Two loop stress-energy tensor for inflationary scalar electrodynamics

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We calculate the expectation value of the coincident product of two field strength tensors at two loop order in scalar electrodynamics on de Sitter background. The result agrees with the stochastic formulation which we have developed in a companion paper [T. Prokopec, N. C. Tsamis, and R. P. Woodard, Ann. Phys. (N.Y.) **323**, 1324 (2008)] for the nonperturbative resummation of leading logarithms of the scale factor. When combined with a previous computation of scalar bilinears [T. Prokopec, N. C. Tsamis, and R. P. Woodard, Classical Quantum Gravity **24**, 201 (2007)], our current result also gives the two loop stress-energy tensor for inflationary scalar electrodynamics. This shows a secular decrease in the vacuum energy which derives from the vacuum polarization induced by the inflationary production of charged scalars.

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I. INTRODUCTION

This is the third and final paper in a series with the goal of establishing a stochastic theory of inflation for scalar quantum electrodynamics (SQED). Stochastic SQED is outlined in Ref. [1]. In this paper and in Ref. [2] we compare the predictions of stochastic SQED with explicit and exact perturbative calculations. In particular, Ref. [2] gives the mostly scalar contribution to the two loop stress energy, and here we calculate the mostly photon contribution.

Stochastic inflationary theory [3–6] was developed primarily in order to capture the effects of small scale stress-energy fluctuations (large momenta) onto large scale fluctuations, which are observed by current large scale structure and cosmological microwave background radiation measurements. Furthermore, stochastic inflation has also been used to study global structure and dynamics [7,8] of inflationary universe models. Even though corrections to stochastic inflationary theories have not yet been systematically studied, these type of studies are feasible within the *in-in* path integral formalism [9]. A phase space formulation of stochastic inflationary theory is given in Ref. [10], where a quantum Liouville equation for the Wigner function was obtained, which is the suitable quantum generalization of the Fokker-Planck equation for the (classical) density function. More recently, a phase space formulation has been used for computing the non-Gaussianity of cosmological perturbations [11–16] in various inflationary models.

We are interested in the stochastic theory of inflation mainly because it captures correctly the leading logarithmic behavior (in the scale factor) of quantum field theories during inflation. In Ref. [1] we have proven that a stochastic theory of scalar electrodynamics (obtained by integrating out the photon field and by subsequently stochasticizing the resulting effective scalar field theory) correctly reproduces the leading secular logarithms to all orders in perturbation theory. The agreement is made explicit by a comparison between the mostly scalar contribution to the two loop stress-energy tensor in de Sitter inflation and the corresponding stochastic prediction. The stochastic theory of inflation goes far beyond perturbation theory, in that it provides an effective means for resummation of the leading logarithm contributions [1]. Unlike full quantum field theories, which cannot be generally solved, stochastic theories are classical, and hence they are relatively easily solved. Indeed, solving a stochastic theory is tantamount to solving the corresponding Fokker-Planck equation for the density function [5].

One of the important applications of stochastic theories is that they permit one to address the question of gravitational backreaction in accelerating space-times [17,18], and can thus answer one of the fundamental unsolved problems of inflationary cosmology: how do quantum effects change the dynamics of space-time during inflation. More broadly, this will teach us something about the infrared sector of quantum gravity and the cosmological constant problem. Ultimately one would like to understand how quantum fluctuations of gravitons affect inflation. The hope is that this question can be answered using a stochastic theory of inflation [19].

Quite a lot is already known about quantum (radiative) effects in inflationary space-times. Perturbative analysis of scalar electrodynamics during de Sitter inflation shows that the photon acquires a mass during inflation [1,20–25], which can be of relevance for generation of the cosmological magnetic fields [20,23,25–27]. The scalar field of SQED also acquires a mass during inflation, but—unlike the photon mass [1]—the scalar mass remains perturbatively small [1]. Similarly, the fermions [28,29] of Yukawa theory acquire mass while the scalars remain light [30,31]. In self-interacting scalar theories, the quantum loop effects during inflation can contribute to the stress energy like a phantom field [32,33], whose equation of state is even more negative than the pressure of a vacuum filled with a positive cosmological term. Although this generally betokens instability [34], the phenomenon in this case is self-limiting [35,36]. The quantum effects of gravitons on the dynamics of fermions and scalars during inflation have also been studied [37–40], and it is now known that gravity induces a logarithmic correction to the fermionic wave function at one loop while no such correction occurs to a massless, minimally coupled scalar. The question of how large quantum corrections to the inflationary observables actually are has been addressed by a number of authors [41–52], but as yet no definite answer can be provided to that question.

This paper is organized as follows. In Sec. II we introduce the relevant scalar and gauge field propagators for de Sitter space and discuss some of their properties. In Sec. III we compute the one loop expectation value of the SQED stress-energy tensor in de Sitter space. In Sec. IV we discuss some general properties of the stress-energy tensor and its constituents. We also employ previous work [2] to give the two loop contributions from the scalar kinetic and potential energies. Section V is the central part of the paper, where we compute the two loop field strength bilinear. In Sec. VI we obtain the total, two loop stress-energy tensor and we discuss the result. Appendixes A, B, and C give identities and integrals useful to readers who wish to reproduce the calculation.

II. THE PROPAGATORS IN DE SITTER SPACE

In this paper we approximate inflationary space-times by de Sitter space. The de Sitter metric with flat spatial sections is

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2), \tag{1}$$

where $a = a(\eta)$ is the scale factor, which in de Sitter space has the form

$$a = -\frac{1}{H\eta} \qquad (\eta < 0). \tag{2}$$

Here H is the Hubble parameter and η denotes conformal time. Furthermore, it is convenient to define a de Sitter invariant function $y = y(x; x')$ between two space-time points x^μ and x'^ν. In flat coordinates (1),

$$\begin{aligned} y &= aa'H^2\Delta x^2, \\ \Delta x^2 &= -(|\eta - \eta'| - i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2 \end{aligned} \tag{3}$$

where $a \equiv a(\eta)$ and $a' \equiv a(\eta')$. There is a simple relation between y and the geodesic distance $\ell = \ell(x; x')$ in de Sitter space,

$$y = 4\sin^2\Big(\frac{H\ell}{2}\Big). \tag{4}$$

We work here in the Schwinger-Keldysh formalism [53–56], which is suitable for describing quantum field theory dynamics. In short, the vertices of Feynman diagrams have two polarities: *plus* (+) and *minus* (−). Because pairs of vertices have four possible polarities, there are four propagators which we denote by $i\Delta_{++}$, $i\Delta_{+-}$, $i\Delta_{-+}$, and $i\Delta_{--}$, respectively. They are obtained from the Feynman propagator $i\Delta \equiv i\Delta_{++}$ by replacing $y \equiv y_{++} = y(x_+; x'_+)$ in Eq. (3) by y_{+-}, y_{-+}, and y_{--}, respectively, defined by

$$\begin{aligned} y_{+-} &= aa'H^2\Delta x^2_{+-}, \\ \Delta x^2_{+-} &= -(\eta - \eta' + i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2, \end{aligned} \tag{5}$$

$$\begin{aligned} y_{-+} &= aa'H^2\Delta x^2_{-+}, \\ \Delta x^2_{-+} &= -(\eta - \eta' - i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2, \end{aligned} \tag{6}$$

$$\begin{aligned} y_{--} &= aa'H^2\Delta x^2_{--}, \\ \Delta x^2_{--} &= -(|\eta - \eta'| + i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2 . \end{aligned} \tag{7}$$

A. The massless minimally coupled scalar propagator

The Feynman propagator for a minimally coupled massless scalar field in de Sitter space obeys the differential equation

$$\sqrt{-g}\Box i\Delta(x; x') = i\delta^D(x - x'), \tag{8}$$

where $\Box = (-g)^{-1/2}\partial_\mu g^{\mu\nu}\sqrt{-g}\partial_\nu$ denotes the scalar d'Alembertian, $g = \det[g_{\mu\nu}]$, and δ^D is the D-dimensional Dirac delta function. The addition of a mass term to (8) results in a de Sitter invariant propagator which was first constructed by Chernikov and Tagirov [57]. However, there is no de Sitter invariant solution in the massless case [58].

The problem can be circumvented by adding de Sitter breaking contributions to the propagator. A natural way of obtaining the form of the de Sitter breaking term is to consider quasi-de Sitter space in which the de Sitter symmetry is mildly broken, such that $\epsilon \equiv -\dot{H}/H^2 \ll 1$ and $\dot{\epsilon} = 0$ (here $\dot{H} \equiv dH/dt$). One can show [59] that the scalar propagator in such a space can be represented as a function of the de Sitter invariant distance suitably rescaled by powers of the two scale factors. Remarkably, in the de Sitter limit when $\epsilon \to 0$, one recovers the massless minimally coupled scalar propagator in D space-time dimensions [33] plus an infinite ϵ-dependent constant. Ignoring the constant, the propagator in de Sitter space can be written in the form

$$i\Delta(x; x') = A(y) + k\ln(aa'), \tag{9}$$

$$A(y) = \frac{H^{D-2}}{(4\pi)^{D/2}}\Big[-\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\pi\cot\Big(\frac{\pi}{2}D\Big) - \sum_{n=-1}^{\infty}\frac{1}{n-\frac{D}{2}+2}\frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-(D/2)+2} + \sum_{n=1}^{\infty}\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n}\Big], \tag{10}$$

$$k = \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}. \tag{11}$$

Note that the function $A = A(y)$ in Eq. (10) obeys the equation

$$(4-y)yA''(y) + D(2-y)A'(y) = (D-1)k. \tag{12}$$

This implies a relation which will have great importance for this computation,

$$(2-y)A'(y) - k = -\frac{K'(y)}{D-2}, \tag{13}$$

where we define $K(y)$ as

$$K(y) \equiv (4-y)yA'(y) + (2-y)k. \tag{14}$$

The expansion of $K(y)$ around $y = 0$ is

$$K(y) = -\frac{H^{D-2}}{(4\pi)^{D/2}}4\Gamma\Big(\frac{D}{2}\Big)\Big\{\Big(\frac{y}{4}\Big)^{1-(D/2)} + \frac{D-2}{2}\Big(\frac{y}{4}\Big)^{2-(D/2)} - \frac{\Gamma(D-1)}{2\Gamma(\frac{D}{2})^2} + \frac{D(D-2)}{8}\Big(\frac{y}{4}\Big)^{3-(D/2)} - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+1)}\frac{y}{4} + \frac{D(D-2)(D+2)}{48}\Big(\frac{y}{4}\Big)^{4-(D/2)} - \frac{\Gamma(D)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+2)}\Big(\frac{y}{4}\Big)^2 + \frac{D-4}{2}\sum_{n=3}^{\infty}\Big(\frac{y}{4}\Big)^n\ln\Big(\frac{y}{4}\Big) + \mathcal{O}((D-4)^2)\Big\}$$
$$= -\frac{H^{D-2}}{(4\pi)^{D/2}}4\Gamma\Big(\frac{D}{2}\Big)\Big\{\Big(\frac{y}{4}\Big)^{1-(D/2)} + \frac{D-4}{2}\frac{\ln(\frac{y}{4})}{1-y/4} + \mathcal{O}((D-4)^2)\Big\}. \tag{15}$$

Note that in $D = 4$ the function $K(y)$ terminates at $y^{1-D/2}$. In fact, up to the terms that are suppressed as $(D-4)$, $K(y)$ is just $-2(D-2)$ times the conformal scalar propagator. Another important relation is

$$(4-y)yK''(y) + D(2-y)K'(y) = (D-2)K(y). \tag{16}$$

Also note that dimensional regularization implies the following coincidence limits ($y \longrightarrow 0$),

$$A(0) = -\frac{H^{D-2}}{(4\pi)^{D/2}}\cos\Big(\frac{\pi D}{2}\Big)\Gamma(D-1)\Gamma\Big(1-\frac{D}{2}\Big), \tag{17}$$

$$A'(0) = \frac{H^{D-2}}{4(4\pi)^{D/2}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}, \tag{18}$$

$$2A'(0) - k = -\frac{H^{D-2}}{2(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} = -\frac{K'(0)}{D-2}. \tag{19}$$

B. The vector propagator

The Lorentz gauge vector propagator in de Sitter space was discussed by Allen and Jacobson [60], and a minor error in their analysis was recently corrected [61]. The propagator obeys the equation

$$\sqrt{-g}(\square g^{\rho\mu} - R^{\rho\mu})i[_\mu\Delta_\nu](x;x') = \delta^\rho_\nu i\delta^D(x-x') + \sqrt{-g}\partial^\rho\partial'_\nu i\Delta(x;x'), \tag{20}$$

where $\square$ denotes the vector d'Alembertian, $R^{\rho\mu}$ is the Ricci tensor, and $i\Delta(x;x')$ represents the scalar propagator (9). The transversality of Eq. (20) follows from the Lorentz gauge condition

$$\sqrt{-g}\nabla^\mu i[_\mu\Delta_\nu](x;x') \equiv \partial_\rho g^{\rho\mu}\sqrt{-g}i[_\mu\Delta_\nu](x;x') = 0,$$
$$\sqrt{-g'}\nabla'^\nu i[_\mu\Delta_\nu](x;x') \equiv \partial'_\sigma g'^{\sigma\nu}\sqrt{-g'}i[_\mu\Delta_\nu](x;x') = 0, \tag{21}$$

where $g^{\rho\mu} \equiv g^{\rho\mu}(x)$, $g'^{\sigma\nu} \equiv g^{\sigma\nu}(x')$, and ∇^μ is the covariant derivative operator.

On de Sitter background the photon propagator can be written in the following de Sitter invariant form:

$$i[_\mu\Delta_\nu](x;x') = B(y)\frac{\partial^2 y}{\partial x^\mu\partial x'^\nu} + C(y)\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x'^\nu}. \tag{22}$$

We can use the Lorentz gauge condition (21) to express $B(y)$ and $C(y)$ in terms of a single function $\gamma(y)$,

$$B(y) = \frac{1}{4(D-1)H^2}\times\{-(4-y)y\gamma'(y) - (D-1)(2-y)\gamma(y)\}, \tag{23}$$

$$C(y) = \frac{1}{4(D-1)H^2}\{(2-y)\gamma'(y) - (D-1)\gamma(y)\}. \tag{24}$$

The propagator equation will follow provided $\gamma(y)$ has the right singularity at $y = 0$ and obeys the differential equation

$$(4-y)y\gamma''(y) + (D+2)(2-y)\gamma'(y) - 2(D-1)\gamma(y) = (D-1)[(2-y)A'(y) - k]. \tag{25}$$

The unique solution for $\gamma(y)$ is [61]

$$\gamma(y) = \frac{D-1}{2(D-3)} \frac{H^{D-2}}{(4\pi)^{D/2}} \Bigg\{ \sum_{n=0}^{\infty} \frac{(n+1)\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2}+1)} \Big[\psi\Big(2-\frac{D}{2}\Big) - \psi\Big(\frac{D}{2}-1\Big) + \psi(n+D-1) - \psi(n+2) \Big] \Big(\frac{y}{4}\Big)^n - \sum_{n=-1}^{\infty} \Big(n-\frac{D}{2}+3\Big) \frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+3)} \Big[\psi\Big(2-\frac{D}{2}\Big) - \psi\Big(\frac{D}{2}-1\Big) + \psi\Big(n+\frac{D}{2}+1\Big) - \psi\Big(n-\frac{D}{2}+4\Big) \Big] \Big(\frac{y}{4}\Big)^{n-(D/2)+2} \Bigg\}. \tag{26}$$

An important simplification for the vector propagator is "the outer leg identity"

$$\partial_{[\mu} i[_{\nu]}\Delta_\rho](x;x') = -\frac{1}{4H^2} \frac{\partial y}{\partial x^{[\mu}} \frac{\partial^2 y}{\partial x^{\nu]} \partial x'^\rho} \times [(2-y)A'(y) - k] = \frac{K'(y)}{4(D-2)H^2} \frac{\partial y}{\partial x^{[\mu}} \frac{\partial^2 y}{\partial x^{\nu]} \partial x'^\rho}. \tag{27}$$

Our convention throughout this paper is that square bracketed indices are antisymmetrized: $H_{[\mu\nu]} \equiv (H_{\mu\nu} - H_{\nu\mu})/2$. The outer leg identity can be easily obtained by making use of Eqs. (22)–(25). Note that Eq. (27) allows us to express the integrals entirely in terms of the A-type propagator (9)–(11), without involving the complicated function $\gamma(y)$. This is a very welcome simplification because the series for $A(y)$ terminates in $D=4$, whereas that for $\gamma(y)$ does not.

III. THE ONE LOOP STRESS-ENERGY TENSOR FOR SQED

A. The SQED action

When expressed in terms of the renormalized fields, the action of scalar electrodynamics is

$$S_{\rm SQED} = \int d^D x \mathcal{L}_{\rm SQED}, \tag{28}$$

$$\mathcal{L}_{\rm SQED} = -(1+\delta Z_2)(\mathcal{D}_\mu \varphi)^* (\mathcal{D}_\nu \varphi) g^{\mu\nu} \sqrt{-g} - \tfrac{1}{4}(1+\delta Z_3) F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} - \delta\xi R \varphi^* \varphi \sqrt{-g} - \tfrac{1}{4}\delta\lambda (\varphi^*\varphi)^2 \sqrt{-g}. \tag{29}$$

Here $\mathcal{D}_\mu \equiv \partial_\mu + ieA_\mu$ is the covariant derivative operator, A_μ is the photon field, φ is the scalar field, $g = \det[g_{\mu\nu}]$ is the determinant of the metric tensor $g_{\mu\nu}$, and $g^{\rho\sigma}$ is its inverse, $g^{\mu\rho} g_{\rho\nu} = \delta^\mu{}_\nu$. Note that we have chosen the renormalized values of the conformal coupling, the scalar mass, and the scalar self-coupling to vanish. There is no need for a mass counterterm because mass is multiplicatively renormalized in dimensional regularization. However, full renormalization does require a conformal counterterm and a scalar self-coupling counterterm. The various one loop counterterms are [1,2,22]

$$\delta Z_2 = -\frac{e^2 H^{D-4}}{(4\pi)^{D/2}} \frac{2(D-1)\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)} + e^2 \delta Z_{2\rm fin} + \mathcal{O}(e^4), \tag{30}$$

$$\delta Z_3 = +\frac{e^2 H^{D-4}}{(4\pi)^{D/2}} \frac{2\Gamma(\frac{D}{2}-1)}{(D-1)(D-3)(D-4)} + e^2 \delta Z_{3\rm fin} + \mathcal{O}(e^4), \tag{31}$$

$$\delta\xi = +\frac{e^2 H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{2(D-3)\Gamma(\frac{D}{2}+1)} \times \Big[-\psi(D-1) + \psi(2) + \psi\Big(\frac{D}{2}-1\Big) - \psi\Big(2-\frac{D}{2}\Big) \Big] + \mathcal{O}(e^4), \tag{32}$$

$$\delta\lambda = +\frac{e^4 H^{D-4}}{(4\pi)^{D/2}} \frac{\Gamma(D+1)}{(D-3)^2 \Gamma(\frac{D}{2}+1)} \times \Bigg\{ \psi'(D-1) + \psi'(2) - \psi'\Big(\frac{D}{2}-1\Big) - \psi'\Big(2-\frac{D}{2}\Big) + \frac{2}{D-3}\Big[-\psi(D-1) + \psi(2) + \psi\Big(\frac{D}{2}-1\Big) - \psi\Big(2-\frac{D}{2}\Big) \Big] + \Big[-\psi(D-1) + \psi(2) + \psi\Big(\frac{D}{2}-1\Big) - \psi\Big(2-\frac{D}{2}\Big) \Big]^2 \Bigg\} + \mathcal{O}(e^6). \tag{33}$$

One can see from these expressions that we have taken the dimensional regularization scale to be about the Hubble constant, $\mu \sim H$. Hence the renormalized electric charge e is defined at this scale. Note that, while the finite parts of the two field strength renormalizations are arbitrary, the finite parts of $\delta\xi$ and $\delta\lambda$ are fixed (at one loop order) by requiring that the scalar effective potential be as flat as possible [1].

From the action (28) one obtains the stress-energy tensor in the standard manner, $T_{\alpha\beta} = -(2/\sqrt{-g})\delta S_{\rm SQED}/\delta g^{\alpha\beta}$. Its expectation value in the presence of the Bunch-Davies vacuum $|\Omega\rangle$ can be broken up into a part derived from the Maxwell field strength and a part derived from the scalar kinetic and potential energies,

$$\begin{aligned} \langle\Omega|T_{\alpha\beta}(x)|\Omega\rangle &= \langle\Omega|T_{\alpha\beta}(x)|\Omega\rangle_{\rm Maxwell} + \langle\Omega|T_{\alpha\beta}(x)|\Omega\rangle_{\rm scalar}, \\ \langle\Omega|T_{\alpha\beta}(x)|\Omega\rangle_{\rm Maxwell} &= (\delta_\alpha{}^\mu\delta_\beta{}^\rho g^{\nu\sigma} - \tfrac{1}{4}g_{\alpha\beta}g^{\mu\rho}g^{\nu\sigma})(1+\delta Z_3)\langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle, \\ \langle\Omega|T_{\alpha\beta}(x)|\Omega\rangle_{\rm scalar} &= (\delta_\alpha{}^\mu\delta_\beta{}^\nu - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu})2(1+\delta Z_2)\langle\Omega|(\mathcal{D}_\mu\varphi(x))^\dagger(\mathcal{D}_\nu\varphi(x))|\Omega\rangle \\ &\quad + 2\delta\xi(G_{\alpha\beta} + g_{\alpha\beta}g^{\mu\nu}\nabla_\mu\nabla_\nu - \nabla_\alpha\nabla_\beta)\langle\Omega|\varphi(x)^\dagger\varphi(x)|\Omega\rangle - \tfrac{1}{4}\delta\lambda g_{\alpha\beta}\langle\Omega|[\varphi(x)^\dagger\varphi(x)]^2|\Omega\rangle. \end{aligned} \tag{34}$$

Here $G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the Einstein curvature tensor, and ∇_μ denotes the covariant derivative. On de Sitter background the various curvatures become

$$\begin{aligned} G_{\alpha\beta} &= -\tfrac{1}{2}(D-1)(D-2)H^2g_{\alpha\beta}, \\ R_{\alpha\beta} &= (D-1)H^2g_{\alpha\beta}, \quad \text{and} \quad R = D(D-1)H^2. \end{aligned} \tag{35}$$

B. One loop stress-energy tensor

At one loop order the various counterterms are irrelevant, and we have

$$\begin{aligned} \langle\Omega|T_{\alpha\beta}|\Omega\rangle_{\rm one\ loop} &= (g_\alpha{}^\mu g_\beta{}^\rho g^{\nu\sigma} - \tfrac{1}{4}g_{\alpha\beta}g^{\mu\rho}g^{\nu\sigma}) \\ &\quad \times \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\rm one\ loop} \\ &\quad + (g_\alpha{}^\mu g_\beta{}^\nu - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu})2\langle\Omega|\partial_\mu\varphi^*\partial_\nu\varphi|\Omega\rangle_{\rm one\ loop}. \end{aligned} \tag{36}$$

The electromagnetic contribution derives from the coincident product of two field strengths,

$$\begin{aligned} \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\rm one\ loop} &= (\partial_\mu\partial'_\rho i[_\nu\Delta_\sigma](x;x'))_{x=x'} + (\partial_\nu\partial'_\sigma i[_\mu\Delta_\rho](x;x'))_{x=x'} \\ &\quad - (\partial_\nu\partial'_\rho i[_\mu\Delta_\sigma](x;x'))_{x=x'} - (\partial_\mu\partial'_\sigma i[_\nu\Delta_\rho](x;x'))_{x=x'}, \end{aligned} \tag{37}$$

while the scalar contribution comes from the scalar kinetic energy,

$$\langle\Omega|\partial_\mu\varphi^*(x)\partial_\nu\varphi(x)|\Omega\rangle_{\rm one\ loop} = (\partial_\mu\partial'_\nu i\Delta(x;x'))_{x=x'}. \tag{38}$$

Making use of the outer leg identity (27) we find that the nonvanishing terms are

$$\begin{aligned} \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\rm one\ loop} &= \frac{1}{(D-2)H^2}\{(\partial'_\rho\partial_{[\mu}y)(\partial_{\nu]}\partial'_\sigma y)K'(y)\}_{x=x'} \\ &= \frac{4H^2K'(0)}{D-2}g_{\rho[\mu}g_{\nu]\sigma}. \end{aligned} \tag{39}$$

When this is inserted into Eq. (19) one arrives at

$$\begin{aligned} \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\rm one\ loop} &= \frac{H^D}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \end{aligned} \tag{40}$$

Similarly, we make use of relation (18) for the scalar propagator (9) to find

$$\begin{aligned} \langle\Omega|\partial_\mu\varphi^*(x)\partial_\nu\varphi(x)|\Omega\rangle_{\rm one\ loop} &= -\frac{H^D}{(4\pi)^{D/2}}\frac{\Gamma(D)}{2\Gamma(\frac{D}{2}+1)}g_{\mu\nu}. \end{aligned} \tag{41}$$

The one loop stress-energy tensor is easily obtained from Eqs. (36), (40), and (41). The electromagnetic contribution is

$$\begin{aligned} \langle\Omega|T_{\alpha\beta}|\Omega\rangle_{\rm one\ loop, Maxwell} &= -\frac{H^D\Gamma(D)}{(4\pi)^{D/2}\Gamma(\frac{D}{2}+1)}\frac{D-4}{4}g_{\alpha\beta}\overset{D\to4}{\longrightarrow}0, \end{aligned} \tag{42}$$

while the scalar contribution reads

$$\begin{aligned} \langle\Omega|T_{\alpha\beta}|\Omega\rangle_{\rm one\ loop, scalar} &= \frac{H^D}{(4\pi)^{D/2}}\frac{\Gamma(D)}{\Gamma(\frac{D}{2}+1)}\frac{D-2}{2}g_{\alpha\beta}\overset{D\to4}{\longrightarrow}\frac{3H^4}{16\pi^2}g_{\alpha\beta}. \end{aligned} \tag{43}$$

Summing the two contributions (42) and (43) results in the final expression (36),

$$\langle\Omega|T_{\alpha\beta}|\Omega\rangle_{\rm one\ loop} = \frac{H^D}{(4\pi)^{D/2}}\frac{\Gamma(D)}{2\Gamma(\frac{D}{2})}g_{\alpha\beta}\overset{D\to4}{\longrightarrow}\frac{3H^4}{16\pi^2}g_{\alpha\beta}. \tag{44}$$

These one loop contributions to the stress-energy tensor in de Sitter space are of the form a constant times $g_{\alpha\beta}$; hence they can be absorbed in the cosmological term.

