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Bachelor Physics & Astronomy

Stochastic cascading transitions

Bachelor Thesis

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Abstract

Coupled climate systems can exhibit cascading transitions, i.e. a transition in one system can cause another transition in some other system. These cascading transitions could help explain the behavior of climate systems, such as the processes behind Dansgaard-Oeschger events or the Mid-Pleistocene Transition. Dekker [1] has created a framework of a leading system coupled to a following system to analyze these cascading transitions in the deterministic case. We add noise to the equation by introducing multiplicative noise in the leading system and additive noise in the following system. We find that noise introduces new behavior, and can even induce transitions in some of these coupled systems.

Contents

1 Introduction

Almost all climate systems on earth are coupled to one or more other climate systems. One might think of an ocean coupled to a coral reef, or a rain forest coupled to the atmosphere. These systems are typically described by physical quantities such as temperature, density, or the concentration of a certain molecule. It's not hard to imagine, for example, that the volume of sea ice might depend on the temperature of the ocean. A sudden change in the temperature of the ocean could then cause a change in the volume of sea ice.

Such transitions in coupled systems might help to explain behavior exhibited by one or both of the systems. Take the Dansgaard-Oeschger (D-O) events for example. These are rapid climate fluctuations that happened in the last glacial period, see Figure 1. Evidence for these events is found in the Greenland ice cores, data of which is shown in the bottom graph of Figure 1. What causes these rapid fluctuations in the oxygen isotopes of the Greenland ice cores is not yet known, analyzing the coupling of related climate systems might shed some light on the processes behind these events.

