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STABILITY OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE DELAYS

GUILING CHEN, DINGSHI LI, ONNO VAN GAANS, SJOERD VERDUYN LUNEL

Abstract. We present new criteria for asymptotic stability of two classes of nonlinear neutral delay differential equations. By using two auxiliary functions on a contraction condition, we extend the results in [12]. Also we give two examples that illustrate our results.

1. Introduction

In recent years there has been an increasing interest in stability results for neutral delay differential equations involving terms of the form $c(t)x(t-r(t))x'(t-r(t))$ (see [3, 15]). In this article, we consider the following two classes of nonlinear neutral delay differential equations

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + b(t)g(x(t-r(t))), \tag{1.1}$$

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} K(t,s)g(x(s))\,ds, \tag{1.2}$$

where $a, b : [0,\infty) \to \mathbb{R}$ are continuous functions, $c : [0,\infty) \to \mathbb{R}$ is continuously differentiable function and $r : [0,\infty) \to (0,\infty)$ is a continuous function, $K(t,s) : [0,\infty)\times[r_0,\infty) \to \mathbb{R}$ is a continuous function, $r_0 = \inf\{t-r(t) : t \ge 0\}$, $g(x) = |x|^\gamma$, $\gamma \ge 1$ is a constant, then g satisfies a locally Lipschitz condition; that is, there exists $L > 0$ and $l > 0$ such that g satisfies

$$|g(x) - g(y)| \le L|x-y| \quad \text{for } x, y \in [-l, l]. \tag{1.3}$$

Ahcene and Rabah [12] studied the special case of (1.1) and (1.2) when $g(x) = x^2$. The results in [12] mainly dependent on the constraint $|\frac{c(t)}{1-r'(t)}| < 1$. However, there are interesting examples where the constraint is not satisfied. It is our aim in this paper to remove this constraint condition and study the stability properties of (1.1) and (1.2).

Recent work of Burton and many others [1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15, 16] has shown the power of the fixed point method in studying stability properties of functional differential equations. The idea of using fixed point method to study properties of solutions seems to have emerged independently several times by different schools of authors. In addition to Burton's work, we would like to mention

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Corduneann [10] and Azbelev and his co-workers [4]. In this paper, we will use the fixed point method to study stability properties of (1.1) and (1.2). In particular, we introduce two auxiliary continuous functions $v(t)$ and $p(t)$ to define an appropriate mapping, and present new criteria for asymptotic stability of equations (1.1) and (1.2) which can be applied in the case $|\frac{c(t)}{1-r'(t)}| \geq 1$ as well.

An initial condition for the differential equation (1.1) is defined by

$$x(t) = \phi(t) \quad \text{for } t \in [r_0, 0], \tag{1.4}$$

where $\phi \in C([r_0,0],\mathbb{R})$. Here $C([r_0,0],\mathbb{R})$ denotes the set of all continuous functions $\varphi : [r_0,0] \to \mathbb{R}$ with the supremum norm $\|\cdot\|$. For $\phi \in C([r_0,0],\mathbb{R})$, we call a continuous function $x(t,\phi)$ to be a solution of (1.1) with initial condition (1.4) if $x : [r_0,a) \to \mathbb{R}$ for some positive constant $a > 0$ satisfies

$$\begin{aligned} &\frac{d}{dt}\Big(x(t) - \frac{c(t)}{2(1-r'(t))}x(t-r(t))^2\Big) \\ &= -a(t)x(t) + b(t)g(x(t-r(t))) - \frac{d}{dt}\Big(\frac{c(t)}{2(1-r'(t))}\Big)x^2(t-r(t)) \end{aligned} \tag{1.5}$$

on $[0,a)$ and $x = \phi$ on $[r_0,0]$. We denote such a solution by $x(t) := x(t,\phi)$. Note that equation (1.5) is in the standard form $\frac{d}{dt}(D(t,x_t)) = f(t,x_t)$ as studied in [13]. According to [13, Theorems 8.1 and 8.3], for each $\phi \in C([r_0,0],\mathbb{R})$, there exists a unique solution $x(t) = x(t,\phi)$ of (1.1) defined on $[0,\infty)$.

Definition 1.1. The zero solution of (1.1) is said to be stable, if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\phi : [r_0,0] \to (-\delta,\delta)$ implies that $|x(t)| < \epsilon$ for $t \geq 0$.

Definition 1.2. The zero solution of (1.1) is said to be asymptotically stable, if it is stable and there exists a $\delta > 0$ such that for any initial function $\phi : [r_0,0] \to (-\delta,\delta)$, the solution $x(t)$ with $x(t) = \phi(t)$ on $[r_0,0]$ tends to zero as $t \to \infty$.

By introducing two auxiliary functions $v(t)$ and $p(t)$ to construct a contraction mapping on a complete metric space, we obtain Theorem 1.3 and Theorem 1.4, which will be proved in Section 2 and Section 3, respectively.

