UTRECHT UNIVERSITY

GRADUATE SCHOOL OF NATURAL SCIENCES

K-theory with Reality

MASTER'S THESIS

Author:
Jorge BECERRA GARRIDO

Supervisor:
dr. Lennart MEIER

Second reader:
dr. Gijs HEUTS

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A mis padres, Isabel y Pedro,
por haber confiado en mi.

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Introduction

One of the main objectives of Algebraic Topology is to associate algebraic invariants to topological spaces. There are plenty and of many different flavours, but some of them – namely cohomology theories – are closely related to other topological objects called *spectra*. These objects arise in a quite natural way, and one can view them as some generalization of spaces: it is possible to do homotopy theory or cohomology theory of spectra, but furthermore they have better properties than spaces.

In this Master's thesis we aim to describe the equivariant version of the former, namely given a (finite) group G, we want to explore cohomology theories for G-spaces (which encode the equivariant phenomena) and the equivariant version of spectra. Our main example, which gives the title of the thesis, will be K-theory with Reality, a slightly different version for $\mathbb{Z}/2$-spaces of the usual complex K-theory.

The main goal of this thesis is to relate real K-theory and K-theory with Reality at the level of spectra. The standard way to do this is using the *homotopy fixed points* spectral sequence, constructed by Dugger in [8]. However, we will present a different version, quite uncommon in the literature, which avoids the computation of the spectral sequence. Therefore, this thesis also has the purpose to present a self-contained solution to this problem.

Throughout this thesis, we denote by Top the category of **compactly generated** topological spaces (that is, weak Hausdorff k-spaces[1]), so that we have a *convenient* category of spaces. This means that Top is complete and cocomplete, (that is, it has all small limits and colimits), and it is cartesian closed, meaning that the direct product $X \times Y$ and the hom-set $\mathrm{Map}(X, Y)$ can be endowed with compactly generated topologies such that there is a natural homeomorphism

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

This category is large enough to contain CW-complexes, metric spaces, topological manifolds and in general locally compact Hausdorff spaces. We will refer to objects in Top simply as "spaces".

Similarly, we denote by Top_* the category of well-pointed spaces, that is, based spaces (X, x_0) with the property that the inclusion $\{x_0\} \hookrightarrow X$ is a Hurewicz cofibration. The suspension Σ and the loop space Ω preserve well-pointedness and the compactly generated topology (see [5, VII.1.9] and [5, §VII.6], so they define the usual suspension-loop adjunction $\Sigma : \mathsf{Top}_* \rightleftarrows \mathsf{Top}_* : \Omega$. It is also worth mentioning that the smash product $\wedge$ also preserves well-pointedness [40, §5.4].

[1]A topological space X is a **k-space** if the closed subsets $A \subset X$ are precisely those for which for any map $u : K \longrightarrow X$ with K compact Hausdorff, $u^{-1}(A)$ is closed in K. A topological space X is **weak Hausdorff** if $u(K) \subset X$ is closed for any map $u : K \longrightarrow X$ with K compact Hausdorff.

Structure of the Thesis

In chapter 1, we firstly introduce G-spaces from a categorical perpective, where G is a finite group. Then we introduce G-CW-complexes and develop homotopy theory from the equivariant perspective, restating the classical results of Whitehead, cellular approximation, etc. We also construct a model structure in the category of G-spaces and prove Elmendorf's theorem, and at the end of the chapter we present Bredon homology and cohomology.

In chapter 2, we start by reviewing complex and real K-theory, recall their basic properties, lift them to cohomology theories and find classifying spaces for them. In the second part, we introduce K-theory with Reality for $\mathbb{Z}/2$-spaces, and mimic the results from the first part of the chapter. We also find a classifying space for it and find two equivariant versions of Bott periodicity. The standard and only reference where one can find a full description of K-theory with Reality is Atiyah's original paper [1], where he uses a not very desirable notation and mostly deals with relative groups instead of the more usual reduced groups. We present here, we believe, a better exposition of K-theory with Reality, following the spirit of Hatcher's *Vector bundles and K-theory* [14], but also emphasizing the role played by classifying spaces.

In chapter 3, we construct the categories of spectra and orthogonal spectra following the modern perspective of *Model categories of diagram spectra* [24]. We briefly discuss enriched category theory and diagram spaces first, and we define (orthogonal) spectra as the category of modules over some monoid object on $\mathcal{D}$-spaces for suitable diagrams. We also discuss homotopy theory of spectra, and we endow both categories with model structures. At last, we introduce the stable homotopy category, and discuss its most important properties.

In chapter 4, we present G-spectra and orthogonal G-spectra: the analogues of spectra for the equivariant setup. As the main example, we lift K-theory with Reality to a $\mathbb{Z}/2$-spectrum. We also discuss some equivariant homotopy theory for G-spectra and describe a model structure on it, inducing the equivariant stable homotopy category. Later we discuss $RO(G)$-graded cohomology theories, and lift K-theory with Reality to a $RO(\mathbb{Z}/2)$-graded one. At the end, we show the main result of this Master's thesis, stating that the homotopy fixed points of the $\mathbb{Z}/2$-spectrum of K-theory with Reality is isomorphic, in the stable homotopy category, to the spectrum of real K-theory.

Contents

Chapter 1

The category of G-spaces

K-theory with Reality deals with spaces with an involution, that is, spaces with a $\mathbb{Z}/2$-action. In order to construct the general framework of equivariant stable homotopy theory, it will be necessary to discuss in the first place about the general theory of spaces with a group action and lay the foundations of equivariant homotopy theory, that is, how to adapt the well-known homotopy theory of spaces when we introduce the action of a group. For the rest of this text, we let G be a finite group. Some of our results are also true for compact Lie groups, but we will not treat this case.

In §1.1 and §1.2, we mostly use [3] and [31], although all proofs we present are original work of the author. In §1.3 we follow the great exposition of [37], and in §1.4 we make use of [3] and [25]. Concretely, in 1.4.4 we fix a mistake that appears in [3, 1.4.8].

1.1 A categorical setup

We can regard our group G as a category with one object $\bullet$ and $\mathrm{Hom}(\bullet, \bullet) := G$: the identity is given by the neutral element $e \in G$ and composition is defined by the product of G.

Definition. The category of G**-spaces** is the functor category $G\mathsf{Top} := \mathsf{Top}^G$.
Similarly, the category of **based** G**-spaces** is the functor category $G\mathsf{Top}_* := \mathsf{Top}_*^G$.

An alternative description is the following: there is a monad

$$G \times - : \mathsf{Top} \longrightarrow \mathsf{Top}$$

with unit η and multiplication μ given by

$$\begin{aligned} \eta_X \colon X &\longrightarrow G \times X \\ x &\longmapsto (e, x), \end{aligned} \qquad \begin{aligned} \mu_X \colon G \times G \times X &\longrightarrow G \times X \\ (g, g', x) &\longmapsto (gg', x). \end{aligned}$$

For the based case, there is a monad[1]

$$G_+ \wedge - : \mathsf{Top}_* \longrightarrow \mathsf{Top}_*$$

with unit $X \longrightarrow G_+ \wedge X$ and multiplication $(G \times G)_+ \wedge X \longrightarrow G_+ \wedge X$ as above.

Proposition 1.1.1 *The category of G-spaces (resp. based G-spaces) is isomorphic to the category of algebras over the monad $G \times -$ (resp. $G_+ \wedge -$).*
Moreover, $G\mathsf{Top}$ and $G\mathsf{Top}_*$ are both complete and cocomplete.

[1] For a space Y, we write $Y_+ := Y \coprod *$.

Proof. Unravelling definitions, we see that in both cases a G-space consists of a topological space X together with a continuous map $G \times X \longrightarrow X$ satisfying the required associativity and unit conditions; and the morphisms are the G-equivariant continuous maps $X \longrightarrow Y$. The second assertion follows from the fact that if $\mathcal{J}$ is a small category and $\mathcal{C}$ is complete and cocomplete, so is $\mathcal{C}^{\mathcal{J}}$ (see [31, §3.3]). The based case is similar. $\square$

We now define some useful concepts in the theory of G-spaces:

Definition. Let $X : G \longrightarrow \mathsf{Top}$ be a G-space.

(a) The **translation grupoid** of X is the category $T_G X$ whose objects are elements $x \in X$ and whose morphisms are $g : x \longrightarrow y$ for $g \in G$ whenever $gx = y$.

(b) The **isotropy subgroup** or **stabilizer** at $x \in X$ is $I_x := \mathrm{Hom}_{T_G X}(x, x)$

(c) The **orbits** $G \cdot x$ of the group action are the elements of $\mathsf{sk}(T_G X)$, the skeleton[2] of $T_G X$.

(d) The **fixed points** of X is $X^G := \lim X$.

(e) The **orbit space** of X is $X/G := \operatorname{colim} X$.

Of course, these definitions coincide with the usual notions on G-spaces,

$$\begin{aligned} I_x = \{g \in G : gx = x\} \quad &, \quad G \cdot x = \{gx \in X : g \in G\}, \\ X^G = \{x \in X : G \cdot x = \{x\}\} \quad &, \quad X/G = X/\sim, \quad x \sim gx. \end{aligned}$$

Every monad gives rise to an adjunction between a category and the category of algebras over the monad. In our case it takes the form

$$\begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & G\mathsf{Top} \arrow[l, bend left, "U"] \end{tikzcd}$$

where U is the obvious forgetful functor and $FX := G \times X$ with the obvious action $g \cdot (g', x) := (gg', x)$. For the based case this is analogous.

For (based) topological spaces X, Y, we let $\mathrm{Map}(X, Y)$ be the space of (based) continuous maps $X \longrightarrow Y$.

Definition. Let X, Y be (based) G-spaces. We will write $\mathrm{Map}_G(X, Y) \subset \mathrm{Map}(UX, UY)$ for the space of G-equivariant maps $X \longrightarrow Y$, endowed with the subspace topology of $\mathrm{Map}(UX, UY)$.

Moreover, we will write $G\mathrm{Map}(X, Y) := \mathrm{Map}(UX, UY)$ for the G-space of maps $X \longrightarrow Y$, where G acts as

$$(g \cdot f)(x) := g^{-1} f(gx).$$

Lemma 1.1.2 $G\mathrm{Map}(X, Y)^G = \mathrm{Map}_G(X, Y)$.

Proof. A map f is a fixed point of $G\mathrm{Map}(X, Y)$ if and only if $g \cdot f = f$, that is, $g^{-1} f(gx) = f(x)$ for all $x \in X$. This expression is rewritten as $f(gx) = g f(x)$, which means that f is G-equivariant. Both topologies agree by definition. $\square$

[2] A category $\mathcal{C}$ is *skeletal* if there is only one object in every isomorphism class. The **skeleton** $\mathsf{sk}\,\mathcal{C}$ of $\mathcal{C}$ is the unique (up to isomorphism) skeletal category equivalent to $\mathcal{C}$.

Example 1.1.3 Let $\varphi : H \longrightarrow G$ be a group homomorphism. This can be viewed as a functor between categories with only one object, which induces a precomposition functor $\varphi^* : \mathsf{Top}^G = G\mathsf{Top} \longrightarrow H\mathsf{Top} = \mathsf{Top}^H$ that we consider as a **restriction of scalars** functor, by analogy with commutative algebra. The left and right Kan extensions of an H-space define left and right adjoints to the restriction of scalars functor, called **induction** and **coinduction**,

$$\begin{tikzcd} H\mathsf{Top} \arrow[r, bend left=60, "\mathrm{ind}_H^G"] \arrow[r, bend right=60, "\mathrm{coind}_H^G"'] & G\mathsf{Top} \arrow[l, "\varphi^*"'] \end{tikzcd}$$

which are given by the **balanced product**

$$\mathrm{ind}_H^G X := G \times_H X := G \times X/\sim \quad , \quad (gh, x) \sim (g, hx)$$

for all $g \in G, h \in H, x \in X$; and

$$\mathrm{coind}_H^G X := \mathrm{Map}_H(\varphi^* G, X).$$

For the induction $\mathrm{ind}_H^G X$, the group G acts via the left action on G, $g \cdot [(g', x)] := [(gg', x)]$. For the coinduction $\mathrm{coind}_H^G X$, the group G acts as $(g \cdot f)(g') := f(g'g)$. It is routine to check that these are well-defined G-actions. The upshot is that given a G-space X and an H-space Y we have bijections

$$\mathrm{Map}_G(G \times_H Y, X) \cong \mathrm{Map}_H(Y, X), \tag{1.1}$$
$$\mathrm{Map}_H(\varphi^* X, Y) \cong \mathrm{Map}_G(X, \mathrm{Map}_H(G, Y)). \tag{1.2}$$

For based G-spaces, this is rewritten as

$$\mathrm{Map}_G(G_+ \wedge_H Y, X) \cong \mathrm{Map}_H(Y, X), \tag{1.3}$$
$$\mathrm{Map}_H(\varphi^* X, Y) \cong \mathrm{Map}_G(X, \mathrm{Map}_H(G_+, Y)). \tag{1.4}$$

If $G = e$ is the trivial group, then viewed as a category it is the discrete category with one single object; thus the unique group homomorphism $H \longrightarrow e$ induces the unique functor $H \longrightarrow e$. In this case, the left and right Kan extensions along this functor are the colimit and limit of the chosen H-space, respectively (see [31, 6.5.1]), so

$$\mathrm{ind}_H^e X = \mathrm{colim}\, X = X/H \qquad , \qquad \mathrm{coind}_H^e X = \lim X = X^H.$$

In conclusion, rewriting G for the nontrivial group for a more familiar notation, we recover the adjunctions

$$\mathrm{Map}_G(Y, X) \cong \mathrm{Map}(Y, X^G), \tag{1.5}$$
$$\mathrm{Map}_G(X, Y) \cong \mathrm{Map}(X/G, Y), \tag{1.6}$$

where X is a G-space and Y has the trivial G-action, both based and unbased.

Example 1.1.4 In the first place observe that if $H \subset G$ is a subgroup of G, then given a G-space X we obtain an H-space by restriction of scalars, that we will also denote as X, so we can consider the H-fixed points of X.

Now consider the **orbit category** $\mathcal{O}_G$, whose objects are the G-spaces of left cosets G/H, and whose morphisms are G-equivariant maps $G/H \longrightarrow G/K$. It is easy to see that there is a G-map $G/H \longrightarrow G/K$ if and only if H is subconjugate of K, ie, there is $\gamma \in G$ such that $\gamma^{-1}H\gamma \subset K$, and such map must be of the form $[g] \mapsto [g\gamma]$.

Then the claim is that taking fixed points with respect to a subgroup of G is the right Kan extension of X along the embedding $G \hookrightarrow \mathcal{O}_G^{op}$ sending the single object to G/e,

$$\begin{tikzcd} G \arrow[rr, "X"] \arrow[dr, hook] & & \mathsf{Top} \\ & \mathcal{O}_G^{op} \arrow[ur, dashed, "\mathrm{Ran}\ X = X^{(-)}"'] & \end{tikzcd}$$

Indeed, by the explicit description of the right Kan extension [31, 6.2.1], the limit that defines $(\mathrm{Ran}\ X)(G/H)$ is given by the limit of the composite $H \longrightarrow G \xrightarrow{X} \mathsf{Top}$, which is X^H. The upshot is that $(\mathrm{Ran}\ X)(G/H) = X^H$ and a morphism $\gamma : G/H \longrightarrow G/K$ as before is sent to $X^K \longrightarrow X^H$, $x \mapsto \gamma x$.

Lemma 1.1.5 *Let X be a (based) G-space. We have the following homeomorphisms:*

1. $X^H \cong \mathrm{Map}_G(G/H, X)$. *For the based case,* $X^H \cong \mathrm{Map}_G((G/H)_+, X)$.
2. $G \times_H X \cong G/H \times X$. *For the based case,* $G_+ \wedge_H X \cong (G/H)_+ \wedge X$
3. $\mathrm{Map}_H(G, X) \cong G\mathrm{Map}(G/H, X)$. *For the based case,* $\mathrm{Map}_H(G_+, X) \cong G\mathrm{Map}((G/H)_+, X)$.

Proof. 1. Every G-equivariant map $f : G/H \longrightarrow X$ corresponds to the element $f([e])$ in X, which is an H-fixed point since $hf([e]) = f([h]) = f([e])$. Conversely, given an H-fixed point $x \in X^H$, we consider the map $f([g]) := gx$, which is well defined as $f([gh]) = ghx = gx$. Both assignments are obviously continuous and it is straightforward to check that one is inverse of each other.

2. By (1.1), giving a G-map $G \times_H X \longrightarrow G/H \times X$ is the same as giving an H-map $X \longrightarrow G/H \times X$. The assignment $x \mapsto ([e], x)$ has as inverse $([g], x) \mapsto (g, g^{-1}x)$, and as before it is easy to do the usual checkings.

3. This is a consequence of the universal property of the quotient. $\square$

In particular, the previous lemma says that $X^G = \lim X \cong \mathrm{Map}_G(*, X)$. As for the rest of limits and colimits, we also have the notion with the *right* homotopy type (see page 95):

Definition. Let X be a G-space. The **homotopy fixed points** of X is $X^{hG} := \mathrm{holim}\ X$.

We aim to give a more explicit description of the homotopy fixed points of a G-space. Consider the simplicial space

$$E_*G : \Delta^{op} \longrightarrow \mathsf{Top} \qquad , \qquad E_nG := G \times \overset{n+1}{\cdots} \times G,$$

where the face and degeneracy maps are the following: write $g_0|g_1|\cdots|g_n|$ for an element of E_nG, also with the bars indexed from 0 to n. Then the face map d_i "deletes" the i-th bar; and the degeneracy map s_i "inserts" $e|$ after the i-th bar. This space is usually called the **bar construction** of the one-point space associated to the free $\dashv$ forgetful adjunction of G-spaces. Write $EG := |E_*G|$ for its geometric realization. Moverover, the multiplication of G factorwise induces a G-action on EG.

Proposition 1.1.6 *For a G-space X, we have that $X^{hG} \cong \mathrm{Map}_G(EG, X)$.*
In the based case, $X^{hG} \cong \mathrm{Map}_G(EG_+, X)$.

Proof. We will use the explicit description of the homotopy limit given in A.4.4. We again view G as a category with only one object $\bullet$. The first observation is that $EG \cong |N(G/\bullet)|$, where $N(G/\bullet)$ denotes the nerve of the slice category $G/\bullet$. Indeed, $N(G/\bullet)_n$ consists of diagrams

$$\begin{tikzcd} \bullet \arrow[r, "h_1"] \arrow[drr, "g_0"'] & \bullet \arrow[r, "h_2"] \arrow[dr, "g_1"] & \bullet \arrow[r, "h_3"] \arrow[d, "g_2"] & \cdots \arrow[r, "h_n"] & \bullet \arrow[dll, "g_n"] \\ & & \bullet & & \end{tikzcd}$$

where the h_i are determined by the g_i (they must satisfy $g_i = g_{i+1}h_{i+1}$), so $N(G/\bullet)_n$ is in bijection with $E_nG = G \times \overset{n+1}{\cdots} \times G$. In particular, it is straightforward to see that this bijection commutes with the face and degeneracy maps of both simplicial spaces, so $N(G/\bullet) \xrightarrow{\cong} E_*G$ is an isomorphism of simplicial spaces.

Now, by A.4.4, we have that

$$X^{hG} \cong \mathrm{eq}\left(X^{EG} \rightrightarrows \prod_G X^{EG} \right),$$

where for $f : EG \longrightarrow X$, the first arrow is the composite

$$EG \xrightarrow{f} X \xrightarrow{g\cdot} X$$

in the factor indexed by $g \in G$; and the second arrow is the composite

$$EG \xrightarrow{g\cdot} EG \xrightarrow{f} X$$

in the factor indexed by $g \in G$. In other words, X^{hG} is the subspace of X^{EG} of maps $f : EG \longrightarrow X$ such that $fg = gf$, that is, the space of G-equivariant maps $\mathrm{Map}_G(EG, X)$. □

1.2 Equivariant homotopy theory

Now that we have described G-spaces, we now continue to develop homotopy theory on them. The first step, of course, is to specify what a homotopy should be.

Definition. Let $f_0, f_1 : X \longrightarrow Y$ be G-equivariant maps and let G act trivially on $I = [0,1]$. A G-**homotopy** from f_0 to f_1 is a map $H : X \times I \longrightarrow Y$ in $G\mathsf{Top}$. In the based case, this is a map $H : X \wedge I_+ \longrightarrow Y$ in $G\mathsf{Top}_*$.

This is the same, of course, as a map $I \longrightarrow \mathrm{Map}_G(X, Y)$; or alternatively, a G-equivariant map $I \longrightarrow G\mathrm{Map}(X, Y)$ (adding disjoint points in the based case).

A natural question any homotopy theorist would ask is: what are the homotopy groups? Of course, to answer this question one should answer first: what objects play the role of spheres, and therefore disks? The following will lead us to the right answer:

Definition. Let A be a G-space. A G**-CW-complex** relative to A is the colimit of a sequence of G-spaces X_n, where X_n arises from X_{n-1} as a pushout

$$\begin{tikzcd} \coprod_{H\subseteq G} J_n^H \times G/H \times \partial D^n \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d] \\ \coprod_{H\subseteq G} J_n^H \times G/H \times D^n \arrow[r] & X_n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $A = X_{-1}$, H varies over all subgroups of G, J_n^H is a discrete space with trivial action and ∂D^n and D^n also have the trivial action.

In the based case, maps are pointed and the space $\coprod_{H\subseteq G} J_n^H \times G/H \times \partial D^n$ should be replaced by $\bigvee_{H\subseteq G}(J_n^H)_+ \wedge (G/H)_+ \wedge \partial D^n$

Examples 1.2.1 Let us now discuss some $\mathbb{Z}/2$-CW-structures for the sphere S^2 endowed with three different $\mathbb{Z}/2$-actions:

(a) If S^2 has the trivial $\mathbb{Z}/2$-action, then all points are fixed and a $\mathbb{Z}/2$-CW-structure is given by one 0-cell $(\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0$ and one 2-cell $(\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^2$, where the attaching map collapses ∂D^2 to D^0, as in the nonequivariant case.

(b) Now consider S^2 with the $\mathbb{Z}/2$-action given by rotation of π with respect to the z axis. The north and south pole, that we denote by n and s, are fixed points, so there are two 0-cells $(\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0$. We also have one 1-cell $(\mathbb{Z}/2)/e \times D^1$ attached by mapping $(\overline{0}, 0)$, $(\overline{1}, 0)$ to n and $(\overline{0}, 1)$, $(\overline{1}, 1)$ to s. Finally, there is one 2-cell $(\mathbb{Z}/2)/e \times D^2$ attached as follows: parametrize the 1-skeleton (homeomorphic to $S^1 \subset \mathbb{C}$ as $\mathbb{Z}/2$-space) with $\theta \in [0, 2\pi]$, where the parameter runs through the copy indexed by $\overline{0}$ from n to s in the first half of the time and the copy indexed by $\overline{1}$ from s to n in the second half of the time. In a similar fashion, ∂D^2 is also parametrized by θ. Then the attaching map $\mathbb{Z}/2 \times \partial D^2 \longrightarrow S^1$ sends

$$(\overline{0}, \theta) \mapsto \theta \qquad , \qquad (\overline{1}, \theta) \mapsto 2\pi - \theta.$$

(c) Consider in third place the sphere S^2 with the antipodal action, so this time there are not fixed points. There is a $\mathbb{Z}/2$-CW-structure with one 0-cell $(\mathbb{Z}/2)/e \times D^0$, one 1-cell $(\mathbb{Z}/2)/e \times D^1$ and one 2-cell $(\mathbb{Z}/2)/e \times D^2$. The north and south poles correspond, of course, to the 0-cells. The 1-cell is attached by sending $(\overline{0}, 0)$, $(\overline{1}, 1)$ to n and $(\overline{0}, 1)$, $(\overline{1}, 0)$ to s. At last, the 2-cell is attached by a map $\mathbb{Z}/2 \times \partial D^2 \longrightarrow S^1$ (where S^1 now has the antipodal action) mapping

$$(\overline{0}, \theta) \mapsto \theta \qquad , \qquad (\overline{1}, \theta) \mapsto \begin{cases} \theta + \pi, & \theta \in [0, \pi], \\ \theta - \pi, & \theta \in [\pi, 2\pi]. \end{cases}$$

Examples 1.2.2 For the sake of exposition, let us allow G to be a compact Lie group for one moment and let us describe some S^1-CW-structures on a couple of S^1-spaces. This is useful to note that for groups not necessarily finite, the topological dimension does not agree with the highest dimension of the attached cells:

(a) Consider the sphere S^2 with the S^1-action given by rotation along the z axis. As before there are two fixed points n and s, which correspond with two 0-cells $S^1/S^1 \times D^0$. There is also one 1-cell $S^1/e \times D^1$ attached to the endpoints in the obvious way.

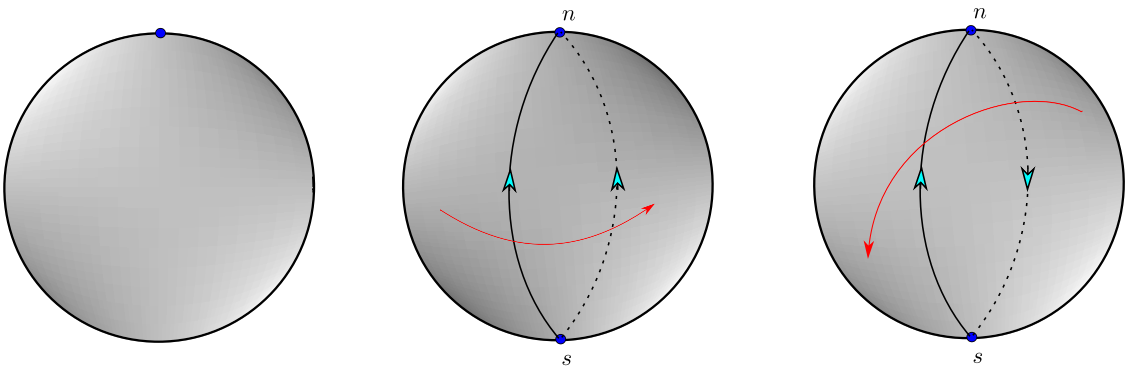

Figure 1.1: $\mathbb{Z}/2$-CW-structures of Example 1.2.1.

(b) Consider the torus $\mathbb{T} = S^1 \times S^1$ with the action given by multiplication on the first factor, so in particular there are not fixed points. A possible S^1-CW-structure is given by one 0-cell $S^1/e \times D^0$ and one 1-cell $S^1/e \times D^1$, where the attaching map collapses the endpoints of D^1 to D^0.

The definition of G-CW-complex suggests what should play the role of spheres: $G/H \times S^n$, so now we have as many spheres as subgroups[3] of G. For G-spaces Y, Y', let ${}_G[Y, Y']$ be the set of G-homotopy classes of G-maps. In the based case, we will denote it as ${}_G[Y, Y']_*$.

Definition. Let X be a pointed G-space and let H be a subgroup of G. The **H-n-th homotopy group** of X is

$$\pi_n^H(X) := {}_G[(G/H)_+ \wedge S^n, X]_*.$$

We have an alternative description of these homotopy groups:

Lemma 1.2.3 *For any pointed G-space X,*

$$\pi_n^H(X) \cong \pi_n(X^H).$$

Proof. The main observation is that in $G\mathsf{Top}_*$ we also have the exponential adjunction: for a pointed G-space X, this says that the functor $X \wedge -$ is left adjoint to $\mathrm{GMap}(X, -)$. In particular we compute

$$\begin{aligned} \pi_n^H(X) &= {}_G[(G/H)_+ \wedge S^n, X]_* \cong {}_G[S^n, \mathrm{GMap}((G/H)_+, X)]_* \overset{1.1.3}{\cong} [S^n, \mathrm{GMap}((G/H)_+, X)^G]_* \\ &\overset{1.1.2}{\cong} [S^n, \mathrm{Map}_G((G/H)_+, X)]_* \overset{1.1.5}{\cong} [S^n, X^H]_* \cong \pi_n(X^H) \end{aligned}$$

□

This allows us to define

Definition. A map of G-spaces $f : X \longrightarrow Y$ is a **weak equivalence** if

$$f_* : \pi_n^H(X) \xrightarrow{\cong} \pi_n^H(Y)$$

is an isomorphism for every subgroup $H \subseteq G$, every $n \geq 0$ and every choice of basepoint. Equivalently, if $f^H : X^H \longrightarrow Y^H$ is a weak equivalence of spaces for all subgroups $H \subseteq G$.

[3] Another plausible choice would have been the representation spheres S^V, the one-point compactification of an orthogonal representation V of G. It turns out that our original choice is the *minimal* one: any other arises as a G-CW-complex as we have just defined it. For a reference see [18].

As in the nonequivariant case, sometimes we are interested that our maps are isomorphisms not for all $n \geq 0$ but only up to some degree. Of course, now we have our homotopy groups indexed also by subgroups of G, so we need a criterion to decide how connected a map should be for every subgroup. This is the equivariant version:

Definition. Let $\mathscr{S}$ be the set of conjugacy classes of subgroups of G, and let $\theta : \mathscr{S} \longrightarrow \mathbb{Z}_{\geq -1}$ be an assignment. A map of G-spaces $f : X \longrightarrow Y$ is **θ-connected** if $f^H : X^H \longrightarrow Y^H$ is $\theta(H)$-connected for all $H \subseteq G$.

Such an assignment $\theta : \mathscr{S} \longrightarrow \mathbb{Z}_{\geq -1}$ can be also used to define the dimension of a G-CW-complex:

Definition. A G-CW-complex is **θ-dimensional** if all cells of orbit type G/H have nonequivariant dimension at most $\theta(H)$.

With these notions of CW-complexes and homotopy groups, we recover all main results of the usual homotopy theory, that we record next:

Theorem 1.2.4 (Equivariant HELP) *Let (X, A) be a θ-dimensional relative G-CW-complex and let $e : Y \longrightarrow Z$ be a θ-connected map between G-CW-complexes. Then given a solid diagram in* $G\mathsf{Top}$

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "e"] \\
X \arrow[r, "f"] \arrow[ur, dashed, "\tilde{g}"] & Z
\end{tikzcd}$$

commuting up to G-homotopy $H : A \times I \longrightarrow Z$, there is a lift $\tilde{g} : X \longrightarrow Y$ which makes the upper triangle commutative and the lower one commutative up to a G-homotopy $\widetilde{H} : X \times I \longrightarrow Z$ which extends H.

In other words, given a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, hook, "i_0"'] & Y \arrow[dr, "e"] & \\
A \times I \arrow[rr, "H"] & & Z, \\
A \arrow[u, hook', "i_1"] \arrow[r, hook] & X \arrow[ur, "f"'] &
\end{tikzcd}$$

it extends to the following commutative diagram, where the dashed arrows exist:

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[dr, hook] \arrow[dd, hook, "i_0"'] & & Y \arrow[ddr, "e"] & \\
& X \arrow[ur, dashed, "\tilde{g}"] \arrow[d, hook, "i_0"] & & \\
A \times I \arrow[r, hook] \arrow[rrr, bend left, "H"] & X \times I \arrow[rr, dashed, "\widetilde{H}"] & & Z \\
A \arrow[u, hook', "i_1"] \arrow[r, hook] & X \arrow[u, hook', "i_1"'] \arrow[urr, "f"'] & &
\end{tikzcd}$$

Proof. The two desired maps $\tilde{g}$ and $\widetilde{H}$ are constructed by induction on the dimension of X and then cell by cell, so we may assume that $X = G/H \times D^n$ and $A = G/H \times \partial D^n$. By the exponential adjunction and 1.1.5, the statement is equivalent to the nonequivariant HELP with $e : Y^H \longrightarrow Z^H$, $X = D^n$ and $A = \partial D^n$ (cf. [26, §10.3]). $\square$

This technical result has two important consequences:

Theorem 1.2.5 *Let X be a G-CW-complex and let $e : Y \longrightarrow Z$ be a θ-connected G-map between G-CW-complexes. Then*

$$e_* : {}_G[X,Y] \longrightarrow {}_G[X,Z]$$

is a bijection if $\dim X < \theta$ *and surjective if* $\dim X = \theta$.

In particular, if e is a weak equivalence and X is any G-CW-complex, then e_* is a bijection.

Proof. Surjectivity is just rephrasing the theorem for $A = \emptyset$. For injectivity, let us suppose that $[e \circ g_0] = [e \circ g_1]$, and let $J = [0,1]$. If $H : X \times J \longrightarrow Z$ is the homotopy between both maps, let $H : (X \times J) \times I \longrightarrow Z$ be as before with the map constant on I. Then applying the previous theorem for the G-CW pair $(X \times J, X \times \partial J)$ (which has one higher dimension), we get a lift $\tilde{g} : X \times J \longrightarrow Y$ of $g = g_0 \cup g_1 : X \times \partial J \longrightarrow Y$, which is our desired homotopy. $\square$

Theorem 1.2.6 (Equivariant Whitehead) *Any θ-connected G-map $e : Y \longrightarrow Z$ between G-CW-complexes of dimension less than θ is a G-homotopy equivalence.*

In particular, if e is a weak equivalence for any G-CW-complexes Y, Z, then it is a G-homotopy equivalence.

Proof. By the previous theorem, the homotopy inverse of e is given by a map $f : Z \longrightarrow Y$ such that $e_*[f] = [\mathrm{Id}_Y]$. $\square$

Theorem 1.2.7 (Equivariant cellular approximation) *Any G-map $f : (X, A) \longrightarrow (Y, B)$ between G-CW-complexes is G-homotopic rel. A to a cellular map.*

Theorem 1.2.8 (Equivariant CW-approximation) *For any G-space X, there is a G-CW-complex* $\mathrm{cw}(X)$ *and a weak equivalence* $\gamma : \mathrm{cw}(X) \xrightarrow{\simeq} X$.

The proofs of these two last results can be found in [25, I.3.4] and [25, I.3.6]

1.3 Elmendorf theorem

We now recover the categorical perspective of the first section. From this part on we will make use of the theory of model categories. A quick review can be found in Appendix A.

The first step is to show that the category of G-spaces has a model structure. This can actually be done in a more general way: let $\mathscr{F}$ be a **family** of subgroups of G, that is, a collection of subgroups containing the trivial one. We will construct a model structure with respect to this family.

Proposition 1.3.1 *Let $\mathscr{F}$ be a family of subgroups of G. There is a model structure on* $G\mathsf{Top}$ *such that a G-map $f : X \longrightarrow Y$ is a weak equivalence (resp. fibration) if for all subgroups $H \in \mathscr{F}$, the map $f^H : X^H \longrightarrow Y^H$ is a weak equivalence (resp. fibration) in* Top.

In $G\mathsf{Top}_*$ *the statement is exactly the same.*

Proof. We want to apply the Model Structure Lifting theorem A.4.3. For that, we use the family of adjunctions given by

$$\begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "G/H\times -"] \arrow[rr, phantom, "\perp"] && G\mathsf{Top} \arrow[ll, bend left, "(-)^H"] \end{tikzcd}$$

for $H \in \mathscr{F}$, which can be obtained simply as composition of the adjunctions

$$\begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\text{trivial}"] \arrow[rr, phantom, "\perp"] && H\mathsf{Top} \arrow[ll, bend left, "(-)^H"] \arrow[rr, bend left, "G\times_H -"] \arrow[rr, phantom, "\perp"] && G\mathsf{Top}. \arrow[ll, bend left, "\text{restr}"] \end{tikzcd}$$

As we already saw, $G\mathsf{Top}$ is complete and cocomplete, so we are left to check the conditions of A.4.3. For the smallness assumptions (1) and (2) from A.4.3, let $G^\infty = \operatorname{colim}_n G^n$ be one of the infinite gluing constructions in question. By the naturality of the adjunction, it is enough to check that

$$\operatorname*{colim}_n \operatorname{Map}(A, (G^k)^H) \longrightarrow \operatorname{Map}(A, (G^\infty)^H)$$

is an isomorphism for all $H \in \mathscr{F}$, where $A = S^{n-1}$ or $A = D^n$. The isomorphism holds as the inclusions

$$S^{n-1} \hookrightarrow D^n \qquad , \qquad D^n \times 0 \hookrightarrow D^n \times I$$

are closed embeddings between T_1-spaces, and both S^{n-1} and D^n are compact spaces. Here we also used that $(-)^H$ preserves filtered colimits of closed inclusions.

For the property (3) of A.4.3, we just need to check that the factor i^H_∞ in question is a weak homotopy equivalence. But this follows from the elementary fact that if Y is the sequential colimit of a sequence pushouts Y_n of diagrams $D^n \times I \hookleftarrow D^n \times 0 \to Y_n$ then Y is weak homotopy equivalent to Y_0.

The assertion claiming what arrows are weak equivalences and what arrows are fibrations is immediate using the adjunction. □

We will write $G\mathsf{Top}_{\mathscr{F}}$ to emphasize that the model structure depends on the family $\mathscr{F}$.

Now observe that the orbit category $\mathcal{O}_G$ from 1.1.4 can be viewed as the full subcategory of $G\mathsf{Top}$ on the objects G/H, with maps $\operatorname{Map}_G(G/H, G/K) \overset{1.1.5}{\cong} (G/K)^H$. As we saw, the right Kan extension of a G-space X defines a functor

$$\begin{aligned} X^{(-)} \colon \mathcal{O}_G^{op} &\longrightarrow \mathsf{Top} \\ G/H &\longmapsto X^H. \end{aligned}$$

Given a G-map $f : X \longrightarrow Y$, it induces a map $f^H : X^H \longrightarrow Y^H$ commuting with the morphisms induced by maps of $\mathcal{O}_G$, so the previous assignment defines itself a functor

$$\begin{aligned} \Phi \colon G\mathsf{Top} &\longrightarrow \mathsf{Top}^{\mathcal{O}_G^{op}} \\ X &\longmapsto X^{(-)}. \end{aligned}$$

Lemma 1.3.2 *Let* $\Theta : \mathsf{Top}^{\mathcal{O}_G^{op}} \longrightarrow G\mathsf{Top}$ *be evaluation at* G/e. *Then* Θ *is left inverse and left adjoint to* Φ,

$$
\begin{tikzcd}
\mathsf{Top}^{\mathcal{O}_G^{op}} \arrow[r, bend left, "\Theta"] \arrow[r, phantom, "\perp"] & G\mathsf{Top} \arrow[l, bend left, "\Phi"]
\end{tikzcd}
$$

Proof. It is clear that $\Theta\Phi(X) = \Theta(X^{(-)}) = X^e = X$ and that the G-action agrees with the one on X. For the adjunction, the desired bijection

$$\mathrm{Map}_G(F(G/e), X) \cong \mathrm{Hom}_{\mathsf{Top}^{\mathcal{O}_G^{op}}}(F, X^{(-)})$$

is the following: any natural transformation $\alpha : F \Rightarrow X^{(-)}$ corresponds to its evaluation at G/e, the G-map $\alpha_{G/e} : F(G/e) \longrightarrow X$. Here $F(G/e)$ inherits a G-space structure as follows: the map $\cdot g : G/e \longrightarrow G/e$, $g' \mapsto g'g$ in $\mathcal{O}_G$ defines a map of spaces $F(\cdot g) : F(G/e) \longrightarrow F(G/e)$. Then for $a \in F(G/e)$, setting $g * a := F(\cdot g)a$ endows $F(G/1)$ with a G-action. The map is indeed G-invariant by the commutativity of the following diagram:

$$
\begin{array}{ccc}
F(G/e) & \xrightarrow{\alpha_{G/e}} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\
F(G/e) & \xrightarrow{\alpha_{G/e}} & X
\end{array}
$$

Conversely, consider the map $\pi : G/e \longrightarrow G/H$ for a subgroup $H \subseteq G$. Then a G-map $f : F(G/e) \longrightarrow X$ corresponds with the natural transformation $\eta : F \Rightarrow X^{(-)}$ such that $\eta_{G/H}$ is defined as the composite

$$F(G/H) \xrightarrow{F\pi} F(G/e) \xrightarrow{f} X.$$

Now the observation is that the image of this map lies on X^H: indeed, for $h \in H$, we have a commutative diagram in $\mathcal{O}_G$

$$
\begin{array}{ccc}
G/e & \xrightarrow{\pi} & G/H \\
\downarrow{\scriptstyle \cdot h} & & \downarrow{\scriptstyle \cdot h = \mathrm{Id}} \\
G/e & \xrightarrow{\pi} & G/H
\end{array}
$$

which induces a diagram

$$
\begin{array}{ccc}
F(G/e) & \xleftarrow{F\pi} & F(G/H) \\
{\scriptstyle h}\uparrow & & \uparrow{\scriptstyle \mathrm{Id}} \\
F(G/e) & \xleftarrow{F\pi} & F(G/H).
\end{array}
$$

Therefore

$$h \cdot (f \circ (F\pi))(a) = f(h \cdot (F\pi)(a)) = (f \circ F\pi)(a),$$

as desired. It is obvious that both assignments are inverses of each other. □

Elmendorf theorem will say that this is more than an adjunction: this is a Quillen equivalence for a suitable model structure on the functor category. Moreover, the result is even true if we restrict to a family of subgroups $\mathscr{F}$ of G.