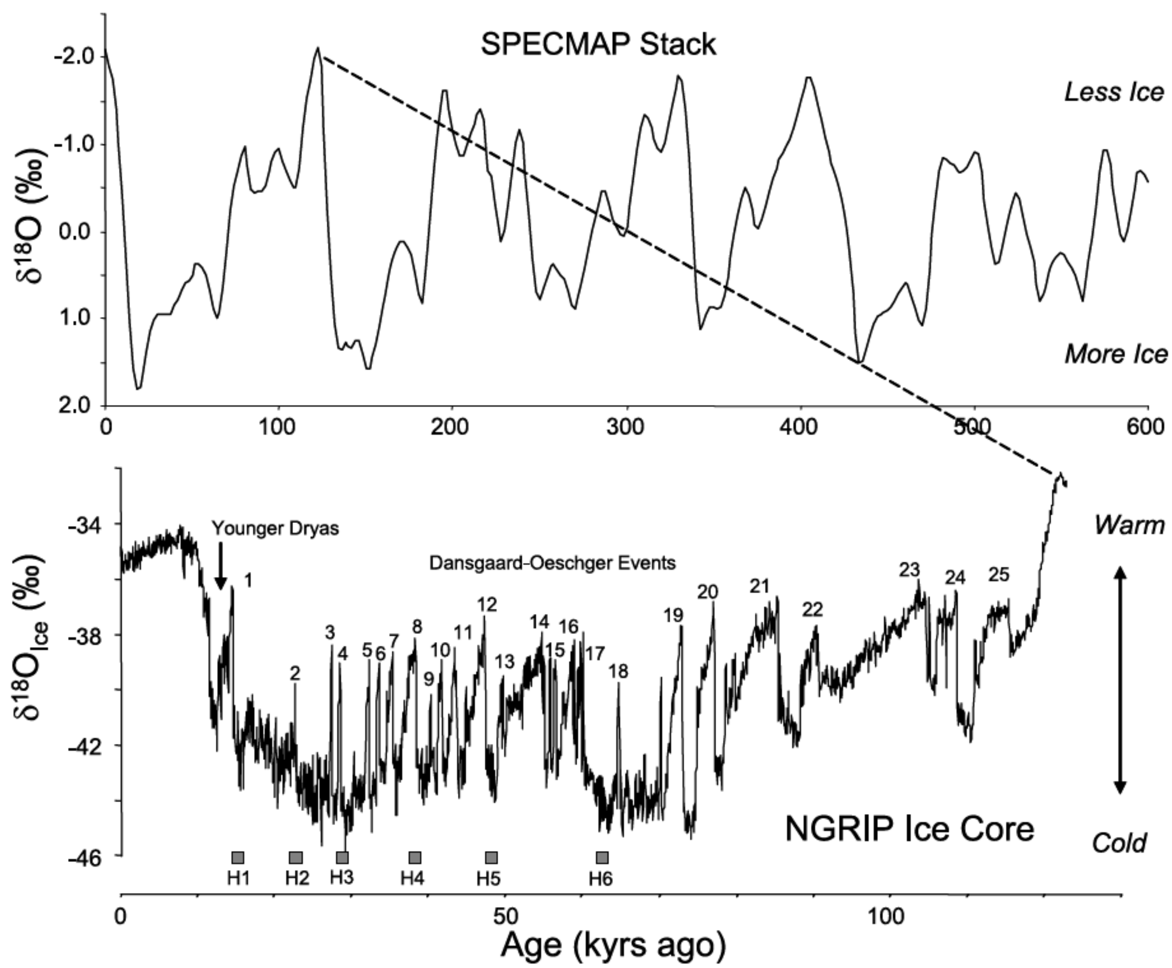

Figure 1: Oxygen isotope records, a proxy for air temperature [2] of the deep sea (top) and ice core (bottom). The rapid fluctuations in the bottom graph are known as Dansgaard-Oeschger events. Figure taken from Clement & Peterson [2].

A more direct example of a transition in a climate system is found in the Mid-Pleistocene Transition (MPT), as seen in Figure 2. The MPT is a change in the period of the glacial cycles. Before the MPT, the cycles mostly followed a 41 kyr periodicity with lower amplitudes. After the MPT, the amplitude increased and the periodicity varied more, with an average cycle length of roughly 100 kyr. This transition is thought to be linked to atmospheric CO2 levels, thus motivating analysis of transitions in coupled climate systems.

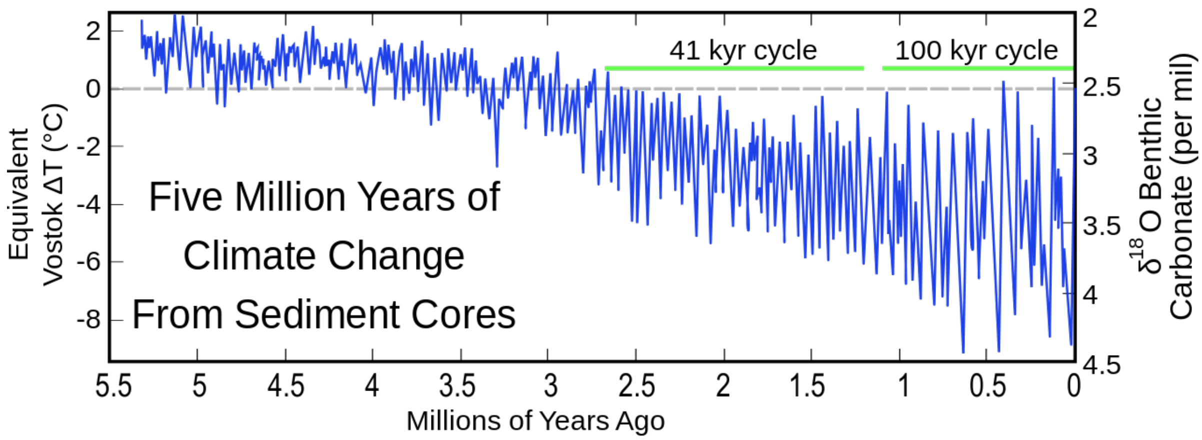

Figure 2: Glacial cycles over the last five million years. The Mid-Pleistocene Transition is the transition in periodicity from 41 kyr to 100 kyr. Figure taken from `Wikipedia.org` [3].

We follow Dekker's [1] reasoning that transitions in one system leading to another transition in some other coupled system, i.e. "tipping cascades", might explain behavior of climate systems. Dekker [1] introduces a mathematical framework to study these tipping cascades using two systems; a leading system and a following system. The following system is linearly dependent on the leading system in such a way that a transition induced in the leading system causes a transition in the following system. Since the leading system is not dependent on the following system, this is where the cascade ends.

However, in reality the leading system can also be dependent on the following system through stochastic backscatter. This means that the following system influences the leading system not in a deterministic way, but through noise. Stochastic backscatter can affect the stability of the leading system and/or following system, making it an important aspect to study regarding transitions.

We look at the influence of noise on cascading transitions and characterize some typical behaviour. We start by introducing stochastic noise in the coupled systems. Next, we analyze the four cases also analyzed by Dekker [1], as well as a Van der Pol oscillator. We then look at typical cases and identify new behaviour compared to the deterministic cases. Lastly, we discuss our results and their relevance.