Theorem 1.3. *Consider the neutral delay differential equation* (1.1) *and suppose the following conditions are satisfied:*

(i) $r(t)$ *is twice differentiable with* $r'(t) \neq 1$ *and* $t - r(t) \to \infty$ *as* $t \to \infty$*;*

(ii) *there exists a bounded function* $p : [r_0,\infty) \to (0,\infty)$ *with* $p(t) = 1$ *for* $t \in [r_0,0]$ *such that* $p'(t)$ *exists on* $[r_0,\infty)$ *and there exists a constant* $\alpha \in (0,1)$ *and* *an arbitrary continuous functions* $v : [r_0, \infty) \to \mathbb{R}$ *such that*

$$
\begin{aligned}
& l\Big\{\big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\big| + \int_0^t |\overline{k}(s) - 2b_1(s)| e^{-\int_s^t v(u)\,du}\,ds\Big\} \\
& + L\int_0^t e^{-\int_s^t v(u)\,du} \frac{|b(s)|p(s-r(s))^\gamma}{p(s)}\,ds \\
& + \int_{t-r(t)}^t \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big|\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^s \big|v(u) - a(u) - \frac{p'(u)}{p(u)}\big|\,du\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}\big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\big||1-r'(s)|\,ds \\
& \le \alpha,
\end{aligned}
\tag{1.6}
$$

where

$$
\overline{k}(s) = \frac{[\overline{c}(s)v(s) + \overline{c}'(s)](1-r'(s)) + \overline{c}(s)r''(s)}{(1-r'(s))^2}, \quad \overline{c}(s) = \frac{c(s)p^2(s-r(s))}{p(s)}, \tag{1.7}
$$

$$
b_1(s) = \frac{c(s)p(s-r(s))p'(s-r(s))}{p(s)}, \tag{1.8}
$$

and the constants l, L *are defined as in* (1.3)*;*

(iii) *and such that*

$$
\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.
$$

Then the zero solution of (1.1) *is asymptotically stable if and only if*

$$
\int_0^t v(s)\,ds \to \infty \quad \text{as } t \to \infty. \tag{1.9}
$$

Theorem 1.4. *Consider the neutral Voterra integro-differential equation* (1.2) *and suppose the following conditions are satisfied:*

(i) $r(t)$ *is twice differentiable,* $r'(t) \ne 1$, $t - r(t) \to \infty$ *as* $t \to \infty$*;*

(ii) *There exists a bounded function* $p : [r_0, \infty) \to (0, \infty)$ *with* $p(0) = 1$ *such that* $p'(t)$ *exists on* $[r_0, \infty)$ *and there exists a constant* $\alpha \in (0, 1)$ *and a continuous functions* $v : [r_0, \infty) \to \mathbb{R}$ *such that*

$$
\begin{aligned}
& l\Big\{\big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\big| + \int_0^t |\overline{k}(s) - 2b_1(s)| e^{-\int_s^t v(u)\,du}\,ds\Big\} \\
& + L\int_0^t e^{-\int_s^t v(u)\,du} \int_{s-r(s)}^s \frac{|K(s,u)|p^\gamma(u)}{p(s)}\,du \\
& + \int_{t-r(t)}^t \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big|\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^s \big|v(u) - a(u) - \frac{p'(u)}{p(u)}\big|\,du\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}\big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\big||1-r'(s)|\,ds \\
& \le \alpha,
\end{aligned}
\tag{1.10}
$$

where $\overline{k}(s)$ *and* $b_1(s)$ *are defined as* (1.7) *and* (1.8), *respectively, the constants* l, L *are defined as in* (1.3)*;*

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.$$

Then the zero solution of (1.2) *is asymptotically stable if and only if*

$$\int_0^t v(s)\,ds \to \infty \quad \text{as } t \to \infty. \tag{1.11}$$

The technique for constructing a contraction mapping comes from an idea in [16]. Our work extends and improves the results in [12, 16].

Remark 1.5. The method applied in this paper can be used to treat more general equations such as

$$\frac{d}{dt}x(t) = -a(t)h(t-r(t)) + \frac{d}{dt}Q(t, x(t-r(t))) + G(t, x(t), x(t-r(t)))$$

(studied by Mesmouli, Ardjouni and Djoudi in [14]) to give more general results.

2. Proof of Theorem 1.3

We start with some preparations. Define

$$\begin{aligned} S_\phi^l = \Big\{&\varphi \in C([r_0,\infty),\mathbb{R}) : \|\varphi\| = \sup_{t\ge r_0} |\varphi(t)| \le l,\ \varphi(t) = \phi(t) \\ &\text{for } t \in [r_0, 0],\ \varphi(t) \to 0 \text{ as } t \to \infty\Big\}. \end{aligned}$$

Then S_ϕ^l is a complete metric space with metric $\rho(x,y) = \sup_{t\ge r_0}\{|x(t)-y(t)|\}$.

