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Modular cocycles and cup product

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A B S T R A C T

The Eichler-Shimura isomorphism relates holomorphic modular cusps forms of positive even integral weight to cohomology classes. The Haberland formula uses the cup product to give a cohomological formulation of the Petersson scalar product. In this paper we extend Haberland's formula to modular cusp forms of positive real weight. This relation is based on the cup product of an Eichler cocycle and a Knopp cocycle.
We may also consider the cup product of two Eichler cocycles. In the classical situation this cup product is almost always zero. However we show evidence that for real weights this cup product may very well be non-trivial. We approach the question whether the cup product is a non-trivial coinvariant by duality with a space of entire modular forms. The cup product yields a bilinear map over $\mathbb{C}$ from pairs of holomorphic modular forms (not necessarily of the same weight, one of them may have large growth at the cusps) to coinvariants in infinite-dimensional modules. To investigate whether this bilinear map is non-trivial we test the result against entire modular forms of a suitable weight. Under some conditions on the weights, this leads to an explicit triple integral, which can be investigated numerically, thus providing evidence that the cup product is non-trivial at least in some situations.

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1. Introduction

In this paper we investigate the cup product of the cohomology classes associated to holomorphic modular form.

In the classical situation of even positive weight Haberland's formula, based on the cup product, gives a cohomological interpretation of the Petersson scalar product of two cusp forms of integral weight. We extend this relation to positive real weight. The Haberland formula is based on the cup product of an Eichler cocycle, which is linear in the cusp form, and a Knopp cocycle, which is conjugate linear in the other cusp form.

The cup product of two Eichler cocycles gives a bilinear form on holomorphic modular forms. The weights may differ, and one of the modular forms has to be a cusp form. In the classical situation of even positive weights this cup product is almost always trivial. We study this cup product for arbitrary real weights. That is more complicated than the classical case, since the coefficient modules have in general infinite dimension. We will give evidence that this cup product may be non-trivial in many cases.

The Eichler-Shimura isomorphism in the classical situation cannot be generalized directly to other real weights. For real weights that are not integers at least 2 the Eichler cocycles determine an injection from the space of all modular forms (without growth conditions) into a parabolic cohomology group $H^1_p(\Gamma; V)$ where V is a specific infinite-dimensional module (Eichler [8], Knopp [10], Knopp-Mawi [11], Choie-

Bruggeman-Diamantis [3]). In [3] we show that the relation between holomorphic modular forms with weight not an integer at least 2 and cohomology is quite similar to the relation between Maass forms and cohomology.

The second cohomology group $H^2(\Gamma; V)$ is zero for all modules V and for all cofinite discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$. However for parabolic cohomology we have

$$H^2_p(\Gamma; V) \cong V_\Gamma = V \Big/ \sum_{\gamma\in\Gamma} V|(1-\gamma)\,, \tag{1.1}$$

the space of coinvariants, provided Γ has cusps; see [4, (11.9)]. (We work with right Γ-modules that are vector spaces over $\mathbb{C}$, and denote the action by a slash.)

The cohomology group $H^2_p(\Gamma;\mathbb{C}) \cong \mathbb{C}$ is used in the cohomological description of the Petersson scalar product on spaces of cusp forms of integral weights at least 2 by Haberland [9]. The cup product is used to go from a pair of 1-cocycles associated to cusp forms to $H^2_p(\Gamma;\mathbb{C}) \cong \mathbb{C}$. This method has been extended and used by Kohnen and Zagier [12, pp. 244–245], Zagier [18, Theorem 1], Cohen [6], Paşol and Popa [17] and Choie, Park and Zagier [5]. The Petersson scalar product of Maass cusp forms can also be formulated in terms of a cup product [4, Theorem 19.1].

Here we investigate the cup product of two cohomology classes associated to two holomorphic modular forms that may have different real weights (one of them should be a cusp form). In general this is a rather abstract exercitation providing a coinvariant in a complicated Γ-module, which is the tensor product $M_1 \otimes M_2$ of two modules of infinite dimension. With an intertwining operator $M_1 \otimes M_2 \to N$ we obtain from the cup product a coinvariant in a simpler module N that may be easier to study. We proceed with two choices. One choice leads to the extension to real weights of the cohomological description of the Petersson scalar product of cusp forms. The other choice leads to a trilinear form on the product of three spaces of modular forms.

We restrict our attention the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. We work with holomorphic modular forms of real weight, with a corresponding multiplier system. To a modular form f of weight $r \in \mathbb{R}$ and multiplier system v we may associate two types of 1-cocycles. The first type is the homogeneous Eichler cocycle

$$c_f(z_1, z_2; t) = \int_{\tau=z_1}^{z_2} f(\tau)\,(\tau - t)^{r-2}\,d\tau \tag{1.2}$$

where the variable t runs through the lower half-plane $\mathfrak{H}^-$, such that $(z-t)^{r-2}$ is well defined. The points z_1 and z_2 are in the upper half-plane $\mathfrak{H}$; if f is a cusp form we may take z_1 and/or z_2 equal to a cusp of Γ.

The other type is the Knopp cocycle, which depends on the modular form in a conjugate-linear way.

$$c_f^K(z_1, z_2; z) = \int_{\tau=z_1}^{z_2} \overline{f(\tau)}\, (\bar\tau - z)^{r-2}\, d\tau\,. \tag{1.3}$$

This cocycle has values in the holomorphic functions on the upper half-plane. It is defined for $z_1, z_2 \in \mathfrak{H}$; if f is a cusp form the points z_1 and z_2 may be cusps.

When dealing with cocycles it is important to indicate the module in which they take their values. Where possible we follow the notations and conventions of [3], which we recall in more detail in Section 2. For the Eichler cocycles we use two modules ${}^-\mathcal{D}^\omega_{v,2-r} \subset {}^-\mathcal{D}^\infty_{v,2-r}$ of holomorphic functions on the lower half-plane $\mathfrak{H}^-$ in which the group Γ acts with weight $2-r$ and multiplier system v. The elements of ${}^-\mathcal{D}^\omega_{v,2-r}$ extend holomorphically into the upper half-plane, the elements of ${}^-\mathcal{D}^\infty_{v,2-r}$ extend to functions in $C^\infty(\mathfrak{H}^- \cup \mathbb{R})$. Both extensions also satisfy some condition at ∞. If $z_1, z_2 \in \mathfrak{H}$ then the values of c_f are in ${}^-\mathcal{D}^\omega_{v,2-r}$. If cusps are involved we need to use ${}^-\mathcal{D}^\infty_{v,2-r}$. The Knopp cocycles take values in similar modules ${}^+\mathcal{D}^\omega_{v^{-1},2-r} \subset {}^+\mathcal{D}^\infty_{v^{-1},2-r}$ of holomorphic functions on $\mathfrak{H}$. See Section 2 for a further discussion.

The association $f \mapsto c_f$ induces an injective linear map from the space $S_r(\Gamma, v)$ of holomorphic modular cusp forms of weight $r > 0$ and corresponding multiplier system v to the parabolic cohomology group $H^1_p(\Gamma; {}^-\mathcal{D}^\infty_{v,2-r})$, and $f \mapsto c_f^K$ induces a conjugate linear map $S_r(\Gamma; v) \to H^1_p(\Gamma; {}^+\mathcal{D}^\infty_{v^{-1},2-r})$.

The space of cusp forms $S_r(\Gamma, v)$ is contained in the much larger space $A_r(\Gamma, v)$ of all modular forms of weight r and multiplier system v (without any condition on the growth at the cusps). This space has infinite dimension for all real weights. The association $f \mapsto c_f$, respectively $f \mapsto c_f^K$, induces a linear map $A_r(\Gamma, v) \to H^1(\Gamma; {}^-\mathcal{D}^\omega_{v,2-r})$, respectively a conjugate linear map $A_r(\Gamma, v) \to H^1(\Gamma; {}^+\mathcal{D}^\omega_{v^{-1},2-r})$. If $r \in \mathbb{Z}_{\geq 2}$ the kernels of these maps have infinite dimension. For $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$ these maps are injective.

Let M_1 and M_2 be Γ-modules. As we will discuss in Section 3 in more detail, there is a cup product construction in parabolic cohomology that extends, under some conditions, to a bilinear map

$$\cup : H^1(\Gamma; M_1) \times H^1_p(\Gamma; M_2) \longrightarrow H^2_p(\Gamma; M_1 \otimes M_2)\,. \tag{1.4}$$

In parabolic cohomology there is a (not canonical) isomorphism $H^2_p(\Gamma; V) \cong V_\Gamma$. The space of coinvariants V_Γ has been defined in (1.1).

We apply this with $M_1 = {}^-\mathcal{D}^\omega_{v_1,2-r_1}$ and M_2 either the Γ-module ${}^-\mathcal{D}^\infty_{v_2,2-r_2}$ or the Γ-module ${}^+\mathcal{D}^\infty_{v_2^{-1},2-r_2}$. Composition with the maps from modular forms to Eichler or Knopp cocycles we get maps

$$\begin{aligned} C_{EE} &: A_{r_1}(\Gamma, v_1) \times S_{r_2}(\Gamma, v_2) \to \Bigl({}^-\mathcal{D}^\omega_{v_1,2-r_1} \otimes {}^-\mathcal{D}^\infty_{v_2,2-r_2} \Bigr)_\Gamma\,, \\ C_{EE}(f_1, f_2) &= [c_{f_1}] \cup [c_{f_2}]\,, \end{aligned} \tag{1.5}$$

which is bilinear, and

$$\begin{aligned} C_{EK} &: A_{r_1}(\Gamma, v_1) \times S_{r_2}(\Gamma, v_2) \to \left({}^-\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^+\mathcal{D}^{\infty}_{v_2^{-1},2-r_2} \right)_{\Gamma}, \\ C_{EK}(f_1, f_2) &= [c_{f_1}] \cup [c^K_{f_2}], \end{aligned} \tag{1.6}$$

which is linear in f_1 and conjugate linear in f_2.

The tensor products in which these maps take their values are large and have a complicated structure. Simplifications are possible.

First we simplify (1.5). Since elements of ${}^-\mathcal{D}^{\omega}_{v_1,2-r_1}$ and of ${}^-\mathcal{D}^{\infty}_{v_2,2-r_2}$ are holomorphic functions on $\mathfrak{H}^-$ their product is also holomorphic on $\mathfrak{H}^-$. Considering actions and the behavior at the boundary we check that multiplication of functions induces a linear intertwining operator of Γ-modules

$$J : {}^-\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^-\mathcal{D}^{\infty}_{v_2,2-r_2} \to {}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}. \tag{1.7}$$

We restrict our attention to the composite bilinear map

$$J \circ C_{EE} : A_{r_1}(\Gamma, v_1) \times S_{r_2}(\Gamma, v_2) \to \left({}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2} \right)_{\Gamma}. \tag{1.8}$$

The module ${}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$ still has infinite dimension, but we are able to do some explicit work.

In the case of an Eichler and a Knopp cocycle in (1.6), there is no such a product construction. The sole case that we can handle further is the case that the weights and multiplier systems are equal. There is a Γ-invariant bilinear form $[\cdot,\cdot]_{2-r}$ on ${}^-\mathcal{D}^{\infty}_{v,2-r} \times {}^+\mathcal{D}^{\infty}_{v^{-1},2-r}$, which induces a Γ-equivariant linear map $D : {}^-\mathcal{D}^{\infty}_{v,2-r} \times {}^+\mathcal{D}^{\infty}_{v^{-1},2-r} \to \mathbb{C}$. The image is the trivial Γ-module $\mathbb{C}$, and hence $\mathbb{C}_{\Gamma} = \mathbb{C}$. With use of $D \circ C_{EK}$ we can extend Haberland's cohomological description of the Petersson scalar product to all real weights. See Section 5.

In case of two Eichler cocycles the coinvariant $(J \circ C_{EE})(f_1, f_2)$ can be represented by many elements of ${}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$. In Section 3 we arrive at the following representative that we consider convenient:

$$\mathrm{cp}(f_1, f_2) = c_{f_1}(\rho - 1, \rho) \cdot c_{f_2}(i, \infty), \tag{1.9}$$

with $\rho = e^{\pi i/3}$ and $\rho - 1 = e^{2\pi i/3}$ the vertices in $\mathfrak{H}$ of the standard fundamental domain of the modular group.

The question is whether $\mathrm{cp}(f_1, f_2)$ represents the trivial coinvariant. In other words, whether there are a and b in the module $M = {}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$ such that $\mathrm{cp}(f_1, f_2) = a|(1-S) + b|(1-T)$ with the generators $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of Γ.

In the classical situation of weights $r_1, r_2 \in \mathbb{Z}_{\geq 2}$ it turns out that $\mathrm{cp}(f_1, f_2)$ represents the trivial coinvariant in ${}^-\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$. See Subsection 7.1. This holds for the modular group; for subgroups of finite index the situation is a bit more complicated. We will give numerical evidence that there are f_1 and f_2 (with real non-integral weights) for which $\mathrm{cp}(f_1, f_2)$ represents a non-trivial coinvariant. (This will be done in Subsection 8.4.)

In Proposition 7.2 we will show that there are modules $M_1 \supset {}^-\mathcal{D}^\infty_{v_1 v_2, 4-r_1-r_2}$ in which the coinvariant represented by $\mathrm{cp}(f_1, f_2)$ is trivial. This is not surprising, since every cocycle is a coboundary if one works with a suitably large module.

For the further investigation of the coinvariant represented by $\mathrm{cp}(f_1, f_2)$ we use the Γ-invariant bilinear form $[\cdot,\cdot]_r$ mentioned above. In Theorem 4.1 we will show that there is for real r and corresponding multiplier system a bilinear form $[\cdot,\cdot]_{2-r}$ on ${}^+\mathcal{D}^{-\infty}_{v^{-1},2-r} \times {}^-\mathcal{D}^\infty_{v,2-r}$. The Γ-module ${}^+\mathcal{D}^{-\infty}_{v^{-1},2-r} \supset {}^+\mathcal{D}^\infty_{v^{-1},2-r}$ consists of the holomorphic functions on $\mathfrak{H}$ with at most polynomial growth at the boundary.

Suppose that we have a Γ-invariant $\beta \in \left({}^+\mathcal{D}^{-\infty}_{v_1^{-1}v_2^{-1},4-r_1-r_2}\right)^\Gamma$. Then

$$\left[\beta, a|(1-T) + b|(1-S)\right]_{2-r} = 0$$

for all $a, b \in {}^-\mathcal{D}^\infty_{v,2-r}$. So if $\left[\beta, \mathrm{cp}(f_1, f_2)\right]_{2-r} \neq 0$ then we know that $\mathrm{cp}(f_1, f_2)$ represents a non-trivial coinvariant.

The nice fact is that we know $\left({}^+\mathcal{D}^{-\infty}_{v_1^{-1}v_2^{-1},4-r_1-r_2}\right)^\Gamma$. It is the space of entire modular forms $M_{4-r_1-r_2}(\Gamma, v_1^{-1}v_2^{-1})$. We put $v_3 = v_1^{-1}v_2^{-1}$ and $r_3 = 4 - r_1 - r_2$, and define a trilinear form

$$\mathbf{T}(f_1, f_2, f_3) = \left[f_3, \mathrm{cp}(f_1, f_2)\right]_{r_3} \tag{1.10}$$

on $A_{r_1}(\Gamma, v_1) \times S_{r_2}(v_2) \times M_{r_3}(\Gamma, v_3)$. If for given f_1, f_2 we can find f_3 such that $\mathbf{T}(f_1, f_2, f_3) \neq 0$, then we know that $\mathrm{cp}(f_1, f_2)$ represents a non-trivial coinvariant.

The drawback is that we do not know whether all Γ-invariant linear forms on ${}^-\mathcal{D}^\infty_{v_3^{-1},r_3}$ arise from $M_{r_3}(\Gamma, v_3)$. So it may happen that $\mathrm{cp}(f_1, f_2)$ represents a non-trivial coinvariant and still $\mathbf{T}(f_1, f_2, f_3)$ vanishes for all entire modular forms f_3.

To investigate the trilinear form $\mathbf{T}$ we can unravel all definitions, and arrive at a complicated description. However, the following theorem relates the rather abstract coinvariant $\mathrm{cp}(f_1, f_2)$ to a much more explicit integral, which gives a formulation suitable for numerical computations.

Theorem 1.1. *Let*

$$f_1 \in A_{r_1}(\Gamma, v_1), \quad f_2 \in S_{r_2}(\Gamma, v_2), \quad f_3 \in M_{r_3}(\Gamma, v_3) \tag{1.11}$$

with

$$r_1 + r_2 + r_3 = 4, \quad r_1 < 2, \quad 0 < r_2 < 2, \quad v_1 v_2 v_3 = 1. \tag{1.12}$$

Then

$$\mathbf{T}(f_1,f_2,f_3) = \frac{(-2i)^{r_3}\,\Gamma(r_3)}{\Gamma(2-r_1)\,\Gamma(2-r_2)} \int\limits_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int\limits_{\tau_2=i}^{\infty} f_2(\tau_2) \cdot \int\limits_{u=0}^{1} f_3\bigl(\tau_1+u(\tau_2-\tau_1)\bigr)\,u^{1-r_2}\,(1-u)^{1-r_1}\,du\,d\tau_2\,d\tau_1 \tag{1.13}$$

$$= \frac{(-2i)^{r_3}\,\Gamma(r_3)}{\Gamma(2-r_1)\,\Gamma(2-r_2)} \int\limits_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int\limits_{\tau_2=i}^{\infty} f_2(\tau_2) \cdot \int\limits_{z=\tau_1}^{\tau_2} f_3(z) \Bigl(\frac{z-\tau_1}{\tau_2-\tau_1}\Bigr)^{1-r_2} \Bigl(\frac{\tau_2-z}{\tau_2-\tau_1}\Bigr)^{1-r_1} \frac{dz}{\tau_2-\tau_1}\,d\tau_2\,d\tau_1\,. \tag{1.14}$$

The paths of integration for τ_1 and τ_2 are geodesic segments. In (1.13) the path of integration over u is the real interval $[0,1]$; in (1.14) the path of z is any path from τ_1 to τ_2 that does not cross $\ell_{\tau_1,\tau_2} \smallsetminus s_{\tau_1,\tau_2}$, where s_{τ_1,τ_2} is the geodesic segment from τ_1 to τ_2, and ℓ_{τ_1,τ_2} is the geodesic line through τ_1 and τ_2.

In (1.14) we may just take the geodesic segment from τ_1 to τ_2 as the path of integration for z.

We state several remarks regarding the above theorem:

Remark 1.2.

(1) This theorem relates the rather abstract coinvariant $\mathrm{cp}(f_1,f_2)$ to a much more explicit integral. In §8.4 we formulate in Proposition 8.5 the trilinear form $\mathbf{T}$ in terms of Fourier coefficients of the f_j. In that way we obtain a formulation suitable for numerical computations.
(2) The restriction on the weight r_2 implies that f_2 is a multiple of the $2r_2$-th power of the Dedekind eta-function. The restrictions $r_1 < 2$ and $r_2 < 2$ are due to our wish to have the inner integral as simple as we can make it.
(3) The triple integral is reminiscent of Manin's iterated integrals, considered for instance in [2]. We have found not direct connection.
(4) In the case of Haberland's formula a cohomological quantity is shown to be related to a quantity in the theory of cusp forms. It would be interesting to find a non-cohomological interpretation of the triple integral.

Overview of the paper. Section 2 recalls definitions over various concepts in some more details than in this introduction. Section 3 discusses the cup product.

In both cases that we consider (two Eichler cocycles, and the combination of an Eichler cocycle and a Knopp cocycle) we use the duality theorem Theorem 4.1 in Section 4. To prove the duality theorem we use principal series representations of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$. This requires further definitions and discussion, which we have put at the end of the paper in Section 9.

In Section 5 we extend to real positive weights Haberland's relation between the Petersson scalar product and the cup product.

Sections 6–8 study the coinvariant represented by $\mathrm{cp}(f_1, f_2)$. Section 6 shows that in the classical context this coinvariant is uninteresting. Section 7 discusses the definition of the trilinear form $\mathbf{T}$ and show explicitly that $\mathrm{cp}(f_1, f_2)$ represents the trivial coinvariant over the module of all holomorphic functions on $\mathfrak{H}^-$.