2 Numerical models

Dekker [1] uses two systems to study cascading transitions: Eq. 1 to represent a fold bifurcation, and Eq. 2 to represent a Hopf bifurcation.

$$\mathrm{d}X = (a_1 X^3 + a_2 X + \phi(t))\,\mathrm{d}t\,,. \tag{1}$$

where a_i, b_i are constants. A fold bifurcation has one equilibrium below the bi-stable regime, a different equilibrium above the bi-stable regime, and both equilibria in the bi-stable regime. The forcing parameter $\phi(t)$ is used to push the system in or over the bi-stable regime, thus inducing a transition in the system. The bi-stable regime is given by $|\phi(t)| < \phi_c$, where ϕ_c is the critical value of the forcing parameter

$$\begin{cases} \mathrm{d}X = (a_1 Y + a_2(\phi(t) - (X^2 + Y^2))X)\,\mathrm{d}t\,, \\ \mathrm{d}Y = (b_1 X + b_2(\phi(t) - (X^2 + Y^2))Y)\,\mathrm{d}t\,. \end{cases} \tag{2}$$

Here a_i, b_i are constants. A Hopf bifurcation behaves differently from a fold bifurcation, in that it doesn't have a bi-stable regime. It has one equilibrium below ϕ_c, and a stable periodic orbit for $\phi(t) > \phi_c$. In this case the transition is not from one constant equilibrium state to another, but from a constant equilibrium state to a periodic equilibrium.

These two systems can be coupled to one another in four different ways: fold-fold, fold-Hopf, Hopf-fold, and Hopf-Hopf, where the systems are denoted as leading-following. Where a transition is induced in the leading system through $\phi(t)$, the subsequent transition in the following system is induced via γ, which then takes the place of $\phi(t)$. Note that γ is linearly dependent on the leading system, so $\gamma(X) = \gamma_1 + \gamma_2 X$. These four cases are entirely deterministic and studied as such by Dekker [1].

We introduce noise by defining $\mathrm{d}W_t$ as a small increment of a Wiener process. Adding the term $\sigma\,\mathrm{d}W_t$ to the right-hand side of equations 1 and 2 will add additive stochastic noise with strength σ, which is what we introduce in the following system. The following system already depends on the leading system via the deterministic coupling γ, so multiplicative noise is not needed. However, the leading system does depend on the following system via noise, so multiplicative noise is in order. We use a quadratic relation to represent the stochastic backscatter, i.e. the term $\sigma Y^2\,\mathrm{d}W_t$, where Y is the state vector of the following system. We use a quadratic relation because non-linear relations are often quadratic, such as the advection term in the Navier-Stokes equations.

The noise strength σ represents the relative influence of noise on the system; $\sigma = 0$ yields the deterministic case as discussed by Dekker [1], while $\sigma \gg 0$ is not a realistic case at all. We use different noise strength for the leading and following systems: σ_1 and σ_2 represent the noise strength for the leading and following systems, respectively. In the next parts we will vary both σ_1 and σ_2 to see the effect of noise on the transitions. We make a distinction between a fold bifurcation as the leading system (section 2.1), and an oscillatory system (Hopf, VDP) as the leading system (section 2.2).

2.1 Fold bifurcation as leading system

We have three cases with a fold bifurcation as leading system: fold-fold, fold-Hopf, and fold-VDP. To force a transition in the leading system, we use a step function $\phi(t)$. This function starts at ϕ_0, where the leading system has only one stationary equilibrium. At $t = 200$ it steps to ϕ_1, forcing a transition in the leading system. At $t = 800$, the forcing parameter drops to ϕ_2 in the bi-stable regime. For the parameter values, see Table 1.

Fold-fold (Eq. 3)	Fold-Hopf (Eq. 4)	Fold-VDP (Eq. 5)
Forcing		
$\phi_0 = -0.2$; $\phi_1 = 0.2$	$\phi_0 = -0.4$; $\phi_1 = 0.4$	$\phi_0 = -0.2$; $\phi_1 = 0.2$
$\phi_2 = 0.$	$\phi_2 = 0.$	$\phi_2 = 0.$
Coupling		
$\gamma(X) = 0.48X$	$\gamma(X) = -0.1 + 0.12X$	$\gamma(X) = -\frac{0.8}{36} - \frac{0.25}{36}X$
Parameters		
$a_1 = -0.5$	$a_1 = -1$	$a_1 = -0.5$
$a_2 = 0.5$	$a_2 = 1$	$a_2 = 0.5$
$b_1 = -0.5$	$b_1 = 1.5$; $b_2 = 0.2$	$b = -\frac{1}{36}$
$b_2 = -1.0$	$c_1 = -1.5$; $c_2 = 0.2$	$c = -\frac{30}{36}$

Table 1: Values of the forcing, coupling, and function parameters. We used the same parameter values as Dekker [1] in the fold-fold case. In the fold-Hopf case, we changed b_i and c_i to increase the amplitude of the oscillation. The values for the Fold-VDP case are derived from Crucifix [4].