In the first place we have to see how we can lift the model structure of Top to a model structure on $\mathsf{Top}^{\mathcal{O}_G^{op}}$. As before we will use theorem A.4.3: let $\mathcal{J}$ be a small category enriched over Top. Then it is easy to see that the functor category $\mathsf{Top}^{\mathcal{J}}$ is also enriched over Top: here $\mathrm{Hom}_{\mathsf{Top}^{\mathcal{J}}}(F,G)$ is a subspace of $\prod_{j\in J}\mathrm{Map}(Fj,Gj)$. Moreover, $\mathsf{Top}^{\mathcal{J}}$ is complete and cocomplete, with limits and colimits constructed objectwise. The first step is to use the Model Structure Lifting theorem A.4.3 to obtain a model structure on $\mathsf{Top}^{\mathcal{J}^{op}}$.

Proposition 1.3.3 *For $j \in \mathcal{J}$, let $\underline{j} := \mathrm{Hom}_{\mathcal{J}}(-,j) : \mathcal{J}^{op} \longrightarrow \mathsf{Top}$; and define $\Theta_j : \mathsf{Top}^{\mathcal{J}^{op}} \longrightarrow \mathsf{Top}$ as evaluation at j. Then there is an adjunction*

$$\begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\underline{j}\times -"] \arrow[rr, phantom, "\perp"] & & \mathsf{Top}^{\mathcal{J}^{op}} \arrow[ll, bend left, "\Theta_j"] \end{tikzcd}$$

Proof. The observation that ignites the proof is that, for a space Y and $F, G \in \mathsf{Top}^{\mathcal{J}^{op}}$, there is an "exponential adjunction"

$$\mathrm{Hom}_{\mathsf{Top}^{\mathcal{J}^{op}}}(F\times Y, G) \cong \mathrm{Hom}_{\mathsf{Top}^{\mathcal{J}^{op}}}(F, \mathrm{Map}(Y, G-))$$

defined objectwise, inherited from the one in Top. Taking $F = \underline{j}$ we obtain

$$\mathrm{Hom}_{\mathsf{Top}^{\mathcal{J}^{op}}}(\underline{j}\times Y, G) \cong \mathrm{Hom}_{\mathsf{Top}^{\mathcal{J}^{op}}}(\underline{j}, \mathrm{Map}(Y, G-)) \overset{\text{Yoneda}}{\cong} \mathrm{Map}(Y, Gj)$$

□

Corollary 1.3.4 *Let $\mathcal{F}$ be a subset of objects of $\mathcal{J}$. Then the adjunctions*

$$\left\{ \begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\underline{j}\times -"] \arrow[rr, phantom, "\perp"] & & \mathsf{Top}^{\mathcal{J}^{op}} \arrow[ll, bend left, "\Theta_j"] \end{tikzcd} \right\}_{k\in\mathcal{F}}$$

define a model structure on $\mathsf{Top}^{\mathcal{J}^{op}}$.

The proof is very similar to 1.3.1 and we omit it. Returning to our original problem, this will allow us to endow the functor category $\mathsf{Top}^{\mathcal{O}_G^{op}}$ with a model structure. This can be done with slight more generality, as we did with the model structure on $G\mathsf{Top}$:

Definition. The **orbit category of** G **with respect to** $\mathscr{F}$ is the full subcategory $\mathcal{O}_{\mathscr{F}} \subset \mathcal{O}_G$ on the objects G/H, where $H \in \mathscr{F}$.

Setting $\mathcal{J} = \mathcal{O}_{\mathscr{F}}$ and $\mathscr{F} = \mathcal{F}$ in the previous corollary, we obtain

Corollary 1.3.5 *Let $\mathscr{F}$ be a family of subgroups of G. The adjunctions*

$$\left\{ \begin{tikzcd} \mathsf{Top} \arrow[rr, bend left, "\underline{G/H}\times -"] \arrow[rr, phantom, "\perp"] & & \mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}} \arrow[ll, bend left, "\Theta_{G/H}"] \end{tikzcd} \right\}_{k\in\mathscr{F}}$$

make $\mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}}$ into a model category.

We now have all necessary ingredients to state Elmendorf theorem. In a nutshell, the theorem states that from the point of view of homotopy theory, the categories $G\mathsf{Top}$ and $\mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}}$ are the same.

Theorem 1.3.6 (Elmendorf) *Let $\mathscr{F}$ be a family of subgroups of G. The adjunction*

$$\begin{tikzcd} \mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}} \arrow[r, bend left, "\Theta"] \arrow[r, phantom, "\perp"] & G\mathsf{Top}_{\mathscr{F}} \arrow[l, bend left, "\Phi"] \end{tikzcd}$$

is a Quillen equivalence. Therefore, there is a equivalence of categories

$$\mathsf{Ho}(G\mathsf{Top}_{\mathscr{F}}) \simeq \mathsf{Ho}(\mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}}).$$

Proof. Let us check in the first place that the former is a Quillen adjunction, more concretely that Φ is right Quillen: given a (trivial) fibration f in $G\mathsf{Top}_{\mathscr{F}}$, $\Phi(f)$ is a (trivial) fibration if and only if $\Theta_{G/H}\Phi(f) = f^H$ is a (trivial) fibration, but this holds by 1.3.1.

Let us check now that the adjunction is a Quillen equivalence: let $Y \in \mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}}$ be a cofibrant object and let $X \in G\mathsf{Top}_{\mathscr{F}}$ be fibrant. We need to show that

$$f' : Y \longrightarrow \Phi(X) \text{ is weak equivalence } \iff \Theta(f') : \Theta(Y) \longrightarrow X \text{ is weak equivalence.}$$

In $G\mathsf{Top}_{\mathscr{F}}$, every object is fibrant, just because in Top the same happens. Factor the unique map $i : \emptyset \longrightarrow Y$ as a cofibration $i_\infty : \emptyset \longrightarrow G^\infty$ followed by a trivial fibration $p_\infty : G^\infty \longrightarrow Y$, where $G^\infty = G^\infty(F\mathcal{I}, i)$ and $F\mathcal{I}$ is the family of all maps $\underline{G/H} \times S^{n-1} \longrightarrow \underline{G/H} \times D^n$ for all subgroups $H \in \mathscr{F}$ and all $n \geq 0$ (see A.4.3). Since Θ is left Quillen, by the Ken Brown lemma A.1.6 we have that Θ preserves weak equivalences between cofibrant objects, thus $\Theta(p_\infty)$ is a weak equivalence. Therefore, by the 2-out-of-3 property, f' is a weak equivalence if and only if so is $f'p_\infty$; so it is enough to show that

$$f : G^\infty \longrightarrow \Phi(X) \text{ is weak equivalence } \iff \Theta(f) : \Theta(G^\infty) \longrightarrow X \text{ is weak equivalence.}$$

Simply by definitions, this statement is equivalent to

$$f_{G/H} \text{ is weak equivalence for all } H \in \mathscr{F} \iff (f_{G/e})^H \text{ is weak equivalence for all } H \in \mathscr{F}.$$

Since f is a natural transformation, there is a commutative square

$$\begin{tikzcd} G^\infty_{G/H} \arrow[r, "f_{G/H}"] \arrow[d] & X^H \arrow[d] \\ G^\infty_{G/e} \arrow[r, "f_{G/e}"] & X, \end{tikzcd}$$

which in particular says that $f_{G/H} = (f_{G/e})^H(\eta_{G^\infty})_{G/H}$, where η is the unit of the adjunction. Thus, by the 2-out-of-3 property, it suffices to show that

$$(\eta_{G^\infty})_{G/H} : G^\infty_{G/H} \longrightarrow (G^\infty_{G/e})^H$$

is a weak equivalence in Top. But this map is in particular an homeomorphism: indeed, it is not hard to check that the map

$$\eta_{\underline{G/H} \times Z} : \underline{G/H} \times Z \longrightarrow \Phi\Theta(\underline{G/H} \times Z)$$

is an isomorphism in $\mathsf{Top}^{\mathcal{O}_{\mathscr{F}}^{op}}$ (cf. [37, 7.7]). Now the claim is immediate using induction and the fact that the functor $(-)^H$ preserves coproducts, pushouts where one arrow is a closed embedding, and filtered colimits of closed inclusions (see [21, 1.2]). $\square$

1.4 Homology and cohomology of G-spaces

By completeness, we will briefly discuss one way to adapt the theory of homology and cohomology for G-spaces. This will be inspired by cellular cohomology, so we will restrict ourselves to G-CW-complexes (for general G-spaces, we just take G-CW-approximations, and this does not depend on the choice of the approximations). Of course, the main problem is to keep track of the G-action. Later in 4.2 we will describe a more general case.

Definition. A **coefficient system** is a functor $M : \mathcal{O}_G^{op} \longrightarrow \mathsf{Ab}$ or $N : \mathcal{O}_G \longrightarrow \mathsf{Ab}$.

Examples 1.4.1 (a) If A is an abelian group, we have the constant coefficient system $\underline{A}$, sending every object to A and every morphism to the identity of A.

(b) If X is a G-space, we saw that there is a functor $X^{(-)} : \mathcal{O}_G^{op} \longrightarrow \mathsf{Top}$. The composition with any functor $\mathsf{Top} \longrightarrow \mathsf{Ab}$ defines a contravariant coefficient system

$$\mathcal{O}_G^{op} \xrightarrow{X^{(-)}} \mathsf{Top} \longrightarrow \mathsf{Ab}.$$

As a particular case, for homotopy groups we have

$$\underline{\pi_n}(X) : \mathcal{O}_G^{op} \longrightarrow \mathsf{Ab}$$

with $\underline{\pi_n}(X)(G/H) := \pi_n^H(X) \cong \pi_n(X^H)$ sending every G-map $G/H \longrightarrow G/K$ to the induced map $\pi_n(X^K) \longrightarrow \pi_n(X^H)$. For homology the story is the same: $\underline{H_n}(X)(G/H) := H_n(X^H; \mathbb{Z})$.

We want to define cohomology with coefficients in a contravariant coefficient system; and homology with coefficients in a covariant system. That is, we want to do homological algebra in the functor categories $\mathsf{Ab}^{\mathcal{O}_G^{op}}$ and $\mathsf{Ab}^{\mathcal{O}_G}$. For that, luckily, we have

Lemma 1.4.2 *The categories* $\mathsf{Ab}^{\mathcal{O}_G^{op}}$ *and* $\mathsf{Ab}^{\mathcal{O}_G}$ *are abelian.*

Proof. The zero object is given by the constant functor with values in the trivial abelian group; and products, coproducts, kernels and cokernels are defined termwise. □

Now let X be a G-CW-complex, and let X_n be its n-skeleton. We consider the coefficient system

$$\underline{C_n}(X) : \mathcal{O}_G^{op} \longrightarrow \mathsf{Ab}$$

sending G/H to $\widetilde{C}_n(X^H) = H_n(X_n^H, X_{n-1}^H; \mathbb{Z})$, the group of n-chains in cellular homology. There is a differential (natural transformation)

$$\underline{\partial_n} : \underline{C_n}(X) \longrightarrow \underline{C_{n-1}}(X)$$

that at G/H is the cellular differential of $C_*(X^H)$, i.e. the connecting homomorphism of the long exact sequence of the triple $(X_n^H, X_{n-1}^H, X_{n-2}^H)$. Moreover, $\underline{\partial_n} \circ \underline{\partial_{n-1}} = 0$.

For a contravariant coefficient system M, consider the abelian group

$$C_G^n(X; M) := \mathrm{Hom}_{\mathsf{Ab}^{\mathcal{O}_G^{op}}}(\underline{C_n}(X), M).$$

This defines a cochain complex of abelian groups $C_G^*(X; M)$ with codifferential $\delta^n := - \circ \underline{\partial_{n+1}}$.

Definition. The **Bredon cohomology** of X with coefficients in M is

$$H^n_G(X; M) := H^n(C^*_G(X; M)).$$

For homology, we proceed in a similar fashion: given a (covariant) coefficient system $N : \mathcal{O}_G \longrightarrow \mathsf{Ab}$, consider the abelian group

$$C^G_n(X; N) := \int^{G/H \in \mathcal{O}^{op}_G} \underline{C_n}(X)(G/H) \otimes N(G/H).$$

The previous coend[4] is also denoted as $\underline{C_n}(X) \otimes_{\mathcal{O}^{op}_G} N$. As before, this defines a chain complex $C^G_*(X; N)$ with differential $\partial_n := \underline{\partial_n} \otimes \mathrm{Id}$.

Definition. The **Bredon homology** of X with coefficients in N is

$$H^G_n(X; N) := H_n(C^G_*(X; N)).$$

Remark 1.4.3 For G-spaces, we have presented here Bredon (co)homology. There is another cohomological invariant that we can associate to a G-space: the **Borel cohomology** of X with coefficients in a ring R is

$$H^n_B(X; R) := H^n(\mathrm{hocolim}\, X; R).$$

One can show, in a similar fashion as 1.1.6, that hocolim $X \cong EG \times_G X$ (this is a balanced product). Recall that EG is contractible, so $X \cong EG \times_G X$ is just a "fattened up" version of $X/G = \mathrm{colim}\, X$.

Example 1.4.4 Let us compute the Bredon cohomology groups, with coefficients in $\underline{\mathbb{Z}}$, of the sphere S^2 with the $\mathbb{Z}/2$-action given by rotation by π from 1.2.1.(b). There we already described a $\mathbb{Z}/2$-CW-structure with two 0-cells $(\mathbb{Z}/2)/(\mathbb{Z}/2) \times D^0$, one 1-cell $(\mathbb{Z}/2)/e \times D^1$ and one 2-cell $(\mathbb{Z}/2)/e \times D^2$. We take this example from [3, 1.4.8], but we fix a mistake that it appears there, with respect to the computation of the cellular differentials of the $\mathbb{Z}/2$-CW-structure (in particular, one does not get the codifferentials for $C^*_{\mathbb{Z}/2}(S^2; \underline{\mathbb{Z}})$ that he claims from the cellular differentials he writes).

In the first place, let us make explicit the morphisms of the category $\mathcal{O}^{op}_{\mathbb{Z}/2}$: this is just

$$\begin{tikzcd} (\mathbb{Z}/2)/e \arrow[loop left, "\mathrm{Id}"] \arrow[loop below, "\tau"] \arrow[r, "\pi"] & (\mathbb{Z}/2)/(\mathbb{Z}/2) \arrow[loop right, "\mathrm{Id}"] \end{tikzcd}$$

where π is the projection and τ is the switching map (which interchanges $\overline{0}$ and $\overline{1}$). The associated "fixed points functor" $\mathcal{O}^{op}_{\mathbb{Z}/2} \longrightarrow \mathsf{Top}$ maps π to the inclusion, and τ to the involution of S^2.

[4]For a bifunctor $H : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{E}$, the **coend** $\int^{\mathcal{C}} H$ is the coequalizer of the diagram

$$\coprod_{f \in \mathrm{arrows}\mathcal{C}} H(\mathrm{cod}\, f, \mathrm{dom}\, f) \rightrightarrows \coprod_{C \in \mathcal{C}} H(C, C) \dashrightarrow \int^{\mathcal{C}} H$$

where $f : C \longrightarrow C'$.

By convenience of notation, let us write in this example only the subgroup when we mean the orbit. Let us compute the coefficient system $\underline{C_n}(S^2) : \mathcal{O}^{op}_{\mathbb{Z}/2} \longrightarrow \mathsf{Ab}$. This is determined by

$$\underline{C_n}(S^2)(e) = H_n((S^2)^e_n, (S^2)^e_{n-1}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0, 1, 2 \\ 0, & \text{else} \end{cases}$$

$$\underline{C_n}(S^2)(\mathbb{Z}/2) = H_n((S^2)^{\mathbb{Z}/2}_n, (S^2)^{\mathbb{Z}/2}_{n-1}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 0 \\ 0, & \text{else} \end{cases}$$

(this is just counting the number of cells). The map induced by π is just either the trivial map $0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ (for $n = 1, 2$) or the identity of $\mathbb{Z} \oplus \mathbb{Z}$ (for $n = 0$). Similarly, τ induces an automorphism of $\mathbb{Z} \oplus \mathbb{Z}$, which is flipping the terms (for $n = 1, 2$) or the identity (for $n = 0$).

Next, we have to compute the differentials $\underline{\partial_n} : \underline{C_n}(S^2) \longrightarrow \underline{C_{n-1}}(S^2)$, which are given by the cellular differential associated to the (nonequivariant) CW-structures that the fixed points spaces inherit. This is depicted in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r, "\underline{\partial_3}(e)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[loop above, "\underline{C_2}(\tau)"] \arrow[r, "\underline{\partial_2}(e)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[loop above, "\underline{C_1}(\tau)"] \arrow[r, "\underline{\partial_1}(e)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[loop above, "\mathrm{Id}"] \\
\cdots \arrow[r] & 0 \arrow[u] \arrow[r, "\underline{\partial_3}(\mathbb{Z}/2)"] & 0 \arrow[u] \arrow[r, "\underline{\partial_2}(\mathbb{Z}/2)"] & 0 \arrow[u] \arrow[r, "\underline{\partial_1}(\mathbb{Z}/2)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[u, equal, "\mathrm{Id}"]
\end{tikzcd}$$

By cellular homology, the (nontrivial) differentials are given by

$$\underline{\partial_2}(e) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad , \quad \underline{\partial_1}(e) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

and both switching maps are given by

$$\tau := \underline{C_2}(\tau) = \underline{C_1}(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We now look at the groups $C^n_{\mathbb{Z}/2}(S^2; \underline{\mathbb{Z}}) = \mathrm{Hom}_{\mathsf{Ab}^{\mathcal{O}^{op}_{\mathbb{Z}/2}}}(\underline{C_n}(S^2), \underline{\mathbb{Z}})$. For $n = 1, 2$, a natural transformation $\underline{C_2}(S^2) \Rightarrow \underline{\mathbb{Z}}$ is determined by a group homomorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$ such that the diagram

$$\begin{tikzcd}
\mathbb{Z} \oplus \mathbb{Z} \arrow[dd, "\tau"'] \arrow[dr, "\varphi"] & \\
& \mathbb{Z} \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes. Such a group homomorphism is determined by the image of $(1, 0)$, so

$$C^1_{\mathbb{Z}/2}(S^2; \underline{\mathbb{Z}}) \cong \mathbb{Z} \quad , \quad C^2_{\mathbb{Z}/2}(S^2; \underline{\mathbb{Z}}) \cong \mathbb{Z}.$$

For $n = 0$, a natural transformation $\underline{C_0}(S^2) \Rightarrow \underline{\mathbb{Z}}$ is determined by a group homomorphism $\varphi : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$, so

$$C^0_{\mathbb{Z}/2}(S^2; \underline{\mathbb{Z}}) \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

hence the cochain complex $C^*_{\mathbb{Z}/2}(S^2;\underline{\mathbb{Z}})$ looks like

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\delta^0} \mathbb{Z} \xrightarrow{\delta^1} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots .$$

Let us finally determine the codifferentials. In $\mathbb{Z}\oplus\mathbb{Z}$, let $e_1 = (1,0)$ and $e_2 = (0,1)$. If $\varphi_n \in C^1_{\mathbb{Z}/2}(S^2;\underline{\mathbb{Z}})$ corresponds with $n \in \mathbb{Z}$, then $\delta^1(n)$ is the image of e_1 by the composite

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\underline{\partial_2}(e)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\varphi_n} \mathbb{Z}$$

which is $\varphi_n(e_1 - e_2) = n - n = 0$, so $\delta^1 = 0$. Similarly, if $\varphi_{(n,m)} \in C^0_{\mathbb{Z}/2}(S^2;\underline{\mathbb{Z}})$ corresponds with $(n,m) \in \mathbb{Z}\oplus\mathbb{Z}$, then $\delta^0(n,m)$ is the image of e_1 by the composite

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\underline{\partial_1}(e)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\varphi_{(n,m)}} \mathbb{Z},$$

so $\delta^0(n,m) = \varphi_{(n,m)}(e_1 - e_2) = n - m$. The upshot is that the Bredon cohomology of S^2 with the involution given by rotation by half-turn is

$$H^n_{\mathbb{Z}/2}(S^2;\underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z}, & n = 0,2 \\ 0, & \text{else} \end{cases}.$$

Chapter 2

Topological K-theory

We will devote the second chapter to discuss topological K-theory in some of its versions. The origin of K-theory was, actually, not in Topology but in Algebra: it was invented by Grothendieck to solve some hard problems of Algebraic Geometry. This idea soon spread to other areas of mathematics, such as Topology, Number Theory or Functional Analysis. The letter K stands for the German word *Klasse* (class).

In Topology, K-theory appeared first in Atiyah and Hirzebruch's *Riemann-Roch theorems for differentiable manifolds* [2]. It was the first generalized cohomology theory deeply studied, and it actually allowed to give relatively easy proofs to problems such that the classification of real division algebras. This was already solved using homology and cohomology, but with a very intricate proof.

We mostly present complex K-theory and K-theory with Reality, where the latter is a slightly different version for spaces endowed with an involution. In §2.1 we follow mostly the approach of [14] but we are also influenced by [19], specially making emphasis in the Grothendieck construction. For §2.2, we collect material from [14], [27], [38], and specially [13] for the last part of the section. Finally, in §2.3 we give a full exposition of K-theory with Reality. We keep close to the standard reference [1], but we present the material following the spirit of [14] that we use in §2.1 and §2.2. We also improve the notation of [1] and use an original one. Proofs in this section are due to the author.

2.1 Complex and real K-theory

In this first section we will briefly recall the general theory of complex (and real) K-theory. This will give us one of the main examples in stable homotopy theory, that we will discuss in chapter 3. Along this section, unless otherwise stated, all spaces will be considered compact Hausdorff.

The first observation is that the forgetful functor $U : \mathsf{Ab} \longrightarrow \mathsf{CMon}$ from abelian groups to commutative monoids has a left-adjoint K,

$$\begin{tikzcd} \mathsf{Ab} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathsf{CMon} \arrow[l, bend right, "K"'] \end{tikzcd}$$

which is given by the **Grothendieck construction**: for a commutative monoid $(M, \oplus, 0)$ we define

$$K(M) := \frac{\mathbb{Z}[M]}{\langle m + m' - m \oplus m' \rangle}.$$

Moreover, if $(M, \oplus, \otimes, 0, 1)$ is a commutative semi-ring, then the multiplication on M induces a multiplication on $K(M)$, $[m] \cdot [m'] := [m \otimes m']$, so the latter is a ring. An alternative description is the following: $K(M)$ is the quotient of $M \times M$ modulo the equivalence relation

$$(m_1, m_2) \sim (m_1', m_2') \iff \text{there is } k \in M \text{ such that } m_1 \oplus m_2' \oplus k = m_2 \oplus m_1' \oplus k.$$

From now on let us fix a compact, Hausdorff space X. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the set $\mathrm{Vect}^\bullet_{\mathbb{F}}(X)$ of isomorphism classes of $\mathbb{F}$-vector bundles of all possible ranks over X is a commutative monoid with respect to the direct sum of vector bundles and the trivial vector bundle $\underline{0} = X \times 0$. Moreover, it is a semi-ring if we also consider the tensor product of vector bundles and the trivial line bundle $\underline{1} = X \times \mathbb{F}$.

Definition. The **complex** K**-theory** of X is the Grothendieck construction of $\mathrm{Vect}^\bullet_{\mathbb{C}}(X)$,

$$K(X) := K(\mathrm{Vect}^\bullet_{\mathbb{C}}(X)),$$

and its **real** K**-theory** is the Grothendieck construction of $\mathrm{Vect}^\bullet_{\mathbb{R}}(X)$,

$$KO(X) := K(\mathrm{Vect}^\bullet_{\mathbb{R}}(X)).$$

By a matter of clarity, from now on we will only refer to complex K-theory, but it is good that the reader keeps in mind that the rest of results until 2.1.4 are also valid for KO-theory.

If $f : X \longrightarrow Y$ is a map of spaces, the pullback of vector bundles induces a morphism in K-theory,

$$f^* : K(Y) \longrightarrow K(X).$$

Definition. The **reduced** K**-theory** of X is

$$\widetilde{K}(X) := \mathrm{coker}(K(*) \longrightarrow K(X))$$

induced by the unique map $X \longrightarrow *$. Since $\mathrm{Vect}^\bullet_{\mathbb{C}}(*) = \mathbb{N}$, $K(*) = \mathbb{Z}$ and $\widetilde{K}(X) = K(X)/\mathbb{Z}$.

Any continuous map also induces a map in reduced K-theory, so they define functors

$$K : \mathsf{CHaus} \longrightarrow \mathsf{Rng} \qquad , \qquad \widetilde{K} : \mathsf{CHaus} \longrightarrow \mathsf{Ab}.$$

Given a vector bundle $E \longrightarrow X$, we will also denote by E its class in $K(X)$ and $\widetilde{K}(X)$. The following proposition is immediate:

Proposition 2.1.1 *Let $E, F \longrightarrow X$ be vector bundles over X.*

1. *In $K(X)$, $E = F$ if and only if $E \oplus \underline{n} \cong F \oplus \underline{n}$ for some $n \in \mathbb{N}$ (they are said* ***stably isomorphic****).*
2. *In $\widetilde{K}(X)$, $E = F$ if and only if $E \oplus \underline{n} \cong F \oplus \underline{m}$ for some $n, m \in \mathbb{N}$.*

If $\mathrm{Vect}^n_{\mathbb{C}}(X)$ denotes the set of isomorphism classes of vector bundles of rank n, there is a map

$$- \oplus \underline{1} : \mathrm{Vect}^n_{\mathbb{C}}(X) \longrightarrow \mathrm{Vect}^{n+1}_{\mathbb{C}}(X).$$

Corollary 2.1.2 *If X is connected, $\widetilde{K}(X) \cong \mathrm{colim}_n \mathrm{Vect}^n_{\mathbb{C}}(X)$.*

Proof. If X is connected, every vector bundle over X has constant rank, so it lies on $\mathrm{Vect}^n_{\mathbb{C}}(X)$ for some $n \geq 0$ and by the previous characterization $\widetilde{K}(X)$ coincides with the colimit. $\square$

For spaces with a preferred basepoint we have a useful property:

Proposition 2.1.3 *Let (X, x_0) be a pointed compact Hausdorff space. There is a split short exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow K(X) \longrightarrow \widetilde{K}(X) \longrightarrow 0$$

so in particular

$$K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}.$$

Proof. Exactness holds by definition, and the splitting is given by a retract $r : K(X) \longrightarrow \mathbb{Z}$, where $r(E - F) := \dim E_{x_0} - \dim F_{x_0}$, which is induced by the inclusion of the basepoint. $\square$

The next result gives some properties of K-theory that reminisce the ones known for singular cohomology (see [14, §2.2]):

Proposition 2.1.4 *The following properties hold for reduced K-theory:*

1. *If $f, g : X \longrightarrow Y$ are homotopic maps between based spaces, then $f^* = g^* : \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$.*
2. *If (X, A) is a pair of pointed spaces, with A closed, there is a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{K}(\Sigma^2 X) \arrow[r] & \widetilde{K}(\Sigma^2 A) \arrow[dll, rounded corners] \\ \widetilde{K}(\Sigma(X/A)) \arrow[r] & \widetilde{K}(\Sigma X) \arrow[r] & \widetilde{K}(\Sigma A) \arrow[dll, rounded corners] \\ \widetilde{K}(X/A) \arrow[r] & \widetilde{K}(X) \arrow[r] & \widetilde{K}(A) \end{tikzcd}$$

3. *If $(X_i)_{i \in I}$ is a collection of pointed spaces, then*

$$\widetilde{K}(\bigvee_{i \in I} X_i) \xrightarrow{\cong} \prod_{i \in I} \widetilde{K}(X_i)$$

is an isomorphism.

If X, Y are spaces, let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be the canonical projections. The map

$$\mu := \pi_1^* \otimes \pi_2^* : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

is called the **external product** .

Now we make the following observation: if x is the tautological line bundle over $\mathbb{CP}^1 \cong S^2$, then $x^2 + 1 = 2x$, so in K-theory $(x-1)^2 = 0$. This induces a ring homomorphim

$$\mathbb{Z}[x]/(x-1)^2 \longrightarrow K(S^2).$$

A fundamental theorem, whose proof can be found in [14, 2.2], is the following:

Theorem 2.1.5 (Product theorem) *Let X be a space. The composite*

$$K(X) \otimes \mathbb{Z}[x]/(x-1)^2 \longrightarrow K(X) \otimes K(S^2) \xrightarrow{\mu} K(X \times S^2)$$

is an isomorphism.

Corollary 2.1.6 *The map $\mathbb{Z}[x]/(x-1)^2 \xrightarrow{\cong} K(S^2)$ is a ring isomorphism. Moreover, the external product*

$$\mu : K(X) \otimes K(S^2) \xrightarrow{\cong} K(X \times S^2)$$

is an isomorphism.

Proof. For the first assertion, take $X = *$ the one-point space. The second follows by the 2-out-of-3 property for isomorphisms. □

Proposition 2.1.7 *Let X, Y be pointed spaces. For $n \geq 0$, there is a split short exact sequence*

$$0 \longrightarrow \widetilde{K}(\Sigma^n(X \wedge Y)) \longrightarrow \widetilde{K}(\Sigma^n(X \times Y)) \longrightarrow \widetilde{K}(\Sigma^n(X \vee Y)) \longrightarrow 0.$$

Proof. The map $\widetilde{K}(\Sigma^n(X \times Y)) \longrightarrow \widetilde{K}(\Sigma^n(X \vee Y)) \cong \widetilde{K}(\Sigma^n X) \oplus \widetilde{K}(\Sigma^n Y)$ has a section $(\Sigma^n \pi_1)^* \oplus (\Sigma^n \pi_2)^*$, so in particular the map is surjective and we conclude by the long exact sequence of the pair $(X \times Y, X \vee Y)$. □

Corollary 2.1.8 $\widetilde{K}(S^n) \cong \begin{cases} \mathbb{Z}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$

Proof. From the general theory of vector bundles, we know that any complex vector bundle over S^1 is trivial, so $\widetilde{K}(S^1) \cong 0$. By the previous corollary, $\widetilde{K}(S^2) \cong \mathbb{Z}$. Now use the previous proposition. □

Now we describe the periodicity theorem. If X, Y are pointed spaces, observe that by 2.1.3 and 2.1.7 we have splittings

$$K(X) \otimes K(Y) \cong (\widetilde{K}(X) \otimes \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}$$

and

$$K(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}.$$

Proposition 2.1.9 *The external product restricts to a homomorphism*

$$\widetilde{\mu} : \widetilde{K}(X) \otimes \widetilde{K}(Y) \longrightarrow \widetilde{K}(X \wedge Y),$$

called the ***reduced external product*** *.*

Proof. For $a \in \widetilde{K}(X) \cong \operatorname{Ker}(K(X) \longrightarrow K(x_0))$ and $b \in \widetilde{K}(Y) \cong \operatorname{Ker}(K(Y) \longrightarrow K(y_0))$, the element $\mu(a \otimes b) = \pi_1^*(a)\pi_2^*(b)$ restricts to zero in $K(X \times \{y_0\})$ and $K(\{x_0\} \times Y)$, so therefore also in $K(X \vee Y)$. This says that it lies in $\widetilde{K}(X \times Y)$, and by 2.1.7 it lies in $\widetilde{K}(X \wedge Y)$ in the above splitting. □

As before, let x represent the tautological line bundle over $S^2 \cong \mathbb{CP}^1$, and let β be the composite

$$\widetilde{K}(X) \xrightarrow{(x-1)\otimes -} \widetilde{K}(S^2) \otimes \widetilde{K}(X) \xrightarrow{\widetilde{\mu}} \widetilde{K}(S^2 \wedge X).$$

Theorem 2.1.10 (Bott periodicity) *The previous homomorphism*

$$\beta : \widetilde{K}(X) \xrightarrow{\cong} \widetilde{K}(S^2 \wedge X)$$

is an isomorphism for all based compact Hausdorff spaces X.

Proof. This is a direct consequence of 2.1.5 and 2.1.9. □

2.2 K-theory as a reduced cohomology theory. Classifying spaces.

In this section we will take a step closer to one of our main constructions, the category of spectra. Here we will give rise to a coarser version of spectra, called Ω-spectra.

Our first observation is that the properties described in 2.1.4 are very similar to the ones of singular cohomology. There are some axioms expressing the properties that a family of functors must satisfy to be called a "cohomology theory":

Definition. A **reduced (generalized) cohomology theory** is a sequence of functors

$$\widetilde{h}^n : \mathsf{Top}_*^{op} \longrightarrow \mathsf{Ab} \qquad , \qquad n \in \mathbb{Z}$$

together with natural isomorphisms $\widetilde{h}^n \circ \Sigma \overset{\cong}{\Longrightarrow} \widetilde{h}^{n-1}$ satisfying:

(i) Homotopic maps induce the same map in $\widetilde{h}^n$.

(ii) Given a based map $f : X \longrightarrow Y$, the sequence

$$\widetilde{h}^n(Cf) \longrightarrow \widetilde{h}^n(Y) \longrightarrow \widetilde{h}^n(X)$$

is exact, where Cf is the reduced mapping cone.

(iii) Given a collection of based spaces $(X_i)_{i\in I}$, the canonical map

$$\widetilde{h}^n(\bigvee_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} \widetilde{h}^n(X_i)$$

is an isomorphism.

Remark 2.2.1 These axioms are slightly different to the classic Eilenberg-Steenrod axioms. In the first place, we do not have any long exact sequence, but one can be constructed with little effort: given a map $f : X \longrightarrow Y$, it is easily verified that the cone of the inclusion $Y \hookrightarrow Cf$ is homotopy equivalent to ΣX. Then there is a long exact sequence

$$\cdots \longrightarrow \widetilde{h}^{n-1}(X) \xrightarrow{\delta^{n-1}} \widetilde{h}^n(Cf) \longrightarrow \widetilde{h}^n(Y) \longrightarrow \widetilde{h}^n(X) \xrightarrow{\delta^n} \widetilde{h}^{n+1}(Cf) \longrightarrow \cdots$$

where the connecting homomorphism is the composite

$$\widetilde{h}^{n-1}(X) \xrightarrow{\cong} \widetilde{h}^n(\Sigma X) \xrightarrow{\cong} \widetilde{h}^n(C(Y \hookrightarrow Cf)) \longrightarrow \widetilde{h}^n(Cf).$$

Concretely, in singular cohomology, this recovers the long exact sequence of a pair (X, A), since by excision

$$\widetilde{H}^n(Ci) \cong H^n(Ci, CA) \cong H^n(Ci - p, CA - p) \cong H^n(X, A)$$

where $i : A \hookrightarrow X$ is the inclusion and p is the cusp of the cone. In second place, we add the adjective *generalized* because we dropped the "dimension axiom" $\widetilde{h}^n(S^0) = 0$ for $n \neq 0$. Under this extra axiom (one talks about *ordinary* cohomology theories), one can show that on pointed CW-complexes all reduced ordinary cohomology theories are naturally isomorphic.

The question now is how we can lift (reduced) K-theory to a reduced cohomology theory. Inspired by 2.1.4, we define for $n \geq 0$

$$\widetilde{K}^{-n}(X) := \widetilde{K}(\Sigma^n X) \quad , \quad \widetilde{K}^{-n}(X, A) := \widetilde{K}^{-n}(X/A).$$

We can also extend this to positive indices using Bott periodicity: for $n \geq 0$ and a pointed space X, define

$$\widetilde{K}^n(X) := \widetilde{K}^{n-2s}(X) \quad , \quad \widetilde{K}^n(X, A) := \widetilde{K}^n(X/A)$$

for $2s > n$. More explicitly,

$$\widetilde{K}^{2n}(X) = \widetilde{K}(X) \quad , \quad \widetilde{K}^{2n+1}(X) = \widetilde{K}(\Sigma X) \quad , \quad \widetilde{K}^n(X, A) = \widetilde{K}^n(X/A)$$

where $A \subset X$ is a closed subspace.

Theorem 2.2.2 *The functors* $\widetilde{K}^n : \mathsf{CHaus}_* \longrightarrow \mathsf{Ab}$ *define a reduced cohomology theory.*

Proof. The homotopy and wedge axioms were noted in 2.1.4, the natural isomorphism with the suspension follows by definition and Bott periodicity, and the exactness axiom holds since $\widetilde{K}(Ci) \cong \widetilde{K}(Ci/CA) \cong \widetilde{K}^n(X/A)$. □

Remark 2.2.3 It is often convenient to lift also K-theory to a unreduced cohomology theory (we will not spell this out, but here one focuses the attention on pairs of spaces instead of pointed spaces). This can be easily done by setting

$$K^n(X) := \widetilde{K}^n(X_+) \quad , \quad K^n(X, A) := \widetilde{K}^n(X, A).$$

One readily checks that for $n = 0$ this definition agrees with our original K-theory, $K^0(X) \cong K(X)$; and moreover $K^1(X) \cong \widetilde{K}^1(X)$. Bott periodicity states, in the unreduced case, that $K^0(X) \cong K^2(X)$.

One fundamental result for reduced singular cohomology is that it is represented (in the homotopy category of pointed CW-complexes) by the Eilenberg-MacLane spaces, that is, there are natural isomorphisms

$$\widetilde{H}^n(-; A) \overset{\cong}{\Longrightarrow} [-, K(A, n)]_*$$

(here A is an abelian group). The suspension isomorphism implies that for any pointed CW-complex X there are natural bijections

$$[X, K(A, n)]_* \cong \widetilde{H}^n(X; A) \cong \widetilde{H}^{n+1}(\Sigma X; A) \cong [\Sigma X, K(A, n+1)]_* \cong [X, \Omega K(A, n+1)]_*$$

which implies that there are weak homotopy equivalences

$$K(A, n) \xrightarrow{\simeq} \Omega K(A, n+1).$$

It will be useful (and extremely important) to give a name to this:

Definition. An **Ω-spectrum** is a sequence of pointed spaces (E_n) with weak homotopy equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.

We will give a more general description of this in the next chapter, but at this point it is important to note that this observation is true for any reduced generalized cohomology theory: any reduced cohomology theory comes from an Ω-spectrum. This is an important result due to Brown that we mention here without further comment (for a proof, see [38]) :

Theorem 2.2.4 (Brown representability) *The functor*

$$\left\{ \quad \Omega\text{-spectra} \quad \right\} \longrightarrow \left\{ \begin{array}{c} \text{reduced generalized} \\ \text{cohomology theories on} \\ \text{pointed CW-complexes} \end{array} \right\},$$

where the assignment sends every Ω-spectrum $E = (E_n)$ to the cohomology theory $\widetilde{E}^n := [-, E_n]_*$, is essentially surjective.

We cannot apply the previous result directly to reduced K-theory as we have only defined it for compact Hausdorff spaces. However, as we will see, there is an Ω-spectrum representing reduced K-theory, that can be used to extend it to a cohomology theory in all pointed CW-complexes.

The following result is key in the theory of principal G-bundles (see [40] for a proof):

Theorem 2.2.5 *If G is a topological group, and X is a paracompact*[1] *space, there exists a space BG (unique up to homotopy equivalence) and a principal G-bundle $EG \longrightarrow BG$ such that*[2]

$$[X, BG] \xrightarrow{\cong} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles over } X \end{array} \right\},$$

where every homotopy class of map $f : X \longrightarrow BG$ corresponds to the principal G-bundle $f^*EG \longrightarrow X$.

Corollary 2.2.6 $\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, BU(n)]$ *for any paracompact space X.*

Proof. It is a well-know result that there is a bijection between complex vector bundles of rank n over X and principal $U(n)$-bundles over X. Then this is a direct consequence of the previous theorem. □

The space $BU(n)$ can be realized as the infinite Grassmanian $G_n(\mathbb{C}^\infty) := \mathrm{colim}_k\, G_n(\mathbb{C}^k)$, where $G_n(\mathbb{C}^k)$ is the space of n-dimensional linear subspaces of $\mathbb{C}^k$.

Now consider X a compact Hausdorff space, and define the function $\dim : \mathrm{Vect}^\bullet_{\mathbb{C}}(X) \longrightarrow [X, \mathbb{N}]$ taking every vector bundle E to the map that takes every point to the dimension of its fibre. The set $[X, \mathbb{N}]$ is a commutative monoid and its Grothendieck construction is precisely $K([X, \mathbb{N}]) = [X, \mathbb{Z}]$. By the universal property of the Grothendieck construction, dim extends to a map $\widehat{\dim} : K(X) \longrightarrow [X, \mathbb{Z}]$. Let $\widehat{K}(X) := \mathrm{Ker}\,\widehat{\dim}$.

Lemma 2.2.7 *Let X be a compact Hausdorff space. There is a split short exact sequence*

$$0 \longrightarrow \widehat{K}(X) \longrightarrow K(X) \longrightarrow [X, \mathbb{Z}] \longrightarrow 0,$$

so $K(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}]$. In particular, if X is a pointed connected space, $\widehat{K}(X) \cong \widetilde{K}(X)$.

---

[1] This includes CW-complexes, compact Hausdorff spaces, topological manifolds, ...

[2] The space EG is the same as in 1.1.6, and $BG := EG/G$. Another possible construction is the following: if $G^{*n} = G * \overset{n}{\cdots} * G$ denotes the n iterated join of G, there are inclusions $G^{*n} \hookrightarrow G^{*(n+1)}$ and $EG := \mathrm{colim}_n\, G^{*n}$. The multiplication of G induces a G-action on it, and as before $BG := EG/G$.

