



What Chern–Simons theory assigns to a point

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Edited by Robion C. Kirby, University of California, Berkeley, CA, and approved October 19, 2017 (received for review June 28, 2017)

We answer the questions, “What does Chern–Simons theory assign to a point?” and “What kind of mathematical object does Chern–Simons theory assign to a point?” Our answer to the first question is representations of the based loop group. More precisely, we identify a certain class of projective unitary representations of the based loop group ΩG . We define the fusion product of such representations, and we prove that, modulo certain conjectures, the Drinfel’d center of that representation category of ΩG is equivalent to the category of positive energy representations of the free loop group LG .[†] The abovementioned conjectures are known to hold when the gauge group is abelian or of type A_1 . Our answer to the second question is bicommutant categories. The latter are higher categorical analogs of von Neumann algebras: They are tensor categories that are equivalent to their bicommutant inside $\text{Bim}(R)$, the category of bimodules over a hyperfinite III_1 factor. We prove that, modulo certain conjectures, the category of representations of the based loop group is a bicommutant category. The relevant conjectures are known to hold when the gauge group is abelian or of type A_n .

Chern–Simons theory | extended field theory | loop groups | Drinfel’d center | conformal nets

The Chern–Simons theories are certain 3D topological quantum field theories introduced by Witten (1). They are parameterized by a compact Lie group G known as the gauge group and by a cohomology class $k \in H^4(BG, \mathbb{Z})$ known as the level of the theory (2–4). The Chern–Simons action

$$S = \frac{1}{4\pi} \int_{M^3} \langle A \wedge dA \rangle_k + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle_k \pmod{2\pi} \quad [1]$$

is a functional of G bundles with connections over compact 3 manifolds. Here, A is the connection form, $\langle \cdot \rangle_k : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$ is a certain metric constructed from the level, and the integral is taken over a global section of the principal bundle.[‡] We point out that not every level $k \in H^4(BG, \mathbb{Z})$ yields a quantum field theory. For example, it is important that $\langle \cdot \rangle_k$ be nondegenerate. In this paper, we deal only with those levels k that satisfy the following positivity condition:

Definition 1. Let G be a compact Lie group. A level $k \in H^4(BG, \mathbb{Z})$ is positive if its image under the Chern–Weil homomorphism $H^4(BG) \rightarrow \text{Sym}^2(\mathfrak{g}^*)^G$ is a positive definite symmetric bilinear form $\langle \cdot \rangle_k : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$.

We write $CS_{G,k}$ for the Chern–Simons theory associated to the gauge group G and the level k . In the case of finite gauge groups, Chern–Simons theory is also known as Dijkgraaf–Witten theory.

It is well known that a G bundle with connection is a critical point of the Chern–Simons action functional (a classical solution of the equations of motion) if and only if the connection is flat. Such bundles are called local systems. We write $\text{Loc}_G(M)$ for the space of gauge equivalence classes of G -local systems on a manifold M . From a mathematical point of view, the formula

$$CS_{G,k}(M) = \int \left\{ \begin{array}{l} \text{Gauge equivalence classes of} \\ G \text{ bundles with connection } A \\ \text{on the 3-manifold } M \end{array} \right\} e^{iS[A]} \mathcal{D}A$$

used to “define” the value of the topological quantum field theory (TQFT) on a 3-manifold M (the value at M of the par-

tion function) is yet to be defined, because the measure $\mathcal{D}A$ remains to be described. However, the quantum Hilbert space $CS_{G,k}(\Sigma)$ associated to a Riemann surface Σ can and has been defined at a mathematical level of precision. Following the prescription of canonical quantization, it is the geometric quantization of $\text{Loc}_G(\Sigma)$ with respect to the natural symplectic structure coming from the Chern–Simons Lagrangian (7, 8) (e.g., see ref. 9, section 6.1 for a discussion of the symplectic structure).

Therefore, from a mathematical perspective, a TQFT that recovers the above quantum Hilbert spaces may be called “Chern–Simons theory.” We explain below that, at least when G is simply connected (and presumably also when it is just connected), the Reshetikhin–Turaev TQFT associated to the modular tensor category $\text{Rep}^k(LG)$ of positive energy representations of the loop group at level k has that property.