2.1.1 Fold-fold

The most intuitive case is the fold-fold case, Eq. 3. We use the same values for a_i and b_i as Dekker [1], as well as the same coupling $\gamma(X)$ parameter. We numerically integrate this dynamical system for different values of noise strength: $0 < \sigma_1, \sigma_2 < 0.5$.

$$\begin{cases} \mathrm{d}X = (a_1X^3 + a_2X + \phi(t))\,\mathrm{d}t + \sigma_1 Y^2\,\mathrm{d}W_1 \\ \mathrm{d}Y = (b_1Y^3 + b_2Y + \gamma(X))\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2 \end{cases} \tag{3}$$

Each time the forcing parameter $\phi(t)$ induces a transition in the leading system, pushing the leading system from the equilibrium below the bi-stable regime (lower equilibrium) to the equilibrium above the bi-stable regime (upper equilibrium). The leading system then induces an equivalent transition in the following system via $\gamma(X)$. This transition in the following system can affect the leading system again through $\sigma_1 Y^2\,\mathrm{d}W_1$, especially when the forcing parameter falls back to $\phi(t) = 0$, in the bi-stable regime. Now the leading system is in the upper equilibrium, but both the lower and upper equilibria are stable since $\phi(t)$ is in the bi-stable regime. In the deterministic case, the system will always stay in the upper equilibrium since there are no perturbations. With noise however, we might see different behavior.

2.1.2 Fold-Hopf

For the second case, fold-Hopf (Eq. 4), we use different values for b_i and c_i than Dekker [1] to increase the amplitude of the oscillation of the Hopf bifurcation. The coupling parameter $\gamma(X)$ is still the same as in Dekker [1]. Since the following system consists of two differential equations, we choose the noise term in the leading system as $\sigma_1(Y^2 + Z^2)\,\mathrm{d}W_t$. We vary the noise strength σ_1 between 0 and 1.5, and σ_2 between 0 and 0.2. We chose to take σ_1 bigger to increase the influence of the following system on the leading system via noise. However, we chose σ_2 smaller because the Hopf bifurcation as a following system seems very vulnerable to noise, and we want to preserve its oscillatory nature. Note that we use the same noise strength σ_2, but different Wiener processes W_2 and W_3 in both differential equations of the following system.

$$\begin{cases} \mathrm{d}X = (a_1X^3 + a_2X + \phi(t))\,\mathrm{d}t + \sigma_1(Y^2 + Z^2)\,\mathrm{d}W_1 \\ \mathrm{d}Y = (b_1Z + b_2(\gamma(X) - (Y^2 + Z^2))Y)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2 \\ \mathrm{d}Z = (b_1Y + b_2(\gamma(X) - (Y^2 + Z^2))Z)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_3 \end{cases} \tag{4}$$

The forcing parameter $\phi(t)$ acts in the same way as with the fold-fold case, pushing the leading system to the upper equilibrium. Consequently, the following system undergoes a transition via $\gamma(X)$ and starts oscillating. In much the same way as with the fold-fold case, we might see the following system cause unexpected behavior in the leading system.

2.1.3 Fold-VDP

The last case with a fold bifurcation as leading system is the fold-VDP case, Eq. 5. The Van der Pol oscillator is used in Crucifix [4] to model the ice volume over the last 800 kyr. It behaves much like a Hopf bifurcation: it has a steady equilibrium below the critical value of the forcing parameter ϕ_c, and oscillates above ϕ_c. The difference is that the steady equilibrium is not as stable as in the Hopf bifurcation. We choose the same a_1 and a_2 as in the fold-fold case, while the values of b and c are derived from Crucifix [4] such that the Van der Pol oscillator behaves like a relaxation oscillator. The term $-\frac{0.8}{36}$ in the coupling $\gamma(X)$ is also derived from Crucifix [4], while the term $-\frac{0.25}{36}X$ is designed to induce the transition in the following system.

Contrary to the fold-Hopf case, we choose the noise in the leading system to only depend on the first equation of the following system, so $\sigma_1 Y^2\,\mathrm{d}W_1$. We make this decision based on the utilization of the VDP oscillator in Crucifix [4]: The first equation is used to model an ice age curve, while the second equation does not appear to have any physical significance. We take σ_1 between 0 and 0.5, and σ_2 between 0 and 0.2.

$$\begin{cases} \mathrm{d}X = (a_1X^3 + a_2X + \phi(t))\,\mathrm{d}t + \sigma_1 Y^2\,\mathrm{d}W_1 \\ \mathrm{d}Y = (bZ + \gamma(X))\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2 \\ \mathrm{d}Z = (c(Z^3/3 - Z - Y))\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_3 \end{cases} \tag{5}$$

After $\phi(t)$ forces the leading system in the upper equilibrium again, the following system transitions via $\gamma(X)$ and starts oscillating. The following system will then start interacting with the leading system through the multiplicative noise.