Proof. A section of $\widehat{\dim}$ can be constructed as follows: any map $f : X \longrightarrow \mathbb{N}$ has compact image, thus finite; say $f(X) = \{n_1, \ldots, n_r\}$. If $X_i = f^{-1}(n_i)$, then $X = \coprod X_i$ and we define a bundle over X by taking the trivial bundles $\underline{n_i}$ at each X_i. This defines a map $[X, \mathbb{N}] \longrightarrow \mathrm{Vect}^{\bullet}_{\mathbb{C}}(X)$ which extends to the desired section by the universal property of K.

The last part follows from the 2-out-of-3 property of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{K}(X) \arrow[r] \arrow[d] & K(X) \arrow[r, "\widehat{\dim}"] \arrow[d, equal] & {[X, \mathbb{Z}]} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{K}(X) \arrow[r] & K(X) \arrow[r] & {[*, \mathbb{Z}]} \arrow[r] & 0
\end{tikzcd}$$

since the third vertical map is an isomorphism when X is connected. □

We can finally describe the space which represents reduced K-theory. The inclusions $U(n) \hookrightarrow U(n+1)$ induce maps on the classifying spaces $BU(n) \hookrightarrow BU(n+1)$ (these are actually embeddings). Set $BU := \mathrm{colim}_n\, BU(n)$.

Proposition 2.2.8 *Let X be a compact Hausdorff space. Then*

$$K(X) \cong [X, BU \times \mathbb{Z}],$$

and if X is pointed,

$$\widetilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*$$

Proof. The first observation is that $\widehat{K}(X) \cong \mathrm{colim}_n \mathrm{Vect}^n_{\mathbb{C}}(X)$ for X compact Hausdorff (compare with 2.1.2). By 2.2.6, we get that

$$\widehat{K}(X) \cong \operatorname*{colim}_n \mathrm{Vect}^n_{\mathbb{C}}(X) \cong \mathrm{colim}[X, BU(n)] \cong [X, BU].$$

An argument of general topology ensures that the colimit commutes with the homotopy classes, as X is compact and the maps which define BU are closed inclusions of T_1 spaces. Therefore, by 2.2.7 we get

$$K(X) \cong \widehat{K}(X) \oplus [X, \mathbb{Z}] \cong [X, BU] \oplus [X, \mathbb{Z}] \cong [X, BU \times \mathbb{Z}],$$

and the last part follows from the computation that $[X, BU \times \mathbb{Z}]_*$ is precisely the kernel of the map

$$K(X) \cong [X, BU \times \mathbb{Z}] \xrightarrow{i^*} [*, BU \times \mathbb{Z}] \cong K(*),$$

which can be done using the fact that our spaces are well-pointed. □

How does this relate with the description of reduced K-theory as a cohomology theory? Bott periodicity 2.1.10 suggests that $BU \times \mathbb{Z}$ and $\Omega^2(BU \times \mathbb{Z})$ might be (weak) homotopy equivalent. This is actually the case, but we will take it for granted here (see [38, 11.60] for a proof):

Theorem 2.2.9 (Topological Bott periodicity) *There is a weak homotopy equivalence*

$$BU \times \mathbb{Z} \simeq \Omega^2 BU,$$

or equivalently,

$$\Omega^2 U \simeq U.$$

The first consequence is that we can extend K-theory to all CW-complexes (not necessarily compact Hausdorff) by setting

$$K(X) := [X, BU \times \mathbb{Z}] \qquad , \qquad \widetilde{K}(X) := [X, BU \times \mathbb{Z}]_*$$

The second, with more importance for us, is that this classifying space forms an Ω-spectrum which represents reduced K-theory: for $n \geq 0$, set

$$K_{2n} := BU \times \mathbb{Z} \qquad , \qquad K_{2n+1} := \Omega BU.$$

Then $\widetilde{K}^n(X) \cong [X, K_n]_*$ and by Bott periodicity $K_n \simeq \Omega K_{n+1}$.

Remark 2.2.10 Along this entire section we have focused our attention on complex K-theory. A very similar discussion can be done about reduced KO-theory with some modifications: for real vector bundles, we should have taken the classifying spaces $BO(n)$ and the colimit BO, concluding that reduced KO-theory is represented by $BO \times \mathbb{Z}$. Bott periodicity states in this case that

$$BO \times \mathbb{Z} \simeq \Omega^8 BO,$$

so $\widetilde{KO}(X) \cong \widetilde{KO}(\Sigma^8 X)$, which gives a 8-periodic reduced cohomology theory. The Ω-spectrum representing reduced KO-theory is in this case $KO_{8n+i} := \Omega^{8-i}(BO \times \mathbb{Z})$, for $0 \leq i < 8$.

2.3 KR-theory

In the previous two sections, we reviewed the foundations of complex and real K-theory. We are now interested in extending these theories to G-spaces. This is possible and it is called **equivariant K-theory**, and was developed by Segal in [36]. However, we will focus our attention on the particular case $G = \mathbb{Z}/2$, and will take a different version of vector bundles over $\mathbb{Z}/2$-spaces. This is called *K-theory with Reality* and was carried out by Atiyah in [1]. This has the advantage that it encodes complex and real K-theory, as we will see. As before, spaces will be assumed compact Hausdorff, unless otherwise stated.

The first observation is that for a $\mathbb{Z}/2$-space X, the action is completely determined by an involution $\tau : X \xrightarrow{\cong} X$, that is, an homeomorphism such that $\tau^2 = \mathrm{Id}$. For $x \in X$, we will usually write $\overline{x} := \tau(x)$, motivated by the standard $\mathbb{Z}/2$-action on $\mathbb{C}$ given by complex conjugation.

Definition. Let X be a $\mathbb{Z}/2$-space. A **Real**[3] **vector bundle** over X is a $\mathbb{Z}/2$-space E together with a $\mathbb{Z}/2$-equivariant map $\pi : E \longrightarrow X$ such that

(*i*) $\pi : E \longrightarrow X$ is a complex vector bundle over X.

(*ii*) The map $\tau_x : E_x \longrightarrow E_{\overline{x}}$ is anti-linear.

The "bar" notation happens to be useful: using this, (*i*) says that $\pi(\overline{e}) = \overline{\pi(e)}$ for $e \in E$ (so the fibre of x maps indeed to the fibre of $\overline{x}$); and moreover for $\lambda \in \mathbb{C}$, we have $\overline{\lambda e} = \overline{\lambda}\overline{e}$. This only differs from Segal's notion of $\mathbb{Z}/2$-vector bundle in the fact that here we ask the involution to be fibrewise anti-linear, instead of linear.

[3]Caution! Capitalized *Real* means something different than *real*.

Examples 2.3.1 1. If X is a $\mathbb{Z}/2$-space, and we consider over $\mathbb{C}^n$ the $\mathbb{Z}/2$-action given by complex conjugation termwise, then the trivial vector bundle $X \times \mathbb{C}^n = \underline{n} \longrightarrow X$ is a Real vector bundle.

2. Let H be the tautological line bundle over $\mathbb{CP}^n$. Then taking complex conjugation componentwise defines $\mathbb{Z}/2$-actions on both spaces and in particular $H \longrightarrow \mathbb{CP}^n$ becomes a Real vector bundle.

3. Let $E \longrightarrow X$ be a real vector bundle in the category of $\mathbb{Z}/2$-spaces, that is, a $\mathbb{Z}/2$-equivariant map between $\mathbb{Z}/2$-spaces which is a real vector bundle over X, with the involution of E being linear. Then its complexification $E \otimes \mathbb{C}$ can be made into a Real vector bundle over X, by extending fibrewise the involution $\tau_x : E_x \otimes \mathbb{C} \longrightarrow E_{\overline{x}} \otimes \mathbb{C}$ anti-linearly.

4. If $E, E' \longrightarrow X$ are Real vector bundles over X, then $E \oplus E'$ and $E \otimes E'$ are again Real vector bundles, with involutions

$$\overline{(e, e')} := (\overline{e}, \overline{e'}) \qquad , \qquad \overline{e \otimes e'} := \overline{e} \otimes \overline{e'}.$$

5. If $f : X \longrightarrow Y$ is a $\mathbb{Z}/2$-map, and $E \longrightarrow Y$ is a Real vector bundle, then the pullback bundle $f^*E \longrightarrow X$ is also a Real vector bundle.

The previous examples show that we have Real trivial bundles, and also that the direct sum and tensor product of Real vector bundles are Real vector bundles. So as in the nonequivariant case, if $\mathrm{Vect}_R^\bullet(X)$ denotes the set of isomorphism classes of Real vector bundles over a $\mathbb{Z}/2$-space X, then

$$(\mathrm{Vect}_R^\bullet(X), \oplus, \otimes, \underline{0}, \underline{1})$$

is a commutative semiring, so we can make a similar construction:

Definition. Let X be a $\mathbb{Z}/2$-space. The ***K*-theory with Reality** of X is the Grothendieck construction

$$KR(X) := K(\mathrm{Vect}_R^\bullet(X)).$$

Observe that if $f : X \longrightarrow Y$ is a $\mathbb{Z}/2$-map, just as in the nonequivariant case, there is an induced map $f^* : KR(Y) \longrightarrow KR(X)$.

Definition. The **reduced *KR*-theory** of X is

$$\widetilde{KR}(X) := \mathrm{coker}(K(*) \longrightarrow K(X))$$

induced by the unique $\mathbb{Z}/2$-map $X \longrightarrow *$, where the one-point space has obviously the trivial action.

As for nonequivariant spaces, we have functors

$$KR : (\mathbb{Z}/2)\mathsf{CHaus} \longrightarrow \mathsf{Rng} \qquad , \qquad \widetilde{KR} : (\mathbb{Z}/2)\mathsf{CHaus} \longrightarrow \mathsf{Ab}.$$

Complex and real K-theory are very easily recovered: a simple observation of Linear Algebra is enough to relate KR-theory with KO-theory:

Lemma 2.3.2 *There is an equivalence of categories*

$$\{\,\mathbb{R}\text{-vector spaces}\,\} \xrightarrow{\ \simeq\ } \left\{ \begin{array}{c} \mathbb{C}\text{-vector spaces with} \\ \text{anti-linear involution} \end{array} \right\}$$

where the assignment sends every real vector space V to $V \otimes \mathbb{C}$.

Proof. The inverse functor maps every complex vector space V with an anti-linear $\mathbb{Z}/2$-action to $V^{\mathbb{Z}/2}$, the fixed points of the action. Observe that this is well-defined: if $e \in V^{\mathbb{Z}/2}$ and $\lambda \in \mathbb{R}$, then $\lambda e \in V^{\mathbb{Z}/2}$ because $\overline{\lambda e} = \lambda \overline{e} = \lambda e$.

On the one hand, if V is a $\mathbb{R}$-vector space, then $(V \otimes \mathbb{C})^{\mathbb{Z}/2} \cong V \otimes \mathbb{R} \cong V$ since the involution only takes place in $\mathbb{C}$.

On the other hand, if V is a $\mathbb{C}$-vector space with anti-linear $\mathbb{Z}/2$-action, then consider the subspace $V' := \{v \in V : \overline{v} = -v\}$. Then it yields that $V = V^{\mathbb{Z}/2} \oplus V'$: the intersection is clearly trivial, and the sum is V since any $v \in V$ can be written as $v = \frac{1}{2}(v + \overline{v}) + \frac{1}{2}(v - \overline{v})$. In particular, multiplication by $i \in \mathbb{C}$ defines a linear isomorphism $\cdot i : V^{\mathbb{Z}/2} \xrightarrow{\cong} V'$, so we get an isomorphism of $\mathbb{C}$-vector spaces with anti-linear involution

$$V \cong V^{\mathbb{Z}/2} \oplus iV^{\mathbb{Z}/2},$$

where on the right-hand side the involution is given by $\overline{(v_1, iv_2)} = (v_1, -iv_2)$. We conclude that

$$V^{\mathbb{Z}/2} \otimes \mathbb{C} \cong V^{\mathbb{Z}/2} \otimes (\mathbb{R} \oplus i\mathbb{R}) \cong V^{\mathbb{Z}/2} \oplus iV^{\mathbb{Z}/2} \cong V.$$

It is routine to check that all isomorphisms involved are natural on spaces, so this certainly defines an equivalence of categories. □

Corollary 2.3.3 *If X has the trivial $\mathbb{Z}/2$-action, then there is an isomorphism of commutative semirings*

$$\begin{aligned} \mathrm{Vect}^{\bullet}_{\mathbb{R}}(X) &\xrightarrow{\cong} \mathrm{Vect}^{\bullet}_{R}(X) \\ E &\longmapsto E \otimes \underline{1} \end{aligned}$$

In particular, $KR(X) \cong KO(X)$.

Proof. If the $\mathbb{Z}/2$-action is trivial, then for a Real vector bundle E we have fibrewise anti-linear involutions $\tau_x : E_x \longrightarrow E_x$, and therefore a bijection by the previous proposition. The morphism obviously preserves direct sums, but also tensor products since for $\mathbb{R}$-vector spaces V, V', we have a canonical isomorphism $(V \otimes_{\mathbb{R}} V') \otimes_{\mathbb{R}} \mathbb{C} \cong (V \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (V' \otimes_{\mathbb{R}} \mathbb{C})$ as complex vector spaces. □

For complex K-theory, we have the following

Proposition 2.3.4 *If X is a compact Hausdorff space, then there is an isomorphism of commutative semirings*

$$\begin{aligned} \mathrm{Vect}^{\bullet}_{R}(X \textstyle\coprod X) &\xrightarrow{\cong} \mathrm{Vect}^{\bullet}_{\mathbb{C}}(X) \\ E &\longmapsto E_{|X} \end{aligned}$$

where $X \coprod X \cong X \times \{\pm i\}$ as $\mathbb{Z}/2$-spaces. In particular, $KR(X \coprod X) \cong K(X)$.

Proof. Let $\overline{E}$ denote the conjugated vector bundle over X. Then the inverse is given by taking every complex vector bundle $p : E \longrightarrow X$ to $E \coprod \overline{E} \longrightarrow X \coprod X$), where $E \coprod \overline{E}$ has also the obvious switching map as involution and the map is just p factorwise. This is clearly $\mathbb{Z}/2$-equivariant, and moreover the action on $E \coprod \overline{E}$ is fibrewise anti-linear, since $E_x \longrightarrow \overline{E}_{\overline{x}} = \overline{E}_x$ is just the anti-linear extension of the identity. □

Remark 2.3.5 Classically, spaces with an involution (ie, $\mathbb{Z}/2$-spaces) are called *Real spaces*; and complex vector spaces with anti-linear involution are called *Real vector spaces*. This is motivated by 2.3.2, and also gives motivation for the name of *K-theory with Reality*.

According to 2.3.3, $KR(*) \cong KO(*) \cong \mathbb{Z}$, so $\widetilde{KR}(X) \cong KR(X)/\mathbb{Z}$ for a $\mathbb{Z}/2$-space X, just as in the nonequivariant case. Moreover, the proof of 2.1.3 is still valid, so we get a splitting

$$KR(X) \cong \widetilde{KR}(X) \oplus \mathbb{Z}$$

and $\widetilde{KR}(X) \cong \mathrm{Ker}(KR(X) \longrightarrow KR(x_0))$ for a pointed $\mathbb{Z}/2$-space (X, x_0).

Trying to mimic the first section of this chapter, now we are aimed to prove a product theorem and a periodicity theorem. For that, the first thing that we need are spheres with a $\mathbb{Z}/2$-action. We will use the following description (though not the standard one): let $p, q \geq 0$ and let $V^{p,q} := \mathbb{R}^p \oplus i\mathbb{R}^q$ with the involution given by complex-conjugation and the standard metric, so this is an orthogonal $\mathbb{Z}/2$-representation. We will denote

$$S^{p,q} := S^{V^{p,q}}$$

for the one-point compactification of $V^{p,q}$, so this is a $(p+q)$-dimensional sphere. In particular, this means that

$$S^{p,q} \cong S(S(V^{p,q})) \cong S(V^{p+1,q})$$

as $\mathbb{Z}/2$-spaces, where $S(V^{p,q})$ denotes the unit sphere of $V^{p,q}$ and S is the unreduced suspension (where the trivial action is taken in the unit interval).

One can further show that all unit spheres arise as the one-point compactification of a $\mathbb{Z}/2$-representation, except for the unit sphere of $i\mathbb{R}^q$, which has the antipodal action (and therefore no fixed points). To include this one also in our notation we let $S^{-1,q+1} := S(i\mathbb{R}^{q+1})$, so that $S^{p,q}$ is a $(p+q)$-dimensional sphere for $p \geq -1$, $q \geq 0$ and $p + q \geq 0$, and the expresion $S^{p,q} \cong S(V^{p+1,q})$ is always valid.

As for the nonequivariant case, if X, Y are $\mathbb{Z}/2$-spaces, we consider

$$\mu := \pi_1^* \otimes \pi_2^* : KR(X) \otimes KR(Y) \longrightarrow KR(X \times Y)$$

the **external product** for KR-theory, where π_1, π_2 denote the projections.

In the product theorem 2.1.5 we had that the isomorphism involved $\mathbb{CP}^1$, which as $\mathbb{Z}/2$-space happens to be homeomorphic to $S^{1,1}$ (just because $\mathbb{CP}^1 \cong S^{\mathbb{R}\oplus i\mathbb{R}}$). With a bit of care, one can check that the proof given for the product theorem for complex K-theory can be amended for the equivariant case, where we replace S^2 by $S^{1,1}$. A proof of this fact can be found in [1, 2.1], but here we just state the result without further comment.

Theorem 2.3.6 (Product theorem) *Let X be a $\mathbb{Z}/2$-space. The composite*

$$KR(X) \otimes \mathbb{Z}[x]/(x-1)^2 \longrightarrow KR(X) \otimes KR(S^{1,1}) \xrightarrow{\mu} KR(X \times S^{1,1})$$

is a ring isomorphism.

Observe that here x represents the tautological line bundle over $\mathbb{CP}^1$, which is a Real vector bundle. The same argument as in the nonequivariant case gives

Corollary 2.3.7 *The map* $\mathbb{Z}[x]/(x-1)^2 \xrightarrow{\cong} KR(S^{1,1})$ *is a ring isomorphism. Moreover, the external product*

$$\mu : KR(X) \otimes KR(S^{1,1}) \xrightarrow{\cong} KR(X \times S^{1,1})$$

is an isomorphism.

Under this result, now Bott periodicity for KR-theory follows exactly by the same reasons as for K-theory: for X, Y pointed $\mathbb{Z}/2$-spaces, the external product for KR-theory restricts to a reduced external product

$$\widetilde{\mu} : \widetilde{KR}(X) \otimes \widetilde{KR}(Y) \longrightarrow \widetilde{KR}(X \wedge Y).$$

Then if β denotes the composite

$$\widetilde{KR}(X) \xrightarrow{(x-1)\otimes -} \widetilde{KR}(S^{1,1}) \otimes \widetilde{KR}(X) \xrightarrow{\widetilde{\mu}} \widetilde{KR}(S^{1,1} \wedge X),$$

we obtain

Theorem 2.3.8 (Equivariant Bott periodicity) *The previous homomorphism*

$$\beta : \widetilde{KR}(X) \xrightarrow{\cong} \widetilde{KR}(S^{1,1} \wedge X)$$

is an isomorphism for all based compact Hausdorff $\mathbb{Z}/2$*-spaces* X.

Higher KR-theory groups

Now we will discuss the "cohomological flavour" of (reduced) KR-theory. The way we obtained higher K-theory groups in the nonequivariant case was by taking the iterated suspension of a space. Of course, in the category of based $\mathbb{Z}/2$-spaces, we do not have the usual suspension, but we do have spheres: for a based $\mathbb{Z}/2$-space X, we consider

$$\Sigma^{p,q} X := S^{p,q} \wedge X$$

for $p \geq -1, q \geq 0$ and $p + q \geq 0$. Then, mimicking K-theory, we simply define the higher KR-theory groups as

$$\widetilde{KR}^{-p,-q}(X) := \widetilde{KR}(\Sigma^{p,q} X) \qquad , \qquad \widetilde{KR}^{-p,-q}(X, A) := \widetilde{KR}^{-p,-q}(X/A)$$

where $A \subset X$ is a closed subspace. With this notation, Bott periodicity states that

$$\widetilde{KR}^{-p,-q}(X) \cong \widetilde{KR}^{-p-1,-q-1}(X)$$

since $S^{p,q} \wedge S^{p',q'} \cong S^{p+p',q+q'}$ as $\mathbb{Z}/2$-spaces (for $p' \geq 0$). We can as well extend these groups to positive integers using Bott periodicity: for $p \geq -1, q \geq 0$ and $p + q \geq 0$, we set

$$\widetilde{KR}^{p,q}(X) := \widetilde{KR}(S^{n-p,n-q} \wedge X) \qquad , \qquad \widetilde{KR}^{p,q}(X, A) := \widetilde{KR}^{p,q}(X/A)$$

for any $n > p, q$ (compare with page 24).

Remark 2.3.9 In the equivariant case we can also lift these higher groups to unreduced ones, as usual by setting

$$KR^{-p,-q}(X) := \widetilde{KR}^{-p,-q}(X_+) \qquad , \qquad KR^{-p,-q}(X,A) := \widetilde{KR}^{-p,-q}(X,A)$$

for a $\mathbb{Z}/2$-space X and $A \subset X$ a closed subspace (and the disjoint point is considered to be fixed). It is important to note that, under this description,

$$\begin{aligned} KR^{-p,-q}(X,A) = \widetilde{KR}^{-p,-q}(X/A) = \widetilde{KR}(S^{p,q} \wedge X/A) &\cong \widetilde{KR}(D^{p,q}/S^{p-1,q} \wedge X/A) \\ &\cong KR(X \times D^{p,q}, X \times S^{p-1,q} \cup A \times D^{p,q}) \end{aligned}$$

for $p,q \geq 0$, where $D^{p,q} = D(V^{p,q})$ is the unit disk of $V^{p,q}$, so that $\partial D^{p,q} \cong S^{p-1,q}$.

If we consider the spheres $S^{p,0}$ with the trivial action, then we can mimic the argument that gives the long exact sequence of a pair in the nonequivariant case.

Proposition 2.3.10 *Let (X, x_0) be a pointed $\mathbb{Z}/2$-space. Then the following properties hold:*

1. *If $f, g : X \longrightarrow Y$ are homotopic, then $f^* = g^* : \widetilde{KR}(Y) \longrightarrow \widetilde{KR}(X)$.*
2. *If (X, A) is a pointed pair of $\mathbb{Z}/2$-spaces, with A closed, there is a long exact sequence*

$$\cdots \longrightarrow \widetilde{KR}^{-2,0}(X) \longrightarrow \widetilde{KR}^{-2,0}(A) \longrightarrow \widetilde{KR}^{-1,0}(X/A) \longrightarrow \widetilde{KR}^{-1,0}(X) \longrightarrow \widetilde{KR}^{-1,0}(A) \longrightarrow \widetilde{KR}^{0,0}(X/A) \longrightarrow \widetilde{KR}^{0,0}(X) \longrightarrow \widetilde{KR}^{0,0}(A)$$

3. *If $(X_i)_{i\in I}$ is a collection of pointed $\mathbb{Z}/2$-spaces, then*

$$\widetilde{KR}(\bigvee_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} \widetilde{KR}(X_i)$$

is an isomorphism.

A mere algebraic consequence of the previous long exact sequence is the following

Corollary 2.3.11 *Let (X, X', X'') be a pointed triple of $\mathbb{Z}/2$-spaces. Then there is a long exact sequence*

$$\cdots \longrightarrow \widetilde{KR}^{-2,0}(X,X'') \longrightarrow \widetilde{KR}^{-2,0}(X',X'') \longrightarrow \widetilde{KR}^{-1,0}(X,X') \longrightarrow \widetilde{KR}^{-1,0}(X,X'') \longrightarrow \widetilde{KR}^{-1,0}(X',X'') \longrightarrow \widetilde{KR}^{0,0}(X,X') \longrightarrow \widetilde{KR}^{0,0}(X,X'') \longrightarrow \widetilde{KR}^{0,0}(X',X'')$$

Corollary 2.3.12 *Let (X, A) be a pointed pair of $\mathbb{Z}/2$-spaces. For every $q \geq 0$ there is a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{KR}^{-2,-q}(X) \arrow[r] & \widetilde{KR}^{-2,-q}(A) \arrow[dll, rounded corners] \\
\widetilde{KR}^{-1,-q}(X/A) \arrow[r] & \widetilde{KR}^{-1,-q}(X) \arrow[r] & \widetilde{KR}^{-1,-q}(A) \arrow[dll, rounded corners] \\
\widetilde{KR}^{0,-q}(X/A) \arrow[r] & \widetilde{KR}^{0,-q}(X) \arrow[r] & \widetilde{KR}^{0,-q}(A)
\end{tikzcd}$$

Proof. Let us see that we can obtain the desired sequence by rewriting the long exact sequence of the triple

$$(X \times D^{0,q}, X \times S^{0,q-1} \cup A \times D^{0,q}, X \times S^{0,q-1} \cup x_0 \times D^{0,q}),$$

where $x_0 \in A \subset X$ is the basepoint. We simply compute

$$\begin{aligned}
\widetilde{KR}^{-p,0}(X \times D^{0,q}, X \times S^{0,q-1} \cup A \times D^{0,q}) &\cong \widetilde{KR}^{-p,0}(D^{0,q}/S^{0,q-1} \wedge X/A) \\
&\cong \widetilde{KR}^{-p,0}(S^{0,q} \wedge X/A) = \widetilde{KR}(S^{p,0} \wedge S^{0,q} \wedge X/A) \\
&\cong \widetilde{KR}^{-p,-q}(X/A) = \widetilde{KR}^{-p,-q}(X, A),
\end{aligned}$$

where we used that, for pairs of $\mathbb{Z}/2$-spaces $(X, A), (Y, B)$, we have

$$\frac{X \times Y}{X \times B \cup A \times Y} \cong X/A \wedge Y/B$$

also as $\mathbb{Z}/2$-spaces. Similarly,

$$\begin{aligned}
\widetilde{KR}^{-p,0}(X \times D^{0,q}, X \times S^{0,q-1} \cup x_0 \times D^{0,q}) &\cong \widetilde{KR}^{-p,0}(S^{0,q} \wedge X) \\
&\cong \widetilde{KR}(S^{p,q} \wedge X/A) \cong \widetilde{KR}^{-p,-q}(X, A).
\end{aligned}$$

Finally, observe that the inclusion $A \times D^{0,q} \hookrightarrow X \times S^{0,q-1} \cup A \times D^{0,q}$ induces, by the universal property of the quotient topology, a continuous bijection

$$\frac{A \times D^{0,q}}{A \times S^{0,q-1} \cup x_0 \times D^{0,q}} \longrightarrow \frac{X \times S^{0,q-1} \cup A \times D^{0,q}}{X \times S^{0,q-1} \cup x_0 \times D^{0,q}}$$

between compact Hausdorff spaces, thus an homeomorphism. Therefore, we obtain

$$\begin{aligned}
&\widetilde{KR}^{-p,0}(X \times S^{0,q-1} \cup A \times D^{0,q}, X \times S^{0,q-1} \cup x_0 \times D^{0,q}) \\
\cong\ &\widetilde{KR}^{-p,0}(A \times D^{0,q}, A \times S^{0,q-1} \cup x_0 \times D^{0,q}) \\
\cong\ &\widetilde{KR}^{-p,0}(S^{0,q} \wedge A) \cong \widetilde{KR}^{-p,-q}(A)
\end{aligned}$$

□

For a pointed pair of $\mathbb{Z}/2$-spaces (X, A), and $n \geq 0$, it is customary to set

$$\widetilde{KR}^{-n}(X, A) := \widetilde{KR}^{-n,0}(X, A) \qquad , \qquad \widetilde{KR}^{n}(X, A) := \widetilde{KR}^{0,-n}(X, A).$$

Corollary 2.3.13 *There is a long exact sequence*

$$\cdots \longrightarrow \widetilde{KR}^{-1}(X) \longrightarrow \widetilde{KR}^{-1}(A) \longrightarrow \widetilde{KR}^{0}(X,A) \longrightarrow \widetilde{KR}^{0}(X) \longrightarrow \widetilde{KR}^{0}(A) \longrightarrow \widetilde{KR}^{1}(X,A) \longrightarrow \widetilde{KR}^{1}(X) \longrightarrow \cdots$$

Proof. It follows from 2.3.8, 2.3.10 and 2.3.12. □

Moreover, from this we can also conclude that $\widetilde{KR}^n$ is a reduced cohomology theory on $(\mathbb{Z}/2)\mathsf{CHaus}$.

Represented KR-theory

We continue now our analogy with nonequivariant K-theory. Our next aim is to find a $\mathbb{Z}/2$-space which represents reduced KR-theory, as we did before. For complex K-theory, we showed that $\widetilde{K}(X) \cong [X, BU \times \mathbb{Z}]_*$. We will see that this classifying space will be enough for our purpose.

The first step is to imitate the notion of principal G-bundle for Γ-spaces, where Γ is a finite group (or a compact Lie group) which also acts on G. Of course, we want to think $\Gamma = \mathbb{Z}/2$ for our goal. Here we will only present the main results, and refer the interested reader to [39, §I.8] for further details about principal G-bundles on Γ-spaces.

Let Γ be a discrete group, G a topological group and let $\alpha : \Gamma \longrightarrow \mathrm{Aut}(G)$ be a group homomorphism. We will denote $\alpha(\gamma)$ by α_γ and additionally require that the map $\Gamma \times G \longrightarrow G$ is continuous.

Definition. Let X be a Γ-space. A (Γ, α)**-equivariant principal** G**-bundle** over X is a principal G-bundle $p : E \longrightarrow X$ over X such that

(*i*) E is a Γ-space and the projection is Γ-equivariant.

(*ii*) $\gamma(eg) = (\gamma e) \cdot \alpha_\gamma(g)$ for all $\gamma \in \Gamma, g \in G$ and $e \in E$.

Example 2.3.14 If $\alpha : \mathbb{Z}/2 \longrightarrow \mathrm{Aut}(U(n))$ acts by complex conjugation, then a $(\mathbb{Z}/2, \alpha)$-equivariant principal $U(n)$-bundle is the same thing as a Real vector bundle of rank n. Indeed, condition (*ii*) means that $\gamma(eg) = (\gamma e) \cdot \overline{g}$ for $g \in U(n)$, which fibrewise means $\overline{\lambda e} = \overline{\lambda}\overline{e}$ for $\lambda \in \mathbb{C}$.

In the equivariant case, we still have a classification theorem as in 2.2.5:

Theorem 2.3.15 *If X is paracompact, then there exists a Γ-space $B(\Gamma, \alpha, G)$ (unique up to Γ-homotopy equivalence) and a (Γ, α)-equivariant principal G-bundle $E(\Gamma, \alpha, G) \longrightarrow B(\Gamma, \alpha, G)$ such that*[4]

$$_{\Gamma}[X, B(\Gamma, \alpha, G)] \xrightarrow{\cong} \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ (\Gamma, \alpha)\textit{-equivariant principal} \\ G\textit{-bundles over } X \end{array} \right\},$$

[4] In the equivariant case, these spaces can be constructed in the following way: in the first place, we consider a family $\{p_i : E_i \longrightarrow \Gamma/\Lambda_i\}_{i \in I}$ of (Γ, α)-equivariant principal G-bundles where E_i is Hausdorff and Γ/Λ_i is an orbit space. If $L := \coprod_i E_i$, then setting $EG := \mathrm{colim}_n\, L^{*n}$ (iterated join) and $BG := EG/G$ gives the desired spaces.

where every Γ-homotopy class of map $f : X \longrightarrow BG$ corresponds to $f^*EG \longrightarrow X$.

For our case of interest, with $\Gamma = \mathbb{Z}/2$ and $G = U(n)$, the involution $U(n) \longrightarrow U(n)$ given by complex conjugation termwise induces an involution $BU(n) \longrightarrow BU(n)$ on classifying spaces, so $BU(n)$ is a $\mathbb{Z}/2$-space. In particular, $BU(n)$ with this involution is a model for $B(\mathbb{Z}/2, \alpha, U(n))$, as the following proposition asserts:

Proposition 2.3.16 $\mathrm{Vect}^n_R(X) \cong {}_{\mathbb{Z}/2}[X, BU(n)]$, *for any paracompact $\mathbb{Z}/2$-space X.*

Proof. We need to check that the bijection of 2.2.6 restricts to the desired correspondence. For recall that there are explicit models for $EU(n)$ and $BU(n)$: if $G_n(\mathbb{C}^k)$ is the space of n-dimensional linear subspaces of $\mathbb{C}^k$ and $E_n(\mathbb{C}^k) = \{(\ell, v) \in G_n(\mathbb{C}^k) \times \mathbb{C}^k : v \in \ell\}$ then $EU(n) = \mathrm{colim}_k\, E_n(\mathbb{C}^k)$ and $BU(n) = \mathrm{colim}_k\, G_n(\mathbb{C}^k)$.

Now observe that there are $\mathbb{Z}/2$-actions on $G_n(\mathbb{C}^k)$ and $E_n(\mathbb{C}^k)$ given by taking complex conjugation componentwise, so in particular both are $\mathbb{Z}/2$-spaces and the projection $E_n(\mathbb{C}^k) \longrightarrow G_n(\mathbb{C}^k)$ is a Real vector bundle. Since this involution is compatible with the inclusions $G_n(\mathbb{C}^k) \subset G_n(\mathbb{C}^{k+1})$ and $E_n(\mathbb{C}^k) \subset E_n(\mathbb{C}^{k+1})$, we conclude that $EU(n) \longrightarrow BU(n)$ is a Real vector bundle. Therefore, the pullback of $EU(n)$ along a $\mathbb{Z}/2$-equivariant map $X \longrightarrow BU(n)$ inherits a $\mathbb{Z}/2$-action such that the projection is $\mathbb{Z}/2$-invariant and the involution is fibrewise anti-linear. □

The last observation is that the $\mathbb{Z}/2$-action on $BU(n)$ induced by complex-conjugation is compatible with the embeddings $BU(n) \hookrightarrow BU(n+1)$, so BU inherits a $\mathbb{Z}/2$-action.

Corollary 2.3.17 *Let X be a compact Hausdorff $\mathbb{Z}/2$-space. Then*

$$KR(X) \cong {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}],$$

where $BU \times \mathbb{Z}$ has the trivial involution on $\mathbb{Z}$. Moreover, if X is also pointed, then

$$\widetilde{KR}(X) \cong {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}]_*$$

Proof. The arguments given in 2.2.7 and 2.2.8 are still valid here. □

The last corollary together with (equivariant) Bott periodicity say that for any compact Hausdorff $\mathbb{Z}/2$-space X,

$${}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}]_* \cong \widetilde{KR}(X) \cong \widetilde{KR}(\Sigma^{1,1}X) \cong {}_{\mathbb{Z}/2}[\Sigma^{1,1}X, BU \times \mathbb{Z}]_* \cong {}_{\mathbb{Z}/2}[X, \Omega^{1,1}(BU \times \mathbb{Z})]_*$$

where $\Omega^{1,1} = G\mathrm{Map}(S^{1,1}, -)$. One can show, as in the nonequivariant case, that $BU \times \mathbb{Z}$ and $\Omega^{1,1}(BU \times \mathbb{Z})$ are related:

Theorem 2.3.18 (Equivariant topological Bott periodicity I) *There is a $\mathbb{Z}/2$-equivariant weak homotopy equivalence*

$$BU \times \mathbb{Z} \simeq \Omega^{1,1}(BU \times \mathbb{Z}).$$

As we did in last section, we can now extend KR-theory for all $\mathbb{Z}/2$-CW-complexes (not necessarily compact Hausdorff), setting

$$KR(X) := {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}] \qquad , \qquad \widetilde{KR}(X) := {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}]_*$$

Surprisingly, for KR-theory, we have one more version of Bott periodicity. Recall from 2.3.3 that if X is a based compact Hausdorff space with trivial G-action, then $\widetilde{KR}(X) = \widetilde{KO}(X)$. In particular this implies

$$\begin{aligned} {}_{\mathbb{Z}/2}[X, \Omega^{8,0}(BU \times \mathbb{Z})]_* &\cong {}_{\mathbb{Z}/2}[\Sigma^{8,0}X, BU \times \mathbb{Z}]_* \cong \widetilde{KR}(\Sigma^{8,0}X) \cong \widetilde{KO}(\Sigma^8 X) \cong \widetilde{KO}(X) \\ &\cong [X, BO \times \mathbb{Z}]_* \cong [X, (BU \times \mathbb{Z})^{\mathbb{Z}/2}]_* \overset{(1.5)}{\cong} {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}]_* \end{aligned}$$

This suggests a relation between $BU \times \mathbb{Z}$ and $\Omega^{8,0}(BU \times \mathbb{Z})$, which is indeed true:

Theorem 2.3.19 (Equivariant topological Bott periodicity II) *There is a $\mathbb{Z}/2$-equivariant weak homotopy equivalence*

$$BU \times \mathbb{Z} \simeq \Omega^{8,0}(BU \times \mathbb{Z}).$$

Remark 2.3.20 One would now like to make the $\widetilde{KR}^{p,q}(X)$ groups into a cohomology theory, but at this point it is not clear how to do it, as we index KR-theory with two indices. Such a cohomology theory will be $RO(\mathbb{Z}/2)$-graded, instead of $\mathbb{Z}$-graded. This roughly means that instead of indexing by integers, we will index by representations of $\mathbb{Z}/2$. We will tackle this in 4.2.4.

Chapter 3

Stable homotopy theory

We will devote the next two chapters to the main part of this Master's thesis: (equivariant) stable homotopy theory, where K-theory and KR-theory will be our most important examples. Before giving a detailed construction and description, we would like to start off with some motivation for what follows, picking ideas from [22], [27] , and [32].

Why do we care about stable homotopy theory?

One of the main goals of Algebraic Topology is the study of algebraic invariants of topological spaces, and more concretely of (finite) CW-complexes, such as homology, cohomology, homotopy groups, K-theory, etc. However, these invariants strongly depend on the dimension of the CW-complex, and more concretely on taking suspensions. We would like to develop some analogues to spaces where the objects are "independent of dimension" and "stable under suspension", for some suitable notions.

In (unstable) homotopy theory we can also encounter examples of such a stable phenomenon: let us look at the table of Figure 3.1, which shows the homotopy groups of the spheres.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/12$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$

Figure 3.1: Homotopy groups of the spheres.

These are, in general, really hard to compute (in fact there is no sphere S^n for $n > 1$ that we know all its homotopy groups) but there is some regularity that we can explain: for instance, the lower left triangle of $\pi_k(S^n) \cong 0$ for $k < n$ is by Cellular Approximation, the upper row of $\pi_k(S^1) \cong 0$ for $k > 1$ is due to covering theory, the diagonal $\pi_n(S^n) \cong \mathbb{Z}$ is by Hurewicz, the two equal rows $\pi_k(S^2) \cong \pi_k(S^3)$ for $k \geq 3$ are given by the Hopf fibration, etc. And there is another regularity shown in the table: diagonals stabilize. This is due to the following stabilization theorem (see [38, 15.46]):

Theorem 3.0.1 (Freudenthal suspension) *Let X be a pointed $(n-1)$-connected space. Then the suspension homomorphism*

$$\pi_{n+k}(X) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma X)$$
$$[S^{n+k} \xrightarrow{f} X] \longmapsto [S^{n+k+1} \xrightarrow{\Sigma f} \Sigma X]$$

is an isomorphism for $k \leq n-2$ and surjective for $k = n-1$.

According with the theorem, $\pi_{n+k}(S^n) \xrightarrow{\cong} \pi_{n+k+1}(S^{n+1})$ is an isomorphism as soon as $n \geq k+2$, so indeed diagonals stabilize. In particular, for any pointed space X, the sequence

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \pi_{k+2}(\Sigma^2 X) \xrightarrow{\Sigma} \pi_{k+3}(\Sigma^3 X) \longrightarrow \cdots$$

will eventually stabilize as $\Sigma^n X$ becomes more and more connected. It is customary to give a name to this stable value.

Definition. Let X be a based space. The ***k*-th stable homotopy group** of X is

$$\pi_k^{st}(X) := \operatorname*{colim}_n \pi_{n+k}(\Sigma^n X).$$

Directly from the definition, we have that $\pi_k^{st}(X) \cong \pi_{k+1}^{st}(\Sigma X)$, just as the suspension isomorphism in a (reduced) homology theory. Of course this is not a coincidence, and we will come back later to this. Moreover, these stable homotopy groups are very related with the class of objects we want to get in stable homotopy theory.

Let us discuss another feature which brings to light this stable phenomenon (and it was in particular the starting point of stable homotopy theory). The following result is also due to Freudenthal (see [32, §3.4]):

Theorem 3.0.2 *Let X be a pointed n-connected CW-complex and let Y be a pointed CW-complex of dimension $\leq 2n$. Then the suspension map*

$$[X, Y]_* \xrightarrow{\Sigma} [\Sigma X, \Sigma Y]_*$$

is a bijection of pointed sets.