The main work to prove Theorem 1.1 is done in Section 8.2. In Section 8.4 we mention how we approach the study of $\mathbf{T}(f_1, f_2, f_3)$ numerically.

2. Modular forms and cohomology

Modular forms. By a holomorphic modular form f of *weight* $r \in \mathbb{R}$ with *multiplier system* v we mean a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ such that

$$f(\gamma z) = v(\gamma)\,(cz+d)^r f(z) \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z})\,. \tag{2.1}$$

We take $\arg(cz+d) \in (-\pi, \pi]$. The multiplier system is a function $v : \Gamma \to \mathbb{C}^*$ such that non-zero solutions of this equation are possible. For the modular group we use $v = v[p]$ with $p \equiv r \bmod 2$, where $v[p]$ is the multiplier system of the $(2p)$-th power of the Dedekind eta-function:

$$v[p]\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{\eta^{2p}(\gamma z)}{(cz+d)^p \eta^{2p}(z)}\,. \tag{2.2}$$

The transformation behavior of the Dedekind eta function has a multiplier system with values in the 24-th roots of unity. See, e.g., [13, Chap. IX]. Hence the multiplier system $v[p]$ depends only on $p \bmod 12$. Any multiplier system suitable for weight r satisfies

$$v\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = e^{-\pi i r}\,. \tag{2.3}$$

We denote by $A_r(v) = A_r(\Gamma, v)$ the space of all holomorphic f satisfying (2.1). We do not impose any further restriction; so $A_r(v)$ has infinite dimension. *Entire modular forms* satisfy the condition $f(z) = \mathrm{O}(1)$ as $\mathrm{Im}\, z \uparrow \infty$; we denote by $M_r(v) = M_r(\Gamma, v)$ the resulting subspace of $A_r(v)$. This space is finite dimensional and zero if $r < 0$. A further restriction is the condition of quick decay $f(z) = \mathrm{O}\big((\mathrm{Im}\, z)^{-a}\big)$ for all $a > 0$, as $\mathrm{Im}\, z \uparrow \infty$. It defines the subspace $S_r(v) = S_r(\Gamma, v) \subset M_r(v)$ of *cusp forms*, which is zero if $r \leq 0$.

Actions. For each weight r and corresponding multiplier system v there are right actions of $\mathrm{PSL}_2(\mathbb{Z}) = \Gamma/\{1, -1\}$; the action $|_{v,r}$ is on functions on the upper half-plane $\mathfrak{H}$, and $|^-_{v^{-1},r}$ on functions on the lower half-plane $\mathfrak{H}^-$ given for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by

$$\begin{aligned} f|_{v,r}\gamma(z) &= v(\gamma)^{-1}\,(cz+d)^{-r}\,f(\gamma z) && (z\in\mathfrak{H})\,,\\ f|^{-}_{v^{-1},r}\gamma(t) &= v(\gamma)\,(ct+d)^{-r}\,f(\gamma t) && (t\in\mathfrak{H}^-)\,. \end{aligned} \tag{2.4}$$

It turns out to be convenient to use the argument conventions $\arg(cz+d)\in(-\pi,\pi]$ for $z\in\mathfrak{H}$, and $\arg(ct+d)\in[-\pi,\pi)$ for $t\in\mathfrak{H}^-$. Hence we use $v(\gamma)^{-1}$ in the action on functions on $\mathfrak{H}$ and $v(\gamma)$ in the action on functions of $\mathfrak{H}^-$. These representations of Γ are trivial on the center $\{1,-1\}$ of Γ. So they are in fact representations of $\mathrm{PSL}_2(\mathbb{Z})\cong\Gamma/\{1,-1\}$.

The property (2.1) of modular forms is invariance under the action $|_{v,r}$ of Γ. See [3, §2.1] for more information.

Eichler cocycles. For $f\in A_r(v)$, $t\in\mathfrak{H}^-$, $z_1,z_2\in\mathfrak{H}$ we put

$$c_f(z_1,z_2;t)\;=\;\int_{z=z_1}^{z_2} f(z)\,(z-t)^{r-2}\,dz\,, \tag{2.5}$$

with $\arg(z-t)\in\big(-\pi/2,3\pi/2\big)$. This defines holomorphic functions on $\mathfrak{H}^-$ satisfying

$$\begin{aligned} c_f(z_1,z_3) &= c_f(z_1,z_2)+c_f(z_2,z_3)\,,\\ c_f(\gamma^{-1}z_1,\gamma^{-1}z_2) &= c_f(z_1,z_2)|^{-}_{v,2-r}\gamma && (\gamma\in\Gamma)\,. \end{aligned} \tag{2.6}$$

So this defines a homogeneous 1-cocycle, called an *Eichler cocycle*. It has values in the holomorphic functions on $\mathfrak{H}^-$ with the action $|^{-}_{v,2-r}$. The corresponding inhomogeneous 1-cocycle is $\gamma\mapsto c_f(\gamma^{-1}z_0,z_0)$; it depends on the choice of a base point $z_0\in\mathfrak{H}$. See for instance [4, §6.1] for a discussion of cohomology based on homogeneous cocycles.

The values of Eichler cocycles are holomorphic functions on $\mathfrak{H}^-$ satisfying further properties. They are in the Γ-module ${}^{-}\mathcal{D}^{\omega}_{v,2-r}$ of holomorphic functions h on $\mathfrak{H}^-$ that have a holomorphic continuation to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ and for which $t\mapsto(i-t)^{2-r}\,h(t)$ is holomorphic in $\frac{-1}{t}$ on a neighborhood of $t=0$ in $\mathbb{C}$. The association $f\mapsto c_f$ induces a linear map

$$A_r(v)\longrightarrow H^1(\Gamma;{}^{-}\mathcal{D}^{\omega}_{v,2-r}) \tag{2.7}$$

which is injective if $r\in\mathbb{R}\smallsetminus\mathbb{Z}_{\ge2}$ [3, Theorem A].

For a cusp form $f\in S_r(v)$ we can form $c_f(z_1,z_2;t)=\int_{z=z_1}^{z_2} f(z)\,(z-t)^{r-2}\,dz$, where z_1 and z_2 may be cusps. These cocycles take values in the larger module ${}^{-}\mathcal{D}^{\infty}_{v,2-r}$ of holomorphic functions h on $\mathfrak{H}^-$ for which h has an extension to $C^\infty(\mathfrak{H}^-\cup\mathbb{R})$ and for which $t\mapsto(i-t)^{2-r}\,h(t)$ is C^∞ in $\frac{-1}{t}$ on a neighborhood of 0 in $\mathfrak{H}^-\cup\mathbb{R}$. The association $f\mapsto c_f$ induces an injective linear map

$$S_r(v)\longrightarrow H^1(\Gamma;{}^{-}\mathcal{D}^{\infty}_{v,2-r})\,. \tag{2.8}$$

This map is bijective if $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\geq 2}$, by Theorem B in [3]. For $r \in \mathbb{Z}_{\geq 2}$ the cocycles take values in the $(r-2)$-dimensional submodule ${}^{-}\mathcal{D}^{\mathrm{pol}}_{v,2-r} \subset {}^{-}\mathcal{D}^{\omega}_{v,2-r}$ of polynomial functions on $\mathbb{C}$ of degree at most $r-2$.

Knopp cocycles. The conjugate linear map ι given by $\iota f(z) = \overline{f(\bar{z})}$ interchanges functions on $\mathfrak{H}$ and $\mathfrak{H}^-$ and intertwines the actions $|_{v,r}$ and $|^{-}_{v^{-1},r}$ (for real r and corresponding multiplier system). This map can be used to define ${}^{+}\mathcal{D}^{\omega}_{v^{-1},2-r} \subset {}^{+}\mathcal{D}^{\infty}_{v^{-1},2-r}$ in terms of the modules ${}^{-}\mathcal{D}^{\omega}_{v,2-r} \subset {}^{-}\mathcal{D}^{\infty}_{v,2-r}$ with analogous descriptions. If $r \in \mathbb{Z}_{\geq 2}$ we have also ${}^{+}\mathcal{D}^{\mathrm{pol}}_{v^{-1},2-r} \subset {}^{+}\mathcal{D}^{\omega}_{v^{-1},2-r}$ consisting of the polynomial functions of degree at most $r-2$.

The Knopp cocycle

$$c^K_f(z_1, z_2) \;=\; \iota c_f(z_1, z_2) \tag{2.9}$$

induces conjugate linear maps

$$\begin{aligned} A_r(v) &\to H^1(\Gamma; {}^{+}\mathcal{D}^{\omega}_{v^{-1},2-r})\,, \\ S_r(v) &\to H^1_p(\Gamma; {}^{+}\mathcal{D}^{\infty}_{v^{-1},2-r})\,. \end{aligned} \tag{2.10}$$

Actually, Knopp [10] considers the resulting map with values in the larger module ${}^{+}\mathcal{D}^{-\infty}_{v^{-1},2-r}$ of holomorphic functions on $\mathfrak{H}$ with at most polynomial growth:

$$h(z) \;=\; \mathrm{O}\big((\mathrm{Im}\, z)^{-B}\big) + \mathrm{O}\big(|z|^{\sigma}\big) \qquad z \in \mathfrak{H} \text{ for some } B > 0 \text{ and } \sigma > 0\,. \tag{2.11}$$

Knopp and Mawi [11] showed that

$$S_r(v) \to H^1_p(\Gamma; {}^{+}\mathcal{D}^{-\infty}_{v^{-1},2-r}) \tag{2.12}$$

is bijective for all $r \in \mathbb{R}$.

3. Cup product of modular cocycles

We recall the cup product construction in parabolic cohomology. Applied to two 1-cocycles with values in possibly different modules it gives a coinvariant in the tensor product of the two modules.

Applied to pairs of cocycles attached to modular forms this construction yields coinvariants in a module that may be large and complicated to investigate. We discuss various choices in which one obtains coinvariants in simpler modules.

3.1. Cup product

We base our discussion of the parabolic cup product on the discussion in [4, §19], where more details can be found.

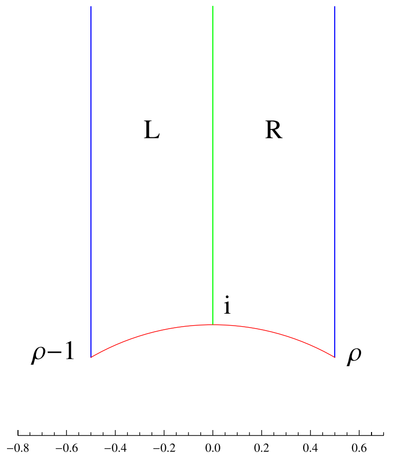


Fig. 1. The standard fundamental domain of the modular group Γ as the union of two triangles. The tessellation of the upper half-plane formed by the Γ-translates of these triangles is the basis of the cup product construction that we use.

We use the tessellation of the upper half plane given by the Γ-translates of the triangles L and R in Fig. 1. This tessellation determines a resolution with which we can compute parabolic cohomology, as discussed in [4, §12.1]. It is based on the spaces $\mathbb{C}[X_i]$, where X_0 is the set of vertices of the tessellation, X_1 a set of oriented edges, and X_2 the set of faces.

In [4, §19.2] the construction of the cup product is based on a diagonal approximation. In the context of general cofinite groups a diagonal approximation as in Table 19.1 of [4] is used. Here we can proceed in a simpler way, and determine for $i = 0, 1, 2$ the diagonal approximation

$$\delta_i : \mathbb{C}[X_i] \to \sum_{j=0}^{i} \mathbb{C}[X_j] \otimes \mathbb{C}[X_{i-j}] \tag{3.1}$$

as a $\mathbb{C}[\Gamma]$-linear map. It has to satisfy the compatibility relations $\partial_j \delta_j = \delta_{j-1}\partial_j$ with the boundary map ∂_j. In dimension 0 we determine it by

$$\delta_0(P) = (P) \otimes (P) \qquad (P \in X_0)\,. \tag{3.2}$$

In dimension 1 we take on the basis edges $e_{\rho,\infty}$, $e_{\rho-1,i}$ and $e_{i,\infty}$

$$\delta_1 e_{P,Q} \;=\; (P) \otimes e_{P,Q} + e_{P,Q} \otimes Q\,, \tag{3.3}$$

use $\delta_1 e_{Q,P} = -\delta_1 e_{P,Q}$, and extend δ_1 $\mathbb{C}[\Gamma]$-linearly. Guided by the diagonal embedding in Table 19.1 in [4] we take

$$\begin{aligned} \delta_2 R \;&=\; (\rho) \otimes R + e_{\rho,i} \otimes e_{i,\infty} + R \otimes (\infty)\,, \\ \delta_2 L \;&=\; (\rho-1) \otimes L - e_{\rho-1,i} \otimes e_{i,\infty} + L \otimes (\infty)\,, \end{aligned} \tag{3.4}$$

and extend this $\mathbb{C}[\Gamma]$-linearly. It takes some work to check the compatibility relation $\partial_2\delta_2 = \delta_1\partial_2$ with the boundary maps.

The cup product of cochains $c_1 \in C^p(\mathbb{C}[X.];V)$ and $c_2 \in C^q(\mathbb{C}[X.];W)$ is defined by

$$(c_1 \cup c_2)(x) \;=\; -(c_1 \otimes c_2)\bigl(\delta_{p+q}(x)\bigr) \qquad \text{for } x \in \mathbb{C}[X_{p+q}]\,. \tag{3.5}$$

The tensor $c_1 \otimes c_2$ sees only the component $\mathbb{C}[X_p] \otimes \mathbb{C}[X_q]$ of

$$(\mathbb{C}[X.] \otimes \mathbb{C}[X.])_{p+q} \;=\; \bigoplus_j \mathbb{C}[X_j] \otimes \mathbb{C}[X_{p+q-j}]\,.$$

This defines $c_1 \cup c_2 \in C^{p+q}(\mathbb{C}[X.]; V \otimes W)$. If both c_1 and c_2 are cocycles, then $c_1 \cup c_2$ is a cocycle. If one is a coboundary and the other a cocycle then the cup product is a coboundary.

In particular for $p = q = 1$ we are interested in

$$\begin{aligned}(c_1 \cup c_2)(L+R) \;&=\; -(c_1 \otimes c_2)\,(e_{\rho,i} \otimes e_{i,\infty} - e_{\rho-1,i} \otimes e_{i,\infty})\\ &=\; -c_1(e_{\rho,i}) \otimes c_2(e_{i,\infty}) + c_1(e_{\rho-1,i}) \otimes c_2(e_{i,\infty})\\ &=\; c_1(e_{\rho-1,\rho}) \otimes c_2(e_{i,\infty})\,.\end{aligned} \tag{3.6}$$

In principle this works when c_1 and c_2 are both parabolic cocycles. However, c_1 is evaluated only on the interior edge $e_{\rho-1,\rho}$, which makes sense for a cocycle that is not parabolic. (See §11.1 in [4].) Now [4, equation (19.7)] states that this results in a well-defined linear map

$$\cup : H^1(\Gamma;V) \otimes H^1_p(\Gamma;W) \to H^2_p(\Gamma; V \otimes W)\,. \tag{3.7}$$

This is not true in the generality in which [4] states it. We need the condition that the space of invariants W^π is zero for all parabolic $\pi \in \Gamma$. For the modular group it suffices that $W^T = \{0\}$.

3.2. Application to modular forms

We apply this to the cocycles attached to two modular forms, $f_1 \in A_{r_1}(v_1)$, $f_2 \in S_{r_2}(v_2)$, with real weights r_j and corresponding multiplier systems, with $r_2 > 0$. We take the cocycle $c_1 = c_{f_1}$ representing a cohomology class in $H^1(\Gamma; {}^-\mathcal{D}^\omega_{v_1,2-r_1})$, and either the parabolic cocycle $c_2 = c_{f_2}$ representing a class in $H^1_p(\Gamma; {}^-\mathcal{D}^\infty_{v_2,2-r_2})$ or $c_2 = c^K(f_2)$ representing a class in $H^1_p(\Gamma; {}^+\mathcal{D}^\infty_{v_2^{-1},2-r_2})$. We need to impose the condition $r_2 \notin \mathbb{Z}_{\geq 2}$ to have $W^T = \{0\}$; see [3, Lemma 12.1]. If $r_2 \in \mathbb{Z}_{\geq 2}$ we assume that $f_1 \in S_{r_1}(v_1)$ as well, to have a well-defined cup product.

In this way the cup product gives coinvariants in the Γ-modules

$${}^-\mathcal{D}^\omega_{v_1,2-r_1} \otimes {}^-\mathcal{D}^\infty_{v_2,2-r_2} \text{ and } {}^-\mathcal{D}^\omega_{v_1,2-r_1} \otimes {}^+\mathcal{D}^\infty_{v_2^{-1},2-r_2}\,,$$

or if $r_2 \in \mathbb{Z}_{\geq 2}$ in

$$^{-}\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^{-}\mathcal{D}^{\mathrm{pol}}_{v_2,2-r_2} \text{ and } {}^{-}\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^{+}\mathcal{D}^{\mathrm{pol}}_{v_2^{-1},2-r_2}\,.$$

These tensor products are large complicated modules. It is hard to determine what it means to have a representative of a coinvariant in such a module. There is hope to get more information if there is an intertwining operator from the tensor product to a simpler module.

We restrict our attention to a number of situations, in which an intertwining operator to a simpler module can be found.

EE. $J : {}^{-}\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^{-}\mathcal{D}^{\infty}_{v_2,2-r_2} \to {}^{-}\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$ by multiplication of functions, under the assumption that $r_2 \notin \mathbb{Z}_{\geq 2}$. The image of the cup product $c_{f_1} \cup c_{f_2}$ is a coinvariant in ${}^{-}\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$ represented by the product

$$\mathrm{cp}(f_1, f_2) \,=\, c_{f_1}(\rho - 1, \rho) \cdot c_{f_2}(i, \infty) \tag{3.8}$$

for $f_1 \in A_{r_1}(v_1)$ and $f_2 \in S_{r_2}(v_2)$.

cEE. $J : {}^{-}\mathcal{D}^{\mathrm{pol}}_{v_1,2-r_1} \otimes {}^{-}\mathcal{D}^{\mathrm{pol}}_{v_2,2-r_2} \to {}^{-}\mathcal{D}^{\mathrm{pol}}_{v_1 v_2,4-r_1-r_2}$ by multiplication of polynomial functions. For $f_j \in S_{r_j}(v_2)$ with $r_1, r_2 \in \mathbb{Z}_{\geq 2}$ we get the coinvariant in ${}^{-}\mathcal{D}^{\mathrm{pol}}_{v_1 v_2,4-r_1-r_2}$ represented by the product $\mathrm{cp}(f_1, f_2)$ in (3.8).

cEK. $J : {}^{+}\mathcal{D}^{\infty}_{v^{-1},2-r} \otimes {}^{-}\mathcal{D}^{\infty}_{v,2-r} \to \mathbb{C}$ by a duality to be discussed in Section 4, for $r = r_1 = r_2 > 0$, $v = v_1 = v_2$. To two cusp forms f_1, f_2 in the same space $S_r(v)$ is associated a number $\left[c^K_{f_2}(i, \infty), c_{f_1}(\rho - 1, \rho)\right]_{2-r}$.

We discuss case cEE in Sections 6, and show in Section 5 how case cEK leads to a generalization of the relation between cup product and Petersson scalar product.

Case EE is the subject of Sections 7 and 8, where we use the duality theorem in Section 4 to go from cup product to a fourfold and a triple integral.

4. Duality theorem

In this section we formulate the duality theorem, and show that a space of entire modular forms gives the continuous Γ-invariant linear forms on ${}^{-}\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$ that we will use for further study of the cup product.

