Modification of the Lifshitz-Kosevich formula for anomalous de Haas–van Alphen oscillations in inverted insulators

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Traditionally, quantum oscillations were interpreted as the hallmark of a Fermi surface. Recently, it was understood that in order to have well defined de Haas–van Alphen oscillations the presence of a Fermi surface is not required. Following the Luttinger approach, we investigate the oscillations of an inverted insulator. Clear quantum oscillations are found and we provide a formula for the Lifshitz-Kosevich amplitude both in the clean as well as weakly disordered regime. We find that the LK formula is strongly modified compared to systems with a Fermi surface.

I. INTRODUCTION

Landau quantization of electrons in magnetic fields is at the heart of many interesting phenomena. Besides being responsible for integer and fractional quantum Hall effects, it also plays an important role in determining the electronic structure of (correlated) metallic states. Information about the Fermi surface, quasiparticle effective masses, and disorder levels can be revealed by quantum oscillation measurements. In the standard experimental setup of quantum oscillations, thermodynamic quantities are measured as functions of the inverse magnetic field and display periodic behavior. The period is set by the cross section of the Fermi surface along the orbitals of electrons in the semiclassical picture. In uncorrelated metals the amplitude of the oscillations of thermodynamic quantities is damped due to thermal, disorder, and interaction broadenings and this damping is described by the Lifshitz-Kosevich (LK) formula [1–3]. There are a few cases in which deviations from the standard LK form have been shown theoretically [4–10]. In particular, it was shown that inverted insulators (Fig. 1) can exhibit well defined anomalous quantum oscillations even in the absence of a Fermi surface. This phenomenon has been discussed for different model systems [10–16].

The main contribution of this work is to apply a Matsubara frequency based approach (in contrast to Euler-Maclaurin approach employed by some other authors [15]) to derive a modified LK formula. We demonstrate that in some limits the modified LK formula can be approximated by analytic expressions.

The paper is organized as follows. In Sec. II we introduce a model system that describes an inverted insulator. We then derive the spectrum of this system subjected to a perpendicular magnetic field. We then calculate the grand potential in Sec. III with and without disorder where we follow the treatment of Luttinger [2]. Details of the disorder treatment can be found in the Appendixes. We continue by deriving some approximate analytic expressions for modified LK formula in Sec. IV and finish with a conclusion (Sec. V).

II. MODEL AND LANDAU LEVEL STRUCTURE

We will first describe the model without the magnetic field and derive the spectrum. Then, we will introduce external magnetic field and find energy values of Landau levels in the limit of large value of Landau level index. We consider a generic two-band model in two spatial dimensions that is described by the following Hamiltonian in momentum basis [17]:

$$H = \frac{1}{2\tilde{m}}\begin{pmatrix} 2\tilde{m}\tilde{\Delta} - |p|^2 & vp^*p^* \\ vpp & -2\tilde{m}\tilde{\Delta} + |p|^2 \end{pmatrix}, \quad (1)$$

where the complex momentum is expressed as $p = p_x + ip_y$ and p^* denotes its complex conjugate. The parameter $\tilde{m}$ is the effective mass and the parameter $\tilde{\Delta}$ may describe the perpendicular electric field. The dimensionless v is the hybridization parameter. The spectrum of the system for $\tilde{\Delta} = 0$ and $v = 0$ consists of two parabolas that touch each other. For $\tilde{\Delta} \neq 0$ the parabolas are pushed into each other and cross each other along a one-dimensional ring in the k space. For $v \neq 0$ the bands are hybridized and the ring shaped crossing develops into a gap. For reasons soon explained, we relabel the parameters in the following way: $\tilde{\Delta} = \Delta\sqrt{1 + v^2}$ and $\tilde{m} = m\sqrt{1 + v^2}$. We also introduce $\delta = v\Delta$. In this way, the energy in the large-momentum limit is fixed to be $\frac{p^2}{2m}$, the size of the gap is δ, and the location of the ring-shaped gap is given by $\sqrt{2m\Delta}$. Analytically, the eigenvalues of this Hamiltonian are given in the new notation by

$$E = \pm\sqrt{\left(\frac{p^2}{2m} - \Delta\right)^2 + \delta^2}. \quad (2)$$

A one-dimensional cut of this spectrum is sketched in Fig. 1. If the chemical potential of the system is equal to the gap $\mu = \delta$ the system has a Fermi surface and it is the circular minimum of the upper branch of the spectrum. The quantity $2\pi m\Delta$ then measures the area enclosed by the Fermi surface in k space. Naturally, this area describes the period of quantum
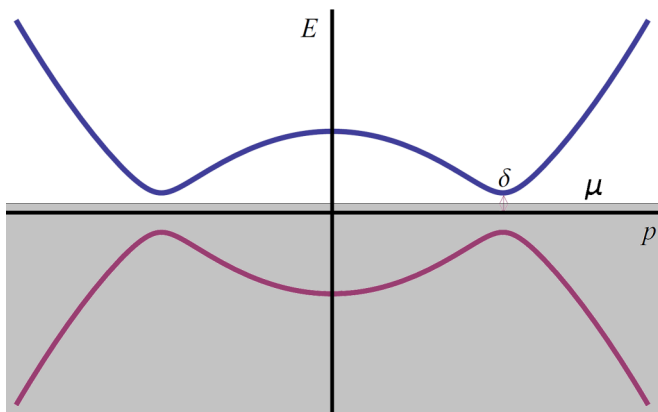
FIG. 1. Sketch of the spectrum of an inverted insulator as described by Eq. (2). It shows a cut along an arbitrary direction in the two-dimensional momentum space. The energy E is plotted on the vertical axis and the momentum on the horizontal axis. We furthermore allow for a chemical potential μ and all states below it are occupied at zero temperature. The system exhibits a gap of size δ.

oscillations [3]. If $\mu < \delta$, the Fermi surface does not exist; however, as we will show, the quantum oscillations are still present and they still have approximately the same frequency. If the effective masses of the upper and lower branch are different, there is a possibility that quantum oscillations will exhibit more than one frequency. We do not consider this situation, however, it has been explored in a strongly particle-hole asymmetric case [18]. We will now move on to Landau level calculation.

General Landau level structure

A perpendicular magnetic field is introduced via minimal coupling and we subsequently specify the Landau gauge, i.e., $\vec{A} = B(0,x,0)^T$. We make an ansatz for the wave function of the type $\Phi(x,y) = e^{ik_y y}\phi_{k_y}(x)$ and introduce the raising and lowering operators $a = \sqrt{\frac{m\omega_c}{2}}(x + \frac{i}{m\omega_c}p_x)$ and $a^\dagger = \sqrt{\frac{m\omega_c}{2}}(x - \frac{i}{m\omega_c}p_x)$ ($[a,a^\dagger] = 1$), where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency. This constitutes a basis change and allows us to rewrite the Hamiltonian as

$$H' = \begin{pmatrix} \tilde{\Delta} - \omega_c \frac{a^\dagger a + 1/2}{\sqrt{1+v^2}} & -\frac{v}{\sqrt{1+v^2}}\omega_c a^\dagger a^\dagger \\ -\frac{v}{\sqrt{1+v^2}}\omega_c aa & -\tilde{\Delta} + \omega_c \frac{a^\dagger a + 1/2}{\sqrt{1+v^2}} \end{pmatrix}. \quad (3)$$

The operators a and $a^\dagger$ act as raising and lowering operators on the harmonic oscillator eigenstates defined by $a|N\rangle = \sqrt{N}|N-1\rangle$ and $a^\dagger|N\rangle = \sqrt{N+1}|N+1\rangle$.

