{g}$ be a dg-Lie algebroid over A such that $\mathfrak{g} \cong \bigoplus_{n<0} A^{\oplus k_n}[n]$ as a graded A-module. Then the map $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g})$ induces an equivalence on cotangent complexes over k*

$$L_{C^*_{\mathrm{poly}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{g})} A \xrightarrow{\ \simeq\ } L_{C^*(\mathfrak{g})} \otimes_{C^*(\mathfrak{g})} A.$$

Proof. Consider the trivial cofibration, followed by a fibration

$$0 \xrightarrow{\ \sim\ } \mathfrak{h} = F\big(\mathfrak{g}[0,-1]\big) \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{g}$$

where $\mathfrak{h}$ is the free dg-Lie algebroid on the map $\mathfrak{g}[0,-1] \longrightarrow \mathfrak{g} \longrightarrow T_A$ from the (contractible) path space of $\mathfrak{g}$. Let V be the free graded $\mathbb{Q}$-vector space spanned by the generators x_i of $\mathfrak{g}$, so that $\mathfrak{g} = A \otimes_{\mathbb{Q}} V$. As a graded Lie algebroid, $\mathfrak{h}$ is then freely generated by the graded $\mathbb{Q}$-vector space $V[0,-1]$. Consequently, the map $\mathfrak{h} \longrightarrow \mathfrak{g}$ is given without differentials by the A-linear extension of a map from the free Lie algebra

$$\mathfrak{h} = A \otimes \mathrm{Lie}\big(V[0,-1]\big) \longrightarrow A \otimes V = \mathfrak{g}$$

which sends V to the generators of $\mathfrak{g}$ and $V[-1]$ to zero. Without differentials, this map has a splitting, induced by the inclusion

$$V \longrightarrow V[0,-1] \longrightarrow \mathrm{Lie}\big(V[0,-1]\big)$$

where the first map (which does not respect differentials) includes V into its path object and the second map is the inclusion of the generators in the free Lie algebra. Using this splitting, we can identify $\mathfrak{h} \longrightarrow \mathfrak{g}$ with

$$\mathfrak{h} = A \otimes (V \oplus W) \xrightarrow{(\mathrm{id},0)} A \otimes V = \mathfrak{g}.$$

Here W is a graded $\mathbb{Q}$-vector space isomorphic to $\mathrm{Lie}\big(V[0,-1]\big)/V$, which is degreewise finite dimensional and concentrated in degrees ≤ -2. Indeed, V is degreewise finite dimensional and concentrated in degrees ≤ -1, so that $\mathrm{Lie}\big(V[0,-1]\big)$ has these properties as well. The map $V \longrightarrow \mathrm{Lie}\big(V[0,-1]\big)$ is an isomorphism in degree -1, since the elements in $V[-1]$ and brackets of elements in $V[0,-1]$ are all of degree ≤ -2.

Let us now consider the commutative diagram of cdgas associated to $\mathfrak{h} \longrightarrow \mathfrak{g}$

$$\begin{array}{ccccc} C^*_{\mathrm{poly}}(\mathfrak{g}) & \longrightarrow & C^*_{\mathrm{poly}}(\mathfrak{h}) & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ C^*(\mathfrak{g}) & \longrightarrow & C^*(\mathfrak{h}) & \xrightarrow[\sim]{} & A. \end{array} \tag{4.11}$$

The right bottom map is a weak equivalence since $\mathfrak{h}$ is cofibrant and weakly contractible. The map $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow C^*_{\mathrm{poly}}(\mathfrak{h})$ can be identified with a map of polynomial algebras

$$A \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}\big(V[1]^\vee\big) \longrightarrow A \otimes_{\mathbb{Q}} \mathrm{Sym}_{\mathbb{Q}}\big((V \oplus W)[1]^\vee\big).$$

It follows that $C^*_{\mathrm{poly}}(\mathfrak{h})$ is freely generated over $C^*_{\mathrm{poly}}(\mathfrak{g})$ by $W[1]^\vee$, which is degreewise finite dimensional and concentrated in degrees ≥ 1. In particular, $C^*_{\mathrm{poly}}(\mathfrak{g}) \longrightarrow C^*_{\mathrm{poly}}(\mathfrak{h})$ is a cofibration of cdgas.

On the other hand, the map $C^*(\mathfrak{g}) \longrightarrow C^*(\mathfrak{h})$ is given without differentials by

$$\mathrm{Hom}_{\mathbb{Q}}\Big(\mathrm{Sym}_{\mathbb{Q}} V[1], A\Big) \longrightarrow \mathrm{Hom}_{\mathbb{Q}}\Big(\mathrm{Sym}_{\mathbb{Q}}(V[1] \oplus W[1]), A\Big).$$

It now follows from Lemma 4.9 that the left square in (4.11) is a (homotopy) pushout square of cdgas. Its image under $L_{(-)} \otimes_{(-)} A$

$$\begin{array}{ccc} L_{C^*_{\mathrm{poly}}(\mathfrak{g})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{g})} A & \longrightarrow & L_{C^*_{\mathrm{poly}}(\mathfrak{h})} \otimes_{C^*_{\mathrm{poly}}(\mathfrak{h})} A \\ \downarrow & & \downarrow \\ L_{C^*(\mathfrak{g})} \otimes_{C^*(\mathfrak{g})} A & \longrightarrow & L_{C^*(\mathfrak{h})} \otimes_{C^*(\mathfrak{h})} A \end{array} \tag{4.12}$$

is a homotopy pushout square as well. Since $C^*(\mathfrak{h}) \longrightarrow A$ is a weak equivalence, the map $L_{C^*(\mathfrak{h})} \longrightarrow L_A$ is a weak equivalence. On the other hand, the map $L_{C^*_{\text{poly}}(\mathfrak{h})\otimes C^*_{\text{poly}}(\mathfrak{h})} A \longrightarrow L_A$ is identified with the projection map

$$\Omega_A \oplus \mathfrak{h}[1]^\vee \longrightarrow \Omega_A$$

by Lemma 4.8. The kernel of this map is contractible, since $\mathfrak{h}$ is a cofibrant contractible dg-A-module. It follows that the right vertical map in Diagram (4.12) is an equivalence, so that the left map is an equivalence as well. □

Proof (of Proposition 4.1). By Lemma 4.4, it suffices to show that the map $\mathfrak{g} \longrightarrow L^\vee_{A/C^*}$ is adjoint to a weak equivalence $L_{A/C^*(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$. This map fits into a sequence of maps

$$L_{A/C^*_{\text{poly}}(\mathfrak{g})} \longrightarrow L_{A/C^*(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$$

classifying the composite map of commutative dg-algebras over A

$$C^*_{\text{poly}}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g}) \xrightarrow{c} A^{\mathfrak{g}} = A \oplus_{\rho^\vee} \mathfrak{g}[1]^\vee \tag{4.13}$$

where c is as in (4.3). The map $L_{A/C^*_{\text{poly}}(\mathfrak{g})} \longrightarrow L_{A/C^*(\mathfrak{g})}$ is an equivalence by Lemma 4.10, so it suffices to show that $L_{A/C^*_{\text{poly}}(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$ is an equivalence. This map can be computed explicitly: the map (4.13) is simply the quotient of $C^*_{\text{poly}}(\mathfrak{g})$ by the augmentation ideal $(\mathfrak{g}[1]^\vee)^2$. Unwinding the definitions, e.g. using the pullback square (3.13), one finds the following description of the classifying map $L_{A/C^*_{\text{poly}}(\mathfrak{g})} \longrightarrow \mathfrak{g}^\vee$: it is the canonical map from the mapping cone of

$$\Omega_{C^*_{\text{poly}}(\mathfrak{g})} \otimes_{C^*_{\text{poly}}(\mathfrak{g})} A \cong \Omega_A \oplus \mathfrak{g}[1]^\vee \longrightarrow \Omega_A$$

to $\mathfrak{g}^\vee$. This map is a weak equivalence, which concludes the proof. □

5. Formal moduli problems

We will now use the results of Section 4 to establish an equivalence between Lie algebroids and formal moduli problems (Definition 1.2):

Theorem 5.1. *Let A be a cofibrant connective commutative dg-k-algebra. Then there is an adjoint pair of functors*

$$\text{MC}\colon \text{LieAlgd}_A \rightleftarrows \text{FMP}_A \colon T_{A/}$$

which is an equivalence whenever A is eventually coconnective.

Theorem 5.1 follows formally from Corollary 4.2, by means of a general procedure due to Lurie [17] that we will briefly recall.

5.1. Generators

Categories of chain complexes or spectra endowed with a certain algebraic structure often admit a presentation in terms of generators and relations.

Definition 5.2. Let Ξ be a locally presentable ∞-category equipped with a collection of right adjoint functors

$$e_\alpha\colon \Xi \longrightarrow \mathrm{Mod}_{\mathbb{S}} := \mathrm{Sp}$$

to the ∞-category of spectra (the reader may replace $\mathrm{Mod}_{\mathbb{S}}$ by the ∞-category $\mathrm{Mod}_{\mathbb{Z}}$ of chain complexes in everything that follows, by Example 5.6). The left adjoint to e_α sends the map $\mathbb{S}[n] \longrightarrow 0$ in $\mathrm{Mod}_{\mathbb{S}}$ to a map in Ξ that we will denote by $K_{\alpha,n} \longrightarrow \emptyset$. We will say that an object $X \in \Xi$ is *good* if it admits a finite filtration

$$\emptyset = X^{(0)} \longrightarrow X^{(1)} \longrightarrow \dots \longrightarrow X^{(n)}$$

where for each i, there is an α and a $n \leq -2$, together with a pushout square

$$\begin{array}{ccc} K_{\alpha,n} & \longrightarrow & X^{(i-1)} \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & X^{(i)}. \end{array} \tag{5.3}$$

Let $\Xi^{\mathrm{good}} \subseteq \Xi$ be the full subcategory on the good objects; it is the smallest subcategory of Ξ which contains $\emptyset$ and is closed under pushouts along the maps $K_{\alpha,n} \longrightarrow \emptyset$ with $n \leq -2$.

Proposition 5.4 *([17, Theorem 1.3.12]). Let (Ξ, e_α) be as in Definition 5.2. Suppose that each e_α preserves small sifted homotopy colimits and that a map f in Ξ is an equivalence if and only if each $e_\alpha(f)$ is an equivalence of spectra. Then the right adjoint functor*

$$\Psi\colon \Xi \longrightarrow \mathrm{PSh}(\Xi^{\mathrm{good}});\ \ X \longmapsto \mathrm{Map}_{\Xi}(-, X)$$

is fully faithful, with essential image consisting of those (space-valued) presheaves F satisfying the following two conditions:

(a) *$F(\emptyset)$ is contractible.*
(b) *For any α and $n \leq -2$, F sends a pushout square of the form (5.3) to a pullback square of spaces.*

This is exactly [17, Theorem 1.3.12], replacing the category Υ^{sm} from [17] by the opposite of Ξ^{good}.

Example 5.5. Let Ξ be a compactly generated stable ∞-category. The collection of mapping spectrum functors $\mathrm{Hom}_{\Xi}(K, -)\colon \Xi \longrightarrow \mathrm{Mod}_{\mathbb{S}}$, for all compact objects K, satisfies the condition of Proposition 5.4. The good objects are exactly the compact objects and a functor $F\colon \Xi^{\omega,\mathrm{op}} \longrightarrow \mathcal{S}$ satisfies conditions (a) and (b) if and only if it is left exact. Proposition 5.4 then reproduces the equivalence

$$\Xi \simeq \mathrm{Ind}(\Xi^{\omega}).$$

Example 5.6. Let A be a connective commutative dg-algebra and let Mod_A be the ∞-category of A-modules. The single functor $e\colon \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{S}}$, forgetting the A-module structure, satisfies the conditions of Proposition 5.4. In this case, the good A-modules can be presented by the dg-A-modules whose underlying graded A-module is free on finitely many generators x_i of degree < 0.

There is an equivalence of ∞-categories

$$\mathrm{Mod}_A^{\mathrm{good,op}} \longrightarrow \mathrm{Mod}_A^{\mathrm{f.p.},\geq 0};\ E \longrightarrow E[1]^{\vee}$$

to the ∞-category of finitely presented connective A-modules, i.e. dg-A-modules generated by finitely many generators of degree ≥ 0. Combining this equivalence with Proposition 5.4, one finds that Mod_A is equivalent to the ∞-category of functors

$$F\colon \mathrm{Mod}_A^{\mathrm{f.p.},\geq 0} \longrightarrow \mathcal{S}$$

that send 0 to a contractible space and preserve pullbacks along the maps $0 \longrightarrow A[n]$ with $n \geq 1$.

5.2. Proof of Theorem 5.1

Let A be a cofibrant commutative dg-k-algebra. By Proposition 2.4 and Example 5.6, the composite forgetful functor

$$\mathrm{LieAlgd}_A \xrightarrow{U} \mathrm{Mod}_A/T_A \xrightarrow{\ker} \mathrm{Mod}_A \xrightarrow{e} \mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp}$$

preserves small limits and sifted colimits, and detects equivalences. The corresponding notion of a *good Lie algebroid* can be described in terms of the (projective) model structure on dg-Lie algebroids as follows: let us say that a dg-Lie algebroid $\mathfrak{g}$ is *very good* if it admits a finite sequence of cofibrations

$$0 = \mathfrak{g}^{(0)} \longrightarrow \cdots \longrightarrow \mathfrak{g}^{(n)} = \mathfrak{g},$$

each of which is the pushout of a (generating) cofibration with $n_i \leq -2$

$$\mathrm{Free}\big(\partial\phi\colon A[n_i] \longrightarrow T_A\big) \longrightarrow \mathrm{Free}\big(\phi\colon A[n_i, n_i+1] \longrightarrow T_A\big). \tag{5.7}$$

Here ϕ is a map from the cone of $A[n_i]$ to T_A, which is determined uniquely by a degree (n_i+1) element of T_A. Then the good Lie algebroids can be presented by the very good dg-Lie algebroids over A.

Remark 5.8. Good Lie algebroids may have nontrivial anchor maps, even though the Lie algebroids in (5.7) have null-homotopic anchor maps.

Lemma 5.9. *Let $\mathfrak{g}$ be a very good dg-Lie algebroid over A. Then the following hold:*

(1) *$\mathfrak{g}$ has a cofibrant underlying dg-A-module.*
(2) *Without the differential, $\mathfrak{g}$ is freely generated by a negatively graded finite-dimensional vector space over T_A.*
(3) *$\mathfrak{g}$ is isomorphic as a graded A-module to $\bigoplus_{n<0} A^{\oplus k_n}[n]$ for some sequence of $k_n \in \mathbb{N}_{\geq 0}$.*

Proof. Assertion (1) is obvious and (3) follows immediately from (2). For (2), note that each pushout along a map (5.7) freely adds a single generator of degree < 0 at the level of graded Lie algebroids. □

Proof (of Theorem 5.1). Let us denote the ∞-category of presheaves satisfying conditions (a) and (b) of Proposition 5.4 by

$$\mathcal{E} \subseteq \mathrm{PSh}\big(\mathrm{LieAlgd}_A^{\mathrm{good}}\big).$$

We then have an equivalence $\mathrm{LieAlgd}_A \simeq \mathcal{E}$, so that it suffices to produce the required adjunction (equivalence) between E and the ∞-category of formal moduli problems. To this end, recall that the good Lie algebroids form the smallest subcategory of $\mathrm{LieAlgd}_A$ that contains 0 and is closed under pushouts against the maps

$$F\big(0\colon A[n] \longrightarrow T_A\big) \longrightarrow 0 \qquad n \leq -2. \tag{5.10}$$

The functor C^* preserves colimits (Proposition 3.1) and sends the above maps to the maps $(\mathrm{id}, 0)\colon A \longrightarrow A \oplus A[-1-n]$, for $n \leq -2$. It follows that C^* restricts to a functor

$$C^*\colon \mathrm{LieAlgd}_A^{\mathrm{good}} \longrightarrow \Big(\mathrm{CAlg}_k^{\mathrm{sm}}/A\Big)^{\mathrm{op}}. \tag{5.11}$$

The restriction of a formal moduli problem along C^* is a presheaf contained in $\mathcal{E}$. We therefore obtain a right adjoint functor

$$T_{A/}\colon \mathrm{FMP}_A \xrightarrow{(C^*)^*} \mathcal{E} \xrightarrow{\simeq} \mathrm{LieAlgd}_A.$$

If A is eventually coconnective, Corollary 4.2 and Lemma 5.9 show that (5.11) is fully faithful. The essential image of C^* contains A and is closed under pullbacks along the maps $(\mathrm{id}, 0)\colon A \longrightarrow A \oplus A[n]$ for $n \geq 1$. Indeed, such pullbacks can dually be computed as pushouts along the images of the maps (5.10). The small extension of A form the smallest subcategory of CAlg_k/A with these two closure properties, so (5.11) is essentially surjective as well. This implies that the functor $T_{A/}$ is an equivalence. □

Example 5.12. Let $f\colon B \longrightarrow A$ be a map of connective commutative k-algebras and consider the formal moduli problem

$$\mathrm{Spec}(B)^\wedge\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ \ \tilde{A} \longmapsto \mathrm{Map}(B, \tilde{A}) \times_{\mathrm{Map}(B,A)} \{f\}.$$

Unwinding the definitions, one sees that the associated Lie algebroid is given by $\mathfrak{D}(B)$, which can be identified with the Lie algebroid $T_{A/B} \longrightarrow T_A$ of (derived) B-linear derivations of A by Remark 3.15.