1. Extended TQFTs

In the functorial approach to quantum field theory, a d -dimensional quantum field theory is a functor from a certain cobordism category, whose objects are $(d - 1)$ -dimensional manifolds and whose morphisms are d -dimensional cobordisms, to the category of vector spaces (10, 11). Extended quantum field theory has been proposed by Lawrence (12), and later Freed (13, 14) and Baez and Dolan (15), as an enhancement of the functorial approach in which one assigns values not only to d - and $(d - 1)$ -dimensional manifolds, but also to $(d - 2)$ -dimensional manifolds, all of the way down to 0-dimensional manifolds.

In his influential paper (14) (also ref. 16), Freed argued that Dijkgraaf–Witten theory fits into the framework of extended TQFT. Using a “categorified path integral,” he computed the value of that theory on the circle and showed that it is $\text{Rep}(D^k(\mathbb{C}[G]))$, the representation category of the k -twisted

Significance

There are two main classes of 3D topological field theories: Turaev–Viro theories, associated to fusion categories, and Reshetikhin–Turaev theories, associated to modular tensor categories. Since the groundbreaking work by Lurie on the cobordism hypothesis, it has been a major open question to know which topological field theories (TFTs) extend down to points. Turaev–Viro theories can be extended down to points. But for most Reshetikhin–Turaev theories, including Chern–Simons theories, this was believed to be impossible (unless one puts them on the boundary of a 4D TFT). The present paper achieves two things: It shows that Reshetikhin–Turaev theories extend down to points, and it puts Turaev–Viro theories and Reshetikhin–Turaev theories on an equal footing by providing a unified language, bicommutant categories, that applies to both.

Author contributions: A.G.H. wrote the paper.

The author declares no conflict of interest.

This article is a PNAS Direct Submission.

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[†]We warn the reader that $\text{Rep}(\Omega G)$ is not a topological boundary condition for Chern–Simons theory.

[‡]When G is not simply connected, principal bundles over 3 manifolds can fail to have global sections and so one cannot use Eq. 1 to define the action. See refs. 2, 5, and 6 for ways to overcome this difficulty.

Drinfel'd double of the group algebra of G . Freed did not extend the theory all of the way down to points, even though the case $k = 0$ of $CS_{G,k}(pt)$ is implicit in his paper.

The representation category of $D^k(\mathbb{C}[G])$ is equivalent to $\text{Vect}_G^k[G]$, the category of k -twisted equivariant vector bundles on G with respect to the adjoint action of the group on itself (17). The latter can in turn be identified with $Z(\text{Vect}^k[G])$, the Drinfel'd center of the category of G -graded vector spaces with k -twisted convolution product (18). Summarizing, for finite gauge group G ,

$$CS_{G,k}(S^1) = \text{Vect}_G^k[G] = \text{Rep}(D^k(\mathbb{C}[G])) = Z(\text{Vect}^k[G]). \quad [2]$$

The latter was taken as evidence in ref. 18 (also ref. 19) for the claim that

$$CS_{G,k}(pt) = \text{Vect}^k[G] \quad (G \text{ finite}). \quad [3]$$

Indeed, it is a general feature of TQFTs that for any manifold M the value on $M \times S^1$ is the center of the value on M (where the meaning of "center" depends on the context and in particular on the dimension of the TQFT). As a special case, the value on S^1 should always be the center of the value on a point. See ref. 20, lemma 3.75 along with the discussion preceding that lemma for a proof in the case of 2D TQFTs, and see the proof of ref. 21, proposition 4.9 for a sketch in the case of 3D TQFTs.

A more direct argument for why $\text{Vect}^k[G]$ deserves to be called $CS_{G,k}(pt)$ can be found in ref. 21. This goes via a certain 3-categorical limit construction which is a sort of discrete path integral. The construction is described in ref. 21, section 8.1, "case $m = 2$ " (compare with ref. 21, example 3.14 for more details on the "case $m = 1$ ").