2.2 Hopf bifurcation or VDP as leading system

We have three cases left: Hopf-fold, Hopf-Hopf, VDP-fold. In these three cases we use a slightly different forcing $\phi(t)$: at $t = 800$, the forcing parameter drops back to ϕ_0 instead of ϕ_2. For the parameter values, see Table 2.

Hopf-fold (Eq. 6)	Hopf-Hopf (Eq. 7)	VDP-fold (Eq. 8)
Forcing		
$\phi_0 = -0.5$; $\phi_1 = 0.5$	$\phi_0 = -0.4$; $\phi_1 = 0.4$	$\phi_0 = \frac{1.05}{36}$; $\phi_1 = \frac{0.8}{36}$
Coupling		
$\gamma(X) = 0.05 + 0.5X$	$\gamma = (X) - 0.05 + 2X$	$\gamma(X) = 0.05 + 0.5X$
Parameters		
$a_1 = 0.05$; $a_2 = 1$	$a_1 = 0.04$; $a_2 = 2$	$a = -\frac{1}{36}$
$b_1 = -0.05$; $b_2 = 1$	$b_1 = -0.04$; $b_2 = 2$	$b = -\frac{30}{36}$
$c_1 = -1$	$c_1 = 0.4$; $c_2 = 1$	$c_1 = -0.5$
$c_2 = 1$	$d_1 = -0.4$; $d_2 = 1$	$c_2 = 0.5$

Table 2: Values of the forcing, coupling, and function parameters. The values for the VDP-fold case are derived from Crucifix [4].

2.2.1 Hopf-fold

The first case with a Hopf bifurcation as the leading system is the Hopf-fold case (Eq. 6). We use the same parameter values and coupling $\gamma(X)$ as Dekker [1]. We take both σ_1 and σ_2 between 0 and 0.2.

$$\begin{cases} \mathrm{d}X = (a_1Y + a_2(\phi(t) - (X^2 + Y^2))X)\,\mathrm{d}t + \sigma_1 Z^2\,\mathrm{d}W_1 \\ \mathrm{d}Y = (b_1X + b_2(\phi(t) - (X^2 + Y^2))Y)\,\mathrm{d}t + \sigma_1 Z^2\,\mathrm{d}W_2 \\ \mathrm{d}Z = (c_1Z^3 + c_2Z + \gamma(X))\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_3 \end{cases} \tag{6}$$

As $\phi(t)$ steps to ϕ_1, it induces a transition in the leading system; the leading systems starts to oscillate. This causes the following system to transition to the upper equilibrium. When $\phi(t)$ drops back to ϕ_0, the leading system stops oscillating but the following system doesn't fall back to the lower equilibrium.

2.2.2 Hopf-Hopf

The last case studied by Dekker [1] is the Hopf-Hopf case, Eq. 7. We again use the same parameter values and coupling $\gamma(X)$. Like the fold-Hopf case, we define the multiplicative noise term as $\sigma_1(U^2 + V^2)\,\mathrm{d}W_1$, and use different Wiener processes for each of the four equations. We vary the noise strength: $0 \leq \sigma_1 < 0.5$, $0 \leq \sigma_2 < 0.2$.

$$\begin{cases} \mathrm{d}X = (a_1 Y + a_2(\phi(t) - (X^2 + Y^2))X)\,\mathrm{d}t + \sigma_1(U^2 + V^2)\,\mathrm{d}W_1 \\ \mathrm{d}Y = (b_1 X + b_2(\phi(t) - (X^2 + Y^2))Y)\,\mathrm{d}t + \sigma_1(U^2 + V^2)\,\mathrm{d}W_2 \\ \mathrm{d}U = (c_1 V + c_2(\gamma(X) - (U^2 + V^2))U)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_3 \\ \mathrm{d}V = (d_1 U + d_2(\gamma(X) - (U^2 + V^2))V)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_4 \end{cases} \tag{7}$$

A transition is induced in the leading system, forcing it to start oscillating. This in turn induces a transition in the following system, which also starts to oscillate. When $\phi(t)$ falls back to ϕ_0, both systems will stop oscillating.