IV. GENERAL PROPERTIES

The one loop stress tensor and its constituents are de Sitter invariant. Their tensor structures are comprised entirely from products of the metric, and the coefficients of these products are constants. That is not true of higher loop contributions. These receive de Sitter breaking contributions from the undifferentiated scalar propagator (9) and from vertex integrations that reach back to the finite initial time. Spatial homogeneity and isotropy are preserved but tensor structures can involve the unit timelike vector $a\delta^0_\mu$, and coefficients can depend upon $\ln(a)$. Of course, it is the temporal growth implied by factors of $\ln(a)$ that gives this computation its physical interest, and it was to reproduce and sum up the leading powers of these logarithms to all orders that the stochastic formulation was developed.

The expectation value of $T_{\mu\nu}$ at a general order derives from the expectation values of four composite operators: $F_{\mu\nu}F_{\rho\sigma}$, $(D_\mu\varphi)^*D_\nu\varphi$, $\varphi^*\varphi$, and $(\varphi^*\varphi)^2$. It is useful to parametrize these expectation values as follows:

$$\langle\Omega|F_{\mu\nu}F_{\rho\sigma}|\Omega\rangle \equiv Eg_{\mu[\rho}g_{\sigma]\nu} + Fa^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}, \quad (45)$$

$$\langle\Omega|(D_\mu\varphi)^* D_\nu\varphi|\Omega\rangle \equiv Jg_{\mu\nu} + La^2\delta^0_\mu\delta^0_\nu, \quad (46)$$

$$\langle\Omega|\varphi^*\varphi|\Omega\rangle \equiv \frac{M}{H^2}, \quad (47)$$

$$\delta\lambda\langle\Omega|(\varphi^*\varphi)^2|\Omega\rangle \equiv N. \quad (48)$$

The expectation value of the stress tensor can be given a similar form,

$$\langle\Omega|T_{\mu\nu}|\Omega\rangle = pg_{\mu\nu} + (\rho + p)a^2\delta^0_\mu\delta^0_\nu. \quad (49)$$

The coefficient functions E, F, J, L, M, N, p, and $(\rho + p)$ all take the form of $H^D/(4\pi)^{D/2}$ times functions of the scale factor and the dimensionless loop counting parameter $e^2H^{D-4}/(4\pi)^{D/2}$. In SQED there can be at most one additional factor of $\ln(a)$ for every additional loop [1], and it turns out that the temporal coefficient functions F, L, and $(\rho + p)$ are always down from this limit by at least one overall power of $\ln(a)$. Note that we have already encountered this pattern at one loop where there are no powers of $e^2H^{D-4}/(4\pi)^{D/2}$, the coefficient functions E, J, and p are constant, and the three temporal coefficient functions accordingly vanish.

In addition to factors of $\ln(a)$, the various coefficient functions generally depend in a complicated way upon inverse powers of a that redshift to zero at late times. This can be illustrated by the two loop result for the expectation value of the stress-energy tensor of a massless, minimally coupled scalar with a quartic self-interaction [32,33]. We report this with a minor change of the renormalization scheme to bring it into conformity with our convention of not employing a mass counterterm,

$$(p)_{\lambda\varphi^4} = -\frac{\delta\Lambda_{\rm fin}}{8\pi G} + \frac{\lambda H^4}{(2\pi)^4}\Big\{-\frac{1}{8}\ln^2(a) - \Big(\frac{15}{32} + \frac{\gamma_E}{32}\Big)\ln(a) + \frac{37}{144} + \frac{\gamma_E}{48} - \frac{1}{24}\sum_{n=1}^\infty \frac{n^2-4}{(n+1)^2}\frac{1}{a^{n+1}}\Big\} + \mathcal{O}(\lambda^2), \quad (50)$$

$$(\rho + p)_{\lambda\varphi^4} = \frac{\lambda H^4}{(2\pi)^4}\Big\{-\frac{1}{12}\ln(a) - \frac{37}{288} - \frac{\gamma_E}{96} + \frac{1}{18}\frac{1}{a^3} - \frac{1}{24}\sum_{n=1}^\infty \frac{n+2}{n+1}\frac{1}{a^{n+1}}\Big\} + \mathcal{O}(\lambda^2). \quad (51)$$

The larger number of infrared logarithms—two for each extra loop in this model, as opposed to one in SQED—is simply explained by having four, as opposed to two, undifferentiated scalars in the interaction [1]. What concerns us at the moment is the terms which fall off like powers of $1/a$. These terms derive from the lower limits of conformal time integrations. Because they are separately conserved, it has been conjectured that these terms can be absorbed into perturbative corrections of the initial state [33]. If so, they are an artifact of the initial state rather than a universal feature of inflationary particle production. Because these sorts of terms fall off at late times, they play no role in the stochastic formalism we are checking. Hence there is no point in struggling to retain these terms, and we shall not bother to do so. This allows vast simplifications from partially integrating without concern for surface terms, which we have already done in evaluating the scalar coefficient functions J, L, and M at two loop order [2], and which we shall be using throughout this work e.g. when extracting d'Alembertians from the inner loop.

An all-orders relation between the scalar coefficient functions is implied by the operator equation [2]

$$\Box(\varphi^*\varphi) = 2g^{\mu\nu}(D_\mu\varphi)^*D_\nu\varphi + \frac{2\delta\xi}{1+\delta Z_2}R\varphi^*\varphi + \frac{\delta\lambda}{1+\delta Z_2}(\varphi^*\varphi)^2. \quad (52)$$

Taking expectation values and substituting Eqs. (46) and (47) gives

$$-M'' - (D-1)M' = 2DJ - 2L + \frac{2D(D-1)\delta\xi}{1+\delta Z_2}M + \frac{1}{1+\delta Z_2}N, \quad (53)$$

where a prime denotes differentiation with respect to $\ln(a)$. A slight rearrangement produces a key result,

$$-2D[(1+\delta Z_2)J + \delta\xi(D-1)M] = (1+Z_2)[M'' + (D-1)M'] - 2(1+\delta Z_2)L + N. \quad (54)$$

It is now time to work out the Maxwell and scalar contributions to p and $(\rho + p)$ in terms of the various coefficient functions. We find the Maxwell contributions from relations (34) and (35),

$$(p)_{\rm Maxwell} = (1+\delta Z_3)[-\tfrac{1}{8}(D-1)(D-4)E + \tfrac{1}{8}(D-3)F], \quad (55)$$

$$(\rho + p)_{\rm Maxwell} = (1+\delta Z_3)\tfrac{1}{4}(D-2)F. \quad (56)$$

The scalar contribution to $(\rho + p)$ is almost as simple,

$$(\rho + p)_{\rm scalar} = (1+\delta Z_2)2L + \delta\xi[-2M'' + 2M']. \quad (57)$$

However, it is best to modify the scalar contribution to the pressure using Eq. (54),

$$(p)_{\rm scalar} = -(D-2)[(1+\delta Z_2)J + \delta\xi(D-1)M] + (1+\delta Z_2)L - 2\delta\xi[M'' + (D-2)M'] - \tfrac{1}{4}N, \quad (58)$$

$$= (1 + \delta Z_2)\left(\frac{D-2}{2D}\right)[M'' + (D-1)M'] - 2\delta\xi[M'' + (D-2)M'] + \frac{2}{D}(1 + \delta Z_2)L + \left(\frac{D-4}{4D}\right)N. \quad (59)$$

This final form (59) is quite significant because it precludes $\ln(a)$ contributions from the scalar at two loop order. This is a key prediction of the stochastic formalism [1].

Stress-energy conservation provides an important accuracy check,

$$\nabla^\nu\langle\Omega|T_{\mu\nu}|\Omega\rangle = -Ha\delta^0_\mu[(\rho + p)' - p' + (D-1)(\rho + p)] = 0. \quad (60)$$

Recall that a prime denotes differentiation with respect to $\ln(a)$, and note that the powers of $1/a$ we ignore should be separately conserved. The Maxwell and scalar contributions are not separately conserved but the Heisenberg equations of motion for SQED relate their divergences to the expectation value of a field strength contracted into a current,

$$\nabla^\nu\langle\Omega|T_{\mu\nu}|\Omega\rangle_{\rm Maxwell} = -\nabla^\nu\langle\Omega|T_{\mu\nu}|\Omega\rangle_{\rm scalar} = -ie(1 + \delta Z_2)g^{\rho\sigma}\langle\Omega|F_{\mu\rho}[\varphi^* D_\sigma\varphi - (D_\sigma\varphi)^*\varphi]|\Omega\rangle. \quad (61)$$

Although we shall not give the derivation, it is simple to evaluate (61) at the order e^2 relevant to this analysis,

$$-ie(1 + \delta Z_2)g^{\rho\sigma}\langle\Omega|F_{\mu\rho}[\varphi^* D_\sigma\varphi - (D_\sigma\varphi)^*\varphi]|\Omega\rangle = -Ha\delta^0_\mu\left\{\frac{e^2H^{2D-4}}{(4\pi)^D}\Gamma(D-1) \times\left[-\frac{8}{D-4} - 10 + \mathcal{O}(a^{-1}, D-4)\right] + \mathcal{O}(e^4)\right\}. \quad (62)$$

This is probably the right point to discuss finiteness. Conventional renormalization only serves to remove divergences from noncoincident 1PI (one particle irreducible) functions. Composite operators such as $F_{\mu\nu}(x)F_{\rho\sigma}(x)$ and the others require additional, composite operator renormalization to remove their divergences. This is evident from the divergence of expression (62). One renormalizes composite operators by adding a series of counteroperators which remove the divergences order by order in the loop expansion [62,63]. As with conventional renormalization, there are ambiguities in finite parts, which are resolved by imposing additional renormalization conditions. There is absolutely no reason to burden an already formidable analysis by bothering with any of this. The stochastic formalism which we are trying to check provides unambiguous predictions for the dimensionally regulated expectation values of composite operators [1], so we may as well leave the regulator on and verify these predictions.

A related point concerns the stress-energy tensor. Unlike the other composite operators, its expectation value is a 1PI function of an enlarged theory; specifically, it represents matter contributions to the 1-graviton 1PI function in gravity + SQED. It must therefore be that divergences in the expectation value of the stress-energy tensor can be absorbed with the addition of purely gravitational counterterms which all degenerate on de Sitter background to changes in the bare cosmological constant. This was shown at two loop order in $\lambda\varphi^4$ theory by explicit computation [32,33], and the ability to change the result by an additional, finite renormalization of the cosmological constant is evident in expression (50). For SQED this means three things:

(i) Divergences in the temporal coefficient functions F and L must cancel when they are formed to give the full $(\rho + p)$;

(ii) The parts of p which contain factors of $\ln(a)$ must be finite; and

(iii) The parts of p which are constant can diverge.

The first two facts provide more accuracy checks; the third means that there is no point in working very hard to determine the constant contributions to p because any constant part of p can be subsumed into a renormalization of the cosmological constant.

It remains to summarize what is already known about the coefficient functions. The Maxwell coefficient functions (45) take the form

$$E = \frac{H^D}{(4\pi)^{D/2}}\left\{\frac{2\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} + \frac{e^2H^{D-4}}{(4\pi)^{D/2}} \times[E_1\ln(a) + E_2 + \mathcal{O}(a^{-1})] + \mathcal{O}(e^4)\right\}, \quad (63)$$

$$F = \frac{H^D}{(4\pi)^{D/2}}\left\{0 + \frac{e^2H^{D-4}}{(4\pi)^{D/2}}[F_1 + \mathcal{O}(a^{-1})] + \mathcal{O}(e^4)\right\}. \quad (64)$$

The point of this paper is to compute the D-dependent numbers E_1, E_2, and F_1. They give the order $e^2H^{D-4}/(4\pi)^{D/2}$ electromagnetic contributions to p and $(\rho + p)$,

$$(p)_{\rm Maxwell} = \frac{H^D}{(4\pi)^{D/2}}\left\{-\frac{(D-4)\Gamma(D)}{4\Gamma(\frac{D}{2}+1)} + \frac{e^2H^{D-4}}{(4\pi)^{D/2}} \times\left[-\frac{1}{8}(D-1)(D-4)E_1\ln(a) - \frac{2\Gamma(D-1)}{D(D-2)(D-3)} - \frac{1}{8}(D-1)(D-4)E_2 + \frac{1}{8}(D-3)F_1 + \mathcal{O}(a^{-1}, D-4)\right] + \mathcal{O}(e^4)\right\}, \quad (65)$$

$$(\rho + p)_{\rm Maxwell} = \frac{H^D}{(4\pi)^{D/2}}\Big\{0 + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}} \times \Big[\frac{1}{4}(D-2)F_1 + \mathcal{O}(a^{-1})\Big] + \mathcal{O}(e^4)\Big\}. \tag{66}$$

Note that Eq. (62) implies a relation between the coefficients E_1 and F_1,

$$\frac{1}{8}(D-1)(D-4)E_1 + \frac{1}{4}(D-1)(D-2)F_1 = -\frac{8\Gamma(D-1)}{D-4} - 20 + \mathcal{O}(D-4). \tag{67}$$

The stochastic formalism predicts [1],

$$E_1 = \frac{8\Gamma^2(D-1)}{(D-3)\Gamma(\frac{D}{2}+1)\Gamma(\frac{D}{2})}\Big[-\psi\Big(2-\frac{D}{2}\Big) + \psi\Big(\frac{D}{2}-1\Big) - \psi(D-1) + \psi(2)\Big], \tag{68}$$

$$= -\frac{16\Gamma(D-1)}{D-4} + 16 + \mathcal{O}(D-4). \tag{69}$$

So we can use Eq. (67) to infer what F_1 should be,

$$F_1 = -\frac{16\Gamma(D-1)}{(D-1)(D-4)} + \mathcal{O}(D-4). \tag{70}$$

No comparable check involves E_2, and we see from (65) that E_2 has no physical relevance because it can be absorbed into a renormalization of the cosmological constant.

The coefficient functions (46) associated with the scalar kinetic energy are [2]

$$J = \frac{H^D}{(4\pi)^{D/2}}\Big\{-\frac{\Gamma(D)}{2\Gamma(\frac{D}{2}+1)} + \delta\xi \times \frac{\Gamma(D)}{\Gamma(\frac{D}{2})}\Big[-2\ln(a) + \pi\cot\Big(\frac{D\pi}{2}\Big)\Big] + \delta Z_2 \times \frac{\Gamma(D)}{2\Gamma(\frac{D}{2}+1)} + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}\Big[\frac{16\Gamma(D-1)}{(D-1)(D-4)} - 17 + 2\pi^2 + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}, \tag{71}$$

$$L = \frac{H^D}{(4\pi)^{D/2}}\Big\{0 + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}\Big[\frac{10\Gamma(D-1)}{(D-1)(D-4)} + \frac{8}{3} + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}. \tag{72}$$

Recall that δZ_2 and $\delta\xi$ are of order $e^2 H^{D-4}/(4\pi)^{D/2}$ and are given in expressions (30) and (32), respectively. The coefficient function M was defined in Eq. (47) and takes the form [2]

$$M = \frac{H^D}{(4\pi)^{D/2}}\Big\{\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big[2\ln(a) - \pi\cot\Big(\frac{\pi D}{2}\Big)\Big] + \delta Z_2\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big[-2\ln(a) + \pi\cot\Big(\frac{\pi D}{2}\Big)\Big] + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}\Big[\Big[\frac{-108\Gamma(D-1)}{(D-1)^2(D-4)} + 40 - \frac{16\pi^2}{3}\Big]\ln(a) + M_1 + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}. \tag{73}$$

The constant M_1 has not been determined but it drops out because only derivatives of M enter expression (59) for the stress-energy tensor. Note that even the lowest order contribution to M contains an infrared logarithm, so the ℓ loop result can have up to ℓ factors of $\ln(a)$, as opposed to only $\ell - 1$ in J. This is entirely in conformity with the rule that each additional loop brings at most one additional factor of $\ln(a)$ [1]. The final coefficient function (48) is

$$N = \frac{H^D}{(4\pi)^{D/2}}\Big\{0 + \frac{\delta\lambda H^{D-4}}{(4\pi)^{D/2}}\frac{2\Gamma^2(D-1)}{\Gamma^2(\frac{D}{2})} \times \Big[2\ln(a) - \pi\cot\Big(\frac{\pi D}{2}\Big)\Big]^2 + \mathcal{O}(e^6)\Big\}. \tag{74}$$

Because $\delta\lambda$ in (33) is of order $[e^2 H^{D-4}/(4\pi)^{D/2}]^2$, the function N does not affect the two loop stress-energy tensor.

We infer the scalar contribution to the stress-energy tensor by substituting these coefficient functions into expressions (57) and (59),

$$(p)_{\rm scalar} = \frac{H^D}{(4\pi)^{D/2}}\Big\{\frac{(D-2)\Gamma(D)}{2\Gamma(\frac{D}{2}+1)} + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}} \times \Big[0\times\ln(a) + \frac{2\Gamma(D-1)}{(D-1)(D-4)} + \frac{94}{3} - 4\pi^2 + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}, \tag{75}$$

$$(\rho + p)_{\rm scalar} = \frac{H^D}{(4\pi)^{D/2}}\Big\{0 + \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}\Big[\frac{8\Gamma(D-1)}{(D-1)(D-4)} + \frac{20}{3} + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}. \tag{76}$$

Note that there are no infrared logarithms in (75) at this order. Note also that (76) obeys the partial conservation identity (61) and (62), which represents an additional check of the two loop scalar stress-energy calculation of Ref. [2].

V. THE TWO LOOP GAUGE FIELD STRENGTH BILINEAR

Three diagrams contribute to the vacuum expectation value (VEV) at two loop order. They are depicted in Fig. 1.

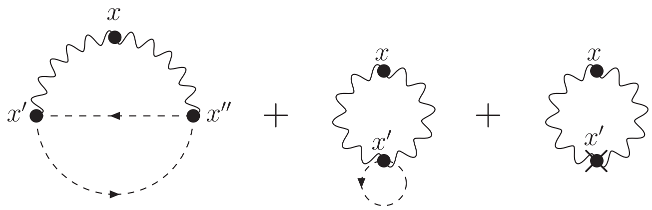

FIG. 1. Two loop contributions to $\langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle$.

A. The counterterm diagram

The diagram on the far right of Fig. 1 derives from the photon field strength renormalization. Covariance and index symmetries dictate its form,

$$I_{\rm c.t.} = 4i\delta Z_3 \int d^Dx'\{\partial_{[\mu}i[_{\nu]}\Delta_\alpha]_{++}(x;x') \times \sqrt{-g'}[g'^{\alpha\beta}g'^{\gamma\delta} - g'^{\alpha\delta}g'^{\beta\gamma}] \times \nabla'_\gamma\nabla'_\delta\partial_{[\rho}i[_{\sigma]}\Delta_\beta]_{++}(x;x') - (+-)\}, \tag{77}$$

where ∇'_μ stands for the covariant derivative operator with respect to x'^μ. To evaluate this diagram we first interchange the order of covariant differentiation in the term proportional to $g'^{\alpha\delta}g'^{\beta\gamma}$ and then use the Lorentz gauge condition to get

$$\sqrt{-g'}[g'^{\alpha\beta}g'^{\gamma\delta} - g'^{\alpha\delta}g'^{\beta\gamma}]\nabla'_\gamma\nabla'_\delta\partial_{[\rho}i[_{\sigma]}\Delta_\beta]_{+\pm}(x;x') = \sqrt{-g'}[g'^{\alpha\beta}\square' - R'^{\alpha\beta}]\partial_{[\rho}i[_{\sigma]}\Delta_\beta]_{+\pm}(x;x'). \tag{78}$$

Now we use the appropriate Schwinger-Keldysh generalizations of the propagator Eq. (20),

$$\sqrt{-g'}[g'^{\alpha\beta}\square' - R'^{\alpha\beta}]i[_\sigma\Delta_\beta]_{++}(x;x') = i\delta^\alpha_\sigma\delta^D(x-x') + \sqrt{-g'}g'^{\alpha\beta}\partial'_\beta\partial_\sigma i\Delta_{++}(x;x'), \tag{79}$$

$$\sqrt{-g'}[g'^{\alpha\beta}\square' - R'^{\alpha\beta}]i[_\sigma\Delta_\beta]_{+-}(x;x') = 0. \tag{80}$$

The antisymmetrized derivative with respect to x^ρ makes the second term in (79) drop out, so the result for the far right diagram (77) of Fig. 1 is [cf. Equations (37), (39), and (40)]

$$\begin{aligned} I_{\rm c.t.} &= -4\delta Z_3\int d^Dx'\partial_{[\mu}i[_{\nu]}\Delta_\alpha]_{++}(x;x')\partial_{[\rho}\delta^\alpha_{\sigma]}\delta^D(x-x') \\ &= -\delta Z_3 \lim_{x'\to x}\{\partial_\mu\partial'_\rho i[_\nu\Delta_\sigma]_{++}(x;x') - \partial_\nu\partial'_\rho i[_\mu\Delta_\sigma]_{++}(x;x') + \partial_\nu\partial'_\sigma i[_\mu\Delta_\rho]_{++}(x;x') - \partial_\mu\partial'_\sigma i[_\nu\Delta_\rho]_{++}(x;x')\} \\ &= 2\delta Z_3 H^2[2A'(0) - k](g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\ &= -\delta Z_3 \times \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\text{one loop}}. \end{aligned} \tag{81}$$

From expression (55) we see that the contribution of $I_{\rm c.t.}$ to the stress-energy tensor is exactly canceled by the contribution from δZ_3 times the one loop result.