When G is a connected Lie group (always assumed compact), extended Chern–Simons theory is generally well understood down to dimension one. In particular, it is widely agreed that the value on the circle should be the category $\text{Rep}^k(LG)$ of positive energy representations of the loop group LG at level k (e.g., ref. 22, section 4). For example, it was shown by Freed and Teleman (23) that $\text{Rep}^k(LG)$ can be obtained directly from $\text{Loc}_G(S^1)$ by a procedure akin to geometric quantization. There is a natural gerbe (the analog of a prequantum line bundle) on $\text{Loc}_G(S^1)$ whose "sections" form a category. Freed and Teleman identified a certain subcategory (the polarized sections) and proved that it is equivalent to $\text{Rep}^k(LG)$.[§]

Let us present a different argument why

$$CS_{G,k}(S^1) = \text{Rep}^k(LG). \quad [4]$$

Recall ref. 7 that the quantum Hilbert space $CS_{G,k}(\Sigma)$ associated to a surface Σ is the geometric quantization of $\text{Loc}_G(\Sigma)$. To perform the geometric quantization, one needs a polarization of $\text{Loc}_G(\Sigma)$, which requires a choice of complex structure on Σ . So one gets a vector bundle over the moduli spaces of Riemann surfaces, which moreover comes with a natural projectively flat connection (7, 8). Starting instead from the right-hand side[¶] of Eq. 4, there is an associated complex modular functor (ref. 25, theorems 5.7.11 and 6.7.12), which is part of the Reshetikhin–Turaev package (ref. 25, chap. 4) (26). That modular functor is the same as the modular functor of Wess–Zumino–Witten conformal blocks studied in ref. 27 [the two agree on genus zero curves by definition of the fusion product on $\text{Rep}^k(Lg)$]. To summarize, both sides of Eq. 4 are related to constructions of vector bundles with connection over the moduli spaces of Riemann surfaces: One comes directly from the Chern–Simons Lagrangian, and the other one comes from the modular tensor category $\text{Rep}^k(LG)$.

[§] Twisted loop groups are expected to show up when G is disconnected (24).

[¶] Here, we consider the incarnation of $\text{Rep}^k(LG)$ as $\text{Rep}^k(Lg)$, and we restrict to the case when G is simply connected. See the next section for a discussion.

Those vector bundles with connection are known to agree (28) (also refs. 29–32): that is our evidence for Eq. 4.

Freed, Hopkins, Lurie, and Teleman (21) have suggested to extend the proposal 3 to the case when the gauge group is a torus T by letting $CS_{T,k}(pt) := \text{Sky}^k[T]$, the category of skyscraper sheaves on T (that is, T -graded vector spaces with grading supported on finitely many points of T and finite dimensional in each degree), with k -twisted associator. As in Eq. 2, the Drinfel'd center[#] of $\text{Sky}^k[T]$ is equivalent to $\text{Sky}_T^k[T]$. But this does not satisfy the desired property $Z(\text{Sky}^k[T]) = \text{Rep}^k(LT)$. It is, however, not too far off: one has $Z(\text{Sky}^k[T]) = \text{Sky}_T^k[T] \cong \text{Rep}^k(LT) \otimes \text{Sky}^k[\mathfrak{t}]$, and the extra factor $\text{Sky}^k[\mathfrak{t}]$ can be interpreted as an anomaly. Specifically, the authors of ref. 21 claim that taking instead a certain relative Drinfel'd center of $\text{Sky}^k[T]$ does recover $\text{Rep}^k(LT)$. We do not know of any functor between $\text{Sky}^k[T]$ and our proposed answer for $CS_{T,k}(pt)$.

For connected nonabelian groups, the category $\text{Sky}^k[G]$ still makes sense but its center $\text{Sky}_G^k[G]$ is very small (the conditions of G equivariance and finite support are almost incompatible) and does not seem very related to $\text{Rep}^k(LG)$.