Remark 5.13. Suppose that A is eventually coconnective, so that $\mathrm{CAlg}_k^{\mathrm{sm}}/A$ is equivalent to the ∞-category of good Lie algebroids over A. If $\mathfrak{g}$ is a Lie algebroid over A, then the formal moduli problem $\mathrm{MC}_\mathfrak{g}$ is given (up to a natural equivalence) by

$$\mathrm{MC}_\mathfrak{g}(B) = \mathrm{Map}_{\mathrm{LieAlgd}_A}\big(\mathfrak{D}(B), \mathfrak{g}\big).$$

By Remark 5.12, one can think of this as the space of flat $\mathfrak{g}$-valued connections on the fiberwise tangent bundle of A over B.

6. Representations and quasi-coherent sheaves

In the previous section, we have seen that there is an equivalence between formal moduli problems over A and Lie algebroids over A. Geometrically, one can think of a formal moduli problem as a map of stacks $x\colon \mathrm{Spec}(A) \longrightarrow X$ that realizes X as an infinitesimal thickening of $\mathrm{Spec}(A)$. Any such stack X gives rise to an ∞-category of *quasi-coherent sheaves* on X in the usual way (Definition 6.13): a quasi-coherent sheaf F is given by a collection of B-modules F_y for every $B \in \mathrm{CAlg}_k^{\mathrm{sm}}/A$ and every $y \in X(B)$, together with a coherent family of equivalences

$$F_{\alpha(y)} \simeq B' \otimes_B F_y$$

for every $\alpha\colon B \longrightarrow B'$. In particular, a quasi-coherent sheaf F on X determines an A-module F_x, by restricting to the canonical point $x \in X(A)$. We will see that F_x carries a representation of $T_{A/X}$. In fact, we will prove the following:

Theorem 6.1. *Let A be eventually coconnective and let X be a formal moduli problem on A with associated Lie algebroid $T_{A/X}$. Then there is a fully faithful, symmetric monoidal left adjoint functor*

$$\Psi_X\colon \mathrm{QC}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}$$

where $\Psi_X(F)$ has underlying A-module given by the restriction F_x to the canonical base-point $x \in X(A)$. Furthermore, the functor Ψ_X induces an equivalence

$$\mathrm{QC}(X)^{\geq 0} \simeq \mathrm{Rep}^{\geq 0}_{T_{A/X}}$$

between the connective quasi-coherent sheaves on X and the $T_{A/X}$-representations whose underlying A-module is connective.

Our proof closely follows the discussion in [17, Section 2.4]. We will begin by considering representations of good Lie algebroids in Section 6.1. Theorem 6.1 then follows from a formal argument described in Section 6.3.

6.1. Representations of good Lie algebroids

Recall from Remark 3.3 that there is a lax monoidal functor sending a $\mathfrak{g}$-representation E to its Chevalley-Eilenberg complex $C^*(\mathfrak{g}, E)$. This functor is part of an adjoint pair

$$K(\mathfrak{g}) \otimes_{C^*(\mathfrak{g})} (-)\colon \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}\colon C^*(\mathfrak{g}, -) \tag{6.2}$$

were $K(\mathfrak{g})$ is the Koszul complex of $\mathfrak{g}$ (see Remark 3.11).

When $\mathfrak{g}$ is A-cofibrant, this adjoint pair is a Quillen pair between the projective model structure on dg-$C^*(\mathfrak{g})$-modules and the model structure on $\mathfrak{g}$-representations. We let $\Phi_{\mathfrak{g}}\colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ denote the left derived functor between ∞-categories. One can think of $\Phi_{\mathfrak{g}}$ as the functor $F \mapsto A \otimes_{C^*(\mathfrak{g})} F$, since $K(\mathfrak{g})$ is equivalent to the $\mathfrak{g}$-representation A.

Lemma 6.3. *Suppose that $\mathfrak{g} \in \mathrm{LieAlgd}_A$ is compact. Then $\Phi_{\mathfrak{g}}\colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ is fully faithful.*

Proof. Let $\mathcal{C}$ be the class of objects in $\mathrm{Mod}_{C^*(\mathfrak{g})}$ for which the derived unit map is an equivalence. Then $\mathcal{C}$ is closed under finite colimits and retracts and contains $C^*(\mathfrak{g})$ because $\Phi_{\mathfrak{g}}(C^*(\mathfrak{g})) \simeq A$. It follows that the unit map is an equivalence for all compact $C^*(\mathfrak{g})$-modules. Since $\mathrm{Mod}_{C^*(\mathfrak{g})} \simeq \mathrm{Ind}(\mathrm{Mod}^{\omega}_{C^*(\mathfrak{g})})$ is the ind-completion of the category of compact $C^*(\mathfrak{g})$-modules, it suffices to show that the right adjoint $C^*(\mathfrak{g}, -)$ preserves filtered colimits. Equivalently, it suffices to verify that $\Phi_{\mathfrak{g}}(C^*(\mathfrak{g})) \simeq A$ is a compact object of $\mathrm{Rep}_{\mathfrak{g}}$. By Remark 3.11, A fits into a cofiber sequence of $\mathcal{U}(\mathfrak{g})$-modules

$$L_{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow A.$$

It suffices to show that $L_{\mathfrak{g}}$ is compact. This follows from the fact that the cotangent complex functor preserves compact objects, since its right adjoint (taking square zero extensions) preserves filtered colimits, which are computed at the level of A-modules (Proposition 2.4). □

Lemma 6.4. *Suppose that $\mathfrak{g} \in \mathrm{LieAlgd}_A$ admits a finite filtration*

$$0 = \mathfrak{g}^{(0)} \longrightarrow \dots \longrightarrow \mathfrak{g}^{(n)} = \mathfrak{g}$$

with the following property: for each i, there is a pushout square of Lie algebroids

$$\begin{array}{ccc} F(V) & \longrightarrow & \mathfrak{g}^{(i-1)} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{g}^{(i)} \end{array}$$

where $F(V)$ is the free dg-Lie algebroid on $0\colon V \longrightarrow T_A$, with V a cofibrant dg-A-module of the form $V = \bigoplus_{n\leq -2} A^{\oplus k_n}[n]$, equipped with some differential.

Let E be a $\mathfrak{g}$-representation whose underlying A-module is connective. Then there exists a map $\bigoplus_\alpha A \longrightarrow E$ in the ∞-category $\mathrm{Rep}_{\mathfrak{g}}$ which induces a surjection on π_0.

Proof. Pick representatives $e_\alpha \in E$ for the generators of $\pi_0(E)$ and consider the associated map of $\mathfrak{g}$-representations $\bigoplus_\alpha \mathcal{U}(\mathfrak{g}) \longrightarrow E$. This map is clearly surjective on π_0, so it suffices to prove that it factors (up to homotopy) as

$$\bigoplus\nolimits_\alpha \mathcal{U}(\mathfrak{g}) \longrightarrow \bigoplus\nolimits_\alpha A \longrightarrow E.$$

Using the cofiber sequence $L_{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{g}) \longrightarrow A$, we therefore have to provide a null-homotopy of each composite map

$$L_{\mathfrak{g}} \longrightarrow \mathcal{U}(\mathfrak{g}) \xrightarrow{e_\alpha} E.$$

By the assumption on $\mathfrak{g}$, the cotangent complex $L_{\mathfrak{g}}$ admits a filtration by $\mathcal{U}(\mathfrak{g})$-modules

$$0 = L_{\mathfrak{g}}^{(0)} \longrightarrow \dots \longrightarrow L_{\mathfrak{g}}^{(n)} = L_{\mathfrak{g}}$$

where each $L_{\mathfrak{g}}^{(i-1)} \longrightarrow L_{\mathfrak{g}}^{(i)}$ has cofiber of the form $\mathcal{U}(\mathfrak{g}) \otimes_A V'$, with V' a cofibrant dg-A-module of the form $V' = \bigoplus_{n\leq -1} A^{\oplus k_n}[n]$. Since all maps from such V' to a connective dg-A-module are null-homotopic, one obtains by induction that any map $L_{\mathfrak{g}} \longrightarrow E$ to a connective $\mathfrak{g}$-representation is null-homotopic, which concludes the proof. □

Corollary 6.5. *Suppose that A is eventually coconnective and let $\mathfrak{g} \in \mathrm{LieAlgd}_A$ be good. Then $\Phi_{\mathfrak{g}} \colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}$ is fully faithful and induces an equivalence*

$$\Phi_{\mathfrak{g}} \colon \mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})} \xrightarrow{\sim} \mathrm{Rep}^{\geq 0}_{\mathfrak{g}}$$

between the full subcategories consisting of connective $C^*(\mathfrak{g})$-modules and representations whose underlying A-module is connective.

Proof. Since $\mathfrak{g}$ is good, it is compact in $\mathrm{LieAlgd}_A$, so that $\Phi_{\mathfrak{g}}$ is fully faithful by Lemma 6.3. The subcategory $\mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})} \subseteq \mathrm{Mod}_{C^*(\mathfrak{g})}$ is the smallest subcategory of $\mathrm{Mod}_{C^*(\mathfrak{g})}$ which is closed under colimits and extensions and which contains $C^*(\mathfrak{g})$. As a consequence, the essential image

$$\mathcal{C} := \Phi_{\mathfrak{g}}\left(\mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})}\right) \subseteq \mathrm{Rep}_{\mathfrak{g}}$$

under $\Phi_{\mathfrak{g}}$ is the smallest subcategory of $\mathrm{Rep}_{\mathfrak{g}}$ which is closed under colimits and extensions and which contains A. Clearly $\mathcal{C}$ is contained in $\mathrm{Rep}_{\mathfrak{g}}$, so it suffices to prove the reverse inclusion.

To this end, let E be a left $\mathcal{U}(\mathfrak{g})$-module whose underlying dg-A-module is connective. We will inductively construct a sequence of left $\mathcal{U}(\mathfrak{g})$-modules $0 = E^{(-1)} \longrightarrow E^{(0)} \longrightarrow \cdots \longrightarrow E$ such that each $E^{(n)} \in \mathcal{C}$ and such that each map $E^{(n)} \longrightarrow E$ induces an isomorphism on homotopy groups in degrees $< n$ and a surjection on π_n. It follows that the map $\mathrm{colim}\, E^{(n)} \longrightarrow E$ is a weak equivalence, so that $E \in \mathcal{C}$.

To construct this sequence, suppose we have constructed $E^{(n-1)}$ and let F be the fiber of the map $E^{(n-1)} \longrightarrow E$. Then F is a left $\mathcal{U}(\mathfrak{g})$-module whose underlying dg-A-module is $(n-2)$-connective. Since $\mathfrak{g}$ is good, a shift of Lemma 6.4 shows that there exists a map $\bigoplus_\alpha A[n-2] \longrightarrow F$ which induces a surjection on π_{n-2}. Now let $E^{(n)}$ be the cofiber of the map $\bigoplus_\alpha A[n-2] \longrightarrow F \longrightarrow E^{(n-1)}$. This cofiber is contained in $\mathcal{C}$ and the five lemma shows that $E^{(n)} \longrightarrow E$ induces an isomorphism on homotopy groups in degrees $< n$ and a surjection on π_n. □

6.2. Naturality

Let us now address the functoriality of $\Phi_{\mathfrak{g}}$ in the dg-Lie algebroid $\mathfrak{g}$. This is somewhat delicate, because the Quillen adjunction

$$K(\mathfrak{g}) \otimes_{C^*(\mathfrak{g})} (-) \colon \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \colon C^*(\mathfrak{g}, -)$$

does not strictly intertwine restriction of $\mathfrak{g}$-representations with induction of $C^*(\mathfrak{g})$-modules. However, this does become true at the level of ∞-categories. To see this, let us make the following definitions (following the discussion in [17, Section 2.4]):

Construction 6.6. Let $\mathrm{Mod}^{\mathrm{dg},\otimes}$ be the category in which

- an object is a tuple $(B, M_1, \ldots, M_m)$, where B is a cdga and each M_i is a dg-B-module.
- a morphism $(B, M_1, \ldots, M_m) \longrightarrow (C, N_1, \ldots, N_n)$ is a map of finite pointed sets $\alpha\colon \langle m\rangle \longrightarrow \langle n\rangle$, a map of cdgas $B \longrightarrow C$ and B-multilinear maps $\bigotimes^{\alpha(i)=j} M_i \longrightarrow N_j$.

Similarly, let $\mathrm{Rep}^{\mathrm{dg},\otimes}$ be the category in which

- an object is a tuple $(\mathfrak{g}, E_1, \ldots, E_m)$, where $\mathfrak{g}$ is an A-cofibrant dg-Lie algebroid and each E_i is a $\mathfrak{g}$-representation.
- a morphism $(\mathfrak{g}, E_1, \ldots, E_m) \longrightarrow (\mathfrak{h}, F_1, \ldots, F_n)$ is a map of finite pointed sets $\alpha\colon \langle m\rangle \longrightarrow \langle n\rangle$, a map of dg-Lie algebroids $\mathfrak{h} \longrightarrow \mathfrak{g}$ and maps of $\mathfrak{h}$-representations $\bigotimes_A^{\alpha(i)=j} E_i \longrightarrow F_j$.

These categories fit into a commuting square

$$\begin{array}{ccc} \mathrm{Rep}^{\mathrm{dg},\otimes} & \xrightarrow{C^*} & \mathrm{Mod}^{\mathrm{dg},\otimes} \\ \downarrow & & \downarrow \\ \left(\mathrm{LieAlgd}_A^{\mathrm{dg},A-\mathrm{cof}}\right)^{\mathrm{op}} \times \mathrm{Fin}_* & \xrightarrow[C^*]{} & \mathrm{CAlg}_k^{\mathrm{dg}} \times \mathrm{Fin}_* \end{array} \tag{6.7}$$

where Fin_* is the category of finite pointed sets. The vertical functors are the obvious projections, which are both cocartesian fibrations. The top functor sends $(\mathfrak{g}, E_1, \ldots, E_m)$ to $\big(C^*(\mathfrak{g}), C^*(\mathfrak{g}, E_1), \ldots, C^*(\mathfrak{g}, E_m)\big)$.

Lemma 6.8. *After inverting the weak equivalences on the left and right of (6.7), one obtains a commuting square of ∞-categories*

$$\begin{array}{ccc} \mathrm{Rep}^{\otimes} & \xrightarrow{C^*} & \mathrm{Mod}^{\otimes} \\ \downarrow & & \downarrow \\ (\mathrm{LieAlgd}_A)^{\mathrm{op}} \times \mathrm{Fin}_* & \xrightarrow[C^*]{} & \mathrm{CAlg}_k \times \mathrm{Fin}_*, \end{array}$$

in which the vertical functors are cocartesian fibrations.