B. The figure 8 diagram

The middle diagram of Fig. 1 is

$$I_\infty = -8ie^2\int d^Dx'\sqrt{-g'}\{\partial_{[\mu}i[_{\nu]}\Delta_\alpha]_{++}(x;x') \times \partial_{[\rho}i[_{\sigma]}\Delta_\beta]_{++}(x;x') - (+-)\}g'^{\alpha\beta}i\Delta(x';x'). \tag{82}$$

Making use of the outer leg identity (27) transforms this to

$$I_\infty = \frac{-ie^2}{2(D-2)^2H^4}\int d^Dx'\sqrt{-g'}i\Delta(x';x')\{[K'(y_{++})]^2 - (+-)\}\frac{\partial y}{\partial x^{[\mu}}\frac{\partial^2 y}{\partial x^{\nu]}\partial x'^\alpha}g'^{\alpha\beta}\frac{\partial^2 y}{\partial x'^\beta\partial x^{[\sigma}}\frac{\partial y}{\partial x^{\rho]}}. \tag{83}$$

Now use the contraction identity [2,61,64],

$$\frac{\partial^2 y}{\partial x^\nu\partial x'^\alpha}g'^{\alpha\beta}\frac{\partial^2 y}{\partial x'^\beta\partial x^\sigma} = 4H^4 g_{\nu\sigma} - H^2\frac{\partial y}{\partial x^\nu}\frac{\partial y}{\partial x^\sigma}. \tag{84}$$

The final term drops out owing to the antisymmetrizations, so the middle diagram becomes

$$\begin{aligned} I_\infty &= \frac{-2ie^2}{(D-2)^2}\int d^Dx'\sqrt{-g'}i\Delta(x';x')\frac{\partial y}{\partial x^{[\mu}}g_{\nu][\sigma}\frac{\partial y}{\partial x^{\rho]}} \times\{[K'(y_{++})]^2 - (+-)\} \\ &= \frac{-2ie^2}{(D-2)^2}\sum_\pm(\pm)\int d^Dx'\sqrt{-g'}i\Delta(x';x') \times\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y_{+\pm}). \end{aligned} \tag{85}$$

From Eqs. (10), (11), and (17) we can read off the coincidence limit of the scalar propagator,

$$\begin{aligned} i\Delta(x';x') &= A(0) + 2k\ln(a') \\ &= \frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times\Big[-\cos\Big(\frac{\pi}{2}D\Big)\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(1-\frac{D}{2}\Big) + 2\ln(a')\Big]. \end{aligned} \tag{86}$$

We will see that the leading logarithm contribution E_1 derives entirely from the $2k\ln(a')$ term in the integral (85).

C. The two leg diagram

The far left diagram of Fig. 1 is the most difficult in any gauge. To simplify it we must derive new identities for contracting adjacent basis tensors into one another. Let us first make the following definitions for the three coordinate separations and their associated length functions,

$$\Delta x^\mu \equiv x^\mu - x'^\mu \Rightarrow y \equiv y(x;x') = aa'H^2\Delta x^2, \tag{87}$$

$$\Delta x'^{\mu} \equiv x'^{\mu} - x''^{\mu} \Rightarrow y' \equiv y(x';x'') = a'a''H^2\Delta x'^2, \tag{88}$$

$$\Delta x''^{\mu} \equiv x^{\mu} - x''^{\mu} \Rightarrow y'' \equiv y(x;x'') = aa''H^2\Delta x''^2. \tag{89}$$

The three fundamental "adjacent" contractions we require are

$$\frac{\partial y}{\partial x'^{\rho}} g'^{\rho\sigma} \frac{\partial y'}{\partial x'^{\sigma}} = H^2\{2y + 2y' - yy' - 2y''\}, \tag{90}$$

$$\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}} g'^{\rho\sigma} \frac{\partial y'}{\partial x'^{\sigma}} = H^2\Big\{(2-y')\frac{\partial y}{\partial x^{\mu}} - 2\frac{\partial y''}{\partial x^{\mu}}\Big\}, \tag{91}$$

$$\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}} g'^{\rho\sigma} \frac{\partial^2 y'}{\partial x'^{\sigma}\partial x''^{\nu}} = -2H^2\frac{\partial^2 y''}{\partial x^{\mu}\partial x''^{\nu}} - H^2\frac{\partial y}{\partial x^{\mu}}\frac{\partial y'}{\partial x''^{\nu}}. \tag{92}$$

For $x''^{\mu} \to x^{\mu}$ these relations degenerate to the well-known identities [2,61,64]

$$\frac{\partial y}{\partial x'^{\rho}} g'^{\rho\sigma} \frac{\partial y}{\partial x'^{\sigma}} = (4-y)yH^2, \tag{93}$$

$$\frac{\partial^2 y}{\partial x^{\mu}\partial x'^{\rho}} g'^{\rho\sigma} \frac{\partial y}{\partial x'^{\sigma}} = H^2(2-y)\frac{\partial y}{\partial x^{\mu}}, \tag{94}$$

$$\frac{\partial^2 y}{\partial x^{\nu}\partial x'^{\alpha}} g'^{\alpha\beta} \frac{\partial^2 y}{\partial x'^{\beta}\partial x^{\sigma}} = 4H^4 g_{\nu\sigma} - H^2\frac{\partial y}{\partial x^{\nu}}\frac{\partial y}{\partial x^{\sigma}}. \tag{95}$$

Note that the last of these is just the contraction identity (84). The contractions on the "other" site x'' are trivial to obtain from these by interchanging x'^{μ} and x''^{μ}, which interchanges y and y'', while leaving y' unchanged. Two other useful identities are

$$\nabla_{\nu}\frac{\partial}{\partial x^{\mu}} y = H^2(2-y)g_{\mu\nu} \quad \text{and} \quad \square y = D(2-y)H^2, \tag{96}$$

where ∇_{ν} is the covariant derivative and $\square$ is the scalar d'Alembertian.

We are now ready to tackle the leftmost (two leg) diagram of Fig. 1,

$$\begin{aligned} I = 8e^2\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'\sqrt{-g'}\,\partial_{[\mu}i[_{\nu]}\Delta_{\alpha}]_{+\pm}(x;x') \\ \times\int d^Dx''\sqrt{-g''}\,\partial_{[\rho}i[_{\sigma]}\Delta_{\gamma}]_{+\pm}(x;x'')g'^{\alpha\beta}g''^{\gamma\delta} \\ \times\{\partial'_{\beta}i\Delta_{\pm\pm}(x';x'')\partial''_{\delta}i\Delta_{\pm\pm}(x';x'') \\ - i\Delta_{\pm\pm}(x';x'')\partial'_{\beta}\partial''_{\delta}i\Delta_{\pm\pm}(x';x'')\}. \end{aligned} \tag{97}$$

The Lorentz gauge condition (21) allows us to partially integrate the ∂'_{β} so that the inner loop consists of just a single term,

$$\begin{aligned} I = 16e^2\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'\sqrt{-g'}\,\partial_{[\mu}i[_{\nu]}\Delta_{\alpha}]_{+\pm}(x;x') \\ \times\int d^Dx''\sqrt{-g''}\,\partial_{[\rho}i[_{\sigma]}\Delta_{\gamma}]_{+\pm}(x;x'') \\ \times g'^{\alpha\beta}g''^{\gamma\delta}\partial'_{\beta}i\Delta_{\pm\pm}(x';x'')\partial''_{\delta}i\Delta_{\pm\pm}(x';x'') + \mathcal{O}(a^{-1}). \end{aligned} \tag{98}$$

The surface terms we have neglected can only involve the initial time surface in the Schwinger-Keldysh formalism [65]. As explained in Sec. III, these surface terms fall off like powers of $1/a$ and play no role in the check we are making of the stochastic formalism. We shall consistently ignore them.

Expression (98) is already a significant simplification because it means we do not have to worry about either the undifferentiated logarithm term or the local contribution. It also simplifies the tensor algebra. The next step is to act with the derivatives on the two inner loop propagators and expand,

$$\begin{aligned} &\partial'_{\beta}i\Delta_{\pm\pm}(x';x'')\partial''_{\delta}i\Delta_{\pm\pm}(x';x'') \\ &= A'^2(y')\frac{\partial y'}{\partial x'^{\beta}}\frac{\partial y'}{\partial x''^{\delta}} + kHa'\delta^0_{\beta}A'(y')\frac{\partial y'}{\partial x''^{\delta}} \\ &\quad + kHa''\delta^0_{\delta}A'(y')\frac{\partial y'}{\partial x'^{\beta}} + k^2H^2a'a''\delta^0_{\beta}\delta^0_{\delta}. \end{aligned} \tag{99}$$

The term proportional to k^2 drops out in the polarity sum, so we really only have three terms from the inner loop.

Let us define $I[f(y)]$ as the indefinite integral of $f(y)$ with respect to y. By making use of the identity

$$\frac{\partial}{\partial x'^{\beta}}\frac{\partial}{\partial x''^{\delta}}I^2[A'(y')^2] = \frac{\partial y'}{\partial x'^{\beta}}\frac{\partial y'}{\partial x''^{\delta}}A'(y')^2 + \frac{\partial^2 y'}{\partial x'^{\beta}\partial x''^{\delta}}I[A'(y')^2], \tag{100}$$

we can rewrite Eq. (99) as

$$\begin{aligned} &\partial'_{\beta}i\Delta_{\pm\pm}(x';x'')\partial''_{\delta}i\Delta_{\pm\pm}(x';x'') \\ &= -\frac{\partial^2 y'}{\partial x'^{\beta}\partial x''^{\delta}}I[A'^2(y')] + \frac{\partial}{\partial x'^{\beta}}\frac{\partial}{\partial x''^{\delta}}I^2[A'^2(y')] \\ &\quad + \frac{\partial}{\partial x''^{\delta}}[kHa'\delta^0_{\beta}A'(y')] + \frac{\partial}{\partial x'^{\beta}}[kHa''\delta^0_{\delta}A'(y')] \\ &\quad + \frac{\partial}{\partial x'^{\beta}}\frac{\partial}{\partial x''^{\delta}}[k^2\ln(a')\ln(a'')]. \end{aligned} \tag{101}$$

Observe that when Eq. (101) is inserted into Eq. (98), all terms but the first vanish upon partial integration as a consequence of the Lorentz condition (21), again up to $1/a$ surface terms. With this, Eq. (98) becomes

$$I = \frac{-e^2}{(D-2)^2 H^4} \sum_{\pm\pm} (\pm)(\pm) \int d^D x' \sqrt{-g'} K'(y_{+\pm}) \times \int d^D x'' \sqrt{-g''} K'(y''_{+\pm}) \frac{\partial y}{\partial x^{[\mu}} \frac{\partial^2 y}{\partial x^{\nu]} \partial x'^{\alpha}} g'^{\alpha\beta} \times \frac{\partial^2 y'}{\partial x'^{\beta} \partial x''^{\delta}} g''^{\delta\gamma} \frac{\partial y''}{\partial x^{[\rho}} \frac{\partial^2 y''}{\partial x^{\sigma]} \partial x''^{\gamma}} \times I[A'^2(y'_{\pm\pm})] + \mathcal{O}(a^{-1}), \tag{102}$$

where we used the outer leg identity (27) twice.

Note that one can add an integration constant to the first term in (101): $I[A'^2(y')] \to I[A'^2(y')] + c_I$, provided the appropriate change is also made in the second term, $I^2[A'^2(y')] \to I^2[A'^2(y')] + c_I y'$. We have just seen that $I^2[A'^2(y')]$ in (101) can be partially integrated to zero as a consequence of the Lorentz condition (21), up to $1/a$ surface terms. Hence it must be that any constant term in $I[A'^2(y')]$ must fall off like powers of $1/a$.

Equation (102) can be further simplified by making use of the identities (84) and (91) and the antisymmetry in μ and ν and in ρ and σ to obtain

$$I = \frac{8e^2 H^2}{(D-2)^2} \sum_{\pm\pm} (\pm)(\pm) \int d^D x' \sqrt{-g'} \int d^D x'' \sqrt{-g''} \frac{\partial}{\partial x^{[\mu}} \times K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}} K(y''_{+\pm}) I[A'^2(y'_{\pm\pm})] + \mathcal{O}(a^{-1}). \tag{103}$$

We shall often encounter integrals of this general form,

$$\sum_{\pm\pm} (\pm)(\pm) \int d^D x' \sqrt{-g'} \frac{\partial}{\partial x^{\mu}} K(y) \times \int d^D x'' \sqrt{-g''} \frac{\partial}{\partial x^{\nu}} K(y'') \times F(y'). \tag{104}$$

Our basic strategy for evaluating them is to extract d'Alembertians for the y'-dependent terms using the procedure described in Appendix A. In addition to d'Alembertians, this procedure will generally also produce a delta function and a finite remainder for which D can be set to 4,

$$F(y') = \frac{\square'}{H^2}[G(y')] + \text{const} \times \delta^D(x' - x'') + R(y'). \tag{105}$$

We partially integrate the $\square'$ to act upon $K(y)$, ignoring the order $1/a$ surface terms. To see the effect of acting $\square'$ on $K(y)$, note first that, up to delta function contributions arising from factors of $y^{1-D/2}$,

$$\frac{\square'}{H^2} f(y) = (4-y) y f''(y) + D(2-y) f'(y). \tag{106}$$

In view of relation (16), and the coefficient of $y^{1-(D/2)}$ in the expansion (15) of $K(y)$, we have

$$\frac{\square'}{H^2} K(y_{++}) = -\frac{2(D-2)}{a^D H^2} i\delta^D(x - x') + (D-2)K(y)$$

$$\text{and} \quad \frac{\square'}{H^2} K(y_{+-}) = (D-2)K(y). \tag{107}$$

Hence the general reduction (105) will produce four sorts of terms:

(1) A less divergent, 2-vertex integral from the factor of $(D-2)K(y)$ in (107);
(2) A potentially divergent, 1-vertex integral from the delta function in (107);
(3) A potentially divergent, 1-vertex integral from the delta function in (105); and
(4) A finite, 2-vertex integral from the remainder $R(y')$ in (105).

Our strategy is to extract another d'Alembertian from the type-1 terms and continue in this way until the limit $D = 4$ can be taken in all 2-vertex integrals. Appendix C explains how to evaluate these. It turns out that the 1-vertex integrals of type-3 can be usefully combined with the figure 8 diagram (85) to cancel an overlapping divergence.

Note that the integral (103) is symmetric under the exchange of the inner legs, $x' \leftrightarrow x''$, under which $y \leftrightarrow y''$ and $y' \leftrightarrow y'$. Because the integral is quartically divergent, it suffices to keep the three most divergent terms in the expansion of $I[A'^2(y'_{\pm\pm})]$. From

$$A'(y) = \frac{H^{D-2}}{(4\pi)^{D/2}} \frac{1}{4} \left\{ -\sum_{n=-1}^{\infty} \frac{\Gamma(n + \frac{D}{2} + 1)}{\Gamma(n+2)} \left(\frac{y}{4}\right)^{n-(D/2)+1} + \sum_{n=1}^{\infty} \frac{\Gamma(n + D - 1)}{\Gamma(n + \frac{D}{2})} \left(\frac{y}{4}\right)^{n-1} \right\} \tag{108}$$

we find

$$I[A'(y')^2] = -\frac{H^{2D-4}}{(4\pi)^D} \frac{\Gamma^2(\frac{D}{2})}{4} \left\{ \frac{1}{D-1} \left(\frac{y'}{4}\right)^{1-D} + \frac{D}{D-2} \left(\frac{y'}{4}\right)^{2-D} + \frac{D(D+1)}{2(D-3)} \left(\frac{y'}{4}\right)^{3-D} - \frac{4\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)(D-2)} \left(\frac{y'}{4}\right)^{1-(D/2)} + \frac{D(D+1)(D+2)}{6(D-4)} \left(\frac{y'}{4}\right)^{4-D} - \frac{D+6}{D-4} \frac{\Gamma(D+1)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+2)} \left(\frac{y'}{4}\right)^{2-(D/2)} + \mathcal{O}(y'^{5-D}, y'^{3-D/2}, y'^1) \right\}. \tag{109}$$

Note that the terms of the order y'^{4-D}, $y'^{2-D/2}$ or higher in Eq. (109) are suppressed by $(D-4)^1$. We are now ready to extract d'Alembertians from $I[A'(y')^2]$ defined in Eq. (109). Making use of relations in Appendix A, we obtain

$$
\begin{aligned}
I[A'(y')^2] = &-\frac{H^{2D-4}}{(4\pi)^D}\frac{\Gamma^2(\frac{D}{2})}{4}\Big\{\frac{\Box'}{H^2}\Big[\frac{2}{(D-1)(D-2)^2}\Big(\frac{y'}{4}\Big)^{2-D}+\frac{2(D+1)}{(D-1)(D-3)(D-4)}\Big(\Big(\frac{y'}{4}\Big)^{3-D}-\Big(\frac{y'}{4}\Big)^{1-(D/2)}\Big)\\
&+\frac{2(D+1)}{(D-4)(D-6)}\Big(\frac{D}{2(D-3)}-\frac{4}{(D-1)(D-4)}\Big)\Big(\frac{y'}{4}\Big)^{4-D}\\
&-\frac{2}{D-4}\Big(\frac{D(D-2)(D+1)}{2(D-1)(D-3)(D-4)}-\frac{4}{D-2}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big)\Big(\frac{y'}{4}\Big)^{2-(D/2)}\Big]\\
&+\frac{2(D+1)}{(D-1)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha')^D}i\delta^D(x'-x'')\\
&+\Big[\frac{D(D+1)(D+2)}{6(D-4)}-\frac{6(D+1)}{D-6}\Big(-\frac{4}{(D-1)(D-4)}+\frac{D}{2(D-3)}\Big)\Big]\Big(\frac{y'}{4}\Big)^{4-D}\\
&+\Big[-\frac{D+6}{D-4}\frac{\Gamma(D+1)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+2)}+\frac{D+2}{2}\Big(\frac{D(D+1)(D-2)}{2(D-1)(D-3)(D-4)}-\frac{4}{D-2}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big)\Big]\Big(\frac{y'}{4}\Big)^{2-(D/2)}\Big\}.
\end{aligned}
\tag{110}
$$

When inserted into the integral (103) the last two lines in Eq. (110) result in the finite, type-4 integral,

$$
\begin{aligned}
I_{\mathrm{fin},1} = &-e^2\frac{H^{2D-2}}{2(4\pi)^D}\Gamma^2\Big(\frac{D}{2}-1\Big)\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'a'^D\int d^Dx''a''^D\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y''_{+\pm})\\
&\times\Big[10\ln\Big(\frac{y'_{\pm\pm}}{4}\Big)+\mathcal{O}(a^{-1},D-4)\Big],
\end{aligned}
\tag{111}
$$

where we dropped the order $D-4$ terms as well as an irrelevant constant. The delta function in (110) leads to the type-3 integral,

$$
I_{\mathrm{sg},1} = -ie^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{\Gamma(\frac{D}{2}-1)(D+1)}{(D-1)(D-3)(D-4)}\sum_{\pm}(\pm)\int d^Dx'a'^D\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y_{+\pm}).
\tag{112}
$$

The d'Alembertian terms, Eq. (110), produce a type-1 integral,

$$
\begin{aligned}
I_1 = &-e^2\frac{H^{2D-2}}{(4\pi)^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'\sqrt{-g'}\int d^Dx''\sqrt{-g''}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y''_{+\pm})\\
&\times\Big[\frac{2}{(D-1)(D-2)^2}\Big(\frac{y'}{4}\Big)^{2-D}+\frac{2(D+1)}{(D-1)(D-3)(D-4)}\Big(\Big(\frac{y'}{4}\Big)^{3-D}-\Big(\frac{y'}{4}\Big)^{1-(D/2)}\Big)\\
&+\frac{2(D+1)}{(D-4)(D-6)}\Big[-\frac{4}{(D-1)(D-4)}+\frac{D}{2(D-3)}\Big]\Big(\frac{y'}{4}\Big)^{4-D}\\
&-\frac{2}{(D-4)}\Big[\frac{D(D-2)(D+1)}{2(D-1)(D-3)(D-4)}-\frac{4}{D-2}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big]\Big(\frac{y'}{4}\Big)^{2-(D/2)}\Big]_{\pm\pm},
\end{aligned}
\tag{113}
$$

and a type-2 integral,

$$
\begin{aligned}
I_{\mathrm{sg},2} = &\,ie^2\frac{H^{2D-4}}{(4\pi)^D}2\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}\\
&\times\Big[\frac{2}{(D-1)(D-2)^2}\Big(\frac{y}{4}\Big)^{2-D}+\frac{2(D+1)}{(D-1)(D-3)(D-4)}\Big(\Big(\frac{y}{4}\Big)^{3-D}-\Big(\frac{y}{4}\Big)^{1-(D/2)}\Big)\\
&+\frac{2(D+1)}{(D-4)(D-6)}\Big[-\frac{4}{(D-1)(D-4)}+\frac{D}{2(D-3)}\Big]\Big(\frac{y}{4}\Big)^{4-D}\\
&-\frac{2}{D-4}\Big[\frac{D(D-2)(D+1)}{2(D-1)(D-3)(D-4)}-\frac{4}{D-2}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big]\Big(\frac{y}{4}\Big)^{2-(D/2)}\Big]_{+\pm}.
\end{aligned}
\tag{114}
$$

We have seen that the total integral (103) breaks up to the sum

$$\mathcal{I} = \mathcal{I}_{\mathrm{fin},1} + \mathcal{I}_{\mathrm{sg},1} + \mathcal{I}_{\mathrm{sg},2} + \mathcal{I}_1 + \mathcal{O}(a^{-1}). \tag{115}$$

The next step is to extract another d'Alembertian from the y'-dependent terms of Eq. (113). The result is

$$\begin{aligned}\mathcal{I}_1 = &-e^2\frac{H^{2D-2}}{(4\pi)^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'\sqrt{-g'}\int d^Dx''\sqrt{-g''}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y''_{+\pm})\\ &\times\bigg\{\frac{\Box'}{H^2}\bigg[\frac{4}{(D-1)(D-2)^2(D-3)(D-4)}\bigg(\Big(\frac{y'}{4}\Big)^{3-D}-\Big(\frac{y'}{4}\Big)^{1-(D/2)}\bigg)\\ &+\frac{4}{(D-1)(D-4)^2(D-6)}\Big(-\frac{4}{(D-2)^2}+\frac{D+1}{D-3}\Big)\Big(\frac{y'}{4}\Big)^{4-D}\\ &-\frac{2}{(D-1)(D-3)(D-4)^2}\Big(\frac{D}{D-2}-2(D+1)\Big)\Big(\frac{y'}{4}\Big)^{2-(D/2)}\bigg]\\ &+\frac{2}{(D-1)(D-2)(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2})(a'H)^D}i\delta^D(x'-x'')\\ &-\frac{20}{3(D-4)^2}+\frac{86}{9(D-4)}-\frac{23}{27}-5\ln\Big(\frac{y'}{4}\Big)+\frac{5}{3}\ln^2\Big(\frac{y'}{4}\Big)+\mathcal{O}(D-4)\bigg\}_{\pm\pm},\end{aligned} \tag{116}$$

where we expanded the terms in the last line in powers of $D-4$. Just like $\mathcal{I}$ in Eq. (115), $\mathcal{I}_1$ can be decomposed into four sorts of integrals,

$$\mathcal{I}_1 = \mathcal{I}_{\mathrm{fin},2} + \mathcal{I}_{\mathrm{sg},3} + \mathcal{I}_{\mathrm{sg},4} + \mathcal{I}_2 + \mathcal{O}(a^{-1}). \tag{117}$$

The contributions of type-4, type-3, and type-2 are, respectively,

$$\begin{aligned}\mathcal{I}_{\mathrm{fin},2} = &-e^2\frac{H^{2D-2}}{2(4\pi)^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\sum_{\pm\pm}(\pm)(\pm)\int d^Dx'a'^D\int d^Dx''a''^D\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y''_{+\pm})\\ &\times\bigg[-10\ln\Big(\frac{y'_{\pm\pm}}{4}\Big)+\frac{10}{3}\ln^2\Big(\frac{y'_{\pm\pm}}{4}\Big)+\mathcal{O}(a^{-1},D-4)\bigg],\end{aligned} \tag{118}$$

$$\mathcal{I}_{\mathrm{sg},3} = -ie^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{2\Gamma(\frac{D}{2}-1)}{(D-1)(D-2)(D-3)(D-4)}\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y_{+\pm}), \tag{119}$$

and

$$\begin{aligned}\mathcal{I}_{\mathrm{sg},4} = &\,ie^2\frac{H^{2D-4}}{(4\pi)^D}4\Gamma^2\Big(\frac{D}{2}\Big)\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}\\ &\times\bigg\{\frac{4}{(D-1)(D-2)^2(D-3)(D-4)}\bigg(\Big(\frac{y}{4}\Big)^{3-D}-\Big(\frac{y}{4}\Big)^{1-(D/2)}\bigg)\\ &+\frac{4}{(D-1)(D-4)^2(D-6)}\Big(-\frac{4}{(D-2)^2}+\frac{D+1}{D-3}\Big)\Big(\frac{y}{4}\Big)^{4-D}\\ &-\frac{2}{(D-1)(D-3)(D-4)^2}\Big(\frac{D}{D-2}-2(D+1)\Big)\Big(\frac{y}{4}\Big)^{2-(D/2)}\bigg\}_{+\pm}.\end{aligned} \tag{120}$$