Employing a further ansatz $\phi_{k_y}(x + k_y/eB)$, where $\phi_{k_y}(x) = \binom{\alpha\langle x|N\rangle}{\beta\langle x|N-2\rangle}$, we reduce the problem to a 2×2 Hamiltonian for a two-component spinor $(\alpha, \beta)^T$, given by

$$H'' = \begin{pmatrix} \tilde{\Delta} - \omega_c \frac{N+1/2}{\sqrt{1+v^2}} & -\frac{v\omega_c}{\sqrt{1+v^2}}\sqrt{N(N-1)} \\ -\frac{v\omega_c}{\sqrt{1+v^2}}\sqrt{N(N-1)} & -\tilde{\Delta} + \omega_c \frac{N-3/2}{\sqrt{1+v^2}} \end{pmatrix}. \quad (4)$$

In the semiclassical regime of quantum oscillations, i.e., $N \gg 1$, the Hamiltonian becomes

$$H'' \approx \begin{pmatrix} \tilde{\Delta} - \omega_c \frac{N}{\sqrt{1+v^2}} & -\omega_c \frac{v}{\sqrt{1+v^2}}N \\ -\omega_c \frac{v}{\sqrt{1+v^2}}N & -\tilde{\Delta} + \omega_c \frac{N}{\sqrt{1+v^2}} \end{pmatrix}. \quad (5)$$

The energies of the quantized Landau levels that follow from this Hamiltonian are

$$E_N = \pm\sqrt{(\omega_c N - \Delta)^2 + \delta^2}. \quad (6)$$

We remind the reader that the magnetic field dependence of these energy levels is expressed via the effective cyclotron frequency $\omega_c = \frac{eB}{m}$.

III. GRAND POTENTIAL AND MODIFIED LIFSHITZ-KOSEVICH FORMULA

A. Clean case

The grand potential in a clean noninteracting system is given by

$$\Omega = -T \operatorname{tr} \ln(-\hat{G}^{-1}), \quad (7)$$

where $\hat{G}$ is the Green function of the noninteracting system and T is the temperature. Using the spectrum defined in Eq. (6) we can write the grand potential as

$$\Omega = -DT \sum_{\omega_n} \sum_{N=0}^{\infty} \sum_{\lambda=\pm} \ln(i\omega_n + \mu + \lambda E_N)$$

$$= -DT \sum_{\omega_n} \sum_{N=0}^{\infty} \ln((i\omega_n + \mu)^2 - E_N^2), \quad (8)$$

with $\omega_n = (2n+1)\pi T$ and the Landau level degeneracy factor $D = c\frac{eBL^2}{2\pi}$, where c counts the number of different species of electrons: spin, valley, etc. Note that we neglect a possible Zeeman effect.

Using the Poisson resummation technique we can decompose the sum over the Landau levels into a sum of integrals which is organized in terms of harmonic oscillations, i.e., $\sum_{N=0}^{\infty} f_N = \frac{1}{2}f_0 + \int_0^{\infty} dx\, f(x) + 2\sum_{l=1}^{\infty} \int_0^{\infty} dx\, f(x)\cos(2\pi lx)$, where $f(x)$ is the function obtained by replacing the integer parameter N by a continuous real variable x. In metals this allows us to perform semiclassical approximations with the control parameter T/ω_c (in the clean and noninteracting limit) leading to the famous Lifshitz-Kosevich formula [1,3]. Most importantly, we can identify the oscillatory part from Eq. (7) as

$$\tilde{\Omega} = 2DT \sum_{\omega_n} \sum_{l=1}^{\infty} \tilde{\Omega}_{l,\omega_n}, \quad \tilde{\Omega}_{l,\omega_n} = -\int_0^{\infty} dx$$

$$\times \ln((i\omega_n + \mu)^2 - E(x)^2)\cos(2\pi lx). \quad (9)$$

Integrating by parts and ignoring the nonoscillatory boundary terms containing, for instance, Landau diamagnetism we obtain

$$\tilde{\Omega}_{l,\omega_n} = \int_0^{\infty} dx \frac{\sin(2\pi lx)}{2\pi l} \frac{d}{dx} \ln((i\omega_n + \mu)^2 - E(x)^2). \quad (10)$$

This can be rewritten as

$$\tilde{\Omega}_{l,\omega_n} = \int_0^{\infty} dx \frac{\sin(2\pi lx)}{2\pi l}\left(\frac{1}{x - x_1} + \frac{1}{x - x_2}\right), \quad (11)$$

where $x_1 = \frac{1}{\omega_c}(\Delta + x_- + isx_+)$ and $x_2 = \frac{1}{\omega_c}(\Delta - x_- - isx_+)$ are the roots of the inverse Green function and where $s = \mathrm{sgn}(\omega_n \mu)$ and

$$x_+ = \sqrt{\frac{\sqrt{(\omega_n^2 + \delta^2 - \mu^2)^2 + 4\mu^2 \omega_n^2} + (\omega_n^2 + \delta^2 - \mu^2)}{2}},$$

$$x_- = \sqrt{\frac{\sqrt{(\omega_n^2 + \delta^2 - \mu^2)^2 + 4\mu^2 \omega_n^2} - (\omega_n^2 + \delta^2 - \mu^2)}{2}}. \quad (12)$$

For the remainder of the paper we assume that Δ is the largest energy scale in the problem, which allows us to make analytical progress. A consequence of this is that the poles of Eq. (11) are sufficiently far into the right half of the complex plane if $\omega_n \ll \Delta$. This allows us to approximate $\int_0^\infty dx \to \int_{-\infty}^\infty dx$ in Eq. (11) as long as the Matsubara frequency ω_n is not too large. However, it can be proven that large ω_n contributions are exponentially suppressed and the approximation is valid. Extending the integration range over the whole real line allows us to use the theory of residues for the integral in Eq. (11) leading to

$$\tilde{\Omega} = 2DT \sum_{\lambda = \pm} \sum_{l=1}^\infty \frac{1}{l} \sum_{\omega_n > 0} e^{-\frac{2\pi l}{\omega_c} x_+} \cos\left(\frac{2\pi l}{\omega_c}(\Delta + \lambda x_-)\right)$$

$$= 4DT \sum_{l=1}^\infty \frac{1}{l} \cos\left(\frac{2\pi l}{\omega_c}\Delta\right) \sum_{\omega_n > 0} e^{-\frac{2\pi l}{\omega_c} x_+} \cos\left(\frac{2\pi l}{\omega_c} x_-\right). \quad (13)$$

We observe that the period of the quantum oscillations is governed by the area enclosed by the above-mentioned [in the discussion below Eq. (2)] circular minimum of the spectrum controlled by the parameter Δ. Since Δ is the largest energy scale, we can treat a part of the expression as an amplitude function that modulates quantum oscillations. Let us rewrite Eq. (13) as

$$\tilde{\Omega} = 4D \sum_{l=1}^\infty \frac{1}{l} \tilde{\Omega}_m \cos\left(\frac{2\pi l}{\omega_c}\Delta\right), \quad (14)$$

where we introduce the quantity

$$\tilde{\Omega}_m = T \sum_{\omega_n > 0} e^{-\frac{2\pi l}{\omega_c} x_+} \cos\left(\frac{2\pi l}{\omega_c} x_-\right), \quad (15)$$

which we call the amplitude function.