Let $z(t) = \phi(t)$ on $[r_0, 0]$, and let $x(t) = p(t)z(t)$ for $t \ge 0$. If z satisfies

$$\begin{aligned} z'(t) = &-\Big(a(t) + \frac{p'(t)}{p(t)}\Big)z(t) + \frac{c(t)p(t-r(t))p'(t-r(t))}{p(t)} z^2(t-r(t)) \\ &+ \frac{c(t)p^2(t-r(t))}{p(t)} z(t-r(t))z'(t-r(t)) \\ &+ \frac{b(t)p(t-r(t))^\gamma}{p(t)} g(z(t-r(t))), \end{aligned} \tag{2.1}$$

then it can be verified that x satisfies (1.5). Since $p(t)$ is a positive bounded function, we only have to prove that the zero solution of (2.1) is asymptotically stable.

If we multiply both sides of (2.1) by $e^{\int_0^t v(s)\,ds}$ and then integrate from 0 to t, we obtain

$$\begin{aligned} z(t) = \phi(0)e^{-\int_0^t v(s)\,ds} &+ \int_0^t \Big(v(s) - a(s) - \frac{p'(s)}{p(s)}\Big)e^{-\int_s^t v(u)\,du} z(s)\,ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \frac{c(s)p(s-r(s))p'(s-r(s))}{p(s)} z^2(s-r(s))\,ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \frac{c(s)p^2(s-r(s))}{p(s)} z(s-r(s))z'(s-r(s))\,ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \frac{b(s)p(s-r(s))^\gamma}{p(s)} g(z(s-r(s)))\,ds. \end{aligned} \tag{2.2}$$

Taking

$$
\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du}\frac{c(s)p^2(s-r(s))}{p(s)}z(s-r(s))z'(s-r(s))\,ds\\
&=\int_0^t e^{-\int_s^t v(u)\,du}\frac{c(s)p^2(s-r(s))}{p(s)}z(s-r(s))z'(s-r(s))\\
&\quad\times(1-r'(s))\frac{1}{1-r'(s)}\,ds
\end{aligned}
\tag{2.3}
$$

and integrating by parts the right-hand side of (2.3), we obtain

$$
\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du}\frac{c(s)p^2(s-r(s))}{p(s)}z(s-r(s))z'(s-r(s))\,ds\\
&=\frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}z^2(t-r(t))-\frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\\
&\quad\times e^{-\int_0^t v(s)\,ds}-\frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\overline{k}(s)z^2(s-r(s))\,ds
\end{aligned}
\tag{2.4}
$$

where $\overline{k}(s)$ is given by (1.7).

Integrating by parts, we obtain

$$
\begin{aligned}
&\int_0^t \Big(v(s)-a(s)-\frac{p'(s)}{p(s)}\Big)e^{-\int_s^t v(u)\,du}z(s)\,ds\\
&=\int_0^t e^{-\int_s^t v(u)\,du}\,d\Big(\int_{s-r(s)}^s\Big(v(u)-a(u)-\frac{p'(u)}{p(u)}\Big)z(u)\,du\Big)\\
&\quad+\int_0^t e^{-\int_s^t v(u)\,du}\Big(v(s-r(s))-a(s-r(s))-\frac{p'(s-r(s))}{p(s-r(s))}\Big)\\
&\quad\times(1-r'(s))z(s-r(s))\,ds\\
&=\int_{t-r(t)}^t\Big(v(s)-a(s)-\frac{p'(s)}{p(s)}\Big)z(s)\,ds\\
&\quad-\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^s\Big(v(u)-a(u)-\frac{p'(u)}{p(u)}\Big)z(u)\,du\,ds\\
&\quad+\int_0^t e^{-\int_s^t v(u)\,du}\Big(v(s-r(s))-a(s-r(s))-\frac{p'(s-r(s))}{p(s-r(s))}\Big)\\
&\quad\times(1-r'(s))z(s-r(s))\,ds
\end{aligned}
\tag{2.5}
$$

Combining (2.2), (2.4) and (2.5), we obtain that a solution of (2.1) has the form

$$
\begin{aligned}
z(t)&=\Big[\phi(0)-\int_{-r(0)}^0\Big(v(s)-a(s)-\frac{p'(s)}{p(s)}\Big)\phi(s)\,ds\\
&\quad-\frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\Big]e^{-\int_0^t v(s)\,ds}\\
&\quad+\frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}z^2(t-r(t))\\
&\quad+\int_{t-r(t)}^t\Big(v(s)-a(s)-\frac{p'(s)}{p(s)}\Big)z(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^s \Big(v(u) - a(u) - \frac{p'(u)}{p(u)}\Big) z(u)\,du\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} \Big(v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\Big) \\
&\times (1-r'(s)) z(s-r(s))\,ds \\
&- \frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du} \left(\overline{k}(s) - 2b_1(s)\right) z^2(s-r(s))\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} \frac{b(s)p(s-r(s))^\gamma}{p(s)} g(z(s-r(s)))\,ds := \sum_{i=1}^7 I_i(t),
\end{aligned}
$$

where $\overline{k}(s)$ and $b_1(s)$ are defined in (1.7) and (1.8), respectively.