The type-1 integral is

$$I_2 = -e^2 \frac{H^{2D-2}}{(4\pi)^D} 2\Gamma^2\Big(\frac{D}{2}\Big) \sum_{\pm\pm} (\pm)(\pm) \int d^D x' \sqrt{-g'} \int d^D x'' \sqrt{-g''} \frac{\partial}{\partial x^{[\mu}} K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}} K(y''_{+\pm})$$
$$\times \Big\{ \frac{4}{(D-1)(D-2)^2(D-3)(D-4)} \Big[\frac{2}{(D-4)(D-6)} \frac{\square'}{H^2} \Big(\frac{y'}{4}\Big)^{4-D} + \frac{2}{(D-4)} \frac{\square'}{H^2} \Big(\frac{y'}{4}\Big)^{2-(D/2)} \Big] + \frac{8}{3(D-4)^2}$$
$$- \frac{35}{9(D-4)} + \frac{89}{27} + \frac{2}{3} \ln\Big(\frac{y'}{4}\Big) - \frac{2}{3} \ln^2\Big(\frac{y'}{4}\Big) + \mathcal{O}(D-4) \Big\}_{\pm\pm}. \tag{121}$$

This time there is no type-3 integral,

$$I_2 = I_{\mathrm{sg},5} + I_{\mathrm{fin},3} + I_{\mathrm{fin},4} + \mathcal{O}(a^{-1}). \tag{122}$$

The contributions of type-4, type-1 (with $D = 4$ taken), and type-3 are, respectively,

$$I_{\mathrm{fin},3} = -e^2 \frac{H^{2D-2}}{2(4\pi)^D} \Gamma^2\Big(\frac{D}{2}\Big) \sum_{\pm\pm} (\pm)(\pm) \int d^D x' a'^D \int d^D x'' a''^D \frac{\partial}{\partial x^{[\mu}} K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}} K(y''_{+\pm})$$
$$\times \Big[\frac{8}{3} \ln\Big(\frac{y'_{\pm\pm}}{4}\Big) - \frac{8}{3} \ln^2\Big(\frac{y'_{\pm\pm}}{4}\Big) + \mathcal{O}(a^{-1}, D-4) \Big] \tag{123}$$

and

$$I_{\mathrm{fin},4} = -e^2 \frac{H^{2D-2}}{2(4\pi)^D} \frac{D-2}{2} \Gamma^2\Big(\frac{D}{2}\Big) \sum_{\pm\pm} (\pm)(\pm) \int d^D x' a'^D \int d^D x'' a''^D \frac{\partial}{\partial x^{[\mu}} K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}} K(y''_{+\pm})$$
$$\times \Big[\frac{4}{3} \ln\Big(\frac{y'_{\pm\pm}}{4}\Big) - \frac{2}{3} \ln^2\Big(\frac{y'_{\pm\pm}}{4}\Big) + \mathcal{O}(a^{-1}, D-4) \Big] \tag{124}$$

and

$$I_{\mathrm{sg},5} = ie^2 \frac{H^{2D-4}}{(4\pi)^D} 4(D-2)\Gamma^2\Big(\frac{D}{2}\Big) \sum_{\pm} (\pm) \int d^D x' \sqrt{-g'} \frac{\partial}{\partial x^{[\mu}} K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}}$$
$$\times \Big\{ \frac{8}{(D-1)(D-2)^2(D-3)(D-4)^2} \Big[\frac{1}{D-6} \Big(\frac{y}{4}\Big)^{4-D} + \Big(\frac{y}{4}\Big)^{2-(D/2)} \Big]_{+\pm} \Big\}. \tag{125}$$

The finite, 2-vertex integrals $I_{\mathrm{fin},1}$, $I_{\mathrm{fin},2}$, $I_{\mathrm{fin},3}$, and $I_{\mathrm{fin},4}$ in Eqs. (111), (119), (123), and (124) can be summed to give

$$I_{\mathrm{fin}} = I_{\mathrm{fin},1} + I_{\mathrm{fin},2} + I_{\mathrm{fin},3} + I_{\mathrm{fin},4} = -2 \frac{e^2 H^{2D-2}}{(4\pi)^D} \Gamma^2\Big(\frac{D}{2} - 1\Big) \sum_{\pm\pm} (\pm)(\pm) \int d^D x' a'^D \int d^D x'' a''^D \frac{\partial}{\partial x^{[\mu}} K(y_{+\pm}) g_{\nu][\sigma} \frac{\partial}{\partial x^{\rho]}}$$
$$\times K(y''_{+\pm}) \Big[\ln\Big(\frac{y'_{\pm\pm}}{4}\Big) + \mathcal{O}(a^{-1}, D-4) \Big] \tag{126}$$

$$= -e^2 \frac{H^{10}}{128\pi^8} \sum_{\pm\pm} (\pm)(\pm) \int d^4 x' a'^4 \int d^4 x'' a''^4 \Big(\frac{\partial}{\partial x^{[\mu}} \frac{1}{y_{+\pm}}\Big) g_{\nu][\sigma} \Big(\frac{\partial}{\partial x^{\rho]}} \frac{1}{y''_{+\pm}}\Big) \ln\Big(\frac{y'_{\pm\pm}}{4}\Big) + \mathcal{O}(a^{-1}, D-4). \tag{127}$$

In the last line we expanded around $D = 4$ and made use of Eq. (15),

$$K(y) \overset{D \to 4}{\longrightarrow} -\frac{H^2}{\pi^2} \frac{1}{y}. \tag{128}$$

In Appendix C we evaluate the integral (127). The result (C35) is a de Sitter invariant which does not contribute a leading logarithm,

$$I_{\mathrm{fin}} = \frac{H^D}{(4\pi)^{D/2}} \times \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}$$
$$\times \{-2 \times g_{\mu[\rho} g_{\sigma]\nu} + \mathcal{O}(a^{-1}, D-4)\}. \tag{129}$$

We are now going to evaluate the 1-vertex integrals $I_{\mathrm{sg},n}$ ($n = 1, 2, \ldots, 5$). In order to do so, we shall need the basic integrals

$$Y_n = \int d^D x' a'^D \Big[\ln^n\Big(\frac{y_{++}}{4}\Big) - \ln^n\Big(\frac{y_{+-}}{4}\Big) \Big] \tag{130}$$
$$(n = 1, 2, 3),$$

$$W_n = \int d^D x' a'^D \ln(a')\Big[\ln^n\Big(\frac{y_{++}}{4}\Big) - \ln^n\Big(\frac{y_{+-}}{4}\Big)\Big] \quad (n = 1, 2), \tag{131}$$

$$\Xi_\alpha = \int d^D x' a'^D \Big[\Big(\frac{y_{++}}{4}\Big)^{-\alpha} - \Big(\frac{y_{+-}}{4}\Big)^{-\alpha}\Big], \tag{132}$$

$$\Lambda_\alpha = \int d^D x' a'^D \ln(a')\Big[\Big(\frac{y_{++}}{4}\Big)^{-\alpha} - \Big(\frac{y_{+-}}{4}\Big)^{-\alpha}\Big]. \tag{133}$$

The Y_n and W_n integrals are evaluated in Appendix B. In the asymptotic regime when $a \to \infty$, these integrals are given by Eqs. (B29)–(B31) and by (B32) and (B33). The Ξ_α integrals can be found in Appendix B 3, Eqs. (B41)–(B51), while the Λ_α integrals ($\alpha = D-4, D-3, D-2, D-1$) are in Appendix B 4, Eqs. (B53)–(B58).

Recall that the single leg integral (85) and (56) is

$$\begin{aligned}
\mathcal{I}_\infty &= \mathcal{I}_{\infty,\mathrm{sg}} + \mathcal{I}_{\infty,\mathrm{fin}},\\
\mathcal{I}_{\infty,\mathrm{sg}} &= ie^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{2\cos(\frac{\pi}{2}D)\Gamma(D-2)\Gamma(1-\frac{D}{2})}{D-2}\\
&\quad\times\sum_\pm(\pm)\int d^Dx'\sqrt{-g'}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}\\
&\quad\times K(y_{+\pm}),\\
\mathcal{I}_{\infty,\mathrm{fin}} &= -ie^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{4\Gamma(D-2)}{(D-2)\Gamma(\frac{D}{2})}\\
&\quad\times\sum_\pm(\pm)\int d^Dx'\sqrt{-g'}\ln(a')\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\\
&\quad\times\frac{\partial}{\partial x^{\rho]}}K(y_{+\pm}).
\end{aligned}\tag{134}$$

The singular part of this integral $\mathcal{I}_{\infty,\mathrm{sg}}$ can be combined with the singular one leg integrals $\mathcal{I}_{\mathrm{sg},1}$ and $\mathcal{I}_{\mathrm{sg},3}$ in Eqs. (112) and (119) to yield

$$\begin{aligned}
\mathcal{I}_{\mathrm{sg},A} &= \mathcal{I}_{\infty,\mathrm{sg}} + \mathcal{I}_{\mathrm{sg},1} + \mathcal{I}_{\mathrm{sg},3}\\
&= ie^2\frac{H^{D-2}}{(4\pi)^{D/2}}\frac{1}{D-2}\Big\{2\cos\Big(\frac{\pi}{2}D\Big)\Gamma(D-2)\Gamma\Big(1-\frac{D}{2}\Big)\\
&\quad-\frac{D\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)}\Big\}\sum_\pm(\pm)\int d^Dx'a'^D\frac{\partial}{\partial x^{[\mu}}\\
&\quad\times K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}}K(y_{+\pm}),
\end{aligned}\tag{135}$$

where $K(y_{+\pm})$ is given by Eq. (15). The terms in curly brackets can be expanded in powers of $D-4$. The result is

$$5 + (D-4)\Big(-4 - \frac{\pi^2}{6} - \frac{5}{2}\gamma_E\Big) + \mathcal{O}((D-4)^2). \tag{136}$$

This means that the most divergent contributions from the two left diagrams in Fig. 1 cancel. This is an example of a general phenomenon of cancellation of overlapping divergences and simplifies our calculation considerably.

A further simplification is facilitated by the identity

$$\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\rho}f(y) = -H^2 g_{\mu\rho}(2-y)I[f](y) + \nabla_\mu\nabla_\rho I^2[f](y), \tag{137}$$

where $I[f](y)$ stands for the indefinite integral with respect to y. By making use of this identity and Eq. (136), we can express the integral in (135) in terms of two relatively straightforward integrals as follows,

$$\begin{aligned}
\mathcal{I}_{\mathrm{sg},A} &= -ie^2\frac{H^D}{(4\pi)^{D/2}}\frac{1}{D-2}\Big\{2\cos\Big(\frac{\pi}{2}D\Big)\Gamma(D-2)\\
&\quad\times\Gamma\Big(1-\frac{D}{2}\Big) - \frac{D\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)}\Big\}\\
&\quad\times\Big[g_{\mu[\rho}g_{\sigma]\nu}\sum_\pm(\pm)\int d^Dx'\sqrt{-g'}(2-y_{+\pm})\\
&\quad\times I[(K')^2](y_{+\pm}) - \frac{\nabla_{[\mu}g_{\nu][\sigma}\nabla_{\rho]}}{H^2}\\
&\quad\times\sum_\pm(\pm)\int d^Dx'\sqrt{-g'}I^2[(K')^2](y_{+\pm})\Big].
\end{aligned}\tag{138}$$

In order to accurately extract both the divergent and finite contributions from this integral, we shall keep the D dimensional form of the terms in K up to $y^{4-D/2}$ and y^2; in higher order terms y^n ($n \geq 3$), we set $D = 4$, since these yield contributions that are suppressed linearly in $D-4$. Making use of Eq. (15) we find

$$\begin{aligned}
K'(y)^2 &= \frac{H^{2D-4}}{(4\pi)^D}\frac{(D-2)^2}{4}\Gamma^2\Big(\frac{D}{2}\Big)\Big\{\Big(\frac{y}{4}\Big)^{-D} + (D-4)\Big(\frac{y}{4}\Big)^{1-D} + \frac{D^2-7D+8}{2}\Big(\frac{y}{4}\Big)^{2-D} + \frac{4\Gamma(D-2)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+1)}\Big(\frac{y}{4}\Big)^{-(D/2)}\\
&\quad+\frac{D(D-2)(D-7)}{6}\Big(\frac{y}{4}\Big)^{3-D} + (D+8)\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+2)}\Big(\frac{y}{4}\Big)^{1-(D/2)} + \mathcal{O}(y^{4-D}, y^{2-(D/2)}, y^0,)\Big\},
\end{aligned}\tag{139}$$

from which it follows that

$$(2-y)I[K'^2](y) = -\frac{H^{2D-4}}{(4\pi)^D}4(D-2)^2\Gamma^2\Big(\frac{D}{2}\Big)\bigg\{\frac{1}{2(D-1)}\Big(\frac{y}{4}\Big)^{1-D} + \bigg[\frac{D-4}{2(D-2)} - \frac{1}{D-1}\bigg]\Big(\frac{y}{4}\Big)^{2-D} + \bigg[\frac{D^2-7D+8}{4(D-3)} - \frac{D-4}{D-2}\bigg]\Big(\frac{y}{4}\Big)^{3-D} + \frac{2\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{y}{4}\Big)^{1-(D/2)} + \bigg[\frac{D(D-2)(D-7)}{12(D-4)} - \frac{D^2-7D+8}{2(D-3)}\bigg]\Big(\frac{y}{4}\Big)^{4-D} + \bigg[\frac{(D-2)^2(D+8)}{2(D-4)} - 2(D+2)\bigg] \times \frac{\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+2)}\Big(\frac{y}{4}\Big)^{2-(D/2)} + \mathcal{O}(y^{5-D}, y^{3-D/2}, y^1,)\bigg\} \tag{140}$$

and

$$I^2[K'^2](y) = \frac{H^{2D-4}}{(4\pi)^D}4(D-2)^2\Gamma^2\Big(\frac{D}{2}\Big)\bigg\{\frac{1}{(D-1)(D-2)}\Big(\frac{y}{4}\Big)^{2-D} + \frac{(D-4)}{(D-2)(D-3)}\Big(\frac{y}{4}\Big)^{3-D} + \frac{D^2-7D+8}{2(D-3)(D-4)}\Big(\frac{y}{4}\Big)^{4-D} + \frac{8\Gamma(D-2)}{(D-4)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big(\frac{y}{4}\Big)^{2-(D/2)} + \mathcal{O}(y^{5-D}, y^{3-D/2}, y^1,)\bigg\}. \tag{141}$$

When these expressions (140) and (141) are inserted into Eq. (138) and one makes use of the integrals (B41)–(B51) evaluated in Appendix B 3, one arrives at the following expression for $I_{\mathrm{sg},A}$ (138),

$$I_{\mathrm{sg},A} = ie^2\frac{H^{3D-4}}{(4\pi)^{3D/2}}8\Gamma^2\Big(\frac{D}{2}\Big)\bigg\{\cos\Big(\frac{\pi}{2}D\Big)\Gamma(D-1)\Gamma\Big(1-\frac{D}{2}\Big) - \frac{2\Gamma(\frac{D}{2}+1)}{(D-3)(D-4)}\bigg\} \times\bigg\{g_{\mu[\rho}g_{\sigma]\nu}\bigg[\frac{1}{2(D-1)}\Xi_{D-1} + \Big(\frac{D-4}{2(D-2)} - \frac{1}{D-1}\Big)\Xi_{D-2} + \Big(\frac{D^2-7D+8}{4(D-3)} - \frac{D-4}{D-2}\Big)\Xi_{D-3} + \frac{2\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Xi_{(D/2)-1} + \Big(\frac{D(D-2)(D-7)}{12(D-4)} - \frac{D^2-7D+8}{2(D-3)}\Big)\Xi_{D-4} + \Big(\frac{(D-2)^2(D+8)}{2(D-4)} - 2(D+2)\Big) \times\frac{\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+2)}\Xi_{(D/2)-2}\bigg] + \frac{\nabla_{[\mu}g_{\nu][\sigma}\nabla_{\rho]}}{H^2}\bigg[\frac{1}{(D-1)(D-2)}\Xi_{D-2} + \frac{D-4}{(D-2)(D-3)}\Xi_{D-3} + \frac{D^2-7D+8}{2(D-3)(D-4)}\Xi_{D-4} + \frac{8\Gamma(D-2)}{(D-4)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Xi_{(D/2)-2}\bigg]\bigg\}. \tag{142}$$

Recalling that the operator

$$\frac{\nabla_{[\mu}g_{\nu][\sigma}\nabla_{\rho]}}{H^2} = a^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}a^2\frac{d^2}{da^2} - g_{\mu[\rho}g_{\sigma]\nu}a\frac{d}{da}, \tag{143}$$

Eq. (142) can be evaluated and expanded in powers of $D-4$. Here we provide an intermediate result, which also shows the terms linear in $(D-4)$,

$$I_{\mathrm{sg},A} = -e^2\frac{H^{2D-4}}{(4\pi)^D}8\frac{\Gamma^3(\frac{D}{2})}{\Gamma(D)}\bigg[\cos\Big(\frac{\pi D}{2}\Big)\Gamma(D-1)\Gamma\Big(1-\frac{D}{2}\Big) - \frac{2\Gamma(\frac{D}{2}+1)}{(D-3)(D-4)}\bigg]\bigg\{g_{\mu[\rho}g_{\sigma]\nu}\bigg[\frac{\Gamma(D)}{\Gamma^2(\frac{D}{2})}\frac{D-6}{2(D-3)(D-4)} - 6 + (D-4)\Big(-\frac{1}{2}\ln^2(a) + \frac{13}{6}\ln(a) + \frac{19}{12} + \frac{\pi^2}{12}\Big)\bigg] + a^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}(D-4)\Big(\ln(a) - \frac{5}{3}\Big)\bigg\} + \mathcal{O}(a^{-1},(D-4)^2). \tag{144}$$

The final result is

$$I_{\mathrm{sg},A} = \frac{H^D}{(4\pi)^{D/2}}\times\frac{e^2H^{D-4}}{(4\pi)^{D/2}}\bigg[\frac{20\Gamma(D-1)}{D-4} - 72 - \frac{4}{3}\pi^2\bigg]g_{\mu[\rho}g_{\sigma]\nu} + \mathcal{O}(a^{-1}, D-4). \tag{145}$$

The second part of the singular contribution to the one leg integral is the sum of Eqs. (114), (120), and (125),

$$
\begin{aligned}
I_{\mathrm{sg},B} &= I_{\mathrm{sg},2} + I_{\mathrm{sg},4} + I_{\mathrm{sg},5} \\
&= ie^2 \frac{H^{2D-4}}{(4\pi)^D} 2\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}\frac{\partial}{\partial x^{[\mu}}K(y_{+\pm})g_{\nu][\sigma}\frac{\partial}{\partial x^{\rho]}} \\
&\quad\times\Big\{\frac{2}{(D-1)(D-2)^2}\Big(\frac{y}{4}\Big)^{2-D} + \frac{2D}{(D-2)(D-3)(D-4)}\Big(\Big(\frac{y}{4}\Big)^{3-D} - \Big(\frac{y}{4}\Big)^{1-(D/2)}\Big) + \frac{D(D+2)}{(D-2)(D-4)(D-6)}\Big(\frac{y}{4}\Big)^{4-D} \\
&\quad+\Big[-\frac{D}{(D-3)(D-4)} + \frac{8}{(D-2)(D-4)}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big]\Big(\frac{y}{4}\Big)^{2-(D/2)}\Big\}_{+\pm}.
\end{aligned}
\tag{146}
$$

This can be expressed in terms of the $\Omega_{\alpha,\beta}$ integrals evaluated in Appendix B 5 (as usual, the expression below is antisymmetrized in both pairs of indices $[\mu,\nu]$ and $[\rho,\sigma]$),

$$
\begin{aligned}
I_{\mathrm{sg},B} &= -ie^2\frac{H^{3D-6}}{(4\pi)^{3D/2}}8\Gamma^2\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)g_{\nu[\sigma}\Big\{\frac{2}{(D-1)(D-2)^2}\Big[\Omega_{D-2,(D/2)-1} + \frac{D-2}{2}\Omega_{D-2,(D/2)-2} \\
&\quad+\frac{D(D-2)}{8}\Omega_{D-2,(D/2)-3} - \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+1)}\Omega_{D-2,-1} + \frac{D(D-2)(D+2)}{48}\Omega_{D-2,(D/2)-4} \\
&\quad-\frac{\Gamma(D)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+2)}\Omega_{D-2,-2}\Big] + \frac{2D}{(D-2)(D-3)(D-4)}\Big[(\Omega_{D-3,(D/2)-1} - \Omega_{(D/2)-1,(D/2)-1}) \\
&\quad+\frac{D-2}{2}(\Omega_{D-3,(D/2)-2} - \Omega_{(D/2)-1,(D/2)-2}) + \frac{D(D-2)}{8}(\Omega_{D-3,(D/2)-3} - \Omega_{(D/2)-3,(D/2)-1}) \\
&\quad-\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}+1)}(\Omega_{D-3,-1} - \Omega_{(D/2)-1,-1})\Big] + \frac{D(D+2)}{(D-2)(D-4)(D-6)} \\
&\quad\times\Big[\Omega_{D-4,(D/2)-1} + \frac{(D-2)}{2}\Omega_{D-4,(D/2)-2}\Big] + \Big[-\frac{D}{(D-3)(D-4)} + \frac{8}{(D-2)(D-4)}\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Big] \\
&\quad\times\Big[\Omega_{(D/2)-1,(D/2)-2} + \frac{D-2}{2}\Omega_{(D/2)-2,(D/2)-2}\Big]\Big\}_{\rho]\mu}.
\end{aligned}
\tag{147}
$$

The result is

$$
\begin{aligned}
I_{\mathrm{sg},B} &= e^2\frac{H^{2D-4}}{(4\pi)^D}\frac{8\Gamma^3(\frac{D}{2})\Gamma(\frac{D}{2}-1)}{\Gamma(D)}\Big\{g_{\nu[\sigma}g_{\rho]\mu}\Big[\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{1}{(3D-10)(D-3)(D-4)} \\
&\quad\times\Big(-\frac{D^2-5D+2}{2(D-1)(D-2)} + \frac{D(2D-7)}{(D-3)(D-4)}\Big) + \frac{1}{2(D-4)} + 3 \\
&\quad+(D-4)\Big(-3\ln^3(a) + \Big(-\frac{67}{6} + \frac{9\pi^2}{2}\Big)\ln(a) - \frac{191}{9} + \frac{5\pi^2}{48} - 18\zeta(3)\Big)\Big] \\
&\quad+a^2\delta^0_{[\rho}g_{\sigma][\nu}\delta^0_{\mu]}(D-4)\Big(-4\ln^2(a) + \frac{50}{3}\ln(a) - \frac{145}{6} + \frac{13\pi^2}{6}\Big)\Big\} + \mathcal{O}(a^{-1},(D-4)^2).
\end{aligned}
\tag{148}
$$

Upon expanding this in powers of $D-4$, one obtains

$$
I_{\mathrm{sg},B} = \frac{H^D}{(4\pi)^{D/2}}\times\frac{e^2H^{D-4}}{(4\pi)^{D/2}}\Big[\frac{8\Gamma(D-1)}{(D-4)^2} - \frac{40\Gamma(D-1)}{(D-1)(D-4)} + \frac{67}{3} - \frac{2}{3}\pi^2\Big]g_{\nu[\sigma}g_{\rho]\mu} + \mathcal{O}(a^{-1},D-4). \tag{149}
$$

What remains to calculate is the integral (134). This integral can be expressed in terms of the Λ_α integrals (B52). First, we make use of Eq. (137) to write (134) as

$$
\begin{aligned}
I_{\infty,\mathrm{fin}} &= ie^2\frac{H^D}{(4\pi)^{D/2}}\frac{4\Gamma(D-2)}{(D-2)\Gamma(\frac{D}{2})}\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}\ln(a')\Big[g_{\mu[\rho}g_{\sigma]\nu}\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}(2-y_{+\pm})I[(K')^2](y_{+\pm}) \\
&\quad-\frac{\nabla_{[\mu}g_{\nu][\sigma}\nabla_{\rho]}}{H^2}\sum_{\pm}(\pm)\int d^Dx'\sqrt{-g'}I^2[(K')^2](y_{+\pm})\Big].
\end{aligned}
\tag{150}
$$