We note that the amplitude function is dependent on both the chemical potential and the temperature in a complicated way. The general character of the amplitude function $\tilde{\Omega}_m$ is that if $x_+ > x_-$ it is mainly an exponential decay function with slight oscillations, which requires $\pi^2 T^2 + \delta^2 > \mu^2$. On the other hand, if $\pi^2 T^2 + \delta^2 < \mu^2$ (following from $x_+ < x_-$), the amplitude is mainly an oscillatory function.

Let us consider a zero-temperature limit of the amplitude function, Eq. (15). The discrete sum over Matsubara frequencies becomes an integral. We change the integration

variable from ω to x_+ from Eq. (12) to obtain the expression

$$\tilde{\Omega}_m = \int_{\sqrt{\max(0, \delta^2 - \mu^2)}}^\infty \frac{dx}{2\pi} \frac{(x^2 + \mu^2)^2 - \mu^2 \delta^2}{(x^2 + \mu^2)^{3/2} \sqrt{x^2 + \mu^2 - \delta^2}}$$

$$\times e^{-\frac{2\pi l}{\omega_c} x} \cos\left(\frac{2\pi l}{\omega_c} \mu \sqrt{\frac{x^2 + \mu^2 - \delta^2}{x^2 + \mu^2}}\right). \quad (16)$$

If we furthermore set the chemical potential to lie in-between bands at $\mu = 0$, the above expression is reduced to

$$\tilde{\Omega}_m = \int_\delta^\infty \frac{x \, dx}{2\pi \sqrt{x^2 - \delta^2}} e^{-\frac{2\pi l}{\omega_c} x} = \frac{l\delta}{\omega_c} K_1\left(\frac{2\pi l \delta}{\omega_c}\right), \quad (17)$$

where $K_1(x)$ is the modified Bessel function of the second kind. In fact, for all values of chemical potential in the gap Eq. (16) is exactly equal to Eq. (17). That is, $\tilde{\Omega}_m(\delta, \mu < \delta) = \tilde{\Omega}_m(\delta, \mu = 0)$. That this is true can be seen by coming back to Eq. (10) and performing the outer Matsubara frequency summation. We can then see that the zero-temperature Fermi distribution functions do not influence the integration bounds if $\mu < \delta$. Since there is no explicit μ in the integrand, there is no overall dependence on μ as long as $\mu < \delta$.

B. Dilute disorder

In this section we will introduce disorder to the system but confine the discussion to the zero-temperature $T = 0$ and zero chemical potential $\mu = 0$ case. We will first derive the fermion Green function in the presence of disorder and then we will derive the oscillation amplitude function in the presence of disorder.

To treat disorder we calculate the self-energy of fermions $\hat{\Sigma}$ that modifies the noninteracting Green function $-\hat{G}^{-1} = -\hat{G}_0^{-1} + \hat{\Sigma}$ that in turn enters grand potential in Eq. (7). We model the disorder by a random scalar potential $V(x)$ that is short ranged, that is, $\langle V(x)V(x')\rangle = \gamma^2 \delta(x - x')$. Here, γ is the disorder coupling constant. We consider the overlapping Landau level limit and thus do not introduce the external magnetic field into the calculation of the self-energy [19]. We follow the self-consistent Born approximation treatment [20] that leads to the following equation for the self-energy Σ in momentum basis:

$$\Sigma_p = \gamma^2 \int \frac{d^2 p'}{(2\pi)^2} \frac{1}{i\omega \sigma_0 - H_{p'} - \Sigma_{p'}}. \quad (18)$$

The above integral is computed and we see that the important part of the self-energy is proportional to the identity matrix. We parametrically express the self-energy as $\Sigma = -\sigma_0 i\omega \tilde{\Sigma}$; then Eq. (18) becomes

$$\tilde{\Sigma} = \frac{1}{2\tau} \frac{(1 + \tilde{\Sigma})}{\sqrt{\delta^2 + \omega^2 (1 + \tilde{\Sigma})^2}}, \quad (19)$$

and we denote $1/\tau = m\gamma^2$. It is apparent from the above equation that for zero gap, $\delta = 0$, the self-energy reduces to a standard expression $\tilde{\Sigma} = \frac{-i \, \mathrm{sgn}\omega}{2\tau}$. For $\delta \neq 0$ the exact calculation of $\tilde{\Sigma}$ involves solving a fourth-order algebraic equation but we will see that this is not required in order to find the amplitude function of the quantum oscillations. Noting the specific matrix form of the self-energy the calculation of the

quantum oscillations mirrors the calculation of the previous subsection. The final result given in Eq. (12) and Eq. (13) remains correct but with the replacement (noting that the self-energy is a positive number)

$$\omega_n \to \omega_n(1 + \tilde{\Sigma}). \tag{20}$$

Using the zero temperature limit of Eq. (15) and plugging in zero chemical potential into Eq. (12) together with Eq. (19) and Eq. (20) we can calculate the oscillation amplitude function,

$$\tilde{\Omega}_m = \int_{\max\left(\delta, \frac{1}{2\tau}\right)}^{\infty} \frac{dx}{2\pi} \frac{x^3 - \delta^2 \frac{1}{2\tau}}{x^2 \sqrt{x^2 - \delta^2}} e^{-\frac{2\pi l}{\omega_c} x}. \tag{21}$$

Detailed derivations of formulas (19) and (21) are included in Appendixes A and B respectively. To sum up, in this section we have derived two modified Lifshitz-Kosevich formulas. One, for clean limit and for finite temperature, chemical potential, and the size of the gap, Eq. (15). Another, for finite disorder strength, finite size of the gap, and zero temperature and chemical potential, Eq. (21).

IV. DISCUSSION

A. Recovery of Lifshitz-Kosevich formula

The only case where we can perform the sum over n analytically in Eq. (15) is when we can approximate $\omega_n^2 - \mu^2 + \delta^2 \approx \omega_n^2 - \mu^2$. This implies that $x_+ = \omega_n$ and $x_- = \mu$. In these cases the amplitude reads

$$\tilde{\Omega}_m^{LK} = T \frac{1}{\sinh\left(\frac{2\pi^2 l T}{\omega_c}\right)} \cos\left(\frac{2\pi l}{\omega_c}\mu\right) \tag{22}$$

and the oscillatory potential is of the standard Lifshitz-Kosevich type [1,3],

$$\tilde{\Omega}^{LK} = DT \sum_{\lambda=\pm} \sum_{l=1}^{\infty} \frac{1}{l \sinh\left(\frac{2\pi^2 l T}{\omega_c}\right)} \cos\left(\frac{2\pi l}{\omega_c}(\Delta + \lambda|\mu|)\right). \tag{23}$$

This is essentially the result of a metal with two Fermi surfaces cut out of the spectrum of our system by a constant energy level surface $E = \mu$.