2. Loop Group Representations

As explained above, it is generally accepted that, for connected gauge groups, the value of Chern–Simons theory on the circle is $\text{Rep}^k(LG)$, the category of positive energy representations of LG at level k . We recall the definition of that category. First, a level $k \in H^4(BG, \mathbb{Z})$ induces by transgression a central extension (ref. 33, theorem 3.7 and ref. 34, proposition 5.1.3)

$$1 \rightarrow U(1) \rightarrow \widetilde{LG} \rightarrow LG \rightarrow 1 \quad [5]$$

of the free loop group $LG = \text{Map}(S^1, G)$. The central extension depends on the level, but we suppress that dependence from the notation.

Definition 2 (35). *Let G be a connected Lie group (always assumed compact) and let $k \in H^4(BG, \mathbb{Z})$ be a positive level. A positive energy representation of LG at level k is a continuous^{||} unitary representation $\widetilde{LG} \rightarrow U(H)$ that extends^{††} to an action of the semidirect product $S^1 \ltimes \widetilde{LG}$ in such a way that the infinitesimal generator of the "energy circle" S^1 has positive spectrum and finite-dimensional eigenspaces. The category of positive energy representations of LG at level k is denoted $\text{Rep}^k(LG)$.*

Here, the semidirect product is taken with respect to a certain action of S^1 on \widetilde{LG} that lifts the "rotate the loops" action on LG .

The category $\text{Rep}^k(LG)$ is known to be a modular tensor category, when equipped with the fusion product^{‡‡}. People have historically preferred to work with a number of alternative categories (hopefully all equivalent). These are (from now on, let us restrict to G simple and simply connected) the categories of representations of the following:

- i) Quantum groups at root of unity (ref. 36 and references therein). We call the resulting braided tensor category $\text{Rep}^{\text{ss}}(U_q\mathfrak{g})$. Here, q is the primitive $m(k + h^\vee)$ th root of unity, where $m \in \{1, 2, 3\}$ is the squared ratio of the lengths of the long roots to short roots. The superscript ^{ss} stands for "semisimplification" and refers to the operation of restricting

[#] This holds for a certain variant of the Drinfel'd center called "continuous" Drinfel'd center.

^{||} for the topology on \widetilde{LG} induced by the C^∞ topology on LG and the strong operator topology on $U(H)$.

^{††} The extension is never unique; the choice of extension is not part of the data of a positive energy representation.

^{‡‡} Defined via the equivalence to $\text{Rep}^k(Lg)$; see item *ii* below.

to the subcategory of tilting modules and then modding out by then negligible morphisms. The fusion product of representations comes from the Hopf algebra structure on $U_q\mathfrak{g}$, and the universal R matrix provides the braiding.

- ii) Affine Lie algebras/vertex operator algebras (ref. 37 and references therein). We call the resulting braided tensor category $\text{Rep}^k(L\mathfrak{g})$. It is the category of integrable highest-weight modules of the affine Lie algebra, equivalently, of the corresponding vertex operator algebra (VOA). The fusion product of representations is given indirectly^{§§}, by defining for each triple of representations W_1, W_2, W_3 the space of intertwiners from $W_1 \boxtimes W_2$ to W_3 .^{¶¶} The associator is defined by means of the Kniznik–Zamolodchikov ordinary differential equation over the moduli space of four-punctured spheres, and the braiding is defined similarly.
- iii) Conformal nets (40, 41)^{##} (also ref. 42, section 4.c and references therein). We call the resulting braided tensor category $\text{Rep}_f(\mathcal{A}_{G,k})$.^{†††} Here, the conformal net $\mathcal{A}_{G,k}$ assigns von Neumann algebras $\mathcal{A}_{G,k}(I)$ to every interval^{‡‡‡} $I \subset S^1$. A representation is a Hilbert space with left actions of all those algebras. The fusion of representations is based on Connes' relative tensor product \boxtimes , also known as Connes fusion. It is given by $H \boxtimes_{\mathcal{A}_{G,k}(\downarrow)} K$, where the right action on H uses the isomorphism $\mathcal{A}_{G,k}(\downarrow)^{\text{op}} \cong \mathcal{A}_{G,k}(\curvearrowright)$ induced by reflection along the horizontal axis.