2.2.3 VDP-fold

For the same reason as in the fold-VDP case, we only let the following system interact with the first equation of the VDP oscillator through noise. We let both σ_1 and σ_2 run from 0 to 0.2.

$$\begin{cases} \mathrm{d}X = (aY + \phi(t))\,\mathrm{d}t + \sigma_1 Z^2\,\mathrm{d}W_1 \\ \mathrm{d}Y = (b(Y^3/3 - Y - X))\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_2 \\ \mathrm{d}Z = (c_1 Z^3 + c_2 Z + \gamma(X))\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_3 \end{cases} \tag{8}$$

The forcing parameter $\phi(t)$ forces the leading system to start oscillating, which forces the following system to transition as well.

3 Results

When σ_1 and σ_2 are very small, we observe behaviour very much like the deterministic case, see the fold-fold case in Figure 3. The forcing parameter forces the leading system to transition to the upper equilibrium, and the following system follows. Both systems stay in the higher equilibrium when the forcing parameter drops to the bi-stable regime. This is typical for all six cases except VDP-fold (Figure 4).

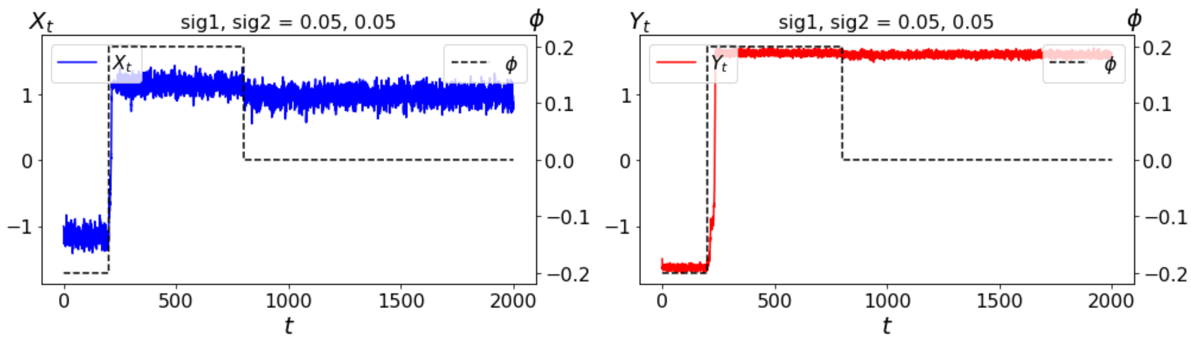

Figure 3: Fold-fold case with low noise strength

In that case the equilibrium state of the Van der Pol oscillator is so unstable that a small perturbation caused by σ_1 will cause the leading system to start oscillating, despite the fact that the forcing parameter is below the critical value ϕ_c. This happens even before $\phi(t)$ should induce the transition at $t = 200$, as seen in the bottom left graph in Figure 4. Note that the period of the oscillation is consistent while the system is in the region where we expect the VDP oscillator to oscillate ($200 < \phi(t) < 800$), but seems to increase immediately after $\phi(t)$ falls back to ϕ_0. The period also seems to lose its consistency, decreasing over time. For larger values of σ_1 and σ_2, the period varies inconsistently across the entire region.