There are three situations in which the above approximation holds: (I) the gap is zero ($\delta = 0$), so the system has a pair of coinciding Fermi surfaces, one for the electrons and one for the holes; (II) the temperature is much larger than the gap, ($T \gg \delta$), so the temperature blurs the effect of the finite gap; (III) the chemical potential is much larger than the gap ($\mu \gg \delta$), so, again, the gap is thermodynamically insignificant [for the last case and for $T = 0$ we will derive an approximate expression, Eq. (26)].

The second modified LK formula for $\mu = 0$, $T = 0$, $1/(2\tau) \neq 0$ [Eq. (21)] becomes very simple in the limit $\delta = 0$. It then reads

$$\tilde{\Omega}^{LK} = D \sum_{l=1}^{\infty} \frac{2\omega_c}{\pi l^2} e^{-\frac{\pi l}{\tau \omega_c}} \cos\left(\frac{2\pi l}{\omega_c}\Delta\right). \tag{24}$$
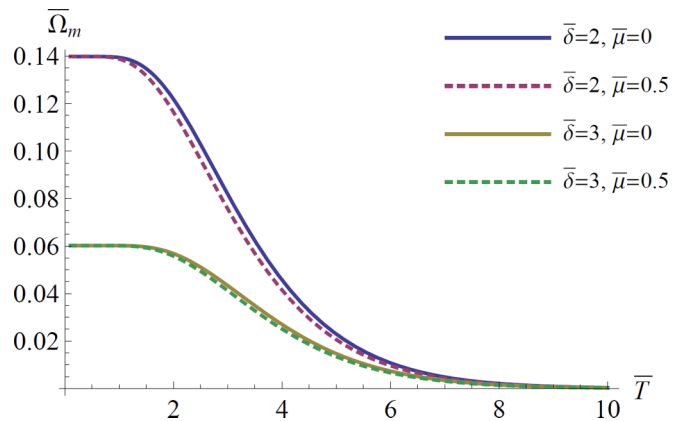

FIG. 2. Amplitude function $\overline{\Omega}_m$ as a function of the dimensionless temperature $\overline{T}$. We observe that the curves decay exponentially for $\overline{T} > \overline{\delta}$ and flatten out for $\overline{T} < \overline{\delta}$. Also, finite chemical potential $\overline{\mu}$ modifies the result only slightly.

We see that, in this case, the oscillation formula reduces to a standard expression with Dingle temperature [3].

B. Analysis of the amplitude function $\tilde{\Omega}_m$, clean case

In the following, we will analyze the modified LK formulas, Eq. (15) and Eq. (21), numerically and will derive some approximate formulas. We begin with Eq. (13).

To analyze the behavior of the amplitude $\tilde{\Omega}_m$ we introduce $y = 2\pi l/\omega_c$. We then analyze the expression

$$\overline{\Omega}_m = \pi \tilde{\Omega}_m y \tag{25}$$

and how it depends on three dimensionless variables, $\overline{T} = \pi T y$, $\overline{\mu} = \mu y$, and $\overline{\delta} = \delta y$. We now explore the behavior of this function, $\overline{\Omega}_m = \overline{\Omega}_m(\overline{T}, \overline{\delta}, \overline{\mu})$.

We first consider the case $\overline{\mu} < \overline{\delta}$ and explore the dependence $\overline{\Omega}_m = \overline{\Omega}_m(\overline{T}, \overline{\delta})$. Usually, the amplitude is damped exponentially with increasing $\overline{T}$ [see Eq. (23) for a specific example]. However, from Fig. 2 we observe that, given that the system is gapped and $\overline{T} < \overline{\delta}$, the amplitude remains constant, instead of decreasing exponentially. We can see that the gap also damps the amplitude because the values of the amplitude for $\overline{T} = 0$ do not coincide. We demonstrate how the amplitude is damped by the gap in Fig. 3. In Fig. 4 we show a contour plot of the amplitude function for $\overline{\mu} = 0$. From Figs. 2 and 3 we can see that small values of $\overline{\mu}$ modify the $\overline{\mu} = 0$ result only slightly and we note that in Fig. 4.

We can now visualize the amplitude $\tilde{\Omega}_m$ as a function of inverse magnetic field y for $\tilde{\mu} = 0$. If we choose specific πT and δ the amplitude function $\overline{\Omega}_m$ takes values along the line $\overline{\delta} = k\overline{T}$ in Fig. 4, where $k = \delta/(\pi T)$.

Now we consider the case $\overline{\mu} > \overline{\delta}$. In principle, in this case chemical potential cuts the spectrum in two rings and quantum oscillations have two frequencies, similar to Eq. (23). The amplitude then must oscillate with a frequency $\sqrt{\overline{\mu}^2 - \overline{\delta}^2}$. We show in Fig. 5 how the amplitude oscillates in the region $\overline{\mu} > \overline{\delta}$ and is suppressed as soon as $\overline{\mu} < \overline{\delta}$ (unless $\overline{T} > \overline{\delta}$). In general it is difficult to describe the behavior when the chemical potential is both nonzero and comparable to all other energy scales, because it is an interpolation between no-Fermi surface
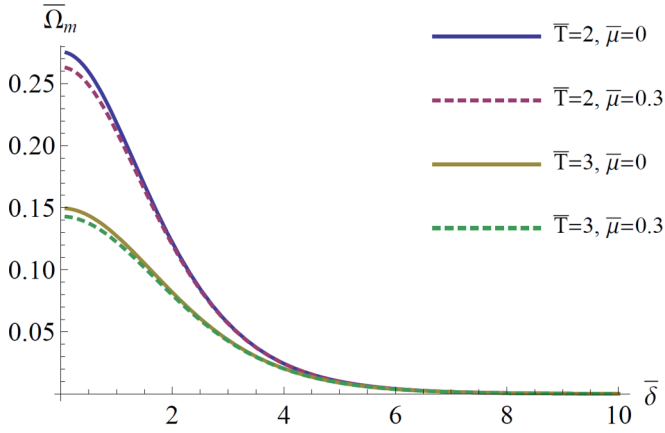
FIG. 3. Amplitude function $\overline{\Omega}_m$ as a function of the dimensionless gap $\overline{\delta}$. We clearly see that for various values of temperature whenever $\overline{\delta} > \overline{T}$ the amplitude converges to the same curve characterized by the gap only (that part of the curve also does not depend on the chemical potential as long as $\overline{\mu} < \overline{\delta}$).

behavior with a single oscillation frequency Δ and two-Fermi surfaces behavior with oscillation frequencies (approximately) $\Delta \pm \mu$.