Second, we make use of Eqs. (140) and (141) and the definition of Λ_α integrals (B52) to obtain [cf. Eq. (142)]

$$\mathcal{I}_{\infty,\mathrm{fin}} = -ie^2 \frac{H^{3D-4}}{(4\pi)^{3D/2}} 16\Gamma(D-1)\Gamma\Big(\frac{D}{2}\Big)\Big\{g_{\mu[\rho}g_{\sigma]\nu}\Big[\frac{1}{2(D-1)}\Lambda_{D-1} + \Big(\frac{D-4}{2(D-2)} - \frac{1}{D-1}\Big)\Lambda_{D-2} + \Big(\frac{D^2-7D+8}{4(D-3)} - \frac{D-4}{D-2}\Big)\Lambda_{D-3} + \frac{2\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Lambda_{(D/2)-1} + \Big(-\frac{D^2-7D+8}{2(D-3)} + \frac{D(D-2)(D-7)}{12(D-4)}\Big)\Lambda_{D-4} + \Big(\frac{(D-2)^2(D+8)}{2(D-4)} - 2(D+2)\Big)\frac{\Gamma(D-2)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+2)}\Lambda_{(D/2)-2}\Big] + \frac{\nabla_{[\mu}g_{\nu][\sigma}\nabla_{\rho]}}{H^2}\Big[\frac{1}{(D-1)(D-2)}\Lambda_{D-2} + \frac{D-4}{(D-2)(D-3)}\Lambda_{D-3} + \frac{D^2-7D+8}{2(D-3)(D-4)}\Lambda_{D-4} + \frac{8\Gamma(D-2)}{(D-4)\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}\Lambda_{(D/2)-2}\Big]\Big\}. \tag{151}$$

This can be evaluated by making use of Eq. (143) and Eqs. (B53)–(B58). Keeping the terms up to order $(D-4)$, we have

$$\mathcal{I}_{\infty,\mathrm{fin}} = e^2 \frac{H^{2D-4}}{(4\pi)^D}\frac{16\Gamma^2(\frac{D}{2})}{D-1}\Big\{g_{\mu[\rho}g_{\sigma]\nu}\Big[\frac{\Gamma(D)}{\Gamma^2(\frac{D}{2})}\frac{1}{(D-3)(D-4)}\Big(\frac{D-6}{2}\ln(a) - \frac{2D-3}{(D-1)(D-2)}\Big) + (-6\ln(a) + 1 - \pi^2) + (D-4)\Big(\frac{7}{12}\ln^3(a) + \frac{13}{12}\ln^2(a) + \Big(\frac{25}{24} - \frac{2\pi^2}{3}\Big)\ln(a) + \frac{1309}{432} - \frac{7\pi^2}{9} + 4\zeta(3)\Big)\Big] + a^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}\Big[-\frac{\Gamma(D)}{\Gamma^2(\frac{D}{2})}\frac{1}{(D-1)(D-3)(D-4)} - 2 + (D-4)\Big(\frac{5}{4}\ln^2(a) - \frac{53}{12}\ln(a) + \frac{119}{18} - \frac{2\pi^2}{3}\Big)\Big]\Big\} + \mathcal{O}(a^{-1}, D-4). \tag{152}$$

Upon rearranging the divergent contributions and dropping the $\mathcal{O}(D-4)$ terms, one obtains

$$\mathcal{I}_{\infty,\mathrm{fin}} = \frac{H^D}{(4\pi)^{D/2}} \times \frac{e^2 H^{D-4}}{(4\pi)^{D/2}}\Big\{\Big[\Big(-\frac{16\Gamma(D-1)}{D-4} + 16\Big)\ln(a) - \frac{40\Gamma(D-1)}{(D-1)(D-4)} + \frac{104}{3} - \frac{16}{3}\pi^2\Big]g_{\mu[\rho}g_{\sigma]\nu} - \frac{16\Gamma(D-1)}{(D-1)(D-4)} \times a^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}\Big\} + \mathcal{O}(a^{-1}, D-4). \tag{153}$$

Summing the integrals (81), (129), (145), (149), and (153) we finally get

$$\langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\text{two loop}} = \mathcal{I}_{\text{c.t.}} + \mathcal{I}_{\text{fin}} + \mathcal{I}_{\text{sg},A} + \mathcal{I}_{\text{sg},B} + \mathcal{I}_{\infty,\text{fin}} + \mathcal{O}(a^{-1}) = -\delta Z_3 \times \langle\Omega|F_{\mu\nu}(x)F_{\rho\sigma}(x)|\Omega\rangle_{\text{one loop}} + \frac{H^D}{(4\pi)^{D/2}} \times \frac{e^2H^{D-4}}{(4\pi)^{D/2}}\Big\{\Big[\Big[-\frac{16\Gamma(D-1)}{D-4} + 16\Big]\ln(a) + \frac{8\Gamma(D-1)}{(D-4)^2} - \frac{20\Gamma(D-1)}{(D-1)(D-4)} - \frac{11}{3} - \frac{22}{3}\pi^2\Big]g_{\mu[\rho}g_{\sigma]\nu} - \frac{16\Gamma(D-1)}{(D-1)(D-4)} \times a^2\delta^0_{[\mu}g_{\nu][\sigma}\delta^0_{\rho]}\Big\} + \mathcal{O}(a^{-1}, D-4). \tag{154}$$

This corresponds to the following values for the D-dependent quantities E_1, E_2, and F_1 defined in expressions (45), (63), and (64),

$$E_1 = -\frac{16\Gamma(D-1)}{D-4} + 16 + \mathcal{O}(D-4), \tag{155}$$

$$E_2 = -\delta Z_3 \times \frac{(4\pi)^{D/2}}{e^2H^{D-4}} \times \frac{2\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)} + \frac{8\Gamma(D-1)}{(D-4)^2} - \frac{20\Gamma(D-1)}{(D-1)(D-4)} - \frac{11}{3} - \frac{22}{3}\pi^2 + \mathcal{O}(D-4), \tag{156}$$

$$F_1 = -\frac{16\Gamma(D-1)}{(D-1)(D-4)} + \mathcal{O}(D-4). \tag{157}$$

Our result for E_1 agrees with the stochastic prediction (69) [1], and our result for F_1 agrees with the prediction (70) based upon conservation.

VI. DISCUSSION

By substituting (155)–(157) into expressions (65) and (66) we obtain two loop results for the field strength contributions to the pressure p and to $(\rho + p)$,

$$(p)_{\rm Maxwell} = \frac{H^D}{(4\pi)^{D/2}}\Big\{-\frac{(D-4)\Gamma(D)}{4\Gamma(\frac{D}{2}+1)} + \frac{e^2H^{D-4}}{(4\pi)^{D/2}} \times\Big[12\ln(a) - \frac{11\Gamma(D-1)}{(D-1)(D-4)} - \frac{1}{3} + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}, \qquad (158)$$

$$(\rho+p)_{\rm Maxwell} = \frac{H^D}{(4\pi)^{D/2}}\Big\{0 + \frac{e^2H^{D-4}}{(4\pi)^{D/2}} \times\Big[-\frac{8\Gamma(D-1)}{(D-1)(D-4)} - \frac{8}{3} + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}. \qquad (159)$$

The $\ln(a)$ term in (158) agrees exactly with the stochastic prediction [1]. Together with the two loop results for the expectation values of $(D_\mu\varphi)^*D_\nu\varphi$ and $\varphi^*\varphi$ [2], this constitutes an impressive verification of the stochastic formalism.

Note that we have not bothered to work out the terms which fall off like powers of $1/a$. Just as for the stress-energy tensor of $\lambda\varphi^4$—expressions (50) and (51)—these terms are separately conserved and play no role at late times. It has been conjectured that the $1/a$ contributions can be subsumed into a perturbative correction of the initial state [33]. Such a correction is in any case inevitable because a free vacuum cannot be a very realistic initial state of an interacting theory.

We have also not bothered to renormalize the composite operators $F_{\mu\nu}F_{\rho\sigma}$, $(D_\mu\varphi)^*D_\nu\varphi$, and $\varphi^*\varphi$. Recall that conventional renormalization only removes divergences from noncoincident 1PI functions. One would need to additionally subtract a series of counteroperators from $F_{\mu\nu}F_{\rho\sigma}$ and the others to remove their divergences, and there would be the usual ambiguities about the finite part [62,63]. This is neither necessary nor desirable. The stochastic formalism, which it was our purpose to check, makes unique predictions for the leading logarithm contributions to the dimensionally regulated expectation values of the three composite operators [1], and these predictions agree in each case.

Accuracy is always an issue in such an intricate computation. Our result essentially consists of three numbers: E_1, E_2, and F_1, which were defined in expressions (45), (63), and (64) and reported in (155)–(157). An obvious check on E_1 is that it agrees with the stochastic formalism for which a compelling and independent theoretical justification exists [1]. A powerful check on F_1 is provided by partial conservation within the electromagnetic sector (61) and (62). No direct check exists for E_2 but it was of course computed using the same reduction strategy and many of the same integrals that produced correct results for E_1 and F_1.

Combining the field strength contributions (158) and (159) with the corresponding scalar results (75) and (76) gives the total for the stress-energy tensor of SQED (49),

$$(p)_{\rm SQED} = \frac{H^D}{(4\pi)^{D/2}}\Big\{\frac{\Gamma(D)}{2\Gamma(\frac{D}{2})} + \frac{e^2H^{D-4}}{(4\pi)^{D/2}} \times\Big[12\ln(a) - \frac{3\Gamma(D-1)}{(D-4)} + 33 - 4\pi^2 + \mathcal{O}(a^{-1}, D-4)\Big] + \mathcal{O}(e^4)\Big\}, \qquad (160)$$

$$(\rho+p)_{\rm SQED} = \frac{H^D}{(4\pi)^{D/2}} \times\Big\{0 + \frac{e^2H^{D-4}}{(4\pi)^{D/2}}[4 + \mathcal{O}(a^{-1}, D-4)] + \mathcal{O}(e^4)\Big\}. \qquad (161)$$

Note that the divergent field strength and scalar contributions to $(\rho+p)$ have canceled. This is required by the fact that the expectation value of $T_{\mu\nu}$ gives the matter contributions to the 1-graviton 1PI function in gravity + SQED. Hence the expectation value of $T_{\mu\nu}$ can be renormalized by purely gravitational counterterms, and all of these degenerate to constants times $g_{\mu\nu}$ on de Sitter background. Because the expectation value of $T_{\mu\nu}$ takes the form $pg_{\mu\nu} + (\rho+p)a^2\delta^0_\mu\delta^0_\nu$, we see that expression (160) for p can contain divergent constants, but the $\ln(a)$ terms in p and all of $(\rho+p)$ must be finite. Because the constant contribution to the pressure can be absorbed into a renormalization of the cosmological constant, this term has no physical significance—which is reassuring because it is the least well checked.

Although this exercise was undertaken to check the stochastic formalism, our results (160) and (161) have considerable physical interest in their own right. After many e-foldings the $\ln(a)$ contribution to the pressure (160) must dominate, and we see that SQED induces a growing, *negative* vacuum energy. Note that there is no simple renormalization group interpretation for (160) and (161) in terms of some time-dependent, running coupling constant, just as there is none for the similar time dependence of the VEV of the stress tensor of massless, minimally coupled $\lambda\varphi^4$ [66]. This is obvious from the fact that the zeroth order energy density and pressure are independent of the coupling constant, while the stress tensor is unaffected by field strength renormalization. The physical interpretation of our result (160) is instead that the vacuum becomes polarized by the inflationary production of charged scalars. Just as a dielectric slab will be drawn into the region between the plates of a charged capacitor, the production of additional scalars seems to be favored by the electric fields of those that came before.

It is natural to wonder how far this progresses, both at higher orders in the loop expansion and in time. Recall that the two loop $\ln(a)$ derived entirely from the field strength contributions. That is not true at higher orders; however, only the field strength contributions act to decrease the vacuum energy [1]. At ℓ loop order, one gets $\ell - 1$ factors of $e^2/(4\pi)^2$, and there can be up to $\ell - 1$ factors of $\ln(a)$ [1]. The leading logarithms at all loops become order 1 after $\ln(a) \sim (4\pi)^2/e^2$ e-foldings. At this point perturbation theory breaks down and one must employ a nonperturbative resummation scheme to evolve further. It was to solve this sort of problem that the stochastic formulation of SQED was developed, and the answer is known [1],

$$\lim_{t\to\infty} p(t) \approx 0.6551 \times \frac{3H^4}{8\pi^2} \approx \frac{\Lambda}{8\pi G} \times 0.2085 \times GH^2. \tag{162}$$

Here $\Lambda = 3H^2$ is the bare cosmological constant and G is Newton's constant.

Note that there is no simple renormalization group interpretation for the late time limit (162). Because scalar QED runs to the trivial fixed point in the far infrared, the renormalization group prediction would be (160) with e^2 set to zero, just as would be the case for the coupling constant of massless, minimally coupled $\lambda\varphi^4$ [66]. The failure of the renormalization group prediction is even more evident when one examines the highly non-Gaussian expectation values of other quantities using Eqs. (171–172) of Ref. [1]. What we are seeing is not a renormalization group flow but rather the effect of inflationary particle production filling an initially empty universe with very long wavelength, charged scalars.

Although (162) is a nonperturbatively large decrease in the vacuum energy, it is suppressed by GH^2 relative to the vacuum energy of the bare cosmological constant. The largest value of GH^2 consistent with the current upper limit on the tensor-to-scalar ratio for anisotropies in the cosmic microwave background is about 10^{-12} [67], so (162) does not represent a significant decrease of the vacuum energy. On the other hand, it is an enormous amount of vacuum energy by current scales. Further, the shift is dynamic; it was caused by inflationary particle production and it would presumably dissipate, on some time scale, after the end of inflation. This may well have important consequences for cosmology [23].

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APPENDIX A: EXTRACTING D'ALEMBERTIANS

Here we list the results of extracting the d'Alembertians from powers of $y/4$ which reduce the degree of divergence by 2.

When a d'Alembertian acts on a nonsingular function $F = F(y)$, one obtains (106),

$$\frac{\square}{H^2} F(y) = (4 - y)yF''(y) + D(2 - y)F'(y). \tag{A1}$$

(A nonsingular function F of $y = y(x; x')$ is a function which, when expanded in powers of y, does not contain the power $y^{1-D/2}$.) Equation (A1) is helpful for establishing the relations

$$\left(\frac{y}{4}\right)^{-\alpha} = -\frac{1}{(\alpha - 1)(\frac{D}{2} - \alpha)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{1-\alpha} + \frac{D - \alpha}{\frac{D}{2} - \alpha} \left(\frac{y}{4}\right)^{1-\alpha} \quad (\alpha \neq D/2), \tag{A2}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^{1-(D/2)} = \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2} - 1)(Ha)^D} i\delta^D(x - x') + \frac{D(D - 2)}{4} \left(\frac{y}{4}\right)^{1-(D/2)}. \tag{A3}$$

For example, for $\alpha = 3D/2 - 3, 3D/2 - 4, 3D/2 - 5, D, D - 1, D - 2, D - 3, D - 4, (D/2) - 1, (D/2) - 2$, these give

$$\left(\frac{y}{4}\right)^{3-(3D/2)} = \left[\frac{2}{(3D - 8)(D - 3)} \frac{\square}{H^2} + \frac{D - 6}{2(D - 3)}\right] \times \left(\frac{y}{4}\right)^{4-(3D/2)}, \tag{A4}$$

$$\begin{aligned}\left(\frac{y}{4}\right)^{4-(3D/2)} = &\left[\frac{2}{(3D - 10)(D - 4)} \frac{\square}{H^2} + \frac{D - 8}{2(D - 4)}\right] \times \left(\frac{y}{4}\right)^{5-(3D/2)} + \frac{2}{(3D - 10)(D - 4)} \\ &\times \left[-\frac{\square}{H^2} + \frac{D(D - 2)}{4}\right]\left(\frac{y}{4}\right)^{1-(D/2)} \\ &+ \frac{2}{(3D - 10)(D - 4)} \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2} - 1)} \frac{i\delta^D(x - x')}{(aH)^D},\end{aligned} \tag{A5}$$

$$\left(\frac{y}{4}\right)^{5-(3D/2)} = \left[\frac{2}{3(D - 4)(D - 5)} \frac{\square}{H^2} + \frac{D - 10}{2(D - 5)}\right] \times \left(\frac{y}{4}\right)^{6-(3D/2)}, \tag{A6}$$

$$\left(\frac{y}{4}\right)^{-D} = \frac{2}{D(D-1)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{1-D}, \quad \text{(A7)}$$

$$\left(\frac{y}{4}\right)^{1-D} = \frac{2}{(D-2)^2} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{2-D} - \frac{2}{D-2} \left(\frac{y}{4}\right)^{2-D}, \quad \text{(A8)}$$

$$\left(\frac{y}{4}\right)^{2-D} = \left[\frac{2}{(D-3)(D-4)} \frac{\square}{H^2} - \frac{4}{D-4}\right] \left(\frac{y}{4}\right)^{3-D} + \frac{2}{(D-3)(D-4)} \left[-\frac{\square}{H^2} + \frac{D(D-2)}{4}\right] \times \left(\frac{y}{4}\right)^{1-(D/2)} + \frac{2}{(D-3)(D-4)} \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)} \times \frac{i\delta^D(x-x')}{(aH)^D}, \quad \text{(A9)}$$

$$\left(\frac{y}{4}\right)^{3-D} = \frac{2}{(D-4)(D-6)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{4-D} - \frac{6}{D-6} \left(\frac{y}{4}\right)^{4-D}, \quad \text{(A10)}$$

$$\left(\frac{y}{4}\right)^{4-D} = \frac{2}{(D-5)(D-8)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{5-D} - \frac{8}{D-8} \left(\frac{y}{4}\right)^{5-D}, \quad \text{(A11)}$$

$$\left(\frac{y}{4}\right)^{1-(D/2)} = -\frac{2}{D-4} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{2-(D/2)} + \frac{D+2}{2} \times \left(\frac{y}{4}\right)^{2-(D/2)}, \quad \text{(A12)}$$

$$\left(\frac{y}{4}\right)^{2-(D/2)} = -\frac{1}{D-6} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{3-(D/2)} + \frac{D+4}{4} \times \left(\frac{y}{4}\right)^{3-(D/2)}. \quad \text{(A13)}$$

When the d'Alembertian acts on a power of y, one gets

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^{-\beta} = -\beta\left(\frac{D}{2} - \beta - 1\right) \left(\frac{y}{4}\right)^{-\beta-1} + \beta(D-\beta-1) \left(\frac{y}{4}\right)^{-\beta} \quad \text{(A14)}$$

which, when applied to $\beta = (D/2) - 1$, $(D/2) - 2$, $(D/2) - 3$, $(D/2) - 4$, -2, -1, yields

$$0 = -\frac{\square}{H^2} \left(\frac{y}{4}\right)^{1-(D/2)} + \frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)(Ha)^D} i\delta^D(x-x') + \frac{D(D-2)}{4} \left(\frac{y}{4}\right)^{1-(D/2)}, \quad \text{(A15)}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^{2-(D/2)} = -\frac{D-4}{2} \left(\frac{y}{4}\right)^{1-(D/2)} + \frac{(D-4)(D+2)}{4} \left(\frac{y}{4}\right)^{2-(D/2)}, \quad \text{(A16)}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^{3-(D/2)} = -(D-6) \left(\frac{y}{4}\right)^{2-(D/2)} + \frac{(D-6)(D+4)}{4} \left(\frac{y}{4}\right)^{3-(D/2)}, \quad \text{(A17)}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^{4-(D/2)} = -\frac{3(D-8)}{2} \left(\frac{y}{4}\right)^{3-(D/2)} + \frac{(D-8)(D+6)}{4} \left(\frac{y}{4}\right)^{4-(D/2)}, \quad \text{(A18)}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right)^2 = -2(D+1) \left(\frac{y}{4}\right)^2 + (D+2) \left(\frac{y}{4}\right), \quad \text{(A19)}$$

$$\frac{\square}{H^2} \left(\frac{y}{4}\right) = -D \left(\frac{y}{4}\right) + \frac{D}{2}. \quad \text{(A20)}$$

Next we make use of the identity (96),

$$\nabla_\mu \nabla_\rho y = (2-y) H^2 g_{\mu\rho} \quad \text{(A21)}$$

to obtain ($\alpha + \beta + 1 \neq D/2$)

$$\partial_\mu \left(\frac{y}{4}\right)^{-\alpha} \partial_\rho \left(\frac{y}{4}\right)^{-\beta} = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)} \nabla_\rho \nabla_\mu \left[\left(\frac{y}{4}\right)^{-(\alpha+\beta)}\right] + \frac{\alpha\beta H^2 g_{\mu\rho}}{D - 2(\alpha+\beta+1)} \times \left[-\frac{1}{(\alpha+\beta)(\alpha+\beta+1)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{-(\alpha+\beta)} + \left(\frac{y}{4}\right)^{-(\alpha+\beta)}\right]. \quad \text{(A22)}$$

The relevant cases are $(\alpha, \beta) = (D/2-1, D-2)$, $(D/2-1, D-3)$, $(D/2-1, D-4)$, $(D/2-1, D-5) \times (D/2-1, D/2-1)$, $(D/2-1, D/2-2)$, $(D/2-1, D/2-3)$, and $(D/2-1, -1)$. From Eq. (A22) we have

$$\partial_\mu \left(\frac{y}{4}\right)^{1-(D/2)} \partial_\rho \left(\frac{y}{4}\right)^{2-D} = \frac{2(D-2)}{3(3D-4)} \nabla_\rho \nabla_\mu \left[\left(\frac{y}{4}\right)^{3-(3/2)D}\right] + H^2 g_{\mu\rho} \left[\frac{1}{3(3D-4)} \frac{\square}{H^2} \left(\frac{y}{4}\right)^{3-(3/2)D} - \frac{D-2}{4} \left(\frac{y}{4}\right)^{3-(3/2)D}\right], \quad \text{(A23)}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{3-D} = \frac{2(D-3)}{3(3D-8)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{4-(3/2)D}\Big] + H^2 g_{\mu\rho}\Big[\frac{1}{3(3D-8)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{4-(3/2)D} - \frac{D-2}{4}\Big(\frac{y}{4}\Big)^{4-(3/2)D}\Big], \tag{A24}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{4-D} = \frac{2(D-2)(D-4)}{(3D-10)(3D-8)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{5-(3/2)D}\Big] + (D-2)H^2 g_{\mu\rho}\Big[\frac{1}{(3D-10)(3D-8)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{5-(3/2)D} - \frac{1}{4}\Big(\frac{y}{4}\Big)^{5-(3/2)D}\Big], \tag{A25}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{5-D} = \frac{2(D-2)(D-5)}{3(3D-10)(D-4)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{6-(3/2)D}\Big] + (D-2)H^2 g_{\mu\rho}\Big[\frac{1}{3(3D-10)(D-4)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{6-(3/2)D} - \frac{1}{4}\Big(\frac{y}{4}\Big)^{6-(3/2)D}\Big], \tag{A26}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{1-(D/2)} = \frac{D-2}{4(D-1)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{2-D}\Big] + H^2 g_{\mu\rho}\Big[\frac{1}{4(D-1)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{2-D} - \frac{D-2}{4}\Big(\frac{y}{4}\Big)^{2-D}\Big], \tag{A27}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{2-(D/2)} = \frac{D-4}{4(D-3)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{3-D}\Big] + H^2 g_{\mu\rho}\Big[\frac{1}{4(D-3)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{3-D} - \frac{D-2}{4}\Big(\frac{y}{4}\Big)^{3-D}\Big], \tag{A28}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big)^{3-(D/2)} = \frac{(D-2)(D-6)}{4(D-3)(D-4)}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{4-D}\Big] + \frac{(D-2)H^2 g_{\mu\rho}}{4}\Big[\frac{1}{(D-3)(D-4)}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{4-D} - \Big(\frac{y}{4}\Big)^{4-D}\Big], \tag{A29}$$

$$\partial_\mu\Big(\frac{y}{4}\Big)^{1-(D/2)}\partial_\rho\Big(\frac{y}{4}\Big) = -\frac{2}{D-4}\nabla_\rho\nabla_\mu\Big[\Big(\frac{y}{4}\Big)^{2-(D/2)}\Big] + H^2 g_{\mu\rho}\Big[\frac{1}{D-4}\frac{\Box}{H^2}\Big(\frac{y}{4}\Big)^{2-(D/2)} - \frac{D-2}{4}\Big(\frac{y}{4}\Big)^{2-(D/2)}\Big]. \tag{A30}$$