This result motivated Spanier to define the **S-category** , with objects pointed CW-complexes and morphisms

$$\mathrm{Hom}_{\mathsf{S}}(X, Y) := \operatorname*{colim}_n [\Sigma^n X, \Sigma^n Y]_*$$

By the suspension-loop adjunction, this is a colimit of groups for $n \geq 1$, which are abelian for $n \geq 2$, so $\mathrm{Hom}_{\mathsf{S}}(X, Y)$ is an abelian group. Moreover, the suspension functor $\Sigma : \mathsf{S} \longrightarrow \mathsf{S}$ is fully faithful (since $\mathrm{Hom}_{\mathsf{S}}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathsf{S}}(\Sigma X, \Sigma Y)$ by definition), but it is not essentially surjective on objects. If we want our desired "stable homotopy category" to be independent of suspensions, then Σ (for some suitable notion) should be an equivalence of categories. One can force this on the S-category by introducing formal desuspensions of spaces, giving rise to the **Spanier-Whitehead category** SW: this has as objects pairs (X, k), where X is a pointed CW-complex and $k \in \mathbb{Z}$; and arrows

$$\mathrm{Hom}_{\mathsf{SW}}((X,k),(Y,r)) := \operatorname*{colim}_n [\Sigma^{k+n}X, \Sigma^{r+n}Y]_*$$

This SW-category is much better than the S-category, in the sense that now $\Sigma : \mathsf{SW} \longrightarrow \mathsf{SW}$ is an equivalence of categories, and moreover it is triangulated, additive and symmetric monoidal. However, it lacks other desirable properties: for instance, it is not large enough to represent cohomology theories. But, respect to this last property, we already saw Brown representability (theorem 2.2.4): every reduced cohomology theory is represented by an Ω-spectrum. Being interested in suspensions rather than loop spaces, it seems sensible to consider the adjoints maps.

Definition. A **spectrum** is a sequence of pointed spaces (E_n) together with structure maps $\Sigma E_n \longrightarrow E_{n+1}$.

Of course, every Ω-spectrum is a spectrum.

This is the starting point to construct the stable homotopy category, and it is possible to do it from this definition (see [27], [38]). However, we will not proceed in this fashion and will take a more modern setup, which will allow us to generalize this notion of spectra to other models and to the equivariant version in a more direct way.

3.1 Diagram spaces

We will follow the modern approach described in *Model categories of diagram spectra* [24] now 20 years ago. Our goal is to define and describe categories $\mathsf{Sp}^{\mathbb{I}}$ and SHC and functors as depicted in the following diagram:

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "(-)_+"] \arrow[d] & \mathsf{Top}_* \arrow[r, "\Sigma^\infty \circ \mathrm{cw}"] \arrow[d] & \mathsf{Sp}^{\mathbb{I}} \arrow[d] & \mathsf{Ab} \arrow[l, "H"'] \arrow[d, "\text{0-th deg}"] \\
\mathsf{Ho(Top)} \arrow[r, "(-)_+"] & \mathsf{Ho(Top_*)} \arrow[r, "\mathbb{L}\Sigma^\infty"] & \mathsf{SHC} \arrow[r, "\pi_*"] \arrow[loop below, "\mathbb{L}\Sigma"] & \mathsf{grAb}
\end{tikzcd}$$

where $\mathbb{L}\Sigma : \mathsf{SHC} \xrightarrow{\simeq} \mathsf{SHC}$ is an equivalence of categories and cw is a CW-approximation functor.

Enriched category theory

We will start our construction of the stable homotopy category with some background on enriched category theory, that we will constantly use. For the elaboration of this section we follow [30, §3.1 – 3.5] for the first part, and [3, §2.3] and [24] for the second, but proofs are original work of the author.

Definition. A **symmetric monoidal category** $(\mathcal{V}, \otimes, *)$ is the data of:

(i) A complete and cocomplete category $\mathcal{V}$,

(ii) A bifunctor $- \otimes - : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$, called **monoidal product**,

(iii) An object $* \in \mathcal{V}$, called the **unit**, together with natural isomorphisms

$$V \otimes W \overset{\cong}{\Longrightarrow} W \otimes V \quad , \quad U \otimes (V \otimes W) \overset{\cong}{\Longrightarrow} (U \otimes V) \otimes W \quad , \quad * \otimes V \overset{\cong}{\Longleftarrow} V \overset{\cong}{\Longrightarrow} V \otimes *$$

which express symmetry, associativity and unit conditions on the monoidal product. These isomorphisms must also satisfy a handful of coherence conditions, see [20, §VII.1]

Examples 3.1.1 One encounters many examples of symmetric monoidal categories in nature:

(a) If R is a commutative ring, the category $(\mathsf{Mod}_R, \otimes_R, R)$ of R-modules together with the tensor product and the ring R (viewed as a module over itself) as unit is symmetric monoidal. In particular, this includes $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ and $(\mathsf{Vect}_k, \otimes_k, k)$.

(b) The categories $(\mathsf{Top}, \times, *)$ and $(\mathsf{Set}, \times, *)$, with the direct product and the one-point space.

(c) The pointed versions of the previous ones $(\mathsf{Top}_*, \wedge, S^0)$ and $(\mathsf{Set}_*, \wedge, S^0)$, with the smash product and the two-points space.

(d) If G is a topological group, then $(G\mathsf{Top}, \times, *)$ as well as its pointed version $(G\mathsf{Top}_*, \wedge, S^0)$ (where S^0 has the trivial action) are symmetric monoidal.

(e) For a fixed space S, the slice category $(\mathsf{Top}/S, \times_S, S)$, with the fibre product and the space S serving as the unit.

We also have preferred functors between symmetric monoidal categories:

Definition. Let $F : (\mathcal{V}, \otimes, *) \longrightarrow (\mathcal{W}, \wedge, \bullet)$ be a functor between symmetric monoidal categories. We say that F is **lax (symmetric) monoidal** if there exist (symmetric), associative and unital natural transformations

$$F(V_1) \wedge F(V_2) \longrightarrow F(V_1 \otimes V_2) \qquad , \qquad \bullet \longrightarrow F(*).$$

When the previous maps are natural isomorphisms, we say that F is **strong (symmetric) monoidal**.

It is usual to find categories where the hom-sets have some extra structure, for instance when they are also objects of the category. There are some subtleties in this concept, so let us be precise about this:

Definition. Let $(\mathcal{V}, \otimes, *)$ be a symmetric monoidal category. A **$\mathcal{V}$-category $\underline{\mathcal{D}}$** or an **enriched category** over $\mathcal{V}$ is the data of:

(*i*) A collection of objects $C, D \ldots \in \underline{\mathcal{D}}$,

(*ii*) For every pair of objects $C, D \in \underline{\mathcal{D}}$, a hom-object $\mathrm{Hom}_{\underline{\mathcal{D}}}(C, D) \in \mathcal{V}$,

(*iii*) For every object $C \in \underline{\mathcal{D}}$, an arrow $\mathrm{Id}_C : * \longrightarrow \mathrm{Hom}_{\underline{\mathcal{D}}}(C, C)$ in $\mathcal{V}$,

(*iv*) For every triple $C, D, E \in \underline{\mathcal{D}}$, an arrow $\circ : \mathrm{Hom}_{\underline{\mathcal{D}}}(D, E) \otimes \mathrm{Hom}_{\underline{\mathcal{D}}}(C, D) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{D}}}(C, E)$ in $\mathcal{V}$,

such that the following diagrams commute[1] for all $C, D, E, F \in \underline{\mathcal{D}}$:

[1]Given an arrow $f : V \otimes W \longrightarrow U$ and an object S in $\mathcal{V}$, the arrow $f \otimes \mathrm{Id}_S : V \otimes W \otimes S \longrightarrow U \otimes S$ is defined as the image of the arrow $(f, \mathrm{Id}_S) : (V \otimes W, S) \longrightarrow (U, S)$ in the category $\mathcal{V} \times \mathcal{V}$ via the functor $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$.

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathcal{D}}}(E,F)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(D,E)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(C,D) \arrow[r, "1\otimes\circ"] \arrow[d, "\circ\otimes 1"] & \mathrm{Hom}_{\underline{\mathcal{D}}}(D,F)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(C,D) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\underline{\mathcal{D}}}(D,E)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(C,D) \arrow[r, "\circ"] & \mathrm{Hom}_{\underline{\mathcal{D}}}(C,F)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathcal{D}}}(C,D)\otimes * \arrow[r, "1\otimes\mathrm{Id}_C"] \arrow[d, "1\otimes\mathrm{Id}_D"] \arrow[dr, "\cong"] & \mathrm{Hom}_{\underline{\mathcal{D}}}(C,D)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(C,C) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\underline{\mathcal{D}}}(C,D)\otimes\mathrm{Hom}_{\underline{\mathcal{D}}}(D,D) \arrow[r, "\circ"] & \mathrm{Hom}_{\underline{\mathcal{D}}}(C,D)
\end{tikzcd}$$

(we have omitted one associativity isomorphism in the first diagram and one symmetry isomorphism in the second).

Example 3.1.2 Let R be a commutative ring. Then Mod_R is enriched over $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$: indeed, for R-modules M, N, the set $\mathrm{Hom}_{\mathsf{Mod}_R}(M,N)$ is an abelian group, and moreover the composite of R-module maps is bilinear, which induces a group homomorphism

$$\mathrm{Hom}_{\mathsf{Mod}_R}(N,P)\otimes\mathrm{Hom}_{\mathsf{Mod}_R}(M,N)\longrightarrow\mathrm{Hom}_{\mathsf{Mod}_R}(M,P).$$

We also have that the forgetful functor $U : \mathsf{Ab} \longrightarrow \mathsf{Set}$ is represented by $\mathbb{Z} \in \mathsf{Ab}$, $\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, -) \cong U$, since any group homomorphism $\mathbb{Z} \longrightarrow A$ is determined by the image of $1 \in \mathbb{Z}$. Therefore, $\mathrm{Id}_M \in \mathrm{Hom}_{\mathsf{Mod}_R}(M,M)$ is represented by a group homomorphism $\mathbb{Z} \longrightarrow \mathrm{Hom}_{\mathsf{Mod}_R}(M,M)$ (sending 1 to Id_M). The diagrams encoding associativity and the unit condition are readily verified.

Definition. Let $(\mathcal{V}, \otimes, *)$ be a symmetric monoidal category enriched over itself, that is, also endowed with **internal homs** $\mathrm{Hom}_{\underline{\mathcal{V}}}(-,-)$ satisfying the above axioms. We say that $\mathcal{V}$ is **closed** if the monoidal product defines a two-variable adjunction using internal homs, so that we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{V}}(U,\mathrm{Hom}_{\underline{\mathcal{V}}}(V,W))\cong\mathrm{Hom}_{\mathcal{V}}(U\otimes V,W)\cong\mathrm{Hom}_{\mathcal{V}}(V,\mathrm{Hom}_{\underline{\mathcal{V}}}(U,W)).$$

Example 3.1.3 The symmetric monoidal categories $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, $(\mathsf{Top}, \times, *)$, $(G\mathsf{Top}, \times, *)$, $(\mathsf{Top}_*, \wedge, S^0)$ and $(G\mathsf{Top}_*, \wedge, S^0)$ are all closed.

However, the category of *all* topological spaces is *not* closed symmetric monoidal. This is the main reason why we choose the category Top of compactly generated spaces as our preferred category of spaces.

Remark 3.1.4 In principle, there is no reason to automatically deduce that a $\mathcal{V}$-category has an underlying category in the usual sense. For instance, it is not clear at all what the hom-sets should, or what the composite should be. Anyway, even if this was the case, this might be trickier than expected. For the too optimistic reader, here you have a discouraging counterexample.

Consider the category $G\mathsf{Top}$ of G-spaces and G-equivariant maps. This category can be enriched in two different ways:

1. $G\mathsf{Top}$ is enriched over $(\mathsf{Top}, \times, *)$, where $\mathrm{Hom}_{\underline{G\mathsf{Top}}}(X,Y) := \mathrm{Map}_G(X,Y)$ endowed with the subspace topology of $\mathrm{Map}(X,Y)$. The identity arrow of a G-space X is the unique map $* \longrightarrow \mathrm{Map}_G(X,X)$ with image $\mathrm{Id}_X : X \longrightarrow X$.

2. $G\mathsf{Top}$ is also enriched over $(G\mathsf{Top}, \times, *)$, setting $\mathrm{Hom}_{\underline{G\mathsf{Top}}}(X,Y) := G\mathrm{Map}(X,Y)$ the G-space of all maps with the G-action $(g \cdot f)(x) := g^{-1}f(gx)$ (see page 2). As before, the identity arrow of a G-space X is the unique G-map $* \longrightarrow G\mathrm{Map}(X,X)$ with image $\mathrm{Id}_X : X \longrightarrow X$.

The naive "underlying category" of $\underline{G\mathsf{Top}}$ obtained by taking the underlying set of $\mathrm{Hom}_{\underline{G\mathsf{Top}}}(X,Y)$ as hom-set returns something larger than expected in the second case, namely the set of all continuous maps. One needs to be a bit more astute (cf. [30, 3.4.5 and 3.4.9] for a proof):

Proposition 3.1.5 *Let $(\mathcal{V}, \otimes, *)$ be symmetric monoidal and let $\underline{\mathcal{D}}$ be a $\mathcal{V}$-category. Then there exists an underlying category $\mathcal{D}$ with same objects as $\underline{\mathcal{D}}$ and hom-sets*

$$\mathrm{Hom}_{\mathcal{D}}(C,D) := \mathrm{Hom}_{\mathcal{V}}(*, \mathrm{Hom}_{\underline{\mathcal{D}}}(C,D)).$$

In particular, if $\mathcal{V}$ is closed symmetric monoidal, the underlying category of $\underline{\mathcal{V}}$ is $\mathcal{V}$.

Example 3.1.6 Let us see how we recover the desired hom-sets in both enrichments of $G\mathsf{Top}$: on the one hand, $\mathrm{Map}(*, \mathrm{Map}_G(X,Y))$ is the set of points of $\mathrm{Map}_G(X,Y)$, i.e., the set of G-equivariant maps. On the other hand, $\mathrm{Map}_G(*, G\mathrm{Map}(X,Y))$ are the fixed points of $G\mathrm{Map}(X,Y)$, as $*$ has the trivial action; and by 1.1.2 we also recover the G-equivariant maps.

Of course, similarly, $G\mathsf{Top}_*$ is enriched over both Top_* and $G\mathsf{Top}_*$.

In order to make explicit the difference between a $\mathcal{V}$-category and its underlying category, the former is usually underlined. However, we will not underline enriched categories and will not distinguish between them and their underlying categories, as it should be clear from the context which one we handle.

There are enriched versions of functors, natural transformations, adjunctions, equivalences of categories, Yoneda lemma, etc. Unless stated otherwise, we will refer to these ones with no further comment. For detail descriptions of all these concepts, we refer to [30, ch. 3].

Diagram spectra

We have now all tools to construct the category of spectra (generalizing the previous provisional definition), which will allow us to construct the so-advertised "stable homotopy category".

Definition. A **diagram** $\mathcal{D}$ is a small symmetric monoidal category $(\mathcal{D}, \otimes, 0)$ enriched over $(\mathsf{Top}_*, \wedge, S^0)$.

The category of **$\mathcal{D}$-spaces** is the category of enriched functors $\mathsf{Top}_*^{\mathcal{D}}$, where Top_* is enriched over itself.

Notation 3.1.7 The $\mathcal{D}$-spaces will be the analogous of "spaces" in stable homotopy theory, so it makes sense to denote them as $X, Y, Z, \ldots$. Moreover, for a $\mathcal{D}$-space X, we will write X_D for $X(D)$.

We now aim to construct a (monoidal) product on $\mathcal{D}$-spaces: if $X, Y \in \mathsf{Top}_*^{\mathcal{D}}$, there is a naive product taking smash termwise,

$$\begin{aligned} X\widetilde{\wedge}Y : \mathcal{D} \times \mathcal{D} &\longrightarrow \mathcal{D} \\ (D,D') &\longmapsto (X\widetilde{\wedge}Y)_{(D,D')} := X_D \wedge Y_{D'}. \end{aligned}$$

We can extend this to a $\mathcal{D}$-space by using the left Kan extension of this functor along the monoidal product $\otimes : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$,

$$\begin{tikzcd} \mathcal{D} \times \mathcal{D} \arrow[r, "X\tilde{\wedge}Y"] \arrow[d, "\otimes"'] & \mathsf{Top}_* \\ \mathcal{D} \arrow[ur, dashed, "X\wedge Y:=\mathrm{Lan}_{\otimes} X\tilde{\wedge}Y"'] & \end{tikzcd}$$

Definition. The previous left Kan extension $X \wedge Y := \mathrm{Lan}_{\otimes} X\tilde{\wedge}Y$ is called **Day convolution**.

Explicitly[2], this is

$$(X \wedge Y)_D = \int^{A,B\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(A \otimes B, D) \wedge X_A \wedge Y_B.$$

Moreover, the universal property of the left Kan extension is rewritten for Day convolution as

$$\mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}\times\mathcal{D}}}(X\tilde{\wedge}Y, Z \circ \otimes).$$

Lemma 3.1.8 *Let $D \in \mathcal{D}$. There is an adjunction*

$$\begin{tikzcd} \mathsf{Top}_* \arrow[r, bend left, "M_D"] \arrow[r, phantom, "\perp"] & \mathsf{Top}_*^{\mathcal{D}} \arrow[l, bend left, "ev_D"] \end{tikzcd}$$

where ev_D is just evaluation at D, $ev_D(X) = X_D$; and for a based space T,

$$M_D T(D') := \mathrm{Hom}_{\mathcal{D}}(D, D') \wedge T$$

*(this is called the **shift desuspension functor**).*

Proof. The bijection

$$\mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(M_D T, X) \cong \mathrm{Hom}_{\mathsf{Top}_*}(T, X_D)$$

is described as follows: given a natural transformation $\alpha : M_D T \Rightarrow X$, the pointed map $\alpha_D : \mathrm{Hom}_{\mathcal{D}}(D, D) \wedge T \longrightarrow X_D$ corresponds, by the exponential adjunction, to a map $\mathrm{Hom}_{\mathcal{D}}(D, D) \longrightarrow \mathrm{Hom}_{\mathsf{Top}_*}(T, X_D)$. The image if the identity Id_D is the image of α by the desired bijection. Its inverse is the following: a pointed map $f : T \longrightarrow X_D$ maps to the natural transformation $\alpha : M_D T \Rightarrow X$ which for an object $D' \in \mathcal{D}$ is the map $\alpha_{D'} : \mathrm{Hom}_{\mathcal{D}}(D, D') \wedge T \longrightarrow X_{D'}$ which by the exponential adjunction corresponds to the pointed map $\mathrm{Hom}_{\mathcal{D}}(D, D') \longrightarrow \mathrm{Hom}_{\mathsf{Top}_*}(T, X_{D'})$ which sends $\varphi : D \longrightarrow D'$ to the composite $T \xrightarrow{f} X_D \xrightarrow{X\varphi} X_{D'}$. $\square$

[2] If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are $\mathcal{V}$-categories, with $\mathcal{C}$ small, $\mathcal{D}$ locally small and $\mathcal{E}$ cocomplete and tensored over $\mathcal{V}$, then the left Kan extension of a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"'] & \end{tikzcd}$$

exists and it is

$$\mathrm{Lan}_K F(D) = \int^{C\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(KC, D) \odot FC,$$

where $\odot$ stands for the copower of $\mathcal{E}$ over $\mathcal{V}$. In the $\mathcal{V}$-category $\mathcal{V}$, the copower is the monoidal product.

Let us now describe an internal hom-object for $\mathcal{D}$-spaces[3]: if X is a $\mathcal{D}$-space and Z is a $(\mathcal{D}\times\mathcal{D})$-space, there is a naive version of function $\mathcal{D}$-space,

$$\widetilde{F}(X,Z)_D := \mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(X, Z_{(D,-)}).$$

In particular, for a $\mathcal{D}$-space Y, the functor $-\widetilde{\wedge}Y$ is left adjoint to $\widetilde{F}(Y,-)$,

$$\mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}\times\mathcal{D}}}(X\widetilde{\wedge}Y, Z) \cong \mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(Y, \widetilde{F}(Y,Z)).$$

Definition. Let X, Y be $\mathcal{D}$-spaces. The **internal function $\mathcal{D}$-space** $F(X,Y)$ is

$$F(X,Y) := \widetilde{F}(X, Y\circ\otimes).$$

Using the definitions and the universal property of the left Kan extension, we get the adjunction

$$\mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(X\wedge Y, Z) \cong \mathrm{Hom}_{\mathsf{Top}_*^{\mathcal{D}}}(X, F(Y,Z)).$$

With this, the following proposition becomes formal (see [15, 3.3.5] for a proof):

Proposition 3.1.9 *The category of $\mathcal{D}$-spaces $(\mathsf{Top}_*^{\mathcal{D}}, \wedge, M_0S^0)$ is closed symmetric monoidal under Day convolution and unit M_0S^0.*

We will use this symmetric monoidal structure on $\mathcal{D}$-spaces to construct the category of spectra.

Definition. Let $(\mathcal{V}, \otimes, *)$ be symmetric monoidal. A **monoid object** is an object R equipped with multiplication and unit maps

$$\mu : R\otimes R \longrightarrow R \qquad , \qquad u : * \longrightarrow R$$

satisfying associativity and unit conditions

$$\mu\circ(\mu\otimes\mathrm{Id}) = \mu\circ(\mathrm{Id}\otimes\mu) \quad , \quad \mu\circ(u\otimes\mathrm{Id}) = \mu\circ(\mathrm{Id}\otimes u) = \mathrm{Id}$$

(here we have omitted the natural isomorphisms of the symmetric monoidal structure). Moreover, the monoid object is **commutative** if the multiplication commutes with the symmetric natural isomorphism τ of $\mathcal{V}$, $\mu\circ\tau = \mu$.

A map of monoid objects $f : R \longrightarrow R'$ is a map in $\mathcal{V}$ which is compatible with the multiplication and unit maps,

$$\varphi\circ\mu = \mu'\circ(\varphi\otimes\varphi) \qquad , \qquad \varphi\circ u = u'.$$

Remark 3.1.10 If $R \in \mathsf{Top}_*^{\mathcal{D}}$ is a monoid object, we have a "unit map" $S^0 \longrightarrow R_0$ of pointed topological spaces coming from evaluating $M_0S^0 \longrightarrow R$ at 0. Similarly, for a pair of objects $D, D' \in \mathcal{D}$, there is a "multiplication map"

$$\mu_{D,D'} : R_D \wedge R_{D'} \longrightarrow R_{D\otimes D'}$$

coming from the universal property of the left Kan extension. Moreover, we can recover the original ones from these:

[3]The following assumes that sk $\mathcal{D}$ is a small category, which ensures that the category of $\mathcal{D}$-spaces $\mathsf{Top}_*^{\mathcal{D}}$ is again a topological category. We will always assume this when necessary.

Proposition 3.1.11 *The category of (commutative) monoid objects on $\mathcal{D}$-spaces is isomorphic to the category of lax (symmetric) monoidal functors $\mathcal{D} \longrightarrow \mathsf{Top}_*$.*

Proof. If R is a monoid object in $\mathcal{D}$-spaces, then by the universal property of the left Kan extension it is equivalent to a natural transformation $R\widetilde{\wedge}R \Rightarrow R \circ \otimes$; and the unit map $M_0S^0 \longrightarrow R$ is equivalent to a map $S^0 \longrightarrow R_0$, by 3.1.8. Such a natural transformation and the latter map define a lax monoidal functor $R : \mathcal{D} \longrightarrow \mathsf{Top}_*$. Under this equivalence, the commutativity condition for a monoid object is rewritten as the symmetry condition for its corresponding lax monoidal functor. □

This proposition immediately gives a very useful criterion:

Corollary 3.1.12 *Let R be a monoid object on $\mathcal{D}$-spaces. Then R is commutative if and only if for any $D, D' \in \mathcal{D}$, the diagram*

$$\begin{tikzcd} R_D \wedge R_{D'} \arrow[r, "\mu_{D,D'}"] \arrow[d, "\mathrm{tw}"'] & R_{D\otimes D'} \arrow[d, "R(\mathrm{tw})"] \\ R_{D'} \wedge R_D \arrow[r, "\mu_{D',D}"] & R_{D'\otimes D} \end{tikzcd}$$

commutes.

Example 3.1.13 A commutative monoid object in $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is a ring. Indeed, the morphisms μ and u give the product and unit element.

The previous example motivates the following

Definition. Let R be a monoid object in $(\mathcal{V}, \otimes, *)$. A (left) R**-module** is an object M together with a multiplication map

$$m : R \otimes M \longrightarrow M$$

satisfying the obvious associativity and unit conditions,

$$\mu \circ (\mathrm{Id} \otimes m) = m \circ (\mu \otimes \mathrm{Id}) \quad , \quad m \circ (u \otimes \mathrm{Id}) = \mathrm{Id}$$

(again we have omitted one natural isomorphism).

A map of R-modules is a map $f : M \longrightarrow M'$ in $\mathcal{V}$ compatible with the multiplication, $f \circ m = m' \circ (\mathrm{Id} \otimes f)$. We denote by Mod_R the category of R-modules in $\mathcal{V}$.

Example 3.1.14 Of course, for a monoid object R in $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ (that is, a ring), an R-module in this sense is an R-module in the algebraic sense.

When a monoid object in $\mathcal{D}$-spaces is commutative, the category of modules inherits a desirable structure (cf. [24, 1.7] for a proof).

Proposition 3.1.15 *Let R be a commutative monoid object in $(\mathsf{Top}_*^{\mathcal{D}}, \wedge, M_0S^0)$. Then there is a product $\wedge_R$ for R-modules defined as the coequalizer of*

$$X \wedge R \wedge Y \underset{\mathrm{Id}\wedge m}{\overset{m\wedge \mathrm{Id}}{\rightrightarrows}} X \wedge Y \dashrightarrow X \wedge_R Y$$

making $(\mathsf{Mod}_R, \wedge_R, R)$ into a closed symmetric monoidal category. Concretely, the internal hom-module $F_R(M, N)$ is defined as the equalizer on $\mathcal{D}$-spaces

$$F_R(X, Y) \dashrightarrow F(X, Y) \underset{\omega}{\overset{m^*}{\rightrightarrows}} F(R \wedge X, Y)$$

where $m^* = F(m, \mathrm{Id})$ and ω is the adjoint map to the composite $F(X, Y) \wedge X \wedge R \xrightarrow{\varepsilon\wedge\mathrm{Id}} Y \wedge R \xrightarrow{m} Y$, and ε is the counit of the adjunction $(- \wedge X) \dashv F(X, -)$.

We will make use of this category of modules over suitable monoid objects to define models of spectra, both in the nonequivariant and the equivariant case.

3.2 Spectra and orthogonal spectra

Along this section we will describe our two preferred models of spectra and see the relation between them. Later in §3.4 we will construct the so-advertised "stable homotopy category" from that. We would like to mention that for this section we have mostly followed [3], [24], [27], [32] and [35], although our text is more detailed and connects the abstract definitions with the explicit ones that appear in the literature. As before, the proofs we include are original work of the author.

Spectra

In the first place, consider $\mathcal{D} = \mathbb{N}$ the discrete category with objects non-negative integers (that is, only identities). Trivially, this category is enriched over Top, and therefore over Top_* by attaching a disjoint point to the hom-spaces. Moreover, $(\mathbb{N}, +, 0)$ is symmetric monoidal under sum with unit $0 \in \mathbb{N}$.

Now we look at the category of $\mathbb{N}$-spaces $\mathsf{Top}_*^{\mathbb{N}}$: there is a (non-commutative) monoid object $\mathcal{S}_{\mathbb{N}} \in \mathsf{Top}_*^{\mathbb{N}}$ given by

$$\mathcal{S}_{\mathbb{N}}(n) := S^n.$$

Here we view $S^n = S^1 \wedge \cdots \wedge S^1$, so that there are preferred homeomorphisms $S^n \wedge S^m \cong S^{n+m}$. The unit of $\mathcal{S}_{\mathbb{N}}$ is given by $M_0 S^0 \longrightarrow \mathcal{S}_{\mathbb{N}}$, only nontrivial when evaluating at 0, which is the identity $M_0 S^0(0) = S^0 \longrightarrow S^0 = \mathcal{S}_{\mathbb{N}}(0)$. The multiplication map $\mathcal{S}_{\mathbb{N}} \wedge \mathcal{S}_{\mathbb{N}} \longrightarrow \mathcal{S}_{\mathbb{N}}$ (here $\wedge$ denotes Day convolution) is given by the preferred homeomophism

$$\mathcal{S}_{\mathbb{N}}(n) \wedge \mathcal{S}_{\mathbb{N}}(m) \xrightarrow{\cong} \mathcal{S}_{\mathbb{N}}(n+m)$$

described before.

Definition. The category of **spectra** is the category $\mathsf{Mod}_{\mathcal{S}_{\mathbb{N}}}$ of $\mathcal{S}_{\mathbb{N}}$-modules in $(\mathsf{Top}_*^{\mathbb{N}}, \wedge, M_0 S^0)$, and it is denoted by $\mathsf{Sp}^{\mathbb{N}}$.

We will spend some time now spelling this out: a spectrum is, therefore, a sequence of pointed spaces $X = (X_n)$ together with a multiplication map $\mathcal{S}_{\mathbb{N}} \wedge X \longrightarrow X$, which by the universal property of the left Kan extension is the same thing as a collection of maps

$$S^n \wedge X_m \longrightarrow X_{n+m}$$

for all $n, m \in \mathbb{N}$. In particular, the associativity condition of the multiplication map implies that these are determined by the maps

$$\sigma_n : \Sigma X_n = S^1 \wedge X_n \longrightarrow X_{n+1}$$

(compare with page 39). A map of spectra $f : X \longrightarrow Y$ is therefore determined by a sequence of pointed maps $f_n : X_n \longrightarrow Y_n$ such that the diagram

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"] & \Sigma Y_n \arrow[d, "\sigma_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes for all $n \geq 0$. Under this description, as expected, we say that a spectrum X is an **Ω-spectrum** when the adjoint maps $X_n \xrightarrow{\simeq} \Omega X_{n+1}$ are weak homotopy equivalences.

Remark 3.2.1 $\mathcal{S}_{\mathbb{N}}$ is *not* a commutative monoid object: indeed, using the criterion 3.1.12, we see that the diagram

$$\begin{array}{ccc} S^n \wedge S^m & \xrightarrow{\cong} & S^{n+m} \\ {\scriptstyle \mathrm{tw}}\downarrow & & \| \\ S^m \wedge S^n & \xrightarrow{\cong} & S^{n+m} \end{array}$$

does not commute, as the symmetry isomorphism $n + m \xrightarrow{\cong} m + n$ in $(\mathbb{N}, +, 0)$ is just the identity. We will solve this problem using orthogonal spectra (to be defined later). However, we can modify *slightly* our category $\mathbb{N}$ so that our monoidal object becomes commutative, and in this way induce our desired symmetric monoidal structure on spectra. This is called **symmetric spectra** and it is due to Jeff Smith [17]: instead of $\mathbb{N}$, we consider now the category Σ with same objects as $\mathbb{N}$ but with morphisms $\mathrm{Hom}_\Sigma(n, n) = \Sigma_n$, the symmetric group, and no arrows $n \longrightarrow m$ if $n \neq m$. This category happens to be symmetric monoidal with sum of integers and block sum of permutations. Similarly, there is a monoid object $\mathcal{S}_\Sigma$ sending n to $S^n = S^1 \wedge \cdots \wedge S^1$, but now the morphisms of Σ make $\mathcal{S}_\Sigma$ commutative. For a detaiedl exposition of symmetric spectra, see [34].

Let us also explicitly describe what it means that the category of $\mathbb{N}$-spaces is closed symmetric monoidal. In the first place, $\mathsf{Top}_*^{\mathbb{N}}$ is complete and cocomplete with limits and colimits created levelwise: for $\mathcal{J} \longrightarrow \mathsf{Top}_*^{\mathbb{N}}$, we have

$$(\operatorname*{colim}_j X_j)_n = \operatorname*{colim}_j (X_j)_n \qquad , \qquad (\lim_j X_j)_n = \lim_j (X_j)_n.$$

It is important to have an explicit description of Day convolution for $\mathbb{N}$-spaces:

Lemma 3.2.2 *Let X, Y be $\mathbb{N}$-spaces. Then*

$$(X \wedge Y)_n = \bigvee_{p+q=n} X_p \wedge Y_q.$$

Proof. Since the category $\mathbb{N}$ only has identities, we have

$$\begin{aligned}
(X \wedge Y)_n &= \int^{p,q \in \mathbb{N}} \mathrm{Hom}_{\mathbb{N}}(p+q, n)_+ \wedge X_p \wedge Y_q \\
&= \mathrm{coeq}\Big(\bigvee_{p,q} \mathrm{Hom}(p+q,n)_+ \wedge X_p \wedge Y_q \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightrightarrows}} \bigvee_{p,q} \mathrm{Hom}(p+q,n)_+ \wedge X_p \wedge Y_q \Big) \\
&= \bigvee_{p,q} \mathrm{Hom}(p+q,n)_+ \wedge X_p \wedge Y_q \\
&= \bigvee_{p+q=n} X_p \wedge Y_q
\end{aligned}$$

□

We can also explicitly describe the internal function $\mathbb{N}$-space: given $\mathbb{N}$-spaces X, Y, the $\mathbb{N}$-space $F(X, Y) \in \mathsf{Top}_*^{\mathbb{N}}$ is given by

$$F(X,Y)_n := \mathrm{Hom}_{\mathsf{Top}_*^{\mathbb{N}}}(X, \mathrm{sh}^n Y) = \prod_{m \geq 0} \mathrm{Map}(X_m, Y_{n+m}),$$

where $\mathrm{sh}^n Y$ is the $\mathbb{N}$-space $(\mathrm{sh}^n Y)_m := Y_{n+m}$.

Lastly, it is also worth mentioning that the category of $\mathbb{N}$-spaces is tensored and cotensored over Top_*: given a pointed space T and an $\mathbb{N}$-space X, there are $\mathbb{N}$-spaces $X \wedge T$ and $F(T, Y)$ defined levelwise,

$$(X \wedge T)_n := X_n \wedge T \qquad , \qquad F(T, X)_n := \mathrm{Map}(T, X_n),$$

and it is readily verified that $(- \wedge T)$ is left adjoint to $F(T, -)$.

Examples 3.2.3 (a) Of course, the monoid object $\mathcal{S}_{\mathbb{N}}$, as a $\mathcal{S}_{\mathbb{N}}$-module, is a spectrum, called the **sphere spectrum** . Trivially, the structure maps are given by the identification $S^1 \wedge S^n \xrightarrow{\cong} S^{n+1}$.

(b) If T is a pointed space, then its **suspension spectrum** $\Sigma^\infty T$ is given by

$$(\Sigma^\infty T)_n := \Sigma^n T = S^n \wedge T$$

with the obvious structure maps $S^1 \wedge (S^n \wedge X) \xrightarrow{\cong} S^{n+1} \wedge T$.

In particular, $\mathcal{S}_{\mathbb{N}} = \Sigma^\infty S^0$, and with more generality, $\Sigma^k \mathcal{S}_{\mathbb{N}} \cong \Sigma^\infty S^k$.

(c) Let A be an abelian group. The **Eilenberg-MacLane spectrum** HA is given by

$$(HA)_n := K(A, n).$$

The structure maps $S^1 \wedge K(A, n) \longrightarrow K(A, n+1)$ are the adjoint maps to the weak homotopy equivalences $K(A, n) \xrightarrow{\simeq} \Omega K(A, n+1)$ (see page 24). Therefore, this is an Ω-spectrum.

(d) Similarly, we have the K **and** KO **spectra**, given by

$$K_n := \Omega^i(BU \times \mathbb{Z}) \qquad , \qquad KO_n := \Omega^j(BO \times \mathbb{Z}),$$

where $0 \le i < 2$, $0 \le j < 8$ and $n + i \equiv 0 (\mathrm{mod}\, 2)$, $n + j \equiv 0 (\mathrm{mod}\, 8)$. These are also Ω-spectra, as we saw.

(e) If T is a pointed space and X is a spectrum, the $\mathbb{N}$-space $X \wedge T$ is a spectrum, with structure maps $(S^1 \wedge X_n) \wedge T \longrightarrow X_{n+1} \wedge T$. Similarly, the $\mathbb{N}$-space $F(T, X)$ is also a spectrum, with structure maps given by the composite

$$S^1 \wedge \mathrm{Map}(T, X_n) \longrightarrow \mathrm{Map}(T, S^1 \wedge X_n) \xrightarrow{(\sigma_n)_*} \mathrm{Map}(T, X_{n+1}),$$

where the first map sends (s, f) to $(t \mapsto (s, f(t))$. This says that $\mathsf{Sp}^{\mathbb{N}}$ is tensored and cotensored over Top_*.

In particular, $\Sigma^\infty T = \mathcal{S}_{\mathbb{N}} \wedge T$.

Orthogonal spectra

Let us now describe another remarkable (and improved) notion of spectra. Now we let $\mathcal{D} = \mathbb{I}$ be the category of finite dimensional real inner product vector spaces, with morphisms isometric isomorphisms. This category is enriched over Top (we have $\mathrm{Hom}_{\mathbb{I}}(V, W) \subset$

$\mathrm{Hom}_{\mathbb{R}-\mathrm{lin}}(V, W)$ with the subspace topology, where in the latter space we consider the topology given by any norm), thus also over Top_* by attaching a disjoint basepoint to the hom-spaces. Moreover, $(\mathbb{I}, \oplus, 0)$ is symmetric monoidal under direct sum and the trivial inner product space $0 \in \mathbb{I}$ as unit.

We now look at the category $\mathsf{Top}_*^{\mathbb{I}}$ of $\mathbb{I}$-spaces: there is a monoid object $\mathcal{S}_{\mathbb{I}}$ given by

$$\mathcal{S}_{\mathbb{I}}(V) = S^V,$$

the one-point compactification of V, where the extra point serves as basepoint. The unit of $\mathcal{S}_{\mathbb{I}}$ is given by $M_0 S^0 \longrightarrow \mathcal{S}_{\mathbb{I}}$, only nontrivial when evaluating at 0, which is the identity $M_0 S^0(0) = S^0 \longrightarrow S^0 = \mathcal{S}_{\mathbb{I}}(0)$ (because the one-point compactification of $0 \in \mathbb{I}$ is S^0). The multiplication map $\mathcal{S}_{\mathbb{I}} \wedge \mathcal{S}_{\mathbb{I}} \longrightarrow \mathcal{S}_{\mathbb{I}}$ is given by the natural isomorphism

$$\mathcal{S}_{\mathbb{I}}(V) \wedge \mathcal{S}_{\mathbb{I}}(W) = S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W} = \mathcal{S}_{\mathbb{I}}(V \oplus W).$$

Now an important observation is that $\mathcal{S}_{\mathbb{I}}$ is a commutative monoid object: indeed, the symmetry isomorphism $V \oplus W \xrightarrow{\cong} W \oplus V$ of $(\mathbb{I}, \oplus, 0)$ is nontrivial, and this makes that the diagram

$$\begin{tikzcd} S^V \wedge S^W \arrow[r, "\cong"] \arrow[d, "\mathrm{tw}"'] & S^{V \oplus W} \arrow[d] \\ S^W \wedge S^V \arrow[r, "\cong"] & S^{W \oplus V} \end{tikzcd}$$

commutes, so we conclude by 3.1.12.

Definition. The category of **orthogonal spectra** is the category $\mathsf{Mod}_{\mathcal{S}_{\mathbb{I}}}$ of $\mathcal{S}_{\mathbb{I}}$-modules in $(\mathsf{Top}_*^{\mathbb{I}}, \wedge, M_0 S^0)$, and it is denoted by $\mathsf{Sp}^{\mathbb{I}}$.

Let us spell this out: unravelling definitions, an orthogonal spectrum is a collection of pointed $O(V)$-spaces $X = (X_V)$ for every $V \in \mathbb{I}$, together with $O(V) \times O(W)$ maps

$$\sigma_{V,W} : S^V \wedge X_W \longrightarrow X_{V \oplus W},$$

called *structure maps*, for every pair of inner product spaces V, W, satisfying the following associativity condition: if U is a third inner product space, then the diagram

$$\begin{tikzcd} S^V \wedge S^W \wedge X_U \arrow[r, "\mathrm{Id} \wedge \sigma_{W,U}"] \arrow[d, "\cong \wedge \mathrm{Id}"'] & S^V \wedge X_{W \oplus U} \arrow[d, "\sigma_{V, W \oplus U}"] \\ S^{V \oplus W} \wedge X_U \arrow[r, "\sigma_{V \oplus W, U}"] & X_{V \oplus W \oplus U} \end{tikzcd}$$

commutes. There are also maps $X_V \longrightarrow X_W$ for every isometric isomorphism $V \longrightarrow W$, but these are not very relevant (we will see why in 3.2.8).

A map of orthogonal spectra $f : X \longrightarrow Y$ is, therefore, a collection of based $O(V)$-maps $f_V : X_V \longrightarrow Y_V$ for every $V \in \mathbb{I}$ which are compatible with the structure maps, that is, for every $V, W \in \mathbb{I}$ the following diagram commutes:

$$\begin{tikzcd} S^V \wedge X_W \arrow[r, "\mathrm{Id} \wedge f_W"] \arrow[d, "\sigma_{V,W}"'] & S^V \wedge Y_W \arrow[d, "\sigma_{V,W}"] \\ X_{V \oplus W} \arrow[r, "f_{V \oplus W}"] & Y_{V \oplus W} \end{tikzcd}$$

As before it is handy to have an explicit description of Day convolution for $\mathbb{I}$-spaces:

Lemma 3.2.4 *Let X, Y be $\mathbb{I}$-spaces. Given $V \in \mathbb{I}$, choose for every $0 \leq p \leq \dim V$ a linear subspace $V_p \subseteq V$ of dimension p. Then*

$$(X \wedge Y)_V = \bigvee_{p=0}^{\dim V} O(V)_+ \wedge_{O(V_p)\times O(V-V_p)} X_{V_p} \wedge Y_{V-V_p},$$

where $V - V_p$ denotes the orthogonal complement of V_p in V.