Next, we discuss the zero-temperature limit of Eq. (16), that is $\overline{\Omega}_m = \overline{\Omega}_m(\overline{T} = 0, \overline{\delta}, \overline{\mu})$. For $\overline{\mu} < \overline{\delta}$ Eq. (17) is the exact result and for $\overline{\mu} > \overline{\delta}$ we can derive the following approximation:

$$\overline{\Omega}_m \approx \frac{1}{2}\sqrt{1 - \frac{\overline{\delta}^2}{\overline{\mu}^2}} \cos{(\sqrt{\overline{\mu}^2 - \overline{\delta}^2})}. \tag{26}$$
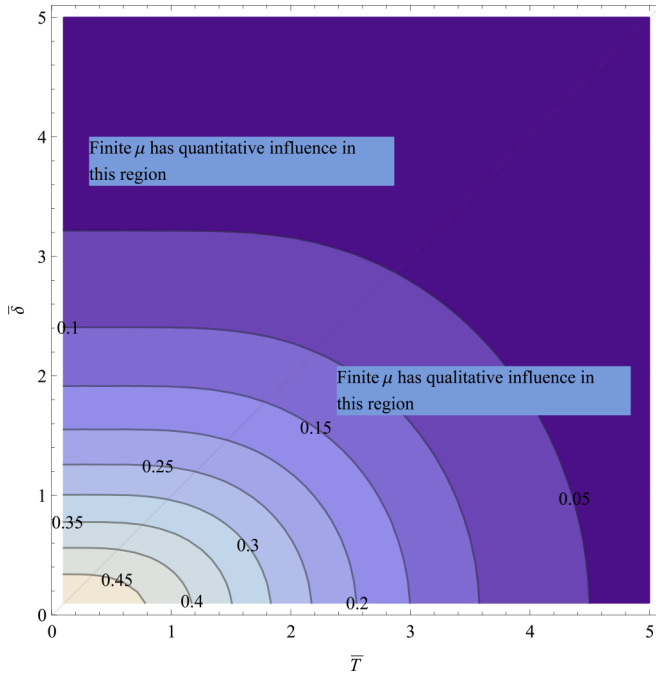

FIG. 4. Contour plot of the amplitude $\overline{\Omega}_m$ as a function of dimensionless gap $\overline{\delta}$ and dimensionless temperature $\overline{T}$ for $\overline{\mu} = 0$. If $\overline{\mu} \neq 0$, the upper side of the graph would not be modified much, whereas the lower side of the graph would be.
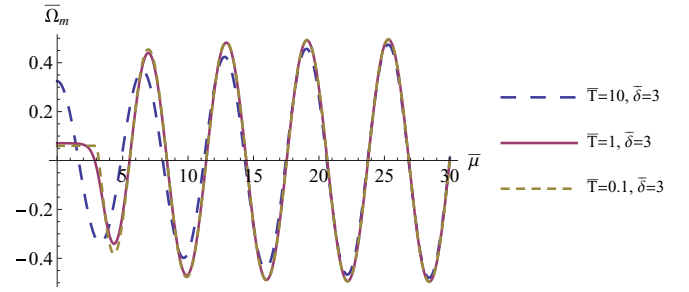

FIG. 5. Plots of $\overline{\Omega}_m$ for different values of $\overline{T}$ but fixed $\overline{\delta}$. We have adjusted the scale to visualize all three functions in one plot because of the severe damping in the large-temperature case. We observe that all three functions oscillate with a period 2π for large $\overline{\mu}$; however, as $\overline{\mu}$ hits the gap $\overline{\delta}$, the function either keeps oscillating (high temperature $\overline{T}$ case) or goes to a constant (low temperature case).

This equation is derived by observing the suppressing exponential in the integrand of Eq. (17). It means that the majority of the contribution comes from low values of $\overline{x}$. We, therefore, neglect x everywhere in the integrand, except the exponent, and perform the integral, resulting in Eq. (17). We can deduce from the above expression that as a function of $\frac{2\pi l}{\omega_c}$ the quantum oscillations do not decay to zero [unlike in the $\overline{\delta} > \mu$ case, Eq. (17)], but oscillate with frequency $\Delta \pm \sqrt{\mu^2 - \delta^2}$. Before moving to our last approximation we will quantify how accurate these approximations are: high-temperature equation Eq. (23) and low-temperature Eq. (17). The validity of these as an approximation to the full solution Eq. (15) is shown in Fig. 6. In order to gauge the faithfulness of the approximations compared to the exact result, Eq. (15), we arbitrarily choose 2% relative accuracy as a criterion. Formula Eq. (26) cannot be evaluated in such a way as it is nonpositive. However, it is almost exact for $\overline{\mu} > 2\pi$ and for $\overline{\mu}$ approaching $\overline{\delta}$ it collapses to the value given by Eq. (17).

These approximations fall short in the parameter region where the temperature is higher than the gap, but it is not yet big enough so that Eq. (22) is accurate. For this purpose we introduce another type of approximation than considered above. It can be shown that if the temperature is sufficiently high, most of the contribution to the amplitude function is contained in the first few terms in Eq. (15). In other words, we propose the following approximation:

$$\tilde{\Omega}_m^{(1)} \approx T \sum_{\omega_n > 0}^{\omega_0} e^{-\frac{2\pi l}{\omega_c}x_+} \cos\left(\frac{2\pi l}{\omega_c}x_-\right), \tag{27}$$

$$\tilde{\Omega}_m^{(2)} \approx T \sum_{\omega_n > 0}^{\omega_1} e^{-\frac{2\pi l}{\omega_c}x_+} \cos\left(\frac{2\pi l}{\omega_c}x_-\right), \tag{28}$$

where $\tilde{\Omega}_m^{(1)}$ contains the first Matsubara mode, while $\tilde{\Omega}_m^{(2)}$ contains the first two of them. Due to the fact that we have increasingly suppressed exponentials in Eq. (15), this approximation gives a good numerical agreement in a large part of the $\overline{\delta} - \overline{T}$ plane as demonstrated in Fig. 7. If we combine the zero-temperature approximation and the two-term approximation, we can describe the amplitude function Eq. (15) for all values $\overline{\delta}$ and $\overline{T}$.
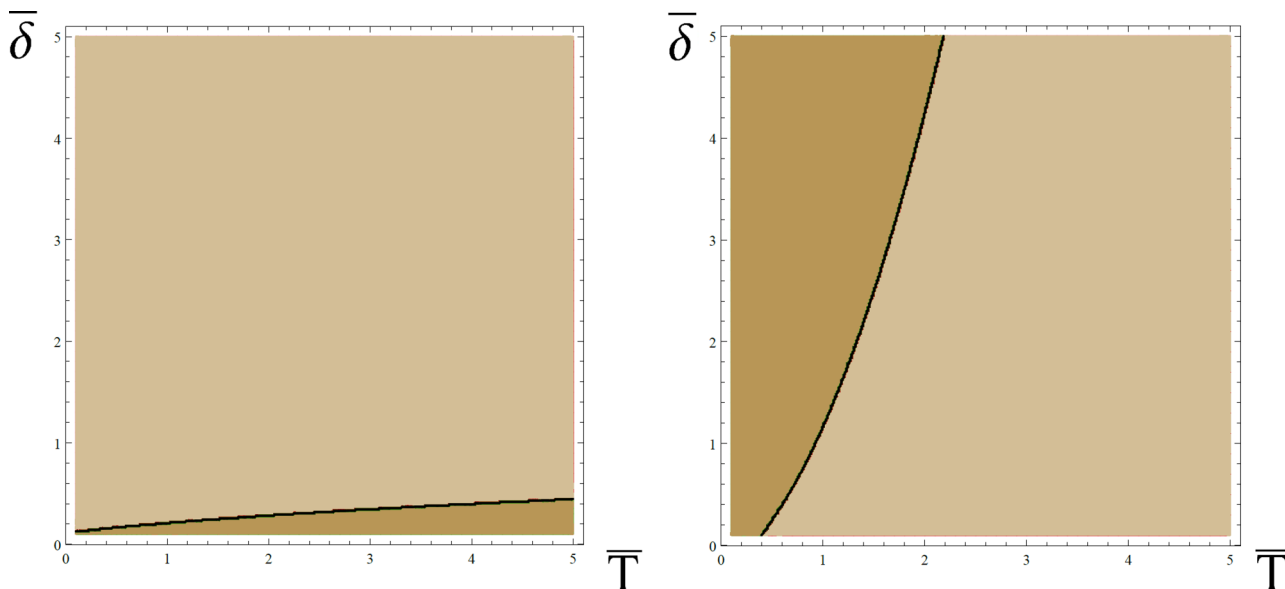
FIG. 6. Validities of formulas Eq. (23) (on the left) and Eq. (17) (on the right) as approximations to formula Eq. (15). The darker region represents a faithful approximation according to the 2% criterion.