The approaches *i* and *ii* have been well studied. In particular, it is known that for every connected group G and level $k \geq 0$ they describe modular tensor categories and that they are additively equivalent to $\text{Rep}^k(LG)$. Moreover, the modular tensor categories obtained via *i* and *ii* are known to be equivalent by combining the works of Finkelberg (43, 44) and of Kazhdan and Lusztig [refs. 45–48 for the simply laced case and refs. 49 and 50 for the nonsimply laced case; the exceptional cases E_6, E_7, E_8 level 1, and E_8 level 2, where the results of Kazhdan and Lusztig do not apply, require an ad hoc analysis (51)].

The approach *iii* is less developed. So far, only the following results appear to be known: For the group $G = SU(n)$, the braided tensor category is modular (52, 53), and its fusion rules agree with those of the corresponding modular tensor categories constructed via *i* and *ii* (ref. 41, section 34)^{§§§} [the latter are well known (36, 55); also ref. 25, section 7.3 and refs. 30, 32, 56]. Even for $G = SU(n)$, the categories constructed via *i* (or *ii*) and *iii* are not known to be equivalent as braided tensor categories, unless $n = 2$. For other Lie groups, the braided tensor category *ii* is not known to be fusion (e.g., the tensor product multiplicities are not known to be finite) and also not known to be additively equivalent to the one constructed via *i* or *ii*. Despite all of the above, the following conjecture is widely believed to be true:

Conjecture 3. For every simple simply connected Lie group G and every level $k \geq 0$, the categories $\text{Rep}^{\text{ss}}(U_q\mathfrak{g})$ [or $\text{Rep}^k(L\mathfrak{g})$] and $\text{Rep}_f(\mathcal{A}_{G,k})$ are equivalent as balanced tensor categories.^{¶¶¶}

For $G = SU(2)$, the above conjecture can be proved as follows.^{###} As mentioned earlier, the fusion rules are known to agree by the work of Wassermann (41). By ref. 58, proposition 8.2.6^{††††}, balanced tensor categories with $SU(2)$ level- k fusion rules are determined by the entries of their T matrix^{††††}. The latter are the exponentials of the conformal weights, both when the modular tensor category comes from a VOA (ref. 59, theorem 4.1) and when it comes from a conformal net (60), and therefore agree.

In the case when G is connected but not simply connected, the VOA and conformal net approaches still make sense (61): The VOA/conformal net associated to a connected Lie group is a simple current extension of the tensor product of one associated to an even lattice and one associated to a simply connected Lie group. The above conjecture can thus be generalized:

Conjecture 4. For every connected Lie group G and every positive level $k \in H^4(BG, \mathbb{Z})$, the category of representations of the corresponding VOA is equivalent to $\text{Rep}_f(\mathcal{A}_{G,k})$ as a balanced tensor category.

When the gauge group is a torus, Conjecture 4 follows from known computations on the VOA side (ref. 62, chap. 12, and ref. 63) and the conformal net side (64, 65), because a modular tensor category all of whose objects are invertible is entirely determined by its fusion rules and by the $U(1)$ -valued quadratic form formed by the entries of its T matrix (ref. 66, proposition 2.14). Conjecture 4 actually follows from Conjecture 3 and the case of tori: The representation category of an extension can be described entirely in terms of the representation category of the original VOA (ref. 67, theorem 3.4) or conformal net (ref. 68, proposition 6.3). So the two conjectures are equivalent.

3. The Value on a Point $\text{Rep}^k(\Omega G)$

Let G be a connected Lie group. It is a general feature of 3D TQFTs that the value on S^1 is the Drinfel'd center of the value on a point.

Thus, following ref. 22, section 4, we take the point of view that a tensor category $T = T_{G,k}$ deserves to be called the value of Chern–Simons theory on a point if its Drinfel'd center $Z(T)$ is equivalent to $\text{Rep}^k(LG)$. The question “What does Chern–Simons theory assign to a point?” therefore reduces to the following:

Question. Find a tensor category $T_{G,k}$ whose Drinfel'd center $Z(T_{G,k})$ is equivalent to the category $\text{Rep}^k(LG)$ of positive energy representations of the loop group at level k .