Proof. All functors in (6.7) preserve weak equivalences, so that they descend to functors between localizations. It suffices to verify that the vertical projections remain cocartesian fibrations after inverting the quasi-isomorphisms. To see this, let $\mathrm{C} \subseteq \mathrm{Rep}^{\mathrm{dg},\otimes}$ be the full subcategory on $(\mathfrak{g}, E_1, \ldots, E_m)$ where each E_i is cofibrant as a dg-A-module. The inclusion $\mathcal{C} \subseteq \mathrm{Rep}^{\mathrm{dg},\otimes}$ induces an equivalence on localizations, with inverse provided by a cofibrant replacement functor. The projection

$$\mathcal{C} \longrightarrow \left(\mathrm{LieAlgd}_A^{\mathrm{dg},A-\mathrm{cof}}\right)^{\mathrm{op}} \times \mathrm{Fin}_*$$

is a cocartesian fibration. For any map $\alpha\colon \langle m\rangle \longrightarrow \langle n\rangle$ and any $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$, the induced functor between fibers is given by

$$\left(\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg},A-\mathrm{cof}}\right)^{\times m} \longrightarrow \left(\mathrm{Rep}_{\mathfrak{h}}^{\mathrm{dg},A-\mathrm{cof}}\right)^{\times n};\ (E_j)_{j\leq m} \longmapsto \left(\bigotimes\nolimits_{\alpha(j)=i} f^*E_j\right)_{i\leq m}.$$

This functor preserves all weak equivalences and induces an equivalence of ∞-categories whenever α is a bijection and f is a weak equivalence. It follows from [11, Proposition 2.1.4] that the induced functor of ∞-categories

$$\mathrm{Rep}^{\otimes} \longrightarrow \mathrm{LieAlgd}_A^{\mathrm{op}} \times \mathrm{Fin}_*$$

is a cocartesian fibration. A similar argument shows that $\mathrm{Mod}^{\otimes} \longrightarrow \mathrm{CAlg}_k \times \mathrm{Fin}_*$ is a cocartesian fibration. □

Remark 6.9. Fix a map $\alpha\colon \langle m\rangle \longrightarrow \langle n\rangle$, a map $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$ and a collection $E_1, \dots, E_m$ of $\mathfrak{g}$-representations that are cofibrant as dg-A-modules. By [11, Proposition 2.1.4], the cocartesian lift of (f, α) in $\mathrm{Rep}^{\otimes}$ with domain $(\mathfrak{g}, E_1, \dots, E_m)$ is the image of

$$(\mathfrak{g}, E_1, \dots, E_m) \longrightarrow \left(\mathfrak{h}, \bigotimes\nolimits_{\alpha(j)=1} f^*E_j, \dots, \bigotimes\nolimits_{\alpha(j)=n} f^*E_j\right)$$

in the ∞-categorical localization of $\mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg},\otimes}$.

Lemma 6.10. *Let* $\mathrm{Mod}_{C^*}^{\otimes} \longrightarrow \left(\mathrm{LieAlgd}_A\right)^{\mathrm{op}} \times \mathrm{Fin}_*$ *denote the base change of the projection* $\mathrm{Mod}^{\otimes} \longrightarrow \mathrm{CAlg}_k \times \mathrm{Fin}_*$ *along the functor* C^**, so that there is a functor of cocartesian fibrations*

$$\begin{array}{ccc} \mathrm{Rep}^{\otimes} & \xrightarrow{\quad C^* \quad} & \mathrm{Mod}_{C^*}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \left(\mathrm{LieAlgd}_A\right)^{\mathrm{op}} \times \mathrm{Fin}_*. & \end{array}$$

This functor admits a left adjoint Φ*, which preserves cocartesian edges.*

In other words, the functors $\Phi_{\mathfrak{g}}$ from Section 6.1 determine a natural (symmetric monoidal) transformation between diagrams of symmetric monoidal ∞-categories.

Proof. For each $\langle m\rangle$ and each dg-Lie algebroid $\mathfrak{g}$, the functor between the fibers

$$C^*\colon \big(\mathrm{Rep}_{\mathfrak{g}}\big)^{\times m} \longrightarrow \mathrm{Mod}_{C^*(\mathfrak{g})}^{\times m}$$

admits a left adjoint $\Phi_{\mathfrak{g}}$: this is just the m-fold product of the left derived functor of the Quillen pair (6.2). By Remark 6.9 and [19, Proposition 7.3.2.11], the existence of the global left adjoint $\Phi\colon \mathrm{Mod}_{C^*}^{\otimes} \longrightarrow \mathrm{Rep}^{\otimes}$, as well as the fact that it preserves cocartesian edges, follows once we verify the following: for any map of dg-Lie algebroids $f\colon \mathfrak{h} \longrightarrow \mathfrak{g}$ and any collection of $C^*(\mathfrak{g})$-modules M_i, the natural map

$$\Phi_{\mathfrak{h}}\big(C^*(\mathfrak{h}) \otimes_{C^*(\mathfrak{g})} M_1 \otimes_{C^*(\mathfrak{g})} \cdots \otimes_{C^*(\mathfrak{g})} M_m\big) \longrightarrow \Phi_{\mathfrak{g}}(M_1) \otimes_A \cdots \otimes_A \Phi_{\mathfrak{g}}(M_m)$$

is an equivalence. Since both functors preserve colimits of modules in each variable, we can reduce to the case where each M_i is equivalent to $C^*(\mathfrak{g})$. In that case, the map can be identified with a map

$$K(\mathfrak{h}) \longrightarrow K(\mathfrak{g}) \otimes_A \cdots \otimes_A K(\mathfrak{g})$$

between Koszul complexes. Since both $K(\mathfrak{g})$ and $K(\mathfrak{h})$ were resolutions of the canonical representation A, the result follows. □

6.3. Quasicoherent sheaves

The left adjoint Φ of Lemma 6.10 corresponds under straightening to a natural transformation

$$\mathrm{LieAlgd}_A^{\mathrm{op}} \underset{\mathrm{Rep}}{\overset{\mathrm{Mod}_{C^*}}{\rightrightarrows}} \mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}} = \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}), \qquad \Phi\colon \mathrm{Mod}_{C^*} \Rightarrow \mathrm{Rep}$$

between diagrams of presentable (closed) symmetric monoidal ∞-categories, with symmetric monoidal left adjoint functors between them. This natural transformation is given pointwise by the left derived functor $\Phi_{\mathfrak{g}}$ of (6.2).

We can precompose with the duality functor $\mathfrak{D}$ (3.2) and obtain a natural transformation of functors

$$\begin{array}{ccccc} \mathrm{CAlg}_k^{\mathrm{sm}}/A & & \longrightarrow & & \mathrm{CAlg}_k \\ \downarrow & & \overset{\Psi}{\Swarrow} & & \downarrow{\scriptstyle \mathrm{Mod}} \\ \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm}}/A, \mathcal{S})^{\mathrm{op}} & \xrightarrow[\mathfrak{D}_!]{} & \mathrm{LieAlgd}_A^{\mathrm{op}} & \xrightarrow[\mathrm{Rep}]{} & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}} \end{array} \qquad (6.11)$$

Here $\mathfrak{D}_!$ is the unique limit-preserving functor which restricts to $\mathfrak{D}$ on the corepresentable functors. For each B in $\mathrm{CAlg}_k^{\mathrm{sm}}/A$, the functor Ψ_B is the composite

$$\Psi_B\colon \mathrm{Mod}_B \longrightarrow \mathrm{Mod}_{C^*\mathfrak{D}(B)} \xrightarrow{\Phi_{\mathfrak{D}(B)}} \mathrm{Rep}_{\mathfrak{D}(B)} \tag{6.12}$$

where the first functor arises from the unit map $B \longrightarrow C^*\mathfrak{D}(B)$.

Definition 6.13. The presentable, symmetric monoidal ∞-category $\mathrm{QC}(X)$ of *quasi-coherent sheaves* on a functor $X\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ is the value on X of the right Kan extension

$$\begin{array}{ccc} \mathrm{CAlg}_k^{\mathrm{sm}}/A & \xrightarrow{\mathrm{Mod}} & \mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}} \\ \downarrow & \nearrow_{\mathrm{QC}} & \\ \mathrm{Fun}\big(\mathrm{CAlg}_k^{\mathrm{sm}}/A, \mathcal{S}\big)^{\mathrm{op}} & & \end{array}$$

This right Kan extension exists by [16, Lemma 5.1.5.5], since $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}}$ has all small limits [19, Proposition 4.8.1.15].

Remark 6.14. The quasi-coherent sheaves of Definition 6.13 are closely related to the quasi-coherent sheaves on a derived (pre)stack $\mathrm{CAlg}_k \longrightarrow \mathcal{S}$, as usually considered in derived algebraic geometry. Indeed, any functor $X\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ determines a derived prestack $f_!X\colon \mathrm{CAlg}_k \longrightarrow \mathcal{S}$, by left Kan extension along the forgetful functor

$$f\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathrm{CAlg}_k.$$

The ∞-category $\mathrm{QC}(X)$ is then equivalent to the ∞-category of quasi-coherent sheaves on $f_!X$ in the usual sense.

By the universal property of QC, we obtain a natural transformation of presentable symmetric monoidal ∞-categories $\Psi_X\colon \mathrm{QC}(X) \longrightarrow \mathrm{Rep}_{\mathfrak{D}_!(X)}$. When X is corepresentable, this is simply the functor Ψ from (6.11). If A is eventually coconnective and X is a formal moduli problem on A, then $\mathfrak{D}_!(X)$ is naturally equivalent to $T_{A/X}$, by Remark 5.13. We thus obtain a natural symmetric monoidal functor

$$\Psi_X\colon \mathrm{QC}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}} \tag{6.15}$$

for any formal moduli problem X. By naturality, the composition

$$\mathrm{QC}(X) \xrightarrow{\Psi_X} \mathrm{Rep}_{T_{A/X}} \longrightarrow \mathrm{Rep}_0 = \mathrm{Mod}_A$$

is naturally equivalent to the functor $x^*\colon \mathrm{QC}(X) \longrightarrow \mathrm{Mod}_A$ that restricts F to the basepoint $x \in X(A)$.

Proof (of Theorem 6.1). We have to prove that for any formal moduli problem X, Ψ_X (6.15) is fully faithful and restricts to an equivalence between $\mathrm{QC}^{\geq 0}(X)$ and $\mathrm{Rep}^{\geq 0}_{T_{A/X}}$. When X is representable by an object $B \in \mathrm{CAlg}^{\mathrm{sm}}_k/A$, this functor is given by the composite (6.12). The first functor is an equivalence by Theorem 5.1, so that the result follows from Corollary 6.5.

The functor QC sends sifted colimits in $\mathrm{Fun}\big(\mathrm{CAlg}^{\mathrm{sm}}/A, \mathcal{S}\big)$ to limits of symmetric monoidal ∞-categories by construction. Lemma 2.13 implies that the same assertion holds for $X \mapsto \mathrm{Rep}_{\mathfrak{D}_!(X)}$. Every formal moduli problem is a sifted colimit of representable functors, since every Lie algebroid is a sifted colimit of good Lie algebroids. It follows that Ψ_X is fully faithful, being a limit of fully faithful functors.

It remains to identify the essential image of $\mathrm{QC}(X)^{\geq 0}$. Recall that a quasi-coherent sheaf F is connective if and only if $f^*F \in \mathrm{Mod}_B$ is connective for any $f \in X(B)$. In terms of its image $\Psi_X(E)$, this means that for any map $f\colon \mathfrak{D}(B) \longrightarrow T_{A/X}$, the restricted representation $f^!\Psi_X(E)$ is the image of a connective B-module. By Corollary 6.5, this is equivalent to $\Psi_X(E)$ being a connective $T_{A/X}$-representation. □

Corollary 6.16. *Let A be eventually coconnective and let X be a formal moduli problem such that $T_{A/X}$ is connective. Then there is an equivalence*

$$\Psi_X\colon \mathrm{QC}(X) \longrightarrow \mathrm{Rep}_{T_{A/X}}.$$

Proof. Since $T_{A/X}$ is connective, its enveloping algebra $\mathcal{U}(T_{A/X})$ is connective as well. It follows that T_A-representations carry a *right complete* t-structure, where $\mathrm{Rep}^{\geq 0}_{T_{A/X}}$ consists of $\mathfrak{g}$-representations whose underlying chain complex is connective [19, Example 2.2.1.3]. Similarly, $\mathrm{QC}(X)$ carries a right complete t-structure where $F \in \mathrm{QC}(X)^{\geq 0}$ if and only if f^*F is a connective chain complex for all $f \in X(B)$.

The functor Ψ_X fits into a sequence of locally presentable ∞-categories and left adjoint functors between them

$$\begin{array}{ccccccc}
\mathrm{QC}(X)^{\geq 0} & \longrightarrow & \mathrm{QC}(X)^{\geq -1} & \longrightarrow & \dots & \longrightarrow & \mathrm{QC}(X) \\
\downarrow{\scriptstyle \Psi_X^{\geq 0}} & & \downarrow{\scriptstyle \Psi_X^{\geq -1}} & & & & \downarrow{\scriptstyle \Psi_X} \\
\mathrm{Rep}^{\geq 0}_{T_{A/X}} & \longrightarrow & \mathrm{Rep}^{\geq -1}_{T_{A/X}} & \longrightarrow & \dots & \longrightarrow & \mathrm{Rep}_{T_{A/X}}.
\end{array}$$

Since $\mathrm{QC}(X)$ and $\mathrm{Rep}_{T_{A/X}}$ are right complete, the horizontal sequences are colimit diagrams, so that the result follows from Theorem 6.1. □

6.4. Deformations of algebras

Theorem 6.1 can be used to study the deformation theory of connective (commutative) A-algebras. Suppose that R is a cofibrant commutative dg-algebra in $\mathrm{Mod}_A^{\mathrm{dg},\geq 0}$, so that R determines an object in the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_A^{\geq 0})$. Consider the functor

$$\mathrm{Def}_R\colon \mathrm{FMP}_A^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty;\ X \longmapsto \mathrm{CAlg}\big(\mathrm{QC}(X)\big) \times_{\mathrm{CAlg}(\mathrm{Mod}_A)} \{R\}$$

sending each formal moduli problem X to the (locally small) ∞-category of commutative algebras in $\mathrm{QC}(X)$, equipped with an equivalence between R and their restriction to the canonical basepoint $x \in X(A)$. One can think of a point in $\mathrm{Def}_R(X)$ as a *deformation* of the commutative algebra along the map of stacks $\mathrm{Spec}(A) \longrightarrow X$.

Note that every such deformation is necessarily connective, since R itself is connective: indeed, Theorem 6.1 implies that $E \in \mathrm{QC}(X)$ is connective if and only if its restriction to $\mathrm{Spec}(A)$ is connective, the latter being the A-module underlying $\Psi_X(E)$. It then follows from Theorem 5.1 and Theorem 6.1 that Def_R can be identified with the functor

$$\mathrm{Act}_R\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty;\ \mathfrak{g} \longmapsto \mathrm{CAlg}(\mathrm{Rep}_{\mathfrak{g}}) \times_{\mathrm{CAlg}(\mathrm{Mod}_A)} \{R\}$$

sending each Lie algebroid $\mathfrak{g}$ to the category of commutative algebras in $\mathrm{Rep}_{\mathfrak{g}}$ whose underlying commutative A-algebra is equivalent to R. In light of Example 2.9, one can think of $\mathrm{Act}_R(\mathfrak{g})$ as the category of actions of $\mathfrak{g}$ on R by derivations.

It follows from Lemma 2.13 that Act_R, and therefore Def_R, preserves all limits. In particular, the restriction of Def_R along the Yoneda embedding gives rise to a formal moduli problem

$$\mathrm{Def}_R\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S};\ B \longmapsto \mathrm{CAlg}(\mathrm{Mod}_B) \times_{\mathrm{CAlg}(\mathrm{Mod}_A)} \{R\}.$$

This takes values in (small) spaces because $\mathrm{Def}_R(A \oplus A[n]) \simeq \Omega\mathrm{Def}_R(A \oplus A[n+1])$ and all ∞-categories involved are locally small. In particular, Def_R is classified by a certain Lie algebroid $\mathfrak{h}$.