C. Analysis of the amplitude function $\tilde{\Omega}_m$, disordered case

The presence of impurities provides an additional damping source for the amplitude function. Although it may appear that the temperature contribution is similar to the disorder contribution, they contribute in a different way, as shown in Fig. 8 due to the presence of the gap. If disorder is weaker than the gap, $1/2\bar{\tau} < \bar{\delta}$, the lower limit of integration is $\bar{\delta}$ in Eq. (21) and we can isolate a perfect linear damping in this region (with respect to $1/2\bar{\tau}$),

$$\overline{\Omega}(\bar{\delta}, \bar{\tau} < \bar{\delta}) = \bar{\delta} f_1(\bar{\delta}) + \frac{1}{2\bar{\tau}} f_2(\bar{\delta}), \qquad (29)$$

where the functions f_1 and f_2 can be read off of Eq. (21). For $1/2\bar{\tau} \gg \bar{\delta}$ the grand canonical potential tends to

$$\overline{\Omega}(\bar{\delta}, \bar{\tau} \gg \bar{\delta}) \approx \frac{1}{2} \exp\left(-\frac{1}{2\bar{\tau}}\right). \qquad (30)$$

The absence of $\bar{\delta}$ from the above equation means that for sufficiently strong disorder the quantum oscillation amplitude is not sensitive to the gap [temperature behaves similarly; see Eq. (22)].

V. CONCLUSION AND OUTLOOK

In this work we have derived two expressions for the oscillatory part of the grand potential. One, as a function of
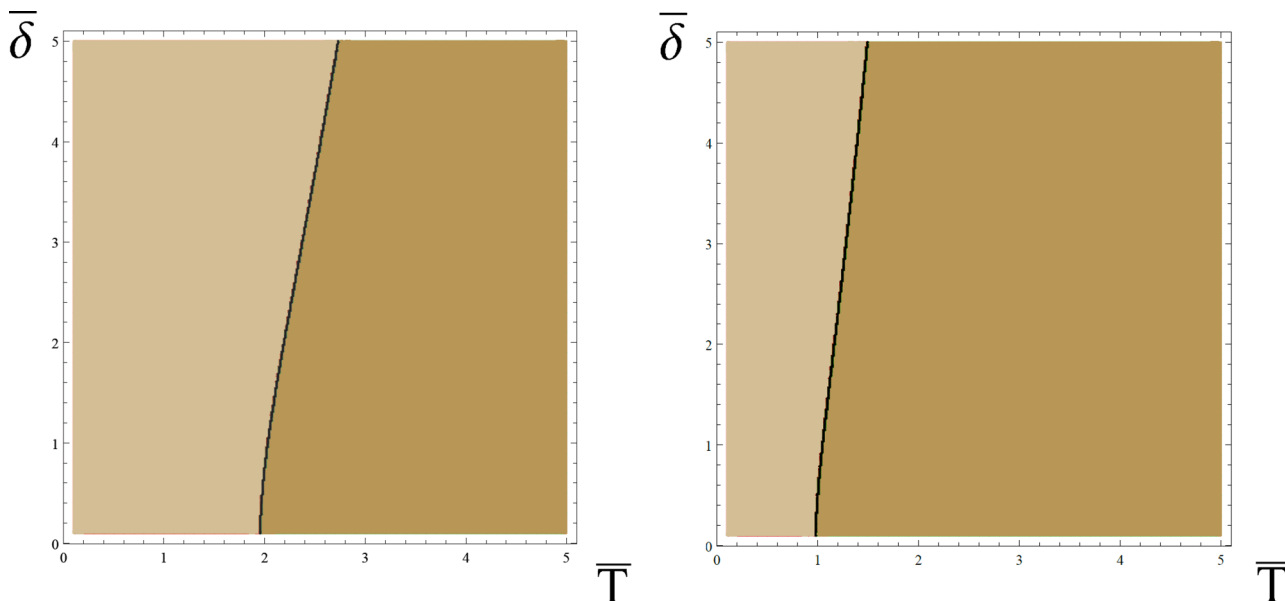

FIG. 7. Validities of formulas Eq. (27) (on the left) and Eq. (28) (on the right) as approximations to formula Eq. (15).
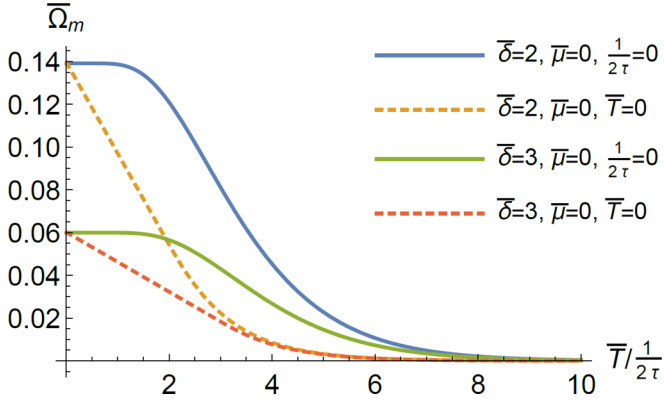
FIG. 8. Comparison of the amplitude function Eq. (15) as a function of temperature (setting the strength of disorder to zero) and as a function of disorder strength (setting the temperature to zero).

magnetic field, temperature, the gap of the system, chemical potential, Eq. (13) describing anomalous quantum oscillations in inverted insulators. Another expression, Eq. (21) describes the same quantity as a function of disorder strength $1/2\tau$ (with $\mu = 0, T = 0$). This expression could be analyzed in different experimental regimes and a determination of system parameters, such as disorder, could be attempted. Also, an independent measurement of the gap could be performed. As already discussed by other authors a candidate system to observe the discussed physics is bilayer graphene [11,14,17] in which the required conditions can be achieved. A possible line of research in the future consists of investigating the effects of inelastic scattering in the present setup and seeing to what extent the LK formula is modified once interactions are included.