4.1. Γ-modules of holomorphic functions

For a real weight r and a corresponding multiplier system we discussed in Section 2 various spaces of holomorphic functions on which Γ acts, related by the conjugate linear transformation ι given by $(\iota f)(z) = \overline{f(\bar{z})}$.

$$\begin{tikzcd} {}^-\mathcal{D}^{\omega}_{v^{-1},r} \arrow[r, hook] \arrow[d, leftrightarrow, "\iota"] & {}^-\mathcal{D}^{\infty}_{v^{-1},r} \arrow[r, hook] \arrow[d, leftrightarrow, "\iota"] & {}^-\mathcal{D}^{-\infty}_{v^{-1},r} \arrow[r, hook] \arrow[d, leftrightarrow, "\iota"] & {}^-\mathcal{D}^{-\omega}_{v^{-1},r} \arrow[d, leftrightarrow, "\iota"] \\ {}^+\mathcal{D}^{\omega}_{v,r} \arrow[r, hook] & {}^+\mathcal{D}^{\infty}_{v,r} \arrow[r, hook] & {}^+\mathcal{D}^{-\infty}_{v,r} \arrow[r, hook] & {}^+\mathcal{D}^{-\omega}_{v,r} \end{tikzcd} \tag{4.1}$$

The modules in the top row consist of holomorphic functions on $\mathfrak{H}^-$, with the action $|^-_{v^{-1},r}$ of Γ, the modules in the bottom row consist of holomorphic function on $\mathfrak{H}$ with the action $|_{v,r}$. If $r \in \mathbb{Z}_{\leq 0}$, there are the $(|r|+1)$-dimensional submodules ${}^-\mathcal{D}^{\mathrm{pol}}_{v^{-1},r} \subset {}^-\mathcal{D}^{\omega}_{v^{-1},r}$ and ${}^+\mathcal{D}^{\mathrm{pol}}_{v,r} \subset {}^+\mathcal{D}^{\omega}_{v,r}$, also related by ι. These submodules consist of the polynomial functions of degree at most $|r|$.

We put

$$\begin{aligned} f^-(t) &= (i-t)^r\, f(t) && \text{for functions } f \text{ on } \mathfrak{H}^-\,, \\ f^+(z) &= (-i-z)^r\, f(z) && \text{for functions } f \text{ on } \mathfrak{H}\,. \end{aligned} \tag{4.2}$$

With this notation we have the following characterizations:

$$\boxed{\begin{array}{ll} {}^\pm\mathcal{D}^{-\omega}_{v^{\pm1},r} & f^\pm \text{ is holomorphic on } \mathfrak{H}^\pm \\ {}^\pm\mathcal{D}^{-\infty}_{v^{\pm1},r} & \exists_{B>0}\ f^\pm(z) = \mathrm{O}\big(|\mathrm{Im}\, z|^{-B}\big) + \mathrm{O}\big(|z|^B\big) \text{ on } \mathfrak{H}^\pm \\ {}^\pm\mathcal{D}^{\infty}_{v^{\pm1},r} & \exists \text{ extension } f^\pm \in C^\infty\big(\mathfrak{H}^\pm \cup \mathbb{P}^1_{\mathbb{R}}\big) \\ {}^\pm\mathcal{D}^{\omega}_{v^{\pm1},r} & \exists \text{ holomorphic extension of } f^\pm \text{ to } U \supset \mathfrak{H}^\pm \cup \mathbb{P}^1_{\mathbb{R}} \\ {}^\pm\mathcal{D}^{\mathrm{pol}}_{v^{\pm1},r} & f^\pm \text{ polynomial function of } \frac{t\mp i}{t\pm i}, \text{ degree at most } |r| \end{array}} \tag{4.3}$$

The actions on ${}^-\mathcal{D}^{\infty}_{v,2-r}$ and ${}^+\mathcal{D}^{\infty}_{v^{-2},2-r}$ are continuous for the topology given by the supremum norms of all derivatives on $\mathbb{P}^1_{\mathbb{R}}$ of the extension of $f^\pm$ to $\mathfrak{H}^\pm \cup \mathbb{P}^1_{\mathbb{R}}$. The derivatives are taken with respect to $\vartheta = -\cot t$ for $t \in \mathbb{P}^1_{\mathbb{R}}$ with $\vartheta \in \mathbb{R} \,/\, \pi\mathbb{Z}$.

The representation spaces ${}^+\mathcal{D}^{\omega}_{v,r}$ and ${}^-\mathcal{D}^{\omega}_{v^{-1},r}$ are the direct limits of spaces of bounded holomorphic functions $f^\pm$ on neighborhoods U_1 of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. The natural topology is obtained by providing these spaces with the supremum norm on U_1.

4.2. Duality

To formulate the duality theorem we use linear operators σ_r:

$$\begin{aligned} \sigma_r &: {}^-\mathcal{D}^{\omega}_{v^{-1},r} \to {}^-\mathcal{D}^{\omega}_{v^{-1},r} && \text{if } r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\leq 0}\,, \\ \sigma_r &: {}^-\mathcal{D}^{\mathrm{pol}}_{v^{-1},r} \to {}^-\mathcal{D}^{\omega}_{v^{-1},r} && \text{if } r \in \mathbb{Z}_{\leq 0}\,. \end{aligned} \tag{4.4}$$

For $r \in \mathbb{R} \smallsetminus \mathbb{Z}_{\leq 0}$ we describe σ_r in terms of the functions $f^- : z \mapsto f(z)\,(i-z)^r$ defined in (4.2).

$$\sigma_r f^-(z) := \frac{1}{\pi}\frac{z+i}{z-i} \int_{\tau\in C} f^-(\tau)\ {}_2F_1\Big(1,1;r;\frac{(\tau-i)(z+i)}{(\tau+i)(z-i)}\Big)\,\frac{d\tau}{\tau^2+1}\,, \tag{4.5}$$

where the hypergeometric function is given on the unit disk by ${}_2F_1(a, b; c; z) := \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$, and $(a)_n$ is the Pochhammer symbol. This function has an analytic extension to $\mathbb{C} \smallsetminus [1, \infty)$. The positively oriented curve C in $\mathfrak{H}$ encircles i and is in the domain of the extension f^-. It should be homotopic to the boundary $\mathbb{P}^1_{\mathbb{R}}$ of $\mathfrak{H}$ in the domain of f^-. The point z is outside the curve C. By definition of ${}^-\mathcal{D}^{\omega}_{v^{-1},r}$ the function is holomorphic on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus K$ for some compact set $K \subset \mathfrak{H}$. Adapting C to K we get a holomorphic function $\sigma_r f^-$ on $\mathbb{P}^1_{\mathbb{C}} \smallsetminus K$.

If we would insist to formulate the relation $g^- = \sigma_r f^-$ in terms of f and g in ${}^-\mathcal{D}^{\omega}_{v^{-1},r}$ the formula would be more complicated:

$$g(z) = (z-i)^{-r} \frac{1}{\pi} \frac{z+i}{z-i} \int\limits_{\tau \in C} f(\tau)\,(i-\tau)^r\; {}_2F_1\Big(1, 1; r; \frac{(\tau-i)(z+i)}{(\tau+i)(z-i)}\Big) \frac{d\tau}{\tau^2+1}\,.$$

To see that this integral makes sense we would need to remember that $\tau \mapsto f(\tau)\,(i-\tau)^r$ extends holomorphically from $\mathfrak{H}^-$ across $\mathbb{P}^1_{\mathbb{R}}$ into a region in $\mathfrak{H}$.

For $f_n^-(t) = \left(\frac{t-i}{t+i}\right)^n$, with $n \leq 0$, we will see in Proposition 9.5 that

$$\sigma_r f_n^- = \frac{|n|!}{(r)_{|n|}} f_{n-1}^-\,. \tag{4.6}$$

If $r \in \mathbb{Z}_{\leq 0}$ then $(r)_{|n|} \neq 0$ for $r \leq n \leq 0$. We use (4.6) to define σ_r for $r \in \mathbb{Z}_{\leq 0}$.

At this point the operators σ_r seem arbitrary. In §9.2 we will see that they arise naturally.

For $h \in {}^+\mathcal{D}^{-\omega}_{v,r}$ and $f \in {}^-\mathcal{D}^{\omega}_{v^{-1},r}$ we consider the integral

$$[h, f]_r = \frac{1}{\pi} \int\limits_{z \in C} h^+(z)\,\sigma_r f^-(z)\,\frac{dz}{(z+i)^2}\,. \tag{4.7}$$

At first sight this integral seems undefined since h and f are holomorphic on disjoint domains, $\mathfrak{H}$ and $\mathfrak{H}^-$ respectively. However, since $f \in {}^-\mathcal{D}^{\omega}_{v^{-1},r}$ the function f^- is holomorphic on $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}} \cup U$ where U is a simply connected, connected neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. Then also $\sigma_r f^-$, for $r \notin \mathbb{Z}_{\leq 0}$, is holomorphic at least on U. The function h^+ is holomorphic on $\mathfrak{H}$, and the integral over a positively oriented curve C in $\mathfrak{H} \cap U$ encircling i makes sense. The precise choice of C does not influence the value of the integral. If $r \in \mathbb{Z}_{\leq 0}$ we need to require that $f \in {}^-\mathcal{D}^{\mathrm{pol}}_{v^{-1},r}$ in order to have a well defined lift $\sigma_r f^-$. In both cases the value of the integral does not change if we add to $\sigma_r f^-$ any holomorphic function on $\mathfrak{H}$.

Theorem 4.1 *(Duality theorem).*

i) *Let $r \notin \mathbb{Z}_{\leq 0}$. The integral in (4.7) defines a non-degenerate Γ-invariant bilinear form on ${}^+\mathcal{D}^{-\omega}_{v,r} \times {}^-\mathcal{D}^{\omega}_{v^{-1},r}$.*

ii) *Let $r \notin \mathbb{Z}_{\leq 0}$. If $h \in {}^{+}\mathcal{D}^{-\infty}_{v,r}$ then the linear form $f \mapsto [h,f]_r$ on ${}^{-}\mathcal{D}^{\omega}_{v^{-1},r}$ extends continuously to ${}^{-}\mathcal{D}^{\infty}_{v^{-1},r}$ for the natural topology on ${}^{-}\mathcal{D}^{\infty}_{v^{-1},r}$ defined in §4.1.*

iii) *Let $r \in \mathbb{Z}_{\leq 0}$. The integral in (4.7) defines a non-degenerate Γ-invariant bilinear form on ${}^{+}\mathcal{D}^{-\omega}_{v,r} \times {}^{-}\mathcal{D}^{\mathrm{pol}}_{v^{-1},r}$. Its restriction to ${}^{+}\mathcal{D}^{\mathrm{pol}}_{v,r} \times {}^{-}\mathcal{D}^{\mathrm{pol}}_{v^{-1},r}$ is non-degenerate.*

iv) *In terms of the expansions*

$$h^{+}(z) = \sum_{n\geq 0} c_n \left(\frac{z-i}{z+i}\right)^{n}, \qquad f^{-}(z) = \sum_{m\leq 0} d_m \left(\frac{z-i}{z+i}\right)^{m},$$

with $d_m = 0$ for $m < |r|$ if $r \in \mathbb{Z}_{\leq 0}$ the duality is given by

$$\big[h,f\big]_r = \sum_{n\geq 0} \frac{n!}{(r)_n}\, c_n\, d_n\,. \tag{4.8}$$

We note that i) and iii) give the bilinear form in terms of a well defined contour integral. The bilinear form in ii) is given by a, less explicit, limit process.

We will give the proof of the duality theorem in §9.3, in the context of principal series representations of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

Use of the duality theorem. At the end of Subsection 3.2 we mentioned three cases in which we want to further investigate the cup product of modular cocycles. In two of them we will use the duality theorem.

In case cEK we associate to $f_1, f_2 \in S_r(v)$ the cup product $c_{f_1}(\rho-1,\rho) \otimes c_{f_2}(i,\infty)$ in ${}^{-}\mathcal{D}^{\infty}_{v,2-r} \otimes {}^{+}\mathcal{D}^{\infty}_{v^{-1},2-r}$. We get a Γ-equivariant linear map from the tensor product to $\mathbb{C}$ generated by $v \otimes w \mapsto [w,v]_{2-r}$ for $r \notin \mathbb{Z}_{\geq 2}$. If $r \in \mathbb{Z}_{\geq 2}$ we use part iii) of the duality theorem. See Section 5.

In the case EE we have $c_{f_1}(\rho-1,\rho) \otimes c_{f_2}(i,\infty) \in {}^{-}\mathcal{D}^{\omega}_{v_1,2-r_1} \otimes {}^{-}\mathcal{D}^{\infty}_{v_2,2-r_2}$ representing the cup product $c_{f_1} \cup c_{f_2}$. The linear map induced by multiplication $v \otimes w \mapsto v\,w$ is an intertwining operator from the tensor product to ${}^{-}\mathcal{D}^{\infty}_{v_1 v_2,4-r_1-r_2}$. If we have Γ-invariant elements $u \in {}^{+}\mathcal{D}^{-\infty}_{v_1^{-1}v_2^{-1},4-r_1-r_2}$, we use the linear form $\varphi \mapsto \big[u,\varphi\big]_{2-r}$ resulting from part iii) of the duality theorem to test whether the coinvariant represented by $\mathrm{cp}(f_1,f_2)$ is non-trivial. See section 7.

5. Cup product and Petersson scalar product

We consider now case EK in Subsection 3.2. After Theorem 4.1 we explained that to two cusp forms $f_1, f_2 \in S_r(v)$ we associate the number

$$[c^{K}_{f_2}(i,\infty), c_{f_1}(\rho-1,\rho)]_{2-r}$$

in the trivial Γ-module $\mathbb{C} \cong \mathbb{C}_\Gamma$. This number turns out to be a multiple of the Petersson scalar product. We extend Haberland's relation in [9] to real weights.

Theorem 5.1. *Let $r > 0$ and let v be a corresponding multiplier system. For all $f_1, f_2 \in S_r(v)$*

$$[c^K_{f_2}(i,\infty), c_{f_1}(\rho-1,\rho)]_{2-r} = -2i\,\big(f_1,f_2\big)_r := -2i \int\limits_{\Gamma\backslash\mathfrak{H}} f_1(\tau)\,\overline{f_2(\tau)}\,y^r\,\frac{dx\,dy}{y^2}\,.$$

Remarks. Haberland [9, §7] gave this relation for even positive weights. With the duality theorem Theorem 4.1 available we can essentially follow Haberland's approach. As far as we know the case of positive weights has not been considered in generality. We note that Neururer's paper [15], Sections 2 and 3, and Cohen's paper [7], Section 3, come close to what we do here.

Proof. We start with the cup product and show that it is a multiple of the Petersson scalar product.

Truncation. The function $c^K_{f_2}$ is in ${}^+\mathcal{D}^\omega_{v^{-1},2-r}$, since the integration is over a compact path in $\mathfrak{H}$. However $c_{f_1} \in {}^-\mathcal{D}^\infty_{v,2-r}$. We use as an approximation the element $c_{f_1}(i, ia) \in {}^-\mathcal{D}^\omega_{v,2-r}$.

Lemma 5.2. *Let $f \in S_r(v)$. Then*

$$\lim_{a\uparrow\infty} c_f(i, ia) = c_f(i,\infty)$$

in the natural topology on ${}^-\mathcal{D}^\infty_{v,2-r}$.

Proof. The topology on ${}^-\mathcal{D}^\infty_{v,2-r}$ is given by the norms associating to φ on $\mathbb{P}^1_{\mathbb{R}}$ the supremum on $\mathbb{P}^1_{\mathbb{R}}$ of the derivatives $\left(\frac{1}{1+t^2}\partial_t\right)^N \varphi^-(t)$. We put $\varphi(t) = c_f(i,\infty;t)$.

$$\varphi_a^-(t) = i\int\limits_{y=1}^{a} f(iy)\left(\frac{iy-t}{i-t}\right)^{r_2-2} dy = i\int\limits_{y=1}^{a} f(iy)\left(\frac{iy/t-1}{i/t-1}\right)^{r_2-2} dy\,,$$

$$\varphi^-(t) = i\int\limits_{y=1}^{\infty} f(iy)\left(\frac{iy-t}{i-t}\right)^{r_2-2} dy = i\int\limits_{y=1}^{\infty} f(iy)\left(\frac{iy/t-1}{i/t-1}\right)^{r_2-2} dy\,.$$

The integrand has the exponentially decreasing factor $f(iy)$ and another factor that is $O(y^{r_2-2})$, uniform in $t \in \mathbb{P}^1_{\mathbb{R}}$. So the integral φ_a^- converges to φ^- in the supremum norm on $\mathbb{P}^1_{\mathbb{R}}$.

Applying the differential operator $\frac{1}{1+t^2}\partial_t$ a number of times gives an integral with a more complicated expression. We do not need to determine these derivatives explicitly. We note that they involve powers of $\frac{1}{1+t^2}$, t, and $\frac{iy-t}{i-t}$, which allows us to handle the derivatives of φ_a^- and φ^- in an analogous way. Thus we conclude that $\varphi_a^- \to \varphi^-$ as $a \uparrow \infty$ in the topology of ${}^-\mathcal{D}^\infty_{v_2,2-r_2}$. $\square$

Lemma 5.3. *For f_1, f_2 as in the theorem*

$$\bigl[c^K_{f_2}(i,\infty), c_{f_1}(\rho-1,\rho)\bigr]_{2-r} \;=\; \lim_{a\uparrow\infty} \bigl[c^K_{f_2}(i,ia), c_{f_1}(\rho-1,\rho)\bigr]_{2-r}\,. \tag{5.1}$$

Proof. Part ii) of Theorem 4.1 gives continuity in the second argument. As long as both arguments are in ${}^{\pm}\mathcal{D}^{\infty}_{v,2-r}$ the transition to $[\iota f, \iota g]_r$ in the theorem gives continuity in the first argument as well. To see this we use part iv) of the duality theorem. □

Rearranging the integrals. We proceed with $\bigl[c^K_{f_2}(i,ia), c_{f_1}(\rho-1,\rho)\bigr]_{2-r}$. The three integrals involved in it all run over compact sets, and we can order them as it pleases us. We take

$$\begin{aligned}&\bigl[c^K_{f_2}(i,ia), c_{f_1}(\rho-1,\rho)\bigr]_{2-r}\\ &\qquad = \int_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int_{\tau_2=i}^{ia} \overline{f_2(\tau_2)}\,\bigl[p_{\tau_2}, q_{\tau_1}\bigr]_{2-r}\, d\tau_1\, d\bar\tau_2\,, \end{aligned}\tag{5.2}$$

$$p_{\tau_2}(z) \;=\; \bigl(\bar\tau_2 - z\bigr)^{r-2}, \qquad q_{\tau_1}(z) \;=\; \bigl(\tau_1 - z\bigr)^{r-2}\,.$$

The inner integral. We note that $p_{\tau_2}(z)$ is holomorphic in $z\in\mathfrak{H}$, and that $z\mapsto(\tau_1-z)^{r-2}$ is holomorphic in $z\in\mathfrak{H}^-$, with an extension $q^-_{\tau_1}$ to a neighborhood of $\mathfrak{H}^-\cup\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. So $[p_{\tau_2}, q_{\tau_1}]_{2-r}$ as in (i) and (iii) of the duality theorem Theorem 4.1 is well-defined, and

$$\bigl[p_{\tau_2}, q_{\tau_1}\bigr]_{2-r} \;=\; \frac{1}{\pi}\int_{z\in C} p^+_{\tau_2}(z)\,\sigma_{2-r} q^-_{\tau_1}(z)\,\frac{dz}{(z+i)^2}\,. \tag{5.3}$$

The closed contour C in $\mathfrak{H}$ encircles the path from τ_1 to i. We have with (4.2):

$$p^+_{\tau_2}(z) \;=\; \Bigl(\frac{\bar\tau_2 - z}{-i-z}\Bigr)^{r-2}, \qquad q^-_{\tau_1}(z) \;=\; \Bigl(\frac{\tau_1 - z}{i-z}\Bigr)^{r-2}\,. \tag{5.4}$$

It is convenient to proceed in disk coordinates $w=\frac{z-i}{z+i}$, $z=i\frac{1+w}{1-w}$, $u_1=\frac{\tau_1-i}{\tau_1-i}$. With $|u_1|<|w|<1$ we find:

$$\begin{aligned} q^-_{\tau_1}(z) &= \Bigl(\frac{u_1-w}{-w(1-u_1)}\Bigr)^{r-2} \;=\; (1-u_1)^{2-r}\bigl(1-u_1/w\bigr)^{r-2}\\ &= (1-u_1)^{2-r}\sum_m \binom{r-2}{m}(-1)^m {u_1}^m\, w^{-m}\,, \end{aligned}\tag{5.5}$$

$$\begin{aligned} \sigma_{2-r} q^-_{\tau_1}(z) &= (1-u_1)^{2-r}\sum_m \binom{r-2}{m}(-1)^m {u_1}^m\,\frac{m!}{(2-r)_m}\,w^{-m-1}\\ &= (1-u_1)^{2-r}\sum_m {u_1}^m\, w^{-m-1}\,. \end{aligned}\tag{5.6}$$

We used (4.6). If $r \notin \mathbb{Z}_{\geq 2}$ the sum runs over $m \geq 0$, and over $0 \leq m \leq r-2$ if $r \in \mathbb{Z}_{\geq 2}$.