Lemma 2.1. *Let $z \in S_\phi^l$ and define an operator by $(Pz)(t) = \phi(t)$ for $t \in [r_0, 0]$ and for $t \geq 0$, $(Pz)(t) = \sum_{i=1}^7 I_i(t)$. If conditions* (i)–(ii) *and 1.9 in Theorem 1.3 are satisfied, then there exists $\delta > 0$ such that for any initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that $P : S_\phi^l \to S_\phi^l$ and P is a contraction with respect to the metric ρ defined on S_ϕ^l.*

Proof. Set $J = \sup_{t\geq 0}\{e^{-\int_0^t v(s)\,ds}\}$, by (iii), J is well defined. Suppose that 1.9 holds. We choose $\delta > 0$ such that

$$
\Big[\delta + \delta\int_{-r(0)}^0 \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big|\,ds + \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\delta^2\Big] J \leq (1-\alpha) l.
$$

Let ϕ be a given small bounded initial function with $\|\phi\| < \delta$, and let $\varphi \in S_\phi^l$, then $\|\varphi\| \leq l$. Since g satisfies a locally Lipschitz condition, from (1.6) in Theorem 1.3, we have

$$
|P\varphi(t)| \leq \Big[\delta + \delta\int_{-r(0)}^0 \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big|\,ds + \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\delta^2\Big] J + \alpha l \leq l.
$$

Thus, $\|P\varphi\| \leq l$.

Next, we show that $P\varphi \to 0$ as $t \to \infty$. It is clear that $I_i(t) \to 0$ for $i = 1, 2, 3, 4, 5, 7$, since $e^{\int_0^t v(s)\,ds} \to \infty$, $t - r(t) \to \infty$ and $\varphi \to 0$ as $t \to \infty$. Now, we prove that $I_6(t) \to 0$ as $t \to \infty$. For $t - r(t) \to \infty$ and $\varphi \to 0$, we obtain that for any $\epsilon > 0$, there is a positive number $T_1 > 0$ such that for $t \geq T_1$, $\varphi(t - r(t)) < \epsilon$, so we have

$$
\begin{aligned}
|I_6(t)| &= \big|\frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du} \left(\overline{k}(s) - 2b_1(s)\right) \varphi^2(s-r(s))\,ds\big| \\
&\leq \frac{1}{2} e^{-\int_{T_1}^t v(u)\,du} \int_0^{T_1} e^{-\int_s^{T_1} v(u)\,du} |\overline{k}(s) - 2b_1(s)| \varphi^2(s-r(s))\,ds \\
&\quad + \frac{1}{2}\int_{T_1}^t e^{-\int_s^t v(u)\,du} \left|\overline{k}(s) - 2b_1(s)\right| \varphi^2(s-r(s))\,ds \\
&\leq \frac{1}{2}\Big(\sup_{t\geq r_0} |\varphi(t)|\Big)^2 e^{-\int_{T_1}^t v(u)\,du} \int_0^{T_1} e^{-\int_s^{T_1} v(u)\,du} |\overline{k}(s) - 2b_1(s)|\,ds \\
&\quad + \frac{1}{2}\epsilon^2 \int_{T_1}^t e^{-\int_s^t v(u)\,du} |\overline{k}(s) - 2b_1(s)|\,ds
\end{aligned}
$$

$$\le \frac{\alpha}{2} l^2 e^{-\int_{T_1}^t v(u)\,du} + \alpha\epsilon.$$

By 1.9, there exists $T_2 > T_1$ such that $t > T_2$ implies $\frac{\alpha}{2} l^2 e^{-\int_{T_1}^t v(u)\,du} < \epsilon$, which implies $I_6(t) \to 0$ as $t \to \infty$. Hence, we have $(P\varphi)(t) \to 0$ as $t \to \infty$.