Proof. For a pair of maps $f : U \longrightarrow U'$ and $g : W \longrightarrow W'$ in $\mathbb{I}$, we write (f, g) for the corresponding pair of arrows in $\mathbb{I} \times \mathbb{I}$. We compute

$$\begin{aligned}
(X \wedge Y)_V &= \int^{U,W\in\mathbb{I}} \mathrm{Hom}_{\mathbb{I}}(U \oplus W, V)_+ \wedge X_U \wedge Y_W \\
&= \mathrm{coeq}\Big(\bigvee_{(f,g)} \mathrm{Hom}_{\mathbb{I}}(U \oplus W, V)_+ \wedge X_{U'} \wedge Y_{W'} \rightrightarrows \bigvee_{U,W} \mathrm{Hom}_{\mathbb{I}}(U \oplus W, V)_+ \wedge X_U \wedge Y_W \Big) \\
&\cong \mathrm{coeq}\Big(\bigvee_{U\subseteq V} O(V)_+ \wedge X_U \wedge Y_{V-U} \underset{\mathrm{Id}\wedge\varphi_2}{\overset{\varphi_1\wedge \mathrm{Id}}{\rightrightarrows}} \bigvee_{U\subseteq V} O(V)_+ \wedge X_U \wedge Y_{V-U} \Big) \\
&\cong \mathrm{coeq}\Big(\bigvee_{p=0}^{\dim V} O(V)_+ \wedge X_{V_p} \wedge Y_{V-V_p} \underset{\mathrm{Id}\wedge\varphi_2}{\overset{\varphi_1\wedge \mathrm{Id}}{\rightrightarrows}} \bigvee_{p=1}^{\dim V} O(V)_+ \wedge X_{V_p} \wedge Y_{V-V_p} \Big) \\
&\cong \bigvee_{p=0}^{\dim V} O(V)_+ \wedge_{O(V_p)\times O(V-V_p)} X_{V_p} \wedge Y_{V-V_p},
\end{aligned}$$

where the morphisms φ_1, φ_2 stand for the $O(U) \times O(V - U)$-actions on $O(V)$ and $X_U \wedge Y_{V-U}$, respectively. We also used that in the coequalizer, all terms corresponding to subspaces of the same dimension are identified. □

In a similar fashion to $\mathbb{N}$-spaces, we also have that the category of $\mathbb{I}$-spaces is tensored and cotensored over Top_*, and it is complete and cocomplete, everything defined termwise. However, the main difference is that this time $\mathcal{S}_{\mathbb{I}}$ is a commutative monoid object, so by 3.1.15 we get

Theorem 3.2.5 *There is a product of orthogonal spectra $\wedge_{\mathbb{I}}$, called* ***smash product*** *, such that*

$$(\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}})$$

is a closed symmetric monoidal category.

In particular, this means that

$$\mathrm{Hom}_{\mathsf{Sp}^{\mathbb{I}}}(X \wedge_{\mathbb{I}} Y, Z) \cong \mathrm{Hom}_{\mathsf{Sp}^{\mathbb{I}}}(X, F(Y, Z)).$$

for all $X, Y, Z \in \mathsf{Sp}^{\mathbb{I}}$, where F denotes the internal hom-orthogonal spectrum. The smash product, defined above as a coequalizer in $\mathbb{I}$-spaces, is hard to state explicitly, but can be handled easily thanks to the following observation:

Proposition 3.2.6 *A map of orthogonal spectra $f : X \wedge_{\mathbb{I}} Y \longrightarrow Z$ is completely determined by a collection of $O(V) \times O(W)$-equivariant maps*

$$f_{V,W} : X_V \wedge Y_W \longrightarrow Z_{V\oplus W} \quad , \quad V, W \in \mathbb{I}$$

which make the diagram

$$\begin{tikzcd}
X_V \wedge S^U \wedge Y_W \arrow[r, "\mathrm{tw}"] \arrow[d, "\mathrm{Id}\wedge\sigma_{U,W}"'] & S^U \wedge X_V \wedge Y_W \arrow[d, "\mathrm{Id}\wedge f_{V,W}"] \arrow[dr, "\sigma_{U,V}\wedge\mathrm{Id}"] & \\
X_V \wedge Y_{U\oplus W} \arrow[d, "f_{V,U\oplus W}"'] & S^U \wedge Z_{V\oplus W} \arrow[d, "\sigma_{U,V\oplus W}"] & X_{U\oplus V} \wedge Y_W \arrow[dl, "f_{U\oplus V,W}"] \\
Z_{V\oplus U\oplus W} \arrow[r] & Z_{U\oplus V\oplus W} &
\end{tikzcd}$$

commute for all $V, U, W \in \mathbb{I}$.

Proof. This follows directly from the universal properties of the coequalizer and Day convolution. □

This additional internal product of orthogonal spectra allows us to define richer structures on spectra. It is worth mentioning

Definition. A (commutative) **orthogonal ring spectrum** is a (commutative) monoid object in $(\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}})$. Sometimes it is also called **strict** ring spectrum.

Examples 3.2.7 (a) The object $\mathcal{S}_{\mathbb{I}}$, viewed as a $\mathcal{S}_{\mathbb{I}}$-module, is called the **sphere orthogonal spectrum** . It has the identifications $S^V \wedge S^W \xrightarrow{\cong} S^{V\oplus W}$ as structure maps. By definition, S^V is a $O(V)$-space and the previous structure maps are $O(V) \times O(W)$-equivariant. Moreover, using the precious proposition one can check that this is an orthogonal ring spectrum.

(b) Let T be a pointed space. The **suspension orthogonal spectrum** $\Sigma^\infty T$ of T is given by

$$(\Sigma^\infty T)_V := S^V \wedge T,$$

endowed with the standard $O(V)$-action on S^V and the trivial action on T. It has the obvious structure maps $S^V \wedge S^W \wedge T \xrightarrow{\cong} S^{V\oplus W} \wedge T$.

In particular, $\mathcal{S}_{\mathbb{I}} = \Sigma^\infty S^0$.

(c) Let us describe the orthogonal version of the Eilenberg-MacLane spectrum HA. One does not have to think very deeply to notice that an abstract $K(A, n)$ does not have in principle an action of the orthogonal group. We have to deal with concrete models of Eilenberg-MacLane spaces.

Let A be a (countable) abelian group. For a pointed set (T, t_0), define the **reduced A-linearization** of T as $\widetilde{A}[T] := A[T]/A[\{t_0\}]$. If T is a pointed space, we can topologize $\widetilde{A}[T]$ with the final topology[4] given by the maps

$$\begin{aligned} A \times \overset{k}{\cdots} \times A \times T \times \overset{k}{\cdots} \times T &\longrightarrow \widetilde{A}[T] \\ (a_1, \dots a_k, x_1, \dots, x_k) &\longmapsto \textstyle\sum_{i=1}^k a_i x_i \end{aligned}$$

[4]We really mean the final or strong topology. In the context of CW-complexes, this is somehow confusing as it is customary to call "weak topology" to the coherent topology given by the closures of the cells. See [41, page 69] for a discussion.

for all $k > 0$, where A has the discrete topology. One can show that for an inner product space $V \in \mathbb{I}$, the reduced A-linearilization $\widetilde{A}[S^V]$ is a $K(A, \dim V)$; but moreover it has an $O(V)$-action inherited from S^V. These are our desired spaces: the **Eilenberg-MacLane orthogonal spectrum** HA is given by $(HA)_V := \widetilde{A}[S^V]$, with structure maps given by

$$\begin{aligned} S^V \wedge \widetilde{A}[S^W] &\longrightarrow \widetilde{A}[S^{V\oplus W}] \\ (v, \textstyle\sum a_i w_i) &\longmapsto \textstyle\sum a_i(v, w_i). \end{aligned}$$

Moreover, if A is a commutative ring, then HA becomes a comutative ring spectrum, with the smash product $HA \wedge_{\mathbb{I}} HA \longrightarrow HA$ determined by the maps

$$\begin{aligned} \widetilde{A}[S^V] \wedge \widetilde{A}[S^W] &\longrightarrow \widetilde{A}[S^{V\oplus W}] \\ (\textstyle\sum_i a_i v_i \, , \, \sum_j b_j w_j) &\longmapsto \textstyle\sum_{i,j} a_i b_j (v_i, w_j). \end{aligned}$$

(d) The category of orthogonal spectra is again tensored and cotensored over Top_*: given $X \in \mathsf{Sp}^{\mathbb{I}}$ and $T \in \mathsf{Top}_*$, we define $X \wedge T$ as the orthogonal spectrum $(X \wedge T)_V := X_V \wedge T$ with trivial $O(V)$-action on T and structure maps $(S^V \wedge X_W) \wedge T \longrightarrow X_{V\oplus W} \wedge T$. Similarly, we define the orthogonal spectrum $F(T, X)$ as $F(T, X)_V := \mathrm{Map}(T, X_V)$, where the $O(V)$-action is inherited from the one on X_V and the structure maps given by the composite

$$S^V \wedge \mathrm{Map}(T, X_W) \longrightarrow \mathrm{Map}(T, S^V \wedge X_W) \xrightarrow{(\sigma_{V,W})_*} \mathrm{Map}(T, X_{V\oplus W}),$$

where the first map sends (s, f) to $(t \mapsto (s, f(t)))$.

Let us finish this exposition of orthogonal spectra with a different, easier description of them. Let $\mathbb{O}$ be the category with objects non negative integers and morphisms $\mathrm{Hom}_{\mathbb{O}}(n, n) := O(n)$ and no morphisms $n \longrightarrow m$ if $n \neq m$. Of course, we think of $n \in \mathbb{O}$ as the euclidean space $\mathbb{R}^n$. The observation that ignites this alternative characterization is the following

Lemma 3.2.8 *There is an equivalence of categories $\mathbb{O} \xrightarrow{\simeq} \mathbb{I}$, where n is mapped to $\mathbb{R}^n$ with the usual inner product. Therefore, the categories of $\mathbb{I}$-spaces and $\mathbb{O}$-spaces are equivalent,* $\mathsf{Top}_*^{\mathbb{I}} \simeq \mathsf{Top}_*^{\mathbb{O}}$.

Proof. The assignment is, by definition, fully faithful, and it is essentially surjective because in $\mathbb{I}$ every morphism is an isomorphism. □

Let us see how we can rephrase orthogonal spectra under this description: there is a monoid object $\mathcal{S}_{\mathbb{O}}$ acting as $\mathcal{S}_{\mathbb{O}}(n) := S^n$ with the standard $O(n)$-action. Observe that in $\mathbb{O}$, the symmetry isomorphism $n + m \cong m + n$ is *not* the identity, but it is given by the isometry

$$\begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix} \in O(n+m),$$

so as expected $\mathcal{S}_{\mathbb{O}}$ is commutative, just by the same argument as for $\mathcal{S}_{\mathbb{I}}$. In a similar fashion, there is an equivalence of categories

$$\mathsf{Mod}_{\mathcal{S}_{\mathbb{I}}} = \mathsf{Sp}^{\mathbb{I}} \simeq \mathsf{Sp}^{\mathbb{O}} := \mathsf{Mod}_{\mathcal{S}_{\mathbb{O}}}, \tag{3.1}$$

so by an abuse of terminology we will also call orthogonal spectra to the latter. Under this equivalence of categories, explicitly, an orthogonal spectrum consists of a sequence $X = (X_n)$ of based $O(n)$-spaces together with $O(n) \times O(m)$-equivariant structure maps

$$\sigma_{n,m} : S^n \wedge X_m \longrightarrow X_{n+m}$$

satisfying the obvious associativity condition. A map of orthogonal spectra $f : X \longrightarrow Y$ is a sequence of pointed $O(n)$-maps $f_n : X_n \longrightarrow Y_n$ such that

$$\begin{tikzcd} S^n \wedge X_m \arrow[r, "\mathrm{Id}\wedge f_m"] \arrow[d, "\sigma_{n,m}"'] & S^n \wedge Y_m \arrow[d, "\sigma_{n,m}"] \\ X_{n+m} \arrow[r, "f_{n+m}"] & Y_{n+m} \end{tikzcd}$$

commutes for $n \geq 0$. Under this terminology, Day convolution of $\mathbb{O}$-spaces becomes (compare with 3.2.4)

$$(X \wedge_{\mathbb{O}} Y)_n = \bigvee_{p=0}^{n} O(n)_+ \wedge_{O(p)\times O(n-p)} X_p \wedge Y_{n-p}.$$

Connection between spectra and orthogonal spectra

We finish this section with a simple but important observation: there is a functor $\mathbb{N} \longrightarrow \mathbb{I}$ mapping $n \in \mathbb{N}$ to the euclidean space $\mathbb{R}^n$ with its standard inner product. Restriction along this functor defines a forgetful functor

$$\mathbb{U} : \mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{Sp}^{\mathbb{N}}.$$

It is straightforward to check that this functor preserves small limits and colimits and tensors and cotensors with pointed spaces, as everything is defined termwise. This functor, which will be key later, allows us to talk about properties of orthogonal spectra that we had in spectra:

Definition. An orthogonal spectrum X is an **orthogonal Ω-spectrum** if $\mathbb{U}X$ is an Ω-spectum.

Moreover, it is possible to show (cf. [24, §2]) that if R is a monoid on $\mathcal{D}$-spaces, the category Mod_R of R-modules is equivalent to the category of diagram spaces over a more complicated diagram $\mathcal{D}_R$: the category $\mathcal{D}_R$ has the same objects as $\mathcal{D}$ and morphisms

$$\mathrm{Hom}_{\mathcal{D}_R}(C, D) := \mathrm{Hom}_{\mathsf{Mod}_R}(M_C S^0 \wedge R, M_D S^0 \wedge R).$$

Furthermore, if R is commutative, then $\mathcal{D}_R$ is symmetric monoidal. In particular, $\mathsf{Sp}^{\mathbb{N}}$ is equivalent to the category of $\mathbb{N}_{\mathcal{S}_{\mathbb{N}}}$-spaces $\mathsf{Top}_*^{\mathbb{N}_{\mathcal{S}_{\mathbb{N}}}}$ and $\mathsf{Sp}^{\mathbb{I}}$ is equivalent to $\mathbb{I}_{\mathcal{S}_{\mathbb{I}}}$-spaces $\mathsf{Top}_*^{\mathbb{I}_{\mathcal{S}_{\mathbb{I}}}}$.

The following proposition creates a left adjoint to $\mathbb{U}$ (see [24, 3.4] for a proof):

Proposition 3.2.9 *Let $\mathcal{C}, \mathcal{D}$ be diagrams and let $\iota : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. If R is a monoid object in $\mathcal{D}$, then ι extends to a functor $\kappa : \mathcal{C}_{\mathbb{U}R} \longrightarrow \mathcal{D}_R$, so the forgetful functor*

$$\mathbb{U} : \mathsf{Top}_*^{\mathcal{D}_R} \longrightarrow \mathsf{Top}_*^{\mathcal{C}_{\mathbb{U}R}}$$

has a left adjoint $\mathbb{P} : \mathsf{Top}_*^{\mathcal{C}_{\mathbb{U}R}} \longrightarrow \mathsf{Top}_*^{\mathcal{D}_R}$, given by a left Kan extension.

Moreover, if R is commutative, then $\mathbb{U}$ is lax symmetric monoidal and κ and $\mathbb{P}$ are strong symmetric monoidal.

For our particular case, this implies

Corollary 3.2.10 *There is an adjunction*

$$\begin{tikzcd} \mathsf{Sp}^{\mathbb{N}} \arrow[r, bend left, "\mathbb{P}"] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^{\mathbb{I}} \arrow[l, bend left, "\mathbb{U}"] \end{tikzcd}$$

This adjunction will be of great importance later.

3.3 Homotopy theory of spectra

We will now discuss some homotopy theory of spectra, and its relation with homology and cohomology theories. Developing these tools will be necessary to obtain the "stable homotopy category". Our main references for this section have been [24] and [32], although we also include results from [27] and [38].

We can define the homotopy groups of a spectrum in a fairly easy way:

Definition. Let X be a spectrum. Its **k-th homotopy group** is

$$\pi_k(X) := \operatorname*{colim}_n \pi_{n+k}(X_n) \qquad , \qquad k \in \mathbb{Z},$$

where the colimit is taken along the sequence

$$\pi_k(X_0) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X_0) \xrightarrow{(\sigma_0)_*} \pi_{k+1}(X_1) \xrightarrow{\Sigma} \pi_{k+2}(\Sigma X_1) \xrightarrow{(\sigma_1)_*} \cdots$$

(of course, it starts as soon as it has positive indexes). If X is an orthogonal spectrum, then $\pi_k(X) := \pi_k(\mathbb{U}X)$.

Observe that, if $f : X \longrightarrow Y$ is a map of spectra, there is a commutative diagram

$$\begin{tikzcd}
\pi_k(X_0) \arrow[r, "\Sigma"] \arrow[d, "(f_0)_*"] & \pi_{k+1}(\Sigma X_0) \arrow[r, "(\sigma_0)_*"] \arrow[d, "(\Sigma f_0)_*"] & \pi_{k+1}(X_1) \arrow[r, "\Sigma"] \arrow[d, "(f_1)_*"] & \cdots \\
\pi_k(Y_0) \arrow[r, "\Sigma"] & \pi_{k+1}(\Sigma Y_0) \arrow[r, "(\sigma_0)_*"] & \pi_{k+1}(Y_1) \arrow[r, "\Sigma"] & \cdots
\end{tikzcd}$$

which induces a map on colimits, thus a map

$$f_* : \pi_k(X) \longrightarrow \pi_k(Y).$$

Definition. A map $f : X \longrightarrow Y$ of spectra is a **π_*-isomorphism** or a **weak equivalence** if it induces an isomorphism in all homotopy groups.

A map $f : X \longrightarrow Y$ of orthogonal spectra is a **π_*-isomorphism** if so is $\mathbb{U}f$.

In particular, these homotopy groups define functors

$$\pi_k : \mathsf{Sp}^{\mathbb{N}} \longrightarrow \mathsf{Ab} \qquad , \qquad \pi_k : \mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{Ab}.$$

Moreover, $\pi_*(X) := \bigoplus_k \pi_k(X)$ is a graded abelian group, so we get functors

$$\pi_* : \mathsf{Sp}^{\mathbb{N}} \longrightarrow \mathsf{grAb} \qquad , \qquad \pi_* : \mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{grAb}.$$

In both cases, the second one is just the composite of $\mathbb{U}$ with the first one.

Recall that, as usual in algebraic topology, we endow grAb with a structure of symmetric monoidal category by setting

$$(A_* \otimes_{\mathbb{Z}} B_*)_n := \bigoplus_{i+j=n} A_i \otimes_{\mathbb{Z}} B_j,$$

with symmetry isomorphism $A_* \otimes_{\mathbb{Z}} B \xrightarrow{\cong} B_* \otimes_{\mathbb{Z}} A_*$, sending $a_i \otimes b_j$ to $(-1)^{ij} b_j \otimes a_i$.

Theorem 3.3.1 *The functor*

$$\pi_* : (\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}}) \longrightarrow (\mathsf{grAb}, \otimes_{\mathbb{Z}}, \mathbb{Z})$$

is lax symmetric monoidal.

Proof sketch. For orthogonal spectra X, Y, there is a natural pairing

$$\pi_*(X) \otimes \pi_*(Y) \longrightarrow \pi_*(X \wedge_{\mathbb{I}} Y)$$

defined as follows: write $X_n = X_{\mathbb{R}^n}$ and similarly for Y. Given maps $f : S^{k+n} \longrightarrow X_n$ and $g : S^{r+m} \longrightarrow Y_m$, we define $f * g$ as the composite

$$S^{k+n} \wedge S^{r+m} \xrightarrow{f \wedge g} X_n \wedge Y_m \xrightarrow{i_{n,m}} (X \wedge_{\mathbb{I}} Y)_{n+m}$$

(the last map is one of the components of the identity of $X \wedge_{\mathbb{I}} Y$ by 3.2.6). This does not depend on the homotopy classes of maps, so it gives a well defined $\mathbb{Z}$-bilinear map $\pi_{k+n}(X_n) \times \pi_{r+m}(X_m) \longrightarrow \pi_{k+n+r+m}((X \wedge_{\mathbb{I}} Y)_{n+m})$. This is compatible with the passage to the colimit up to a sign, caused by a twist map $\mathrm{tw} : S^r \wedge S^n \longrightarrow S^n \wedge S^r$. Therefore, we can define the above pairing as

$$[f] \cdot [g] := (-1)^{nr}[f * g]$$

(see [32, 6.11] for the reason of the choice of the sign). The unit $\mathbb{Z} \longrightarrow \pi_*(\mathcal{S}_{\mathbb{I}}) = \pi_*^{st}$ is just given by the identification $\mathbb{Z} \xrightarrow{\cong} \pi_1(S^1) \cong \pi_0^{st}$. □

Examples 3.3.2 (a) If T is a based space, then $\pi_k(\Sigma^\infty T) = \pi_k^{st}(T)$, just by definition, as the structure maps of $\Sigma^\infty T$ are natural isomorphisms.

In particular, $\pi_k(\mathcal{S}_{\mathbb{N}}) = \pi_k^{st}(S^0)$.

(b) If X is an Ω-spectrum, then

$$\pi_k(X) \cong \begin{cases} \pi_k(X_0), & k \geq 0 \\ \pi_0(X_{-k}), & k \leq 0. \end{cases}$$

Indeed, by the suspension-loop adjunction, the composite

$$\pi_{k+n}(X_n) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma X_n) \xrightarrow{(\sigma_n)_*} \pi_{n+k+1}(X_{n+1})$$

is the same as

$$\pi_{k+n}(X_n) \xrightarrow{(\widetilde{\sigma_n})_*} \pi_{n+k}(\Omega X_{n+1}) \cong \pi_{n+k+1}(X_{n+1}),$$

where $\widetilde{\sigma}_n$ is the adjoint map of σ.

Proposition 3.3.3 *If X is a spectrum, then*

$$\pi_k(X) \cong \pi_{k+1}(X \wedge S^1).$$

Proof. Indeed, we have the chain of isomorphisms

$$\begin{aligned}
\pi_k(X) &\cong \operatorname*{colim}_n(\pi_k(X_0) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X_0) \xrightarrow{(\sigma_0)_*} \pi_{k+1}(X_1) \xrightarrow{\Sigma} \pi_{k+2}(\Sigma X_1) \longrightarrow \cdots) \\
&\cong \operatorname*{colim}_n(\pi_{k+1}(\Sigma X_0) \longrightarrow \pi_{k+2}(\Sigma X_1) \longrightarrow \pi_{k+3}(\Sigma X_2) \longrightarrow \cdots) \\
&\cong \operatorname*{colim}_n(\pi_{k+1}(X_0 \wedge S^1) \longrightarrow \pi_{k+2}(X_1 \wedge S^1) \longrightarrow \pi_{k+3}(X_2 \wedge S^1) \longrightarrow \cdots) \\
&\cong \pi_{k+1}(X \wedge S^1),
\end{aligned}$$

where only the third one needs explanation: though all groups are isomorphic, the maps do not agree under these isomorphisms, as they differ by a twist map $\mathrm{tw} : S^1 \wedge S^1 \longrightarrow S^1 \wedge S^1$ of degree -1. However, the colimit is still the same as we can go two steps at a time. $\square$

Corollary 3.3.4 *A map $f : X \longrightarrow Y$ of spectra is a π_*-isomorphism if and only if the map $f \wedge \mathrm{Id} : X \wedge S^1 \longrightarrow Y \wedge S^1$ is a π_* isomorphism.*

For X a spectrum, we let $\Sigma X := X \wedge S^1$ and $\Omega X := F(S^1, X)$. Let us write $\eta : X \longrightarrow \Omega\Sigma X$ and $\varepsilon : \Sigma\Omega X \longrightarrow X$ for the unit and counit of the adjunction

$$\mathrm{Hom}_{\mathsf{Sp}^{\mathbb{I}}}(X \wedge S^1, Y) \cong \mathrm{Hom}_{\mathsf{Sp}^{\mathbb{I}}}(X, F(S^1, Y)).$$

Proposition 3.3.5 *The maps $\eta : X \longrightarrow \Omega\Sigma X$ and $\varepsilon : \Sigma\Omega X \longrightarrow X$ are π_*-isomorphisms.*

Proof. There are commutative diagrams

$$\begin{tikzcd}
\pi_{n+k}(X_n) \arrow[r, "(\eta_n)_*"] \arrow[d, "- \wedge S^1"] & \pi_{k+n}(\Omega\Sigma X_n) \arrow[d, "\cong"] \\
\pi_{n+k+1}(X_n \wedge S^1) \arrow[r, "\cong"] & \pi_{k+n}(\Omega(X_n \wedge S^1))
\end{tikzcd}$$

for all $n + k \geq 0$, which are compatible for varying n, so they induce a commutative square in the colimits,

$$\begin{tikzcd}
\pi_{n+k}(X) \arrow[r, "\eta_*"] \arrow[d, "- \wedge S^1"] & \pi_{k+n}(\Omega\Sigma X) \arrow[d, "\cong"] \\
\pi_{n+k+1}(X \wedge S^1) \arrow[r, "\cong"] & \pi_{k+n}(\Omega(X \wedge S^1)).
\end{tikzcd}$$

We conclude since $- \wedge S^1$ is an isomorphism by 3.3.3. For the counit of the adjunction the argument is similar. $\square$

Corollary 3.3.6 *A map of spectra $f : X \longrightarrow Y$ is a π_*-isomorphism if and only if $\Omega f : \Omega X \longrightarrow \Omega Y$ is a π_*-isomorphism.*

Spectra and homology theories

We conclude this section discussing the relation between spectra and homology theories. In chapter 2, we said that given an Ω-spectrum E, we can define a reduced cohomology theory on pointed CW-complexes given by

$$\widetilde{E}^n(T) := [T, E_n]_*$$

Moreover, Brown representability (theorem 2.2.4) stated that all reduced cohomology theories on CW-complexes arise in this way. The question one may ask is: can an Ω-spectrum, or with more generality a spectrum, induce a homology theory? The answer turns out to be positive, in the strongest form.

Theorem 3.3.7 *If E is a spectrum, the assignment*

$$\widetilde{E}_n(T) := \pi_n(E \wedge T)$$

defines a reduced (generalized) homology theory on pointed spaces.

Proof sketch. Of course, two homotopic maps of spectra $f, g : X \longrightarrow Y$ induce the same map $f_* = g_* : \pi_k(X) \longrightarrow \pi_k(Y)$ as the maps f_n, g_n are homotopic for all n. On the other hand, the suspension isomorphism follows from 3.3.3 as it implies $E \wedge \Sigma T \cong \Sigma(E \wedge T)$.

For a map of spectra $f : X \longrightarrow Y$, the mapping cone Cf is defined levelwise, $(Cf)_n = Cf_n$, and we have that the sequence

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{i_*} \pi_k(Cf)$$

is exact: the composition is clearly trivial, thus if $\sigma : S^{k+n} \longrightarrow Y_n$ in the kernel of i_*, then $S^{k+n} \longrightarrow Y_n \longrightarrow Cf_n$ is nullhomotopic (after possibly enlarging n), so we can extend it to a map of pairs $(D^{k+n+1}, S^{k+n}) \longrightarrow (Cf_n, Y_n)$, which produced a map $D^{k+n+1}/S^{k+n} \cong S^{n+k+1} \longrightarrow \Sigma X_n \cong Cf_n/Y_n$. The composite $S^{n+k+1} \longrightarrow \Sigma X_n \longrightarrow \Sigma Y_n$ is the desired lift. Therefore, for our particular case, we just apply this to a map of spectra $E \wedge T \longrightarrow E \wedge T'$.

Finally, for the additivity axiom, we note that for spectra X, Y we get a split sequence $\pi_k(E) \longrightarrow \pi_k(X \vee Y) \longrightarrow \pi_k(Y)$ (here the wedge sum of spectra is defined levelwise with the obvious structure maps) as the mapping cone of $X \longrightarrow X \vee Y$ is levelwise homotopy equivalent to Y. We obtain $\pi_k(E \wedge (\vee_i T_i)) \cong \pi_k(\vee_i (E \wedge T_i))$ by a transfinite argument. For a complete proof, see [38, 8.33]. □

Examples 3.3.8 (a) The reduced homology theory that the sphere spectrum $\mathcal{S}_{\mathbb{N}}$ defines gives the stable homotopy groups of spaces: indeed, $\mathcal{S}_{\mathbb{N}} = \Sigma^\infty S^0$ and

$$\pi_k(\mathcal{S}_{\mathbb{N}} \wedge T) = \operatorname*{colim}_n \pi_{n+k}(S^n \wedge T) = \operatorname*{colim}_n \pi_{n+k}(\Sigma^n T) = \pi_k^{st}(T)$$

for a based space T. In particular,

$$\pi_k^{st}(\bigvee_i T_i) \cong \bigoplus_i \pi_k^{st}(T_i).$$

It is important to note that the usual homotopy groups are way too far from satisfying the additivity axiom. See [27, 2.13] for a concrete example.

(b) For the Eilenberg-MacLane spectrum HA, the homology theory that it defines is naturally isomorphic to singular homology,

$$\pi_k(HA \wedge -) \cong \widetilde{H}_k(-; A).$$

Indeed, by 3.3.7 the functor $\pi_k(HA \wedge -)$ defines a reduced generalized homology theory. But it also satisfies the dimension axiom, as $HA \wedge S^0 \cong HA$, so using 3.3.2.(b) we see that $\pi_k(HA \wedge S^0) \cong 0$ for $k > 0$ and $\pi_0(HA \wedge S^0) \cong A$. We conclude by the uniqueness of the Eilenberg-Steenrod axioms.

(c) For the K-theory spectrum, $\pi_k(K \wedge -)$ defines the so-called *K-homology*, used in index theory.

3.4 The stable homotopy category

In this section, we finally construct the "stable homotopy category" SHC and describe its properties. This will arise as the homotopy category of spectra or orthogonal spectra, so we will be dealing with model structures in the following. In this section, we mostly used [24] to construct a model structure on spectra; and [22] to collect the properties of SHC. With exception of 3.4.5, all proofs are due to the author.

The first step is to endow the category of $\mathcal{D}$-spaces with a model structure called the *level model structure*, where $\mathcal{D}$ is a diagram. This will be, in particular, cofibrantly generated, just as the category of spaces, although we will not insist on that.

Definition. Let $f : X \longrightarrow Y$ be a map of $\mathcal{D}$-spaces.

1. We say that f is a **level equivalence** if $f_D : X_D \longrightarrow Y_D$ is a weak homotopy equivalence for all $D \in \mathcal{D}$.

2. We say that f is a **level fibration** if $f_D : X_D \longrightarrow Y_D$ is a Serre fibration for all $D \in \mathcal{D}$.

3. We say that f is a **level trivial fibration** if it is a level equivalence and a level fibration.

4. We say that f is a **q-cofibration** if it has the left lifting property with respect to level trivial fibrations.

5. We say that f is a **level trivial q-cofibration** if it is a level equivalence and a q-cofibration.

We can use once again the Model Structure Lifting theorem A.4.3 and the family of adjunctions

$$M_D : \mathsf{Top}_* \rightleftarrows \mathsf{Top}_*^{\mathcal{D}} : \mathrm{ev}_D$$

from 3.1.8, for all $D \in \mathcal{D}$, to give a model structure in $\mathcal{D}$-spaces.

Theorem 3.4.1 *The category $\mathsf{Top}_*^{\mathcal{D}}$ of $\mathcal{D}$-spaces has a model structure, called the **level model structure**, with respect to level equivalences, level fibrations and q-cofibrations.*

The proof follows by a standard application of the small object argument A.4.2, and it is formally similar to 1.3.1. As mentioned before, the categories of spectra and orthogonal spectra are equivalent to some categories of $\mathcal{D}$-spaces for some more complicated diagrams $\mathbb{N}_{\mathcal{S}_\mathbb{N}}$ and $\mathbb{I}_{\mathcal{S}_\mathbb{I}}$. This is still applicable, so we also obtain model structures on $\mathsf{Sp}^\mathbb{N}$ and $\mathsf{Sp}^\mathbb{I}$. However, these are not the ones we are looking for, as they do not contain information about the stable phenomena. Anyway, the following observation will be useful.

Lemma 3.4.2 *The following holds for maps of spectra or orthogonal spectra:*

1. *A homotopy equivalence is a level equivalence.*

2. *A level equivalence is a π_*-isomorphism.*

3. *For Ω-spectra, a π_*-isomorphism is a level equivalence.*

Proof. Only the last one is not trivial. Suppose that $f : X \longrightarrow Y$ is a π_*-isomorphism of Ω-spectra (for the orthogonal case it is similar). By example 3.3.2.(b), $\pi_i(X_n) \cong \pi_{i-n}(X_0) \cong \pi_{i-n}(X)$, so $f_* : \pi_i(X_n) \longrightarrow \pi_i(Y_n)$ is also an isomorphism for all i and all n. $\square$

We now strive to describe a more interesting model structure in spectra and orthogonal spectra, called the *stable model structure*, which will reflect the stable phenomena:

Definition. Let $f : X \longrightarrow Y$ be a map of spectra or orthogonal spectra.

1. We say that f is a **stable equivalence** if the map $f^* : [Y, Z] \longrightarrow [X, Z]$ is a bijection for all (orthogonal) Ω-spectra Z.

2. We say that f is a **trivial q-cofibration** if it is a stable equivalence and a q-cofibration.

3. We say that f is a **q-fibration** if it has the right lifting property with respect to trivial q-cofibrations.

4. We say that f is a **trivial q-fibration** if it is a stable equivalence and a q-fibration.

Here we introduced stable equivalences, but we could have just omitted them, because of the following

Proposition 3.4.3 *A map of spectra or ortogonal spectra is a stable equivalence if and only if it is a π_*-isomorphism.*

The proof of the previous proposition is quite elaborated and we refer to the interested reader to [24, 8.7]. We finally arrive to the desired model structure in (orthogonal) spectra:

Theorem 3.4.4 *There are model structures in $\mathsf{Sp}^{\mathbb{N}}$ and $\mathsf{Sp}^{\mathbb{I}}$, called the **stable model structures**, with respect to π_*-isomorphisms, q-fibrations and q-cofibrations.*

Now recall from 3.2.10 that we have an adjunction between spectra and orthogonal spectra

$$\begin{tikzcd} \mathsf{Sp}^{\mathbb{N}} \arrow[r, bend left, "\mathbb{P}"] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^{\mathbb{I}}, \arrow[l, bend left, "\mathbb{U}"] \end{tikzcd}$$

where $\mathbb{U}$ was simply forgetful. We want to show that this adjunction induces an equivalence of categories, with the stable model structures on them.

Lemma 3.4.5 *The previous adjunction is a Quillen adjunction.*

Proof sketch. The functor $\mathbb{U}$ preserves q-fibrations and trivial q-fibrations (so it is right Quillen). This follows immediately from [24, 9.5 and 9.9], which characterize trivial q-fibrations as level trivial fibrations, and q-fibrations $p : E \longrightarrow B$ as level fibrations with the extra property that the diagram

$$\begin{tikzcd} E_n \arrow[r, "\tilde{\sigma}"] \arrow[d, "p_n"] & \Omega E_{n+1} \arrow[d, "\Omega p_{n+1}"] \\ B_n \arrow[r, "\tilde{\sigma}"] & \Omega B_{n+1} \end{tikzcd}$$

is a homotopy pullback for all $n \geq 0$. □

Lemma 3.4.6 *The functor $\mathbb{U} : \mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{Sp}^{\mathbb{N}}$ creates π_*-isomorphisms.*

Proof. The homotopy groups of ortogonal spectra are defined precisely using $\mathbb{U}$, so in particular this functor reflects π_*-isomorphisms, thus it also creates them. □

We only need one more ingredient, whose proof can be found in [24, 10.3].

Lemma 3.4.7 *The unit of the adjunction $\eta : X \longrightarrow \mathbb{U}\mathbb{P}X$ is a π_*-isomorphism for all cofibrant (orthogonal) spectra X.*

Theorem 3.4.8 *The adjunction $\mathbb{P} : \mathsf{Sp}^{\mathbb{N}} \rightleftarrows \mathsf{Sp}^{\mathbb{I}} : \mathbb{U}$ is a Quillen equivalence. Therefore, it induces an adjoint equivalence of categories*

$$\mathsf{Ho}(\mathsf{Sp}^{\mathbb{N}}) \simeq \mathsf{Ho}(\mathsf{Sp}^{\mathbb{I}}).$$

Proof. It follows from A.3.7, 3.4.6 and 3.4.7. □

We finally arrive to the category we have been chasing after the whole chapter:

Definition. The **stable homotopy category** is the homotopy category of spectra (or equivalently, orthogonal spectra), and it is denoted by SHC.

Properties of the stable homotopy category

We will spend the last part of this chapter outlining the properties of the stable homotopy category. This category is desirable since, as we will see, it has properties which are far from being true in spaces.

One of the motivations of stable homotopy theory was to find some analogues to spaces which do not depend on taking suspensions. This holds in the stable homotopy category. Recall that we set $\Sigma := - \wedge S^1$ and $\Omega := F(S^1, -)$ for (orthogonal) spectra.

Proposition 3.4.9 *The adjunction*

$$\begin{tikzcd} \mathsf{Sp}^{\mathbb{N}} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^{\mathbb{N}} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

is a Quillen equivalence, so it induces mutually inverse equivalences of categories

$$\mathbb{L}\Sigma : \mathsf{SHC} \xrightarrow{\simeq} \mathsf{SHC} \quad , \quad \mathbb{R}\Omega : \mathsf{SHC} \xrightarrow{\simeq} \mathsf{SHC}.$$

Proof. We first show that it is a Quillen adjuntion: by 3.3.4, we have that Σ preserves π_*-isomorphisms, so let us show that it preserves q-cofibrations: given a q-cofibration $f : X \longrightarrow Y$, consider the lifting problem

$$\begin{tikzcd} X \wedge S^1 \arrow[r] \arrow[d, "\Sigma f"'] & E \arrow[d, "p"] \\ Y \wedge S^1 \arrow[r] & B \end{tikzcd}$$

where p is a level trivial fibration. Termwise, this diagram is equivalent to

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & \Omega E_n \arrow[d, "\Omega p_n"] \\ Y_n \arrow[r] & \Omega B_n. \end{tikzcd}$$

The vertical right-hand arrow Ωp_n is again a trivial fibration, as Ω preserves weak homotopy equivalences and Serre fibrations. Therefore, there is a lift $h_n : Y_n \longrightarrow \Omega E_n$ which induces the desired lift in the first diagram.

We conclude that it is a Quillen equivalence by A.3.7, 3.3.5 and 3.3.6. □

This explains the self-equivalence of the commutative diagram on page 39. Let us explain the rest of the arrows in that diagram:

Proposition 3.4.10 *There is a Quillen adjunction*

$$\begin{tikzcd} \mathsf{Top}_* \arrow[r, bend left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & \mathsf{Sp}^{\mathbb{N}}, \arrow[l, bend left, "\Omega^\infty"] \end{tikzcd}$$

where Ω^∞ is evaluation at $0 \in \mathbb{N}$. In particular, it induces adjoint functors

$$\mathbb{L}\Sigma^\infty : \mathsf{Ho}(\mathsf{Top}_*) \longrightarrow \mathsf{SHC} \quad , \quad \mathbb{R}\Omega^\infty : \mathsf{SHC} \longrightarrow \mathsf{Ho}(\mathsf{Top}_*).$$

Proof. In the first place, let us show that there is an adjunction. If T is a pointed space and X is a spectrum, the desired bijection

$$\mathrm{Hom}_{\mathsf{Sp}^{\mathbb{N}}}(\Sigma^\infty T, X) \xrightarrow{\cong} \mathrm{Map}(T, X_0)$$

takes every map of spectra $f : \Sigma^\infty T \longrightarrow X$ to $f_0 : T \longrightarrow X_0$. Its inverse is constructed as follows: given $g : T \longrightarrow X_0$, we set $f_0 := g$ and inductively define $f_n : \Sigma^n T \longrightarrow X_n$ as the adjoint map of the composite $\Sigma^{n-1}T \longrightarrow X_{n-1} \longrightarrow \Omega X_n$, where the second map is the adjoint of the structure map $\sigma_{n-1} : \Sigma X_{n-1} \longrightarrow X_n$. It is easy to check that this defines a map of spectra $f : \Sigma^\infty T \longrightarrow X$ and that both assignments are inverses of each other.