We take $u_2 = \frac{\tau_2 - i}{\tau_2 + i}$. Then $\bar\tau_2 = -i\frac{\bar u_2 + 1}{\bar u_2 - 1}$ and

$$p^+_{\tau_2}(z) = \Bigl(\frac{1 - w\bar u_2}{1 - \bar u_2}\Bigr)^{r-2} = (1-\bar u_2)^{2-r} \sum_n \binom{r-2}{n} (-1)^n \bar u_2^n w^n . \tag{5.7}$$

Here $0 \leq n \leq r-2$ if $r \in \mathbb{Z}_{\geq 2}$, and $n \geq 0$ otherwise. With use of part iv) of the duality theorem we get

$$\begin{aligned}\bigl[p_{\tau_2}, q_{\tau_1}\bigr]_{2-r} &= (1-\bar u_2^{-1})^{2-r}\,(1-\bar u_2)^{2-r}\,\bigl(1 - u_1 \bar u_2\bigr)^{r-2}\\ &= (2i)^{2-r}\,\bigl(\tau_1 - \bar\tau_2\bigr)^{r-2} .\end{aligned} \tag{5.8}$$

Thus we find for the quantity in (5.2):

$$= (2i)^{2-r} \int_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int_{\tau_2=i}^{ia} \overline{f_2(\tau_2)}\,\bigl(\tau_1 - \bar\tau_2\bigr)^{r-2}\, d\bar\tau_2\, d\tau_1 . \tag{5.9}$$

Limit as $a \uparrow \infty$. The exponential decay of cusp forms shows that the limit of the quantity in (5.9) exists. With (5.1) we get

$$\begin{aligned}&\bigl[c^K_{f_2}(i,\infty), c_{f_1}(\rho-1,\rho)\bigr]_{r-2}\\ &\qquad= (2i)^{2-r} \int_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int_{\tau_2=i}^{\infty} \overline{f_2(\tau_2)}\,\bigl(\tau_1 - \bar\tau_2\bigr)^{r-2}\, d\bar\tau_2\, d\tau_1\\ &\qquad= (2i)^{2-r} \int_{\tau_1=\rho-1}^{\rho} \int_{\tau_2=i}^{\infty} \omega(\tau_1,\tau_2) ,\end{aligned} \tag{5.10}$$

with the differential form on $\mathfrak{H} \times \mathfrak{H}$

$$\omega(\tau_1,\tau_2) = f_1(\tau_1)\,\overline{f_2(\tau_2)}\,(\tau_1 - \bar\tau_2)^{r-2}\, d\bar\tau_2\, d\tau_1 . \tag{5.11}$$

This differential form is invariant for the diagonal action of Γ. Hence we have also

$$\bigl[c^K_{f_2}(i,\infty), c_{f_1}(\rho-1,\rho)\bigr]_{r-2} = -(2i)^{2-r} \int_{\tau_1=i}^{\rho} \int_{\tau_2=0}^{\infty} \omega(\tau_1,\tau_2) . \tag{5.12}$$

Partial integration. We proceed as in [9, §7.2]. The function

$$F_2(\tau_1) = \int_{\tau_2=\tau_1}^{\infty} \overline{f_2(\tau_2)}\,(\tau_1 - \bar\tau_2)^{r-2}\, d\bar\tau_2 \qquad (\tau_1 \in \mathfrak{H}) , \tag{5.13}$$

is not holomorphic, but satisfies

$$\partial_{\bar\tau_1} F_2(\tau_1) \;=\; -\overline{f_2(\tau_1)}\big(\tau_1-\bar\tau_1\big)^{r-2} \;=\; -(2i)^{r-2}\,(\operatorname{Im}\tau_1)^{r-2}\,\overline{f_2(\tau_1)}\,, \tag{5.14}$$

and has the following transformation behavior under $\gamma \in \Gamma$:

$$F_2|_{v^{-1},2-r}\gamma(z) \;=\; F_2(z) + c^K_{f_2}(\infty,\gamma^{-1}\infty)(z)\,. \tag{5.15}$$

We use that

$$\begin{aligned} d\Big(f_1(\tau)\,F_2(\tau)\,d\tau\Big) &= -f_1(\tau)\big(\partial_{\bar\tau}F_2(\tau)\big)\,d\tau\,d\bar\tau\\ &= -(2i)^{r-1}\,f_1(x+iy)\,\overline{f_2(x+iy)}\,y^{r-2}\,dx\,dy\,, \end{aligned}$$

to compute the Petersson scalar product on the fundamental domain $\mathfrak{F}_1 = R \cup TL$ (not the standard fundamental domain), with R and L as in Fig. 1 on p. 306. We use Stoke's theorem to get:

$$\begin{aligned} \big(f_1,f_2\big)_r &= -(2i)^{1-r}\int_{\partial\mathfrak{F}_1} f_1(\tau_1)\,F_2(\tau_1)\,d\tau_1\\ &= -(2i)^{1-r}\Bigg(\int_{i+1}^{\infty} - \int_{i}^{\infty} + \int_{i}^{\rho} - \int_{i+1}^{\rho}\Bigg)\,f_1(\tau_1)\,F_2(\tau_1)\,d\tau_1\,. \end{aligned}$$

The transformation behavior in (5.15) implies that the first two integrals cancel each other, and that the remaining integrals give

$$\begin{aligned} -(2i)^{1-r}&\int_{\tau_1=i}^{\rho} f_1(\tau_1)\,c^K_{f_2}(\infty,0)(\tau_1)\,d\tau_1\\ &= (2i)^{1-r}\int_{\tau_1=i}^{\rho}\int_{\tau_2=0}^{\infty}\omega(\tau_1,\tau_2)\,. \end{aligned}$$

Comparison with (5.12) completes the proof of Theorem 5.1. □

6. Coinvariants of polynomial functions

The case cEE in the discussion of Subsection 3.2 considers the product

$$\operatorname{cp}(f_1,f_2) \;=\; c_{f_1}(\rho-1,\rho)\cdot c_{f_2}(i,\infty)$$

for two cusp forms $f_1 \in S_{r_1}(v_1)$, $f_f \in S_{r_2}(v_2)$ with weights $r_j \in \mathbb{Z}_{\geq 2}$ and the v_j multiplier systems corresponding to r_j. It represents a coinvariant in the finite dimensional module

${}^-\mathcal{D}^{\mathrm{pol}}_{v_1v_2,4-r_1-r_2}$. If we take trivial multiplier systems this is the classical situation of the cup product of two Eichler cohomology classes.

It turns out that in this classical context the cup product of two Eichler cocycles is uninteresting.

Proposition 6.1. *For a weight $r \in \mathbb{Z}_{\geq 2}$ and a corresponding multiplier system v*

$$\left({}^-\mathcal{D}^{\mathrm{pol}}_{v,2-r}\right)_\Gamma \;=\; \begin{cases} \mathbb{C} & \text{if } r=2 \text{ and } v=1\,,\\ \{0\} & \text{otherwise}\,. \end{cases} \tag{6.1}$$

Proof. For integral weights multiplier systems are characters. For any polynomial p of the form $p(t) = t^a +$ lower degree terms, we have

$$p|^-_{v,2-r}(1-T) \;=\; \bigl(1-v(T)\bigr)\, t^a + \text{lower degree terms}\,.$$

So if $v(T) \neq 1$ we obtain by induction on the degree that ${}^-\mathcal{D}^{\mathrm{pol}}_{v,2-r}|^-_{v,2-r}(1-T) = {}^-\mathcal{D}^{\mathrm{pol}}_{v,2-r}$.

The value on T determines the multiplier system, hence we are left with the case $v=1$. For p of degree a as above we have

$$p|^-_{1,2-r}(1-T) = -at^{a-1} + \text{lower degree terms}\,.$$

By induction on a we conclude that $t^a \in {}^-\mathcal{D}^{\mathrm{pol}}_{1,2-r}|^-_{1,2-r}(1-T)$ for $a=0,\ldots,r-3$. Finally we note that

$$\begin{aligned} t^{r-2}|^-_{1,2-r}(1-S) \;&=\; t^{r-2} - v(S)^{-1}\, t^{r-2}\,(-1/t)^{r-2}\\ &=\; t^{r-2} - 1\cdot(-1)^{r-2} \in t^{r-2} + {}^-\mathcal{D}^{\mathrm{pol}}_{1,2-r}|^-_{1,2-r}(1-T)\,. \qquad \square \end{aligned}$$

This implies that in case cEE the polynomial $\mathrm{cp}(f_1,f_2)$ indicated above represents the trivial coinvariant in ${}^-\mathcal{D}^{\mathrm{pol}}_{v_1v_2,4-r_1-r_2}$ if $r_1>2$ or $r_2>2$, and also if $r_1=r_2=2$ and $v_1v_2\neq 1$.

For the modular group the sole multiplier system v for which $S_2(v)\neq\{0\}$ is determined by $v(T)=e^{\pi i/3}$. (The corresponding space of cusp forms is spanned by η^4.) This multiplier system does not satisfy $v^2=1$, and hence it is understandable that in the classical context the cup product of two Eichler cocycles of modular forms is uninteresting.

7. Coinvariants associated to two Eichler cocycles

In the case cEE in the previous section it turned out that for modular forms (on the full modular group $\mathrm{SL}_2(\mathbb{Z})$) with integral weight at least 2 the cup product leads to the trivial coinvariant. In the case EE in §3.2 we consider a modular form $f_1 \in A_{r_1}(v_1)$ and $f_2 \in S_{r_2}(v_2)$ and form the coinvariant in ${}^-\mathcal{D}^{\infty}_{v_1v_2,4-r_1-r_2}$ represented by the following product of two values of Eichler cocycles

$$\mathrm{cp}(f_1, f_2) = c_{f_1}(\rho - 1, \rho) \cdot c_{f_2}(i, \infty)\,.$$

Under the assumption $r_2 \notin \mathbb{Z}_{\geq 2}$ this coinvariant represents the image of the cup product

$$c_{f_1} \cup c_{f_2} \in H^2_p\big(\Gamma; {}^-\mathcal{D}^{\infty}_{v_1, 2-r_1} \otimes {}^-\mathcal{D}^{\infty}_{v_2, 2-r_2}\big) \cong \big({}^-\mathcal{D}^{\infty}_{v_1, 2-r_1} \otimes {}^-\mathcal{D}^{\infty}_{v_2, 2-r_2}\big)_\Gamma$$

under the map in cohomology corresponding to the linear map

$${}^-\mathcal{D}^{\infty}_{v_1, 2-r_1} \otimes {}^-\mathcal{D}^{\infty}_{v_2, 2-r_2} \to {}^-\mathcal{D}^{\infty}_{v_1 v_2, 4-r_1-r_2}$$

induced by $v \otimes w \mapsto v\, w$.

To simplify the formulas we use the multiplier system $v_3 = v_1^{-1} v_2^{-1}$ corresponding to the weight $r_3 = 4 - r_1 - r_2$. Since Γ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the space of coinvariants is the quotient of the infinite-dimensional module ${}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}$ by the submodule

$${}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3} |^-_{v_3^{-1}, r_3} (1 - S) + {}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3} |^-_{v_3^{-1}, r_3} (1 - T)\,.$$

It is hard to understand this submodule. Even the question whether $\mathrm{cp}(f_1, f_2)$ represents the trivial coinvariant is hard to answer.

We use the fact that each Γ-invariant linear form β on ${}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}$ is trivial on the submodule, and induces a linear form on $\big({}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}\big)_\Gamma$. The duality theorem Theorem 4.1 has the following consequence:

Corollary 7.1. *Let $r_3 \geq 0$, and let $c \in {}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}$ represent a coinvariant $[c] \in \big({}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}\big)_\Gamma$. If there exists an entire modular form $h \in M_{r_3}(v_3)$ for which $\big[h, c\big]_{r_3} \neq 0$, then the coinvariant $[c]$ is non-trivial.*

Proof. The space of entire modular forms $M_{r_3}(v_3)$ is characterized by Γ-invariance and polynomial growth at the cusps. Polynomial growth near the boundary implies polynomial growth at the cusps, so $\big({}^+\mathcal{D}^{-\infty}_{v_3, r_3}\big)^\Gamma \subset M_{r_3}(v_3)$. With use of the Fourier expansion one checks that elements of $M_{r_3}(v_3)$ have polynomial growth near $\mathbb{R}$. So $M_{r_3}(v_3)$ is equal to $\big({}^-\mathcal{D}^{-\infty}_{v_3^{-1}, r_3}\big)^\Gamma$.

Suppose that c represents the trivial coinvariant. Then $c \in {}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3} |^-_{v_3^{-1}, r_3} (1 - S) + {}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3} |^-_{v_3^{-1}, r_3} (1 - T)$, and any Γ-invariant linear form α on ${}^-\mathcal{D}^{\infty}_{v_3^{-1}, r_3}$ satisfies $\alpha(c) = 0$. In particular this would mean $[h, c]_{r_3} = 0$. Hence c in the corollary represents a non-trivial coinvariant. $\square$

Remark. We do not know whether ${}^-\mathcal{D}^{\infty}_{v^{-1}, r} |^-_{v^{-1}, r} (1 - S) + {}^-\mathcal{D}^{\infty}_{v^{-1}, r} |^-_{v^{-1}, r} (1 - T)$ is closed in ${}^-\mathcal{D}^{\infty}_{v^{-1}, r}$ for the natural topology on ${}^-\mathcal{D}^{\infty}_{v^{-1}, r}$. So if $[h, c]_r = 0$ for all $h \in M_r(v)$, there still might be a non-continuous Γ-invariant linear form α for which $\alpha(c) \neq 0$. The condition in the corollary is sufficient but not necessary for non-triviality of $[c]$.

Trilinear form. We proceed with $r_1, r_2, r_3 \in \mathbb{R}$ and corresponding multiplier systems v_1, v_2, v_3 satisfying

$$r_1 + r_2 + r_3 \,=\, 4 \qquad r_2 > 0\,, \qquad r_3 \geq 0 \qquad v_1 v_2 v_3 \,=\, 1 \tag{7.1}$$

and consider the trilinear form

$$\mathbf{T}(f_1, f_2, f_3) \,=\, \bigl[f_3, \mathrm{cp}(f_1, f_2)\bigr]_{r_3} \tag{7.2}$$

on $A_{r_1}(v_1) \times S_{r_2}(v_2) \times M_{r_3}(v_3)$.

Our aim in the next section is to reformulate $\mathbf{T}(t_1, f_2, f_3)$ more explicitly (under stronger conditions than (7.1)) such that it can be computed numerically.

The trilinear form $\mathbf{T}$ makes sense without the condition $r_2 \notin \mathbb{Z}_{\geq 2}$. Under this additional condition we know that if $\mathbf{T}(f_1, f_2, f_3) \neq 0$ for some choice of (f_1, f_2, f_3), then $\mathrm{cp}(f_1, f_2)$ represents a non-trivial coinvariant.

7.1. Triviality over larger modules

In this subsection we give ourselves the task to describe the 2-cocycle corresponding to $\mathrm{cp}(f_1, f_2)$ as a 2-coboundary over some larger Γ-module, or equivalently, to write

$$\mathrm{cp}(f_1, f_2) \,=\, A_1|(T-1) + A_2|(S-1)$$

with some A_1 and A_2 in a larger Γ-module. The following result establishes this for two larger modules.

Proposition 7.2. *Let $f_1 \in A_{r_1}(v_1)$, $f_2 \in S_{r_2}(v_2)$ with $r_1 \in \mathbb{R}$ and $r_2 > 0$. As above we use $r_3 = 4 - r_1 - r_2$, and multiplier systems satisfying $v_3 = v_1^{-1} v_2^{-1}$.*

i) *There are fairly explicit functions A_1, A_2 in the module of all real-analytic functions on $\mathfrak{H}^-$ with the action $|^-_{v_3^{-1}, r_3}$ such that*

$$\mathrm{cp}(f_1, f_2) \,=\, A_1|^-_{v_3^{-1}, r_3}(T-1) + A_2|^-_{v_3^{-1}, r_3}(S-1)\,. \tag{7.3}$$

ii) *There are $B_1, B_2 \in {}^-\mathcal{D}^{-\omega}_{v_3^{-1}, r_3}$ such that*

$$\mathrm{cp}(f_1, f_2) \,=\, B_1|^-_{v_3^{-1}, r_3}(T-1) + B_2|^-_{v_3^{-1}, r_3}(S-1)\,. \tag{7.4}$$

The proof takes the remainder of this subsection. It depends on various other results, which may be considered interesting independently. In the course of the proof the meaning of *fairly explicit* will become clear.

Lemma 7.3. *Let V be a linear space of functions on the lower half-plane containing the holomorphic functions and stable under multiplication by holomorphic functions.*

Let $c_1 \in Z^1(\Gamma; {}^-\mathcal{D}^\infty_{v_2,2-r_2})$ and $c_2 \in Z^1_p(\Gamma; {}^-\mathcal{D}^\infty_{v_2,2-r_2})$. If there exist elements $q \in V$ such that

$$q|^-_{v_2,2-r_2} T = q|^-_{v_2,2-r_2} S\,, \qquad \text{and} \qquad q|^-_{v_2,2-r_2}(S-1) = c_1(\rho-1,\rho)\,, \tag{7.5}$$

then

$$\begin{aligned} &c_1(\rho-1,\rho)\, c_2(i,\infty) \\ &\quad = \Bigl(q\, c_2(\rho,\infty)\Bigr)\,|^-_{v_3^{-1},r_3}\,(T-1) + \Bigl(\bigl(q|^-_{v_1,2-r_1} S\bigr)\, c_2(\rho-1,i)\Bigr)\,|^-_{v_3^{-1},r_3}\,(S-1)\,. \end{aligned} \tag{7.6}$$

Remark. We consider (7.6) to be 'fairly explicit' in the data c_1, c_2, and q.

Proof. By the following computation, starting at the right hand side in (7.6). To save space we denote by $|$ the actions $|_{v_1,2-r_1}$ and $|^-_{v_2,2-r_2}$ on the separate factors

$$\begin{aligned} &= \bigl(q|T\bigr)\, c_2(\rho-1,\infty) - q\, c_2(\rho,\infty) + q\, c_2(\rho,i) - (q|S)\, c_2(\rho-1,i) \\ &= (q|S)\,\bigl(c_2(\rho-1,\infty) - c_2(\rho-1,i)\bigr) - q\,\bigl(c_2(\rho,\infty) - q\, c_2(\rho,i)\bigr) \\ &= \bigl(q|S-q\bigr)\, c_2(i,\infty) \;=\; c_1(\rho-1,\rho)\, c_2(i,\infty)\,. \qquad \square \end{aligned}$$

Remark. We found the relation by using the fact that the linear map in (3.7) preserves cohomology classes. So the cup product $dp \cup c_2$ should be a coboundary. Evaluating $p \otimes c_2$ on $\delta_1 \partial\,(L+R)$ leads to (7.6).

If we try to work with the cup product $c_1 \cup dp_2$ the function p_2 would have to have values in a module in which there are no non-trivial invariants for the element $T \in \Gamma$. Cocycles of modular forms do not become trivial in modules satisfying this condition.