Finally, we show that P is a contraction mapping. In fact, for $\varphi, \eta \in S_\phi^l$, using condition (1.6) in Theorem 1.3, we obtain

$$\begin{aligned}
&\big|(P\varphi)(t) - (P\eta)(t)\big| \\
&\le 2l \left|\frac{p^2(t-r(t))}{2p(t)} \frac{c(t)}{1-r'(t)}\right| \|\varphi - \eta\| + \int_{t-r(t)}^t \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big| \|\varphi - \eta\|\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s \big|v(u) - a(u) - \frac{p'(u)}{p(u)}\big|\, \|\varphi - \eta\|\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} \left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| \\
&\quad \times |1 - r'(s)| \|\varphi - \eta\|\,ds \\
&\quad + l \int_0^t e^{-\int_s^t v(u)\,du} |\bar{k}(s) - 2b_1(s)|\, \|\varphi - \eta\|\,ds \\
&\quad + L \int_0^t e^{-\int_s^t v(u)\,du} \frac{|b(s)| p(s-r(s))^\gamma}{p(s)} \|\varphi - \eta\|\,ds \\
&\quad \le \alpha \|\varphi - \eta\|.
\end{aligned}$$

Therefore, $P : S_\phi^l \to S_\phi^l$ is a contraction mapping. □

Proof of Theorem 1.3. Let P be defined as in Lemma 2.1. By the contraction mapping principle, P has a unique fixed point z in S_ϕ^l which is a solution of (2.1) with $z(t) = \phi(t)$ on $[r_0, 0]$ and $z(t) \to 0$ as $t \to \infty$.

To prove stability, let $\epsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{L, \epsilon\}$. Replacing l with m in S_ϕ^l, we obtain that there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (2.1) with $z(t) = \phi(t)$ on $[r_0, 0]$ satisfies $|z(t)| \le m < \epsilon$ for all $t \ge r_0$. This shows that the zero solution of (2.1) is asymptotically stable if 1.9 holds.

Conversely, we suppose that 1.9 fails. Then by (iii), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} v(s)\,ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \le \int_0^{t_n} v(s)\,ds \le M \tag{2.6}$$

for all $n \ge 1$. To simplify our expressions, we define

$$\begin{aligned}
w(s) &= l|\bar{k}(s) - 2b_1(s)| + |v(s)| \int_{s-r(s)}^s \big|v(u) - a(u) - \frac{p'(u)}{p(u)}\big|\,du \\
&\quad + \frac{L|b(s)|p(s-r(s))^\gamma}{p(s)} \\
&\quad + \big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\big| |1 - r'(s)|
\end{aligned}$$

for all $s \geq 0$. By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} v(u)\,du} w(s)\,ds \leq \alpha. \tag{2.7}$$

Combining (2.6) and (2.7), we have $\int_0^{t_n} e^{\int_0^s v(u)\,du} w(s)\,ds \leq \alpha e^{\int_0^{t_n} v(u)\,du} \leq \alpha e^M$, which yields that $\int_0^{t_n} e^{\int_0^s v(u)\,du} w(s)\,ds$ is bounded. Hence, there exists a convergent subsequence, we assume that $\lim_{k\to\infty} \int_0^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds = \gamma$ for some $\gamma \in \mathbb{R}^+$. We choose a positive integer k_1 so large that $\lim_{k\to\infty} \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds \leq \frac{\delta_0}{4J}$ for all $n_k > n_{k_1}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$.

Now, we consider the solution $z(t) = z(t, t_{n_{k_1}}, \phi)$ of (2.1) with $\phi(t_{n_{k_1}}) = \delta_0$ and $\phi(s) \leq \delta_0$ for $s \leq t_{n_{k_1}}$, and we may choose ϕ such that $|z(t)| \leq 1$ for $t \geq t_{n_{k_1}}$ and

$$\begin{aligned} &\phi(t_{n_{k_1}}) - \int_{t_{n_{k_1}} - r(t_{n_{k_1}})}^{t_{n_{k_1}}} \Big(v(s) - a(s) - \frac{p'(s)}{p(s)}\Big)\phi(s)\,ds \\ &- \frac{p^2(t_{n_{k_1}} - r(t_{n_{k_1}}))}{2p(t_{n_{k_1}})} \frac{c(t_{n_{k_1}})}{1 - r'(t_{n_{k_1}})} \phi^2(t_{n_{k_1}} - r(t_{n_{k_1}})) \geq \frac{1}{2}\delta_0. \end{aligned} \tag{2.8}$$

So, it follows from (2.8) with $z(t) = (Pz)(t)$ that for $k \geq k_1$,

$$\begin{aligned} &\Big| z(t_{n_k}) - \frac{p^2(t_{n_k} - r(t_{n_k}))}{2p(t_{n_k})} \frac{c(t_{n_k})}{1 - r'(t_{n_k})} z^2(t - r(t_{n_k})) \\ &- \int_{t_{n_k} - r(t_{n_k})}^{t_{n_k}} \Big[v(s) - a(s) - \frac{p'(s)}{p(s)}\Big] z(s)\,ds \Big| \\ &\geq \frac{1}{2}\delta_0 e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} - \int_{t_{n_{k_1}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}} v(u)\,du} w(s)\,ds \\ &= e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \Big[\frac{1}{2}\delta_0 - e^{-\int_0^{t_{n_{k_1}}} v(u)\,du} \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds\Big] \\ &\geq e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \Big[\frac{1}{2}\delta_0 - J \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds\Big] \\ &\geq \frac{1}{4}\delta_0 e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \geq \frac{1}{4}\delta_0 e^{-2M} > 0. \end{aligned} \tag{2.9}$$