Remark: By Lurie's classification of extended topological field theories (ref. 69, theorem 1.4.9), $CS_{G,k}(pt)$ must also be a fully dualizable object in a symmetric monoidal 3-category. We postpone the discussion of this 3-category until the next section.

We argue that the category of representations of the based loop group at level k offers an answer to the above question and hence deserves to be called the value of Chern–Simons theory on a point: $Z(\text{Rep}^k(\Omega G)) = \text{Rep}^k(LG)$. However, as explained in the previous section, there is more than one possible meaning for $\text{Rep}^k(LG)$. So we need to be a little bit more precise.

Let $H_0 \in \text{Rep}^k(LG)$ be the vacuum representation of LG at level k (the unit for the fusion product). Given an interval

^{§§}The fusion $W_1 \boxtimes W_2$ can also be described directly (38), as the graded dual of a judiciously chosen subspace of the algebraic dual of $W_1 \otimes W_2$. It is using that approach that the strongest results were obtained.

^{¶¶}See ref. 39, section 9.3 for the equivalence between the affine Lie algebra and VOA approaches.

^{##}Ref. 40 deals with all simply connected gauge groups and defines the braiding, but does not compute the fusion rules. Ref. 41 deals only with the case $G = SU(n)$, whose fusion rules it computes, but it does not discuss the braiding on the category of representations.

^{†††}The subscript f means that we take only representations which are finite direct sums of irreducible ones. We reserve the notation $\text{Rep}(\mathcal{A}_{G,k})$ for a category where infinite direct sums are allowed.

^{‡‡‡}An interval is a subset homeomorphic to $[0, 1]$.

^{§§§}The work of Wassermann does not exclude the possibility of “exotic” representations of $\mathcal{A}_{SU(n),k}$ that do not come from representation of the affine Lie algebra. Those can indeed be excluded by combining ref. 53, theorem 3.5.1 plus the equation on line 2 of p. 18 with ref. 52, theorem 33 plus corollary 39. See ref. 54, section 3.2 for an alternative proof when $n \geq 3$.

^{¶¶¶}A balanced tensor category is a tensor category with a braiding and a twist (57).

^{###}This argument, as well as the one below for tori, was communicated by Marcel Bischoff.

^{††††}That proposition applies only to the levels $k \geq 2$. See ref. 58, p. 387 for a discussion of the (easier) case $k = 1$.

^{‡‡‡‡}Also known as conformal spins or balancing phases.

$I \subset S^1$, we write $L_I G \subset LG$ for the subgroup of loops with support in that interval, we write $\widetilde{L_I G}$ for the corresponding central extension (induced by Eq. 5), and we write $\mathcal{A}_{G,k}(I)$ for the von Neumann algebra generated by $\widetilde{L_I G}$ inside the bounded operators on H_0 . The assignment $\mathcal{A}_{G,k} : I \mapsto \mathcal{A}_{G,k}(I)$ is a conformal net. From now on, we take $\text{Rep}_f(\mathcal{A}_{G,k})$ as our new working definition of $\text{Rep}^k(LG)$.^{§§§§} We will actually be working with the slightly larger category $\text{Rep}(\mathcal{A}_{G,k}) = \text{Rep}_f(\mathcal{A}_{G,k})^{\otimes_{\text{Vec}_f, d}}$ (ref. 54, section 3.2). The latter can also be described as the category of unitary representations of \widetilde{LG} with the property that, for every interval $I \subset S^1$, the action of $\widetilde{L_I G}$ extends to an action of $\mathcal{A}_{G,k}(I)$ (ref. 54, theorem 26). We call such representations locally normal, and we denote them $\text{Rep}_{l.n.}^k(LG)$.