Proof of part i) of Proposition 7.2. The space V^{an} of all real-analytic functions on $\mathfrak{H}^-$ satisfies the condition in Lemma 7.3. We put for $t \in \mathfrak{H}^-$

$$Q_{f_1}(t) = \int_{\tau=\rho}^{\bar t} f_1(\tau)\,\bigl(\tau - t\bigr)^{r_1-2}\, d\tau\,. \tag{7.7}$$

The presence of $\bar t$ as limit of integration makes Q_{f_1} non-holomorphic. It is real analytic. A direct computation shows that for all $\gamma \in \Gamma$

$$Q_{f_1}|^-_{v_1,2-r_1}(\gamma-1) = \int_{\tau=\gamma^{-1}\rho}^{\rho} f_1(\tau)\,(\tau-t)^{r_1-2}\, d\tau\,. \tag{7.8}$$

So the group cocycle $\gamma \mapsto \psi_\gamma^\rho = c_{f_1}(\gamma^{-1}\rho, \rho)$ associated to f_1 with base point ρ becomes a coboundary in the module $\big(V^{\mathrm{an}}, |^-_{v_2,2-r}\big)$. It satisfies $\psi^\rho_{TS} = 0$ since $ST^{-1}\rho = \rho$. So $Q_{f_1}|^-_{v_2,2-r_2} TS = Q_{f_1}$. We can take $q = Q_{f_1}$ in Lemma 7.3.

The construction of Q_{f_1} may also be considered *fairly explicit.* □

Proposition 7.4. *For all $r_1 \in \mathbb{R}$ and corresponding multiplier system the space of automorphic functions has a decomposition*

$$A_{r_1}(v_1) \;=\; S_{r_1}(v_1) \oplus X_r(v_1)\,, \tag{7.9}$$

where $X_{r_1}(v_1)$ is the space of $f_1 \in A_{r_1}(v_1)$ for which there is an element $q_{f_1} \in {}^-\mathcal{D}^{-\infty}_{v_1,2-r_1}$ such that

$$c_{f_1}(\gamma^{-1}\rho, \rho) \;=\; q_{f_1}|^-_{v_1,2-r_1}(\gamma - 1) \qquad \textit{for all } \gamma \in \Gamma\,. \tag{7.10}$$

Proof. This is a consequence of the theorem of Knopp and Mawi [11] which gives a bijection

$$S_{r_1}(v_1) \longrightarrow H^1(\Gamma; {}^-\mathcal{D}^{-\infty}_{v_1,2-r_1})\,, \tag{7.11}$$

given by assigning to a cusp form f_1 the cohomology class represented by the group cocycle $\gamma \mapsto \psi^\infty = c_{f_1}(\gamma^{-1}\infty, \infty)$. Actually, they state the result with Knopp cocycles $\iota\psi^\infty$.

The natural maps associated to extension of modules give linear maps

$$A_{r_1}(v_1) \to H^1(\Gamma; {}^-\mathcal{D}^{\omega}_{v_1,2-r_1}) \to H^1(\Gamma; {}^-\mathcal{D}^{\infty}_{v_1,2-r_1}) \to H^1(\Gamma; {}^-\mathcal{D}^{-\infty}_{v_1,2-r_1}) \tag{7.12}$$

sending f_1 to the cohomology class $\big[\psi^\rho\big]$. For cusp forms f_1 the classes of ψ^ρ and ψ^∞ coincide in the last two of the modules in (7.12). We define $X_{r_1}(v_1)$ as the kernel of $A_{r_1}(v_1) \to H^1(\Gamma; {}^-\mathcal{D}^{-\infty}_{v_1,2-r_1})$. The theorem of Knopp and Mawi implies the statements in the proposition. □

Remark. In principle the construction of q_{f_1} might be traced by analyzing the proofs in [10] and [11]. We would not call the result *fairly explicit.*

Proposition 7.5. *Let $f_1 \in A_{r_1}(v_1)$ have at most exponential growth $\mathrm{O}\big(e^{A\,\mathrm{Im}\,z}\big)$ with some $A \in R$ at the cusp. Then there are elements $q \in {}^-\mathcal{D}^{-\omega}_{v_1,2-r_1}$ such that*

$$c_{f_1}(\gamma^{-1}\rho, \rho) \;=\; q_{f_1}|^-_{v_1,2-r_1}(\gamma - 1) \qquad \textit{for all } \gamma \in \Gamma\,. \tag{7.13}$$

Proof. Theorem C in [3] implies that such q exist if the automorphic form f_1 has a $(2-r)$-harmonic lift. For exponentially growing modular forms the existence of such harmonic lifts follows from Theorem 1.1 in [1]. □

Remark. The proof of the existence of harmonic lifts is highly non-explicit. Here we need it only for cusp forms. Then we may work with the cocycle $\psi_{f_1}^{\infty}$. The corresponding functions $q_{f_1}^{\infty}$ are mock modular forms with shadow (a multiple of) f_1. The difference $q_{f_1}^{\infty} - q_{f_1}$ can be made more or less explicit.

Proof of part ii) of Proposition 7.2. The function q in Lemma 7.3 is provided by Proposition 7.4 for $f_1 \in X_{r-1}(v_1)$. For cusp forms we use Proposition 7.5. $\square$

8. Triple integral

Our aim in this section is to prove Theorem 1.1, which gives an expression as a threefold iterated integral for the trilinear form $\mathbf{T}(f_1, f_2, f_3)$ introduced in (7.2). We reformulate the triple integral in Theorem 1.1 in Proposition 8.5. This formulation is sufficiently explicit to allow numerical computations, in Section 8.4, that suggest that for many choices of the weights the trilinear form $(f_1, f_2, f_3) \mapsto \mathbf{T}(f_1, f_2, f_3)$ is non-zero.

Throughout this section we consider three modular forms $f_1 \in A_{r_1}(v_1)$, $f_2 \in S_{r-2}(v_2)$, $f_3 \in M_{r_3}(v_3)$, and assume that the weights r_j and corresponding multiplier systems v_j satisfy the conditions in (7.1).

From Subsection 8.2 on we work under the stronger conditions

$$r_1 < 2\,, \quad 0 < r_2 < 2\,, \quad r_1 + r_2 + r_3 = 4\,, \quad v_1 v_2 v_3 = 1\,. \tag{8.1}$$

The triple integral as stated in Theorem 1.1 is absolutely convergent if $r_1, r_2 < 2$. We expect that analytic continuation in (r_1, r_2) is possible. Then the modular forms should be families depending on the weight. (This makes sense if we multiply the modular forms with powers of the Dedekind eta-function.) Here we do not pursue the analytic continuation.

8.1. Truncation and fourfold integral

We start without the additional assumption $r_1, r_2 < 2$.

Truncation. Like in the proof of Theorem 5.1 we approximate $c_{f_2}(i, \infty)$ by $c_{f_2}(i, ia)$ with $a > 0$, and put

$$\mathbf{T}_a(f_1, f_2, f_3) = \Big[f_3, c_{f_1}(\rho - 1, \rho) \cdot c_{f_2}(i, ia) \Big]_{r_3}\,. \tag{8.2}$$

Lemma 8.1. *Under the assumptions* (7.1)

$$\mathbf{T}(f_1, f_2, f_3) = \lim_{a \to \infty} \mathbf{T}_a(f_1, f_2, f_3),. \tag{8.3}$$

Proof. Lemma 5.2 states that $c_{f_2}(i, ia)$ approximates $c_{f_2}(i, \infty)$ as $a \uparrow \infty$ in the natural topology on ${}^{-}\mathcal{D}^{\infty}_{v_2, 2-r_2}$. The multiplication $\varphi \mapsto c_{f_1}(\rho - 1, \rho)\varphi$ is a continuous map ${}^{-}\mathcal{D}^{\infty}_{v_2, 2-r_2} \to {}^{-}\mathcal{D}^{\infty}_{v_1 v_2, 4-r_1-r_2}$. Since the $\varphi \mapsto [h, \varphi]_{r_3}$ is continuous the lemma follows. $\square$

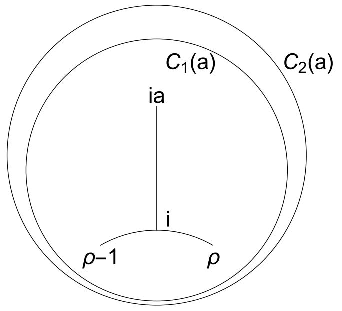


Fig. 2. Paths of integration in Proposition 8.2.

Fourfold integral. The quantity $\mathbf{T}_a(f_1, f_2, f_3)$ has the advantage that it has an expression involving four integrals over compact sets: the two Eichler integrals from $\rho - 1$ to ρ and from i to ia, and the two integrals over compact cycles in the duality theorem Theorem 4.1 and in the definition of $\sigma_r f$. This is quite complicated. However, the compact domains of integration allow us to choose any order of integration that suits us.

Proposition 8.2. *Under the assumptions* (7.1) *and* $r_3 > 0$*:*

$$\begin{aligned}\mathbf{T}(f_1,f_2,f_3) = \lim_{a\to\infty} \int_{\tau_1=\rho-1}^{\rho} f_1(\tau_1) \int_{\tau_2=i}^{ia} f_2(\tau_2)\, \frac{1}{\pi} \int_{z\in C_2(a)} f_3(z)\,(-z-i)^{r_3} \\ \cdot \frac{1}{\pi} \int_{\tau\in C_1(a)} {}_2F_1\Bigl(1,1;r_3;\frac{(\tau-i)(z+i)}{(\tau+i)(z-i)}\Bigr) \Bigl(\frac{\tau_1-\tau}{i-\tau}\Bigr)^{r_1-2} \Bigl(\frac{\tau_2-\tau}{i-\tau}\Bigr)^{r_2-2} \\ \cdot \frac{d\tau}{\tau^2+1}\,\frac{dz}{z^2+1}\, d\tau_2\, d\tau_1\,.\end{aligned} \tag{8.4}$$

We take $C_2(a)$ *and* $C_1(a)$ *as positively oriented circles* $\bigl|\frac{z-i}{z+i}\bigr| = c_2(a)$ *and* $\bigl|\frac{\tau-i}{\tau+i}\bigr| = c_1(a)$ *with* $c_1(a) < c_2(a) < 1$ *such that* $C_1(a)$ *encircles paths from* $\rho-1$ *to* ρ *and from* i *to* ia*. See Fig. 2.*

Proof. We unravel the definitions in Theorem 4.1, and in (4.5), (4.2) and (2.5), and combine them as the limit of a fourfold integral. We use that $r_1 + r_2 = 4 - r_3$. For $\tau \in C_1(a)$ we can combine $(\tau_j - \tau)^{r_j-2}$ and $(i-\tau)^{2-r_j}$ to get $\bigl(\frac{\tau_j-\tau}{i-\tau}\bigr)^{r_j-2}$ as a holomorphic function on $\mathbb{P}^1_{\mathbb{C}}$ minus a path from τ_j to i. Actually, in (4.5) we need the following form of the Eichler integrals

$$c_{f_j}(x_1,x_2)^-(\tau) = (i-\tau)^{2-r_j}\, c_{f_j}(x_1,x_2;\tau)\,,$$

since these functions are holomorphic on a neighborhood of $\mathfrak{H}^- \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. They are directly given by

$$c_{f_j}(x_1,x_2)^-(\tau) \;=\; \int_{\tau=x_1}^{x_2} f_j(\tau_j)\,\Bigl(\frac{\tau_j-\tau}{i-\tau}\Bigr)^{r_j-2}\,d\tau\,. \tag{8.5}$$

We need $r_3 > 0$, since the hypergeometric function in formula (4.5) is not defined at $r_3 = 0$. $\square$

8.2. Triple integral

In this subsection we prove Theorem 1.1. To the assumptions (7.1) we add in (8.1) the conditions $r_1 < 2$, $r_2 < 2$. These additional conditions restrict the triples (f_1, f_2, f_3) considerably. We have $r_3 > 0$, and moreover f_2 can only be a multiple of η^{2r_2}, hence $v_2 = v[r_2]$.

The plan of the proof is to modify the fourfold integral in Proposition 8.2. We start with the two inner integrals, which describe $\bigl[f_3, \kappa(\tau_1,\tau_2;\cdot)\bigr]_{r_3}$, with the product of modified Eichler kernels

$$\kappa(\tau_1,\tau_2;\tau) \;=\; \Bigl(\frac{\tau_1-\tau}{i-\tau}\Bigr)^{r_1-2}\,\Bigl(\frac{\tau_2-\tau}{i-\tau}\Bigr)^{r_2-2}\,. \tag{8.6}$$

Lemma 9.9, derived in the context of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$, gives a simpler expression for the two innermost integrals.

The main work is the transformation is the simplification of this expression. We use Kummer relation for the hypergeometric function, and go over from a closed contour to a segment in $\mathfrak{H}$ as the path of integration.

Combining the resulting integral with the integrals over τ_1 and τ_2 finishes the proof.

Standing assumptions. In stating lemmas we work with the standard assumptions for the f_j, the r_j, and the v_j. From Lemma 8.4 onwards we assume $r_1 < 2$, $r_2 < 2$.

Lemma 8.3. *Let $\tau_1 \neq \tau_2$. Then*

$$\begin{aligned}\bigl[f_3,\kappa(\tau_1,\tau_2;\cdot)\bigr]_{r_3} \;&=\; \frac{(2i)^{2-r_2}}{\pi}\bigl|c\tau_1+d\bigr|^{2r_1-2}\,(\tau_2-\bar\tau_1)^{r_2-2}\\ &\quad\cdot \int_{z\in C} f_3(z)\,(-i-z)^{r_3}\,\Bigl(\frac{z+i}{z-\bar\tau_1}\Bigr)^{1-r_3}\\ &\quad\cdot {}_2F_1\Bigl(1,2-r_2;r_3;\frac{(\tau_2-\tau_1)(z-\bar\tau_1)}{(\tau_2-\bar\tau_1)(z-\tau_1)}\Bigr)\,\frac{dz}{(z-\tau_1)(z+i)}\,,\end{aligned} \tag{8.7}$$

where C is a positively oriented curve in $\mathfrak{H}$ encircling all singularities of the integrand. The matrix $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is determined by the conditions $g\tau_1 = i$ and $g\tau_2 \in i(1,\infty)$, and $\arg(ci+d) \in [0,\pi)$.

Proof. This is Lemma 9.9 with $h^+(z) = f_3(z)\,(-i-z)^{r_3}$. $\square$

Relation (8.7) is complicated, since it depends directly on τ_1 and τ_2, and indirectly via the matrix g. In the following result we use the lower row (c,d) of g again.

Lemma 8.4. *If $r_1, r_2 < 2$ the integral in (8.7) is equal to:*

$$\begin{aligned} &-\pi\, 2^{2-r_1}\, e^{-\pi i(r_1/2+r_3)}\, \mathsf{B}(2-r_1, 2-r_2)^{-1}\, (\tau_2-\bar\tau_1)^{2-r_2}\, \left|c\tau_1+d\right|^{2-2r_1} \\ &\qquad \cdot \int_{z=\tau_1}^{\tau_2} f_3(z) \left(\frac{z-\tau_1}{\tau_2-\tau_1}\right)^{1-r_2} \left(\frac{\tau_2-z}{\tau_2-\tau_1}\right)^{1-r_1} \frac{dz}{\tau_2-\tau_1}\,. \end{aligned} \tag{8.8}$$

The path of integration can be chosen along the geodesic segment s_{τ_1,τ_2} from τ_1 to τ_2, or be deformed in such a way that it does not cross the geodesic ℓ_{τ_1,τ_2} through τ_1 and τ_2 in points of $\ell_{\tau_1,\tau_2} \smallsetminus s_{\tau_1,\tau_2}$.

We use the beta-function $\mathsf{B}(a,b) = \Gamma(a)\,\Gamma(b)/\Gamma(a+b)$.

Proof. We rewrite for $z \in \mathfrak{H}$

$$(-i-z)^{r_3} \left(\frac{z+i}{z-\bar\tau_1}\right)^{1-r_3} = e^{-\pi i r_3}\,(z+i)\,(z-\bar\tau_1)^{r_3-1}\,.$$

In the resulting expression for the integral in (8.7) we go over to the disk coordinate $w = \frac{z-\tau_1}{z-\bar\tau_1}$, $z = \frac{\tau_1 - w\bar\tau_1}{1-w}$, and put $q = \frac{\tau_2-\tau_1}{\tau_2-\bar\tau_1}$. We choose the curve C such that $|w| = q_1$ with $|q| < q_1 < 1$. We write $f_3[w] = f_3(z)$. The integral is equal to

$$\begin{aligned} &= e^{-\pi i r_3}(\tau_1-\bar\tau_1)^{r_3-1} \\ &\qquad \cdot \int_{|w|=q_1} f_3[w]\ {}_2F_1\bigl(1, 2-r_2; r_3; q/w\bigr)\,(1-w)^{-r_3}\,\frac{dw}{w} \\ &= e^{-\pi i r_3}(\tau_1-\bar\tau_1)^{r_3-1} \\ &\qquad \cdot \int_{|u|=q_1/|q|} f_3\bigl[qu\bigr]\ {}_2F_1\bigl(1, 2-r_2; r_3; u^{-1}\bigr)\,(1-qu)^{-r_3}\,\frac{du}{u}\,. \end{aligned} \tag{8.9}$$

We use Kummer relations (1), (7) §6.5 and (9), (13), (17), (21) §6.4 in [14, §6.5] to get, under the additions conditions $r_1, r_2 \neq 1$:

$$\begin{aligned} {}_2F_1\bigl(1, 2-r_2; r_3; z\bigr) &= \frac{\Gamma(r_2-1)\,\Gamma(r_3)}{\Gamma(2-r_1)}\,(-z)^{r_2-2}(1-1/z)^{1-r_1} \\ &\qquad + \frac{r_3-1}{r_2-1}\,z^{-1}\ {}_2F_1\bigl(1, 2-r_3; r_2; 1/z\bigr) \end{aligned} \tag{8.10}$$

$$\begin{aligned} &= \frac{\Gamma(r_1-1)\,\Gamma(r_3)}{\Gamma(2-r_2)}\,z^{1-r_3}\,(1-z)^{1-r_1} \\ &\qquad + \frac{r_3-1}{1-r_1}\ {}_2F_1\bigl(1, 2-r_2; r_1; 1-z\bigr)\,. \end{aligned} \tag{8.11}$$

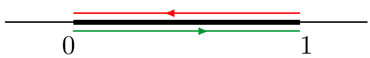
We use that in both cases one hypergeometric function specializes to a simpler expression.

Relation (8.11) is valid for $z \in (0,1)$ and extends to a relation between holomorphic extensions on $\mathbb{C} \smallsetminus (-\infty,0] \cup [1,\infty))$. In relation (8.10) there are no z for which both hypergeometric series are in their domain of absolute convergence. The relations hold for the holomorphic extension for $z \in (-1,0)$, and extend to $\mathbb{C} \smallsetminus [0,\infty)$.

The two singularities of the integrand in (8.9) are determined by the behavior of the hypergeometric function at $u=0$ and at $u=1$. The Kummer relations show that

$$ {}_2F_1\bigl(1,2-r_2;r_3;1/u\bigr) \ll \begin{cases} |u|^{\min(1,2-r_2)} & \text{as } u \to 0\,, \\ |u-1|^{\min(0,1-r_1)} & \text{as } u \to 1\,. \end{cases} \tag{8.12} $$

Together with u^{-1} in $\frac{du}{u}$ this implies that the original path of integration $|u| = q_1/|q|$ can be moved closely to the interval $[0,1]$. We have to determine the contributions of the limits of the integrals on both sides of the interval.



We have to consider the integrand in (8.9) at points $u = x \pm i\varepsilon$ with $x \in (0,1)$ and $\varepsilon \downarrow 0$. The factors $f_3[qu]$ and $(1-qu)^{-r_3}$ are holomorphic at the points $u \in (0,1)$.