On the other hand, suppose that the solution of (2.1) $z(t) = z(t, t_{n_{k_1}}, \phi) \to 0$ as $t \to \infty$. Since $t_{n_k} - r(t_{n_k}) \to \infty$ as $k \to \infty$, and (ii) holds, we have

$$\begin{aligned} &z(t_{n_k}) - \frac{p^2(t_{n_k} - r(t_{n_k}))}{2p(t_{n_k})} \frac{c(t_{n_k})}{1 - r'(t_{n_k})} z^2(t - r(t_{n_k})) \\ &- \int_{t_{n_k} - r(t_{n_k})}^{t_{n_k}} \Big[v(s) - a(s) - \frac{p'(s)}{p(s)}\Big] z(s)\,ds \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

which contradicts (2.9). Hence condition 1.9 is necessary for the asymptotic stability of the zero solution of (2.1).

Since $p(t)$ is a positive bounded function, from the above arguments we obtain that 1.9 is a necessary and sufficient condition for the asymptotic stability of the zero solution of (1.1). $\square$

When $g(x) = x^2$ in (1.1), we have the following result.

Corollary 2.2. *Suppose the following conditions are satisfied:* (i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a bounded function $p : [r_0, \infty) \to (0, \infty)$ with $p(0) = 1$ such that $p'(t)$ exists on $[r_0, \infty)$, and there exists a constant $\alpha \in (0, 1)$ and an arbitrary continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$
\begin{aligned}
& l\Big\{\big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\big| + \int_0^t |\overline{k}(s) - 2\overline{b}(s)| e^{-\int_s^t v(u)\,du}\,ds\Big\} \\
& + \int_{t-r(t)}^t \big|v(s) - a(s) - \frac{p'(s)}{p(s)}\big|\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s \big|v(u) - a(u) - \frac{p'(u)}{p(u)}\big|\,du\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du} \big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\big| |1-r'(s)|\,ds \\
& \leq \alpha,
\end{aligned} \tag{2.10}
$$

where $\overline{k}(s)$ is defined as in (1.7),

$$
\overline{b}(s) = \frac{b(s)p^2(s-r(s)) + c(s)p(s-r(s))p'(s-r(s))}{p(s)}, \tag{2.11}
$$

and $l > 0$ is defined as in (1.3)*;*

(iii) *and such that*

$$
\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.
$$

Then the zero solution $x(t, \phi)$ of (1.1) *is asymptotically stable if and only if*

$$
\int_0^t v(s)\,ds \to \infty \quad \text{as } t \to \infty. \tag{2.12}
$$

3. Proof of Theorem 1.4

We start with some preparations. Define

$$
\begin{aligned}
S_\phi^l = \Big\{ & \varphi \in C([r_0, \infty), \mathbb{R}) : \|\varphi\| = \sup_{t \geq r_0} |\varphi(t)| \leq l,\ \varphi(t) = \phi(t) \\
& \text{for } t \in [r_0, 0],\ \varphi(t) \to 0 \text{ as } t \to \infty \Big\}.
\end{aligned}
$$

Then S_ϕ^l is complete metric space with metric $\rho(x, y) = \sup_{t \geq r_0}\{|x(t) - y(t)|\}$.

Let $z(t) = \phi(t)$ on $[r_0, 0]$, and let $x(t) = p(t)z(t)$, for $t \geq 0$, from (1.2), we obtain

$$
\begin{aligned}
z'(t) &= -\big(a(t) + \frac{p'(t)}{p(t)}\big)z(t) + \frac{c(t)p(t-r(t))p'(t-r(t))}{p(t)} z^2(t-r(t)) \\
&\quad + \frac{c(t)p^2(t-r(t))}{p(t)} z(t-r(t))z'(t-r(t)) \\
&\quad + \int_{t-r(t)}^t \frac{p^\gamma(s)K(t,s)}{p(t)} g(z(s))\,ds.
\end{aligned} \tag{3.1}
$$

Since $p(t)$ is bounded, we only need to prove that the zero solution of (3.1) is asymptotically stable.

If we multiply both sides of (3.1) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$
\begin{aligned}
z(t) &= \Big\{\phi(0) - \int_{-r(0)}^{0} \Big(v(s) - a(s) - \frac{p'(s)}{p(s)}\Big)\phi(s)\,ds \\
&\quad - \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\Big\} e^{-\int_0^t v(s)\,ds} \\
&\quad + \frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)} z^2(t-r(t)) \\
&\quad + \int_{t-r(t)}^{t} \Big[v(s) - a(s) - \frac{p'(s)}{p(s)}\Big] z(s)\,ds \\
&\quad - \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^{s} \Big(v(u) - a(u) - \frac{p'(u)}{p(u)}\Big) z(u)\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} \Big(v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\Big) \\
&\quad \times (1-r'(s)) z(s-r(s))\,ds \\
&\quad - \frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du} \big(\overline{k}(s) - 2b_1(s)\big) z^2(s-r(s))\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} \int_{s-r(s)}^{s} \frac{K(s,u)p^{\gamma}(u)}{p(s)} g(z(u))\,du\,ds := \sum_{i=1}^{7} I_i(t)
\end{aligned}
$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as in (1.7) and (1.8) respectively.