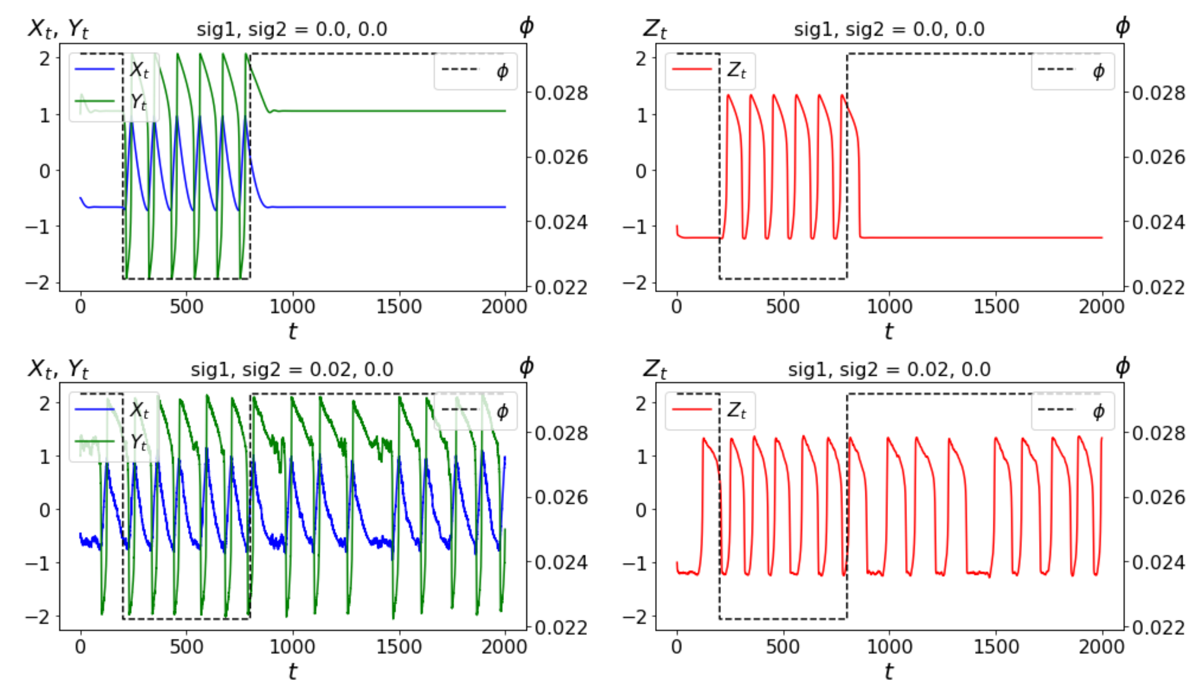

Figure 4: VDP-fold case with no noise (top) and with small σ_1 (bottom). Note that the leading system starts oscillating before the forcing parameter crosses the critical value.

When σ_1 and σ_2 are large, we see both systems behave erratically. This is to be expected when the noise amplitude is so high that the stochastic movements of the systems far exceed the equilibrium states. In the fold-fold case this leads to behavior like seen in Figure 5:

the noise amplitude is simply too high, no behavior other than that caused by noise is discernible. This is typical for all cases with a fold bifurcation as leading system, in cases with oscillatory leading systems it's often hard to distinguish between the steady and the oscillatory equilibrium due to the high amplitude of noise.