We use the Kummer relation (8.11). The points $1-1/x$ are in $(-\infty,0)$. So the hypergeometric function ${}_2F_1\bigl(1,2-r_2,r_1;1-1/u\bigr)$ is holomorphic on the interval $u=(0,1)$, and the contributions from the integrals on both sides cancel each other. The factor $(1/u)^{1-r_3}$ in the first term of (8.11) is the same on both sides of the interval. The imaginary part of $1-1/u$ at $u = x \pm i\varepsilon$ is $\frac{\pm i\varepsilon}{x^2+\varepsilon^2}$. This means that the factor $(1-1/u)^{1-r_1}$ should be computed as $\bigl(\frac{1-u}{u}\bigr)^{1-r_1} e^{\pm \pi i(1-r_1)}$. The integral in (8.9) is equal to

$$ \begin{aligned} & e^{-\pi i r_3}\,(\tau_1-\bar\tau_1)^{r_3-1} \int\limits_{u=0}^{1} f_3[qu]\,(1-qu)^{-r_3}\,\frac{\Gamma(r_1-1)\,\Gamma(r_3)}{\Gamma(2-r_2)}\,u^{r_3-1} \\ & \qquad \cdot \bigl(1/u-1\bigr)^{1-r_1}\,\frac{du}{u}\,\Bigl(-e^{\pi i(1-r_1)} + e^{-\pi i(1-r_1)}\Bigr) \\ & = 2\pi i\, e^{-\pi i r_3}\,(\tau_1-\bar\tau_1)^{r_3-1}\,\frac{\Gamma(r_3)}{\Gamma(2-r_1)\,\Gamma(2-r_2)} \int\limits_{u=0}^{1} f_3[qu]\,\bigl(1-qu\bigr)^{-r_3} \\ & \qquad \cdot u^{1-r_2}\,(1-u)^{1-r_1}\,du\,. \end{aligned} $$

This integral is holomorphic in r_1 and r_2 under the conditions $r_1 < 2$, $0 < r_2 < 2$, $r_3 = 4-r_1-r_2 > 0$. The integral in the left hand side of the relation in the lemma is holomorphic in r_1 and r_2 as well. So we can drop the assumptions that r_1 and r_2 are not integral.

As u runs from 0 to 1, the image $z = \frac{\tau_1-\bar\tau_1 qu}{1-qu}$ runs from τ_1 to $\frac{\tau_1-\bar\tau_1 q}{1-q} = \tau_2$ along the geodesic segment s_{τ_1,τ_2} in $\mathfrak{H}$. Carrying out the backward substitutions $u = w/q$ and $w = \frac{z-\tau_1}{z-\bar\tau_1}$ we arrive at

$$\begin{aligned} 2\pi i\, e^{-\pi i r_3} (\tau_1-\bar\tau_1)^{r_3-1}\, \mathsf{B}(2-r_1,2-r_2)^{-1} \int_{z=\tau_1}^{\tau_2} f_3(z) \left(\frac{z-\bar\tau_1}{\tau_1-\bar\tau_1}\right)^{r_3} \\ \cdot \left(\frac{(z-\tau_1)(\tau_2-\bar\tau_1)}{(z-\bar\tau_1)(\tau_2-\tau_1)}\right)^{1-r_2} \left(\frac{(\tau_2-z)(\tau_1-\bar\tau_1)}{(z-\bar\tau_1)(\tau_2-\tau_1)}\right)^{1-r_1} \\ \cdot \frac{(\tau_2-\bar\tau_1)(\tau_1-\bar\tau_1)}{(\tau_2-\tau_1)(z-\bar\tau_1)^2}\, dz\,, \end{aligned} \tag{8.13}$$

with the beta-function $\mathsf{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. The path of integration is the geodesic segment s_{τ_1,τ_2}. To handle the powers we note that $z-\bar\tau_1$ and $\tau_1-\bar\tau_1$ are both in the upper half-plane. So with the standard choice of the argument in $(-\pi,\pi)$ we have

$$(\tau_1-\bar\tau_1)^{r_3-1} \left(\frac{z-\bar\tau_1}{\tau_1-\bar\tau_1}\right)^{r_3} = (\tau-\bar\tau_1)^{-1} (z-\bar\tau_1)^{r_3}\,.$$

In $\left(\frac{(z-\tau_1)(\tau_2-\bar\tau_1)}{(z-\bar\tau_1)(\tau_2-\tau_1)}\right)^{1-r_2}$, coming from u^{1-r_2}, the arguments of the quotients $\frac{z-\tau_1}{\tau_2-\tau_1}$ and $\frac{z-\bar\tau_1}{\tau_2-\bar\tau_1}$ are equal, and contained in $(-\pi,\pi)$. So we can split up this power correspondingly. The factor coming from $(1-u)^{1-r_1}$ can be handled analogously. This leads to the following:

$$\begin{aligned} &2\pi i\, e^{-\pi i r_3}\, \mathsf{B}(2-r_1,2-r_2)^{-1} \int_{z=\tau_1}^{\tau_2} f_3(z)\,(z-\bar\tau_1)^{r_3-2} \left(\frac{z-\tau_1}{\tau_2-\tau_1}\right)^{1-r_2} \left(\frac{z-\bar\tau_1}{\tau_2-\bar\tau_1}\right)^{r_2-1} \\ &\qquad \cdot \left(\frac{\tau_2-z}{\tau_2-\tau_1}\right)^{1-r_1} \left(\frac{z-\bar\tau_1}{\tau_1-\bar\tau_1}\right)^{r_1-1} \frac{\tau_2-\bar\tau_1}{\tau_2-\tau_1}\, dz \\ &\quad = -\pi\, 2^{2-r_1}\, e^{-\pi i (r_1/2+r_3)}\, \mathsf{B}(2-r_1,2-r_2)^{-1}\, (\tau_2-\bar\tau_1)^{2-r_2}\, \left|c\tau_1+d\right|^{2-2r_1} \\ &\qquad \cdot \int_{z=\tau_1}^{\tau_2} f_3(z) \left(\frac{z-\tau_1}{\tau_2-\tau_1}\right)^{1-r_2} \left(\frac{\tau_2-z}{\tau_2-\tau_1}\right)^{1-r_1} \frac{dz}{\tau_2-\tau_1}\,. \end{aligned} \tag{8.14}$$

In the second step we decomposed the powers of quotients in which $\bar\tau_1$ occurs. This is possible since both numerator and denominator are in the upper half-plane. We also used that $\frac{di-b}{a-ic} = \tau_1$, hence $\tau_1-\bar\tau_1 = \frac{2i}{a^2+c^2}$, and $c\tau_1+d = \frac{1}{a+ic}$.

The resulting integrand is holomorphic in z, and we can deform the path of integration. The two powers in the integrand are multi-valued and have singularities at τ_1 and τ_2, respectively. In the statement of the lemma we choose a region on which the integrand is well-defined. If τ_2 tends to τ_1 the limit of the integral exists. $\square$

Completion of the proof of Theorem 1.1. For $\tau_1 \neq \tau_2$ we obtain from Lemmas 8.3 and 8.4:

$$\begin{aligned}
\bigl[f_3, \kappa(\tau_1,\tau_2;\cdot)\bigr]_{r_3} &= -(2i)^{2-r_2}\,|c\tau_1+d|^{2r_1-2+2-2r_1}\,(\tau_2-\bar\tau_1)^{r_2-2+2-r_2}\\
&\quad\cdot 2^{2-r_1}\,e^{-\pi i(r_1/2+r_3)}\,\mathsf{B}(2-r_1,2-r_2)^{-1}\\
&\quad\cdot \int_{z=\tau_1}^{\tau_2} f_3(z)\left(\frac{z-\tau_1}{\tau_2-\tau_1}\right)^{1-r_2}\left(\frac{\tau_2-z}{\tau_2-\tau_1}\right)^{1-r_1}\frac{dz}{\tau_2-\tau_1}\\
&= (-2i)^{r_3}\,\mathsf{B}(2-r_1,2-r_2)^{-1}\\
&\quad\cdot \int_{z=\tau_1}^{\tau_2} f_3(z)\left(\frac{z-\tau_1}{\tau_2-\tau_1}\right)^{1-r_2}\left(\frac{\tau_2-z}{\tau_2-\tau_1}\right)^{1-r_1}\frac{dz}{\tau_2-\tau_1}\\
&= (-2i)^{r_3}\,\mathsf{B}(2-r_1,2-r_2)^{-1}\\
&\quad\cdot \int_{u=0}^{1} f_3\bigl(\tau_1+u(\tau_2-\tau_1)\bigr)\,u^{1-r_2}\,(1-u)^{1-r_1}\,du\,.
\end{aligned}$$

The quantities $\bar\tau_1$, c and d, which are not holomorphic in τ_1 and τ_2 cancel in this result. The second version shows that the whole expression is holomorphic in $(\tau_1,\tau_2)\in\mathfrak{H}^2$.

These expressions can be inserted for $\bigl[f_3,\kappa(\tau_1,\tau_2)\bigr]_{r_3}$ in

$$\mathbf{T}_a(f_1,f_2,f_3) = \int_{\tau_2=i}^{ia} f_2(\tau_2)\int_{\tau_1=\rho-1}^{\rho} f_1(\tau)\,\bigl[f_3,\kappa(\tau_1,\tau_2)\bigr]_{r_3}\,d\tau_1\,d\tau_2\,.$$

The values $\bigl[f_3,\kappa(\tau_1,\tau_2)\bigr]_{r_3}$ have at most polynomial growth in τ_2, since f_3 has at most polynomial growth. The values of $f_1(\tau)$ stay bounded. The exponential decay of the cusp form $f_2(\tau_2)$ as τ_2 moves up to infinity, ensure that the limit as $a\uparrow\infty$ exists. The limit is given by the same expression with ia replaced by ∞. Lemma 8.1 implies that the resulting limit is equal to $\mathbf{T}(f_1,f_2,f_3)$.

The path of integration of the integral over z can be deformed, provided we take care not to cross singularities of the integrand. The discontinuities of the powers of quotients in (1.14) can be chosen to occur along the geodesic half-lines indicated in the theorem. □

8.3. Reformulations of the triple integral

8.3.1. Integration over a 3-cycle

The triple integral in (1.14) is based on the choice of the pairs $(\rho-1,\rho)$ and (i,∞) determining the limits of integration in the outer integral.

Let us put

$$Y_3 = \bigl\{(\tau_1,\tau_2,z)\in\mathfrak{H}^3 \;:\; z\notin \ell_{\tau_1,\tau_2}\smallsetminus s_{\tau_1,\tau_2}\bigr\}\,, \tag{8.15}$$

with ℓ_{τ_1,τ_2} and s_{τ_1,τ_2} as indicated in Theorem 1.1. On Y_3 we have the holomorphic 3-form

$$\Omega(\tau_1, \tau_2, z) \;=\; \Bigl(\frac{z-\tau_1}{\tau_2-\tau_1}\Bigr)^{1-r_2} \Bigl(\frac{\tau_2-z}{\tau_2-\tau_1}\Bigr)^{1-r_1} \frac{dz\, d\tau_2\, d\tau_1}{e\tau_2-\tau_1}\,. \tag{8.16}$$

One can check that its transformation behavior is such that

$$f_1(\tau-1)\, f_2(\tau_2)\, f_3(z)\, \Omega(\tau_1, \tau_2, z)$$

is Γ-invariant for the diagonal action of Γ on $\mathfrak{H}^3$. Up to the factor

$$\frac{(-2i)^{r_3}}{\mathsf{B}(2-r_1, 2-r_2)}$$

the triple integral $\mathbf{T}(f_1, f_2, f_3)$ is given by a specific 3-dimensional cycle in Y_3. Cutting up the cycle and using the Γ-invariance we can give alternative formulations of the triple integral.

8.3.2. Triple integral expressed in Fourier coefficients

We turn to the question whether $\mathbf{T}(f_1, f_2, f_3)$ in Theorem 1.1 might be zero for all choices of the modular forms f_j. The expression (1.13) for the triple integral is fairly explicit, but for numerical purposes the following version is more convenient.

Proposition 8.5. *Let $r_1, r_2, r_3, p_1, p_2, p_3 \in \mathbb{R}$ satisfy*

$$\begin{aligned} &r_1 < 2, \quad 0 < r_2 < 2, \quad r_3 > 0 \quad r_1 + r_2 + r_3 \;=\; 4\,,\\ &p_j \equiv r_j \bmod 2\,, \qquad p_1 + p_2 + p_3 \;=\; 0\,. \end{aligned} \tag{8.17}$$

We consider modular forms $f_1 \in A_{r_1}\bigl(v[p_1]\bigr)$, $f_2 \in S_{r_2}\bigl(v[p_2]\bigr)$, $f_3 \in M_{r_3}\bigl(v[p_3]\bigr)$ given by their Fourier expansions

$$f_j(z) \;=\; \sum_{m\geq 0} a_j(m)\, e^{2\pi i (m+p_j/12) z}\,. \tag{8.18}$$

Then

$$\begin{aligned} \mathbf{T}(f_1, f_2, f_3) \;=\; &\frac{(-2i)^{r_3}}{\mathsf{B}(2-r_1, 2-r_2)} \sum_{m_1, m_2, m_3 \geq 0} a_1(m_1)\, a_2(m_2)\, a_3(m_3)\\ &\cdot \Psi_{r_1, r_2}\bigl(\tfrac{12m_1+p_1}{12}, \tfrac{12m_2+p_2}{12}, \tfrac{12m_3+p_3}{12}\bigr)\,, \end{aligned} \tag{8.19}$$

where we use

$$\Psi_{r_1,r_2}(\mu_1,\mu_2,\mu_3) = \frac{e^{-2\pi\mu_2-\pi\sqrt{3}(\mu_1+\mu_3)}}{2\pi i}\int_{u=0}^{1} u^{1-r_2}\,(1-u)^{1-r_1}\,\frac{e^{-\pi(2-\sqrt{3})\mu_3 u}}{\mu_2+\mu_3 u} \cdot S\bigl(\pi(\mu_1+(1-u)\mu_3)\bigr)\,du \tag{8.20}$$

$$= \frac{e^{-2\pi\mu_2-\pi\sqrt{3}(\mu_1+\mu_3)}}{2\pi i}\,\mu_3^{1-r_3}\int_{u=0}^{\mu_3} u^{1-r_2}\,(\mu_3-u)^{1-r_1} \cdot \frac{e^{-\pi(2-\sqrt{3})u}}{\mu_2+u}\,S\bigl(\pi(\mu_1+\mu_3-u)\bigr)\,du\,, \tag{8.21}$$

$$S(x) = \frac{\sin x}{x} \qquad \textit{with smooth continuation at } x=0\,. \tag{8.22}$$

Remark. Both versions of the integral for Ψ_{r_1,r_2} are trivially equivalent. Version (8.21) seems slightly simpler for numerical integration. The factor $S(\cdots)$ may oscillate, depending on $\mu_1 \in \mathbb{R}$ and $\mu_3 > 0$.

Proof. The absolute convergence of (1.13) and the uniform absolute convergence of the Fourier expansions of the modular forms f_j in the domains occurring in (1.13) allow us to interchange the order of integration and summation. We compute the resulting integral for a triple of individual Fourier terms. Inserting it gives the triple sum (8.19).

Let $\mu_1 \in \mathbb{R}$, $\mu_2 > 0$, $\mu_3 \geq 0$. We have to consider

$$\int_{\tau_1=\rho-1}^{\rho} e^{2\pi i\mu_1\tau_1}\int_{\tau_2=i}^{\infty} e^{2\pi i\mu_2\tau_2}\int_{u=0}^{1} e^{2\pi i\mu_3(\tau_1+u(\tau_2-\tau_1))} \cdot u^{1-r_2}\,(1-u)^{1-r_1}\,du\,d\tau_2\,d\tau_1\,.$$

The absolute convergence allows us to carry out the integration of τ_2 and τ_1 first. That brings us to the integral

$$= \frac{e^{-\pi(2\mu_2+\sqrt{3}(\mu_1+\mu_3))}}{2\pi i}\int_{u=0}^{1} u^{1-r_2}\,(1-u)^{1-r_1}\,\frac{e^{-\pi(2-\sqrt{3})\mu_3 u}}{\mu_2+u\mu_3} \cdot \frac{\sin\pi(\mu_1+(1-u)\mu_3)}{\pi(\mu_1+(1-u)\mu_3)}\,du\,. \tag{8.23}$$

Except for a change of variables, we do not see a way for further analytic treatment of this integral. $\square$

Table 1
Computations of $\mathbf{T}(f_1, f_2, f_2)$, without the factor in front of the integral in (8.19), and the factor $(2\pi i)^{-1}$ in (8.21).

r_1	$r_2 = .2$	$r_2 = .6$	$r_2 = 1.3$	$r_2 = 1.8$
$-.3$	7.911485			
$-.7$	3.983793	3.811007		
-1.1	2.530185	2.706137	5.777819	
-1.5	1.784794	2.070295	5.008281	
-2.4	0.993019	1.313467	3.934317	22.868919

8.4. Numerical approach

Choice of modular forms. For a numerical computation we consider $f_1 = \eta^{2r_1}$, $f_2 = \eta^{2r_2}$, and $f_3 = E_4\,\eta^{-2(r_1+r_2)}$, with $r_1 < -r_2$ and $0 < r_2 < 2$. In this situation we can apply Proposition 8.5.

Approach to compute the triple integral. We used `GP/Pari` [16] for the computation.

The Fourier coefficients of E_4 are known in terms of divisor sums, and the Fourier expansion of powers of the Dedekind eta-function has the form

$$\eta^{2r}(z) \;=\; \sum_{m\geq 0} p_m(r)\, e^{2\pi i(m+r/12)z}\,, \tag{8.24}$$

with polynomials p_m of degree m in $\mathbb{Q}[r]$, which can be symbolically computed.

For given (r_1, r_2) we start with a computation of the three lists of Fourier coefficients up to a given order, and store them for use later on. Then we have to compute the terms in the triple series in (8.19). For the evaluation of Φ_{r_1,r_2} we use the routine `intnum` of Pari. The integrand may have problematic behavior at the end points $u = 0$ and $u = 1$. This can be indicated in the arguments of `intnum`. We can prescribe a desired precision.

Actual computations. The Pari computations of Φ_{r_1,r_2} give consistent results under increase of the precision. We give in Table 1 a number of computations for f_1, f_2, f_3 as indicated at the start of this subsection.

These results give evidence that *the triple integral is non-zero, and the cup product* $\mathrm{cp}(f_1, f_2)$ *is non-trivial at least in some cases.*

9. Universal covering group and principal series representation

In the previous sections we work with the discrete subgroup $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{R})$, and its action on spaces of holomorphic functions. Some of the results that we need to prove can be more conveniently considered in terms of the universal covering group of $\mathrm{SL}_2(\mathbb{R})$ and its principal series representations. See the Appendix in [3] for a further discussion.

9.1. The universal covering group

The universal covering group $\tilde{G}$ of $G = \mathrm{SL}_2(\mathbb{R})$ can be obtained from the Iwasawa decomposition $G = PK$ of $G = \mathrm{SL}_2(\mathbb{R})$, where $K = \mathrm{SO}(2)$ and P consists of all upper triangular matrices in G. This gives a description of the analytic variety G as the product of a simply connected space $P \cong \mathfrak{H}$, and the space $\mathrm{SO}(2) \cong S^1$. The circle S^1 has as simply connected covering the line $\mathbb{R}$. This gives $\tilde{G}$ as the space $\mathfrak{H} \times \mathbb{R}$. The group operations of G can be lifted in a unique way to $\tilde{G}$. In this way we get a central extension

$$0 \to \mathbb{Z} \to \tilde{G} \to G \to 1\,.$$

The center $\tilde{Z}$ of $\tilde{G}$ is isomorphic to $\mathbb{Z}$ and covers the center $\{1, -1\}$ of G. In [3, (A.2)] a useful section $g \mapsto \tilde{g}$ of the homomorphism $\tilde{G} \to G$ is indicated. We note that it cannot be a group homomorphism.

The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is covered by a discrete subgroup $\tilde{\Gamma}$ of $\tilde{G}$, generated by two elements $\tilde{t} = \widetilde{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)}$ and $\tilde{s} = \widetilde{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}$. The relations between $\tilde{t}$ and $\tilde{s}$ are generated by $\tilde{t}\tilde{s}^2 = \tilde{s}^2\tilde{t}$ and $\tilde{t}\tilde{s}\tilde{t}\tilde{s}\tilde{t} = \tilde{s}$. All multiplier systems $v[p]$ correspond to a character χ_p of $\tilde{\Gamma}$. See [3, (A.10)].