Lemma 3.1. *Let $z \in S_\phi^l$ and define an operator by $(Pz)(t) = \phi(t)$ for $t \in [r_0, 0]$ and for $t \geq 0$, $(Pz)(t) = \sum_{i=1}^{7} I_i(t)$. If conditions* (i)-(iii) *in Theorem 1.4 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that $P : S_\phi^l \to S_\phi^l$ and P is a contraction mapping with respect to the metric defined on S_ϕ^l.*

Proof. Set $J = \sup_{t\geq 0}\{e^{-\int_0^t v(s)\,ds}\}$, by (iii), J is well defined. Suppose that (iii) holds. Using the similar arguments as as the proof of Theorem 1.3, we obtain that $P\varphi \in S_\phi^l$ for $\varphi \in S_\phi^l$. Now, we show that P is a contraction mapping. In fact, for $\varphi, \eta \in S_\phi^l$, by using condition (1.10) in Theorem 1.4, we obtain that

$$
\begin{aligned}
&\big|(P\varphi)(t) - (P\eta)(t)\big| \\
&\leq \Big|\frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}\Big| 2l\|\varphi - \eta\| \\
&\quad + \int_{t-r(t)}^{t} \Big|v(s) - a(s) - \frac{p'(s)}{p(s)}\Big|\,\|\varphi - \eta\|\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^{s} \Big|v(u) - a(u) - \frac{p'(u)}{p(u)}\Big|\,\|\varphi - \eta\|\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} \Big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\Big| \\
&\quad \times |1-r'(s)|\|\varphi - \eta\|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t |\overline{k}(s)-2b_1(s)|2l\|\varphi-\eta\| \\
&+L\int_0^t e^{-\int_s^t v(u)\,du}\int_{s-r(s)}^{s}\frac{|K(s,u)|p^{\gamma}(u)}{p(s)}\,du\cdot\|\varphi-\eta\|\,ds \\
&\le \alpha\|\varphi-\eta\|.
\end{aligned}
$$

Hence, we obtain that $P: S_\phi^l \to S_\phi^l$ is a contraction mapping. □

Proof of Theorem 1.4. Let P be defined as in Lemma 3.1. By the contraction mapping principle, P has a unique fixed point z in S_ϕ^l which is a solution of (1.2) with $z(t)=\phi(t)$ on $[r_0,0]$ and $z(t)\to 0$ as $t\to\infty$.

Let $\epsilon>0$ be given, then we choose $m>0$ so that $m<\min\{l,\epsilon\}$. Replacing l with m in S_ϕ^l, we obtain there is a $\delta>0$ such that $\|\phi\|<\delta$ implies that the unique solution of (1.2) with $z(t)=\phi(t)$ on $[r_0,0]$ satisfies $|z(t)|\le m<\epsilon$ for all $t\ge r_0$. This shows that the zero solution of (1.2) is asymptotically stable if 1.11 holds.

Following the similar arguments as the proof of Theorem 1.3, we obtain that 1.11 is necessary for the asymptotic stability of the zero solution of (1.2). The proof is complete. □

When $g(x)=x^2$, we have the following corollary.

Corollary 3.2. *Suppose the following conditions are satisfied:*

(i) *the delay $r(t)$ is twice differentiable, $r'(t)\neq 1$, $t-r(t)\to\infty$ as $t\to\infty$;*

(ii) *there exists a bounded function $p:[r_0,\infty)\to(0,\infty)$ with $p(0)=1$ such that $p'(t)$ exists on $[r_0,\infty)$, and there exists a constant $\alpha\in(0,1)$, a constant $l>0$ and a continuous functions $v:[r_0,\infty)\to\mathbb{R}$ such that*

$$
\begin{aligned}
&l\Big\{\Big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\Big|+\int_0^t\Big[|\overline{k}(s)-2b_1(s)| \\
&\quad+2\int_{s-r(s)}^{s}\Big|\frac{K(s,u)p^2(u)}{p(s)}\Big|\,du\Big]e^{-\int_s^t v(u)\,du}\,ds\Big\} \\
&\quad+\int_{t-r(t)}^{t}\Big|v(s)-a(s)-\frac{p'(s)}{p(s)}\Big|\,ds \\
&\quad+\int_0^t e^{-\int_s^t v(u)\,du}\Big|v(s-r(s))-a(s-r(s))-\frac{p'(s-r(s))}{p(s-r(s))}\Big||1-r'(s)|\,ds \\
&\quad+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^{s}\Big|v(u)-a(u)-\frac{p'(u)}{p(u)}\Big|\,du\,ds\le\alpha,
\end{aligned}
\tag{3.2}
$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as in (1.7) and (1.8);