APPENDIX B: SOME USEFUL INTEGRALS

1. The Y integrals

Here we evaluate the basic integrals of the form

$$Y_n = \int d^Dx' a'^D\Big[\ln^n\Big(\frac{y_{++}}{4}\Big) - \ln^n\Big(\frac{y_{+-}}{4}\Big)\Big] \quad (n = 0, 1, 2, 3). \tag{B1}$$

In order to perform the integrations, the following relations are needed,

$$\ln(\mu^2\Delta x_{++}^2) - \ln(\mu^2\Delta x_{+-}^2) = 2\pi i\theta(\Delta\eta)\theta(\Delta\eta^2 - r^2), \tag{B2}$$

$$\ln^2(\mu^2\Delta x_{++}^2) - \ln^2(\mu^2\Delta x_{+-}^2) = 4\pi i\theta(\Delta\eta)\theta(\Delta\eta^2 - r^2)\ln|\mu^2(\Delta\eta^2 - r^2)|, \tag{B3}$$

$$\ln^3(\mu^2\Delta x_{++}^2) - \ln^3(\mu^2\Delta x_{+-}^2) = 6\pi i\theta(\Delta\eta)\theta(\Delta\eta^2 - r^2)\Big[\ln^2|\mu^2(\Delta\eta^2 - r^2)| - \frac{\pi^2}{3}\Big], \tag{B4}$$

where $r = \| \vec{x} - \vec{x}' \|$, $\mu^2 = aa'H^2/4$ and we made use of

$$\Delta x_{++}^2 = -(|\eta - \eta'| - i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2, \tag{B5}$$

$$\Delta x_{+-}^2 = -(\eta - \eta' + i\epsilon)^2 + \| \vec{x} - \vec{x}' \|^2. \tag{B6}$$

We shall also need the polylogarithm functions, defined as

$$\mathrm{Li}_n(z) \equiv \mathrm{PolyLog}[n, z] = \sum_{i=1}^{\infty}\frac{z^i}{i^n} \qquad (n = 2, 3). \tag{B7}$$

Note that the natural logarithm is a special case of the Li function, $\ln(1-z) = -\mathrm{Li}_1(z)$. Moreover, we shall make use of the properties of polylogarithms,

$$\mathrm{Li}_2(z) = -\mathrm{Li}_2(1-z) + \ln\Big(\frac{1}{z}\Big)\ln(1-z) + \frac{\pi^2}{6}, \tag{B8}$$

$$\mathrm{Li}_2(z) + \ln(z)\ln(1-z) = \mathrm{Li}_2\Big(1 - \frac{1}{z}\Big) + \frac{1}{2}\ln^2(z) + \frac{\pi^2}{6} \quad (z \notin [-\infty, 0]), \tag{B9}$$

$$\mathrm{Li}_3\Big(1-\frac{1}{z}\Big) = \sum_{n=0}^{\infty}\frac{(1-\frac{1}{z})^n}{n^3} = \zeta(3) - \frac{\pi^2}{6}\frac{1}{z} + \Big(\frac{3}{4} - \frac{\pi^2}{12} - \frac{1}{2}\ln\Big(\frac{1}{z}\Big)\Big)\frac{1}{z^2} + \mathcal{O}\Big(\frac{\ln(z)}{z^3}\Big) \qquad (|z| \gg 1). \tag{B10}$$

Of course,

$$Y_0 = 0. \tag{B11}$$

The simplest nontrivial integral ($n = 1$) in Eq. (B1) is

$$Y_1 = \int_{\eta_0}^{\eta} d\eta' a'^D \int_0^{\Delta\eta} d^{D-1}x'[2\pi i] = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\int_1^a \frac{da'}{a'}\Big(1-\frac{a'}{a}\Big)^{D-1}, \tag{B12}$$

where η_0 denotes an initial conformal time for which we choose $a(\eta_0) = 1$, $\Delta\eta = \eta - \eta_0$, and we made use of the surface area of the $D-2$ dimensional sphere,

$$S^{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})} = \frac{2(4\pi)^{(D/2)-1}\Gamma(\frac{D}{2})}{\Gamma(D-1)}. \tag{B13}$$

The integral (B12) can be expressed in terms of a hypergeometric function. Since we are ultimately interested in expanding in powers of $D-4$, it is more convenient to expand the integrand in powers of $D-4$,

$$\Big(1-\frac{a'}{a}\Big)^{D-1} = \Big(1-\frac{a'}{a}\Big)^3 + (D-4)\Big(1-\frac{a'}{a}\Big)^3\ln\Big(1-\frac{a'}{a}\Big) + \frac{1}{2}(D-4)^2\Big(1-\frac{a'}{a}\Big)^3\ln^2\Big(1-\frac{a'}{a}\Big) + \mathcal{O}((D-4)^3) \tag{B14}$$

and then to integrate. The result is

$$\begin{aligned} Y_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\Big[\ln(a) - \frac{11}{6} + \frac{3}{a} - \frac{3}{2a^2} + \frac{1}{3a^3}\Big] + (D-4)\Big[\Big(\frac{49}{36} - \frac{\pi^2}{6} - \frac{11}{6a} + \frac{7}{12a^2} - \frac{1}{9a^3}\Big) \\ + \Big(-\frac{11}{6} + \frac{3}{a} - \frac{3}{2a^2} + \frac{1}{3a^3}\Big)\ln\Big(1-\frac{1}{a}\Big) + \mathrm{Li}_2\Big(\frac{1}{a}\Big)\Big] + \frac{(D-4)^2}{2}\Big[\Big(-\frac{251}{108} + \frac{49}{18a} - \frac{17}{36a^2} + \frac{2}{27a^3}\Big) - \ln(a)\ln^2\Big(1-\frac{1}{a}\Big) \\ + \Big(\frac{49}{18} - \frac{\pi^2}{3} - \frac{11}{3a} + \frac{7}{6a^2} - \frac{2}{9a^3}\Big)\ln\Big(1-\frac{1}{a}\Big) + \Big(-\frac{11}{6} + \frac{3}{a} - \frac{3}{2a^2} + \frac{1}{3a^3}\Big)\ln^2\Big(1-\frac{1}{a}\Big) + 2\mathrm{Li}_2\Big(\frac{1}{a}\Big)\ln\Big(1-\frac{1}{a}\Big) \\ + 2\mathrm{Li}_3\Big(1-\frac{1}{a}\Big)\Big]\Big\}. \end{aligned} \tag{B15}$$

The next integral in Eq. (B1) is Y_2. With the help of Eqs. (B3), (B14), and (B83), we can write ($x = r/\Delta\eta$, $v = a'/a$, $u = 1 - v$)

$$\begin{aligned} Y_2 &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D-1)}2\int_1^a\frac{da'}{a'}\Big(1-\frac{a'}{a}\Big)^{D-1}\int_0^1 x^{D-2}dx\Big[\ln\Big(\frac{aa'H^2\Delta\eta^2}{4}\Big) + \ln(1-x^2)\Big] \\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}2\int_1^a\frac{da'}{a'}\Big(1-\frac{a'}{a}\Big)^3\Big\{\Big[2\ln\Big(1-\frac{a'}{a}\Big) - \ln\Big(\frac{a'}{a}\Big) - \frac{8}{3}\Big] \\ &\quad + (D-4)\Big[2\ln^2\Big(1-\frac{a'}{a}\Big) - \ln\Big(1-\frac{a'}{a}\Big)\ln\Big(\frac{a'}{a}\Big) - \frac{8}{3}\ln\Big(1-\frac{a'}{a}\Big) + \Big(\frac{20}{9} - \frac{\pi^2}{4}\Big)\Big] + \mathcal{O}((D-4)^2)\Big\} \\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}2\int_0^{1-(1/a)}\frac{du}{1-u}u^3\Big\{\Big[2\ln(u) - \ln(1-u) - \frac{8}{3}\Big] \\ &\quad + (D-4)\Big[2\ln^2(u) - \ln(u)\ln(1-u) - \frac{8}{3}\ln(u) + \Big(\frac{20}{9} - \frac{\pi^2}{4}\Big)\Big] + \mathcal{O}((D-4)^2)\Big\}, \end{aligned} \tag{B16}$$

where we introduced a new variable, $u = 1 - a'/a$. These integrals can be integrated to give

$$Y_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\bigg[\ln^2(a) + \bigg(-\frac{16}{3} + \frac{6}{a} - \frac{3}{a^2} + \frac{2}{3a^3}\bigg)\ln(a) + \bigg(\frac{21}{2} - \frac{2\pi^2}{3} - \frac{52}{3a} + \frac{53}{6a^2} - \frac{2}{a^3}\bigg) + \bigg(-\frac{22}{3} + \frac{12}{a} - \frac{6}{a^2} + \frac{4}{3a^3}\bigg)\ln\bigg(1 - \frac{1}{a}\bigg) + 4\mathrm{Li}_2\bigg(\frac{1}{a}\bigg)\bigg] + (D-4)\bigg[\bigg(\frac{40}{9} - \frac{\pi^2}{2} - \frac{11}{3a} + \frac{7}{6a^2} - \frac{2}{9a^3} + \bigg(-\frac{13}{3} + \frac{6}{a} - \frac{3}{a^2} + \frac{2}{3a^3}\bigg)\ln\bigg(1 - \frac{1}{a}\bigg)\bigg)\ln(a) + \bigg(-\frac{313}{18} + \frac{43\pi^2}{36} + \frac{461}{18a} - \frac{3\pi^2}{2a} - \frac{94}{9a^2} + \frac{3\pi^2}{4a^2} + \frac{20}{9a^3} - \frac{\pi^2}{6a^3}\bigg) + \bigg(\frac{287}{18} - \frac{4\pi^2}{3} - \frac{74}{3a} + \frac{67}{6a^2} - \frac{22}{9a^3}\bigg)\ln\bigg(1 - \frac{1}{a}\bigg) + \bigg(-4\ln(a) - \frac{22}{3} + \frac{12}{a} - \frac{6}{a^2} + \frac{4}{3a^3}\bigg)\ln^2\bigg(1 - \frac{1}{a}\bigg) + 2\ln(a)\mathrm{Li}_2\bigg(\frac{1}{a}\bigg) - \frac{5}{3}\mathrm{Li}_2\bigg(\frac{1}{a}\bigg) + 8\ln\bigg(1 - \frac{1}{a}\bigg)\mathrm{Li}_2\bigg(\frac{1}{a}\bigg) - 2\zeta(3) + 2\mathrm{Li}_3\bigg(\frac{1}{a}\bigg) + 8\mathrm{Li}_3\bigg(1 - \frac{1}{a}\bigg)\bigg]\bigg\}. \tag{B17}$$

The last integral in Eq. (B1) is Y_3. Similarly as in the case of Y_2, we take account of Eq. (B4) and get

$$\begin{aligned} Y_3 &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D-1)}3\int_1^a da' a'^2\bigg(\frac{1}{a'} - \frac{1}{a}\bigg)^3\int_0^1 x^2 dx\bigg[\ln^2\bigg(\frac{aa'H^2\Delta\eta^2}{4}\bigg) + 2\ln\bigg(\frac{aa'H^2\Delta\eta^2}{4}\bigg)\ln(1-x^2) + \ln^2(1-x^2) - \frac{\pi^2}{3} + \mathcal{O}(D-4)\bigg] \\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}3\int_1^a\frac{da'}{a'}\bigg(1 - \frac{a'}{a}\bigg)^3\bigg[4\ln^2\bigg(1 - \frac{a'}{a}\bigg) - 4\ln\bigg(\frac{a'}{a}\bigg)\ln\bigg(1 - \frac{a'}{a}\bigg) + \ln^2\bigg(\frac{a'}{a}\bigg) - \frac{16}{3}\bigg(2\ln\bigg(1 - \frac{a'}{a}\bigg) - \ln\bigg(\frac{a'}{a}\bigg)\bigg) + \frac{104}{9} - \frac{2\pi^2}{3}\bigg] \\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}3\int_0^{1-(1/a)}\frac{du}{1-u}u^3\bigg[4\ln^2(u) - 4\ln(u)\ln(1-u) + \ln^2(1-u) - \frac{16}{3}(2\ln(u) - \ln(1-u)) + \frac{104}{9} - \frac{2\pi^2}{3}\bigg]. \end{aligned} \tag{B18}$$

Upon performing the integrals in (B18), we finally obtain

$$Y_3 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\ln^3(a) + \bigg[-8 + \frac{9}{a} - \frac{9}{2a^2} + \frac{1}{a^3}\bigg]\ln^2(a) + \bigg[\frac{104}{3} - 2\pi^2 - \frac{52}{a} + \frac{53}{2a^2} - \frac{6}{a^3}\bigg]\ln(a) + \bigg[-12\zeta(3) - \frac{2495}{36} + \frac{16\pi^2}{3} + \frac{115}{a} - \frac{6\pi^2}{a} - \frac{237}{4a^2} + \frac{3\pi^2}{a^2} + \frac{122}{9a^3} - \frac{2\pi^2}{3a^3}\bigg] + \bigg[-22 + \frac{36}{a} - \frac{18}{a^2} + \frac{4}{a^3}\bigg]\ln\bigg(1 - \frac{1}{a}\bigg)\ln(a) - 12\ln^2\bigg(1 - \frac{1}{a}\bigg)\ln(a) + 12\mathrm{Li}_2\bigg(\frac{1}{a}\bigg)\ln(a) + \bigg[63 - 4\pi^2 - \frac{104}{a} + \frac{53}{a^2} - \frac{12}{a^3} + 24\mathrm{Li}_2\bigg(\frac{1}{a}\bigg)\bigg]\ln\bigg(1 - \frac{1}{a}\bigg) + \bigg[-22 + \frac{36}{a} - \frac{18}{a^2} + \frac{4}{a^3}\bigg]\ln^2\bigg(1 - \frac{1}{a}\bigg) - 10\mathrm{Li}_2\bigg(\frac{1}{a}\bigg) + 24\mathrm{Li}_3\bigg(1 - \frac{1}{a}\bigg) + 12\mathrm{Li}_3\bigg(\frac{1}{a}\bigg)\bigg\}. \tag{B19}$$

2. The *W* integrals

The following class of basic integrals we need [these integrals we calculate only up to order $(D-4)^2$] is of the form

$$W_n = \int d^D x' a'^D \ln(a')\bigg[\ln^n\bigg(\frac{y_{++}}{4}\bigg) - \ln^n\bigg(\frac{y_{+-}}{4}\bigg)\bigg] \quad (n = 0, 1, 2). \tag{B20}$$

The trivial integral is $W_0 = 0$. The first nontrivial integral can be written as

$$W_1 = \int d^D x' a'^D \ln(a')[2i\pi\theta(\Delta\eta)\theta(\Delta\eta^2 - r^2)]. \tag{B21}$$

After the radial integration is performed, this integral can be reduced to

$$W_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\int_1^a\frac{da'}{a'}\left(1-\frac{a'}{a}\right)^3\ln(a')\left[1+(D-4)\ln\left(1-\frac{a'}{a}\right)\right]+\mathcal{O}((D-4)^2)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\int_{1/a}^1\frac{dv}{v}(1-v)^3[\ln(v)+\ln(a)][1+(D-4)\ln(1-v)]+\mathcal{O}((D-4)^2). \quad \text{(B22)}$$

This can be integrated to give

$$W_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\bigg[\frac{1}{2}\ln^2(a)-\frac{11}{6}\ln(a)+\frac{85}{36}-\frac{3}{a}+\frac{3}{4a^2}-\frac{1}{9a^3}\bigg]$$
$$+(D-4)\bigg[\left(\frac{49}{36}-\frac{\pi^2}{6}\right)\ln(a)+\left(-\frac{395}{108}+\frac{11\pi^2}{36}+\frac{151}{36a}-\frac{11}{18a^2}+\frac{2}{27a^3}\right)+\left(\frac{85}{36}-\frac{3}{a}+\frac{3}{4a^2}-\frac{1}{9a^3}\right)\ln\left(1-\frac{1}{a}\right)$$
$$-\frac{11}{6}\mathrm{Li}_2\left(\frac{1}{a}\right)-\mathrm{Li}_3\left(\frac{1}{a}\right)+\zeta(3)\bigg]\bigg\}+\mathcal{O}((D-4)^2). \quad \text{(B23)}$$

The second integral W_2 in (B20) we only need to leading order in the $D-4$ expansion. We begin with

$$W_2 = \int d^Dx'a'^D\ln(a')\left[4i\pi\theta(\Delta\eta)\theta(\Delta\eta^2-r^2)\ln\left(\frac{aa'H^2(\Delta\eta^2-r^2)}{4}\right)\right]. \quad \text{(B24)}$$

After the radial integration this integral becomes

$$W_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}2\int_1^a\frac{da'}{a'}\left(1-\frac{a'}{a}\right)^3\ln(a')\left[2\ln\left(1-\frac{a'}{a}\right)-\ln\left(\frac{a'}{a}\right)-\frac{8}{3}\right]+\mathcal{O}(D-4)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}2\int_{1/a}^1\frac{dv}{v}(1-v)^3[\ln(v)+\ln(a)]\left[2\ln(1-v)-\ln(v)-\frac{8}{3}\right]+\mathcal{O}(D-4). \quad \text{(B25)}$$

When integrated this gives

$$W_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\frac{1}{3}\ln^3(a)-\frac{8}{3}\ln^2(a)+\left(\frac{21}{2}-\frac{2\pi^2}{3}-\frac{6}{a}+\frac{3}{2a^2}-\frac{2}{9a^3}\right)\ln(a)$$
$$+\left(-\frac{895}{54}+\frac{11\pi^2}{9}+\frac{187}{9a}-\frac{89}{18a^2}+\frac{20}{27a^3}\right)+\left(\frac{85}{9}-\frac{12}{a}+\frac{3}{a^2}-\frac{4}{9a^3}\right)\ln\left(1-\frac{1}{a}\right)-\frac{22}{3}\mathrm{Li}_2\left(\frac{1}{a}\right)-4\mathrm{Li}_3\left(\frac{1}{a}\right)+4\zeta(3)\bigg]\bigg\}$$
$$+\mathcal{O}(D-4). \quad \text{(B26)}$$

We shall now act with d'Alembertians on the basic integrals (B15), (B17), and (B19). For simplicity, we neglect the terms that are suppressed as $\ln^2(a)/a$ or more since they vanish in the limit when $a\to\infty$. Since Y_n are functions of the scale factor only, the d'Alembertian operator simplifies to

$$\frac{\square}{H^2}\to -\frac{1}{a^DH^2}\partial_0a^{D-2}\partial_0 = -a^{2-D}\frac{d}{da}a^D\frac{d}{da}. \quad \text{(B27)}$$

Considering that the d'Alembertian does not increase the power of a,

$$\frac{\square}{H^2}\left(\frac{\ln^2(a)}{a}\right)=\frac{(D-2)\ln^2(a)}{a}-\frac{2(D-3)\ln(a)}{a}-\frac{2}{a},\qquad \frac{\square}{H^2}\left(\frac{\ln(a)}{a}\right)=\frac{(D-2)\ln(a)}{a}-\frac{D-3}{a},\qquad \frac{\square}{H^2}\left(\frac{1}{a}\right)=\frac{D-2}{a}, \quad \text{(B28)}$$

we can use the asymptotic form of the basic integrals (B12), (B17), and (B19) to calculate the late time behavior. In the limit when $a\to\infty$, the integrals become

$$Y_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\left\{\left(\ln(a)-\frac{11}{6}\right)+(D-4)\left(\frac{49}{36}-\frac{\pi^2}{6}\right)+(D-4)^2\left(-\frac{251}{216}+\zeta(3)\right)+\mathcal{O}\left(\frac{\ln(a)}{a}\right)+\mathcal{O}((D-4)^3)\right\}, \quad \text{(B29)}$$

$$Y_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\bigg(\ln^2(a) - \frac{16}{3}\ln(a) + \frac{21}{2} - \frac{2\pi^2}{3} + \mathcal{O}\bigg(\frac{\ln(a)}{a}\bigg)\bigg) + (D-4)\bigg[\bigg(\frac{40}{9} - \frac{\pi^2}{2}\bigg)\ln(a) + \bigg(-\frac{313}{18} + \frac{43\pi^2}{36} + 6\zeta(3)\bigg) + \mathcal{O}\bigg(\frac{\ln(a)}{a}\bigg)\bigg] + \mathcal{O}((D-4)^2)\bigg\}, \tag{B30}$$

$$Y_3 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\ln^3(a) - 8\ln^2(a) + \bigg(\frac{104}{3} - 2\pi^2\bigg)\ln(a) + \bigg(-\frac{2495}{36} + \frac{16\pi^2}{3} + 12\zeta(3)\bigg) + \mathcal{O}\bigg(\frac{\ln^2(a)}{a}\bigg) + \mathcal{O}(D-4)\bigg\}. \tag{B31}$$

The asymptotic forms of the $W_n (n = 1, 2)$ integrals (B23) and (B26) are

$$W_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\bigg[\frac{1}{2}\ln^2(a) - \frac{11}{6}\ln(a) + \frac{85}{36}\bigg] + (D-4)\bigg[\bigg(\frac{49}{36} - \frac{\pi^2}{6}\bigg)\ln(a) - \frac{395}{108} + \frac{11\pi^2}{36} + \zeta(3)\bigg] + \mathcal{O}((D-4)^2)\bigg\} \tag{B32}$$

and

$$W_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{\frac{1}{3}\ln^3(a) - \frac{8}{3}\ln^2(a) + \bigg(\frac{21}{2} - \frac{2\pi^2}{3}\bigg)\ln(a) + \bigg(-\frac{895}{54} + \frac{11\pi^2}{9} + 4\zeta(3)\bigg) + \mathcal{O}(D-4)\bigg\}. \tag{B33}$$

The asymptotic forms for the corresponding (nonvanishing) d'Alembertians are

$$\frac{\Box}{H^2}Y_1 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}[-(D-1) + \mathcal{O}(1/a) + \mathcal{O}((D-4)^3))], \tag{B34}$$

$$\frac{\Box}{H^2}Y_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{[-6\ln(a) + 14] + (D-4)\bigg[-2\ln(a) - 8 + \frac{3\pi^2}{2}\bigg] + \mathcal{O}\bigg(\frac{\ln(a)}{a}\bigg) + \mathcal{O}((D-4)^2)\bigg\}, \tag{B35}$$

$$\bigg(\frac{\Box}{H^2}\bigg)^2 Y_2 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{18 + 12(D-4) + \mathcal{O}\bigg(\frac{\ln(a)}{a}\bigg) + \mathcal{O}((D-4)^2)\bigg\}, \tag{B36}$$

$$\frac{\Box}{H^2}Y_3 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg[-9\ln^2(a) + 42\ln(a) - 88 + 6\pi^2 + \mathcal{O}\bigg(\frac{\ln^2(a)}{a}\bigg) + \mathcal{O}(D-4)\bigg], \tag{B37}$$

$$\bigg(\frac{\Box}{H^2}\bigg)^2 Y_3 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}[54\ln(a) - 108 + \mathcal{O}(\ln^2(a)/a) + \mathcal{O}(D-4)], \tag{B38}$$

$$\bigg(\frac{\Box}{H^2}\bigg)^3 Y_3 = i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}[-162 + \mathcal{O}(\ln^2(a)/a) + \mathcal{O}(D-4)], \tag{B39}$$

and similarly for W_n's.

3. The Ξ integrals

We are now ready to calculate the intermediate integrals that we need for evaluation of the 1-vertex integrals. They are of the general form

$$\Xi_\alpha = \int d^D x' a'^D \bigg[\bigg(\frac{y_{++}}{4}\bigg)^{-\alpha} - \bigg(\frac{y_{+-}}{4}\bigg)^{-\alpha}\bigg] \tag{B40}$$

with $\alpha = \frac{3D}{2} - 6, \frac{3D}{2} - 5, \frac{3D}{2} - 4, 3 - \frac{3D}{2}, D-2, D-3, D-4, \frac{D}{2} - 1$, and $\frac{D}{2} - 2$. All of the Ξ_α integrals can be represented in terms of the basic integrals Y_n given in Eqs. (B29)–(B31) and the associated d'Alembertians (B34)–(B39).