Assuming *Conjectures 3 and 4*, we prove that, provided one replaces $\text{Rep}_f(\mathcal{A}_{G,k})$ by the slightly larger category $\text{Rep}_{l.n.}^k(LG) = \text{Rep}(\mathcal{A}_{G,k})$, the category of locally normal representations of the based loop group at level k (*Main Definition* below) satisfies

$$Z(\text{Rep}_{l.n.}^k(\Omega G)) = \text{Rep}_{l.n.}^k(LG). \quad [6]$$

We take this as evidence for our claim that $\text{Rep}_{l.n.}^k(LG)$ is the value of Chern–Simons theory on a point. (Of course, if one adopts the Platonic point of view that $CS_{G,k}(pt)$ is something that exists and whose value we are trying to compute, then our reasoning according to which, since Eq. 6 holds, it must be the case that

$$CS_{G,k}(pt) = \text{Rep}_{l.n.}^k(\Omega G) \quad (G \text{ connected}) \quad [7]$$

does not constitute a proof, as there might exist other tensor categories with that same Drinfel’d center.)

Let $\Omega G \subset LG$ be the subgroup consisting of loops $\gamma : S^1 \rightarrow G$ such that $\gamma(1) = e$ and such that all of the higher derivatives of γ vanish at that point, and let $\widetilde{\Omega G}$ be the corresponding central extension, inherited from ref. 5.

Main Definition. A unitary representation of $\widetilde{\Omega G}$ on a Hilbert space is a locally normal representation of ΩG at level k if for every interval $I \subset S^1$ such that the base point $1 \in S^1$ is not in the interior of I , the action of $\widetilde{L_I G}$ extends to an action of the von Neumann algebra $\mathcal{A}(I) := \mathcal{A}_{G,k}(I)$. We write $\text{Rep}_{l.n.}^k(\Omega G)$ for the category of locally normal representations of ΩG at level k . The monoidal structure on $\text{Rep}_{l.n.}^k(\Omega G)$ is given by $(H, K) \mapsto H \boxtimes_{\mathcal{A}(\searrow)} K$, as in the definition of fusion of representations of conformal nets. The actions of $\mathcal{A}(\searrow)$ on H and of $\mathcal{A}(\curvearrowright)$ on K induce actions of those same algebras on the fused Hilbert space, which in turn uniquely extend to an action of $\widetilde{\Omega G}$ (ref. 70, lemma 4.4, and ref. 54, theorem 31).

Note that since the algebras $\mathcal{A}_{G,k}(I)$ are type III factors (ref. 40, theorem 2.13; and ref. 60, proposition 1.2), the following is an equivalent^{¶¶¶¶} description of the category $\text{Rep}^k(\Omega G)$: A representation of the based loop group is locally normal if either it is zero or, for every interval $I \subset S^1$ such that the base point is not in the interior of I , its restriction to $\widetilde{L_I G}$ is equivalent to H_0 , the vacuum representation at level k .

We conjecture that our notion of locally normal representation of ΩG at level k admits the following alternative description: a Hilbert space with a strongly continuous action of $\widetilde{\Omega G}$ that extends to $\mathbb{R} \times \widetilde{\Omega G}$ in such a way that the spectrum of the infinitesimal generator of \mathbb{R} is positive. Here, the semidi-

rect product is taken with respect to loop reparameterizations by Möbius transformations that fix the base point $1 \in S^1$. Such a description would be attractive because it is directly parallel to the classical definition (35) of positive energy representation of LG at level k .

Remark 5: One possible objection to our proposal 7 is that the category $\text{Rep}_{l.n.}^k(\Omega G)$ is too big (it is neither rigid nor even abelian; in particular, it falls completely outside of the framework of ref. 71). This is, however, completely unavoidable. By the results of ref. 72, section 5.5 (also ref. 73), if there is a fusion category T such that $Z(T) \cong \text{Rep}^k(LG)$, then the latter must have central charge divisible by 8, a fact which holds only in very few cases.

4. Conformal Nets and Bicommutant Categories

The motivating examples come from loop groups, but our results apply to any conformal net with appropriate finiteness conditions.