Recall from Example 2.3 that associated to the commutative dg-algebra R in $\mathrm{Mod}_A^{\mathrm{dg}}$ is a dg-Lie algebroid $\mathrm{At}_{\mathbb{E}_\infty}(R)$ of compatible derivations of A and R (here $\mathbb{E}_\infty$ denotes the commutative operad). Since R is a cofibrant commutative dg-algebra, $\mathrm{At}_{\mathbb{E}_\infty}(R)$ is a fibrant dg-Lie algebroid. There is an obvious representation of $\mathrm{At}_{\mathbb{E}_\infty}(R)$ on R by means of derivations, which is classified by a map

$$\phi\colon \mathrm{At}_{\mathbb{E}_\infty}(R) \longrightarrow \mathfrak{h}.$$

We have the following (folklore) result, cf. [31] (see also [10]):

Proposition 6.17. *The map ϕ is an equivalence for any cofibrant connective dg-A-algebra R. In other words,* $\mathrm{At}_{\mathbb{E}_\infty}(R)$ *classifies the formal moduli problem* Def_R.

Proof. Let $\mathfrak{s}_n$ be the free Lie algebroid on the map $0\colon A[n] \longrightarrow T_A$, with natural maps $i\colon 0 \longrightarrow \mathfrak{s}_n$ and $r\colon \mathfrak{s}_n \longrightarrow 0$. It suffices to verify that the maps ϕ_* in the following commuting diagram are bijections for all n:

$$\begin{array}{ccccc}
\pi_1\mathrm{Map}\big(\mathfrak{s}_n, \mathrm{At}_{\mathbb{E}_\infty}(R)\big) & \xrightarrow{\phi_*} & \pi_1\mathrm{Map}\big(\mathfrak{s}_n, \mathfrak{h}\big) & \xrightarrow{\cong} & \pi_1\mathrm{Act}_R(\mathfrak{s}_n) =: M_1 \\
\downarrow \cong & & \downarrow \cong & & \psi \downarrow \cong \\
\pi_0\mathrm{Map}(\mathfrak{s}_{n+1}, \mathrm{At}_{\mathbb{E}_\infty}(R)) & \xrightarrow[\phi_*]{} & \pi_0\mathrm{Map}\big(\mathfrak{s}_{n+1}, \mathfrak{h}\big) & \xrightarrow[\cong]{} & \pi_0\mathrm{Act}_R(\mathfrak{s}_{n+1}) =: M_0.
\end{array}$$

The dg-Lie algebroid $\mathrm{At}_{\mathbb{E}_\infty}(R)$ is fibrant and the kernel of its anchor map is the Lie algebra of A-linear derivations of R. The abelian group $\pi_1\mathrm{Map}\big(\mathfrak{s}_n, \mathrm{At}_{\mathbb{E}_\infty}(R)\big)$ can therefore be identified with $\pi_{n+1}\mathrm{Der}_A(R, R)$.

On the other hand, M_1 can *also* be identified with $\pi_{n+1}\mathrm{Der}_A(R, R)$. To see this, let $r^!R \in \mathrm{CAlg}(\mathrm{Rep}_{\mathfrak{s}_n})$ be the trivial representation of $\mathfrak{s}_n$ on R. We can identify M_1 with the set of homotopy classes of maps $\alpha\colon r^!R \longrightarrow r^!R$ such that the restriction $i^!(\alpha)$ is the identity map in $\mathrm{CAlg}(\mathrm{Mod}_A)$. Using that the restriction functors $r^!$ and $i^!$ have a right adjoints $r_!$ and $i_!$, such maps can be identified with sections of

$$r_!r^!(R) \xrightarrow{r_!\eta r^!} r_!i_!i^!r^!(R) = R$$

in the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_A)$. By [24, Theorem 7.11], we can present the ∞-categories $\mathrm{CAlg}(\mathrm{Mod}_A)$ and $\mathrm{CAlg}(\mathrm{Rep}_{\mathfrak{s}_n})$ by the model categories of commutative algebras in $\mathrm{Mod}_A^{\mathrm{dg}}$ and $\mathrm{Rep}_{\mathfrak{s}_n}^{\mathrm{dg}}$. Using this, the right adjoint $r_!$ can be computed at the chain level as

$$r_!(r^!R) \simeq \mathbb{R}\mathrm{Hom}_{\mathcal{U}(\mathfrak{s}_n)}(A, r^!R) \simeq C^*\big(\mathfrak{s}_n, r^!R\big).$$

In fact, since $\mathfrak{s}_n$ is a free Lie algebroid, the discussion of Section 3.2 shows that restriction along $A[n] \longrightarrow \mathfrak{s}_n$ induces a weak equivalence of commutative dg-A-algebras

$$C^*(\mathfrak{s}_n, r^!R) \xrightarrow{\sim} R \oplus \mathrm{Hom}_A\big(A[n+1], r^!R\big) \cong R \oplus R[-n-1]$$

to the *square zero extension* of R by a shifted copy of itself. We conclude that M_1 is isomorphic to the set of homotopy classes of sections in $\mathrm{CAlg}(\mathrm{Mod}_A)$ of

$$R \oplus R[-n-1] \longrightarrow R.$$

The set of such sections is indeed isomorphic to $\pi_{n+1}\mathrm{Der}_A(R, R)$, as asserted.

Consider an element $v \in \mathrm{Der}_A(R, R) \cong M_1$, corresponding to a map $v\colon r^!R \longrightarrow r^!R$ in $\mathrm{CAlg}(\mathrm{Rep}_{\mathfrak{s}_n})$. Unwinding the definitions, the image $\psi(v) \in M_0$ is given by (the equivalence class of) the $\mathfrak{s}_{n+1}$-action on R where the generator of $\mathfrak{s}_{n+1}$ acts by v. This representation is exactly the image of the map $v\colon A[n+1] \longrightarrow \mathfrak{s}_{n+1} \longrightarrow \mathrm{At}(R)$ under the map ϕ_*. We conclude that ϕ_* is indeed an isomorphism. □

Remark 6.18. The exact same proof shows that for any colored simplicial operad $\mathcal{P}$ and a cofibrant $\mathcal{P}$-algebra R, the formal moduli problem Def_R of deformations of R is classified by the Atiyah Lie algebroid $\mathrm{At}_{\mathcal{P}}(R)$ of Example 2.2. For example, this provides an explicit dg-Lie algebroid classifying the deformations of modules, associative algebras, or diagrams thereof.

7. Representations and pro-coherent sheaves

Let X be a formal moduli problem on A with associated Lie algebroid $T_{A/X}$. We have seen in the previous section that the ∞-category of $T_{A/X}$-representations is an *extension* of the ∞-category of quasi-coherent sheaves on X. The purpose of this section is to provide a geometric description of this extension when A is coherent (Definition 7.2), in terms of *pro-coherent sheaves* on X (Definition 7.10).

Theorem 7.1. *Suppose that A is coherent and eventually coconnective and let $X\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ be a formal moduli problem. Then there is a natural fully faithful left adjoint functor*

$$\mathrm{Rep}_{T_{A/X}} \hookrightarrow \mathrm{QC}^!(X)$$

from the ∞-category of (quasi-coherent) $T_{A/X}$-representations to the ∞-category of pro-coherent sheaves on X. Its essential image consists of pro-coherent sheaves on X whose restriction to A is quasi-coherent.

In [5, Vol. II, Ch. 8, §4], the ∞-category of representations of a Lie algebroid is defined to be the ∞-category $\mathrm{IndCoh}(X)$ of ind-coherent sheaves on the associated formal moduli problem. In the setting of [5], this ∞-category is equivalent to $\mathrm{QC}^!(X)$ by Serre duality (see Remark 7.11).

Section 7.1 discusses the definition of the ∞-category of pro-coherent sheaves on a formal moduli problem. In Section 7.2, we relate this to the ∞-category of Lie algebroid representations and prove Theorem 7.1.

7.1. Pro-coherent sheaves on formal moduli problems

Let us start by recalling the following definitions:

Definition 7.2 *([19, Definition 7.2.4.16]).* Let A be a connective commutative dg-algebra over $\mathbb{Q}$. We will say that A is *coherent* if $\pi_0(A)$ is a coherent ring and each $\pi_n(A)$ is a finitely presented $\pi_0(A)$-module.

Definition 7.3 *([4, Definition 2.2.2]).* A module E over a connective commutative dg-algebra A is called *coherent* if it satisfies the following conditions:

1. It is *almost of finite presentation*: for every n there exists a map from a perfect A-module $E_n \longrightarrow E$ whose cone is n-connective. Equivalently, E can be presented by a graded-free dg-A-module with generators in homological degrees $\geq N$ for some integer N, and finitely many generators in each degree.
2. It is eventually coconnective, i.e. $\pi_n(E) = 0$ for $n \gg 0$. Note that (1) already implies that $\pi_n(E) = 0$ for $n \ll 0$.

When A is coherent, this is equivalent to the condition that each $\pi_n(E)$ is finitely presented over $\pi_0(A)$ and that $\pi_n(E) = 0$ for $n \ll 0$ and $n \gg 0$ [19, Proposition 7.2.4.17]. We will denote the ∞-category of coherent A-modules by Coh_A.

Definition 7.4. For any connective cdga A, we define the ∞-category of *pro-coherent sheaves* on A to be

$$\mathrm{QC}^!(A) := \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}}),$$

even though this is technically its opposite category.

Remark 7.5. When A is eventually coconnective, there is a fully faithful inclusion $\mathrm{Perf}_A \subseteq \mathrm{Coh}_A$. This induces a fully faithful embedding $\mathrm{Mod}_A \subseteq \mathrm{QC}^!(A)$ of the quasi-coherent sheaves on A (i.e. A-modules) into the pro-coherent sheaves on A, via

$$\mathrm{Mod}_A \simeq \mathrm{Ind}(\mathrm{Perf}_A) \xrightarrow[\sim]{\mathrm{Ind}((-)^\vee)} \mathrm{Ind}(\mathrm{Perf}_A^{\mathrm{op}}) \hookrightarrow \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}}). \tag{7.6}$$

Unraveling the definitions, the inclusion $\mathrm{Mod}_A \longrightarrow \mathrm{QC}^!(A)$ can be described as follows. If $E \in \mathrm{Mod}_A$ is an A-module, then the corresponding object in $\mathrm{Ind}\big(\mathrm{Coh}_A^{\mathrm{op}}\big)$ is the finite-limit preserving functor

$$\mathrm{Coh}_A \longrightarrow \mathcal{S};\ \ F \longmapsto \Omega^\infty\big(E \otimes_A F\big).$$

Here $\Omega^\infty(V)$ denotes the space underlying an A-module V, e.g. obtained by applying the Dold-Kan correspondence to the connective cover $\tau_{\geq 0}(V)$.

Remark 7.7 *(Serre duality).* Suppose that A is a connective cdga such that $\pi_0(A)$ is Noetherian and each $\pi_n(A)$ is a finitely generated $\pi_0(A)$-module. If A admits a *dualizing complex* ω_A in the sense of [18, Definition 4.2.5], then there is an equivalence of ∞-categories [18, Theorem 4.2.7]

$$\mathrm{Hom}_A(-, \omega_A) \colon \mathrm{Coh}_A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Coh}_A.$$

Taking ind-completions, we obtain an equivalence

$$\mathrm{Ind}\big(\mathrm{Coh}_A^{\mathrm{op}}\big) \xrightarrow{\sim} \mathrm{Ind}\big(\mathrm{Coh}_A\big)$$

between the ∞-category of pro-coherent sheaves and the ∞-category of ind-coherent sheaves studied extensively by Gaitsgory [4] (see also Section 9 of [4]).

By [18, Theorem 4.3.5], A admits a dualizing complex if and only if $\pi_0(A)$ admits a dualizing complex in the classical sense [7, Chapter V.2]. In particular, this is the case when $\pi_0(A)$ is a complete Noetherian local ring, or when $\pi_0(A)$ is locally finitely presented over a field [7, Chapter V.10]. The affines used in the work of Gaitsgory–Rozenblyum [5] have this last property.

For us, the importance of the ∞-category of pro-coherent sheaves stems from the following basic result:

Lemma 7.8. *Let A be a coherent commutative dg-k-algebra and let $f\colon B \longrightarrow A$ be an iterated square zero extension by coherent A-modules. Consider the thick subcategory $\mathcal{C} \subseteq \mathrm{Mod}_B$ generated by the modules of the form f_*F, where $F \in \mathrm{Coh}_A$ is a coherent A-module. Then B is coherent and $\mathcal{C}$ coincides with Coh_B.*

Proof. By assumption, the map f decomposes as

$$B = B_n \longrightarrow B_{n-1} \longrightarrow \dots \longrightarrow B_0 = A,$$

where each map is a square zero extension by a coherent A-module. Proceeding by induction along this tower, we can reduce to the case where $f\colon B \longrightarrow A$ is a square zero extension by a coherent A-module I. In this case, $\pi_0(f)\colon \pi_0(B) \longrightarrow \pi_0(A)$ is a square zero extension by a finitely presented $\pi_0(A)$-module. One easily sees that any finitely presented $\pi_0(A)$-module is finitely presented over $\pi_0(B)$ and that $\pi_0(B)$ is a coherent ring (see e.g. [2, Lemma 3.25 and Lemma 3.26]). Using the long exact sequence of $I \longrightarrow B \longrightarrow A$ and the fact that all $\pi_n(I)$ and $\pi_n(A)$ are finitely presented over $\pi_0(A)$ (hence over $\pi_0(B)$), one deduces that B is coherent.

For every coherent A-module F, its homotopy groups are finitely presented over $\pi_0(A)$ and hence over $\pi_0(B)$. This means that f_*F is a coherent B-module, so that $\mathcal{C} \subseteq \mathrm{Coh}_B$. It remains to be verified that $\mathcal{C}$ exhausts Coh_B. Using Postnikov towers, one sees that Coh_B is the thick subcategory generated by the (discrete) finitely presented $\pi_0(B)$-modules. It therefore suffices to show that $\mathcal{C}$ contains any such finitely presented $\pi_0(B)$-module E. Note that E fits into a short exact sequence of finitely presented $\pi_0(B)$-modules

$$0 \longrightarrow K \longrightarrow E \longrightarrow \pi_0(A) \otimes_{\pi_0(B)} E \longrightarrow 0$$

where K is a quotient of $\ker(\pi_0(f)) \otimes_{\pi_0(B)} E$. In particular, $\ker(\pi_0(f)) \subseteq \pi_0(B)$ acts trivially on K, so that both K and $\pi_0(A) \otimes_{\pi_0(B)} E$ are contained in $f_*(\mathrm{Coh}_A)$. It follows that E is contained in $\mathcal{C}$, as desired. □

Given a functor $X\colon \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$ on the ∞-category of iterated square zero extensions of A by coherent A-modules, its category of pro-coherent sheaves is obtained by gluing, using Lemma 7.8.

Construction 7.9. Suppose that A is coherent and eventually coconnective. It follows from Lemma 7.8 that for any map $f\colon B \longrightarrow B'$ between small extensions of A by coherent A-modules, the functor

$$f_*\colon \mathrm{Mod}_{B'} \longrightarrow \mathrm{Mod}_B$$

preserves coherent modules. Consequently, there is a functor

$$\mathrm{Coh}\colon \big(\mathrm{CAlg}^{\mathrm{sm,coh}}/A\big)^{\mathrm{op}} \longrightarrow \mathrm{StCat}_\infty^{\mathrm{ex}}$$

sending each small extension $B \longrightarrow A$ to the stable ∞-category of coherent B-modules, and each map $f\colon B \longrightarrow B'$ to the exact functor f_*. Consider the composite functor

$$\big(\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A\big)^{\mathrm{op}} \xrightarrow{\ \mathrm{Coh}^{\mathrm{op}}\ } \mathrm{Cat}_\infty \xrightarrow{\ \mathrm{Ind}\ } \mathrm{Pr}^{\mathrm{L}}$$

sending each small extension $B \longrightarrow A$ to the ∞-category $\mathrm{Ind}\big(\mathrm{Coh}_B^{\mathrm{op}}\big)$ of pro-coherent sheaves on B. Every map $f\colon B \longrightarrow B'$ of small extensions over A is sent to the adjoint pair

$$f_* = \mathrm{Ind}(f_*)\colon \mathrm{Ind}\big(\mathrm{Coh}_{B'}^{\mathrm{op}}\big) \rightleftarrows \mathrm{Ind}\big(\mathrm{Coh}_B^{\mathrm{op}}\big) : f^!.$$

By construction, the functor f_* preserves compact objects. It follows that the right adjoint $f^!$ preserves all colimits and admits a further right adjoint $f_!$ (this is discussed in much greater generality in [4]).