Characters of $\tilde{\Gamma}$ correspond to multiplier systems. See [3, (A.10)]. Let ρ be a representation of $\tilde{G}$ in some vector space V. One says that this representation has a *central character* if the center acts by $\tilde{s}^2 \mapsto e^{-\pi i q}$ for some $q \in \mathbb{C}$. The character χ_p of $\tilde{\Gamma}$ corresponding to the multiplier system $v[p]$ also satisfies $\chi_p : \tilde{s}^2 \mapsto e^{-\pi i p}$. Then $\chi_p^{-1} \otimes \rho$ is a representation of $\tilde{\Gamma}$ that is trivial on $\tilde{Z}$. Hence it induces a representation of Γ. We will see in Proposition 9.3 that the representations ${}^{+}\mathcal{D}_{v,r}^{-\omega}$ and ${}^{-}\mathcal{D}_{v^{-1},r}^{\omega}$ are of this form. So it is useful to understand some representations of $\tilde{\Gamma}$ that have a central character.

9.2. Principal series representation

In [3, §A.2] *principal series representations* of the universal covering group $\tilde{G}$ are discussed. We use realizations depending on parameters for $s, p \in \mathbb{C}$.

$$\mathcal{V}^{\omega}(s,p) \;\subset\; \mathcal{V}^{\infty}(s,p) \;\subset\; \mathcal{V}^{-\infty}(s,p) \;\subset\; \mathcal{V}^{-\omega}(s,p)\,. \tag{9.1}$$

The space $\mathcal{V}^{\omega}(s,p)$ consists of holomorphic functions on some neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$, or equivalently of the real-analytic functions on $\mathbb{P}^1_{\mathbb{R}}$, so that the space of the representation $\mathcal{V}^{\omega}(s,p)$ is the space $C^{\omega}(\mathbb{P}^1_{\mathbb{R}})$ of real-analytic functions on $\mathbb{P}^1_{\mathbb{R}}$, the space $\mathcal{V}^{\infty}(s,p)$ is $C^{\infty}(\mathbb{P}^1_{\mathbb{R}})$, the space $\mathcal{V}^{-\infty}(s,p)$ is the space of distributions on $\mathbb{P}^1_{\mathbb{R}}$, and the space $\mathcal{V}^{-\omega}(s,p)$ is the space of hyperfunctions. Hyperfunctions on $\mathbb{P}^1_{\mathbb{R}}$ are represented by holomorphic functions on $U \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ where U is some neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. Two representatives h_1 on $U_1 \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ and h_2 on $U_2 \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ are equivalent if $f_1 - f_2$ on $(U_1 \cap U_2) \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ is the restriction of a holomorphic function on $U_1 \cap U_2$. The natural inclusion in (9.1) sends

$\varphi \in \mathcal{V}^\omega(s,p)$ to the hyperfunction with representative h that is zero on $\mathfrak{H}^-$ and equal to φ on $\mathfrak{H} \cap \mathrm{dom}(\varphi)$.

The action of $\tilde{G}$ is given by the same formulas in all these spaces. It is indicated in [3, (A.23)]. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\arg(ci+d) \in (-\pi,\pi)$ it has the form

$$\begin{aligned} \varphi|^{\mathrm{ps}}_{s,p}\tilde{g}\,(t) \;&:=\; (a-ic)^{p/2-s}\left(\frac{t-i}{t-g^{-1}i}\right)^{s-p/2} \\ &\quad\cdot (a+ic)^{-s-p/2}\left(\frac{t+i}{t-g^{-1}(-i)}\right)^{s+p/2}\varphi(gt)\,. \end{aligned} \tag{9.2}$$

The powers of $\frac{\tau-x_1}{\tau-x_2}$ with x_1 and x_2 both in $\mathfrak{H}$ or both in $\mathfrak{H}^-$ are holomorphic on $\mathbb{P}^1_{\mathbb{C}}$ minus a path from x_1 to x_2. They are determined by the choice of $\arg\frac{\tau-x_1}{\tau-x_2} = 0$ at $\tau = \infty$.

The central character is determined by $\tilde{s}^2 \mapsto e^{-\pi i p}$. Since each element of $\tilde{G}$ can be written as the product of an integral power of $\tilde{s}^2$ and an element $\tilde{g}$, we have a complete description of the action.

9.2.1. Disk coordinates

The spaces in (9.1) have an alternative characterization in *disk coordinates* in the variable $w = \frac{z-i}{z+i}$ on $\mathbb{P}^1_{\mathbb{C}}$. To $|w| = 1$ corresponds the real projective line $\mathbb{P}^1_{\mathbb{R}}$, the upper half-plane is determined by $|w| < 1$, and the lower half-plane by $|w| > 1$ (including $w = \infty$, which corresponds to $z = -i$).

Elements of $\varphi \in \mathcal{V}^\omega(s,p)$ have a *polar expansion*

$$\varphi[w] \;=\; \sum_{n\in\mathbb{Z}} c_n\, w^n\,, \tag{9.3}$$

with $c_n = \mathrm{O}\big((1+\varepsilon)^{-|n|}\big)$ for some $\varepsilon > 0$. For $\mathcal{V}^\infty(s,p)$ the condition is weaker: $c_n = \mathrm{O}\big((1+|n|)^{-A}\big)$ for all $A \geq 0$. The distributions in $\mathcal{V}^{-\infty}(s,p)$ have a polar expansion with coefficients satisfying $d_n = \mathrm{O}\big((1+|n|)^B\big)$ for some $B > 0$. This gives a duality between $C^\infty(\mathbb{P}^1_{\mathbb{R}})$ and the space $C^{-\infty}(\mathbb{P}^1_{\mathbb{R}})$ of distributions by the bilinear form

$$\Big\langle \sum_n d_n w^n, \sum_n c_n w^n \Big\rangle \;=\; \sum_n d_n\, c_{-n}\,. \tag{9.4}$$

In this way each distribution determines a continuous linear form on $C^\infty(\mathbb{P}^1_{\mathbb{R}})$ for the natural topology given by all supremum norms of $h \in \mathbb{P}^1_{\mathbb{R}}$ of the derivatives $\big(\frac{1}{w}\partial_w\big)^k h$ for $k \geq 0$.

A representative of a hyperfunction is given by convergent Laurent series

$$\begin{cases} \sum_{n\in\mathbb{Z}} c_n^+ w^n & \text{for } 1-\varepsilon < |w| < 1\,, \\ -\sum_{n\in\mathbb{Z}} c_n^-\, w^n & \text{for } 1 < |w| < 1+\varepsilon\,. \end{cases} \tag{9.5}$$

One can go over to the representative

$$\begin{aligned}&\begin{cases}\frac{c_0}{2}+\sum_{n\geq 1} c_n\, w^n & \text{for } |w|<1\,,\\ -\frac{c_0}{2}-\sum_{n\leq -1} c_n\, w^n & \text{for } |w|>1\,,\end{cases}\\ &c_n \;=\; c_n^+ + c_n^-\,.\end{aligned} \tag{9.6}$$

This is the unique representative f that is holomorphic on $\mathfrak{H}\cup\mathfrak{H}^-$ and satisfies $f(i)+f(-i)=0$.

9.2.2. Duality

Let h, respectively f, represent an element of $\mathcal{V}^{-\omega}(s,p)$, respectively $\mathcal{V}^{\omega}(1-s,-p)$. Denote $\tilde h(w)=h\bigl(\frac{z-i}{z+i}\bigr)$ and $\tilde f(w)=f\bigl(\frac{z-i}{z+i}\bigr)$ the corresponding functions in disk coordinates. Following [4, §2.1] we may consider

$$\begin{aligned}\langle h,f\rangle \;&=\; \frac{1}{2\pi i}\Bigl(\int\limits_{|w|=c} - \int\limits_{w=c^{-1}}\Bigr)\tilde h(w)\,\tilde f(w)\,\frac{dw}{w}\\ &=\frac{1}{\pi}\Bigl(\int\limits_{z\in C_+} - \int\limits_{z\in C_-}\Bigr) h(z)\, f(z)\,\frac{dz}{z^2+1}\,.\end{aligned} \tag{9.7}$$

The constant $c\in(0,1)$ is such that $\tilde f$ is holomorphic on the region $c\leq |w|\leq c^{-1}$ and $\tilde h$ is holomorphic on the regions $c\leq|w|<1$ and $1<|w|\leq c^{-1}$. This corresponds to contours $C_\pm$ in $\mathfrak{H}^\pm$ such that f is holomorphic on the region in $\mathbb{P}^1_{\mathbb{C}}$ between C_+ and C_-, the contours included, and h is holomorphic on this region with $\mathbb{P}^1_{\mathbb{R}}$ excluded. In this way the choice of the contour does not influence the value of $\langle h,f\rangle$. Moreover, if we replace h by another representative of the same hyperfunction we get the same value, so we have obtained a duality between $\mathcal{V}^{-\omega}(s,p)$ and $\mathcal{V}^{\omega}(1-s,-p)$. One may check that if we take $h\in\mathcal{V}^{-\infty}(s,p)$ then we get the duality in (9.4). Comparison with (9.2) shows that for all $\alpha\in\mathcal{V}^{-\omega}(s,p)$, $f\in\mathcal{V}^{\omega}(1-s,-p)$

$$\bigl\langle \alpha|^{\mathrm{ps}}_{s,p} g, f|^{\mathrm{ps}}_{1-s,-p} g\bigr\rangle \;=\; \langle\alpha,f\rangle \qquad \text{for all } g\in\tilde G\,. \tag{9.8}$$

In Theorem 4.1 we used brackets $[\cdot,\cdot]_r$ for a bilinear Γ-invariant duality between ${}^+\mathcal{D}^{-\omega}_{v,r}$ and ${}^-\mathcal{D}^{\omega}_{v^{-1},r}$. Here we use the brackets $\langle\cdot,\cdot\rangle$ for a more general duality between hyperfunctions and analytic vectors. It is invariant for all principal series actions of $\tilde G$. In §9.3 we will derive the specialized bilinear form $[\cdot,\cdot]_r$ from the general duality $\langle\cdot,\cdot\rangle$.

9.2.3. Identifications

One can relate the Γ-modules ${}^-\mathcal{D}^{\pm\omega}_{v^{-1},r}$ and ${}^+\mathcal{D}^{\pm\omega}_{v,r}$ to submodules of principal series representations of $\tilde G$.

Definition 9.1. Let $r \in \mathbb{R}$.

a) ${}^{+}\mathcal{D}^{-\omega}(r/2, r) \subset \mathcal{V}^{-\omega}(r/2, r)$ consists of the hyperfunctions with a representative that is holomorphic on $\mathfrak{H}$ and zero on $\mathfrak{H}^{-}$.
b) ${}^{-}\mathcal{D}^{-\omega}(r/2, -r) \subset \mathcal{V}^{-\omega}(r/2, -r)$ consists of the hyperfunctions with a representative that is holomorphic on $\mathfrak{H}^{-}$ and zero on $\mathfrak{H}$.

Application of (9.2) to these representatives gives functions of the same type. So the subspaces ${}^{+}\mathcal{D}^{-\omega}(r/2, r)$ of $\mathcal{V}^{-\omega}(r/2, r)$ and ${}^{-}\mathcal{D}^{-\omega}(r/2, -r)$ of $\mathcal{V}^{-\omega}(r/2, -2)$ are in fact submodules for these choices of the parameters (s, p) in the principal series representations. To check it we use that

$$G_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \ : \ \arg(ci + d) \in (-\pi, \pi) \right\} \tag{9.9}$$

is an open neighborhood of 1 in $\mathrm{SL}_2(\mathbb{R})$, which is mapped by the lift $g \mapsto \tilde{g}$ to an open neighborhood of 1 in $\tilde{G}$. Hence representations of the connected Lie group $\tilde{G}$ are determined by their behavior on the $\tilde{g}$ with $g \in G_0$, or even g in a small neighborhood of 1 in $\mathrm{SL}_2(\mathbb{R})$.

Definition 9.2. The submodules ${}^{+}\mathcal{D}^{\omega}(r/2, r) \subset \mathcal{V}^{-\omega}(r/2, r)$ and ${}^{-}\mathcal{D}^{\omega}(r/2, -r) \subset \mathcal{V}^{-\omega}(r/2, -r)$ are defined by the condition that the representative h in Definition 9.1 has the property that the restriction of h to $\mathfrak{H}^{\pm}$ extends holomorphically to a neighborhood of $\mathfrak{H}^{\pm} \cup \mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$.

Of course the extension of the restriction to the half-plane on which h is zero is trivial. It is an easy check with (9.2) that this extension property is preserved by the action of $\tilde{G}$.

The six multiplier systems v on Γ for a given real weight r are $v[p]$ in (2.2), where $p \in \mathbb{R}/12\mathbb{Z}$ satisfies $p \equiv r \bmod 2$. These multiplier systems correspond to characters χ of $\tilde{\Gamma}$ by the relation $\chi(\tilde{\gamma}) = v(\gamma)$ for $\gamma \in \Gamma$. The central character of χ is determined by $\chi(s^2) = e^{-\pi i r}$, the same as for $\mathcal{V}(s, p)$. Hence the representation $\chi^{-1} \otimes \mathcal{V}(s, r)$ is a representation of $\tilde{\Gamma}$ that is trivial on the center $\tilde{Z}$ of $\tilde{\Gamma}$. So it is in fact a representation of $\tilde{\Gamma}/\tilde{Z} \cong \mathrm{PSL}_2(\mathbb{Z}) \cong \Gamma/\{1, -1\}$. We can view it as a representation of Γ that is trivial on -1.

Proposition 9.3.

i) *For a multiplier system of Γ for the real weight r, corresponding to the character χ of $\tilde{\Gamma}$ the following equivalences hold:*

$$\begin{aligned} \chi^{-1} \otimes {}^{+}\mathcal{D}^{-\omega}(r/2, r) &\cong {}^{+}\mathcal{D}^{-\omega}_{v,r} \\ \chi \otimes {}^{-}\mathcal{D}^{-\omega}(r/2, -r) &\cong {}^{-}\mathcal{D}^{-\omega}_{v^{-1},r} \end{aligned} \tag{9.10}$$

ii) *Under these equivalences the submodules obtained by replacing* $-\omega$ *by* ω *are equivalent as well.*
iii) *If* $r \in \mathbb{Z}_{\leq 0}$ *the submodule* ${}^{+}\mathcal{D}^{\mathrm{pol}}_{v,r}$ *corresponds to the submodule of* $\chi^{-1} \otimes {}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r)$ *spanned by the functions* $w \mapsto w^q$ *with integers* q *satisfying* $0 \leq q \leq |r|$.
iv) *If* $r \in \mathbb{Z}_{\leq 0}$ *the submodule* ${}^{-}\mathcal{D}^{\mathrm{pol}}_{v^{-1},r}$ *corresponds to the submodule of* $\chi \otimes {}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r)$ *spanned by the functions* $w \mapsto w^{-q}$ *with integers* q *satisfying* $0 \leq q \leq |r|$.

Proof. Let h^+ be the holomorphic function on $\mathfrak{H}$ that together with 0 on $\mathfrak{H}^-$ represents a given element of ${}^{+}\mathcal{D}^{-\omega}(r/2, r)$. The description in (9.2) implies for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0$ that

$$h^+|^{\mathrm{ps}}_{r/2,r}\tilde{g}(z) \;=\; (a+ic)^{-r}\Bigl(\frac{z+i}{z-g^{-1}(-i)}\Bigr)^{r}\, h^+(gz)\,. \tag{9.11}$$

Now put $h(z) = (-i-z)^{-r}\, h^+(z)$.

$$(-i-z)^{-r}\,\bigl(h^+|^{\mathrm{ps}}_{r/2,r}\tilde{g}\bigr)(z) \;=\; (cz+d)^{-r}\, h(gz)\,. \tag{9.12}$$

To check this relation we first take $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ near to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then we can take apart the powers of products and quotients. The resulting formula extends to $g \in G_0$ by analyticity.

In a similar way $f(z) = (i-z)^{-r} f^-(z)$ on $\mathfrak{H}^-$ leads to the other isomorphism in part i).

Part ii) is obtained by checking the definitions, where the holomorphy at ∞ requires a bit of care.

The submodule ${}^{-}\mathcal{D}^{\mathrm{pol}}_{v^{-1},r}$ consists of the polynomial functions of degree at most $|r|$. Some computations on the basis of $f^-(t) = (i-t)^{|r|}\, f(t)$ and $w = \frac{t-i}{t+i}$ show that polynomials in t of degree at most $|r|$ correspond to polynomials in w^{-1} of degree at most $|r|$. $\square$

So the Γ-modules that are of interest in this paper occur in principal series representations of the universal covering group. We note that we have used the functions h^+ and f^- already in the description in (4.3).

9.2.4. Intertwining operators

In the parameters (s, p) of the principal series representations the parameter p is essentially determined modulo 2. There is an algebraic isomorphism $\ell : \mathcal{V}^{-\omega}(s, p) \to \mathcal{V}^{-\omega}(s, p-2)$, given by multiplication by $w = \frac{z-i}{z+i}$. It preserves the subspaces of distribution vectors, smooth vectors and analytic vectors. See [3, (A.24, p. 157].

A more subtle relation exists between $\mathcal{V}^{-\omega}(s, p)$ and $\mathcal{V}^{-\omega}(1-s, p-2)$. For general complex values of s these spaces are isomorphic. For the values of s in which we are interested the isomorphism breaks down, but there is a relation.

Proposition 9.4.

i) *For each* $r \in \mathbb{R}$ *here is an intertwining operator of* $\tilde{G}$*-modules*

$$J_r : \mathcal{V}^\omega(1 - r/2, 2 - r) \to \mathcal{V}^\omega(r/2, -r) \tag{9.13}$$

given by the integral transformation

$$J_r h(z) = \frac{1}{\pi} \int_{\tau \in C} h(\tau) \Big(\frac{2i(z-\tau)}{(\tau+i)(z-i)} \Big)^{-r} \frac{d\tau}{(\tau+i)^2}\,, \tag{9.14}$$

where the positively oriented cycle in $\mathfrak{H}$ *is inside the domain of* h*, encircles* $-i$*, and* z *is outside* C*.*

ii) *If* $r \notin \mathbb{Z}_{\leq 0}$ *the image of* J_r *is the module* ${}^{-}\mathcal{D}^\omega(r/2, -r)$*, and the kernel is* ${}^{+}\mathcal{D}(1 - r/2, 2 - r)$*.*

iii) *If* $r \in \mathbb{Z}_{\leq 0}$ *then the image of* J_r *is the module* ${}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r)$*.*

Proof. Any $h \in \mathcal{V}^\omega(1 - r/2, 2 - r)$ has an expansion $h = \sum_{m\in\mathbb{Z}} c_m\, w^m$ in the disk coordinate $w = \frac{z-i}{z+i}$, where $c_m = \mathrm{O}\big((1+\varepsilon)^{-|m|}\big)$ for some $\varepsilon > 0$.

If we formulate the integral in (9.14) in disk coordinates w and v, with the substitution $\tau = i\frac{1+v}{1-v}$, and insert the expansion we arrive at

$$J_r : \sum_{m\in\mathbb{Z}} c_m w^m \mapsto \sum_{m\leq -1} c_m \frac{(r)_{-m-1}}{(-m-1)!}\, w^{m+1}\,. \tag{9.15}$$

To show that J_r is an intertwining operator we have to check that

$$(J_r h)|^{\mathrm{ps}}_{r/2,-r} g = J_r\big(h|^{\mathrm{ps}}_{1-r/2,2-r} g\big) \qquad \text{for all } g \in \tilde{G}\,.$$

It suffices to do this for $\tilde{g}$ with g in a small neighborhood of 1 in $\mathrm{SL}_2(\mathbb{R})$, for which we do not have to worry about taking apart powers of products and quotients. We leave this computation to the reader. (An alternative would be to look at the Lie algebra action on the weight vectors w^m.) This completes the proof of part i).