(iii) *and such that*

$$\liminf_{t\to\infty}\int_0^t v(s)\,ds>-\infty.$$

Then the zero solution of (1.2) with a small initial function ϕ is asymptotically stable if and only if

$$\int_0^t v(s)\,ds\to\infty \quad \text{as } t\to\infty. \tag{3.3}$$

4. Examples

Example 4.1. Consider the nonlinear neutral differential equation

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + b(t)x^2(t-r(t)) \tag{4.1}$$

for $t \geq 0$, where $a(t) = \frac{2}{t+1}$, $c(t) = 0.95$, $r(t) = 0.05t$, $l = 1$, $b(t)$ satisfies $|\overline{k}(s) - 2\overline{b}(s)| \leq \frac{0.3}{s+1}$, then the zero solution of (4.1) is asymptotically stable.

Proof. We check condition (2.10) in Corollary 2.2, choosing $v(t) = \frac{1.5}{t+1}$ and $p(t) = \frac{1}{t+1}$, we obtain that

$$\begin{aligned}
& l\Big\{\Big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\Big| + \int_0^t e^{-\int_s^t v(u)\,du}|\overline{k}(s) - 2\overline{b}(s)|\,ds\Big\} \\
& + \int_{t-r(t)}^t \Big|v(s) - a(s) - \frac{p'(s)}{p(s)}\Big|\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \Big|v(u) - a(u) - \frac{p'(u)}{p(u)}\Big|\,du\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}\Big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\Big||1-r'(s)|\,ds \\
& < 0.36 + 0.026 + 0.026 + 0.33 + 0.2 = 0.941 < 1,
\end{aligned}$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{1.5}{s+1}\,ds = 1.5\ln(t+1) \to \infty$ as $t \to \infty$, $p(t) \leq 1$, so the conditions of Corollary 2.2 are satisfied. Therefore, the zero solution of (4.1) is asymptotically stable. □

Note that $|\frac{c(t)}{1-r'(t)}| = 1$; therefore the result in [12] is not applicable.

Example 4.2. Consider the nonlinear neutral Volterra integral equation

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^t K(t,s)x^2(s)\,ds \tag{4.2}$$

for $t \geq 0$, where $a(t) = \frac{2.5}{t+0.1}$, $c(t) = \frac{(0.95t+0.1)^2}{t+0.1}$, $r(t) = 0.05t$, $l = 1$, $K(t,s) = \frac{1}{t+0.1}$, then the zero solution of (4.2) is asymptotically stable.

Proof. We check the condition (3.2) in Corollary 3.2, choosing $v(t) = \frac{2}{t+0.1}$ and $p(t) = \frac{0.1}{t+0.1}$, we obtain that

$$\begin{aligned}
& l\Big|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\Big| \\
& + l\int_0^t \Big[|\overline{k}(s) - 2b_1(s)| + 2\int_{s-r(s)}^s \Big|\frac{K(s,u)p^2(u)}{p(s)}\Big|\,du\Big]e^{-\int_s^t v(u)\,du}\,ds \\
& + \int_{t-r(t)}^t \Big|v(s) - a(s) - \frac{p'(s)}{p(s)}\Big|\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \Big|v(u) - a(u) - \frac{p'(u)}{p(u)}\Big|\,du\,ds \\
& + \int_0^t e^{-\int_s^t v(u)\,du}\Big|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\Big|\,|1-r'(s)|\,ds \\
& < 0.106 + 0.416 + 0.026 + 0.026 + 0.25 = 0.824 < 1,
\end{aligned}$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{2}{s+0.1}\,ds = 2\ln(t+0.1) \to \infty$ as $t \to \infty$, $p(t) \leq 1$, so the conditions of Corollary 3.2 are satisfied. Therefore, the zero solution of (4.2) is asymptotically stable. $\square$

Note that $|\frac{c(t)}{1-r'(t)}| = \frac{(0.95t+0.1)^2}{0.95(t+0.1)} \to \infty$ as $t \to \infty$; therefore the result in [12] is not applicable.

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Guiling Chen

Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, China

E-mail address: `guiling@home.swjtu.edu.cn`

Dingshi Li (corresponding author)

Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, China

E-mail address: `lidingshi2006@163.com`

Onno van Gaans
Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA, Leiden, The Netherlands
E-mail address: vangaans@math.leidenuniv.nl

Sjoerd Verduyn Lunel
Mathematical Institute, Utrecht University, P.O. Box 80010, 3508 TA, Utrecht, The Netherlands
E-mail address: S.M.VerduynLunel@uu.nl