Definition 7.10. Let A be coherent and eventually coconnective. For any functor $X\colon \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$, we define the stable presentable ∞-category $\mathrm{QC}^!(X)$ of *pro-coherent sheaves* on X to be the value on X of the left Kan extension

$$\begin{array}{ccc} \big(\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A\big)^{\mathrm{op}} & \xrightarrow{\ \mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}})\ } & \mathrm{Pr}^{\mathrm{L}} \\ \big\downarrow & \overset{\mathrm{QC}^!}{\nearrow} & \\ \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A, \mathcal{S}) & & \end{array}$$

This exists by [16, Lemma 5.1.5.5], since Pr^{L} has all small colimits [16]. Informally, a pro-coherent sheaf on X is given by a collection of pro-coherent sheaves F_y over $B \in \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A$ for every $y \in X(B)$, together with a coherent family of equivalences for every $f\colon B \longrightarrow B'$

$$F_{f(y)} \simeq f^! F_y.$$

For a functor $X\colon \mathrm{CAlg}_k^{\mathrm{sm}}/A \longrightarrow \mathcal{S}$ defined on the full subcategory $\mathrm{CAlg}_k^{\mathrm{sm}}/A \subseteq \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A$, we define $\mathrm{QC}^!(X)$ in a similar fashion.

Remark 7.11 *(Serre duality).* Suppose that we are in the 'locally almost of finite type' situation considered in [4,5], i.e. that A is eventually coconnective, coherent and with $\pi_0(A)$ locally finitely presented over a field. In this case, these properties are shared by each object in $\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A$. Work of Gaitsgory [4, Theorem 9.1.4] then provides a natural equivalence of functors

$$\big(\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A\big)^{\mathrm{op}} \underset{\mathrm{Ind}(\mathrm{Coh})}{\overset{\mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}})}{\rightrightarrows}} \mathrm{Pr}^{\mathrm{L}}, \quad \sim \Downarrow \mathbb{D} \tag{7.12}$$

This natural equivalence is given on objects by Serre duality (see [4, Section 9.5], cf. Remark 7.7). By Kan extension, one then obtains for any $X\colon \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$ a Serre duality equivalence

$$\mathbb{D}\colon \mathrm{QC}^!(X) \xrightarrow{\sim} \mathrm{IndCoh}(X).$$

Here the ∞-category $\mathrm{IndCoh}(X)$ agrees with the ∞-category of ind-coherent sheaves on X considered in [4, Section 10] and [5] (by an argument as in Remark 6.14).

7.2. Koszul duality

Recall that the left adjoint Φ from Lemma 6.10 corresponds under straightening to a natural transformation between diagrams

$$\mathrm{Rep}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \qquad \mathrm{Mod}_{C^*}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$$

of locally presentable ∞-categories and left adjoint functors between them. When evaluated on a map of A-cofibrant dg-Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, this natural transformation is given by the commuting square of left adjoints

$$\begin{array}{ccc} \mathrm{Mod}_{C^*(\mathfrak{h})} & \xrightarrow{\Phi} & \mathrm{Rep}_{\mathfrak{h}} \\ {\scriptstyle f^*:=C^*(\mathfrak{g})\otimes_{C^*(\mathfrak{h})}-}\big\downarrow & & \big\downarrow{\scriptstyle f^!} \\ \mathrm{Mod}_{C^*(\mathfrak{g})} & \xrightarrow[\Phi]{} & \mathrm{Rep}_{\mathfrak{g}}. \end{array}$$

Let us pass to the associated diagram of right adjoint functors and next take opposite categories. The natural transformation Φ then determines a natural transformation between two diagrams of large ∞-categories and left adjoint functors between them

$$\mathrm{Rep}^{\mathrm{op}}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{L}} \qquad \mathrm{Mod}_{C^*}^{\mathrm{op}}\colon \mathrm{LieAlgd}_A^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{L}}.$$

The value of this natural transformation on a map of Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is given by

$$\begin{array}{ccc} \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{op}} & \xrightarrow{C^*(\mathfrak{g},-)} & \mathrm{Mod}_{C^*(\mathfrak{g})}^{\mathrm{op}} \\ {\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\ \mathrm{Rep}_{\mathfrak{h}}^{\mathrm{op}} & \xrightarrow[C^*(\mathfrak{h},-)]{} & \mathrm{Mod}_{C^*(\mathfrak{h})}^{\mathrm{op}} \end{array} \tag{7.13}$$

where $f_!$ is the right adjoint to $f^!$, given by coinduction. For every A-cofibrant dg-Lie algebroid $\mathfrak{g}$, taking the A-linear dual of a $\mathfrak{g}$-representation determines a left Quillen functor

$$(-)^\vee\colon \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg}} \longrightarrow \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{dg,op}}$$

whose right adjoint is given by $(-)^\vee$ as well. For every map $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, this functor intertwines induction and coinduction, i.e. there is a natural commuting diagram

$$\begin{array}{ccc} \mathrm{Rep}_{\mathfrak{g}} & \xrightarrow{(-)^\vee} & \mathrm{Rep}_{\mathfrak{g}}^{\mathrm{op}} \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_!} \\ \mathrm{Rep}_{\mathfrak{h}} & \xrightarrow[(-)^\vee]{} & \mathrm{Rep}_{\mathfrak{h}}^{\mathrm{op}} \end{array} \tag{7.14}$$

at the level of ∞-categories. We therefore obtain a natural transformation

$$\mu\colon \mathrm{Rep}_* \xrightarrow{(-)^\vee} \mathrm{Rep}^{\mathrm{op}} \xrightarrow{C^*} \mathrm{Mod}_{C^*}^{\mathrm{op}} \tag{7.15}$$

between functors $\mathrm{LieAlgd}_A \longrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{L}}$. Here Rep_* denotes the functor sending a map of Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ to the induction functor $f_*\colon \mathrm{Rep}_{\mathfrak{g}} \longrightarrow \mathrm{Rep}_{\mathfrak{h}}$.

Proposition 7.16. *Suppose that A is cofibrant, coherent and eventually coconnective, and let $\mathfrak{g}$ be a good Lie algebroid over A. Then the left adjoint functor μ (7.15) restricts to a fully faithful functor*

$$\mu\colon \mathrm{Rep}^{\omega}_{\mathfrak{g}} \longrightarrow \mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}$$

on the compact $\mathfrak{g}$-representations. Its essential image is the thick subcategory of $\mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}$ generated by $f_*(A)$.

Proof. The right adjoint of the functor μ is given by

$$\nu\colon \mathrm{Mod}^{\mathrm{op}}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}_{\mathfrak{g}};\ M \longrightarrow \Phi(M)^{\vee}.$$

We will first show that for any compact $\mathfrak{g}$-representation E, the unit map $E \longrightarrow \nu\mu(E)$ is an equivalence. To this end, let $f\colon 0 \longrightarrow \mathfrak{g}$ be the initial map and note that the forgetful functor $f^!\colon \mathrm{Rep}_{\mathfrak{g}} \longrightarrow \mathrm{Mod}_A$ detects equivalences and preserves filtered colimits. It follows that $\mathrm{Rep}^{\omega}_{\mathfrak{g}} \subseteq \mathrm{Rep}_{\mathfrak{g}}$ is the thick subcategory generated by the $\mathfrak{g}$-representations

$$f_*(F) = \mathcal{U}(\mathfrak{g}) \otimes_A F$$

where $F \in \mathrm{Mod}^{\omega}_A$ is a compact object in the ∞-category of A-modules. It therefore suffices to check that the unit map $f_*F \longrightarrow \nu\mu(f_*F)$ is an equivalence for any such free $\mathfrak{g}$-representation f_*F. We may assume that $\mathfrak{g}$ is a very good dg-Lie algebroid and, up to retracts, that F is a cofibrant dg-A-module of the form

$$F = \bigoplus_{n\leq n_0} A^{\oplus i_n}[n]. \tag{7.17}$$

(In the present situation the above sum can be taken finite, but in the tame setting of Section 8 we cannot assume this). To see that $f_*F \longrightarrow \nu\mu(f_*F)$ is an equivalence, note that it decomposes as

$$f_*F \longrightarrow \big((f_*F)^{\vee}\big)^{\vee} \xrightarrow{\ \epsilon\ } \Phi\big(C^*(\mathfrak{g}, (f_*F)^{\vee})\big)^{\vee}$$

where ϵ is the unit of the adjoint pair (Φ, C^*). The PBW theorem and Lemma 5.9 imply that $\mathcal{U}(\mathfrak{g}) \cong \bigoplus_{n\leq 0} A^{\oplus k_n}[n]$ as a graded A-module. It follows that f_*F is a cofibrant dg-A-module of the form

$$f_*F = \mathcal{U}(\mathfrak{g}) \otimes_A F \cong \bigoplus_{n\leq n_0} A^{\oplus m_n}[n].$$

Its linear dual $(f_*F)^{\vee}$ is then a dg-A-module of the form $\bigoplus_{n\leq n_0} A^{\oplus m_n}[-n]$. In particular, $(f_*F)^{\vee}$ is eventually connective, so that ϵ is an equivalence by Corollary 6.5. Furthermore, since A is eventually coconnective, the argument used in the proof of Corollary 4.2 shows that the biduality map $f_*F \longrightarrow (f_*F)^{\vee\vee}$ is an equivalence. We conclude that μ is fully faithful on the compact objects of $\mathrm{Rep}_{\mathfrak{g}}$.

The essential image of the compact objects is the smallest thick subcategory of $\mathrm{Mod}_{C^*(\mathfrak{g})}$ containing the images $\mu(f_*F)$ with $F \in \mathrm{Mod}_A^{\omega}$. Because

$$\mu(f_*F) \simeq C^*(\mathfrak{g}, (f_*F)^{\vee}) \simeq C^*(\mathfrak{g}, f_!(F^{\vee})) \simeq f_*(F^{\vee})$$

by the commuting diagrams (7.13) and (7.14), and because Mod_A^{ω} is generated by A itself, we find that the essential image agrees with the thick subcategory of $\mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}$ generated by $f_*(A)$. □

For any good Lie algebroid $\mathfrak{g}$, Proposition 7.16 provides a fully faithful inclusion

$$\mathrm{Ind}(\mu)\colon \mathrm{Rep}_{\mathfrak{g}} \simeq \mathrm{Ind}\big(\mathrm{Rep}_{\mathfrak{g}}^{\omega}\big) \hookrightarrow \mathrm{Ind}\big(\mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}\big) = \mathrm{QC}^!\big(C^*(\mathfrak{g})\big).$$

Its essential image can also be identified using the following observation:

Lemma 7.18. *Suppose that A is coherent and eventually coconnective and let $f\colon B \longrightarrow A$ be an object of $\mathrm{CAlg}_k^{\mathrm{sm}}/A$. Then the following stable subcategories of $\mathrm{Ind}(\mathrm{Coh}_B^{\mathrm{op}})$ are equivalent:*

($\mathcal{C}_1$) *The smallest stable subcategory that is closed under colimits and contains $f_*(A)$.*
($\mathcal{C}_2$) *The subcategory of those $E \in \mathrm{Ind}(\mathrm{Coh}_B^{\mathrm{op}})$ such that $f^!(E)$ is contained in the full subcategory $\mathrm{Mod}_A \subseteq \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}})$ (see Diagram (7.6)).*

Proof. Note that $f^!$ and the inclusion $\mathrm{Mod}_A \subseteq \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}})$ of (7.6) both preserve colimits. It follows that $\mathcal{C}_2$ is closed under colimits. Consequently, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ as soon as $f^!f_*(A)$ is quasi-coherent, i.e. an A-module.

Assuming this for the moment, we also find that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Indeed, by Lemma 7.8, $\mathrm{Ind}(\mathrm{Coh}_B^{\mathrm{op}})$ admits a set of compact generators of the form $f_*(F)$, with $F \in \mathrm{Coh}_A^{\mathrm{op}}$. Consequently, $f^!\colon \mathrm{Ind}(\mathrm{Coh}_B^{\mathrm{op}}) \longrightarrow \mathrm{Ind}(\mathrm{Coh}_A^{\mathrm{op}})$ detects equivalences and preserves colimits, so that the adjoint pair $(f_*, f^!)$ is monadic [19, Theorem 4.7.3.5]. This means that every $E \in \mathrm{Ind}(\mathrm{Coh}_B^{\mathrm{op}})$ arises as the colimit of its bar resolution

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; (f_*f^!)^2(E) \;\substack{\longrightarrow\\ \longrightarrow}\; (f_*f^!)(E) \dashrightarrow E.$$

If $f^!(E)$ is contained in Mod_A, then each term in the bar resolution is contained in $\mathcal{C}_1 \subseteq \mathcal{C}_2$, so that $E \in \mathcal{C}_1$.

To see that $f^!f_*(A)$ is a quasi-coherent A-module, note that it can be described by the limit-preserving functor $\mathrm{Coh}_A \longrightarrow \mathcal{S}$ sending F to

$$\mathrm{Map}_{\mathrm{Coh}_{C^*(\mathfrak{g})}}\big(f_*(A), f_*(F)\big) \simeq \mathrm{Map}_{\mathrm{Mod}_A}\big(f^*f_*(A), F\big).$$

Our goal is to argue that this functor is of the form described in Remark 7.5. To this end, let us assume that A is modeled by a cofibrant dg-algebra and that $B = C^*(\mathfrak{g})$ arises from a very good dg-Lie algebroid $\mathfrak{g}$. Diagrams (7.13) and (7.14) then show that $f_*(A) \in \mathrm{Coh}_B$ can be modeled at the chain level by

$$\mu\big(\mathcal{U}(\mathfrak{g})\big) = C^*\big(\mathfrak{g}, \mathcal{U}(\mathfrak{g})^\vee\big).$$

Because $\mathfrak{g}$ is a very good dg-Lie algebroid over A, the PBW theorem shows that $\mathcal{U}(\mathfrak{g})^\vee \cong \mathrm{Sym}_A(\mathfrak{g}^\vee)$ is a connective dg-A-module, which is freely generated without differential by finitely many generators in each degree. In particular, it is cofibrant as a dg-A-module. It follows that $C^*\big(\mathfrak{g}, \mathcal{U}(\mathfrak{g})^\vee\big)$ is a free graded $C^*(\mathfrak{g})$-module with generators in degrees ≥ 0, and is therefore cofibrant as a dg-$C^*(\mathfrak{g})$-module. We conclude that

$$f^*f_*(A) \simeq A \otimes_{C^*(\mathfrak{g})} C^*\big(\mathfrak{g}, \mathcal{U}(\mathfrak{g})^\vee\big) \cong \mathcal{U}(\mathfrak{g})^\vee.$$

Note that we can model every coherent A-module F by a graded-free dg-A-module with finitely many generators in each degree and no generators in degrees $\ll 0$. Since $\mathcal{U}(\mathfrak{g})^\vee$ has the same finiteness properties, there is an isomorphism of dg-A-modules

$$\mathrm{Hom}_A\big(\mathcal{U}(\mathfrak{g})^\vee, F\big) \cong \mathcal{U}(\mathfrak{g}) \otimes_A F.$$

Both $\mathcal{U}(\mathfrak{g})^\vee$ and $\mathcal{U}(\mathfrak{g})$ are cofibrant dg-A-modules, so the mapping complex and the tensor product already compute their derived functors. Applying the functor $\Omega^\infty\colon \mathrm{Mod}_A \longrightarrow \mathcal{S}$ taking underlying spaces (e.g. applying the Dold-Kan correspondence to connective covers), we find that the pro-coherent sheaf $f^!f_*(A)$ is given by the limit-preserving functor $\mathrm{Coh}_A \longrightarrow \mathcal{S}$

$$F \mapsto \mathrm{Map}_A\big(\mathcal{U}(\mathfrak{g})^\vee, F\big) \simeq \Omega^\infty\big(\mathcal{U}(\mathfrak{g}) \otimes_A F\big).$$

By Remark 7.5, this means that $f^!f_*(A)$ is contained in $\mathrm{Mod}_A \subseteq \mathrm{Ind}\big(\mathrm{Coh}_A^{\mathrm{op}}\big)$. □

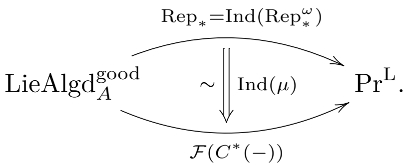
Corollary 7.19. *Suppose that A is cofibrant, coherent and eventually coconnective, and let $\mathfrak{g}$ be a good Lie algebroid over A. Then the functor $\mu\colon \mathrm{Rep}_{\mathfrak{g}}^{\omega} \longrightarrow \mathrm{Coh}_{C^*(\mathfrak{g})}$ of Proposition 7.16 induces a natural equivalence*

$$\mathrm{Ind}(\mu)\colon \mathrm{Rep}_{\mathfrak{g}} \xrightarrow{\ \sim\ } \mathrm{QC}^!\big(C^*(\mathfrak{g})\big) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A.$$

Proof. Passing to ind-completions, Proposition 7.16 identifies $\mathrm{Rep}_{\mathfrak{g}}$ with the ind-completion of the thick subcategory of $\mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}$ generated by $f_*(A)$. This ind-completion can be identified with the full subcategory $\mathrm{C}_1 \subseteq \mathrm{QC}^!(C^*(\mathfrak{g}))$ of Lemma 7.18. Lemma 7.18 shows that this subcategory is equivalent to the ∞-category of pro-coherent sheaves whose restriction to A is quasi-coherent, i.e. to the fiber product of ∞-categories $\mathrm{QC}^!(C^*(\mathfrak{g})) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A$. $\square$

We will now deduce Theorem 7.1 from the functoriality of the equivalence provided by Corollary 7.19.