Let us begin with $\alpha = \frac{3D}{2} - 6$ which, when expanded in powers of $D-4$, yields

$$\Xi_{(3D/2)-6} = \int d^D x' a'^D \Big[\Big(\frac{y_{++}}{4}\Big)^{-(3/2)(D-4)} - \Big(\frac{y_{+-}}{4}\Big)^{-(3/2)(D-4)}\Big]$$
$$= -\frac{3}{2}(D-4)Y_1 + \frac{9}{8}(D-4)^2 Y_2 - \frac{9}{16}(D-4)^3 Y_3 + \mathcal{O}((D-4)^4)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{(D-4)\Big[-\frac{3}{2}\ln(a) + \frac{11}{4}\Big] + (D-4)^2\Big[\frac{9}{8}\ln^2(a) - 6\ln(a) + \frac{469}{48} - \frac{\pi^2}{2}\Big] + (D-4)^3$$
$$\times \Big[-\frac{9}{16}\ln^3(a) + \frac{9}{2}\ln^2(a) + \Big(-\frac{29}{2} + \frac{9\pi^2}{16}\Big)\ln(a) + \Big(\frac{12\,191}{576} - \frac{53\pi^2}{32} - \frac{3}{2}\zeta(3)\Big)\Big] + \mathcal{O}((D-4)^4)\Big\}. \tag{B41}$$

The next integral we need corresponds to $\alpha = \frac{D}{2} - 2$,

$$\Xi_{(D/2)-2} = \int d^D x' a'^D \Big[\Big(\frac{y_{++}}{4}\Big)^{-(D-4)/2} - \Big(\frac{y_{+-}}{4}\Big)^{-(D-4)/2}\Big]$$
$$= -\frac{1}{2}(D-4)Y_1 + \frac{1}{8}(D-4)^2 Y_2 - \frac{1}{48}(D-4)^3 Y_3 + \mathcal{O}((D-4)^4)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{(D-4)\Big[-\frac{1}{2}\ln(a) + \frac{11}{12}\Big] + (D-4)^2\Big[\frac{1}{8}\ln^2(a) - \frac{2}{3}\ln(a) + \frac{91}{144}\Big]$$
$$+ (D-4)^3\Big[-\frac{1}{48}\ln^3(a) + \frac{1}{6}\ln^2(a) + \Big(-\frac{1}{6} - \frac{\pi^2}{48}\Big)\ln(a) + \Big(-\frac{257}{1728} + \frac{11\pi^2}{288}\Big)\Big] + \mathcal{O}((D-4)^4)\Big\}. \tag{B42}$$

The following integral is $\Xi_{(3D/2)-5}$, which we can rewrite with the help of Eq. (A6) as

$$\Xi_{(3D/2)-5} = \Big[\frac{2}{3(D-4)(D-5)}\frac{\Box}{H^2} + \frac{D-10}{2(D-5)}\Big]\Xi_{(3D/2)-6}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-3 - \frac{25}{4}(D-4) + (D-4)^2\Big[-\frac{85}{16} - \frac{3\pi^2}{8}\Big]$$
$$+ (D-4)^3\Big[-\frac{27}{16}\ln^3(a) + \frac{189}{16}\ln^2(a) + \Big(-\frac{69}{2} + \frac{27\pi^2}{16}\Big)\ln(a) + \Big(\frac{8377}{192} - \frac{151\pi^2}{32} - \frac{9}{2}\zeta(3)\Big)\Big]$$
$$+ \mathcal{O}((D-4)^4)\Big\}. \tag{B43}$$

The next integral we calculate is $\alpha = \frac{D}{2} - 1$. With the help of Eq. (A12), we arrive at

$$\Xi_{(D/2)-1} = \Big[-\frac{2}{D-4}\frac{\Box}{H^2} + \frac{D+2}{2}\Big]\Xi_{(D/2)-2}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-3 - \frac{7}{4}(D-4) + (D-4)^2\Big[\frac{11}{16} - \frac{\pi^2}{8}\Big]$$
$$+ (D-4)^3\Big[-\frac{1}{16}\ln^3(a) + \frac{7}{16}\ln^2(a) + \Big(-\frac{1}{6} - \frac{\pi^2}{16}\Big)\ln(a) + \Big(-\frac{89}{192} + \frac{7\pi^2}{96}\Big)\Big] + \mathcal{O}((D-4)^4)\Big\}. \tag{B44}$$

In order to evaluate the integral $\alpha = \frac{3D}{2} - 4$, one needs to subtract the divergent contribution according to Eq. (A5). The result is

$$\Xi_{(3D/2)-4} = \Big[\frac{2}{(3D-10)(D-4)}\frac{\Box}{H^2} + \frac{D-8}{2(D-4)}\Big]\Xi_{(3D/2)-5} + \frac{2}{(3D-10)(D-4)}\Big[-\frac{\Box}{H^2} + \frac{D(D-2)}{4}\Big]\Xi_{(D/2)-1}$$
$$+ i\frac{1}{H^D}\frac{2}{(3D-10)(D-4)}\frac{(4\pi)^{(D/2)}}{\Gamma(\frac{D}{2}-1)} \tag{B45}$$

$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{12 + (D-4)\bigg[4 + \frac{\pi^2}{2}\bigg] + (D-4)^2\bigg[\frac{13}{4}\ln^3(a) - \frac{65}{8}\ln^2(a) + \bigg(\frac{61}{6} - \frac{7\pi^2}{2}\bigg)\ln(a) + \bigg(-\frac{19}{4} + \frac{13\pi^2}{3} + \zeta(3)\bigg)\bigg] + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{2}{(3D-10)(D-4)} + \mathcal{O}((D-4)^3)\bigg\}. \tag{B46}$$

Furthermore, for $\alpha = \frac{3D}{2} - 3$ we make use of Eq. (A4) to obtain

$$\Xi_{(3D/2)-3} = \bigg[\frac{2}{(3D-8)(D-3)}\frac{\Box}{H^2} + \frac{D-6}{2(D-3)}\bigg]\Xi_{(3D/2)-4}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{-12 + (D-4)\bigg[14 - \frac{\pi^2}{2}\bigg] + (D-4)^2\bigg[-\frac{13}{4}\ln^3(a) - \frac{13}{2}\ln^2(a) + \bigg(\frac{107}{24} + \frac{7\pi^2}{2}\bigg)\ln(a) + \bigg(-\frac{115}{8} + \frac{5\pi^2}{3} - 9\zeta(3)\bigg)\bigg] + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{D-6}{(D-3)(3D-10)(D-4)} + \mathcal{O}((D-4)^3)\bigg\}. \tag{B47}$$

The next class of integrals begins with $\alpha = D - 4$. The integral is of the form

$$\Xi_{D-4} = -(D-4)Y_1 + \frac{1}{2}(D-4)^2Y_2 - \frac{1}{6}(D-4)^3Y_3 + \mathcal{O}((D-4)^4)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{(D-4)\bigg[-\ln(a) + \frac{11}{6}\bigg] + (D-4)^2\bigg[\frac{1}{2}\ln^2(a) - \frac{8}{3}\ln(a) + \frac{35}{9} - \frac{\pi^2}{6}\bigg] + (D-4)^3\bigg[-\frac{1}{6}\ln^3(a) + \frac{4}{3}\ln^2(a) + \bigg(-\frac{32}{9} + \frac{\pi^2}{12}\bigg)\ln(a) + \bigg(\frac{217}{54} - \frac{7\pi^2}{24}\bigg)\bigg] + \mathcal{O}((D-4)^4)\bigg\}. \tag{B48}$$

With the help of Eq. (A10) we calculate the next integral,

$$\Xi_{D-3} = \bigg[\frac{2}{(D-4)(D-6)}\frac{\Box}{H^2} - \frac{6}{D-6}\bigg]\Xi_{D-4}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{-3 - 4(D-4) + (D-4)^2\bigg[-1 - \frac{\pi^2}{4}\bigg] + (D-4)^3\bigg[-\frac{1}{2}\ln^3(a) + \frac{7}{2}\ln^2(a) + \bigg(-8 + \frac{\pi^2}{4}\bigg)\ln(a) + \bigg(8 - \frac{11\pi^2}{12}\bigg)\bigg] + \mathcal{O}((D-4)^4)\bigg\}. \tag{B49}$$

The next integral corresponds to $\alpha = D = 2$, and can be calculated by means of Eq. (A9). The result is

$$\Xi_{D-2} = \bigg[\frac{2}{(D-3)(D-4)}\frac{\Box}{H^2} - \frac{4}{D-4}\bigg]\Xi_{D-3} + \frac{2}{(D-3)(D-4)}\bigg[-\frac{\Box}{H^2} + \frac{D(D-2)}{4}\bigg]\Xi_{(D/2)-1} + \frac{2}{(D-3)(D-4)}\frac{(4\pi)^{D/2}}{\Gamma(\frac{D}{2}-1)}\frac{i}{H^D}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{12 + (D-4)\bigg[4 + \frac{\pi^2}{2}\bigg] + (D-4)^2\bigg[\frac{7}{4}\ln^3(a) - \frac{35}{8}\ln^2(a) + \bigg(-\frac{1}{6} - \frac{5\pi^2}{4}\bigg)\ln(a) + \bigg(\frac{25}{12} + \frac{53\pi^2}{24}\bigg)\bigg] + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{2}{(D-3)(D-4)} + \mathcal{O}((D-4)^3)\bigg\}. \tag{B50}$$

The final integral corresponds to $\alpha = D - 1$, and it evaluates to

$$\Xi_{D-1} = \bigg[\frac{2}{(D-2)^2}\frac{\Box}{H^2} - \frac{2}{D-2}\bigg]\Xi_{D-2}$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\bigg\{-12 + (D-4)\bigg[2 - \frac{\pi^2}{2}\bigg] + (D-4)^2\bigg[-\frac{7}{4}\ln^3(a) - \frac{7}{2}\ln^2(a) + \bigg(\frac{193}{24} + \frac{5\pi^2}{4}\bigg)\ln(a) + \bigg(\frac{37}{24} - \frac{\pi^2}{12}\bigg)\bigg] - \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{4}{(D-2)(D-3)(D-4)} + \mathcal{O}((D-4)^3)\bigg\}. \tag{B51}$$

4. The Λ integrals

There is one more class of integrals to be evaluated which are based on the basic integrals (B20) that are needed to evaluate the 1-vertex integral contributing to the central diagram in Fig. 1. The general form of these integrals is

$$\Lambda_\alpha = \int d^Dx' a'^D \ln(a')\Big[\Big(\frac{y_{++}}{4}\Big)^{-\alpha} - \Big(\frac{y_{+-}}{4}\Big)^{-\alpha}\Big] \tag{B52}$$

with $\alpha = D-1$, $D-2$, $D-3$, $D-4$, $\frac{D}{2}-1$, and $\frac{D}{2}-2$. All of these integrals can be represented in terms of the W_n functions given in Eqs. (B23) and (B26). We give these integrals to order $(D-4)^2$, which is also what we need for the evaluation of the integral (134).

When expanded in powers of $D-4$, the $\Lambda_{\alpha=D-4}$ integral can be written as

$$\begin{aligned}\Lambda_{D-4} &= \int d^Dx' a'^D \ln(a')\Big[\Big(\frac{y_{++}}{4}\Big)^{-(D-4)} - \Big(\frac{y_{+-}}{4}\Big)^{-(D-4)}\Big] = -(D-4)W_1 + \frac{1}{2}(D-4)^2W_2 + \mathcal{O}((D-4)^3)\\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{(D-4)\Big[-\frac{1}{2}\ln^2(a) + \frac{11}{6}\ln(a) - \frac{85}{36}\Big] + (D-4)^2\Big[\frac{1}{6}\ln^3(a) - \frac{4}{3}\ln^2(a) + \Big(\frac{35}{9} - \frac{\pi^2}{6}\Big)\ln(a)\\ &\quad - \frac{125}{27} + \frac{11\pi^2}{36} + \zeta(3)\Big] + \mathcal{O}((D-4)^3)\Big\}.\end{aligned} \tag{B53}$$

Similarly, for the other integrals that we need, we have

$$\begin{aligned}\Lambda_{(D/2)-2} &= -\frac{D-4}{2}W_1 + \frac{(D-4)^2}{8}W_2 + \mathcal{O}((D-4)^3)\\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{(D-4)\Big[-\frac{1}{4}\ln^2(a) + \frac{11}{12}\ln(a) - \frac{85}{72}\Big]\\ &\quad + (D-4)^2\Big[\frac{1}{24}\ln^3(a) - \frac{1}{3}\ln^2(a) + \frac{91}{144}\ln(a) - \frac{35}{144}\Big] + \mathcal{O}((D-4)^3)\Big\},\end{aligned} \tag{B54}$$

$$\begin{aligned}\Lambda_{D-3} &= \Big[\frac{2}{(D-4)(D-6)}\frac{\Box}{H^2} - \frac{6}{(D-6)}\Big]\Lambda_{D-4} + \mathcal{O}((D-4)^3)\\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\Big[-3\ln(a) + \frac{9}{2}\Big] + (D-4)\Big[-4\ln(a) + 6 - \frac{\pi^2}{2}\Big]\\ &\quad + (D-4)^2\Big[\frac{1}{2}\ln^3(a) - \frac{7}{2}\ln^2(a) + \Big(7 - \frac{\pi^2}{2}\Big)\ln(a) - 7 + \frac{\pi^2}{2} + 3\zeta(3)\Big] + \mathcal{O}((D-4)^3)\Big\},\end{aligned} \tag{B55}$$

$$\begin{aligned}\Lambda_{(D/2)-1} &= \Big[-\frac{2}{D-4}\frac{\Box}{H^2} + \frac{D+2}{4}\Big]\Lambda_{(D/2)-2}\\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-3\ln(a) + \frac{9}{2} + (D-4)\Big[-\frac{7}{4}\ln(a) + \frac{3}{4}\Big] + (D-4)^2\Big[\frac{1}{8}\ln^3(a) - \frac{7}{8}\ln^2(a) + \frac{49}{48}\ln(a) - \frac{1}{18}\Big]\\ &\quad + \mathcal{O}((D-4)^3)\Big\},\end{aligned} \tag{B56}$$

$$\begin{aligned}\Lambda_{D-2} &= \Big[\frac{2}{(D-3)(D-4)}\frac{\Box}{H^2} - \frac{4}{(D-4)}\Big]\Lambda_{D-3} + \frac{2}{(D-3)(D-4)}\Big[-\frac{\Box}{H^2} + \frac{D(D-2)}{4}\Big]\Lambda_{(D/2)-1}\\ &\quad + i\frac{(4\pi)^{D/2}}{H^D}\frac{1}{\Gamma(\frac{D}{2}-1)}\frac{2}{(D-3)(D-4)}\ln(a) + \mathcal{O}((D-4)^3)\\ &= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{[12\ln(a) - 12 + 2\pi^2] + (D-4)\Big[-\frac{3}{2}\ln^3(a) + \frac{15}{4}\ln^2(a) + \Big(\frac{1}{3} + 2\pi^2\Big)\ln(a) - \frac{43}{72}\\ &\quad + \pi^2 - 12\zeta(3)\Big] + (D-4)^2\Big[-\frac{1}{24}\ln^3(a) + \frac{61}{12}\ln^2(a) - \frac{2323}{144}\ln(a) + \frac{2533}{144} - 2\pi^2\Big]\\ &\quad + \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{2}{(D-3)(D-4)}\ln(a)\Big\} + \mathcal{O}((D-4)^3),\end{aligned} \tag{B57}$$

$$\Lambda_{D-1} = \Big[\frac{2}{(D-2)^2}\frac{\square}{H^2} - \frac{2}{D-2}\Big]\Lambda_{D-2} + \mathcal{O}((D-4)^3)$$
$$= i\frac{(4\pi)^{D/2}}{H^D}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{[-12\ln(a) - 6 - 2\pi^2] + (D-4)\Big[\frac{3}{2}\ln^3(a) + 3\ln^2(a) + \Big(-\frac{13}{12} - 2\pi^2\Big)\ln(a) + \frac{169}{72} - 3\pi^2 + 12\zeta(3)\Big] + (D-4)^2\Big[-\frac{17}{24}\ln^3(a) - \frac{361}{48}\ln^2(a) + \Big(\frac{169}{144} + \pi^2\Big)\ln(a) + \frac{233}{288} + 4\pi^2 - 6\zeta(3)\Big] - \frac{\Gamma(D)}{\Gamma(\frac{D}{2})^2}\frac{2}{(D-3)(D-4)}\ln(a) - \frac{\Gamma(D)}{\Gamma(\frac{D}{2})^2}\frac{2(D-1)}{(D-2)(D-3)(D-4)} + \mathcal{O}((D-4)^3)\Big\}. \quad (B58)$$

5. The Ω integrals

We can now use these results to calculate the following class of integrals we need, which are of the form

$$\Omega_{\alpha,\beta} = \int d^D x' a'^D \Big[\partial_\mu\Big(\frac{y_{++}}{4}\Big)^{-\alpha}\partial_\rho\Big(\frac{y_{++}}{4}\Big)^{-\beta} - \partial_\mu\Big(\frac{y_{+-}}{4}\Big)^{-\alpha}\partial_\rho\Big(\frac{y_{+-}}{4}\Big)^{-\beta}\Big]$$
$$= \Big[\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}\nabla_\mu\nabla_\rho + \frac{\alpha\beta H^2 g_{\mu\rho}}{D-2(\alpha+\beta+1)} \times\Big(-\frac{1}{(\alpha+\beta)(\alpha+\beta+1)}\frac{\square}{H^2} + 1\Big)\Big]\Xi_{\alpha+\beta}, \quad (B59)$$

where we made use of Eq. (A22) and assumed $\alpha + \beta + 1 \neq \frac{D}{2}$. These integrals can be evaluated with a help of Eqs. (A23)–(A30) and the Ξ_γ integrals (B41)–(B50). We need the integrals corresponding to the pairs $(\alpha, \beta) = (\frac{D}{2} - 1, D-2)$, $(\frac{D}{2} - 1, D-2)$, $(\frac{D}{2} - 1, D-3)$, $(\frac{D}{2} - 1, D-4)$, $(\frac{D}{2} - 1, D-5)$, $(\frac{D}{2} - 1, D-6)$, $(\frac{D}{2} - 2, D-2)$, $(\frac{D}{2} - 2, D-3)$, $(\frac{D}{2} - 2, D-4)$, $(\frac{D}{2} - 5, D-5)$, $(\frac{D}{2} - 1, \frac{D}{2} - 2)$, $(\frac{D}{2} - 1, \frac{D}{2} - 3)$, $(\frac{D}{2} - 2, \frac{D}{2} - 2)$.

Note that when $\nabla_\mu\nabla_\rho$ acts on a function of time (or equivalently of a, which is a scalar function), one obtains

$$\frac{\nabla_\mu\nabla_\rho}{H^2} = -g_{\mu\rho}a\partial_a + (a^2\delta^0_\mu\delta^0_\rho)a^2\partial_a^2, \quad (B60)$$

$$\frac{\square}{H^2} = g^{\mu\rho}\frac{\nabla_\mu\nabla_\rho}{H^2} = -a^{2-D}\partial_a a^D\partial_a. \quad (B61)$$

Making use of these relations and of Eq. (A22) (and assuming that $\alpha + \beta + 1 \neq \frac{D}{2}$), Eq. (B59) reduces to

$$\Omega_{\alpha,\beta} = H^2 g_{\mu\rho}\Big[-\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}a\partial_a + \frac{\alpha\beta}{D-2(\alpha+\beta+1)} \times\Big(\frac{a^{2-D}\partial_a a^D\partial_a}{(\alpha+\beta)(\alpha+\beta+1)} + 1\Big)\Big]\Xi_{\alpha+\beta}, + H^2(a^2\delta^0_\mu\delta^0_\rho)\frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}a^2\partial_a^2\Xi_{\alpha+\beta}. \quad (B62)$$

Note that the last term breaks the Lorentz symmetry. The $\Omega_{\alpha,\beta}$ integrals are symmetric under $\alpha \rightleftarrows \beta$.

Since the 1-vertex integrals are at least linearly divergent in $D-4$, to extract the finite contribution, we only need terms up to linear order in the $D-4$ expansion.

We begin with the integrals of the form $\alpha = \frac{D}{2} - n$, $\beta = D - n$ [$\alpha + \beta = \frac{3D}{2} - (m+n)$; $m = 1, 2, 3, 4$; $n = 2, 3, 4, 5$]. We get

$$\Omega_{(D/2)-1,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{6 + (D-4)\Big(-4 + \frac{\pi^2}{4}\Big) - \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{(D-2)(D-6)}{4(D-3)(3D-10)(D-4)}\Big\} + \mathcal{O}((D-4)^2), \quad (B63)$$

$$\Omega_{(D/2)-1,D-3} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-6 + (D-4)\Big(-5 - \frac{\pi^2}{4}\Big) - \frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{D-2}{2(3D-10)(D-4)}\Big\} + \mathcal{O}((D-4)^2), \quad (B64)$$

$$\Omega_{(D/2)-1,D-4} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{3}{2} + \frac{31}{8}(D-4)\Big] + \mathcal{O}((D-4)^2), \quad (B65)$$

$$\Omega_{(D/2)-1,D-5} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{1}{2}+\frac{7}{8}(D-4)\Big] + i(a^2\delta^0_\mu\delta^0_\rho)\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[-1+(D-4)\Big(\frac{3}{2}\ln(a)-\frac{7}{2}\Big)\Big] + \mathcal{O}((D-4)^2), \tag{B66}$$

$$\Omega_{(D/2)-2,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-6(D-4)+(D-4)^2\Big(1-\frac{\pi^2}{4}\Big)-\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\frac{D-2}{2(3D-10)(D-3)}\Big\} + \mathcal{O}((D-4)^3), \tag{B67}$$

$$\Omega_{(D/2)-2,D-3} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\frac{3}{4}+\frac{37}{16}(D-4)+(D-4)^2\Big(\frac{185}{64}+\frac{3\pi^2}{32}\Big)\Big\} + \mathcal{O}((D-4)^3), \tag{B68}$$

$$\Omega_{(D/2)-2,D-4} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-\frac{1}{4}(D-4)^2\Big\} + ia^2\delta^0_\mu\delta^0_\rho\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\frac{1}{2}(D-4)^2\Big\} + \mathcal{O}((D-4)^3), \tag{B69}$$

$$\Omega_{(D/2)-3,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-\frac{3}{D-4}-\frac{25}{4}+(D-4)\Big(-\frac{73}{16}-\frac{3\pi^2}{8}\Big) + (D-4)^2\Big[-\frac{27}{16}\ln^3(a)+\frac{135}{32}\ln^2(a)+\Big(-\frac{33}{8}+\frac{27\pi^2}{16}\Big)\ln(a)+\frac{1009}{192}-\frac{35\pi^2}{16}-\frac{9}{2}\zeta(3)\Big] + \mathcal{O}((D-4)^3)\Big\}, \tag{B70}$$

$$\Omega_{(D/2)-3,D-3} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\frac{1}{2}+\frac{11}{8}(D-4)+(D-4)^2\Big(\frac{133}{96}+\frac{\pi^2}{16}\Big)\Big\} + i(a^2\delta^0_\mu\delta^0_\rho)\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[-1+(D-4)\Big(\frac{3}{2}\ln(a)-\frac{9}{2}\Big) + (D-4)^2\Big(-\frac{9}{8}\ln^2(a)+\frac{27}{4}\ln(a)-\frac{67}{6}+\frac{3\pi^2}{8}\Big)\Big] + \mathcal{O}((D-4)^3), \tag{B71}$$

$$\Omega_{(D/2)-4,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{2+5(D-4)+(D-4)^2\Big(\frac{31}{6}-\frac{\pi^2}{16}\Big)\Big\} + i(a^2\delta^0_\mu\delta^0_\rho)\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[-4+(D-4)(6\ln(a)-17) + (D-4)^2\Big(-\frac{9}{2}\ln^2(a)+\frac{51}{2}\ln(a)-\frac{253}{6}+\frac{3\pi^2}{2}\Big)\Big] + \mathcal{O}((D-4)^3). \tag{B72}$$