On the other hand, the reduced suspension preserves weak homotopy equivalences in the category Top_* of well-pointed based (compactly generated) spaces [40, 6.7.10], so trivially Σ^∞ preserves weak equivalences. Therefore, to see that this functor is left Quillen, we just have to check that it preserves cofibrations. The argument is similar to 3.4.9: if $g : A \longrightarrow T$ is a cofibration of spaces, then consider the lifting problem

$$\begin{tikzcd}
\Sigma^n A \arrow[r] \arrow[d, "\Sigma^n g"'] & E_n \arrow[d, "p_n"] \\
\Sigma^n T \arrow[r] & B_n
\end{tikzcd}$$

where p_n is a trivial fibration. By the suspension-loop adjunction, this diagram is equivalent to

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "g"'] & \Omega^n E_n \arrow[d, "\Omega^n p_n"] \\
T \arrow[r] \arrow[ur, dashed] & \Omega^n B_n
\end{tikzcd}$$

where the dashed arrow exists since $\Omega^n E_n \longrightarrow \Omega^n B_n$ is a trivial fibration. $\square$

The following result is obvious from the definitions:

Corollary 3.4.11 *The following diagrams are commutative:*

$$\begin{tikzcd}
\mathsf{Top}_* \arrow[r, "\Sigma \circ \mathrm{cw}"] \arrow[d] & \mathsf{Top}_* \arrow[d] \\
\mathsf{Ho}(\mathsf{Top}_*) \arrow[r, "\mathbb{L}\Sigma"] \arrow[d, "\mathbb{L}\Sigma^\infty"'] & \mathsf{Ho}(\mathsf{Top}_*) \arrow[d, "\mathbb{L}\Sigma^\infty"] \\
\mathsf{SHC} \arrow[r, "\mathbb{L}\Sigma"] & \mathsf{SHC}
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\mathsf{Top}_* \arrow[r, "\Omega"] \arrow[d] & \mathsf{Top}_* \arrow[d] \\
\mathsf{Ho}(\mathsf{Top}_*) \arrow[r, "\mathbb{R}\Omega"] & \mathsf{Ho}(\mathsf{Top}_*) \\
\mathsf{SHC} \arrow[r, "\mathbb{R}\Omega"] \arrow[u, "\mathbb{R}\Omega^\infty"] & \mathsf{SHC} \arrow[u, "\mathbb{R}\Omega^\infty"']
\end{tikzcd}$$

The last piece which remains to be explained from the diagram of page 39 is that $\pi_0(HA) \cong A$. This follows immediately from 3.3.2.(b).

The stable homotopy category is neither complete nor cocomplete. However, it has arbitrary products and coproducts (coming from $\mathsf{Sp}^{\mathbb{N}}$): given a family of spectra $(X_i)_{i \in I}$, the coproduct $\bigvee_i X^i$ and the product $\prod_i X^i$ are defined levelwise,

$$\left(\bigvee_i X^i\right)_n := \bigvee_i X^i_n \quad , \quad \left(\prod_i X^i\right)_n := \prod_i X^i_n$$

with the obvious structure maps in the first case, $\Sigma(\vee_i X_n^i) \cong \vee_i \Sigma X_n^i \xrightarrow{\vee_i \sigma} \vee_i X_{n+1}^i$; and the map $\Sigma \prod_i X_n^i \longrightarrow \prod_i X_{n+1}^i$ which in the i-th factor is the composite

$$\Sigma \left(\prod_i X_n^i \right) \xrightarrow{\Sigma \mathrm{pr}_i} \Sigma X_n^i \xrightarrow{\sigma_i} X_{n+1}^i.$$

The stable homotopy category also has the following desirable property:

Proposition 3.4.12 SHC *is an* ***additive*** *category, that is,*

1. $\mathrm{Hom}_{\mathsf{SHC}}(X, Y)$ *is an abelian group and composition is bilinear.*
2. *Finite products coincide with finite coproducts, in other words, the canonical map*

$$\bigvee_{i=1}^{n} X_i \longrightarrow \prod_{i=1}^{n} X_i$$

is an isomorphism in SHC.

See [34, II.1.10] for a proof. We can also have a look at the hom-groups of the stable homotopy category:

Proposition 3.4.13 *Let T, T' be finite CW-complexes, let X be a spectrum and let Y be an Ω-spectrum. Then we have the following group isomorphisms:*

1. $\mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty T, X) \cong \mathrm{colim}_n [\Sigma^n T, X_n]_*$
2. $\mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty T, \Sigma^\infty T') \cong \mathrm{colim}_n [\Sigma^n T, \Sigma^n T']_*$
3. $\mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty T, Y) \cong [T, Y_0]_*$
4. $\pi_k(X) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^k \mathcal{S}_{\mathbb{N}}, X) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty S^k, X)$.
5. $\pi_k^{st}(T) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty S^k, \Sigma^\infty T)$.

Proof. The observation that ignites the proof is that the fibrant objects in $\mathrm{Sp}^{\mathbb{N}}$ are precisely the Ω-spectra, and that if T is a CW-complex, then $\Sigma^\infty T$ is cofibrant. In particular, a fibrant replacement of a spectrum X can be obtained as

$$(RX)_k := \mathrm{hocolim}_n\, \Omega^n X_{n+k}$$

where the structure maps are inherited from the ones of X (see [27, 4.9]).

For 1. we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty T, X) &\overset{A.2.8}{\cong} [\Sigma^\infty T, RX] \overset{3.4.10}{\cong} [T, (RX)_0]_* = [T, \mathrm{hocolim}_n \Omega^n X_n]_* \\ &\cong \operatorname*{colim}_n [T, \Omega^n X_n]_* \cong \operatorname*{colim}_n [\Sigma^n T, X_n]_* \end{aligned}$$

where we use the compactness of T. Here the first $[-,-]$ denotes homotopy classes of maps of spectra.

2. is a direct consequence of 1., and 3. follows also from 1. since $[\Sigma^n, Y_n]_* \cong [T, \Omega Y_n]_* \cong [T, Y_0]_*$. For 4. we compute

$$\begin{aligned} \pi_n(X) &\cong \operatorname*{colim}_n [S^{n+k}, X_n]_* \overset{1.}{\cong} \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty S^k, X) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^\infty \Sigma^k S^0, X) \\ &\cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^k \Sigma^\infty S^0, X) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^k \mathcal{S}_{\mathbb{N}}, X) \end{aligned}$$

and 5. follows directly from 2. $\square$

Next, recall that the category $(\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}})$ is closed symmetric monoidal. We would like to pass this extra-structure to its homotopy category, SHC. This is a formal consequence of the interaction between model categories and symmetric monoidal categories. In what is left we follow [16].

Definition. Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be model categories. We say that an adjunction of two variables $\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ is **Quillen** if given cofibrations $f : C \longrightarrow C'$ in $\mathcal{C}$ and $g : D \longrightarrow D'$ in $\mathcal{D}$, the induced map

$$C' \otimes D \cup_{C \otimes D} C \otimes D' \longrightarrow C' \otimes D'$$

is a cofibration on $\mathcal{E}$ which is trivial if either f or g is.

The left adjoint $\otimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}$ is called a **Quillen bifunctor**.

Definition. A **symmetric monoidal model category** is a closed symmetric monoidal category $(\mathcal{V}, \otimes, *)$ endowed with a model structure such that both are compatible in the following way:

(*i*) The monoidal product $\otimes : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ is a Quillen bifunctor.

(*ii*) If $q : Q(*) \longrightarrow *$ is a cofibrant replacement for the unit $*$, then the natural map $q \otimes \mathrm{Id} : Q(*) \otimes X \longrightarrow * \otimes X$ is a weak equivalence for all X cofibrant.

Example 3.4.14 The categories $(\mathsf{Top}, \times, *)$, $(\mathsf{Top}_*, \wedge, S^0)$, $(\mathsf{sSet}, \times, \underline{\Delta}^0)$, $(\mathsf{Ch}_R, \otimes, R)$, $(\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}})$ are symmetric monoidal model categories (see [16, §4.2]).

The following theorem formally gives the desired structure in SHC (see [16, 4.3.2]):

Theorem 3.4.15 *Let $(\mathcal{V}, \otimes, *)$ be a symmetric monoidal model category. Then the homotopy category inherits a structure of closed symmetric monoidal category*

$$(\mathsf{Ho}(\mathcal{V}), \otimes^{\mathbb{L}}, *),$$

where the monoidal product $\otimes^{\mathbb{L}}$ is the total left derived functor of $\otimes$. In particular,

$$j : (\mathcal{V}, \otimes, *) \longrightarrow (\mathsf{Ho}(\mathcal{V}), \otimes^{\mathbb{L}}, *)$$

is lax symmetric monoidal.

Corollary 3.4.16 *The stable homotopy category inherits a closed symmetric monoidal structure*

$$(\mathsf{SHC}, \wedge_{\mathbb{I}}^{\mathbb{L}}, \mathcal{S}_{\mathbb{I}}).$$

This extra-structure allows us to define additional objects in the stable homotopy category. For instance, a **ring spectrum** is a monoid object in $(\mathsf{SHC}, \wedge_{\mathbb{I}}^{\mathbb{L}}, \mathcal{S}_{\mathbb{I}})$. Obviously, any orthogonal ring spectrum gives rise to a ring spectrum (though both notions are not equivalent). We will talk more about this in the equivariant case.

We finish off the list of properties of SHC by stating an important additional structure that it has: the stable homotopy category carries a structure of triangulated category. This will be certainly useful for our main goal in §4.3.

Theorem 3.4.17 *The stable homotopy category* SHC *is a triangulated category, equipped with the suspension self-equivalence $\mathbb{L}\Sigma : \mathsf{SHC} \xrightarrow{\simeq} \mathsf{SHC}$ and taking as class of distinguished triangles the closure under isomorphisms of the images under $j : \mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{SHC}$ of the canonical cofibre sequences*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X,$$

where f is a cofibration of cofibrant objects and Z is the pushout of the diagram $* \leftarrow X \xrightarrow{f} Y$.

The proof is a formal consequence of A.5.3.(4). For the reader interested in a explicit proof, see [34, II.2.9].

Remark 3.4.18 We would like to wrap up this chapter with an important observation. We have induced the stable homotopy category SHC from spectra $\mathsf{Sp}^{\mathbb{N}}$ or equivalently, orthogonal spectra $\mathsf{Sp}^{\mathbb{I}}$, since in 3.4.8 we saw that they were Quillen equivalent. If one considers the symmetric spectra that we discussed in 3.2.1, or $\mathscr{W}$-spaces [24, 4.6], then they have model structures such that their homotopy categories are equivalent to SHC. One can show that, actually, symmetric spectra and $\mathscr{W}$-spaces are Quillen equivalent to orthogonal spectra [24, 10.5]. With this in mind, one may ask whether given two categories inducing SHC, they must be Quillen equivalent. The answer turns out to be positive (see [33]):

Theorem 3.4.19 (Rigity, Schwede 2007) *Let $\mathcal{C}$ be a stable model category. If $\mathsf{Ho}(\mathcal{C})$ and SHC are equivalent as triangulated categories, then $\mathcal{C}$ and $\mathsf{Sp}^{\mathbb{I}}$ are Quillen equivalent.*

Chapter 4

Equivariant stable homotopy theory

In the last chapter we defined spectra and constructed the stable homotopy category, and we said that its objects are some analogues to spaces. The goal of this chapter is to construct objects which are analogue, in the same way, to G-spaces. We will define them in a very similar way and will see that we adapt the stable homotopy theory developed in §3.3 to the equivariant setup, in a similar fashion as we adapted homotopy theory of spaces to G-spaces in chapter 1. We will also see how we can lift K-theory with Reality to this stable setup, and at the end we will generalize the notion of G-fixed points and relate KR-theory with KO-theory at the level of spectra, with an original proof of the author.

4.1 G-spectra

This first section runs parallel to §3.1 – §3.3 altogether, so we will try to be a little bit more direct to avoid repeating ourselves. In this section we have mostly used [23] and [35], but we present G-spectra and orthogonal G-spectra differently, as a category of modules, by analogy with chapter 3. The description of KR-theory as a $\mathbb{Z}/2$-spectrum is also due to the author.

Remark 4.1.1 Before starting, we would like to do the following observation: to define spectra in the nonequivariant setup, we always considered enriched categories and enriched functors over $(\mathsf{Top}_*, \wedge, S^0)$. For a Top_*-category $\underline{\mathcal{C}}$, the internal hom-space $\mathrm{Hom}_{\underline{\mathcal{C}}}(X, Y)$ has as underlying set the hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of its underlying category $\mathcal{C}$. If we want to run the equivariant analogous version, we will have to replace Top_* by $(G\mathsf{Top}_*, \wedge, S^0)$. This seems inoffensive but there is a subtle point: in a similar fashion as 3.1.6, for a category $\underline{\mathcal{C}}$ enriched over $G\mathsf{Top}_*$, the hom-set of $\mathcal{C}$ is not anymore the underlying set of the internal hom of $\underline{\mathcal{C}}$, but rather the underlying set of its G-fixed points space $\mathrm{Hom}_{\underline{\mathcal{C}}}(X, Y)^G$. By ease of notation, we will keep dropping the underline for enriched categories, but the reader should be aware of the difference between a $G\mathsf{Top}_*$-category and its underlying category, as well as of the difference between the internal hom-G-space of the enriched category and the hom set of its underlying category.

Diagram G-spaces

Definition. A **G-diagram** $\mathcal{D}$ is a small symmetric monoidal category $(\mathcal{D}, \otimes, 0)$ enriched over $(G\mathsf{Top}_*, \wedge, S^0)$.

The category of **$\mathcal{D}$-G-spaces** is the category of enriched functors $G\mathsf{Top}_*^{\mathcal{D}}$ (which is also enriched over $G\mathsf{Top}_*$).

As in the chapter 3, we can make the category of $\mathcal{D}$-G-spaces symmetric monoidal: given $\mathcal{D}$-G-spaces X, Y, the naive "external" product

$$\begin{aligned} X\widetilde{\wedge}Y : \mathcal{D}\times\mathcal{D} &\longrightarrow \mathcal{D} \\ (D, D') &\longmapsto (X\widetilde{\wedge}Y)_{(D,D')} := X_D \wedge Y_{D'}. \end{aligned}$$

can be promoted to an "internal" product using **Day convolution**, that is, the left Kan extension of $X\widetilde{\wedge}Y$ along the monoidal product $\otimes : \mathcal{D}\times\mathcal{D} \longrightarrow \mathcal{D}$,

$$\begin{tikzcd} \mathcal{D}\times\mathcal{D} \arrow[r, "X\widetilde{\wedge}Y"] \arrow[d, "\otimes"'] & G\mathsf{Top}_* \\ \mathcal{D} \arrow[ur, dashed, "X\wedge Y:=\mathrm{Lan}_{\otimes}X\widetilde{\wedge}Y"'] & \end{tikzcd}$$

The equivalence relation that defines the coend of the left Kan extension is G-invariant, which ensures that it produces a well-defined $\mathcal{D}$-G-space $X \wedge Y$ (see [23, page 34]. Day convolution is determined by the universal property of the left Kan extension, that in this context states that there is an homeomorphism of based G-spaces

$$\mathrm{Hom}_{G\mathsf{Top}_*^{\mathcal{D}}}(X\wedge Y, Z) \cong \mathrm{Hom}_{G\mathsf{Top}_*^{\mathcal{D}\times\mathcal{D}}}(X\widetilde{\wedge}Y, Z\circ\otimes).$$

Similarly to the nonequivariant case, the unit of the symmetric monoidal structure is given by the left adjoint of the evaluation at $0 \in \mathcal{D}$, coming from the adjunction (analogous to 3.1.8)

$$\begin{tikzcd} G\mathsf{Top}_* \arrow[r, bend left, "M_D"] \arrow[r, phantom, "\perp"] & G\mathsf{Top}_*^{\mathcal{D}} \arrow[l, bend left, "ev_D"] \end{tikzcd}$$

for any $D \in \mathcal{D}$. Explicitly this is

$$M_D T(D') := \mathrm{Hom}_{\mathcal{D}}(D, D') \wedge T$$

for a based G-space T and $D' \in \mathcal{D}$. The internal hom-G-space that will give us an adjoint for Day convolution is defined as in chapter 3: if X is a $\mathcal{D}$-G-space and Z is a $(\mathcal{D}\times\mathcal{D})$-space, there is a naive version of function space,

$$\widetilde{F}(X, Z)_D := \mathrm{Hom}_{G\mathsf{Top}_*^{\mathcal{D}}}(X, Z_{(D,-)}),$$

and the **internal function $\mathcal{D}$-G-space** $F(X, Y)$ is defined as

$$F(X, Y) := \widetilde{F}(X, Y\circ\otimes).$$

This provides an homeomorphism of based G-spaces

$$\mathrm{Hom}_{G\mathsf{Top}_*^{\mathcal{D}}}(X\wedge Y, Z) \cong \mathrm{Hom}_{G\mathsf{Top}_*^{\mathcal{D}}}(X, F(Y, Z)).$$

In summary, we have the following

Proposition 4.1.2 *The category of $\mathcal{D}$-G-spaces $(G\mathsf{Top}_*^{\mathcal{D}}, \wedge, M_0S^0)$ is closed symmetric monoidal under Day convolution and unit M_0S^0.*

We will take advantage of this structure to define G-spectra and orthogonal G-spectra simply as modules over some monoid object, as usual. However, an important observation when we want to mimic spectra in the equivariant case is that we can no longer index over natural numbers: if we want to deal with spheres with G-actions, we are forced to take representation spheres in any case. What we can control is the representations we want to include. This will be called the *universe*.

Warning. From now on all representations of the group G will be considered real, finite-dimensional and orthogonal (that is, through isometries).

Definition. A **universe** $\mathscr{U}$ is a sum of countably many copies of each G-inner product space in some set of irreducible representations of G that includes the trivial representations.

We say that $\mathscr{U}$ is **complete** if it contains, up to isomorphism, all irreducible representations of G, and **trivial** if it only contains the trivial ones.

From now on we fix a universe $\mathscr{U}$. Let us describe the analogous versions of spectra and orthogonal spectra for the equivariant case, although this time it will be a bit trickier. Since we assume that G is a finite group, G has finitely many irreducible representations in our universe, say $V_1, \ldots, V_r$ (in particular, all of them are contained in the left regular representation $\rho_G = \mathbb{R}[G]$). We now define $\mathbb{J}_G$ as the discrete category whose objects are ordered sequences

$$V_1^{\oplus k_1} \oplus \cdots \oplus V_r^{\oplus k_r},$$

where $k_1, \ldots, k_r \geq 0$. We can also think of its objects as r-tuples of natural numbers $(k_1, \ldots, k_r)$. This is also enriched over $G\mathsf{Top}_*$, just by considering the trivial G-action on the internal hom-spaces, where we have attached a disjoint point. This category turns out to be symmetric monoidal under sum of integers termwise, with the trivial representation as unit, here represented by the r-tuple $(0, \ldots, 0)$.

We also consider the enriched category $\mathbb{I}_G$, with objects finite-dimensional G-inner product subspaces $V \subset \mathscr{U}$, and arrows G-linear isometric isomorphisms. It is enriched over $G\mathsf{Top}_*$ with G acting by conjugation in the hom-spaces (we have implicitly attached a disjoint point to the hom-spaces). Moreover, this category is symmetric monoidal under direct sum and the trivial 0-dimensional representation as unit.

Remark 4.1.3 There is a canonical functor $\mathbb{J}_G \longrightarrow \mathbb{I}_G$ which is an embedding (faithful functor injective on objects), since different decompositions give rise to non-isomorphic representations. Therefore, the category $\mathbb{J}_G$ can be viewed as a subcategory of $\mathbb{I}_G$.

Now we can look, as usual, at the categories $G\mathsf{Top}_*^{\mathbb{J}_G}$ and $G\mathsf{Top}_*^{\mathbb{I}_G}$. There are monoid objects $\mathcal{S}_{\mathbb{J}_G}$ and $\mathcal{S}_{\mathbb{I}_G}$ defined as

$$\mathcal{S}_{\mathbb{J}_G}(W) := S^W \qquad , \qquad \mathcal{S}_{\mathbb{I}_G}(V) := S^V$$

for every $W = V_1^{\oplus k_1} \oplus \cdots \oplus V_r^{\oplus k_r}$ and $V \subset \mathscr{U}$. In particular, the latter is a commutative monoid object, by the same diagram as in the nonequivariant case (see page 49). Notice that these representation spheres have an $(O(V) \times G)$-action.

Definition. The category of G**-spectra** is the category $G\mathsf{Sp}^{\mathbb{J}} := \mathsf{Mod}_{\mathcal{S}_{\mathbb{J}_G}}$ of $\mathcal{S}_{\mathbb{J}_G}$-modules over $G\mathsf{Top}_*^{\mathbb{J}_G}$; and the category of **orthogonal** G**-spectra** is the category $G\mathsf{Sp}^{\mathbb{I}} := \mathsf{Mod}_{\mathcal{S}_{\mathbb{I}_G}}$ of $\mathcal{S}_{\mathbb{I}_G}$-modules over $G\mathsf{Top}_*^{\mathbb{I}_G}$.

Observe that $\mathbb{J}_G$ and $\mathbb{I}_G$ depend on the choice of the universe $\mathscr{U}$. If $\mathscr{U}$ is the trivial universe, then we will use the name of **naive** (orthogonal) G-spectra. If $\mathscr{U}$ is a complete universe, then we will use the name of **genuine** (orthogonal) G-spectra.

Moreover, just as in the previous chapter, there is a forgetful functor

$$\mathbb{U} : G\mathsf{Sp}^{\mathbb{I}} \longrightarrow G\mathsf{Sp}^{\mathbb{J}}$$

induced by $\mathbb{J}_G \longrightarrow \mathbb{I}_G$.

Remark 4.1.4 The categories $G\mathsf{Sp}^{\mathbb{J}}$ and $G\mathsf{Sp}^{\mathbb{I}}$ are tensored and cotensored over $G\mathsf{Top}_*$, just as in the nonequivariant case. The only changes to make to the same claim that we made for spectra and Top_* are that for a based G-space T and an orthogonal G-spectrum X, the space $X_V \wedge T$ has the diagonal G-action, and the space $F(T, X_V) = G\mathrm{Map}(T, X_V)$ has a G-action given by conjugation, $(g \cdot f)(t) := gf(g^{-1}t)$. If we let $\Sigma^V := - \wedge S^V$ and $\Omega^V := F(S^V, -)$, then the former is left adjoint to the latter.

Let us spell out the definitions of G-spectra and orthogonal G-spectra: a G-spectrum is a collection of pointed G-spaces $X = (X_V)$ for every $V \in \mathbb{J}_G$, together with based G-maps

$$\sigma_{V,W} : S^V \wedge X_W \longrightarrow X_{V \oplus W}$$

called *structure maps*, satisfying the evident associativity condition (see page 49). A map of G-spectra $f : X \longrightarrow Y$ is a collection of based G-maps $f_V : X_V \longrightarrow Y_V$ that commute with the structure maps. Using this description, we can define

Definition. An **Ω-G-spectrum** is a G-spectrum X such that the adjoints of the structure maps

$$X_W \longrightarrow \Omega^V X_{V \oplus W}$$

are weak homotopy equivalences.

Similarly, we see that an orthogonal G-spectrum consists of a collection of $(O(V) \times G)$-based spaces $X = (X_V)$ for every $V \in \mathbb{I}_G$, together with based maps

$$\sigma_{V,W} : S^V \wedge X_W \longrightarrow X_{V \oplus W}$$

which are G-equivariant and $(O(V) \times O(W))$-equivariant, satisfying the same associativity condition as before. A map of orthogonal G-spectra $f : X \longrightarrow Y$ is a collection of based $(O(V) \times G)$-maps $f_V : X_V \longrightarrow Y_V$ which commute with the structure maps.

Since $\mathcal{S}_{\mathbb{I}_G}$ is a commutative monoid object, the evident equivariant version of 3.1.15 gives

Theorem 4.1.5 *There is a product of orthogonal G-spectra $\wedge_{\mathbb{I}}$, called **smash product**, such that*

$$(G\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}_G})$$

is a closed symmetric monoidal category.

Example 4.1.6 (K-theory with Reality) We highlight now the main example of equivariant spectrum for us: the $\mathbb{Z}/2$-spectrum KR of (reduced) K-theory with Reality. Recall from 2.3.17 that reduced KR-theory is represented by $BU \times \mathbb{Z}$,

$$\widetilde{KR}(X) \cong {}_{\mathbb{Z}/2}[X, BU \times \mathbb{Z}]_*$$

where BU has the $\mathbb{Z}/2$-action induced by complex conjugation and $\mathbb{Z}$ has the trivial action. Moreover, we also have the equivariant version of Bott periodicity (theorem 2.3.18), stating that there is a $\mathbb{Z}/2$-equivariant weak homotopy equivalence

$$\Omega^{1,1}(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}.$$

We will make use of this to define a $\mathbb{Z}/2$-spectrum KR: to begin with, observe that the only real (orthogonal) irreducible representations of $\mathbb{Z}/2$ are $\mathbb{R}$ with the trivial action and $i\mathbb{R}$ with complex conjugation (the latter is usually called the *sign representation*, and it appears in the literature as σ). Indeed, every real irreducible representation of a finite abelian group has dimension one, and being orthogonal these are the only possibilities. Therefore, every real representation of $\mathbb{Z}/2$ is isomorphic to one of the form

$$V^{p,q} = \mathbb{R}^p \oplus i\mathbb{R}^q.$$

Now observe that

$$\widetilde{KR}^{p,q}(X) = \widetilde{KR}(S^{n-p,n-q} \wedge X) \cong {}_{\mathbb{Z}/2}[S^{n-p,n-q} \wedge X, BU \times \mathbb{Z}]_* \cong {}_{\mathbb{Z}/2}[X, \Omega^{n-p,n-q}(BU \times \mathbb{Z})]_*$$

where $p, q \geq 0$ and $n \geq p, q$. Trying to mimic the complex K-theory spectrum, this motivates us to define the $\mathbb{Z}/2$-spectrum KR as

$$KR(\mathbb{R}^p \oplus i\mathbb{R}^q) := \Omega^{n-p,n-q}(BU \times \mathbb{Z})$$

for $n := \max(p, q)$. The structure maps are fully determined, by associativity, by the case $S^{1,1} \wedge KR(\mathbb{R}^p \oplus i\mathbb{R}^q) \longrightarrow KR(\mathbb{R}^{p+1} \oplus i\mathbb{R}^{q+1})$, which is the map whose adjoint is the weak equivalence

$$BU \times \mathbb{Z} \xrightarrow{\simeq} \Omega^{1,1}(BU \times \mathbb{Z})$$

given by Bott periodicity (theorem 2.3.18).

Of course, we also have the standard examples, the equivariant versions of the ones given in chapter 3. This time we will only discuss the ortogonal case, since for G-spectra they are obtained by neglect of structure.

Examples 4.1.7 (a) The object $\mathcal{S}_{\mathbb{I}_G}$, viewed as a $\mathcal{S}_{\mathbb{I}_G}$-module, is called the **sphere orthogonal G-spectrum**. It has the identifications $S^V \wedge S^W \xrightarrow{\cong} S^{V \oplus W}$ as structure maps. The space S^V has a $(O(V) \times G)$-action inherited from V, and the previous structure maps are G-equivariant and $(O(V) \times O(W))$-equivariant.

(b) Let T be a pointed G-space. The **suspension orthogonal G-spectrum** $\Sigma^\infty T$ of T is given by

$$(\Sigma^\infty T)_V := S^V \wedge T,$$

endowed with the standard $(O(V) \times G)$-action on S^V and the trivial action of $O(V)$ on T. It has the obvious structure maps $S^V \wedge S^W \wedge T \xrightarrow{\cong} S^{V \oplus W} \wedge T$.

In particular, $\mathcal{S}_{\mathbb{I}_G} = \Sigma^\infty S^0$, and with more generality, $\Sigma^V \mathcal{S}_{\mathbb{I}_G} \cong \Sigma^\infty S^V$, for a G-representation V.

(c) Let us describe the equivariant orthogonal version of the Eilenberg-MacLane spectrum HA. In 3.2.7.(c) we already saw how to obtain concrete models of Eilenberg-MacLane spaces endowed with actions of orthogonal groups. The question now is: how to endow these with an action of the group G? The answer turns out to be: we simply impose an (additive) G-action on A. Formally, this means that instead of taking an abelian group, we must take a $\mathbb{Z}[G]$-module A, where $\mathbb{Z}[G]$ denotes the **group ring**[1] of G over $\mathbb{Z}$. Then A can

[1]Given a ring R and a group G, the free R-module $R[G]$ has an additional structure of ring with multiplication given by $(\sum_i a_i g_i)(\sum_j b_j h_j) := \sum_{i,j}(a_i b_j)(g_i h_j)$, and it is called the **group ring** of G over R.

be viewed as an abelian group with an action of G which is additive, that is, in the sense that $(ng)a = n(ga)$.

This is enough to mimic the nonequivariant case: the reduced A-linearlization of a sphere representation $\widetilde{A}[S^V]$ inherits an $O(V)$-action, $\varphi \cdot (ax) = a(\varphi x)$, as in the nonequivariant case; but also a G-action, where the group acts diagonally on the reduced A-linearlization, that is, $g \cdot (ax) := (ga)(gx)$ (cf. [35, 2.13]). In particular, the underlying nonequivariant space of $\widetilde{A}[S^V]$ is a $K(A, \dim V)$. Setting

$$(HA)_V := \widetilde{A}[S^V]$$

gives the the **Eilenberg-MacLane orthogonal G-spectrum** HA, where the structure maps

$$\begin{aligned} S^V \wedge \widetilde{A}[S^W] &\longrightarrow \widetilde{A}[S^{V\oplus W}] \\ (v, \textstyle\sum a_i w_i) &\longmapsto \textstyle\sum a_i(v, w_i) \end{aligned}$$

are $(O(V) \times O(W))$-equivariant and G-equivariant. One can further show that this is an orthogonal Ω-G-spectrum.

Remark 4.1.8 There is a simplified notion of orthogonal G-spectra which produces an equivalent category, although this time the equivalence is not trivial. This is Schwede's preferred notion of orthogonal G-spectra, and it is fully explained in [35]. This is similar to the equivalence of categories $\mathsf{Sp}^{\mathbb{O}} \simeq \mathsf{Sp}^{\mathbb{I}}$, but less evident.

Consider the category $\mathbb{O}$ from chapter 3, whose objects are non-negative integers and it has arrows $\mathrm{Hom}_{\mathbb{O}}(n, n) := O(n)$ and no morphisms $n \longrightarrow m$ if $n \neq m$. This is symmetric monoidal and also enriched over $G\mathsf{Top}_*$, where the internal homs have the trivial G-action. If $\mathcal{S}_{\mathbb{O}}(n) := S^n$ with the trivial G-action, then the category of $\mathcal{S}_{\mathbb{O}}$-modules on $G\mathsf{Top}_*^{\mathbb{O}}$ is denoted as $G\mathsf{Sp}^{\mathbb{O}}$. Explicitly, an object X of $G\mathsf{Sp}^{\mathbb{O}}$ is a sequence of based $(O(n) \times G)$-spaces and based structure maps $S^n \wedge X_m \longrightarrow X_{n+m}$ which are $(O(n) \times O(m))$-equivariant and G-equivariant, where G acts trivially on S^n.

Then the claim is that there is a non-trivial equivalence of categories

$$G\mathsf{Sp}^{\mathbb{I}} \simeq G\mathsf{Sp}^{\mathbb{O}}.$$

An orthogonal G-spectrum $X \in G\mathsf{Sp}^{\mathbb{I}}$ maps to the object which takes the values $X_n := X(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the trivial G-representation. Its inverse is given by mapping an object $X = (X_n)$ to the orthogonal G-spectrum that for a G-orthogonal representation V takes the value

$$X(V) := \mathrm{Hom}_{\mathbb{I}_G}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n.$$

For a proof of the equivalence, see [35, 2.7].

Homotopy theory of G-spectra

Let us discuss briefly some highlights of the homotopy theory of G-spectra, focusing on the equivariant phenomena. The first question one may ask is how to define homotopy groups of G-spectra. In chapter 1 we already saw that we should consider homotopy groups indexed also by subgroups of G in order to keep track the action of the group. Let us face now the stable version of what we did there.

Observe that for a spectrum X, its n-th homotopy group is

$$\pi_k(X) = \operatorname*{colim}_n \pi_{n+k}(X_n) = \operatorname*{colim}_n [\Sigma^n S^k, X_n]_* = \operatorname*{colim}_n [S^k, \Omega^n X_n]_*$$

This motivates the following

Definition. Let X be a G-spectrum, let H be a subgroup of G and let $n \in \mathbb{Z}$. The ***H*-*n*-th homotopy group** of X is

$$\pi_n^H(X) := \operatorname*{colim}_V \pi_n^H(\Omega^V X_V) \qquad , \qquad n \geq 0,$$
$$\pi_{-n}^H(X) := \operatorname*{colim}_{V \supset \mathbb{R}^n} \pi_0^H(\Omega^{V-\mathbb{R}^n} X_V) \qquad , \qquad n > 0,$$

where $\mathbb{R}^n$ stands for the trivial representation of dimension n, and $V - \mathbb{R}^n$ stands for the orthogonal complement of $\mathbb{R}^n$ in V.

For an orthogonal G-spectrum X, we define $\pi_n^H(X) := \pi_n^H(\mathbb{U}X)$.

A map of G-spectra $f : X \longrightarrow Y$ induces compatible maps $\Omega^V X_V \longrightarrow \Omega^V Y_V$, so there is an induced map in colimits

$$f_* : \pi_n^H(X) \longrightarrow \pi_n^H(Y).$$

Definition. A map of G-spectra is a **π_*-isomorphism** if it induces isomorphisms in all homotopy groups.

A map $f : X \longrightarrow Y$ of orthogonal G-spectra is a π_*-isomorphism if $\mathbb{U}f$ is a π_*-isomorphism.

Most of the results that we presented in §3.3 can be reformulated for G-spectra (compare with 3.3.5):

Lemma 4.1.9 *Let $V \in \mathbb{J}_G$ and let X be a G-spectrum. Then the unit $\eta : X \longrightarrow \Omega^V \Sigma^V X$ and counit $\varepsilon : \Sigma^V \Omega^V X \longrightarrow X$ of the adjunction $\Sigma^V : G\mathsf{Sp}^{\mathbb{J}} \rightleftarrows G\mathsf{Sp}^{\mathbb{J}} : \Omega^V$ are π_*-isomorphisms.*

Proof. We only treat the case of the unit (see [32, 9.15] for the counit). Write $V^j := \oplus_{k=1}^j V$, for $j \in \mathbb{N} \cup \{\infty\}$. We can write the universe $\mathscr{U}$, up to isomorphism, as $\mathscr{U}' \oplus V^\infty$. If we choose an expanding sequence of G-representations U_i' with colimit $\mathscr{U}'$, then we can rewrite

$$\pi_n^H(X) = \operatorname*{colim}_W \pi_n^H(\Omega^W X_W) \cong \operatorname*{colim}_{i,j} \pi_n^H(\Omega^{U_i' \oplus V^j} X_{U_i' \oplus V^j}),$$
$$\pi_n^H(\Omega^V \Sigma^V X) = \operatorname*{colim}_W \pi_n^H(\Omega^W \Omega^V \Sigma^V X_W) \cong \operatorname*{colim}_{i,j} \pi_n^H(\Omega^{U_i' \oplus V^j} \Omega^V \Sigma^V X_{U_i' \oplus V^j}).$$

The unit $\eta : X \longrightarrow \Omega^V \Sigma^V X$ induces a map from the first colimit to the second; and the structure maps $\Sigma^V X_{U_i' \oplus V^j} \longrightarrow X_{U_i' \oplus V^{j+1}}$ induce a map from the second to the first. These maps are inverse of each other. For negative indexes the argument is similar. □

Theorem 4.1.10 *Let $f : X \longrightarrow Y$ be a π_*-isomorphism of G-spectra, let A be a G-CW-complex and B a finite G-CW-complex. Then*

1. *$f \wedge \mathrm{Id} : X \wedge A \longrightarrow Y \wedge A$ is a π_*-isomorphism.*
2. *$f_* : F(B, X) \longrightarrow F(B, Y)$ is a π_*-isomorphism.*

The proof is quite involved and we refer to [23, §III.3]. But it allows us to show (compare with 3.3.4)

Corollary 4.1.11 *Let $V \in \mathbb{J}_G$. A map $f : X \longrightarrow Y$ of G-spectra is a π_*-isomorphism if and only if $\Sigma^V f : \Sigma^V X \longrightarrow \Sigma^V Y$ is a π_*-isomorphism.*

Proof. The direct implication follows from 4.1.10.(1). For the converse, taking $A = S^V$ in 4.1.10.(2), we have that $\Omega^V \Sigma^V X \longrightarrow \Omega^V \Sigma^V Y$ is a π_*-isomorphism. Now the result follows from 4.1.9 by the naturality of the unit. □

Remark 4.1.12 It is possible to extend the $\mathbb{Z}$-graded homotopy groups of G-spectra to groups indexed by G-representations. Indeed, in nonequivariant spectra we have that $\pi_n(X) \cong \pi_0(\Omega^n X)$. If spheres in equivariant spectra are given by representation spheres, then we can define
$$\pi_V^H(X) := \pi_0^H(\Omega^V X).$$
Moreover, since Ω^V and Σ^V are π_*-inverses of each other by 4.1.9, we can go further and define also homotopy groups indexed by "negative" representations,
$$\pi_{-V}^H(X) := \pi_0^H(\Sigma^V X).$$

The equivariant stable homotopy category

To finish the analogy of G-spectra with the nonequivariant case, we will discuss how to induce the homotopy category of G-spectra. As in the last chapter, we will also use the general machinery of model categories. All of the statements and most of the proofs are identical in the equivariant case, so we will make this subsection shorter and refer to [23, III.4] for the extra arguments necessary to generalize our proofs of §3.4.

Definition. Let $f : X \longrightarrow Y$ be a map of G-spectra or orthogonal G-spectra.

1. We say that f is a **level trivial fibration** if each map $f_V : X_V \longrightarrow Y_V$ is a trivial fibration in $G\mathsf{Top}_*$, that is, if $f_V^H : X_V^H \longrightarrow Y_V^H$ is a weak homotopy equivalence and a Serre fibration for all $H \subset G$ and any G-representation V.

2. We say that f is a **q-cofibration** if it has the left lifting property with respect to level trivial fibrations.

3. We say that f is a **trivial q-cofibration** if it is a π_*-isomorphism and a q-cofibration.

4. We say that f is a **q-fibration** if it has the right lifting property with respect to trivial q-cofibrations.

5. We say that f is a **trivial q-fibration** if it is a π_*-isomorphism and a q-fibration.

With these definitions, we obtain a similar result as in nonequivariant spectra (see [23, III.4.2]):

Theorem 4.1.13 *There are model structures on $G\mathsf{Sp}^{\mathbb{J}}$ and $G\mathsf{Sp}^{\mathbb{I}}$, called the* ***stable model structures****, with respect to π_*-isomorphisms, q-fibrations and q-cofibrations.*

Example 4.1.14 Let T be a G-CW-complex. Then it is easy to see that the tensor and cotensor of $G\mathsf{Top}_*$ with orthogonal G-spectra gives an adjunction

$$\begin{tikzcd} G\mathsf{Sp}^{\mathbb{I}} \arrow[r, bend left, "-\wedge T"] \arrow[r, phantom, "\perp"] & G\mathsf{Sp}^{\mathbb{I}}, \arrow[l, bend left, "F(T,-)"] \end{tikzcd}$$

which is Quillen.

Moreover, by a similar argument to that of 3.2.10, the forgetful $\mathbb{U} : G\mathsf{Sp}^{\mathbb{I}} \longrightarrow G\mathsf{Sp}^{\mathbb{J}}$ admits a left-adjoint,

$$\begin{tikzcd} G\mathsf{Sp}^{\mathbb{J}} \arrow[rr, bend left, "\mathbb{P}"] \arrow[rr, phantom, "\perp"] & & G\mathsf{Sp}^{\mathbb{I}}, \arrow[ll, bend left, "\mathbb{U}"] \end{tikzcd}$$

which is given by a left Kan extension. This adjunction induces an equivalence of categories, with the stable model structures on them, just as in the nonequivariant case (cf. [23, III.4.16]).

Theorem 4.1.15 *The previous adjunction $\mathbb{P} : G\mathsf{Sp}^{\mathbb{J}} \rightleftarrows G\mathsf{Sp}^{\mathbb{I}} : \mathbb{U}$ is a Quillen equivalence. Therefore, it induces an adjoint equivalence of categories*

$$\mathsf{Ho}(G\mathsf{Sp}^{\mathbb{J}}) \simeq \mathsf{Ho}(G\mathsf{Sp}^{\mathbb{I}}).$$

Definition. The **equivariant stable homotopy category** is the homotopy category of G-spectra (or equivalently, orthogonal G-spectra), and it is denoted by $G\mathsf{SHC}$.

This category has exactly the same properties as the nonequivariant stable homotopy category SHC. Substituting Top by $G\mathsf{Top}$, and suspension Σ by Σ^V for a finite-dimensional G-representation V, we obtain a commutative diagram

$$\begin{tikzcd} \mathsf{GTop} \arrow[r, "(-)_+"] \arrow[d] & \mathsf{GTop}_* \arrow[r, "\Sigma^\infty \circ \mathrm{cw}"] \arrow[d] & G\mathsf{Sp}^{\mathbb{I}} \arrow[d] & \mathsf{Ab} \arrow[l, "H"'] \arrow[d, "\text{0-th deg}"] \\ \mathsf{Ho}(G\mathsf{Top}) \arrow[r, "(-)_+"] & \mathsf{Ho}(G\mathsf{Top}_*) \arrow[r, "\mathbb{L}\Sigma^\infty"] & G\mathsf{SHC} \arrow[r, "\pi_*"] \arrow[loop below, "\mathbb{L}\Sigma^V"] & \mathsf{grAb} \end{tikzcd}$$

where $\mathbb{L}\Sigma^V : G\mathsf{SHC} \xrightarrow{\simeq} G\mathsf{SHC}$ is an equivalence of categories and cw is a G-CW-approximation functor. The homotopy category $G\mathsf{SHC}$ turns out to be triangulated, as it follows formally from A.5.3.(4). Moreover, since $(G\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}_G})$ is closed symmetric monoidal, applying 3.4.15 we get

Corollary 4.1.16 *The equivariant stable homotopy category inherits a closed symmetric monoidal structure*

$$(G\mathsf{SHC}, \wedge_{\mathbb{I}}^{\mathbb{L}}, \mathcal{S}_{\mathbb{I}_G}).$$

This allows us to define a concept that will be key for our further purpose of relating the $\mathbb{Z}/2$-spectrum KR with the (nonequivariant) spectrum KO.