APPENDIX A: SELF-CONSISTENT BORN APPROXIMATION CALCULATION OF IMPURITY GREEN FUNCTION

The SCBA equation we are trying to solve is

$$\Sigma_p = \gamma^2 \int \frac{d^2 p'}{(2\pi)^2} (i\omega\sigma_0 - H_{p'} - \Sigma_{p'})^{-1}, \qquad \text{(A1)}$$

where the Hamiltonian of the system is the following:

$$H_p = \left[\tilde{\Delta} - \frac{p_x^2 + p_y^2}{2\tilde{m}} \right] \sigma_z + \frac{v}{2\tilde{m}}(p_x^2 - p_y^2)\sigma_x + \frac{p_x p_y v}{\tilde{m}}\sigma_y.$$

1. Non-self-consistent lowest order perturbation

We neglect the self-energy on the right hand side of Eq. (A1) in this section. We need to invert a matrix of the form

$$(a\sigma_0 - b\sigma_x - c\sigma_y - d\sigma_z)^{-1} = \frac{a\sigma_0 + b\sigma_x + c\sigma_y + d\sigma_z}{a^2 - b^2 - c^2 - d^2},$$

where σ_0 is a 2×2 identity matrix and $\sigma_{x,y,z}$ are Pauli matrices. For our case the expression $(i\omega\sigma_0 - H_{p'})^{-1}$ is expressed by choosing

$$a = i\omega,$$

$$b = \frac{v}{2\tilde{m}}(p_x^2 - p_y^2),$$

$$c = \frac{p_x p_y v}{\tilde{m}},$$

and

$$d = \tilde{\Delta} - \frac{p_x^2 + p_y^2}{2\tilde{m}}.$$

We rename p' to p and drop p from the left hand side because it is obvious that the self-energy is not going to depend on the momentum. We also go to cylindrical coordinates in the integrand by denoting $p_x^2 + p_y^2 = p^2$ and we observe that terms proportional to σ_x and σ_y terms vanish due to the symmetry of the integrand. Then, Eq. (A1) becomes

$$\Sigma = \gamma^2 \int \frac{d^2 p}{(2\pi)^2} \frac{i\omega\sigma_0 + \left(\tilde{\Delta} - \frac{p^2}{2\tilde{m}}\right)d\sigma_z}{(i\omega)^2 - \tilde{\Delta}^2 + \frac{\tilde{\Delta}}{\tilde{m}}p^2 - p^4\frac{(1+v^2)}{4\tilde{m}^2}}.$$

Like in the main text, we change variables,

$$\tilde{\Delta} = \Delta\sqrt{1 + v^2},$$

$$\tilde{m} = m\sqrt{1 + v^2},$$

and

$$v = \delta/\Delta$$

to arrive at

$$\Sigma = \gamma^2 \int \frac{d^2 p}{(2\pi)^2} \frac{(i\omega)\sigma_0 + \left(\sqrt{\Delta^2 + \delta^2} - \frac{p^2}{2m\sqrt{1+v^2}}\right)\sigma_z}{(i\omega)^2 - \left(\frac{p^2}{2m} - \Delta\right)^2 - \delta^2}.$$

We go to cylindrical coordinates explicitly and change the integration variable to $x = p^2/2m$ to obtain

$$\Sigma = -\frac{m\gamma^2}{2\pi} \int_0^\infty dx \frac{i\omega\sigma_0 + \left(\sqrt{\Delta^2 + \delta^2} - \frac{x}{\sqrt{1+v^2}}\right)\sigma_z}{\delta^2 + \omega^2 + (x - \Delta)^2}.$$

We compute this integral for the σ_0 part first; it gives

$$\Sigma_0 = -\frac{m\gamma^2}{2\pi} i\omega\sigma_0 \int_0^\infty dx \frac{1}{\delta^2 + \omega^2 + (x - \Delta)^2}$$

$$= -\frac{m\gamma^2}{2\pi} i\omega\sigma_0 \frac{\pi + 2\arctan\frac{\Delta}{\sqrt{\delta^2 + \omega^2}}}{2\sqrt{\delta^2 + \omega^2}}.$$

Throughout this paper we assume that Δ is always much bigger than other energy scales; therefore, the arctan function can be simplified to $\pi/2$. This then results in

$$\Sigma_0 = -\frac{m\gamma^2}{2\pi} i\omega\sigma_0 \frac{\pi}{\sqrt{\delta^2 + \omega^2}}.$$

Next, the z part of the self-energy,

$$\Sigma_z = -\frac{m\gamma^2}{2\pi\sqrt{\delta^2 + \Delta^2}} \int_0^\infty dx \frac{(\Delta^2 + \delta^2 - x\Delta)}{\delta^2 + \omega^2 + (x - \Delta)^2},$$

does not converge, it actually diverges logarithmically. First, we use the fact that Δ is the biggest energy scale to approximate

$$\Sigma_z \approx -\frac{m\gamma^2}{2\pi} \int_0^\infty dx \frac{\Delta - x}{\delta^2 + \omega^2 + (x - \Delta)^2},$$

where we neglected δ^2/Δ term in comparison to Δ. Then, we can calculate this integral analytically, by substituting the cutoff $\Lambda \gg \Delta$ for the upper range of integration,

$$\Sigma_z = \frac{m\gamma^2}{4\pi} \log \frac{\delta^2 + \omega^2 + (\Lambda - \Delta)^2}{\delta^2 + \omega^2 + \Delta^2} \approx \frac{m\gamma^2}{2\pi} \log\left(\frac{\Lambda}{\Delta} - 1\right).$$

Being proportional to the σ_z matrix, it renormalizes the $\tilde{\Delta}$ part of the original Hamiltonian. We can absorb this term by redefining Δ. Since Δ does not influence the amplitude of the quantum oscillations, we can omit this term. The final self-energy thus reads

$$\Sigma = -\frac{m\gamma^2}{2\pi} i\omega\sigma_0 \frac{\pi}{\sqrt{\delta^2 + \omega^2}}.$$

If $\delta = 0$, this reduces to

$$\Sigma = -i\frac{m\gamma^2}{2}\sigma_0 \text{sgn}(\omega).$$

Usually we would denote this self-energy as

$$\Sigma = -i\frac{1}{2\tau}\sigma_0 \text{sgn}(\omega), \tag{A2}$$

so we identify a parameter

$$\frac{1}{2\tau} = \frac{m\gamma^2}{2}$$

that gives an energy scale for the influence of disorder.