Proof (of Theorem 7.1). We will follow the proof of Theorem 6.1. Corollary 7.19 provides a natural equivalence

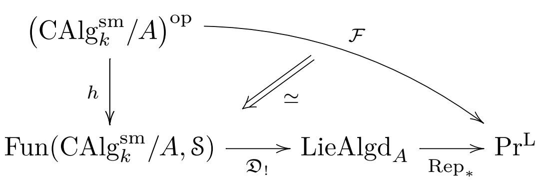


Here Rep_* sends a good Lie algebroid to its ∞-category of representations and a map $\mathfrak{g} \longrightarrow \mathfrak{h}$ to the induction functor. The functor $\mathcal{F}\colon \big(\mathrm{CAlg}_k^{\mathrm{sm}}/A\big)^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\mathrm{L}}$ sends a map of small extensions $f\colon B \longrightarrow B'$ to the adjoint pair

$$f_* \times \mathrm{id}\colon \mathrm{QC}^!\big(B'\big) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A \rightleftarrows \mathrm{QC}^!\big(B\big) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A\colon f^! \times \mathrm{id}.$$

These functors are indeed adjoint by Lemma 7.18: this implies that both f_* and $f^!$ preserve the category of pro-coherent sheaves whose restriction to A is quasi-coherent.

Precomposing with the duality functor $\mathfrak{D}$ (3.2), we obtain a natural diagram



commuting up a natural equivalence, given by the inverse of

$$\mathrm{Rep}_{\mathfrak{D}(B)} \xrightarrow{\mathrm{Ind}(\mu)} \mathrm{QC}^!\big(C^*\mathfrak{D}(B)\big) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A \longrightarrow \mathrm{QC}^!(B) \times_{\mathrm{QC}^!(A)} \mathrm{Mod}_A.$$

The second functor arises from the unit map $B \longrightarrow C^*\mathfrak{D}(B)$, which is an equivalence by Corollary 4.2.

Now consider the left Kan extension of the functor $\mathcal{F}$ along the Yoneda embedding h. By its universal property, there is a natural transformation of this left Kan extension to the functor $\mathrm{Rep}_{\mathfrak{D}_!}$. When $X = \mathrm{colim}_{\alpha \in \mathcal{C}_X} h(B_\alpha)$ is the colimit of a diagram of corepresentables, this natural transformation is given at X by

$$\lim_{\mathcal{C}_X}\left(\mathrm{QC}^!(B_\alpha)\times_{\mathrm{QC}^!(A)}\mathrm{Mod}_A\right)\xrightarrow{\sim}\lim_{\mathcal{C}_X}\left(\mathrm{Rep}_{\mathfrak{D}(B_\alpha)}\right)\longrightarrow \mathrm{Rep}_{\mathfrak{D}_!(X)}. \tag{7.20}$$

Here we take the limits of the diagram of ∞-categories and *right adjoint* functors $f^!$ between them. When X is a formal moduli problem, we can proceed as in the proof of Theorem 6.1: we can assume that $\mathcal{C}_X$ is sifted, so that the second functor in (7.20) is an equivalence by Lemma 2.13. Furthermore, $\mathcal{C}_X$ is in particular contractible, so that the limit of the constant diagram on Mod_A is just Mod_A itself (and similarly for $\mathrm{QC}^!(A)$). We conclude that the domain of (7.20) can be identified with

$$\lim_{\mathcal{C}_X}\left(\mathrm{QC}^!(B_\alpha)\right)\times_{\mathrm{QC}^!(A)}\mathrm{Mod}_A\simeq \mathrm{QC}^!(X)\times_{\mathrm{QC}^!(A)}\mathrm{Mod}_A.$$

Since $\mathfrak{D}_!(X)\simeq T_{A/X}$ for any formal moduli problem, we obtain a natural diagram

$$\mathrm{Rep}_{T_{A/X}}\xleftarrow{\sim}\mathrm{QC}^!(X)\times_{\mathrm{QC}^!(A)}\mathrm{Mod}_A\hookrightarrow \mathrm{QC}^!(X)$$

where the left functor is an equivalence and the right functor is fully faithful. This proves Theorem 7.1. $\square$

8. Pro-coherent Lie algebroids

Theorem 7.1 provides a geometric interpretation of the ∞-category of representations of a Lie algebroid in terms of pro-coherent sheaves on the corresponding formal moduli problem. This puts forward the theory of pro-coherent sheaves as a variant of the theory of quasi-coherent sheaves that is much more well-behaved from the point of view of deformation theory. For example, for every map of small extensions $B\longrightarrow B'$, the restriction functor

$$f^!\colon \mathrm{QC}^!(B)\longrightarrow \mathrm{QC}^!(B')$$

admits both a left and right adjoint, corresponding roughly to induction and coinduction of Lie algebroid representations under Koszul duality. In contrast, the functor $f^*\colon \mathrm{QC}(B)\longrightarrow \mathrm{QC}(B')$ only admits a right adjoint.

However, the relation between Lie algebroid representations and pro-coherent sheaves discussed in Section 7 is not perfect: because we have only discussed Lie algebroid representations on (quasi-coherent) A-modules, we can only give a Lie-algebraic description of those pro-coherent sheaves whose restriction to A is quasi-coherent. One would expect the full ∞-category of pro-coherent sheaves on X to correspond to the ∞-category of pro-coherent representations of $T_{A/X}$.

The purpose of this section is to describe some refinements of the results from the previous sections to the setting of *pro-coherent Lie algebroids* and their representations.

Our main observation is that (in certain cases) pro-coherent Lie algebroids can be conveniently described in terms of the *tame* homotopy theory of dg-Lie algebroids. This homotopy theory is transferred from the tame model structure on dg-A-modules and its weak equivalences form a *strict subclass* of the quasi-isomorphisms. Unfortunately, the resulting ∞-category is only well-behaved when A is coherent (Definition 7.2) and bounded (see Warning 8.14).

Definition 8.1. A connective commutative dg-k-algebra A is (strictly) *bounded* if it is concentrated in degrees $[0, n]$, for some $n \geq 0$. Note that being bounded is not invariant under quasi-isomorphisms; a connective dg-algebra is *eventually coconnective* if and only if it is quasi-isomorphic to a bounded dg-algebra.

With respect to the tame homotopy theory, a dg-Lie algebroid is A-cofibrant if its underlying graded A-module is graded-projective. In particular, the analyses of Section 3 and Section 4, which essentially only rely on the PBW theorem, can be applied almost verbatim to the tame setting. Consequently, we obtain the following analogues of Theorem 5.1 and Theorem 7.1:

Theorem 8.2. *Suppose that A is a cofibrant, bounded and coherent commutative dg-k-algebra. Then there is an equivalence of ∞-categories*

$$\mathrm{MC}\colon \mathrm{LieAlgd}^{!}_{A} \rightleftarrows \mathrm{FMP}^{!}_{A} : T_{A/}$$

between the ∞-category of pro-coherent Lie algebroids over A and the ∞-category of pro-coherent formal moduli problems $\mathrm{CAlg}^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$ (see Section 1).

Theorem 8.3. *Let A be a cofibrant, bounded and coherent commutative dg-k-algebra and let X be a pro-coherent formal moduli problem on A. Then there is an equivalence*

$$\mathrm{QC}^{!}(X) \xrightarrow{\sim} \mathrm{Rep}^{!}_{T_{A/X}}$$

between the ∞-categories of pro-coherent sheaves on X and pro-coherent $T_{A/X}$-representations. For any map $f\colon X \longrightarrow Y$ of pro-coherent formal moduli problems, this equivalence identifies the functor $f^{!}\colon \mathrm{QC}^{!}(Y) \longrightarrow \mathrm{QC}^{!}(X)$ with the restriction functor from $T_{A/Y}$-representations to $T_{A/X}$-representations.

Example 8.4. Suppose that $A = k$ is coherent (e.g Noetherian) and eventually coconnective. Then one can always find a (strictly) bounded model for A to which the theorems apply. In this case, Theorem 8.2 provides an equivalence between *Lie algebras* in $\mathrm{Ind}(\mathrm{Coh}_k^{\mathrm{op}})$, defined by explicit point-set models, and formal moduli problems $\mathrm{CAlg}_k^{\mathrm{sm,coh}}/k \longrightarrow \mathcal{S}$.

Example 8.5. The main type of dg-algebra A to which the above theorems apply is as follows. Suppose that k is eventually coconnective and Noetherian, i.e. $\pi_0(k)$ is Noetherian and each $\pi_n(k)$ is finitely generated over $\pi_0(k)$. We can assume that k is modeled at the chain level by a bounded cdga. Suppose that A is a commutative dg-k-algebra which is free without differential, such that:

(i) A has finitely many generators.
(ii) Each generator is either in degree 0 or in odd degree.

Then A is cofibrant over k, bounded and coherent. In particular, if k is eventually coconnective and Noetherian, then the derived zero locus of any map $\mathbb{A}^n_k \longrightarrow \mathbb{A}^m_k$ over $\mathrm{Spec}(k)$ can be modeled by a commutative dg-algebra A with these properties.

We will start in Section 8.1 by recalling the tame model structure on dg-A-modules. Section 8.2 describes the relation between this model structure and the ∞-category of pro-coherent sheaves on A. We then discuss the induced homotopy theory on dg-Lie algebroids (in Section 8.3) and their representations (in Section 8.4). Most importantly, we describe how one can modify the proofs of Theorem 5.1 and Theorem 7.1 to establish Theorem 8.2 and Theorem 8.3.

8.1. Tame dg-A-modules

Recall that for any commutative dg-algebra A, the category of dg-A-modules can be endowed with the *tame*, or *contraderived* model structure (see e.g. [1,25]). In this model structure, a map of dg-A-modules $E \longrightarrow F$ is

- a fibration if it is degreewise surjective.
- a cofibration if it is a monomorphism, whose cokernel is projective as a graded A-module.
- a weak equivalence if for any graded-projective dg-A-module P, the map on hom-complexes $\mathrm{Hom}_A(P, E) \longrightarrow \mathrm{Hom}_A(P, F)$ is a quasi-isomorphism.

We will denote the associated ∞-category by $\mathrm{Mod}^!_A$ and refer to it as the ∞-category of *tame dg-A-modules*. This terminology is supposed to emphasize that $\mathrm{Mod}^!_A$ depends on the explicit cdga A, i.e. quasi-isomorphic cdgas may have non-equivalent ∞-categories of tame dg-A-modules.

The tame model structure is stable and symmetric monoidal for the usual tensor product over A, and the usual projective model structure is a (symmetric monoidal) right Bousfield localization of the tame model structure. It follows that there is a fully faithful, symmetric monoidal left adjoint functor of ∞-categories

$$\mathrm{Mod}_A \longrightarrow \mathrm{Mod}^!_A.$$

Furthermore, $\mathrm{Mod}_A^!$ is a presentable ∞-category, since the tame model structure is combinatorial:

Lemma 8.6. *Let $\mathcal{T}^{\leq n}$ be the set of graded-free dg-A-modules T satisfying the following two conditions:*

(i) *T has no generators in (homological) degree $> n$*
(ii) *T has finitely many generators in each individual degree.*

Let $\mathcal{T} = \bigcup_n \mathcal{T}^{\leq n}$. Then the collection of cone inclusions $\{T \longrightarrow T[0,1] : T \in \mathcal{T}\}$ is a set of generating cofibrations for the tame model structure. In particular, the tame model structure is cofibrantly generated.

Proof. Suppose that $p\colon E \longrightarrow F$ is a map with the right lifting property against all $T \longrightarrow T[0,1]$ with $T \in \mathcal{T}$. To see that p is a trivial fibration, let $\mathcal{C} \subseteq \mathrm{Mod}_A^{\mathrm{dg}}$ be the full subcategory of graded-projective dg-A-modules P for which

$$\mathrm{Hom}_A(P, E) \longrightarrow \mathrm{Hom}_A(P, F)$$

is a trivial fibration. Then $\mathcal{T} \subseteq \mathcal{C}$ and we have to show that $\mathcal{C}$ contains all graded-projective dg-A-modules. To this end, let us make the following observations:

(1) $\mathcal{C}$ is closed under retracts.
(2) Let $P \longrightarrow Q$ be a cofibration such that P and Q/P are contained in $\mathcal{C}$. Then Q is contained in $\mathcal{C}$.
(3) If $\{P_0 \longrightarrow \dots \longrightarrow P_\alpha \longrightarrow \dots\}$ is a (transfinite) sequence of cofibrations between objects in $\mathcal{C}$, then the colimit $\operatorname{colim} P_\alpha$ is contained in $\mathcal{C}$ as well.

By (1), it suffices to verify that $\mathcal{C}$ contains all graded-free dg-A-modules. To see this, suppose that Q is a graded-free dg-A-module, with set of generators $S = \{y_i\}$. Let $\mathcal{B} \subseteq P(S)$ be the poset of subsets $S' \subseteq S$ which generate a sub dg-A-module of Q that is contained in $\mathcal{C}$. It follows from (3) that any chain in $\mathcal{B}$ has an upper bound. By Zorn's lemma, $\mathcal{B}$ admits a maximal element S_0. Let $P = A\langle S_0\rangle$ be its A-linear span and consider the quotient Q/P. We claim that Q/P is trivial, so that $Q \in \mathcal{C}$.

Indeed, Q/P has a set of generators $S \setminus S_0$. If there exists a generator $y \in S \setminus S_0$ of degree n, one can find a set $S' \subseteq S \setminus S_0$ containing y such that

- $A\langle S'\rangle$ is closed under the differential.
- S' contains finitely many generators in each degree $\leq n$ and no generators in degrees above n.

Indeed, one can proceed inductively, in each step choosing (finitely many) generators whose A-linear span includes the differentials of the (finitely many) generators chosen before.