Definition. A (commutative) **ring G-spectrum** is a (commutative) monoid object in the equivariant stable homotopy category $(G\mathsf{SHC}, \wedge_{\mathbb{I}}^{\mathbb{L}}, \mathcal{S}_{\mathbb{I}_G})$.

We reserve the name of **orthogonal ring G-spectrum** for monoids objects in $(G\mathsf{Sp}^{\mathbb{I}}, \wedge_{\mathbb{I}}, \mathcal{S}_{\mathbb{I}_G})$, although they also appear in the literature as *strict* ring G-spectra. Of course, every orthogonal ring G-spectrum gives rise to a orthogonal G-spectrum (but the converse is not true). For instance, the orthogonal sphere G-spectrum $\mathcal{S}_{\mathbb{I}_G}$ is a (orthogonal) ring G-spectrum, and if A is a $\mathbb{Z}[G]$-algebra (so A is itself a ring), then the orthogonal Eilenberg-MacLane G-spectrum is also an (orthogonal) ring G-spectrum.

It is also worth mentioning that, with similar arguments to 3.4.13 one shows

Proposition 4.1.17 *Let X be a G-spectrum, let T, T' be finite G-CW-complexes and let $H \subset G$. Then*

1. $\mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^\infty T, X) \cong \mathrm{colim}_{V\ G}[\Sigma^V T, X_V]_*$
2. $\mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^\infty T, \Sigma^\infty T') \cong \mathrm{colim}_{V\ G}[\Sigma^V T, \Sigma^V T']_*$
3. $\pi_k^G(X) \cong \mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^k \mathcal{S}_{\mathbb{I}_G}, X) \cong \mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^\infty S^k, X)$.
4. $\pi_V^G(X) \cong \mathrm{Hom}_{\mathsf{SHC}}(\Sigma^V \mathcal{S}_{\mathbb{I}_G}, X) \cong \mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^\infty S^V, X)$.
5. $\pi_k^H(X) \cong \mathrm{Hom}_{G\mathsf{SHC}}(\Sigma^\infty \Sigma^k (G/H)_+, X)$*, and similar for any G-representation V.*

4.2 $RO(G)$-graded cohomology theories

We now continue the question that we deferred at the end of §2.3. What should be the preferred cohomology theories for G-spaces? If we understand G-spheres as the representation spheres S^V, then the $\mathbb{Z}$-graduation of a usual cohomology theory does not encode the equivariant phenomena, since the suspension isomorphism $\widetilde{h}^n(X) \cong \widetilde{h}^{n+k}(S^k \wedge X)$ only uses trivial representations. Therefore, using representation spheres forces us to grade on G-representations. Moreover, one can be a bit more flexible and allow formal differences of representations. In the literature, this has the technical misnomer[2] of $RO(G)$-graded cohomology theories.

This section is based in [3, §3.3] and [25, ch. XIII]. Example 4.2.4 lifting KR-theory to a $RO(\mathbb{Z}/2)$-graded cohomology theory is original work of the author.

Let $(\mathrm{Repr}_{\mathbb{R}}(G), \otimes, 0, \otimes, \mathbb{R})$ be the semiring of real, finite-dimensional and orthogonal representations of a finite group G, with the direct sum and tensor product of representations.

Definition. The real **representation ring** of G is the Grothendieck construction of $\mathrm{Repr}_{\mathbb{R}}(G)$,

$$RO(G) := K(\mathrm{Repr}_{\mathbb{R}}(G)).$$

Remark 4.2.1 Roughly speaking, a $RO(G)$-graded cohomology theory should be a collection of functors $E^\alpha : \mathsf{Ho}(G\mathsf{Top}_*) \longrightarrow \mathsf{Ab}$ indexed by $\alpha \in RO(G)$ with a suspension isomorphism of the form $E^\alpha(X) \cong E^{\alpha+V}(\Sigma^V X)$ for any G-representation V, such that for fixed $\alpha \in RO(G)$, the functor E^α satisfies the rest of axioms of a cohomology theory. However, defining this in terms of isomorphism classes of representations is too imprecise.

Let us fix a universe $\mathscr{U}$ and consider the category $\mathbb{I}_G$ subject to the chosen universe. We call a pair of maps $f, g : V \longrightarrow W$ on $\mathbb{I}_G$ **homotopic** if they induce the same map $\Sigma^\infty S^V \longrightarrow \Sigma^\infty S^W$ in $G\mathsf{SHC}$. We write $\mathsf{Ho}(\mathbb{I}_G)$ for the naive homotopy category, with same objects as $\mathbb{I}_G$ and morphisms homotopy classes of maps.

For every G-representation W, define the functor

$$\begin{aligned} \Sigma^W : \mathsf{Ho}(\mathbb{I}_G) \times \mathsf{Ho}(G\mathsf{Top}_*) &\longrightarrow \mathsf{Ho}(\mathbb{I}_G) \times \mathsf{Ho}(G\mathsf{Top}_*) \\ (V, X) &\longmapsto (V \oplus W, \Sigma^W X). \end{aligned}$$

[2] In words of Peter May [25], *there are several ways to be precise about $RO(G)$-graded cohomology theories, and there are several ways to be imprecise. The latter are better presented in the literature than the former.*

Definition. A $RO(G,\mathscr{U})$**-graded cohomology theory** is a functor

$$E^* : \mathsf{Ho}(\mathbb{I}_G) \times \mathsf{Ho}(G\mathsf{Top}_*)^{op} \longrightarrow \mathsf{Ab},$$

where we will usually write $E^*(V,X) = E^V(X)$, together with a collection of natural isomorphisms $\sigma^W : E^* \overset{\cong}{\Longrightarrow} E^* \circ \Sigma^W$, written

$$E^V(X) \xrightarrow{\cong} E^{V\oplus W}(\Sigma^W X)$$

for every G-representation W, satisfying the following axioms:

(i) Given a map $f : X \longrightarrow Y$ of based G-spaces, the sequence

$$E^V(Cf) \longrightarrow E^V(Y) \longrightarrow E^V(X)$$

is exact for every $V \in \mathbb{I}_G$.

(ii) Given a collection of based G-spaces $(X_i)_{i\in I}$, the canonical map

$$E^V(\bigvee_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} E^V(X_i)$$

is an isomorphism for every $V \in \mathbb{I}_G$.

(iii) We have $\sigma^0 = \mathrm{Id}$ and the family $\{\sigma^W\}$ is transitive, $\sigma^{W'} \circ \sigma^W = \sigma^{W\oplus W'}$.

(iv) Given a map $\alpha : W \longrightarrow W'$ in $\mathbb{I}_G$, then the following diagram commutes:

$$\begin{tikzcd}
E^V(X) \arrow[r, "\sigma^W"] \arrow[d, "\sigma^{W'}"'] & E^{V\oplus W}(\Sigma^W X) \arrow[d, "E^{\mathrm{Id}\oplus\alpha}(\mathrm{Id})"] \\
E^{V\oplus W'}(\Sigma^{W'} X) \arrow[r, "(\bar{\alpha}\wedge \mathrm{Id})^*"] & E^{V\oplus W'}(\Sigma^W X)
\end{tikzcd}$$

for any V, where $\bar{\alpha} : S^W \longrightarrow S^{W'}$ is the induced map on the one-point compactifications.

For a complete universe $\mathscr{U}$, we will usually drop it from the notation, so we will write $RO(G,\mathscr{U}) = RO(G)$.

Remark 4.2.2 We now explain the reason of the name of $RO(G,\mathscr{U})$-graded cohomology theory. If E^* is such a theory, we can extend it to **formal differences** $V \ominus W$: given homotopy classes of G-representations V, W, set

$$E^{V\ominus W}(X) := E^V(\Sigma^W X).$$

Now the observation is that this definition is compatible with the usual equivalence relation of formal differences (which identifies $V \ominus W$ with $V' \ominus W'$ if there is a G-linear isometric isomorphism $\alpha : V \oplus W' \xrightarrow{\cong} V' \oplus W$), in the sense that such an α determines an explicit isomorphism

$$E^{V\ominus W}(X) \xrightarrow{\cong} E^{V'\ominus W'}(X)$$

defined as the left vertical arrow in the following diagram of isomorphisms,

$$\begin{tikzcd}
E^V(\Sigma^W X) \arrow[r, "\sigma^{W'}"] \arrow[d] & E^{V\oplus W'}(\Sigma^{W\oplus W'} X) \arrow[d, "E^\alpha(\Sigma^{\mathrm{tw}}\mathrm{Id})"] \\
E^{V'}(\Sigma^{W'} X) \arrow[r, "\sigma^W"] & E^{V'\oplus W}(\Sigma^{W'\oplus W} X)
\end{tikzcd}$$

where $\mathrm{tw} : W \oplus W' \longrightarrow W' \oplus W$ is the twist isomorphism.

Remark 4.2.3 Given a $RO(G,\mathscr{U})$-graded cohomology theory E^*, we can obtain a $\mathbb{Z}$-graded cohomology theory by restricting to the trivial representations, setting $E^n := E^{\mathbb{R}^n}$, where now the suspension isomorphism is determined by $\sigma^{\mathbb{R}}$.

Example 4.2.4 Let us explain now how to make (reduced) KR-theory into a $RO(\mathbb{Z}/2)$-graded cohomology theory. Recall that any $\mathbb{Z}/2$-equivariant representation of $\mathbb{Z}/2$ is, up to isomorphism, of the form $V^{p,q} = \mathbb{R}^p \oplus i\mathbb{R}^q$. Then we define

$$\begin{aligned} \widetilde{KR} : \mathsf{Ho}(\mathbb{I}_{\mathbb{Z}/2}) \times \mathsf{Ho}((\mathbb{Z}/2)\mathsf{Top}_*)^{op} &\longrightarrow \mathsf{Ab} \\ (V^{p,q}, X) &\longmapsto \widetilde{KR}^{p,q}(X). \end{aligned}$$

This assignment is clearly functorial on X, but some words must be said with respect to p,q. In the first place, note that any $\mathbb{Z}/2$-equivariant isometric isomorphism $\alpha : V^{p,q} \longrightarrow V^{p',q'}$ must satisfy $p = p', q = q'$. Therefore, given an isometric isomorphism $\alpha : V^{p,q} \longrightarrow V^{p,q}$, the induced map $\alpha_* : \widetilde{KR}^{p,q}(X) \longrightarrow \widetilde{KR}^{p,q}(X)$ is defined by the composite displayed in the below diagram, where $n > 2p, 2q$:

$$\begin{tikzcd} \widetilde{KR}^{p,q}(X) \arrow[r, "\alpha_*"] \arrow[d, "\cong"] & \widetilde{KR}^{p,q}(X) \arrow[d, "\cong"] \\ \widetilde{KR}(S^{n-2p,n-2q} \wedge S^{p,q} \wedge X) & \widetilde{KR}(S^{n-2p,n-2q} \wedge S^{p,q} \wedge X) \arrow[l, "(\mathrm{Id}\wedge\bar{\alpha}\wedge\mathrm{Id})^*"'] \end{tikzcd} \tag{4.1}$$

The suspension isomorphism is given by the composite of isomorphisms

$$\widetilde{KR}^{p+p',q+q'}(S^{p',q'} \wedge X) \cong \widetilde{KR}(S^{n-p-p',n-q-q'} \wedge S^{p',q'} \wedge X) \cong \widetilde{KR}(S^{n-p,n-q} \wedge X) \cong \widetilde{KR}^{p,q}(X).$$

The properties (*i*) and (*ii*) follow from the theory developed in chapter 2 for KR-theory, (*iii*) follows by the definition of the suspension isomorphism and the commutativity of the diagram

$$\begin{tikzcd} \widetilde{KR}^{r,s}(X) \arrow[rr, "\sigma^{p,q}"] \arrow[dr, "\sigma^{p+p',q+q'}"'] & & \widetilde{KR}^{r+p,s+q}(\Sigma^{p,q}X) \arrow[dl, "\sigma^{p',q'}"] \\ & \widetilde{KR}^{r+p+p',s+q+q'}(\Sigma^{p+p',q+q'}X) & \end{tikzcd}$$

and for (*iv*) it sufficies to observe that for every $\mathbb{Z}/2$-equivariant isometric isomorphism $\alpha : V^{p,q} \longrightarrow V^{p,q}$, the map

$$(\mathrm{Id} \wedge \alpha)_* : \widetilde{KR}^{r+p,s+q}(\Sigma^{p,q}X) \longrightarrow \widetilde{KR}^{r+p,s+q}(\Sigma^{p,q}X)$$

is exactly the inverse of the map

$$(\bar{\alpha} \wedge \mathrm{Id})^* : \widetilde{KR}^{r+p,s+q}(\Sigma^{p,q}X) \longrightarrow \widetilde{KR}^{r+p,s+q}(\Sigma^{p,q}X)$$

under the isomorphisms displayed in the diagram (4.1).

In the equivariant case, G-spectra have a relation with $RO(G,\mathscr{U})$-graded cohomology theories similar to the one that, in the nonequivariant case, spectra have with $\mathbb{Z}$-graded cohomology theories.

Proposition 4.2.5 *Any Ω-G-spectrum E gives rise to a $RO(G)$-graded cohomology theory on based G-spaces, defined by*

$$E^V(X) := {}_G[X, E_V]_*$$

See [25, XIII.2.2]. The surprising result is that, just as in the nonequivariant case, any $RO(G)$-graded cohomology theory arises in this way:

Theorem 4.2.6 (Brown representability) *The functor*

$$\{\ \Omega\text{-}G\text{-spectra}\ \} \longrightarrow \left\{ \begin{array}{c} RO(G)\text{-graded} \\ \text{cohomology theories on} \\ \text{pointed } G\text{-CW-complexes} \end{array} \right\}$$

is essentially surjective.

The proof of this can be found in [25, XIII.3.2].

Remark 4.2.7 (Mackey functors) One sensible question at this point is whether any $\mathbb{Z}$-graded cohomology theory on based G-spaces extends to a $RO(G)$-graded cohomology theory. One particular example of $\mathbb{Z}$-graded cohomology we studied in chapter 1 was Bredon cohomology $H^n_G(-; M)$, where M is a coefficient system, that is, a functor $\mathcal{O}_G^{op} \longrightarrow \mathsf{Ab}$. We can easily define the "stable" analogue of this functor. In the first place, define the **Burnside category** of G as the full subcategory $\mathscr{B}_G$ of $G\mathsf{SHC}$ on the objects $\Sigma^\infty G/H_+$. Then a **Mackey functor** is a functor

$$\mathscr{B}_G^{op} \longrightarrow \mathsf{Ab}.$$

There is an obvious functor $\mathcal{O}_G \longrightarrow \mathscr{B}_G$, so every Mackey functor gives rise to a coefficient system. The converse is not always true, and more concretely, one can show that Bredon cohomology with coefficients in M extends to a $RO(G)$-graded cohomology theory if and only if M extends to a Mackey functor [25, IX.5.2].

4.3 The homotopy fixed points of the KR-spectrum

In this last section, we aim to develop some extra tools to present one remarkable result, which relates the $\mathbb{Z}/2$-spectrum KR of K-theory with Reality with the nonequivariant spectrum KO of real K-theory.

Fixed points of G-spectra

In chapter 1, given a (based) G-space, we defined its fixed points $X^G := \lim X$ and its homotopy fixed points $X^{hG} := \text{holim } X$. One may ask if we can mimic this and also have fixed points functors from (orthogonal) G-spectra to (orthogonal) spectra. For this part we mostly look at [23, §V.3–4] and [3, §3.5].

We showed in 1.1.5 and 1.1.6 that (homotopy) fixed points can be expressed as

$$X^H \cong \mathrm{Map}_G(G/H_+, X) \qquad , \qquad X^{hG} \cong \mathrm{Map}_G(EG_+, X),$$

where $H \subseteq G$ is a subgroup. Since the category of G-spectra is cotensored over G-spaces, this motivates the following definitions:

Definition. The **categorical fixed points** is the functor

$$(-)^G : G\mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{Sp}^{\mathbb{I}}$$

given by taking G-fixed points termwise,

$$(X^G)_V := (X_V)^G$$

(we can evaluate in any real inner-product space V viewed as a trivial G-representation, since the universe $\mathscr{U}$ contains all trivial representations). The structure maps are given by the fixed point maps of the structure maps of X,

$$S^V \wedge (X_W)^G \longrightarrow (X_{V\oplus W})^G.$$

For a subgroup $H \subseteq G$, we define

$$X^H := F(G/H_+, X)^G,$$

where we use that the category of G-spectra is cotensored over G-spaces.

The following important result is immediate from the definitions:

Proposition 4.3.1 *Let X be an orthogonal Ω-G-spectrum. Then*

$$\pi_n^H(X) \cong \pi_n(X^H)$$

for all $n \in \mathbb{Z}$.

Recall that, given a based space, we can always view it as a based G-space with the trivial action, and it is left adjoint to taking fixed points (see 1.1.3). We would like to do the same for spectra, but we have to be a little bit careful.

For the choice of a trivial universe $\mathscr{U}$, write $G\mathsf{Sp}^{\mathbb{I}}_{\text{naive}}$ for the category of naive orthogonal G-spectra. There is a functor

$$\text{triv} : \mathsf{Sp}^{\mathbb{I}} \longrightarrow G\mathsf{Sp}^{\mathbb{I}}_{\text{naive}}$$

which maps every spectrum X to a G-spectrum with the property that $\text{triv}(X)_{\mathbb{R}^n} = X_{\mathbb{R}^n}$ viewed with the trivial G-action. For any other trivial G-representation V, it takes the value

$$\text{triv}(X)_V = \text{Hom}_{\mathbb{I}_G}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_{\mathbb{R}^n}.$$

Then the categorical fixed points functor plays the role of the fixed points for G-spaces:

Lemma 4.3.2 *Fix a trivial universe $\mathscr{U}$. There is a (Quillen) adjunction*

$$\begin{tikzcd} G\mathsf{Sp}^{\mathbb{I}}_{\text{naive}} \arrow[r, phantom, "\perp"] \arrow[r, bend left, leftarrow, "\text{triv}"] & \mathsf{Sp}^{\mathbb{I}} \arrow[l, bend left, "(-)^G"] \end{tikzcd}$$

Furthermore, for any choice of universe $\mathscr{U}$, $(-)^G : G\mathsf{Sp}^{\mathbb{I}} \longrightarrow \mathsf{Sp}^{\mathbb{I}}$ is also right Quillen, so we have a right derived functor

$$\mathbb{R}(-)^G \overset{\text{notation}}{=} (-)^G : G\mathsf{SHC} \longrightarrow \mathsf{SHC}.$$

One can find this in [23, V.3.4]. We also state some properties of the categorical fixed points, which can be found in the [23, V.3.7 and V.3.8]:

Proposition 4.3.3 *The categorical fixed points functor satisfies the following properties:*

1. $(\Sigma^\infty A)^G \cong \Sigma^\infty(A^G)$*, for a based G-space A.*
2. $X^G \wedge_{\mathbb{I}} Y^G \cong (X \wedge_{\mathbb{I}} Y)^G$*, for* Ω*-G-spectra* X, Y*.*

We will also make use of two more fixed points functors. By convenience, we will treat them directly at the level of the homotopy categories. Recall that for a finite group G, the G-space EG can be realized as a G-CW-complex, so by 4.1.14 the functors $- \wedge EG_+$ and $F(EG_+, -)$ are left and right Quillen, respectively. We again abuse of notation and we will write $- \wedge EG_+$ for $\mathbb{L}(- \wedge EG_+)$ and $F(EG_+, -)$ for $\mathbb{R}F(EG_+, -)$, so we get

$$\begin{aligned} - \wedge EG_+ &: G\mathsf{SHC} \longrightarrow G\mathsf{SHC} \\ F(EG_+, -) &: G\mathsf{SHC} \longrightarrow G\mathsf{SHC} \end{aligned}$$

Definition. The **homotopy fixed points** functor $(-)^{hG}$ is the composite

$$G\mathsf{SHC} \xrightarrow{F(EG_+,-)} G\mathsf{SHC} \xrightarrow{(-)^G} \mathsf{SHC},$$

so we will write $X^{hG} := F(EG_+, X)^G$ for a G-spectrum $X \in G\mathsf{SHC}$.

The main theorem

We will spend the rest of the chapter showing the main result of this Master's thesis, which was its goal. Here we relate the equivariant spectrum KR with KO.

Theorem 4.3.4 *In the stable homotopy category* SHC*, the homotopy fixed points of the* $\mathbb{Z}/2$*-spectrum KR is isomorphic to the spectrum KO,*

$$KR^{h\mathbb{Z}/2} \cong KO.$$

To the author's knowledge, the proof we present of 4.3.4 has not appeared before. The strategy of the proof was outlined to the author by Lennart Meier. We will split the proof of the theorem in several steps. To start with, recall that the categories that we are dealing with, namely Top_*, $G\mathsf{Top}_*$, $\mathsf{Sp}^{\mathbb{I}}$ and $G\mathsf{Sp}^{\mathbb{I}}$, are pointed model categories (meaning that they are model categories with the property that the unique map between the initial and terminal object is an isomorphism). In particular, there are fibre and cofibre sequences, as discussed at the end of §A.5.

In the first place, observe that there is a pair of Quillen adjunctions (see 3.4.10)

$$\begin{tikzcd} G\mathsf{Top}_* \arrow[r, bend left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & G\mathsf{Sp}^{\mathbb{I}} \arrow[l, bend left, "\Omega^\infty"] \arrow[r, bend left, "X\wedge_{\mathbb{I}}-"] \arrow[r, phantom, "\perp"] & G\mathsf{Sp}^{\mathbb{I}} \arrow[l, bend left, "F(X,-)"] \end{tikzcd}$$

whose composite is therefore a Quillen adjunction. The left adjoint of this new adjunction is precisely the functor $X \wedge - : G\mathsf{Top}_* \longrightarrow G\mathsf{Sp}^{\mathbb{I}}$ obtained by the tensor of $G\mathsf{Sp}^{\mathbb{I}}$ over $G\mathsf{Top}_*$. As usual, we will also denote $\mathbb{L}(X \wedge -)$ by $X \wedge -$.

Proposition 4.3.5 *Let X be a G-spectrum.*

1. *The functor $X \wedge - : \mathsf{Ho}(G\mathsf{Top}_*) \longrightarrow G\mathsf{SHC}$ preserves cofibre sequences.*
2. *The categorical fixed point functor $(-)^G : G\mathsf{SHC} \longrightarrow \mathsf{SHC}$ preserves cofibre sequences.*

Proof. The functor of 1. is precisely the left derived functor of a left Quillen, so the result follows formally from A.5.3.(1). For 2., one has that the functor from the statement is the right derived functor of a right Quillen, so by A.5.3.(1) it preserves fibre sequences. However, by A.5.3.(5), fibre sequences and cofibre sequences are the same, so the result follows. □

Now, fixed a finite group G, consider the universal space EG, and consider the collapse map $EG_+ \longrightarrow S^0$ in $\mathsf{Ho}(G\mathsf{Top}_*)$. By A.5.3.(1), this map can be extended to a cofibre sequence in $\mathsf{Ho}(G\mathsf{Top}_*)$

$$EG_+ \longrightarrow S^0 \longrightarrow \widetilde{EG}. \tag{4.2}$$

Given X a G-spectrum, let ε be the composite

$$X \xrightarrow{\cong} F(S^0, X) \longrightarrow F(EG_+, X),$$

where the first map is the canonical identification and the second map is induced by the collapse map $EG_+ \longrightarrow S^0$. If we smash the G-spectra X and $F(EG_+, X)$ with (4.2), we get a commutative diagram in $G\mathsf{SHC}$

$$\begin{tikzcd}
X \wedge EG_+ \arrow[r] \arrow[d, "\varepsilon\wedge \mathrm{Id}"] & X \arrow[r] \arrow[d, "\varepsilon"] & X \wedge \widetilde{EG} \arrow[d, "\varepsilon\wedge \mathrm{Id}"] \\
F(EG_+, X) \wedge EG_+ \arrow[r] & F(EG_+, X) \arrow[r] & F(EG_+, X) \wedge \widetilde{EG}
\end{tikzcd} \tag{4.3}$$

where both rows are cofibre sequences by 4.3.5.(1).

Proposition 4.3.6 *Let $\mathbb{U} : G\mathsf{SHC} \longrightarrow \mathsf{SHC}$ be the forgetful functor and let $f : X \longrightarrow Y$ be a map of G-spectra such that $\mathbb{U}f : \mathbb{U}X \longrightarrow \mathbb{U}Y$ is an isomorphism of nonequivariant spectra. Then*

$$f \wedge \mathrm{Id} : X \wedge EG_+ \longrightarrow Y \wedge EG_+$$

is an isomorphism of G-spectra.

The previous result can be found in [12, I.1.1]. For us this has the following important consequence:

Corollary 4.3.7 *The map $\varepsilon \wedge \mathrm{Id} : X \wedge EG_+ \longrightarrow F(EG_+, X) \wedge EG_+$ is an isomorphism.*

Now we want to take categorical fixed points to the diagram (4.3). We will introduce some terminology first.

Definition. Let X be a G-spectrum. The **homotopy orbits** of X is the spectrum

$$X_{hG} := (X \wedge EG_+)^G.$$

Since the left vertical map is an isomorphism by 4.3.7, we will abuse of notation and we will also denote $(F(EG_+, X) \wedge EG_+)^G$ by X_{hG}. Therefore, after taking fixed points in (4.3), the lower left horizontal map becomes $N : X_{hG} \longrightarrow X^{hG}$, which reminds to the norm map that defines Tate cohomology. Together with 4.3.5.(2), this motivates the following

Definition. Let X be a G-spectrum. The **Tate construction** of X is the cofibre of the map $N : X_{hG} \longrightarrow X^{hG}$,

$$X^{tG} := (F(EG_+, X) \wedge \widetilde{EG})^G.$$

We need one last ingredient, for which we specialize to the case $G = \mathbb{Z}/2$:

Definition. The **geometric fixed points** functor $(-)^{\Phi\mathbb{Z}/2}$ is the composite

$$G\mathsf{SHC} \xrightarrow{-\wedge\widetilde{E\mathbb{Z}/2}} G\mathsf{SHC} \xrightarrow{(-)^{\mathbb{Z}/2}} \mathsf{SHC}$$

Note that by 4.1.10.(1), the functor $-\wedge\widetilde{E\mathbb{Z}/2} : G\mathsf{Sp}^{\mathbb{I}} \longrightarrow G\mathsf{Sp}^{\mathbb{I}}$ preserves π_*-isomorphisms, so its left derived functor $G\mathsf{SHC} \longrightarrow G\mathsf{SHC}$ is just smashing with $\widetilde{E\mathbb{Z}/2}$. The geometric fixed points can be defined with way more generality, but for our purpose we will just restrict to this case. A very detailed exposition of this can be found in [35, §7].

Under these considerations, taking categorical fixed points to the diagram (4.3), we obtain

Theorem 4.3.8 *Let X be a $\mathbb{Z}/2$-spectrum. Then there is a commutative diagram in* SHC

$$\begin{array}{ccccc} X_{h\mathbb{Z}/2} & \longrightarrow & X^{\mathbb{Z}/2} & \longrightarrow & X^{\Phi\mathbb{Z}/2} \\ \| & & \downarrow & & \downarrow \\ X_{h\mathbb{Z}/2} & \xrightarrow{N} & X^{h\mathbb{Z}/2} & \longrightarrow & X^{t\mathbb{Z}/2} \end{array}$$

*called the **Tate diagram**, where rows are cofibre sequences.*

The existence of this Tate diagram appears in [4] and [6]. Until now, we constructed the Tate diagram for any $\mathbb{Z}/2$-spectrum. In order to show 4.3.4, we now specialize to $X = KR$, so we get a commutative diagram in SHC

$$\begin{array}{ccccc} KR_{h\mathbb{Z}/2} & \longrightarrow & KR^{\mathbb{Z}/2} & \longrightarrow & KR^{\Phi\mathbb{Z}/2} \\ \| & & \downarrow & & \downarrow \\ KR_{h\mathbb{Z}/2} & \longrightarrow & KR^{h\mathbb{Z}/2} & \longrightarrow & KR^{t\mathbb{Z}/2} \end{array} \tag{4.4}$$

where rows are cofibre sequences. This means that the rows of this diagram are distinguished triangles in the triangulated structure of SHC.

Proposition 4.3.9 *$KR^{\mathbb{Z}/2} \cong KO$ (in SHC).*

Proof. We simply compute that termwise the spaces are weak homotopy equivalent. For $n \geq 0$, $0 \leq j, p < 8$ and $p + j \equiv 0 \mod 8$ we get

$$\begin{aligned} KR^{\mathbb{Z}/2}(\mathbb{R}^{8n+j}) &= KR(\mathbb{R}^{8n+j})^{\mathbb{Z}/2} = \Omega^{0,8n+j}(BU\times\mathbb{Z})^{\mathbb{Z}/2} \overset{2.3.19}{\simeq} \Omega^{8n+8,8n+j}(BU\times\mathbb{Z})^{\mathbb{Z}/2} \\ &\overset{2.3.18}{\simeq} \Omega^{p,0}(BU\times\mathbb{Z})^{\mathbb{Z}/2} = G\mathrm{Map}(S^{p,0}, BU\times\mathbb{Z})^{\mathbb{Z}/2} \overset{1.1.2}{\cong} \mathrm{Map}_G(S^{p,0}, BU\times\mathbb{Z}) \\ &\overset{(1.5)}{\cong} \mathrm{Map}(S^p, BO\times\mathbb{Z}) = \Omega^p(BO\times\mathbb{Z}) = KO(\mathbb{R}^{8n+j}). \end{aligned}$$

Here we used that $BU(n)^{\mathbb{Z}/2} = BO(n)$, which can be checked using Grassmannians; and that taking fixed points commutes with filtered colimits of closed inclusions [21, 1.2], so $BU^{\mathbb{Z}/2} = BO$. Therefore, if we choose the weak equivalence $BO\times\mathbb{Z} \xrightarrow{\simeq} \Omega^8(BO\times\mathbb{Z})$ as the underlying nonequivariant weak equivalence of 2.3.19, it is readily verified that the structure maps coincide. □

The last step of the proof is to show that the right vertical map of (4.4) is an isomorphism.

Proposition 4.3.10 *KR is a ring $\mathbb{Z}/2$-spectrum (in $(\mathbb{Z}/2)\mathsf{SHC}$).*

The last proposition is a known result, but found really rarely in the literature (it appears stated in [10, 5.2] and [6, 2.2.4]). Roughly, the tensor product of Real vector bundles makes KR into a multiplicative $RO(\mathbb{Z}/2)$-graded cohomology theory, which induces a map $KR \wedge_{\mathbb{I}}^{\mathbb{L}} KR \longrightarrow KR$. However, with the tools that we have developed, we are still far from giving a precise argument, so we will just blackbox the proof.

The following general result of the Tate diagram is key for our argument:

Proposition 4.3.11 *If X is a ring $\mathbb{Z}/2$-spectrum, then so are $X^{\mathbb{Z}/2}$, $X^{h\mathbb{Z}/2}$, $X^{\Phi\mathbb{Z}/2}$ and $X^{t\mathbb{Z}/2}$. Moreover, the right-hand square of the Tate diagram*

$$\begin{tikzcd} X^{\mathbb{Z}/2} \arrow[r] \arrow[d] & X^{\Phi\mathbb{Z}/2} \arrow[d] \\ X^{h\mathbb{Z}/2} \arrow[r] & X^{t\mathbb{Z}/2} \end{tikzcd}$$

is a commutative diagram of ring $\mathbb{Z}/2$-spectra.

The proof can be found in [12, I.3.5]. For us, it has the consequence that $KR^{\Phi\mathbb{Z}/2} \longrightarrow KR^{t\mathbb{Z}/2}$ is a map of ring $\mathbb{Z}/2$-spectra.

In the following, we denote by 0 the zero object of the triangulated category SHC.

Lemma 4.3.12 *Let X be a ring spectrum. If there is a ring map $0 \longrightarrow X$, then $X = 0$.*

Proof. Let $\mathcal{S}_{\mathbb{I}}$ be the unit of SHC, as in 3.4.16. Given a map of ring spectra $0 \longrightarrow X$, the commutativity of the diagram

$$\begin{tikzcd} & \mathcal{S}_{\mathbb{I}} \arrow[dl, "0"'] \arrow[dr, "\eta"] & \\ 0 \arrow[rr] & & X \end{tikzcd}$$

implies that the unit map $\eta : \mathcal{S}_{\mathbb{I}} \longrightarrow X$ must be the zero map. But this means that the identity of X must be the zero map, by the unit condition of the multiplication map

$$\begin{tikzcd} X \arrow[r, "\cong"] \arrow[drr, "\mathrm{Id}"'] & \mathcal{S}_{\mathbb{I}} \wedge_{\mathbb{I}}^{\mathbb{L}} X \arrow[r, "\eta\wedge\mathrm{Id}"] & X \wedge_{\mathbb{I}}^{\mathbb{L}} X \arrow[d, "\mu"] \\ & & X \end{tikzcd}$$

so $X = 0$. $\square$

The last ingredient is the following key fact about the KR spectrum:

Proposition 4.3.13 *$KR^{\Phi\mathbb{Z}/2} \cong 0$ (in SHC).*

Proof. Let $a_\sigma : S^0 \hookrightarrow S^{0,1}$ be the $\mathbb{Z}/2$-equivariant inclusion from S^0 (with the trivial action) to $S^{0,1} = S^\sigma$. It is readily verified that this map is not $\mathbb{Z}/2$-nullhomotopic: any $\mathbb{Z}/2$-equivariant map $H : S^0 \wedge I_+ \cong I \longrightarrow S^{0,1}$ must be the constant map. We can smash this map with the sphere $\mathbb{Z}/2$-spectrum $\mathcal{S}_{\mathbb{I}_{\mathbb{Z}/2}} = \mathcal{S}$, which yields a map $a_\sigma : \mathcal{S} \wedge S^0 \cong \mathcal{S} \longrightarrow \mathcal{S} \wedge S^{0,1}$. By 4.1.17.(4) it corresponds with a nonzero element

$$a_\sigma \in \pi_{0,0}^{\mathbb{Z}/2}(\mathcal{S} \wedge S^{0,1}) = \pi_{0,-1}^{\mathbb{Z}/2}(\mathcal{S}),$$

where $\pi_{p,q}^{\mathbb{Z}/2} = \pi_{\mathbb{R}^p \oplus i\mathbb{R}^q}^{\mathbb{Z}/2}$ and $\pi_{-p,-q}^{\mathbb{Z}/2} = \pi_{-\mathbb{R}^p \oplus i\mathbb{R}^q}^{\mathbb{Z}/2}$ (see 4.1.12).

Now we make the following observation: since $\mathcal{S}$ is a $\mathbb{Z}/2$-ring spectrum, its homotopy groups $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(\mathcal{S}) := \bigoplus_{p,q} \pi_{p,q}^{\mathbb{Z}/2}(\mathcal{S})$ form a graded ring. Moreover, as any $\mathbb{Z}/2$-spectrum X is a $\mathcal{S}$-module, $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(X) := \bigoplus_{p,q} \pi_{p,q}^{\mathbb{Z}/2}(X)$ has a structure of graded $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(\mathcal{S})$-module, and if X is a ring $\mathbb{Z}/2$-spectrum, then $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(X)$ is a graded $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(\mathcal{S})$-algebra.

The first claim now is that

$$\pi_{p,q}^{\mathbb{Z}/2}(KR \wedge \widetilde{E\mathbb{Z}/2}) \cong \operatorname{colim}(\pi_{p,q}^{\mathbb{Z}/2}(KR) \xrightarrow{\cdot a_\sigma} \pi_{p,q-1}^{\mathbb{Z}/2}(KR) \xrightarrow{\cdot a_\sigma} \cdots). \tag{4.5}$$

Indeed, there is a explicit description of $\widetilde{E\mathbb{Z}/2}$, namely $S^{\infty i} := \operatorname{colim}_n S^{0,n}$. Since homotopy groups commute with filtered colimits, we compute

$$\pi_{p,q}^{\mathbb{Z}/2}(KR \wedge \widetilde{E\mathbb{Z}/2}) \cong \operatorname*{colim}_n \pi_{p,q}^{\mathbb{Z}/2}(KR \wedge S^{0,n}) = \operatorname*{colim}_n \pi_{p,q-n}^{\mathbb{Z}/2}(KR),$$

where the maps in the last colimit are precisely multiplication by a_σ, since they are induced by the inclusions $S^{0,n} \hookrightarrow S^{0,n+1}$, that can be seen as $a_\sigma \wedge \mathrm{Id} : S^0 \wedge S^{0,n} \longrightarrow S^{0,1} \wedge S^{0,n}$. This concludes the first claim.

Our second claim is that the colimit in (4.5) is zero precisely because the 3-fold composite of the map a_σ is the zero map, that is, multiplication by a_σ^3 is zero. For let $\bar{v}$ be the composite

$$\mathcal{S} \longrightarrow KR \xrightarrow{\cong} \Omega^{1,1}KR$$

of the unit map given by the $\mathbb{Z}/2$-ring spectrum structure of KR (see 4.3.10) and the obvious isomorphism given by Bott Periodicity (theorem 2.3.18). As before, this map corresponds with an element $\bar{v} \in \pi_{1,1}^{\mathbb{Z}/2}(KR)$. In particular, this element is invertible in the graded ring $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(KR)$, with inverse $\bar{v}^{-1} \in \pi_{-1,-1}^{\mathbb{Z}/2}(KR)$ given by the composite

$$\Omega^{1,1}\mathcal{S} \longrightarrow \Omega^{1,1}KR \xrightarrow{\cong} KR.$$

Multiplying a_σ with $\bar{v}$ we get an element

$$a_\sigma \bar{v} \in \pi_{1,0}^{\mathbb{Z}/2}(KR) \cong \pi_1(KR^{\mathbb{Z}/2}) \overset{4.3.9}{\cong} \pi_1(KO) \cong \widetilde{KO}(S^1) \cong \mathbb{Z}/2,$$

where $\mathbb{Z}/2$ is generated by η, which is represented by the Möbius bundle in $\widetilde{KO}(S^1)$. If η^3 denotes its 3-fold reduced external product, then $\eta^3 \in \widetilde{KO}(S^3)$, but this group is trivial (see [38, 11]), so $\eta^3 = 0$. The upshot is that, anycase, $(a_\sigma \bar{v})^3 = 0$, and since $\bar{v}$ is invertible, a_σ^3 must act as zero in $\pi_{\bullet,\bullet}^{\mathbb{Z}/2}(KR)$. This finishes the second claim.

Therefore, we see that the colimit (4.5) must be trivial, so in particular we get that

$$\pi_n(KR^{\Phi\mathbb{Z}/2}) \cong \pi_n((KR \wedge \widetilde{E\mathbb{Z}/2})^{\mathbb{Z}/2}) \cong \pi_{n,0}^{\mathbb{Z}/2}(KR \wedge \widetilde{E\mathbb{Z}/2}) \cong 0,$$

what implies that $KR^{\Phi\mathbb{Z}/2} \cong 0$ in SHC. □

We are finally ready to show the main theorem:

Proof of 4.3.4. By 4.3.12 and 4.3.13, we get that the map $KR^{\Phi\mathbb{Z}/2} \longrightarrow KR^{t\mathbb{Z}/2}$ is the zero map between the zero spectra, thus in particular isomorphism. Since $KR^{\mathbb{Z}/2} \cong KO$ by 4.3.9, applying the two-out-of-three property A.5.2 for triangulated categories to the Tate diagram (4.4) yields the result. □

Appendix A

Model categories

Here we will review one useful tool of Algebraic Topology, namely model categories. They were first introduced in 1967 by Quillen [29] as a generalization of a well-known situation in topological spaces: there are three distinguished classes of arrows: weak homotopy equivalences, Serre fibrations and (retracts of) inclusions of relative cell complexes, satisfying some relations between them, for example

- If two out of three arrows of a composite are weak equivalences, so is the third.
- A map is a Serre fibration if and only if it has the right lifting property with respect any (retract) of a inclusion of a cell complex which is also a weak equivalence.
- A map is a inclusion of a relative cell complex if and only if it has the left lifting property with respect to Serre fibrations which are also weak equivalences.

These are properties of spaces which are stated only in terms of their arrows. The purpose of the theory of model categories is to axiomatize these properties and to be able to extrapolate formal consequences to other categories.

For the elaboration of the appendix, we mostly follow Hovey's excellent expository piece *Model Categories* [16]. Besides, we also make use of [9] and [28] in §A.1 – §A.3 (for instance, our definition of the homotopy category is different than Hovey's). In §A.4 we also follow [7] and [37], and in §A.5 [11] and [34].

A.1 Model structures

Definition. Let $\mathcal{C}$ be a category, and let Map $\mathcal{C}$ be the category with objects arrows of $\mathcal{C}$ and morphisms commutative squares.