In part ii) we have $r \notin \mathbb{Z}_{\leq 0}$ and the statements concerning the kernel and the image of J_r follow from (9.15). For part iii) we have $r = -a$ with $a \in \mathbb{Z}_{\geq 0}$. So $(r)_{-m-1}$ is nonzero if and only if $-m-1$ or $-a + |m| - 2 \leq -1$. This leads to possibly non-zero terms in the image in (9.15) with $-a \leq m + 1 \leq 0$. □

A lift of the intertwining operator J_r in (9.13) is a linear map $\sigma : \mathrm{Im}\, J_r \to \mathcal{V}^\omega(1 - r/2, 2-r)$ such that $J_r \circ \sigma$ is the identity on $\mathrm{Im}\, J_r$. A lift is in general not an intertwining operator. With (9.15) it is not hard to describe a lift.

Proposition 9.5.

i) *For* $r \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ *we define* $\sigma_r\varphi$ *for* $\varphi \in {}^{-}\mathcal{D}^{\omega}(r/2, -r)$

$$\sigma_r\varphi(z) = \frac{1}{\pi}\frac{z+i}{z-i}\int_{\tau\in C}\varphi(\tau)\,{}_2F_1\Big(1,1;r;\frac{(\tau-i)(z+i)}{(\tau+i)(z-i)}\Big)\frac{d\tau}{\tau^2+1}, \tag{9.16}$$

where C *is a positively oriented closed curve in* $\mathbb{P}^1_{\mathbb{C}} \setminus \{i, -i\}$ *homotopic to* $\mathbb{P}^1_{\mathbb{R}}$ *in the domain of* φ*, and where* z *is outside* C*.*
The resulting linear map $\sigma_r : {}^{-}\mathcal{D}^{\omega}(r/2, -r) \to \mathcal{V}^{\omega}(1 - r/2, 2 - r)$ *is a lift of* J_r*.*

ii) *For* $n \in \mathbb{Z} \leq 0$ *we denote by* φ_n *the element of* ${}^{-}\mathcal{D}^{\omega}(r/2, -r)$ *given by* $\varphi_n(w) = w^n$ *(in disk coordinates). Then*

$$\sigma_r\varphi_n = \frac{|n|!}{(r)_{|n|}}\varphi_{n-1} \tag{9.17}$$

for all $r \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$*.*

iii) *For* $r \in \mathbb{Z}_{\leq 0}$ *we define* σ_r *on* ${}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r)$ *by use of* (9.17) *for* $r \leq n \leq 0$ *on the basis elements* φ_n*. The resulting map* $s_r : {}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r) \to \mathcal{V}^{\omega}(1 - r/2, 2 - r)$ *is a lift of* J_r*.*

Proof. Comparison with (9.15) shows that (9.17) in part ii) describes a lift of J_r for $r \notin \mathbb{Z}_{\leq 0}$, and gives part iii) as well.

Insertion of the power series of the hypergeometric function in (9.16) and interchanging the order of summation and integration leads to (9.17). This gives part i). □

Freedom in the section. We have given one section σ_r of J_r with a simple description in the disk coordinate. We can add to σ_r an arbitrary map from ${}^{-}\mathcal{D}^{\omega}(r/2, -r)$, respectively ${}^{-}\mathcal{D}^{\mathrm{pol}}(r/2, -r)$, to ${}^{+}\mathcal{D}^{\omega}(1 - r/2, 2 - r)$ to obtain another section.

We note that for all $g \in \tilde{G}$ and $\varphi \in {}^{-}\mathcal{D}^{\omega}(r/2, -r)$

$$\big(\sigma_r\varphi\big)|^{\mathrm{ps}}_{1-r/2,2-r}g \quad \text{and} \quad \sigma_r\big(\varphi|^{\mathrm{ps}}_{r/2,-r}g\big) \tag{9.18}$$

have the same image under J_r. Hence their difference is in ${}^{+}\mathcal{D}^{\omega}(1 - r/2, 2 - r)$.

9.3. Duality

We formulate the duality Theorem 4.1 in the context of suitable principal series representations.

Theorem 9.6.

i) *Let* $r \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$*. There is a non-degenerate* $\tilde{G}$*-invariant bilinear form* $[\cdot,\cdot]_r$ *on* ${}^{+}\mathcal{D}^{-\omega}(r/2, r) \times {}^{-}\mathcal{D}^{\omega}(r/2, -r)$ *given by*

$$[\alpha, f^-]_r = \frac{1}{\pi} \int_{z\in C} h^+(z) \left(\sigma_r f^-\right)(z) \frac{dz}{(z+i)^2} . \tag{9.19}$$

Here h^+ is the holomorphic function on $\mathfrak{H}$ that together with the zero function on $\mathfrak{H}^-$ represents the hyperfunction $\alpha \in {}^+\mathcal{D}^{-\omega}(r/2, r)$, and $f^- \in {}^-\mathcal{D}^{\omega}(r/2, -r)$. The cycle C is homotopic to $\mathbb{P}^1_{\mathbb{R}}$ in the intersection of $\mathfrak{H}$ and the domain of $\sigma_r f^-$.
The value of the bilinear form does not change if we add to $\sigma_r f^-$ any holomorphic function on $\mathfrak{H}$.

ii) *Let $r \in \mathbb{Z}_{\leq 0}$. The integral in (9.19), now with $f \in {}^-\mathcal{D}^{\mathrm{pol}}(r/2, -r)$ defines a $\tilde{G}$-invariant bilinear form on ${}^+\mathcal{D}^{-\omega}(r/2, r) \times {}^-\mathcal{D}^{\mathrm{pol}}(r/2, -r)$. The bilinear form is non-degenerate when restricted to ${}^+\mathcal{D}^{\mathrm{pol}}(r/2, r) \times {}^-\mathcal{D}^{\mathrm{pol}}(r/2, -r)$.*

We first show how this theorem implies Theorem 4.1, and next prove the present version.

Proof of Theorem 4.1. Parts i) and iii) are identical to part i) in Theorem 4.1 with use of the identifications in Proposition 9.3.

For part ii) we note that if $h \in {}^+\mathcal{D}^{-\infty}_{v,r}$ then the coefficients in the polar expansion (9.3) of h^+ satisfy $c_n \ll (1+n)^B$ for some $B > 0$. The natural topology on ${}^-\mathcal{D}^{\infty}_{v^{-1},r}$ given by the supremum norms of the derivatives with respect to $-\cot t$ for $t \in \mathbb{P}^1_{\mathbb{R}}$ can also be described by the seminorms $\sum_{m\leq 0} d_m w^m \mapsto \sup_{m\leq 0} |d_m|\, m^A$ for all $A > 0$. A computation shows that

$$\Big[\sum_{n\geq 0} c_n\, w^n, \sum_{m\leq 0} d_m\, w^{-m}\Big]_r = \sum_{n\geq 0} \frac{n!}{(r)_n}\, c_n d_m . \tag{9.20}$$

This gives part iv) of the theorem, and shows the linear form $f \mapsto [h, f]_r$ extends continuously to ${}^-\mathcal{D}^{\infty}_{v^{-1},r}$. □

Proof of Theorem 9.6. We start with the duality in (9.8), in the situation

$$\langle\cdot,\cdot\rangle : \mathcal{V}^{-\omega}(r/2, r-2) \times \mathcal{V}^{\omega}(1-r/2, 2-r) \to \mathbb{C} .$$

There is an algebraic isomorphism $\ell : \mathcal{V}^{-\omega}(s, p) \to \mathcal{V}^{-\omega}(s, p-2)$, given by multiplication by $w = \frac{z-i}{z+i}$. It preserves the subspaces of distribution vectors, smooth vectors and analytic vectors. See the characterization in terms of the polar expansion in §9.2.1. Applying it to ${}^+\mathcal{D}^{-\omega}(r/2, r) \subset \mathcal{V}^{-\omega}(r/2, r)$ we get a submodule of $\mathcal{V}^{-\omega}(r/2, r-2)$. With use of the expansion in the disk coordinate w this submodule can be checked to be orthogonal to ${}^+\mathcal{D}^{\omega}(1-r/2, 2-r)$. So we get a $\tilde{G}$-invariant duality

$${}^+\mathcal{D}^{-\omega}(r/2, r) \times \Big(\mathcal{V}^{\omega}(1-r/2, 2-r) \Big/ {}^+\mathcal{D}^{\omega}(1-r/2, 2-r)\Big) \to \mathbb{C} .$$

If $r \notin \mathbb{Z}_{\leq 0}$ the section σ_r of the intertwining operator J_r in Proposition 9.5 induces an isomorphism

$$ {}^{-}\mathcal{D}^{\omega}(r/2,-r) \cong \Big(\mathcal{V}^{\omega}(1-r/2,2-r) \Big/ {}^{+}\mathcal{D}^{\omega}(1-r/2,2-r) \Big) \,. $$

Combining this we obtain for $\alpha \in {}^{+}\mathcal{D}^{-\omega}(r/2,r)$ and $\varphi \in {}^{-}\mathcal{D}^{\omega}(r/2,-r)$

$$ [\alpha,\varphi]_r \;=\; \langle \ell\alpha, \sigma_r\varphi \rangle \,. \tag{9.21} $$

Since $\alpha \in {}^{+}\mathcal{D}^{-\omega}(r/2,r)$, it has a representative given by a holomorphic function ψ on $\mathfrak{H}$ and by 0 on $\mathfrak{H}^-$. The hyperfunction $\ell\alpha$ is represented by $z \mapsto \frac{z-i}{z+i}\, h^+(z)$.

To see that the bilinear form is non-degenerate we note that in terms of $w = \frac{z-i}{z+i}$ we have

$$ \langle w^a, w^{-b} \rangle \;=\; \frac{a!}{(r)_a}\, \delta_{a,b} \,. \qquad \square \tag{9.22} $$

If $r \in \mathbb{Z}_{\leq 0}$, the lift σ_r is defined only on ${}^{-}\mathcal{D}^{\mathrm{pol}}(r/2,-r)$. The bilinear form is defined on ${}^{+}\mathcal{D}^{-\omega}(r/2,r) \times {}^{-}\mathcal{D}^{\mathrm{pol}}(r/2,-r)$. In (9.21) we see that it is non-degenerate on ${}^{+}\mathcal{D}^{\mathrm{pol}}(r/2,r) \times {}^{-}\mathcal{D}^{\mathrm{pol}}(r/2,-r)$.

9.4. Variants of the integral in the duality theorem

In the discussion in §8.2 of the triple integral we need an alternative way to describe the integrals in the duality theorem. The proofs are easiest in the context of the universal covering group $\tilde{G}$.

The function

$$ \kappa(\tau_1,\tau_2;\tau) \;=\; \Big(\frac{\tau_1-\tau}{i-\tau}\Big)^{r_1-2} \Big(\frac{\tau_2-\tau}{i-\tau}\Big)^{r_2-2} \tag{9.23} $$

is holomorphic on $\mathfrak{H}^3$ minus the set for which τ is on the union of geodesic segment $s_{i,\tau_1} \cup s_{i,\tau_2}$. The function $\tau \mapsto \kappa(\tau_1,\tau_2,\tau)$ extends holomorphically to $\mathbb{P}^1_{\mathbb{R}}$ and $\mathfrak{H}^-$, and determines an element of ${}^{-}\mathcal{D}^{\omega}(r_3/2,-r_3)$.

For functions holomorphic on the product of $\mathfrak{H}^2$ and some neighborhood of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$ we define two actions of $\tilde{G}$:

$$ \begin{aligned} |^{+}_{r_1,r_2,r_3} g \;&=\; |_{2-r_1} g \otimes |_{2-r_2} g \otimes |^{\mathrm{ps}}_{1-r_3/2,2-r_3} g \,, \\ |^{-}_{r_1,r_2,r_3} g \;&=\; |_{2-r_1} g \otimes |_{2-r_2} g \otimes |^{\mathrm{ps}}_{r_3/2,-r_3} g \,. \end{aligned} \tag{9.24} $$

The action $|^{\mathrm{ps}}_{s,p}$ in the third factor of the tensor product is given in (9.2). We define the other actions by taking for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ near 1 in $\mathrm{SL}_2(\mathbb{R})$

$$ h|_{2-r_1}\tilde{M} \otimes |_{2-r_2}\tilde{M}(\tau_1,\tau_2,z) \;=\; (c\tau_1+d)^{r_1-2}\,(c\tau_2+d)^{r_2-2}\, h(M\tau_1, M\tau_2, z) \,. $$

This determines an action of the universal covering group.

Lemma 9.7. *Let $r_1, r_2 \in \mathbb{R}$, and put $r_3 = 4 - r_1 - r_2$.*

i) $\kappa|^-_{2-r_1,2-r_2,r_3} g = \kappa$ *for all* $g \in \tilde{G}$ *and all* $r_3 \in \mathbb{R}$.
ii) *For given* $\tau_1, \tau_2 \in \mathfrak{H}^2$, $r_3 \notin \mathbb{Z}_{\le 0}$ *and* $g \in \tilde{G}$

$$(\sigma_{r_3}\kappa)|^+_{2-r_1,2-r_2,r_3} g\,(\tau_1,\tau_2,z) - \sigma_{r_3}\kappa(\tau_1,\tau_2,z)$$

extends as a holomorphic function of z on $\mathfrak{H}$.

Proof. Part i) is a consequence of the properties of Eichler kernels. It can be checked for $g = \tilde{M}$ near to 1 by a direct computation. Since $\tilde{G}$ is connected this implies the invariance for all $g \in \tilde{G}$.

For part ii) we use Proposition 9.5. It gives the operator σ_{r_3} as a lift of the intertwining operator J_{r_3}. Hence for each $h \in {}^-\mathcal{D}^\omega(r_3/2, -r_3)$ and each $g \in \tilde{G}$

$$\begin{aligned} J_{r_3}\Big((\sigma_{r_3}h)|^{\mathrm{ps}}_{1-r_3/2,2-r_3} g\Big) \;&=\; \big(J_{r_3}\sigma_{r_3}h\big)\,|^{\mathrm{ps}}_{r_3/2,-r_3}\, g \;=\; h|^{\mathrm{ps}}_{r_3/2,-r_3} g\\ &=\; J_{r_3}\Big(\sigma_{r_3}\big(h|_{r_3/2,-r_3} g\big)\Big)\,. \end{aligned}$$

So $(\sigma_{r_3}h)|^{\mathrm{ps}}_{1-r_3/2,2-r_3} g - \sigma_{r_3}\big(h|_{r_3/2,-r_3} g\big)$ is in the kernel of J_{r_3}, hence holomorphic on $\mathfrak{H}$. The lift σ_{r_3} concerns only the third coordinate. Modulo functions of z extending holomorphically to $\mathfrak{H}$ we have

$$\begin{aligned} (\sigma_{r_3}\kappa)|_{2-r_1} g \otimes |_{2-r_2} g \otimes |^{\mathrm{ps}}_{1-r_3/2,1-r_3} g \;&=\; \sigma_{r_3}\big(\kappa(|_{2-r_1} g \otimes |_{2-r_2} g)\big)|^{\mathrm{ps}}_{1-r_3/2,2-r_3} g\\ \equiv\; \sigma_{r_3}\big(\kappa(|_{2-r_1} g \otimes g_{2-r_2} \otimes g^{\mathrm{ps}}_{1-r_3/2,2-r_3} g)\big) \;&=\; \sigma_{r_3}\kappa\,, \end{aligned} \tag{9.25}$$

with use in the last step of the invariance of κ given in part i). $\square$

Lemma 9.8. *For $p \ge 1$, $r_3 = 4 - r_1 - r_2 \notin \mathbb{Z}_{\le 0}$*

$$\sigma_{r_3}\kappa(i, ip; \cdot)(z) \;=\; 2^{2-r_2}\,(p+1)^{r_2-2}\,\frac{z+i}{z-i}\;{}_2F_1\Big(1, 2-r_2; r_3; \frac{(p-1)(z+i)}{(p+1)(z-i)}\Big)\,, \tag{9.26}$$

for z outside a curve from i to ip.

Proof. We use disk coordinates $w = \frac{\tau-i}{\tau+i}$, $u = \frac{z-i}{z+i}$ and $v = \frac{ip-i}{ip+i} = \frac{p-1}{p+1}$. We have $v \in (0,1)$ and should take C_1 corresponding to $|w| = c_1$ with $v < c_1 < 1$. Then $\sigma_{r_3}\kappa(i, ip; \cdot)(z)$ is applicable for $|u| > c_1$.

We have

$$\sigma_{r_3}\kappa(i, ip; \cdot)(z) \;=\; \frac{1}{\pi}\frac{1}{u}\int\limits_{|w|=c_1} 1^{r_1-2}\,\Big(\frac{1-v/w}{1-v}\Big)^{r_2-2}\;{}_2F_1\big(1, 1; r_3; w/u\big)\,\frac{1}{2i}\,\frac{dw}{w}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\,\frac{1}{u}(1-v)^{2-r_2} \int_{|w|=c_1} \sum_{n\geq 0} \frac{(2-r_2)_n}{n!}\,\frac{v^n}{w^n} \sum_{m\geq 0} \frac{m!}{(r_3)_m}\,\frac{w^m}{u^m}\,\frac{dw}{w}\\
&= u^{-1}(1-v)^{2-r_2}\,{}_2F_1\big(1,2-r_2;r_3;v/u\big)\\
&= 2^{2-r_2}\,(p+1)^{r_2-2}\,\frac{z+i}{z-i}\,{}_2F_1\Big(1,2-r_2;r_3;\frac{(p-1)(z+i)}{(p+1)(z-i)}\Big). \qquad \square
\end{aligned}
$$

Lemma 9.9. *Let $r_3 = 4-r_1-r_2 \notin \mathbb{Z}_{\leq 0}$, and $\tau_1 \neq \tau_2$ in $\mathfrak{H}$. For $\alpha \in {}^{+}\mathcal{D}^{-\omega}(r_3/2, r_3)$ represented by the holomorphic function h^+ on $\mathfrak{H}$ and zero on $\mathfrak{H}^-$, and $f^-(\tau) = \kappa(\tau_1,\tau_2;\tau)$*

$$
\begin{aligned}
\big[\alpha, f^-\big]_{r_3} &= \frac{(2i)^{2-r_2}}{\pi}\,\big|c\tau_1+d\big|^{2r_1-2}\,(\tau_2-\bar\tau_1)^{r_2-2}\int_{z\in C} h^+(z)\\
&\quad\cdot\Big(\frac{z+i}{z-\bar\tau_1}\Big)^{1-r_3}\,{}_2F_1\Big(1,2-r_2;r_3;\frac{\tau_2-\tau_1}{\tau_2-\bar\tau_1}\,\frac{z-\bar\tau_1}{z-\tau_1}\Big)\,\frac{dz}{(z-\tau_1)(z+i)}\,,
\end{aligned}
\tag{9.27}
$$

where C is a wide closed curve in $\mathfrak{H}$ encircling all singularities of the integrand in $\mathfrak{H}$, and where $g = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and p satisfy $g\tau_1 = i$, $g\tau_2 = ip$, $p \geq 1$, and $\arg(ci+d) \in [0,\pi)$.

Remarks. The matrix g is unique for given different τ_1 and τ_2.

This is a hybrid formula. The right hand side depends directly on τ_1 and τ_2, and indirectly via c and d. The integrand is not holomorphic in τ_1 and τ_2.

Proof. In the integral for $[\alpha, f^-]_{r_3}$ in Theorem 9.6 we can replace $\sigma_{r_3}\kappa(\tau_1,\tau_2;\tau)$ by $(\sigma_{r_3}\kappa)|^{+}_{2-r_1,2-r_2,r_3}\tilde g\,(\tau_1,\tau_2;\tau)$, by Lemma 9.7. The choice of g is such that we can apply the explicit formula in Lemma 9.8. We try to write the result in terms of $\tau_1 = g^{-1}i$ and $\tau_2 = g^{-1}(ip)$ as far as possible. We use

$$
\begin{aligned}
a+ic &= \frac{1}{c\bar\tau_1+d} & \qquad a-ic &= \frac{1}{2\tau_1+d}\\
p+1 &= -i(g\tau_2 - g\bar\tau_1) & \qquad p-1 &= -i(g\tau_2 - g\tau_1)
\end{aligned}
$$

After some computations we arrive at the formula in the lemma. We handle powers of products and quotients by first taking τ_1 and τ_2 near to each other, and hence g near to the unit matrix. We observe that both sides of the equality are real-analytic in τ_1 and τ_2. $\square$

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