2. Self-consistent calculation

We return to the SCBA equation,

$$\Sigma = \gamma^2 \int \frac{d^2p}{(2\pi)^2} (i\omega\sigma_0 - H_p - \Sigma)^{-1}.$$

We insert the Σ found in the last section, Eq. (A2),

$$\Sigma = -\frac{1}{2\tau} i\omega\sigma_0 \frac{1}{\sqrt{\delta^2 + \omega^2}},$$

on the right hand side,

$$\gamma^2 \int \frac{d^2p}{(2\pi)^2} ((i\omega)\sigma_0 - H_p - \Sigma)^{-1}$$

$$= \gamma^2 \int \frac{d^2p}{(2\pi)^2} \left(i\omega\left(1 + \frac{1}{2\tau}\frac{1}{\sqrt{\delta^2 + \omega^2}}\right)\sigma_0 - H_p \right)^{-1}.$$

The bracket that multiplies $i\omega$ is a positive factor that rescales ω in the integral but leaves the rest of the integral the same. This means that the previous calculation can be applied here. In fact, we parametrize the self-energy this way,

$$\Sigma = -\sigma_0 i\omega\tilde{\Sigma}.$$

The self-consistent equation becomes

$$-\sigma_0 i\omega\tilde{\Sigma} = \gamma^2 \int \frac{d^2p}{(2\pi)^2} (i\omega(1 + \tilde{\Sigma})\sigma_0 - H_p)^{-1}.$$

We define

$$\tilde{\omega} = \omega(1 + \tilde{\Sigma}) \tag{A3}$$

and the SCBA equation becomes

$$-\sigma_0 i\omega\tilde{\Sigma} = \gamma^2 \int \frac{d^2p}{(2\pi)^2} (i\tilde{\omega}\sigma_0 - H_p)^{-1}.$$

From the previous section we know the functional form of the above integral; it is

$$-\sigma_0 i\omega\tilde{\Sigma} = -\frac{1}{2\tau} i\tilde{\omega}\sigma_0 \frac{1}{\sqrt{\delta^2 + \tilde{\omega}^2}}.$$

We restore $\tilde{\omega}$ and simplify,

$$\tilde{\Sigma} = \frac{1}{2\tau} \frac{(1 + \tilde{\Sigma})}{\sqrt{\delta^2 + \omega^2(1 + \tilde{\Sigma})^2}}.$$

This is our algebraic self-consistency equation for the self-energy. We verify that, if $\delta = 0$, we reproduce $\tilde{\Sigma} = 1/2\tau|\omega|$. For other values of δ, the result is going to be different. For convenience, we denote $\tilde{\Sigma} = \epsilon$ and $\frac{1}{2\tau} = t$ to have

$$\epsilon = t\frac{1 + \epsilon}{\sqrt{\delta^2 + \omega^2(1 + \epsilon)^2}}, \tag{A4}$$

which is a fourth order algebraic equation.

APPENDIX B: ZERO-TEMPERATURE GRAND POTENTIAL Ω

The result for zero-temperature grand potential in the clean limit ($t = 0$) and for finite gap δ is

$$\Omega = \frac{1}{2\pi} \int_0^\infty e^{-\sqrt{\delta^2 + \omega^2}} d\omega,$$

where the variable $2\pi l/\omega_c$ is set to one. Now, taking into account SCBA self-energy contribution to the Green function, Eq. (A3), the grand potential reads

$$\Omega(t) = \frac{1}{2\pi} \int_0^\infty e^{-\sqrt{\delta^2 + \omega^2(1 + \epsilon(\omega))^2}} d\omega,$$

where we note that ϵ depends on ω via Eq. (A4). We change the variable of integration from ω to y,

$$y = \sqrt{\delta^2 + \omega^2(1 + \epsilon(\omega))^2}, \tag{B1}$$

and the integral is transformed to

$$\Omega(t) = \int_a^b e^{-y} \frac{dy}{|dy/d\omega|}, \tag{B2}$$

where a and b are the integration limits to be determined and we also need to compute the Jacobian.

To determine the bounds of integration, we first derive limiting expressions of Eq. (A4) for $\omega \to 0$ and $\omega \to \infty$. For $\omega \to 0$, there are two cases. If $\delta > t$, then $\epsilon \to t/(\delta - t)$. If $\delta < t$, then $\epsilon \to \sqrt{t^2 - \delta^2}/\omega$. For $\omega \to \infty$ it follows that $\epsilon \to t/\omega$ for any values of δ and t. Using these results and Eq. (B1) we find the limits of integration a and b,

$$y(\omega = 0, t < \delta) = \delta, \tag{B3}$$

$$y(\omega = 0, t > \delta) = t, \tag{B4}$$

$$y(\omega = \infty) = \infty. \tag{B5}$$

We move on to calculate the Jacobian of the integral. We consider $y = y(\omega)$ and $\epsilon = \epsilon(\omega)$ as functions of ω. Using Eqs. (A4) and (B1) we can eliminate ω,

$$y(\omega) = t \frac{1 + \epsilon(\omega)}{\epsilon(\omega)}. \tag{B6}$$

Differentiating both sides with respect to ω gives

$$\frac{dy(\omega)}{d\omega} = -\frac{t}{\epsilon(\omega)^2} \frac{d\epsilon(\omega)}{d\omega}. \tag{B7}$$

We differentiate both sides of Eq. (A4) with respect to ω to obtain

$$\frac{d\epsilon(\omega)}{d\omega} = t \frac{\delta^2 \frac{d\epsilon(\omega)}{d\omega} - \omega(1 + \epsilon(\omega))^3}{(\delta^2 + \omega^2(1 + \epsilon(\omega))^2)^{3/2}}. \tag{B8}$$

We eliminate $\epsilon(\omega)$ between Eq. (B1) and Eq. (B6) and express ω in terms of $y(\omega)$,

$$\omega = \frac{|y(\omega) - t|\sqrt{y(\omega)^2 - \delta^2}}{y(\omega)}. \tag{B9}$$

We substitute Eq. (B9) to eliminate ω from the nominator of Eq. (B8). Also, using Eq. (B6) we substitute $\epsilon(\omega)$ into the nominator of Eq. (B8). Finally, we simplify the denominator of Eq. (B8) using Eq. (B1). From the resulting equation, we algebraically find $d\epsilon/d\omega$,

$$\frac{d\epsilon}{d\omega} = -\frac{s(y - t)y^2 \sqrt{y^2 - \delta^2} t}{(y - t)^2 (y^3 - \delta^2 t)}, \tag{B10}$$

where $s(x) = \mathrm{sgn}(x)$. Finally, we substitute Eq. (B10) and $\epsilon(\omega)$ from Eq. (B6) into Eq. (B7) to obtain the Jacobian,

$$\frac{dy}{d\omega} = \frac{s(y - t)y^2 \sqrt{y^2 - \delta^2}}{y^3 - \delta^2 t}. \tag{B11}$$

The bounds in Eqs. (B3)–(B5) and Eq. (B11) are substituted into Eq. (B2) to obtain

$$\Omega(t) = \frac{1}{2\pi} \int_{\max(\delta, t)}^{\infty} e^{-\frac{2\pi l y}{\omega_c}} dy \frac{|y^3 - \delta^2 t|}{y^2 \sqrt{y^2 - \delta^2}},$$

where we have restored the magnetic field dependence in the argument of the exponent. We observe that both for $\delta > t$ and for $\delta < t$ the expression $y^3 - \delta^2 t$ is always positive for the appropriate ranges of integration, so we can drop the modulus and thus arrive at Eq. (21).

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