The resulting dg-A-module $A\langle S'\rangle \subseteq Q/P$ is contained in $\mathcal{T}$. Using (2), it follows that $A\langle S_0 \cup S'\rangle \subseteq Q$ is contained in $\mathcal{C}$ as well. This contradicts maximality of S_0, so we conclude that $S \setminus S_0$ is empty and $Q/P = 0$. $\square$

Remark 8.7. Let E be a dg-A-module and consider its connective cover

$$\tau_{\geq 0}E = (\ldots \longrightarrow E_1 \longrightarrow Z_0(E) \longrightarrow 0)$$

Then the map $\tau_{\geq 0}E \longrightarrow E$ is a tame weak equivalence if and only if $[T, E] = 0$ for any $T \in \mathcal{T}^{\leq -1}$. We will say that E is a *connective* tame dg-A-module if it satisfies these equivalent conditions. The full subcategory $\mathrm{Mod}_A^{!,\geq 0}$ on the connective tame dg-A-modules determines an accessible t-structure on $\mathrm{Mod}_A^!$.

One can easily verify that a map between connective tame dg-A-modules is a tame weak equivalence if and only if it is a quasi-isomorphism. It follows that the fully faithful inclusion $\mathrm{Mod}_A \longrightarrow \mathrm{Mod}_A^!$ induces an equivalence $\mathrm{Mod}_A^{\geq 0} \longrightarrow \mathrm{Mod}_A^{!,\geq 0}$ on connective objects.

8.2. Tame modules and pro-coherent sheaves

We do not know of a good homotopy theoretic interpretation of the ∞-category of tame dg-modules over a general dg-algebra A. However, when A is *bounded and coherent*, the ∞-category $\mathrm{Mod}_A^!$ is very well-behaved, and can be identified with the ∞-category of pro-coherent sheaves on A (Corollary 8.9). In particular, $\mathrm{Mod}_A^!$ is a compactly generated ∞-category, and we have an explicit identification of its compact generators, analogous to the case where A is discrete, as treated in e.g. [22]:

Proposition 8.8. *Suppose that A is bounded and coherent and let $\mathcal{K}^{\leq 0}$ be the set of (weak equivalence classes of) dg-A-modules E such that*

- (i) *E is contained in the set $\mathcal{T}^{\leq 0}$ of Lemma 8.6, i.e. graded-free, with generators of degree ≤ 0 and finitely many generators in each degree.*
- (ii) *its A-linear dual $E^\vee = \mathrm{Hom}_A(E, A)$ is eventually coconnective, i.e. $\pi_i(E^\vee)$ vanishes for $i \gg 0$.*

Then $\mathcal{K}^{\leq 0}$ is a set of compact generators for $\mathrm{Mod}_A^!$.

Corollary 8.9. *Let A be bounded and coherent and consider the functor*

$$(-)^\vee \colon \mathrm{Mod}_A^! \longrightarrow \mathrm{Mod}_A^{\mathrm{op}}$$

sending a tame dg-A-module to its A-linear dual. This functor restricts to an equivalence between the full subcategory $\mathcal{K}^{\leq 0} \subseteq \mathrm{Mod}^!_A$ *and the opposite of the full subcategory* $\mathrm{Coh}^{\geq 0}_A \subseteq \mathrm{Mod}_A$ *of connective coherent A-modules. In particular, there is an equivalence of stable* ∞*-categories*

$$\mathrm{Mod}^!_A \simeq \mathrm{Ind}\big(\mathrm{Coh}^{\mathrm{op}}_A\big) = \mathrm{QC}^!(A).$$

Proof. The functor $(-)^\vee\colon \mathrm{Mod}^{\mathrm{dg}}_A \longrightarrow \mathrm{Mod}^{\mathrm{dg,op}}_A$ is a left Quillen functor from the tame model structure to the projective model structure, whose right adjoint is given by $(-)^\vee$ as well. For any (cofibrant) object $E \in \mathcal{K}^{\leq 0}$, the dual $E^\vee$ is graded-free, with finitely many generators in each nonnegative degree. In particular, it is cofibrant. Since A is bounded, the derived unit map $E \longrightarrow E^{\vee\vee}$ is an isomorphism. The full subcategory of $\mathrm{Mod}^!_A$ on the compact generators is therefore equivalent to its essential image under $(-)^\vee$ in $\mathrm{Mod}^{\mathrm{op}}_A$. Unwinding the definitions, this essential image is exactly the opposite of $\mathrm{Coh}^{\geq 0}_A$. □

Example 8.10. Let $f\colon A \longrightarrow B$ be a weak equivalence of bounded coherent cdgas and consider the Quillen pair $f^!\colon \mathrm{Mod}^{\mathrm{dg}}_A \leftrightarrows \mathrm{Mod}^{\mathrm{dg}}_B\colon f_!$ between the tame model structures, where $f^!$ sends E to $B \otimes_A E$. This Quillen pair is a Quillen equivalence. Indeed, note that $f^!$ preserves the compact generators from Proposition 8.8, since $f^!(E)^\vee \cong B \otimes_A (E^\vee)$ for any $E \in \mathcal{K}^{\leq 0}$. On compact objects, $f^!$ is therefore given by the composite of equivalences

$$f^!\colon \mathrm{Mod}^{!,\omega}_A \xrightarrow{(-)^\vee} \mathrm{Coh}^{\mathrm{op}}_A \xrightarrow{B\otimes_A(-)} \mathrm{Coh}^{\mathrm{op}}_B \xrightarrow{(-)^\vee} \mathrm{Mod}^{!,\omega}_B$$

This implies that $f^!\colon \mathrm{Mod}^!_A \longrightarrow \mathrm{Mod}^!_B$ is an equivalence as well.

The remainder of this section is devoted to the proof of Proposition 8.8.

Lemma 8.11. *Suppose that A is bounded and that E is an object in* $\mathcal{K}^{\leq 0}$*. Let* n_0 *be an integer such that* $\pi_i(E^\vee) = 0$ *for all* $i \geq n_0$*. Then the following hold:*

(1) *For any indexing set I, the map* $\bigoplus_I \mathrm{Hom}_A(E, A) \longrightarrow \mathrm{Hom}_A(E, \bigoplus_I A)$ *is an isomorphism.*
(2) *Let F be a dg-A-module which is graded-free, with generators* x_i*. Let* $F^{(n)}$ *be the quotient of F by the submodule generated by the* x_i *of degree* $< n$*. Then for all* $n > 1$*:*

$$\pi_0\mathrm{Hom}_A(E, F^{(n)}) = 0.$$

(3) *For every* $m \leq -n_0$*, the map*

$$\pi_0\mathrm{Hom}_A(E, F) \longrightarrow \pi_0\mathrm{Hom}_A(E, F^{(m)})$$

is an isomorphism.

Proof. For (1), we can write $E = \bigoplus_{n\geq 0} A^{\oplus i_n}[-n]$ at the level of graded A-modules. The map $\bigoplus_I \mathrm{Hom}_A(E, A) \longrightarrow \mathrm{Hom}_A(E, \bigoplus_I A)$ is then given by

$$\bigoplus_I \prod_{n\geq 0} A^{\oplus i_n}[n] \longrightarrow \prod_{n\geq 0} \bigoplus_I A^{\oplus i_n}[n].$$

This map is an isomorphism in each degree, since A is concentrated in nonnegative degrees.

For assertion (2), note that $F^{(n)}$ is graded-free on the generators x_i of degree $\geq n$. Since E only has generators in degree ≤ 0, the complex $\mathrm{Hom}_A(E, F^{(n)})$ is trivial in degree 0 when $n > 1$.

For (3), consider the tower of fibrations between graded-free dg-A-modules

$$F \longrightarrow \dots \longrightarrow F^{(-2)} \longrightarrow F^{(-1)} \longrightarrow F^{(0)}.$$

The natural map $F \longrightarrow \lim_{n\leq 0} F^{(n)}$ to the (homotopy) limit of this tower can be identified with the map

$$\bigoplus_n \bigoplus_{I_n} A[n] \longrightarrow \prod_n \bigoplus_{I_n} A[n].$$

Here I_n denotes the set of generators of F of degree n. Because A is bounded, this map is an isomorphism.

For every $m < -n_0$, the fiber of the map $\mathrm{Hom}_A(E, F^{(m)}) \longrightarrow \mathrm{Hom}_A(E, F^{(m+1)})$ can be identified with

$$\mathrm{Hom}_A\Big(E, \bigoplus_{I_m} A[m]\Big) \cong \bigoplus_{I_m} \mathrm{Hom}_A\big(E, A[m]\big) \cong \bigoplus_{I_m} E^\vee[m].$$

By our assumption on E, the homotopy groups of this complex vanish in all degrees ≥ -1. This implies that the map

$$\pi_0\mathrm{Hom}_A(E, F) \cong \pi_0\Big(\lim_{n\leq 0} \mathrm{Hom}_A(E, F^{(n)})\Big) \longrightarrow \pi_0\mathrm{Hom}_A(E, F^{(m)})$$

is an isomorphism for all $m \leq -n_0$. $\square$

Lemma 8.12. *Suppose that A is bounded and coherent. Let $\mathcal{C} \subseteq \mathrm{Mod}^!_A$ be the smallest full subcategory of $\mathrm{Mod}^!_A$ that contains the objects in $\mathcal{K}^{\leq 0}$ and is closed under colimits and extensions. Then $\mathcal{C} = \mathrm{Mod}^!_A$.*

Proof. By Lemma 8.6, it suffices to show that $\mathcal{C}$ contains all objects $T \in \mathcal{T}$, which are of the form $\bigoplus_{n\geq n_0} A^{\oplus k_n}[-n]$ without differential. The dual of such a T is given by the (projectively) cofibrant dg-A-module $\bigoplus_{n\geq n_0} A^{\oplus k_n}[n]$ and the map $T \longrightarrow T^{\vee\vee}$ is an isomorphism. Consider the Postnikov tower of $T^\vee$

$$T^\vee \longrightarrow \dots \longrightarrow \tau_{\leq n}(T^\vee) \longrightarrow \tau_{\leq n-1}(T^\vee) \longrightarrow \cdots \longrightarrow \pi_0(T^\vee).$$

Because A is coherent, each $\pi_i(T^\vee)$ is a coherent A-module and admits an almost finite resolution $Y_i \xrightarrow{\sim} \pi_i(T^\vee)$ [19, Proposition 7.2.4.17]. One can use these resolutions to resolve the entire Postnikov tower by

$$T^\vee \longrightarrow \dots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0$$

where $P_n = \bigoplus_{i=0}^n Y_i[i]$, equipped with a certain differential. The sequence of P_n becomes stationary in each individual degree, so that there is a homotopy equivalence $T^\vee \longrightarrow P_\infty = \bigoplus_i Y_i[i]$. Taking the dual, one finds that $T = T^{\vee\vee}$ is homotopy equivalent to the colimit of the sequence

$$P_0^\vee \longrightarrow P_1^\vee \longrightarrow \dots .$$

This is a sequence of cofibrations whose associated graded consists of the $Y_i[i]^\vee$. Each $Y_i[i]^\vee$ is contained in $\mathcal{K}$, because its dual is $Y_i[i] \simeq \pi_i(T^\vee)[i]$. This implies that the (homotopy) colimit T is contained in the category $\mathcal{C}$. $\square$

Proof (of Proposition 8.8). By Lemma 8.12, the objects of $\mathcal{K}^{\leq 0}$ generate $\mathrm{Mod}^!_A$. To prove that any $E \in \mathcal{K}^{\leq 0}$ is compact, it suffices to show that for any set S and any collection of graded-free dg-A-modules $\{P_\alpha\}_{\alpha\in S}$, the map

$$\bigoplus_\alpha \pi_0\mathrm{Hom}_A(E, P_\alpha) \longrightarrow \pi_0\mathrm{Hom}_A\Big(E, \bigoplus_\alpha P_\alpha\Big)$$

is an isomorphism. Using the filtration of Lemma 8.11, we obtain commuting squares

$$\begin{array}{ccc}
\bigoplus_\alpha \pi_0\mathrm{Hom}_A(E, P_\alpha) & \longrightarrow & \pi_0\mathrm{Hom}_A\Big(E, \bigoplus_\alpha P_\alpha\Big) \\
\downarrow & & \downarrow \\
\bigoplus_\alpha \pi_0\mathrm{Hom}_A(E, P_\alpha^{(n)}) & \xrightarrow{\phi_n} & \pi_0\mathrm{Hom}_A\Big(E, \bigoplus_\alpha P_\alpha^{(n)}\Big)
\end{array}$$

for all $n \in \mathbb{Z}$. We will prove by decreasing induction on n that ϕ_n is an isomorphism; this proves that the top map is an isomorphism, because the vertical maps become isomorphisms for all m smaller than a certain n_0, by Lemma 8.11. We can start the induction at $n = 2$, where both objects are zero by Lemma 8.11.

For the inductive step, suppose that ϕ_n is an isomorphism. To prove that ϕ_{n-1} is an isomorphism as well, let F_α denote the fiber of the map $P_\alpha^{(n-1)} \longrightarrow P_\alpha^{(n)}$. It suffices to check that the map

$$\bigoplus_\alpha \mathrm{Hom}_A(E, F_\alpha) \longrightarrow \mathrm{Hom}_A\Big(E, \bigoplus F_\alpha\Big)$$

is a quasi-isomorphism. But each F_α is just given by a direct sum $\bigoplus A[n-1]$, so the result follows from (a shift of) Lemma 8.11. □

8.3. Pro-coherent Lie algebroids

Suppose that A is an eventually coconnective, coherent commutative k-algebra and let $\mathrm{CAlg}_k^{\mathrm{sm,coh}}/A$ be the ∞-category of iterated square zero extensions of A by connective coherent A-modules. Recall from the introduction that a *pro-coherent formal moduli problem* is a functor

$$X\colon \mathrm{CAlg}_k^{\mathrm{sm,coh}}/A \longrightarrow \mathcal{S}$$

satisfying the Schlessinger conditions: $X(A)$ is contractible and X preserves pullbacks along the maps $A \longrightarrow A \oplus E[1]$, where $E \in \mathrm{Coh}_A^{\geq 0}$.

The purpose of this section is to show that such pro-coherent formal moduli problems can be described (in suitable cases) in terms of Lie algebroids. To this end, note that for any connective commutative dg-algebra A, the tame model structure on dg-A-modules can be transferred to a semi-model structure on dg-Lie algebroids, by [23, Remark 4.26]:

Proposition 8.13. *The category of dg-Lie algebroids over A carries the* tame *semi-model structure, in which a map is a weak equivalence (fibration) if and only if it is a tame weak equivalence (fibration). The forgetful functor* $\mathrm{LieAlgd}_A^{\mathrm{dg}} \longrightarrow \mathrm{Mod}_A^{\mathrm{dg}}/T_A$ *is a right Quillen functor to the tame model structure, which preserves cofibrant objects and all sifted homotopy colimits.*

Warning 8.14. The tame homotopy theory of dg-Lie algebroids is only well-behaved when A is both cofibrant over k (Remark 2.7) and bounded coherent (Proposition 8.8). There is some conflict between cofibrancy and being bounded: for example, A cannot have any generators of degree 2 (cf. Example 8.5).

Definition 8.15. Let A be a cofibrant connective cdga over k, which is bounded and coherent. The ∞-category of *pro-coherent Lie algebroids* over A is the ∞-categorical localization

$$\mathrm{LieAlgd}_A^! := \mathrm{LieAlgd}_A^{\mathrm{dg}}\Big[\{\text{tame w.e.}\}^{-1}\Big].$$

This is a locally presentable ∞-category by the argument of Remark 2.5.

Note that the projective semi-model structure on dg-Lie algebroids is a right Bousfield localization of the tame semi-model structure, so that there is a fully faithful left adjoint functor

$$\mathrm{LieAlgd}_A \hookrightarrow \mathrm{LieAlgd}^!_A.$$

The remainder of this section is devoted to the proof of Theorem 8.2, relating pro-coherent Lie algebroids over a cofibrant, bounded and coherent dg-k-algebra A to pro-coherent formal moduli problems.

Proof (of Theorem 8.2). The proof of Theorem 8.2 is essentially a copy of the proof of Theorem 5.1. We will therefore only give an outline of the argument, pointing out the changes that need to be made to the statements appearing in the previous sections.