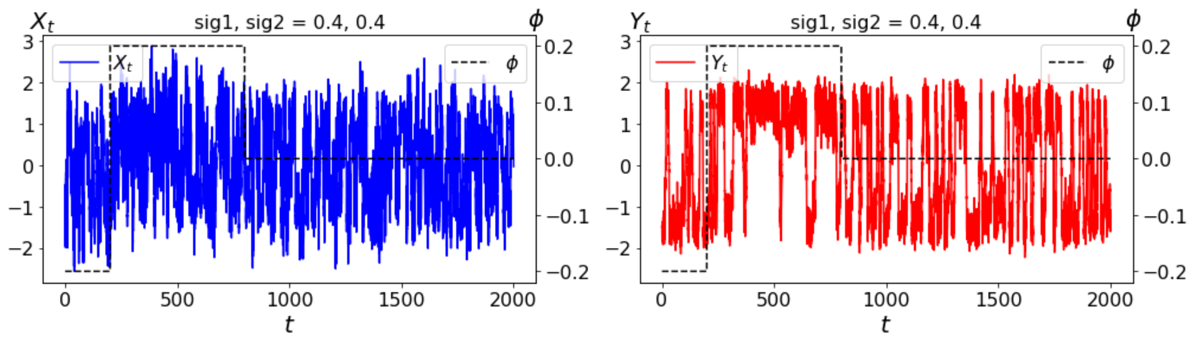

Figure 5: Fold-fold case with high noise strength

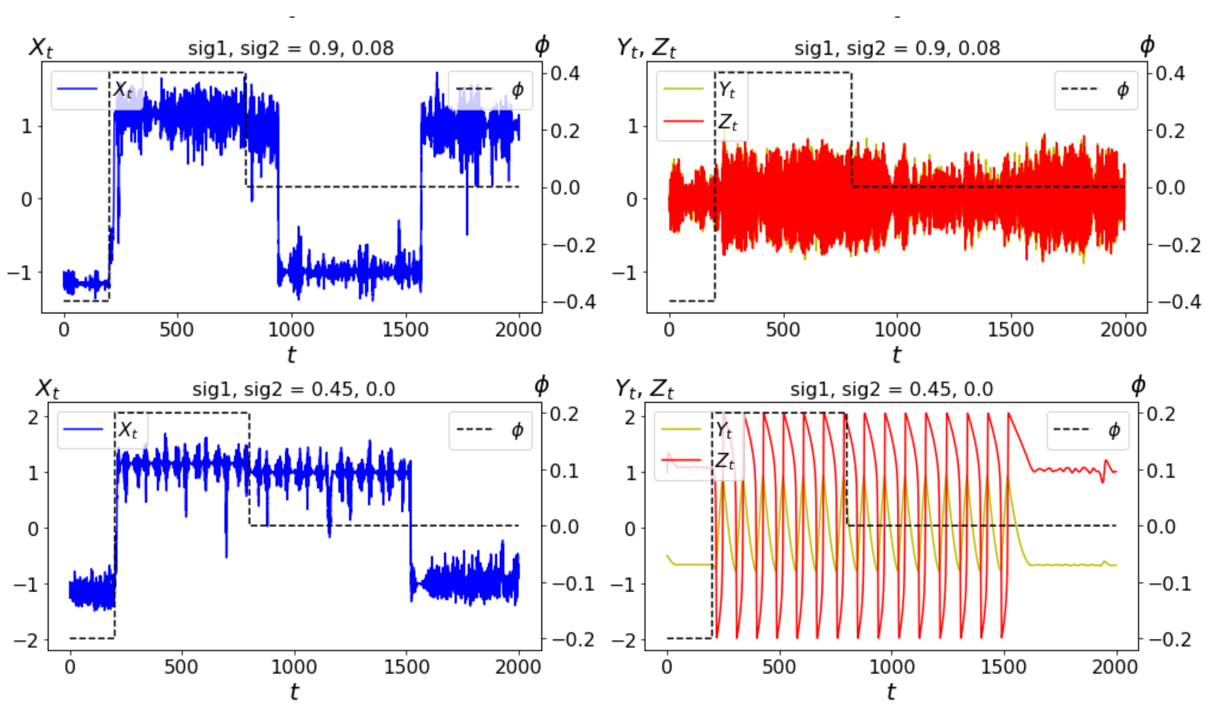

Figure 6: Fold-Hopf case with $\sigma_1 = 0.9$, $\sigma_2 = 0.08$ (top) and the fold-VDP case with $\sigma_1 = 0.45$, $\sigma_2 = 0$ (bottom)

However, when we choose σ_1, σ_2 high enough to influence the systems but low enough to not turn the systems into chaos, we do see interesting new behaviour. In the cases of fold-Hopf and fold-VDP (Figure 6), we see the leading system drop back to the lower equilibrium when the forcing parameter is in the bi-stable regime. For increasing noise strength we even observe the leading system "hop" between both equilibria, with the hopping frequency increasing as the noise strength grows. This behavior is consistently displayed in both cases for high enough σ_1 and σ_2, with the hopping becoming erratic for large σ_1 and σ_2. Note that the VDP oscillator in the bottom right graph of Figure 6, acting as the following system, stops oscillating as soon as the leading system drops to the lower equilibrium. This is because the noise strength of the following system, σ_2, is equal to zero. For nonzero values of σ_2 the VDP oscillator will not stop oscillating, and even start oscillating before any transition is induced, much like the VDP-fold case. The fact that the following system does not stop oscillating for $\sigma_2 > 0$ allows the leading system in those cases to hop between equilibria, just like the fold-Hopf case.

4 Discussion

We have analyzed six different coupled systems (fold-fold, fold-Hopf, fold-VDP, Hopf-fold, Hopf-Hopf, VDP-fold) to study noise induced cascading transitions. We found that with low noise strength, most cases behave like their deterministic counterparts, the only exception being the VDP-fold case. With high noise strength, we can no longer see any of the characteristic behavior of the systems. The influence of noise has far exceeded the dynamics of the systems, thus we can no longer speak of equilibrium states. With the noise strength somewhere in between however, we found new behavior in the fold-Hopf and fold-VDP cases. We observed the leading (fold) system falling back to the lower equilibrium while $\phi(t)$ is in the bi-stable region, and with increasing noise strength we saw the leading system hop between equilibria with higher frequency.

Compared to the deterministic cases analyzed by Dekker [1], where the systems do not transition back after being forced into the first transition, we do observe additional transitions in two cases (fold-Hopf, fold-VDP). These additional transitions are caused by the coupling of the following system to the leading system through noise, e.g. the term $\sigma_1 Y^2 \, dW_1$ in equation 3. The square of the following system, together with the noise strength σ_1, determines the amplitude of the noise. An increase in noise amplitude can lead to a spontaneous transition in the bi-stable region, where spontaneous transitions are not possible in the deterministic case. The noise strength of the following system also plays an important role; increasing σ_2 can directly affect the amplitude of the following system $Y(t)$ as it oscillates, which can then affect the noise amplitude in the leading system via $\sigma_1 Y^2 \, dW_1$.

The fact that noise can induce transitions in these coupled systems, implies that noise is an important aspect of the dynamics of these systems. It might be able to help explain transitions in real climate systems, such as Dansgaard-Oeschger events and the Mid-Pleistocene Transition. If we think back to how the volume of sea ice is dependent on the temperature of the ocean, it's easy to see how the temperature of the ocean will also depend on the volume of sea ice. And how a change in volume of sea ice caused by a change in ocean temperature, will in turn have a small but notable effect on the ocean temperature. This small but notable interaction could very well be stochastic, and have big consequences.

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