The following integrals are also useful,

$$\Omega_{-1,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{-\frac{6}{D-4}-11+(D-4)\Big(-6-\frac{\pi^2}{2}\Big) + (D-4)^2\Big(-\ln^3(a)+\frac{5}{2}\ln^2(a)+\Big(2+\frac{\pi^2}{2}\Big)\ln(a)-2-\frac{4\pi^2}{3}\Big)\Big\} + \mathcal{O}((D-4)^3), \tag{B73}$$

$$\Omega_{-1,D-3} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{1}{2}+\frac{7}{6}(D-4)+(D-4)^2\Big(\frac{5}{6}+\frac{\pi^2}{24}\Big)\Big] + i\delta^0_\mu\delta^0_\rho\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[-1+(D-4)\Big(\ln(a)-\frac{11}{3}\Big)+(D-4)^2\Big(-\frac{1}{2}\ln^2(a)+\frac{11}{3}\ln(a)-\frac{56}{9}+\frac{\pi^2}{12}\Big)\Big] + \mathcal{O}((D-4)^3), \tag{B74}$$

$$\Omega_{-2,D-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[2+\frac{11}{3}(D-4)+(D-4)^2\Big(2+\frac{\pi^2}{3}\Big)\Big] + i\delta^0_\mu\delta^0_\rho\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[-4+(D-4)\Big(4\ln(a)-\frac{38}{3}\Big)+(D-4)^2\Big(-2\ln^2(a)+\frac{38}{3}\ln(a)-\frac{176}{9}+\frac{\pi^2}{3}\Big)\Big] + \mathcal{O}((D-4)^3). \tag{B75}$$

The next class of Ω integrals is of the form $\alpha = \frac{D}{2} - m$, $\beta = \frac{D}{2} - n$ $[\alpha + \beta = D - (n+m); m, n = 1, 2, 3]$. Since the coefficient of these integrals is not singular when $D = 4$, actually we need only the $\mathcal{O}((D-4)^0)$ contribution. The integrals are

$$\Omega_{(D/2)-1,(D/2)-1} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big\{\Big[-6+(D-4)\times\Big(-5-\frac{\pi^2}{4}\Big)\Big]-\frac{\Gamma(D)}{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}\times\frac{(D-2)}{2(D-3)(D-4)}+\mathcal{O}((D-4)^2)\Big\}, \tag{B76}$$

$$\Omega_{(D/2)-1,(D/2)-2} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{3}{2}+\frac{11}{4}(D-4)\Big] + \mathcal{O}((D-4)^2), \tag{B77}$$

$$\Omega_{(D/2)-1,(D/2)-3} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{1}{2}+\frac{2}{3}(D-4)\Big] + i(a^2\delta^0_\mu\delta^0_\rho)\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\times\Big[-1+(D-4)\Big(\ln(a)-\frac{8}{3}\Big)\Big] + \mathcal{O}((D-4)^2), \tag{B78}$$

$$\Omega_{(D/2)-2,(D/2)-2} = 0 + \mathcal{O}((D-4)^2), \tag{B79}$$

$$\Omega_{-1,(D/2)-1} = ig_{\mu\rho}\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\Big[\frac{1}{2}+\frac{11}{24}(D-4)\Big] + i(a^2\delta^0_\mu\delta^0_\rho)\frac{(4\pi)^{D/2}}{H^{D-2}}\frac{\Gamma(\frac{D}{2})}{\Gamma(D)}\times\Big[-1+(D-4)\Big(\frac{1}{2}\ln(a)-\frac{11}{6}\Big)\Big] + \mathcal{O}((D-4)^2). \tag{B80}$$

6. The radial integrals

The radial integrals are of the form

$$J_n = \int_0^1 dx x^{D-2}[\ln(1-x^2)]^n \tag{B81}$$

and when evaluated result in

$$J_0 = \frac{1}{D-1}, \tag{B82}$$

$$J_1 = \frac{1}{D-1}\Big[-\psi\Big(\frac{D+1}{2}\Big)+\psi(1)\Big] = \frac{1}{D-1}\Big[\Big(\ln(4)-\frac{8}{3}\Big)+(D-4)\Big(\frac{20}{9}-\frac{\pi^2}{4}\Big) + \mathcal{O}((D-4)^2)\Big], \tag{B83}$$

$$J_2 = \frac{1}{D-1}\Big[\frac{\pi^2}{6}+\Big(\psi\Big(\frac{D+1}{2}\Big)-\psi(1)\Big)^2-\psi'\Big(\frac{D+1}{2}\Big)\Big] = \frac{1}{D-1}\Big[\frac{104}{9}-\frac{\pi^2}{3}-\frac{16}{3}\ln(4)+\ln^2(4)+\mathcal{O}(D-4)\Big], \tag{B84}$$

$$J_3 = \frac{1}{D-1}\Big[-\frac{640}{9}+\frac{8\pi^2}{3}-\pi^2\ln(4)+\frac{104}{3}\ln(4) - 8\ln^2(4)+\ln^3(4)+12\zeta(3)\Big]+\mathcal{O}(D-4). \tag{B85}$$

APPENDIX C: THE FINITE TWO LEG INTEGRAL

In this appendix we evaluate the two leg finite integral (127). The relevant part of the integral is of the form

$$\mathcal{I}_{\mu\rho} = \sum_{\pm\pm}(\pm)(\pm)\int d^4x' a'^4\int d^4x'' a''^4\Big(\frac{\partial}{\partial x^\mu}\frac{1}{y_{+\pm}}\Big)\times\Big(\frac{\partial}{\partial x^\rho}\frac{1}{y''_{+\pm}}\Big)\ln\Big(\frac{y'_{\pm\pm}}{4}\Big). \tag{C1}$$

The integral can be naturally split into spatial and timelike parts, each of which we evaluate separately. The spatial part of (C1) can be written as

$$\mathcal{I}_{ij} = \frac{4\delta_{ij}}{3a^2H^4}\sum_{\pm\pm}(\pm)(\pm)\int\frac{d^4x'a'^3}{(\Delta x_{+\pm})^4}\int\frac{d^4x''a''^3}{(\Delta x''_{+\pm})^4}\times\Delta\vec{x}\cdot\Delta\vec{x}''\ln\Big(\frac{a'a''H^2}{4}\Delta x'^2{}_{\pm\pm}\Big). \tag{C2}$$

It is natural to introduce the coordinates $r = \|\vec{x}-\vec{x}'\|$, $r'' = \|\vec{x}-\vec{x}''\|$, in terms of which $\|\vec{x}'-\vec{x}''\|^2 = r^2 + r''^2 - 2rr''\cos(\theta)$, where $\theta = \angle(\vec{x}-\vec{x}', \vec{x}-\vec{x}'')$. The angular integral over θ can be easily performed. The result is

$$I_{ij} = \frac{16\pi^2}{3}\frac{\delta_{ij}}{a^2H^4}\sum_{\pm\pm}(\pm)(\pm)\int_{\eta_0}^{\eta} d\eta' a'^3 \int_{\eta_0}^{\eta'} d\eta'' a''^3 \int_{-\infty}^{\infty}\frac{drr}{(r^2-\Delta\eta^2_{+\pm})^2}\int_0^\infty \frac{dr'' r''}{(r''^2-\Delta\eta''^2{}_{+\pm})^2} \times \bigg\{(r^2+r''^2-\Delta\eta'^2)[(r+r'')^2-\Delta\eta'^2]\bigg(\ln\bigg[\frac{H^2}{4}((r+r'')^2-\Delta\eta'^2)\bigg]-1\bigg) - \frac{1}{2}[(r+r'')^2-\Delta\eta'^2]^2\bigg(\ln\bigg[\frac{H^2}{4}((r+r'')^2-\Delta\eta'^2)\bigg]-\frac{1}{2}\bigg)\bigg\}, \tag{C3}$$

where we also used the symmetry of the resulting integrand under the exchange $r \to -r$ to extend the limit of integration of the r integral from $[0, \infty)$ to $(-\infty, \infty)$. Finally, we made use of the fact that temporal integrals extend from $\eta_0 = -1/H$ to η, and that the symmetry allows us to constrain the η'' integral up to η', thus gaining a factor 2. With this we have achieved an important time ordering, $\eta'' \le \eta' \le \eta$, which we will find useful below. Next we split the integral (C3) into two integrals, $I_{ij} = I^{(1)}_{ij} + I^{(2)}_{ij}$, as follows,

$$I^{(1)}_{ij} = \frac{16\pi^2}{3}\frac{\delta_{ij}}{a^2H^4}\int_{\eta_0}^{\eta} d\eta' a'^3 \int_{\eta_0}^{\eta'} d\eta'' a''^3 \times \sum_{x''=\pm}(\pm)\int_0^\infty \frac{dr'' r''}{(r''^2-\Delta\eta''^2{}_{+\pm})^2}J^{(1)}, \tag{C4}$$

$$I^{(2)}_{ij} = \frac{16\pi^2}{3}\frac{\delta_{ij}}{a^2H^4}\int_{\eta_0}^{\eta} d\eta' a'^3 \int_{\eta_0}^{\eta'} d\eta'' a''^3 \times \sum_{x''=\pm}(\pm)\int_0^\infty \frac{dr'' r''^2}{(r''^2-\Delta\eta''^2{}_{+\pm})^2}J^{(2)}, \tag{C5}$$

where

$$J^{(1)} = \sum_{x'=\pm}(\pm)\int_{-\infty}^{\infty}\frac{drr}{(r^2-\Delta\eta^2_{+\pm})^2}\bigg\{[(r+r'')^2-\Delta\eta'^2{}_{\pm\pm}]^2 \times \bigg(\frac{1}{2}\ln\bigg[\frac{H^2}{4}((r+r'')^2-\Delta\eta'^2{}_{\pm\pm})\bigg]-\frac{3}{4}\bigg)\bigg\}, \tag{C6}$$

$$J^{(2)} = \sum_{x'=\pm}(\pm)\int_{-\infty}^{\infty}\frac{drr^2}{(r^2-\Delta\eta^2_{+\pm})^2}\bigg\{[(r+r'')^2-\Delta\eta'^2{}_{\pm\pm}] \times \bigg(-2\ln\bigg[\frac{H^2}{4}((r+r'')^2-\Delta\eta'^2{}_{\pm\pm})\bigg]+2\bigg)\bigg\}. \tag{C7}$$

Before we begin evaluation of these integrals, we recall the $i\epsilon$ prescription,

$$\begin{aligned}
\Delta x^2_{++} &= \|\Delta\vec{x}\|^2 - (|\Delta\eta| - i\epsilon)^2 \to \|\Delta\vec{x}\|^2 - (\Delta\eta - i\epsilon)^2,\\
\Delta x^2_{+-} &= \|\Delta\vec{x}\|^2 - (\Delta\eta + i\epsilon)^2,\\
\Delta x^2_{-+} &= \|\Delta\vec{x}\|^2 - (\Delta\eta - i\epsilon)^2,\\
\Delta x^2_{--} &= \|\Delta\vec{x}\|^2 - (|\Delta\eta| + i\epsilon)^2 \to \|\Delta\vec{x}\|^2 - (\Delta\eta + i\epsilon)^2,
\end{aligned} \tag{C8}$$

where the implication follows as a result of the time ordering $\eta \ge \eta' \ge \eta''$ in Eqs. (C4) and (C5). Analogous prescriptions hold for the other two intervals. Based on this, we arrive at the $i\epsilon$ prescriptions as given in Table I. Note that $\Delta\eta''$ and $\Delta\eta'$ have identical $i\epsilon$ prescriptions and that they are completely specified by the *signature* of the x'' vertex. On the other hand, the $i\epsilon$ prescriptions of $\Delta\eta$ are completely given in terms of the signature of the x' vertex.

We shall now use this fact to perform the r integrals (C6) and (C7). The first integral $J^{(1)}$ can be performed by making use of the Dirac identity,

$$\frac{r}{[r^2-(\Delta\eta-i\epsilon)^2]^2} - \frac{r}{[r^2-(\Delta\eta+i\epsilon)^2]^2} = \frac{1}{4\Delta\eta}2\pi i\bigg[\frac{\partial}{\partial r}\delta(r+\Delta\eta) + \frac{\partial}{\partial r}\delta(r-\Delta\eta)\bigg]. \tag{C9}$$

The derivative $\partial/\partial r$ can be moved by partial integration to the integrand, such that the result of integration of (C6) is

$$J^{(1)} = -\frac{\pi i}{\Delta\eta}\bigg\{(r''+\Delta\eta)[((r''+\Delta\eta)^2-\Delta\eta'^2)] \times \bigg[\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2]\bigg)-1\bigg] + (r''-\Delta\eta)[((r''-\Delta\eta)^2-\Delta\eta'^2)] \times \bigg[\ln\bigg(\frac{H^2}{4}[(r''-\Delta\eta)^2-\Delta\eta'^2]\bigg)-1\bigg]\bigg\}. \tag{C10}$$

Analogously, to evaluate $J^{(2)}$ we can use the following Dirac identity,

TABLE I. The $i\epsilon$ prescriptions of the intervals $\Delta\eta = \eta - \eta'$, $\Delta\eta'' = \eta - \eta''$, and $\Delta\eta' = \eta' - \eta''$ as a function of the signature $\pm$ of the vertices x, x', and x''.

x	x'	x''	$\Delta\eta$	$\Delta\eta''$	$\Delta\eta'$
+	+	+	$-i\epsilon$	$-i\epsilon$	$-i\epsilon$
+	+	−	$-i\epsilon$	$+i\epsilon$	$+i\epsilon$
+	−	+	$+i\epsilon$	$-i\epsilon$	$-i\epsilon$
+	−	−	$+i\epsilon$	$+i\epsilon$	$+i\epsilon$

$$\frac{r^2}{[r^2-(\Delta\eta-i\epsilon)^2]^2}-\frac{r^2}{[r^2-(\Delta\eta+i\epsilon)^2]^2}=\frac{\pi i}{2}\bigg\{\bigg[\frac{\partial}{\partial r}\delta(r-\Delta\eta)-\frac{\partial}{\partial r}\delta(r+\Delta\eta)\bigg]-\frac{1}{\Delta\eta}[\delta(r-\Delta\eta)+\delta(r+\Delta\eta)]\bigg\}, \tag{C11}$$

with whose help one can evaluate $J^{(2)}$. The result is

$$J^{(2)}=\frac{\pi i}{2}\bigg\{4(r''+\Delta\eta)\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2]\bigg)-4(r''-\Delta\eta)\ln\bigg(\frac{H^2}{4}[(r''-\Delta\eta)^2-\Delta\eta'^2]\bigg)$$
$$+\frac{2}{\Delta\eta}[(r''+\Delta\eta)^2-\Delta\eta'^2]\bigg[\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2]\bigg)-1\bigg]$$
$$+\frac{2}{\Delta\eta}[(r''-\Delta\eta)^2-\Delta\eta'^2]\bigg[\ln\bigg(\frac{H^2}{4}[(r''-\Delta\eta)^2-\Delta\eta'^2]\bigg)-1\bigg]\bigg\}. \tag{C12}$$

The symmetry of the integrals $J^{(1)}$ and $J^{(2)}$ under $r''\to -r''$ allows one to extend the r'' integration to $-\infty$. Upon inserting these two integrals into Eqs. (C4) and (C5) and summing them, we obtain

$$\mathcal{I}_{ij}=-i\frac{16\pi^3}{3}\frac{\delta_{ij}}{a^2H^4}\int_{\eta_0}^{\eta}d\eta' a'^3\int_{\eta_0}^{\eta'}d\eta'' a''^3\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr'' r''}{(r''^2-\Delta\eta''^2{}_{+\pm})^2}\bigg\{[\Delta\eta^2-r''^2-\Delta\eta'^2{}_{+\pm}]$$
$$\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg)-[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg\}. \tag{C13}$$

It is convenient to rewrite this expression in terms of two radial integrals as follows,

$$\mathcal{I}_{ij}=-i\frac{16\pi^3}{3}\frac{\delta_{ij}}{a^2H^4}\int_{\eta_0}^{\eta}d\eta' a'^3\int_{\eta_0}^{\eta'}d\eta'' a''^3[L^{(1)}+L^{(2)}], \tag{C14}$$

where

$$L^{(1)}=\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}dr''\frac{r''}{(r''^2-\Delta\eta''^2{}_{+\pm})^2}\times[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\times\bigg\{\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg)-1\bigg\}, \tag{C15}$$

$$L^{(2)}=-2\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}dr''\frac{r''^2}{(r''^2-\Delta\eta''^2{}_{+\pm})^2}[r''+\Delta\eta]\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg). \tag{C16}$$

$L^{(1)}$ can be partially integrated to give

$$L^{(1)}=\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr''}{r''^2-\Delta\eta''^2{}_{+\pm}}[r''+\Delta\eta]\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg). \tag{C17}$$

The first term in $L^{(2)}$ can be rewritten as

$$-2\frac{r''^2}{(r''^2-\Delta\eta''^2)^2}=-\frac{1}{r''^2-\Delta\eta''^2}-\frac{1}{2}\frac{1}{(r''-\Delta\eta'')^2}-\frac{1}{2}\frac{1}{(r''+\Delta\eta'')^2}, \tag{C18}$$

resulting in four simple integrals, $L^{(1)}+L^{(2)}=L_A+L_B+L_C+L_D$, where

$$L_A=-\frac{1}{2}\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr''}{r''-\Delta\eta''{}_{+\pm}}\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2_{+\pm}]\bigg), \tag{C19}$$

$$L_B=-\frac{1}{2}\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr''}{r''+\Delta\eta''_{+\pm}}\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'_{+\pm}]\bigg), \tag{C20}$$

$$L_C=-\frac{1}{2}\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr''(\Delta\eta+\Delta\eta'')}{(r''-\Delta\eta''_{+\pm})^2}\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg), \tag{C21}$$

$$L_D=\frac{1}{2}\sum_{x''=\pm}(\pm)\int_{-\infty}^{\infty}\frac{dr''\Delta\eta'}{(r''+\Delta\eta''_{+\pm})^2}\times\ln\bigg(\frac{H^2}{4}[(r''+\Delta\eta)^2-\Delta\eta'^2{}_{+\pm}]\bigg), \tag{C22}$$

where we made use of $r''+\Delta\eta=r''-\Delta\eta''+\Delta\eta+\Delta\eta''=r''+\Delta\eta''-\Delta\eta'$. All of these integrals are simple to evaluate by contour integration. For example, upon writing the $i\epsilon$ prescription explicitly (e.g. from Table I), L_A can be written as

$$L_A = -\frac{1}{2}\int_{-\infty}^{\infty} dr''\left\{\frac{\ln[r'' + \Delta\eta - \Delta\eta' + i\epsilon]}{r'' - \Delta\eta'' + i\epsilon} + \frac{\ln[r'' + \Delta\eta'' - i\epsilon]}{r'' - \Delta\eta'' + i\epsilon} - \frac{\ln[r'' + \Delta\eta - \Delta\eta' - i\epsilon]}{r'' - \Delta\eta'' - i\epsilon} - \frac{\ln[r'' + \Delta\eta'' + i\epsilon]}{r'' - \Delta\eta'' - i\epsilon}\right\}. \tag{C23}$$

This is straightforward to evaluate,

$$L_A = 2\pi i[\ln(2\Delta\eta'')]. \tag{C24}$$

Similarly, the remaining integrals in Eqs. (C20)–(C22) evaluate to

$$L_B = 2\pi i[-\ln(2\Delta\eta')], \tag{C25}$$

$$L_C = 2\pi i\left[\frac{1}{2} + \frac{1}{2}\frac{\Delta\eta}{\Delta\eta''}\right], \tag{C26}$$

$$L_D = 2\pi i[-\tfrac{1}{2}]. \tag{C27}$$

Upon combining these integrals, we can write the spatial integral I_{ij} (C14) as follows,

$$I_{ij} = -\frac{32\pi^4}{3}\frac{\delta_{ij}}{a^2H^4}\int_{\eta_0}^{\eta} d\eta' a'^3 \int_{\eta_0}^{\eta'} d\eta'' a''^3 \times \left[\ln\left(\frac{\Delta\eta'}{\Delta\eta''}\right) - \frac{1}{2}\frac{\Delta\eta}{\Delta\eta''}\right]. \tag{C28}$$

Upon changing the variables to $v = a'/a$ and $w = a''/a$, this reduces to

$$I_{ij} = -\frac{32\pi^4}{3}\frac{a^2\delta_{ij}}{H^6}\int_{1/a}^{a} dv v \int_{1/a}^{v} dw w \times \left[\ln\left(1 - \frac{w}{v}\right) - \ln(1-w) - \frac{1}{2}\frac{w(1-v)}{v(1-w)}\right]. \tag{C29}$$

These integrals are straightforward to do. The result is

$$I_{ij} = \frac{\pi^4}{H^6}a^2\delta_{ij}\left[1 - \frac{8}{3a^3} + \frac{5}{3a^4} - \frac{4}{3a^4}\ln(a)\right]. \tag{C30}$$

This is the final result for the spatial part of the finite 2-vertex integral.

The de Sitter invariant form of the integrand in Eq. (C1) indicates that the leading order contribution to the integral $I_{\mu\nu}$ should be de Sitter invariant, such that when combined with the 00 component of $I_{\mu\nu}$ one expects to get $I_{\mu\nu} = \frac{\pi^4}{H^6}g_{\mu\nu}$ plus terms suppressed by powers of the scale factor a. A detailed evaluation confirms this expectation. Albeit more cumbersome, the evaluation of I_{00} closely resembles the evaluation of I_{ij}, and here we present only the major steps.

From Eq. (C1) it follows that

$$I_{00} = \frac{1}{H^2}\sum_{\pm\pm}(\pm)(\pm)\int d^4x' a'^3\left[\frac{1}{\Delta x^2_{+\pm}} + \frac{2\eta\Delta\eta}{(\Delta x_{+\pm})^4}\right]\int d^4x'' a''^3\left[\frac{1}{\Delta x''^2{}_{+\pm}} + \frac{2\eta\Delta\eta''}{(\Delta x''_{+\pm})^4}\right]\ln\left(\frac{a'a''H^2}{4}\Delta x'^2{}_{\pm\pm}\right). \tag{C31}$$

Upon performing the r integral, one obtains

$$I_{00} = i\frac{16\pi^3}{H^2}\sum_{x''=\pm}(\pm)\int_{\eta_0}^{\eta} d\eta' a'^3\int_{\eta_0}^{\eta'} d\eta'' a''^3\int_{-\infty}^{\infty} dr''\left[\frac{r''}{r''^2 - \Delta\eta''^2{}_{+\pm}} + \frac{2\eta r''\Delta\eta''}{(r''^2 - \Delta\eta''^2{}_{+\pm})^2}\right] \times\left\{-\frac{1}{2}[(r''+\Delta\eta)^2 - \Delta\eta'^2{}_{+\pm}]\left[\ln\left(\frac{a'a''H^2}{4}\right) + \ln((r''+\Delta\eta)^2 - \Delta\eta'^2{}_{+\pm}) - 1\right] - \eta[r''+\Delta\eta]\left[\ln\left(\frac{a'a''H^2}{4}\right) + \ln((r''+\Delta\eta)^2 - \Delta\eta'^2{}_{+\pm})\right]\right\}. \tag{C32}$$

Performing the r'' integral requires more work,

$$I_{00} = -\frac{32\pi^4}{H^2}\int_{\eta_0}^{\eta} d\eta' a'^3\int_{\eta_0}^{\eta'} d\eta'' a''^3\Big\{[\Delta\eta\Delta\eta''\ln[a'a''H^2(\Delta\eta'')^2] - \Delta\eta(\Delta\eta'' + \Delta\eta')] + \eta[(\Delta\eta + \Delta\eta'')\ln[a'a''H^2(\Delta\eta'')^2] - 2\Delta\eta'] + \eta^2\left[\ln[a'a''H^2\Delta\eta''\Delta\eta'] + 1 + \frac{1}{2}\frac{\Delta\eta}{\Delta\eta''}\right]\Big\}. \tag{C33}$$

By making the substitutions $v = a'/a$ and $w = a''/a$, this can be reduced to a set of relatively simple integrals. The result of integration is

$$I_{00} = \frac{\pi^4 a^2}{H^6}\left\{-\left(1-\frac{1}{a}\right)\left(1+\frac{17}{a}-\frac{23}{a^2}+\frac{17}{a^3}\right)+\frac{4}{a^2}\left(4-\frac{1}{a^2}\right)\ln(a)-\frac{32}{a^2}\left(1-\frac{1}{a}\right)^2\ln\left(1-\frac{1}{a}\right)\right\}. \tag{C34}$$

When the two parts I_{ij} in Eq. (C30) and I_{00} in Eq. (C34) are combined, the leading order contribution is de Sitter invariant,

$$I_{\mu\nu} = \frac{\pi^4}{H^6} g_{\mu\nu} \tag{C35}$$

plus terms suppressed as powers of scale factors, in accordance with the expectation.

From this result we see that the finite integral $I_{\rm fin}$ does not contribute a leading logarithm to the two loop photon field strength bilinear.

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