1. A map $f : A \longrightarrow B$ is a **retract** of a map $g : C \longrightarrow D$ if f is a retract of g in Map $\mathcal{C}$; in other words, if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{Id}"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{Id}"'] & D \arrow[r] & B \end{tikzcd}$$

2. A **functorial factorization** in $\mathcal{C}$ is a pair (α, β) of functors such that $\beta \circ \alpha = \mathrm{Id}_{\mathrm{Map}\,\mathcal{C}}$
3. Given maps $i : A \longrightarrow B$ and $p : X \longrightarrow Y$, we say that i has the **left lifting property** with respect to p, or that p has the **right lifting property** with respect to i, if for every solid commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a dashed arrow making the triangles commute.

Definition. A **model category** is a category $\mathcal{C}$ with three distinguished subcategories called

- weak equivalences,
- fibrations,
- cofibrations,

together with two functorial factorizations $(\alpha, \beta), (\gamma, \delta)$ satisfying the following properties (call a map an **acyclic** or **trivial (co)fibration** of it is a (co)fibration and a weak equivalence):

(i) (*2-out-of-3*) If two out of three of maps f, g, gf are weak equivalences, so is the third.

(ii) (*Retracts*) If f is a retract of g, and g is in one of the preferred subcategories, so is f.

(iii) (*Lifting*) Trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to trivial fibrations.

(iv) (*Factorization*) Any morphism f factors in two different ways: as a cofibration $\alpha(f)$ and a trivial fibration $\beta(f)$; or as a trivial cofibration $\gamma(f)$ and a fibration $\delta(f)$.

(v) (*(Co)limits*) The category $\mathcal{C}$ is complete and cocomplete.

Examples A.1.1 All examples here require long proofs, but we include them here for the record.

(a) The category Top of *all* topological spaces with the classical or *Quillen model structure*:
   - weak equivalences: weak homotopy equivalences
   - fibrations: Serre fibrations,
   - cofibrations: retracts of relative cell complexes inclusions $A \hookrightarrow X$.

(b) The category Top of *all* topological spaces with the *Strøm model structure*:
   - weak equivalences: homotopy equivalences,
   - fibrations: Hurewicz fibrations,
   - cofibrations: Hurewicz cofibrations.

(c) The category Ch_R of chain complexes over a ring R, with
   - weak equivalences: quasi-isomorphisms,
   - fibrations: maps levelwise surjective,

- cofibrations: maps levelwise injective with projective cokernel.

(d) The category sSet of simplicial sets with

- weak equivalences: maps whose geometric realization is a weak homotopy equivalence,
- fibrations: Kan fibrations,
- cofibrations: maps levelwise injective.

(e) The category cdga of commutative differential graded algebras with

- weak equivalences: quasi-isomorphisms,
- fibrations: surjective maps,
- cofibrations: retracts of relative Sullivan algebra inclusions.

(f) If $\mathcal{C}$ is a model category and $*$ is the terminal object, then $\mathcal{C}_* := */\mathcal{C}$ the category under $*$, called the **pointed category**, inherits a model structure.

As any model category is complete and cocomplete, it has an initial object $\emptyset$ and a terminal object $*$.

Definition. We say that an object $X \in \mathcal{C}$ is **cofibrant** if $\emptyset \longrightarrow X$ is a cofibration; and that X is **fibrant** if $X \longrightarrow *$ is a fibration.

Remark A.1.2 If $X \in \mathcal{C}$, applying the functorial factorization (α, β) to the unique map $\emptyset \longrightarrow X$ we get a factorization

$$\emptyset \longrightarrow QX \xrightarrow{q_X} X$$

for some cofibrant object QX. Similarly applying (γ, δ) to the unique map $X \longrightarrow *$ we get a factorization

$$X \xrightarrow{r_X} RX \longrightarrow *$$

for some fibrant object RX. This defines functors

$$Q, R : \mathcal{C} \longrightarrow \mathcal{C}$$

called the **cofibrant** and **fibrant replacement functors**; and natural transformations $q : Q \Rightarrow \mathrm{Id}$ and $r : \mathrm{Id} \Rightarrow R$.

Lemma A.1.3 (Retract Argument) *If $f = p \circ i$ is a factorization of a map f and it has the left lifting property with respect to p, then f is a retract of i.*

Similarly, if f has the right lifting property with respect to i, then f is a retract of p

In the definition of a model category we have pointed three distinguished classes of maps. However, it is redundant: the class of cofibrations is determined by the other two; and the same happens with the class of fibrations. This is the content of the following

Proposition A.1.4 *Let $\mathcal{C}$ be a model category.*

1. *A map is a cofibration (trivial cofibration) if and only if it has the left lifting property with respect all trivial fibrations (fibrations).*

2. *A map is a fibration (trivial fibration) if and only if it has the right lifting property with respect all trivial cofibrations (cofibrations).*

Corollary A.1.5 *Fibrations and trivial fibrations are stable under base change.*
Dually, cofibrations and trivial cofibrations are stable under cobase change

Lemma A.1.6 (Ken Brown) *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between model categories. If F takes trivial (co)fibrations between (co)fibrant objects to weak equivalences, then it preserves weak equivalences between (co)fibrant objects.*

A.2 The homotopy category

Given a model category $\mathcal{C}$, we want to make weak equivalences into isomorphisms. For that, we can formally invert those morphisms, in a similar way as in commutative algebra: given a ring R and a multiplicative system $S \subset R$, we can construct the *localization* $R[S^{-1}]$ (ring of fractions), where now the elements of S, under the *morphism of localization* $j : R \longrightarrow R[S^{-1}]$, are invertible. We can mimic this constructions for categories:

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subcategory of $\mathcal{C}$. The **localization** of $\mathcal{C}$ with respect to $\mathcal{W}$ (if it exists) is a category $\mathcal{C}[\mathcal{W}^{-1}]$ together with a functor $j : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$, called **functor of localization**, satisfying

(i) $j(w)$ is an isomorphism for all arrow w in $\mathcal{W}$.

(ii) If $\mathcal{D}$ is another category and $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor with the property that $F(w)$ is an isomorphism for all arrow w in $\mathcal{W}$, then there exists a unique functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ such that $\overline{F} \circ j = F$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "j"'] & \mathcal{D} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, dashed, "\overline{F}"'] & \end{tikzcd}$$

Theorem A.2.1 *The localization of a category with respect to any subcategory exists and it is unique up to isomorphism.*

Concretely, the localization $\mathcal{C}[\mathcal{W}^{-1}]$ has the same objects as $\mathcal{C}$ and morphisms equivalence classes of finite strings of composable arrows $(f_1, \ldots, f_r)$, where f_i is either an arrow of $\mathcal{C}$ or the reversal w_i^{-1} of an arrow w_i in $\mathcal{W}$, subject to the relations

$$\mathrm{Id}_C = (\mathrm{Id}_C) \quad , \quad (f, g) = (f \circ g) \quad , \quad (w, w^{-1}) = \mathrm{Id}_{\mathrm{dom}\, w} \quad , \quad (w^{-1}, w) = \mathrm{Id}_{\mathrm{codom}\, w}.$$

Definition. Let $\mathcal{C}$ be a model category and let $\mathcal{W}$ be the subcategory of weak equivalences. The **homotopy category** $\mathsf{Ho}\,\mathcal{C}$ of $\mathcal{C}$ is the localization of $\mathcal{C}$ with respect $\mathcal{W}$,

$$\mathsf{Ho}\,\mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}].$$

Let us denote by $\mathcal{C}_c, \mathcal{C}_f, \mathcal{C}_{cf}$ the full subcategories of cofibrant, fibrant and cofibrant fibrant objects of $\mathcal{C}$, respectively.

Proposition A.2.2 *The inclusions $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_c \hookrightarrow \mathcal{C}$ and $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}_f \hookrightarrow \mathcal{C}$ induce equivalences of categories*

$$\mathsf{Ho}\,\mathcal{C}_{cf} \xrightarrow{\simeq} \mathsf{Ho}\,\mathcal{C}_c \xrightarrow{\simeq} \mathsf{Ho}\,\mathcal{C} \qquad , \qquad \mathsf{Ho}\,\mathcal{C}_{cf} \xrightarrow{\simeq} \mathsf{Ho}\,\mathcal{C}_f \xrightarrow{\simeq} \mathsf{Ho}\,\mathcal{C}.$$

This definition of the homotopy category is formally very desirable but quite inexplicit. We aim now to give a more down-to-Earth description of this category (and in particular of its arrows).

Definition. Let $\mathcal{C}$ be a model category.

1. A **cylinder object** for $B \in \mathcal{C}$ is a factorization of the fold map $\mathrm{Id} \coprod \mathrm{Id} : B \coprod B \longrightarrow B$ as a cofibration $i_0 \coprod i_1 : B \coprod B \longrightarrow B'$ and a weak equivalence $s : B' \longrightarrow B$.

2. A **path object** for $X \in \mathcal{C}$ is a factorization of the diagonal map $(\mathrm{Id}, \mathrm{Id}) : X \longrightarrow X \times X$ as a weak equivalence $r : X \longrightarrow X'$ and a fibration $(p_0, p_1) : X' \longrightarrow X \times X$.

By the factorization axiom, there is always a cylinder object $B \times I$ for B with the extra property that $s : B \times I \longrightarrow B$ is a trivial fibration. Similarly, there is always a path object X^I for X with the propety that $X \longrightarrow X^I$ is a trivial cofibration.

Definition. Let $f, g : B \longrightarrow X$ be maps in $\mathcal{C}$.

1. A **left homotopy** from f to g is a map $H : B' \longrightarrow X$ from some cylinder object B' for B, such that $H \circ i_0 = f$, $H \circ i_1 = g$.

2. A **right homotopy** from f to g is a map $K : B \longrightarrow X'$ from some path object X' for X, such that $p_0 \circ K = f$, $p_1 \circ K = g$.

3. We say that f and g are **homotopic** when they are left and right homotopic.

4. We say that f is a **homotopy equivalence** when there is $h : X \longrightarrow B$ such that $f \circ h$ and $h \circ f$ are homotopic to the identity.

Proposition A.2.3 *Let $\mathcal{C}$ be a model category and let $f, g : B \longrightarrow X$ be maps in $\mathcal{C}$.*

1. *Left (right) homotopies are always preserved under composition by the left (right).*

2. *If X is fibrant (B cofibrant), then left (right) homotopies are also preserved under composition by the right (left).*

3. *If B is cofibrant (X fibrant), then left (right) homotopy is an equivalence relation in $\mathrm{Hom}_{\mathcal{C}}(B, X)$. We write $[B, X]^\ell$ and $[B, X]^r$ for the sets of left (right) homotopy classes of maps $B \longrightarrow X$.*

4. *If B is cofibrant and $h : X \longrightarrow Y$ is a trivial fibration or a weak equivalence of fibrant objects, then h induces a bijection*
$$h_* : [B, X]^\ell \xrightarrow{\cong} [B, Y]^\ell.$$

Corollary A.2.4 *If B is cofibrant and X is fibrant, then the left and right homotopy relations coincide and they are equivalence relations on $\mathrm{Hom}_{\mathcal{C}}(B, X)$.*

We write $[B, X]$ for the set of homotopy classes of maps $B \longrightarrow X$.

Corollary A.2.5 *In $\mathcal{C}_{cf}$, the homotopy relation on morphisms is an equivalence relation compatible with composition, thus the quotient category $\mathcal{C}_{cf}/\sim$ exists.*

The natural functor $\mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$ sends homotopy equivalences to isomorphisms, but we care about weak equivalences! We are lucky since

Proposition A.2.6 *In $\mathcal{C}_{cf}$, a map is a weak equivalence if and only if it is a homotopy equivalence.*

Corollary A.2.7 *Let $j : \mathcal{C}_{cf} \longrightarrow \mathsf{Ho}\,\mathcal{C}_{cf}$ and $\pi : \mathcal{C}_{cf} \longrightarrow \mathcal{C}_{cf}/\sim$ be the canonical functors. Then the unique functor*
$$\bar{j} : \mathcal{C}_{cf}/\sim \xrightarrow{\cong} \mathsf{Ho}\,\mathcal{C}_{cf}$$
such that $\bar{j} \circ \pi = j$ is an isomorphism of categories.

We gather all these results in the

Theorem A.2.8 (Fundamental of model categories) *Let $\mathcal{C}$ be a model category, let $j : \mathcal{C} \longrightarrow \mathsf{Ho}\,\mathcal{C}$ be the morphism of localization and let Q, R be the cofibrant and fibrant replacement functors. Then*

1. *The inclusion $\mathcal{C}_{cf} \hookrightarrow \mathcal{C}$ induces an equivalence of categories*

$$\mathcal{C}_{cf}/\sim \xrightarrow{\simeq} \mathsf{Ho}\,\mathcal{C}.$$

2. *There are natural bijections*

$$\mathrm{Hom}_{\mathsf{Ho}\,\mathcal{C}}(X,Y) \cong [QRX, QRY] \quad , \quad \mathrm{Hom}_{\mathsf{Ho}\,\mathcal{C}}(X,Y) \cong [RQX, RQY],$$

$$\mathrm{Hom}_{\mathsf{Ho}\,\mathcal{C}}(X,Y) \cong [QX, RY],$$

and in particular, if X is cofibrant and Y is fibrant, then

$$\mathrm{Hom}_{\mathsf{Ho}\,\mathcal{C}}(X,Y) \cong [X,Y].$$

3. *The functor $j : \mathcal{C} \longrightarrow \mathsf{Ho}\,\mathcal{C}$ identifies left or right homotopic maps.*

4. *Given a map f in $\mathcal{C}$, it holds that f is a weak equivalence if and only if $j(f)$ is an isomorphism.*

A.3 Quillen functors and derived functors

As usual in mathematics, given some objects with a structure, we want to study the morphisms which preserve the structure. In our particular case, we are also interested for functors between model categories which induce functors between the homotopy categories.

Definition. Let $\mathcal{C}, \mathcal{D}$ be model categories.

1. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is **left Quillen** if it is left adjoint and preserves cofibrations and trivial cofibrations.

2. A functor $U : \mathcal{D} \longrightarrow \mathcal{C}$ is **right Quillen** if it is right adjoint and preserves fibrations and trivial fibrations.

Lemma A.3.1 *Every left (right) Quillen functor preserves weak equivalences between cofibrant (fibrant) objects.*

Proposition A.3.2 *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction between model categories. Then F is left Quillen if and only if U is right Quillen.*

Definition. An adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between model categories is **Quillen** if F is left Quillen (or equivalently, if U is right Quillen).

Examples A.3.3 (a) The adjunction $|\cdot| : \mathsf{sSet} \rightleftarrows \mathsf{Top} : \mathcal{S}$ is Quillen.

(b) The adjunction $(-)_+ : \mathcal{C} \rightleftarrows \mathcal{C}_* : U$ is Quillen.

(c) If $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction, so is $F_* : \mathcal{C}_* \rightleftarrows \mathcal{D}_* : U_*$.

Now we face the following question: given a left Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, how can we come up with an induced functor $\mathbb{L}F : \mathsf{Ho}\,\mathcal{C} \longrightarrow \mathsf{Ho}\,\mathcal{D}$ between the homotopy categories? One could think: consider the composite $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{j} \mathsf{Ho}\,\mathcal{D}$ and then appeal to the universal property of the localization. But there is a problem: for this we need that the functor preserves weak equivalences, but a left Quillen functor only preserves weak equivalences between cofibrant objects! We cannot apply this argument directly, but there is an easy solution: by the 2-out-of-3 property, $Q : \mathcal{C} \longrightarrow \mathcal{C}_c$ preserves weak equivalences, so the composite $\mathcal{C} \xrightarrow{Q} \mathcal{C}_c \xrightarrow{j} \mathsf{Ho}\,\mathcal{C}_c$ sends weak equivalences to isomorphisms, hence by the universal property of the localization we get a functor $\mathsf{Ho}\,Q : \mathsf{Ho}\,\mathcal{C} \longrightarrow \mathsf{Ho}\,\mathcal{C}_c$. Similarly, let $F_{|\mathcal{C}_c} : \mathcal{C}_c \longrightarrow \mathcal{D}$ be the restriction of F to $\mathcal{C}_c$. Now this morphism preserves weak equivalences (since all objects are cofibrant), thus by the universal property of the localization we get a functor $\mathsf{Ho}\,F_{|\mathcal{C}_c} : \mathsf{Ho}\,\mathcal{C}_c \longrightarrow \mathsf{Ho}\,\mathcal{D}$.

Definition. Let $\mathcal{C}, \mathcal{D}$ be model categories.

1. Given a left Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, the **total left derived functor** of F is the functor $\mathbb{L}F : \mathsf{Ho}\,\mathcal{C} \longrightarrow \mathsf{Ho}\,\mathcal{D}$ defined as the composite
$$\mathsf{Ho}\,\mathcal{C} \xrightarrow{\mathsf{Ho}\,Q} \mathsf{Ho}\,\mathcal{C}_c \xrightarrow{\mathsf{Ho}\,F_{|\mathcal{C}_c}} \mathsf{Ho}\,\mathcal{D},$$
and given a natural transformation $\alpha : F \Rightarrow F'$ of left Quillen functors, the **total derived natural transformation** $\mathbb{L}\alpha$ is given by $(\mathbb{L}\alpha)_X := \alpha_{QX}$.

2. Given a right Quillen functor $U : \mathcal{D} \longrightarrow \mathcal{C}$, the **total right derived functor** of U is the functor $\mathbb{R}U : \mathsf{Ho}\,\mathcal{D} \longrightarrow \mathsf{Ho}\,\mathcal{C}$ defined as the composite
$$\mathsf{Ho}\,\mathcal{D} \xrightarrow{\mathsf{Ho}\,R} \mathsf{Ho}\,\mathcal{D}_f \xrightarrow{\mathsf{Ho}\,U_{|\mathcal{D}_f}} \mathsf{Ho}\,\mathcal{C},$$
and given a natural transformation $\alpha : U \Rightarrow U'$ of right Quillen functors, the **total derived natural transformation** $\mathbb{R}\alpha$ is given by $(\mathbb{R}\alpha)_X := \alpha_{RX}$.

Remark A.3.4 The total derived natural transformation is functorial, that is, if $\alpha : F \Rightarrow F'$ and $\beta : F' \Rightarrow F''$ are natural transformations between left Quillen functors, then $\mathbb{L}(\beta \circ \alpha) = \mathbb{L}\beta \circ \mathbb{L}\alpha$ and $\mathbb{L}\mathrm{Id}_F = \mathrm{Id}_{\mathbb{L}F}$; and similarly for right Quillen functors.

However, taking the left or right derived functors is *not* a functorial assignment, for instance $\mathbb{L}\mathrm{Id}_{\mathcal{C}} = \mathsf{Ho}\,Q$. However, one can show that it is functorial up to natural isomorphism.

Theorem A.3.5 *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. Then the left and right derived functors define an adjunction between the homotopy categories,*
$$\begin{tikzcd} \mathsf{Ho}\,\mathcal{C} \arrow[rr, bend left, "\mathbb{L}F"] \arrow[rr, phantom, "\perp"] && \mathsf{Ho}\,\mathcal{D} \arrow[ll, bend left, "\mathbb{R}U"] \end{tikzcd}$$
*called the **derived adjunction**.*

One of the key properties of Quillen adjunctions is that, for some of them, they will induce an equivalence of categories at the level of the homotopy categories:

Definition. A **Quillen equivalence** is a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ with the extra property that for every $X \in \mathcal{C}$ cofibrant and $Y \in \mathcal{D}$ fibrant, the bijection
$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, UY)$$
makes weak equivalences correspond with weak equivalences.

Theorem A.3.6 *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. Then the following are equivalent:*

1. *$F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen equivalence.*
2. *$\mathbb{L}F : \mathsf{Ho}\,\mathcal{C} \rightleftarrows \mathsf{Ho}\,\mathcal{D} : \mathbb{R}U$ is an adjoint equivalence of categories.*
3. *The composites*

$$X \xrightarrow{\eta} UFX \xrightarrow{Ur_{FX}} URFX \quad , \quad X \text{ cofibrant},$$
$$FQUX \xrightarrow{Fq_{UX}} FQX \xrightarrow{\varepsilon} X \quad , \quad X \text{ fibrant},$$

are weak equivalences.

Another useful criterion that we will use is

Corollary A.3.7 *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction. If U creates*[1] *the weak equivalences of $\mathcal{D}$, and $\eta : B \longrightarrow UFB$ is a weak equivalence for all cofibrant objects B in $\mathcal{C}$, then $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen equivalence.*

A.4 The model structure of topological spaces

We will describe some relevant facts of the Quillen model structure of Top, where

- weak equivalences are weak homotopy equivalences,
- fibrations are Serre fibrations
- cofibrations are retracts of inclusions of relative cell complexes $A \hookrightarrow X$.

For every space, obviously every map $X \longrightarrow *$ is a Serre fibration, so every object is fibrant in Top. Moreover, from the third item one gets that CW-complexes are cofibrant, so from A.2.8 we obtain

Corollary A.4.1 *If X is a CW-complex and Y is any space, then maps $X \longrightarrow Y$ in $\mathsf{Ho}(\mathsf{Top})$ correspond bijectively to the usual set of homotopy classes of maps,*

$$\mathrm{Hom}_{\mathsf{Ho}(\mathsf{Top})}(X,Y) \cong [X,Y].$$

The small object argument

Another remarkable issue that we will briefly explain is the factorization of every map in two possible ways (axiom (*iv*)). This is due to a general statement called the *small object argument*.

Definition. Let $\mathcal{C}$ be a cocomplete category and let

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

be a sequential diagram with $X := \mathrm{colim}_n X_n$. An object A of $\mathcal{C}$ is called **small relatively to** X if the natural map

$$\underset{n}{\mathrm{colim}}\, \mathrm{Hom}_{\mathcal{C}}(A, X_n) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, X)$$

is an isomorphism.

[1]A functor $U : \mathcal{D} \longrightarrow \mathcal{C}$ **creates** weak equivalences if given objects $D, D' \in \mathcal{D}$ such that FD, FD' are weak equivalent, then D, D' are also weak equivalent.

Definition. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a set of morphisms of $\mathcal{C}$. The class of morphisms of $\mathcal{C}$ with the right lifting property with respect to all morphisms in $\mathcal{I}$ is called the class of **$\mathcal{I}$-injectives**, and denoted as $\mathcal{I}$-inj.

Let $\mathcal{I} = \{f_i : A_i \longrightarrow B_i\}$ be a set of morphisms, and fix $f : X \longrightarrow Y$ a morphism in $\mathcal{C}$. Set $G^0(\mathcal{I}, f) := X$ and $p_0 := f$. We will construct objects $G^k(\mathcal{I}, f)$ and morphisms $p_k : G^k(\mathcal{I}, f) \longrightarrow Y$ and $i_k : G^k(\mathcal{I}, f) \longrightarrow G^{k+1}(\mathcal{I}, f)$.

For every $k \in \mathbb{N}$, let $S_k(i)$ be the set of pairs of arrows (g, h) in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{g} & G^k(\mathcal{I}, f) \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle p_k} \\ B_i & \xrightarrow{h} & Y \end{array}$$

commute. Inductively, we define $G^{k+1}(\mathcal{I}, f)$ as the pushout of the diagram

$$\begin{array}{ccc} \coprod\limits_i \coprod\limits_{(g,h)\in S_k(i)} A_i & \xrightarrow{\coprod g} & G^k(\mathcal{I}, f) \\ \downarrow{\scriptstyle \coprod f_i} & & \downarrow{\scriptstyle i_k} \\ \coprod\limits_i \coprod\limits_{(g,h)\in S_k(i)} B_i & \longrightarrow & G^{k+1}(\mathcal{I}, f) \end{array}$$

We get maps $i_k : G^k(\mathcal{I}, f) \longrightarrow G^{k+1}(\mathcal{I}, f)$, and using the universal property of the pushout, maps $p_k : G^k(\mathcal{I}, f) \longrightarrow Y$.

Definition. The **infinite gluing construction** of f with respect to $\mathcal{I}$ is the space

$$G^\infty(\mathcal{I}, f) := \operatorname{colim}_k G^k(\mathcal{I}, f).$$

Let us denote by $i_\infty : G^0(\mathcal{I}, f) \longrightarrow G^\infty(\mathcal{I}, f)$ the natural projection to the colimit, and let $p_\infty : G^\infty(\mathcal{I}, f) \longrightarrow Y$ be the map induced from the colimit out of the morphisms p_k.

Theorem A.4.2 (Small object argument) *In the previous situation, f factors as $f = p_\infty \circ i_\infty$, where i_∞ has the left lifting property with respect to every map in $\mathcal{I}$-inj. Moreover, if for all i, A_i is small relatively to $G^\infty(\mathcal{I}, f)$, then p_∞ is in $\mathcal{I}$-inj.*

Under this construction, it is not hard to show that we can factorize every continuous map $f : X \longrightarrow Y$ between topological spaces as a cofibration followed by a trivial fibration using the inclusions

$$\mathcal{I} = \{S^n \hookrightarrow D^n : n \geq 0\},$$

and as a trivial cofibration followed by a fibration using the inclusions at 0

$$\mathcal{J} = \{i_0 : D^n \hookrightarrow D^n \times I : n \geq 0\}.$$

The infinite gluing construction has an important application for us: it allows to state a criterion to lift the model structure of Top to another category using a family of adjunctions.

Theorem A.4.3 (Model Structure Lifting) *Let $\mathcal{D}$ be a complete and cocomplete category, let*

$$\left\{ \begin{tikzcd} \mathsf{Top} \arrow[r, bend left, "F_k"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G_k"] \end{tikzcd} \right\}_{k\in\mathcal{F}}$$

be a family of adjunctions, and write

$$F\mathcal{I} := \{F_k(i) : k \in \mathcal{F}, i \in \mathcal{I}\} \qquad , \qquad F\mathcal{J} := \{F_k(j) : k \in \mathcal{F}, j \in \mathcal{J}\}.$$

Suppose that for all $f : D \longrightarrow D'$ in $\mathcal{D}$ and $k \in \mathcal{F}$,

1. *The object $F_k(S^{n-1})$ is small relative to the Infinite Gluing Construction $G^\infty(F\mathcal{I}, f)$.*
2. *The object $F_k(D^n)$ is small relative to the Infinite Gluing Construction $G^\infty(F\mathcal{J}, f)$.*
3. *The factor $G_k(i_\infty) : G_k(D) \longrightarrow G_k(G^\infty(F\mathcal{J}, f))$ is a weak equivalence in* Top *for all $k \in \mathcal{F}$.*

Then the choices

- ***Weak equivalences:*** *Maps f in $\mathcal{D}$ such that $G_k(f)$ is a weak equivalence in* Top *for all $k \in \mathcal{F}$.*
- ***Fibrations:*** *Maps f in $\mathcal{D}$ such that $G_k(f)$ is a fibration in* Top *for all $k \in \mathcal{F}$.*
- ***Cofibrations:*** *Maps with the LLP with respect to maps in $F\mathcal{I}$-inj.*

make $\mathcal{D}$ into a model category.

The proof is not very hard and can be found in [37, 5.1].

Homotopy limits and colimits

We now briefly discuss homotopy limits and colimits. Let $\mathcal{J}$ be a small category. One can endow the functor category $\mathsf{Top}^{\mathcal{J}}$ with the *projective model structure*, where a map $X \longrightarrow Y$ in $\mathsf{Top}^{\mathcal{J}}$ is a weak equivalence or a fibration if and only if so is every $X_j \longrightarrow Y_j$. In particular, the adjunction

$$\mathrm{colim} : \mathsf{Top}^{\mathcal{J}} \rightleftarrows \mathsf{Top} : \mathrm{const}$$

turns out to be Quillen, so the left derived functor $\mathbb{L}\,\mathrm{colim}$ exists. In general, let $\mathcal{C}$ be a cofibrantly generated model category, so we endow $\mathcal{C}^{\mathcal{J}}$ with the projective model structure; and let $\mathcal{D}$ be a combinatorial model category, so we endow $\mathcal{D}^{\mathcal{J}}$ with the *injective model structure* (weak equivalences and cofibrations are defined objectwise). Denote by j all localization functors.

Definition. The **homotopy colimit** is a functor hocolim : $\mathcal{C}^{\mathcal{J}} \longrightarrow \mathcal{C}$ which is a *model* for $\mathbb{L}\,\mathrm{colim}$, that is, which has the property that for any $F : \mathcal{J} \longrightarrow \mathcal{C}$,

$$j(\mathrm{hocolim}\ F) = (\mathbb{L}\,\mathrm{colim})jF.$$

The **homotopy limit** is a functor holim : $\mathcal{D}^{\mathcal{J}} \longrightarrow \mathcal{D}$ which is a *model* for $\mathbb{R}\lim$,

$$j(\mathrm{holim}\ F) = (\mathbb{R}\lim)jF.$$

By definition, it is clear that it is unique up to weak equivalence. For the existence, it is enough to compose the functors colim or lim with the cofibrant or fibrant replacement functors Q, R, respectively.

Whereas Top is cofibrantly generated (so hocolim always exists), it is not combinatorial. However, one can always define the homotopy limit of a functor $F : \mathcal{J} \longrightarrow \mathsf{Top}$ as the totalization of the cosimplicial replacement of F (see [7, 5.7] for details), which still has the property of being weak homotopy invariant. The following proposition will provide us a formula to compute homotopy limits in Top. Recall that for a category $\mathcal{J}$, its **nerve** $N\mathcal{J}$ is the simplicial set which in dimension n consists of all possible strings of n composable arrows $(j_0 \longrightarrow j_1 \longrightarrow \cdots \longrightarrow j_n)$. The face and degeneracy maps consist of considering the composite of two maps, or including the identity of an object, respectively. In the following proposition we use $Y^X = \mathrm{Map}(X, Y)$ for spaces X, Y (see [7, 5.7]).

Proposition A.4.4 *If* $X : \mathcal{J} \longrightarrow \mathsf{Top}$ *, then*

$$\mathrm{holim}_{\mathcal{J}} X \cong \mathrm{eq} \left(\prod_k X_k^{|N(\mathcal{J}/k)|} \rightrightarrows \prod_{i \to j} X_j^{|N(\mathcal{J}/i)|} \right)$$

where the first arrow maps $(f_k : |N(\mathcal{J}/k)| \longrightarrow X_k)_{k \in \mathcal{J}}$ *to the composite*

$$|N(\mathcal{J}/i)| \xrightarrow{f_i} X_i \xrightarrow{X(i \to j)} X_j$$

in the factor indexed by $i \longrightarrow j$*; and the second arrow maps* (f_k) *to the composite*

$$|N(\mathcal{J}/i)| \longrightarrow |N(\mathcal{J}/j)| \xrightarrow{f_j} X_j$$

in the factor indexed by $i \longrightarrow j$*, where* $|N(\mathcal{J}/i)| \longrightarrow |N(\mathcal{J}/j)|$ *is induced by the functor* $\mathcal{J}/i \longrightarrow \mathcal{J}/j$ *induced by* $i \longrightarrow j$*.*

A.5 Triangulated categories

We briefly recall the definition and basic properties of a triangulated category. This is a topic at first independent of model categories, but we explain their relation at the end of the section.

Definition. A category $\mathcal{C}$ is **semiadditive** if it has finite products and coproducts and such that the canonical map

$$\coprod_{i=1}^{n} C_i \longrightarrow \prod_{i=1}^{n} C_i$$

is an isomorphism. In particular, this implies the existence of a zero object.

Every semiadditive category is naturally enriched over the category CMon of commutative monoids: given maps $f, g : C \longrightarrow D$, there is a binary operation $f + g$ defined as the composite

$$C \xrightarrow{(\mathrm{Id},\mathrm{Id})} C \times C \cong C \coprod C \xrightarrow{(f,0) \coprod (0,g)} D \times D \cong D \coprod D \xrightarrow{\mathrm{Id} \coprod \mathrm{Id}} D,$$

where $0 : C \longrightarrow D$ denotes the unique morphism which factors through the zero object. Moreover, this zero morphism is the neutral element.

Definition. A semiadditive category $\mathcal{C}$ is **additive** if the previous enrichment over CMon takes values in Ab, that is, if every morphism has an additive inverse.

Now let $\mathcal{C}$ be a category and $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ a self-functor. A **triangle** with respect to Σ is a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

and we will refer to it as a triple (f, g, h). A morphism of triangles (f, g, h), (f', g', h') is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

Definition. A **triangulated category** is an additive category $\mathcal{C}$ together with a self-equivalence $\Sigma : \mathcal{C} \longrightarrow \mathcal{C}$ and a collection of triangles, called the class of **distinguished triangles**, satisfying the following axioms:

(*i*) The class of distinguished triangles is closed under isomorphism.

(*ii*) For every object X, the triangle $0 \longrightarrow X \xrightarrow{\mathrm{Id}} X \longrightarrow 0$ is distinguished.

(*iii*) If the triangle (f, g, h) is distinguished, so is $(g, h, -\Sigma f)$.

(*iv*) If the rows of the following solid diagram are distinguished and the left-hand square commutes, then the dashed arrow exists and the entire diagram commutes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

(*v*) For every pair of composable morphisms $f : X \longrightarrow Y$, $f' : Y \longrightarrow D$, there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"] & B \arrow[r, "g"] \arrow[d, "f'"] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "f'f"] & D \arrow[r, "g''"] \arrow[d, "g'"] & E \arrow[r, "h''"] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\
 & F \arrow[r, "\mathrm{Id}"] \arrow[d, "h'"] & F \arrow[r, "h'"] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma g"] & \Sigma C &
\end{tikzcd}$$

such that the triangles $(f, g, h), (f', g', h'), (f'f, g'', h'')$ and $(x, y, (\Sigma g) \circ h')$ are distinguished.

In particular, every morphism f is part of a distinguished triangle.

From axioms (*ii*) and (*iv*) it follows that the composite of two consecutive maps in a distinguished triangle is the 0 map.

Proposition A.5.1 *Let $\mathcal{C}$ be triangulated and let $A \in \mathcal{C}$. Then the functor*

$$\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathsf{Ab}$$

takes distinguished triangles to exact sequences.

Proof. By axiom (*iii*), it is enough to see that given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

then

$$\mathrm{Hom}_{\mathcal{C}}(A, X) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{C}}(A, Y) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{C}}(A, Z)$$

is exact. By the previous observation, the composite $g_* \circ f_* = 0$. Conversely, if $\varphi : A \longrightarrow Y$ is a morphism such that $g \circ \varphi = 0$, then we have the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & \Sigma A \arrow[r, "-\mathrm{Id}"] \arrow[d, dashed] & \Sigma A \arrow[d, "\Sigma a"] \\
Y \arrow[r, "f"] & Z \arrow[r, "g"] & \Sigma X \arrow[r, "-\Sigma f"] & \Sigma Y
\end{tikzcd}$$

where the rows are distinguished by the axioms. Moreover, (iii) produces the dashed arrow, which must be $\Sigma\alpha$ for a unique $\alpha : A \longrightarrow X$ provided that Σ is fully faithful. □

Corollary A.5.2 (2-out-of-3 property) *Given a commutative diagram with distinguished rows*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X',
\end{tikzcd}$$

if two out of three of the vertical maps a, b, c are isomorphisms, so is the third.

Proof. Without loss of generality assume that a, b are isomorphisms. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ for any A, we have a commutative diagram of abelian groups

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, X) \arrow[r, "f_*"] \arrow[d, "a_*"] & \mathrm{Hom}_{\mathcal{C}}(A, Y) \arrow[r, "g_*"] \arrow[d, "b_*"] & \mathrm{Hom}_{\mathcal{C}}(A, Z) \arrow[r, "h_*"] \arrow[d, "c_*"] & \mathrm{Hom}_{\mathcal{C}}(A, \Sigma X) \arrow[r, "(\Sigma f)_*"] \arrow[d, "(\Sigma a)_*"] & \mathrm{Hom}_{\mathcal{C}}(A, \Sigma Y) \arrow[d, "(\Sigma b)_*"] \\
\mathrm{Hom}_{\mathcal{C}}(A, X') \arrow[r, "f'_*"] & \mathrm{Hom}_{\mathcal{C}}(A, Y') \arrow[r, "g'_*"] & \mathrm{Hom}_{\mathcal{C}}(A, Z') \arrow[r, "h'_*"] & \mathrm{Hom}_{\mathcal{C}}(A, \Sigma X') \arrow[r, "(\Sigma f')_*"] & \mathrm{Hom}_{\mathcal{C}}(A, \Sigma Y')
\end{tikzcd}$$

By the previous proposition, rows are exact, so we conclude by the five lemma for abelian groups. □

Pointed model categories

To finish this appendix, we want to describe how we can get a triangulated category from a pointed model category. Giving precise definitions and the exact statements of theorems would take us quite a lot of pages, so we will only overview the most important aspects. Recall that a category with initial and terminal objects is **pointed** if the unique map between them is an isomorphism, and it is called the zero object.

From now on we fix $\mathcal{C}$ a pointed model category, with zero object $*$.

Definition. Let $f : X \longrightarrow Y$ be a map in $\mathcal{C}$. The **cofibre** of f is the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & Z. \end{tikzcd}$$

Similarly, the **fibre** of f is the pullback of the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "e"'] & * \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

In a pointed model category, one can show that there is always an adjunction

$$\begin{tikzcd} \mathsf{Ho}(\mathcal{C}) \arrow[rr, bend left, "\Sigma"] \arrow[rr, phantom, "\perp"] & & \mathsf{Ho}(\mathcal{C}) \arrow[ll, bend left, "\Omega"] \end{tikzcd}$$

with the property that $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{C})}(\Sigma^n X, Y)$ and $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{C})}(X, \Omega^n Y)$ are groups, which are abelian if $n \geq 2$.

Definition. A **cofibre sequence** in $\mathsf{Ho}(\mathcal{C})$ is a diagram

$$X \longrightarrow Y \longrightarrow Z$$

in $\mathsf{Ho}(\mathcal{C})$ together with a right coaction of ΣX on Z which is isomorphic in $\mathsf{Ho}(\mathcal{C})$ to a diagram of the form $A \xrightarrow{f} B \xrightarrow{g} C$ where f is a cofibration of cofibrant objects in $\mathcal{C}$ with cofibre g and where C has a particular right coaction of ΣA (described in [16, 6.2.1]).

Similarly, a **fibre sequence** in $\mathsf{Ho}(\mathcal{C})$ is a diagram

$$X \longrightarrow Y \longrightarrow Z$$

in $\mathsf{Ho}(\mathcal{C})$ together with a right action of ΩZ on X which is isomorphic to a diagram of the form $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration of fibrant objects with fibre i and where F has a particular right action of ΩB (also described in [16, 6.2.1]).

We now collect the main properties of pointed model categories, that we use in §4.3. These can be found in [16, §6.3, §6.4 and §7.1].

Theorem A.5.3 *Let $\mathcal{C}$ be a pointed model category.*

1. *Any map $X \longrightarrow Y$ in $\mathsf{Ho}(\mathcal{C})$ is part of a cofibre sequence $X \longrightarrow Y \longrightarrow Z$ and a fibre sequence $W \longrightarrow X \longrightarrow Y$.*

2. *Given a cofibre sequence $X \longrightarrow Y \longrightarrow Z$, there is a boundary map $\partial : Z \longrightarrow \Sigma X$ such that $Y \longrightarrow Z \longrightarrow \Sigma X$ is a cofibre sequence.*

3. *If $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction between pointed model categories, then $\mathbb{L}F$ preserves cofibre sequences, and $\mathbb{R}U$ preserves fibre sequences.*

4. *If $\Sigma : \mathsf{Ho}(\mathcal{C}) \longrightarrow \mathsf{Ho}(\mathcal{C})$ is an equivalence of categories, then $\mathsf{Ho}(\mathcal{C})$ is a triangulated category. In this case, one says that $\mathcal{C}$ is a* ***stable model category****.*

5. *If $\mathcal{C}$ is a stable model category, then fibre sequences and cofibre sequences are the same thing.*

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Index of Notation

Top_* Category of compactly generated well-pointed spaces, page vii

$\underline{\mathcal{D}}$ Enriched category, page 40

$\wedge$ Day convolution of $\mathcal{D}$-spaces, page 43

$\wedge_{\mathbb{I}}$ Smash product, page 50

$\wedge_{\mathbb{I}}^{\mathbb{L}}$ Smash product in SHC, page 63

$\wedge_G$ Balanced product, page 3

BG Classifying space of G, page 25

EG Universal bundle of G, page 25

$G\mathsf{SHC}$ Equivariant stable homotopy category, page 73

$G\mathsf{Sp}^{\mathbb{I}}$ Category of orthogonal G-spectra, page 67

$H_G^n(-;M)$ Bredon cohomology, page 15

HA Eilenberg-MacLane spectrum, page 51

$K(X)$ Complex K-theory, page 20

$KO(X)$ Real K-theory, page 20

$KR(X)$ K-theory with Reality, page 28

$RO(G)$ Representation ring, page 74

$S^{p,q}$ One-point compactification of $\mathbb{R}^p \oplus i\mathbb{R}^q$, page 30

$V^{p,q}$ $\mathbb{R}^p \oplus i\mathbb{R}^q$, page 30

X^G Categorical fixed points, page 78

X^{hG} Homotopy fixed points, page 79

X^{tG} Tate construction, page 81

X_+ Attaching a disjoint basepoint, page 1

${}_G[-,-]$ G-homotopy classes of G-maps, page 7

${}_G[-,-]_*$ Basepoint preserving G-homotopy classes of based G-maps, page 7

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