Step 1: Chevalley-Eilenberg complex. For any cofibrant commutative dg-algebra A, the results of Section 2 and Section 3 hold verbatim for the tame semi-model structure on dg-Lie algebroids, as long as the condition of being 'A-cofibrant' is taken with respect to the tame model structure. In particular (cf. Proposition 3.1), the Chevalley-Eilenberg complex is the left adjoint in an adjoint pair

$$C^*\colon \mathrm{LieAlgd}^!_A \rightleftarrows \left(\mathrm{CAlg}_k/A\right)^{\mathrm{op}} : \mathfrak{D}. \tag{8.16}$$

Here CAlg_k/A is the usual ∞-category of commutative k-algebras over A, i.e. cdgas up to quasi-isomorphism. The right adjoint sends $B \longrightarrow A$ to the dual of the map between cotangent complexes $L_A \longrightarrow L_{A/B}$ (over k); this dual is now computed in pro-coherent sheaves, rather than Mod_A (i.e. with respect to the tame model structure).

Step 2: Koszul duality. Let A be a cofibrant dg-k-algebra and suppose that $\mathfrak{g}$ is a dg-Lie algebroid whose underlying graded A-module is of the form

$$\mathfrak{g} = \bigoplus_{n<0} A^{\oplus k_n}[n]. \tag{8.17}$$

Note that all such $\mathfrak{g}$ are (tamely) A-cofibrant, so that the value of the left adjoint in (8.16) is still computed by the Chevalley-Eilenberg complex $C^*(\mathfrak{g})$ (i.e. one does not have to cofibrantly replace $\mathfrak{g}$). Lemma 4.4 then identifies the unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$ of the adjunction (8.16) with

$$\mathfrak{g} \longrightarrow L^\vee_{C^*(\mathfrak{g})/A}.$$

Here the dual is taken in with respect to the tame model structure. For $\mathfrak{g}$ as in (8.17), $C^*(\mathfrak{g})$ is a connective cdga, so that $L_{C^*(\mathfrak{g})/A}$ is a connective dg-A-module. In particular, it is contained in the subcategory $\mathrm{Mod}_A \subseteq \mathrm{QC}^!(A)$ (Remark 8.7). The computations of Section 4 then provide an equivalence $L_{C^*(\mathfrak{g})/A} \simeq \mathfrak{g}^\vee$ in $\mathrm{Mod}_A \subseteq \mathrm{QC}^!(A)$.

It follows that the derived unit map $\mathfrak{g} \longrightarrow \mathfrak{D}C^*(\mathfrak{g})$ can be identified with the map of dg-A-modules $\mathfrak{g} \longrightarrow \mathfrak{g}^{\vee\vee}$ (cf. Proposition 4.1). Because A is bounded, this map is an isomorphism. Consequently (cf. Corollary 4.2), the left adjoint C^* in (8.16) is fully faithful on all tame dg-Lie algebroids of the form (8.17).

Step 3: axiomatic deformation theory. Under our assumption that A is bounded and coherent, the ∞-categories $\mathrm{Mod}^!_A$ and $\mathrm{LieAlgd}^!_A$ are compactly generated, by Proposition 8.8 and Proposition 8.13. We are therefore in the position to apply the machinery developed by Lurie [17] (cf. Section 5). The good pro-coherent Lie algebroids are now modeled by those dg-Lie algebroids that are obtained from finitely many pushouts along

$$\mathrm{Free}\big(\partial\phi\colon K[-2] \longrightarrow T_A\big) \longrightarrow \mathrm{Free}\big(\phi\colon K[-2,-1] \longrightarrow T_A\big) \tag{8.18}$$

with K in the subcategory $\mathcal{K}^{\leq 0} \subseteq \mathrm{Mod}^!_A$ of compact generators, as in Proposition 8.8. In particular (cf. Lemma 5.9), they are all of the form (8.17), so that we obtain a fully faithful functor

$$C^*\colon \mathrm{LieAlgd}^{!,\mathrm{good}}_A \hookrightarrow \big(\mathrm{CAlg}_k/A\big)^{\mathrm{op}}.$$

The functor C^* sends the maps (8.18) to the maps $A \longrightarrow A \oplus E[1]$, where $E = K^\vee$ is the A-linear dual of an object in $\mathcal{K}^{\leq 0}$. By Proposition 8.8, these are exactly the maps $A \longrightarrow A \oplus E[1]$ where E is a connective coherent A-module. Arguing as in the proof of Theorem 5.1, it follows that C^* yields an equivalence between the good pro-coherent Lie algebroids over A and the ∞-category $\mathrm{CAlg}^{\mathrm{sm,coh}}_k/A$ of small extensions of A by coherent A-modules. Theorem 8.2 now follows by applying Proposition 5.4. □

8.4. Pro-coherent representations

Suppose that A is a bounded, coherent and cofibrant dg-k-algebra. Just as the tame model structure on dg-Lie algebroids over A provides a good model for the ∞-category of pro-coherent Lie algebroids, one can describe their ∞-categories $\mathrm{Rep}^!_{\mathfrak{g}}$ of *pro-coherent representations* using the tame model structure as well.

Lemma 8.19. *Let $\mathfrak{g}$ be a (tame) A-cofibrant dg-Lie algebroid. Then the following assertions hold:*

(1) *The category $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ carries a combinatorial model structure whose weak equivalence (fibrations) are the tame weak equivalences (fibrations) of dg-A-modules. For every $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$, induction and restriction form a Quillen pair*

$$f_* = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} -\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{h}}\colon f^!$$

which is a Quillen equivalence when f is a tame weak equivalence between (tame) A-cofibrant dg-Lie algebroids.

(2) *The category* $\mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}$ *carries a model structure whose weak equivalences (cofibrations) are the tame weak equivalences (cofibrations) of dg-A-modules. This is a monoidal model category, which is Quillen equivalent to the model category from (1). Every* $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ *between A-cofibrant dg-Lie algebroids, induces a symmetric monoidal left Quillen functor*

$$f^!\colon \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{h}} \longrightarrow \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}}.$$

In particular, it gives rise to a symmetric monoidal left adjoint functor $f^!\colon \mathrm{Rep}^!_{\mathfrak{h}} \longrightarrow \mathrm{Rep}^!_{\mathfrak{g}}$ *between closed symmetric monoidal* ∞*-categories.*

(3) *The functor of* ∞*-categories (e.g. constructed via fibrations, as in Section 6.2)*

$$\mathrm{Rep}^!\colon \mathrm{LieAlgd}^{!,\mathrm{op}}_A \longrightarrow \mathrm{Pr}^{\mathrm{L}}_{\mathrm{sym.mon.}}/\mathrm{Mod}^!_A$$

preserves all limits.

Proof. The PBW filtration also applies to dg-Lie algebroids which are A-cofibrant for the tame model structure on dg-A-modules. The proof of Lemma 2.10 can therefore be applied to show (1) (the existence of the model structure follows from Quillen's path object argument). Similarly, the proof of Lemma 2.11 yields (2).

For (3), one applies the same proof as in Lemma 2.13. Indeed, let $\mathrm{BiMod}^!_A$ be the ∞-category associated to the category of dg-A-bimodules (over $\mathbb{Q}$), endowed with the model structure transferred from the *tame* model structure on left dg-A-modules. This is a cofibrantly generated monoidal model category, with generating cofibrations with cofibrant domains. By [19, Theorem 4.1.8.4], the ∞-category $\mathrm{Alg}(\mathrm{BiMod}^!_A)$ can be modeled by the transferred model structure on associative algebras in this monoidal model category. Forgetting the symmetric monoidal structure, the functor $\mathrm{Rep}^!$ now decomposes as

$$\mathrm{LieAlgd}^{!,\mathrm{op}}_A \xrightarrow{\ \mathcal{U}\ } \mathrm{Alg}\big(\mathrm{BiMod}^!_A\big)^{\mathrm{op}} \xrightarrow{\ \mathrm{LMod}\ } \widehat{\mathrm{Cat}}_\infty/\mathrm{Mod}_A.$$

The second functor preserves limits by [19, Theorem 4.8.5.11]. To see that the first functor preserves sifted limits, it suffices to show that $\mathcal{U}\colon \mathrm{LieAlgd}^!_A \longrightarrow \mathrm{BiMod}^!_A$ preserves sifted colimits. This follows from [23, Theorem 4.22, Remark 4.26]. As in the proof of Lemma 2.13, the fact that $\mathcal{U}$ also preserves finite products follows from the fact that $\mathcal{U}(F(V)) \cong T_A(V)$ for any cofibrant dg-A-module. □

In the remainder of this section, we will outline the proof of Theorem 8.3, identifying pro-coherent sheaves on a pro-coherent formal moduli problem X with pro-coherent representations of $T_{A/X}$.

Proof (of Theorem 8.3). The proof of Theorem 7.1 essentially carries over verbatim to this situation. In fact, the argument simplifies because we can work with the ∞-category $\mathrm{QC}^!(X)$ of all pro-coherent sheaves on a formal moduli problem, rather than the subcategory of pro-coherent sheaves that restrict to quasi-coherent sheaves on A. Again, let us only indicate the structure of the proof.

Step 1: representations of good Lie algebroids. For every dg-Lie algebroid whose underlying dg-A-module is tamely cofibrant, the adjunction

$$K(\mathfrak{g}) \otimes_{C^*(\mathfrak{g})} (-)\colon \mathrm{Mod}^{\mathrm{dg}}_{C^*(\mathfrak{g})} \rightleftarrows \mathrm{Rep}^{\mathrm{dg}}_{\mathfrak{g}} \colon C^*(\mathfrak{g}, -)$$

is a Quillen adjunction between the usual projective model structure on dg-$C^*(\mathfrak{g})$-modules and the model structure on $\mathfrak{g}$-representations described in Lemma 8.19(1). Let $\Phi_{\mathfrak{g}}\colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}^!_{\mathfrak{g}}$ denote the left derived functor between ∞-categories.

When $\mathfrak{g} \in \mathrm{LieAlgd}^!$ is compact, $\Phi_{\mathfrak{g}}(C^*(\mathfrak{g})) = K(\mathfrak{g}) \simeq A$ is a compact object in $\mathrm{Rep}^!_{\mathfrak{g}}$, from which one deduces that $\Phi_{\mathfrak{g}}$ is fully faithful (cf. Lemma 6.3). Moreover, $\Phi_{\mathfrak{g}}$ identifies $\mathrm{Mod}^{\geq 0}_{C^*(\mathfrak{g})}$ with the full subcategory of connective pro-coherent $\mathfrak{g}$-representations (cf. Lemma 6.4 and Corollary 6.5).

Step 2: naturality. The constructions of Section 6.2 carry over to the tame setting verbatim (taking 'A-cofibrant' with respect to the tame model structure everywhere). In particular, they show that the functor $\Phi_{\mathfrak{g}}\colon \mathrm{Mod}_{C^*(\mathfrak{g})} \longrightarrow \mathrm{Rep}^!_{\mathfrak{g}}$ depends functorially on the tame Lie algebroid $\mathfrak{g}$.

Step 3. As in the beginning of Section 7.2, consider the (natural) right adjoint to $\Phi_{\mathfrak{g}}$ and precompose this with the natural functor $(-)^\vee\colon \mathrm{Rep}^!_{\mathfrak{g}} \longrightarrow \mathrm{Rep}^{!,\mathrm{op}}_{\mathfrak{g}}$ taking A-linear duals. The result is a natural transformation μ (cf. Diagram 7.15), whose value on a map of pro-coherent Lie algebroids $f\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is given by

$$\begin{array}{ccccc} \mathrm{Rep}^!_{\mathfrak{g}} & \xrightarrow{(-)^\vee} & \mathrm{Rep}^{!,\mathrm{op}}_{\mathfrak{g}} & \xrightarrow{C^*(\mathfrak{g},-)} & \mathrm{Mod}^{\mathrm{op}}_{C^*(\mathfrak{g})} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Rep}^!_{\mathfrak{h}} & \xrightarrow[(-)^\vee]{} & \mathrm{Rep}^{!,\mathrm{op}}_{\mathfrak{h}} & \xrightarrow[C^*(\mathfrak{h},-)]{} & \mathrm{Mod}^{\mathrm{op}}_{C^*(\mathfrak{h})}. \end{array} \tag{8.20}$$

Here the left vertical functor is induction along $\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ and the right vertical functor restricts a module along $C^*(\mathfrak{h}) \longrightarrow C^*(\mathfrak{g})$.

Step 4: Koszul duality. Let $\mathfrak{g}$ be a good pro-coherent Lie algebroid and let $f\colon 0 \longrightarrow \mathfrak{g}$ denote the zero map. The proof of Proposition 7.16 shows that the functor μ is fully faithful on the compact objects in $\mathrm{Rep}^!_{\mathfrak{g}}$. Note that $\mathrm{Rep}^!_{\mathfrak{g}}$ has a set of compact generators given by the free representations $f_*(E) = \mathcal{U}(\mathfrak{g}) \otimes_A E$, where $E \in \mathrm{Mod}^{!,\omega}_A$ is a compact object in the tame model structure on dg-A-modules. By naturality (8.20), the natural transformation μ sends such a free representation to the $C^*(\mathfrak{g})$-module $f_*(E^\vee)$. By Proposition 8.8, the duals $E^\vee$ of the compact tame dg-A-modules coincide with the coherent A-modules. Since the thick subcategory of $\mathrm{Mod}_{C^*(\mathfrak{g})}$ generated by $f_*(\mathrm{Coh}_A)$ coincides with $\mathrm{Coh}_{C^*(\mathfrak{g})}$ by Lemma 7.8, we obtain an equivalence

$$\mu\colon \mathrm{Rep}^{!,\omega}_{\mathfrak{g}} \longrightarrow \mathrm{Coh}^{\mathrm{op}}_{C^*(\mathfrak{g})}$$

for any good pro-coherent Lie algebroid $\mathfrak{g}$. This equivalence is natural in $\mathfrak{g}$, i.e. it intertwines induction of representations with the direct image of coherent modules.

Step 5: globalizing. Finally, we proceed as in the proofs of Theorem 6.1 and Theorem 7.1: passing to ind-completions and precomposing with the duality functor $\mathfrak{D}$ (8.16), we obtain a diagram

$$\begin{array}{ccccc} \left(\mathrm{CAlg}^{\mathrm{sm,coh}}_k/A\right)^{\mathrm{op}} & & \xrightarrow{\quad\mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}})=\mathrm{QC}^!\quad} & & \mathrm{Pr}^{\mathrm{L}} \\ \downarrow & & \swarrow\!\!\swarrow \sim & & \| \\ \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{sm,coh}}_k/A, \mathcal{S}) & \xrightarrow[\mathfrak{D}_!]{} & \mathrm{LieAlgd}^!_A & \xrightarrow[\mathrm{Rep}^!_*]{} & \mathrm{Pr}^{\mathrm{L}} \end{array} \tag{8.21}$$

commuting up to a natural equivalence, given by *inverse* of the composite equivalence

$$\mathrm{Rep}^!_{\mathfrak{D}(B)} \simeq \mathrm{Ind}\Big(\mathrm{Rep}^{!,\omega}_{\mathfrak{D}(B)}\Big) \xrightarrow{\mathrm{Ind}(\mu)} \mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}}_B) \xrightarrow{\sim} \mathrm{Ind}(\mathrm{Coh}^{\mathrm{op}}_{C^*\mathfrak{D}(B)}).$$

For every $X\colon \mathrm{CAlg}^{\mathrm{sm,coh}}_k/A \longrightarrow \mathcal{S}$, the ∞-category $\mathrm{QC}^!(X)$ is defined via left Kan extension (see Definition 7.10), so that we obtain a natural functor

$$\mathrm{QC}^!(X) \longrightarrow \mathrm{Rep}^!_{\mathfrak{D}_!(X)}.$$

When X is corepresentable, this functor is simply the natural equivalence from Diagram (8.21). Furthermore, the domain and codomain of this natural functor both preserve sifted colimits in X: for $\mathrm{Rep}^!_{\mathfrak{D}_!(X)}$ this follows from the fact that $\mathfrak{D}_!$ preserves colimits by definition and that $\mathrm{Rep}^!$ preserves sifted colimits by Lemma 8.19. Since any pro-coherent formal moduli problem X is a sifted colimit of corepresentable functors, the above functor becomes an equivalence. The theorem then follows from the fact that $\mathfrak{D}_!(X) \simeq T_{A/X}$ for any $X \in \mathrm{FMP}^!_A$. $\square$

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