For every conformal net \mathcal{A} , we consider the following tensor category $T_{\mathcal{A}}$. The objects of $T_{\mathcal{A}}$ are Hilbert spaces equipped with compatible actions of the algebras $\mathcal{A}(I)$ for every interval $I \subset S^1$ such that the base point of $1 \in S^1$ is not in the interior of I . Such representations are known in the conformal net literature as solitons (74–77). When \mathcal{A} is $\mathcal{A}_{G,k}$, the conformal net associated to a loop group, this recovers the category $\text{Rep}_{l.n.}^k(\Omega G)$ of locally normal representations of the based loop group (ref. 54, theorem 31).

Recall (52) that there is an invariant $\mu(\mathcal{A}) \in \mathbb{R}_+ \cup \{\infty\}$ of a conformal net, called the μ -index. By definition, it is the Jones–Kosaki index (78, 79) of the subfactor $\mathcal{A}(I_1) \otimes \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \otimes \mathcal{A}(I_4))'$, where I_1, I_2, I_3 , and I_4 are intervals that cover the circle

as follows:  and the prime denotes the commutant, taken on the vacuum sector of the conformal net. It has the property (52, 77) that^{####}

$$\mu(\mathcal{A}) < \infty \Leftrightarrow \text{Rep}(\mathcal{A}) \text{ is fusion} \Leftrightarrow \text{Rep}(\mathcal{A}) \text{ is modular} \quad [8]$$

(see ref. 42, section 3, and ref. 80, section 3C for an alternative proof).

Our main result states that, for conformal nets with finite μ -index, the Drinfel’d center of $T_{\mathcal{A}}$ is equivalent to the category of representations of the conformal net:

Main Theorem. *If \mathcal{A} has finite μ -index, then there is an equivalence of balanced tensor categories $Z(T_{\mathcal{A}}) \cong \text{Rep}(\mathcal{A})$.*

The proof of the *Main Theorem* is the content of our companion paper (70) [see ref. 70, remark 10 for a discussion of the balanced structure on $Z(T_{\mathcal{A}})$ and on $\text{Rep}(\mathcal{A})$].^{††††}

It is widely expected that the conformal nets associated to loop groups have finite μ -index (in which case the above *Main Theorem* could be applied to them), but this remains an open problem. At the moment, the best result in that direction is the one of Xu (53), based on the work of Wassermann (41), according to which the conformal nets associated to $SU(n)$ have finite μ -index.

Going back to the special case of Chern–Simons theory, and using the fact that $\text{Rep}(\mathcal{A}_{G,k}) = \text{Rep}_{l.n.}^k(LG)$, we have the following corollaries of our *Main Theorem*:

Corollary 6. *For $G = SU(n)$, the Drinfel’d center $Z(\text{Rep}_{l.n.}^k(\Omega G))$ of the category of locally normal representations of the based loop*

^{§§§§}Note that, even as an abelian category, $\text{Rep}_f(\mathcal{A}_{G,k})$ is not known, in general, to be equivalent to the category defined in *Definition 2*. See ref. 54, section 3.2 for some partial results.

^{¶¶¶¶}Provided one restricts to separable Hilbert spaces.

^{####}The last two conditions in Eq. 8 are phrased in a somewhat sloppy way. The correct way to formulate them is to say that $\text{Rep}(\mathcal{A}) \cong \mathcal{C}^{\otimes_{\text{Vec}_f, d}}$ Hilb for some fusion (modular) tensor category \mathcal{C} .

^{††††}The braiding on $\text{Rep}(\mathcal{A})$ defined in ref. 80, section 3B has not been compared with the one used in ref. 52. We can therefore not exclude the possibility that, when $\mu(\mathcal{A}) < \infty$, the category $\text{Rep}(\mathcal{A})$ has two distinct modular structures. In ref. 70, we prove our *Main Theorem* for the braided structure (52).

group at level k is equivalent as a balanced tensor category to $\text{Rep}_{1,n}^k(LG)$, the category of locally normal representations of the free loop group at level k .

Proof: $\mu(\mathcal{A}_{SU(n),k}) < \infty$ by ref. 53. \square

Corollary 7. For every connected Lie group G for which Conjecture 4 holds, there is an equivalence of balanced tensor categories between $Z(\text{Rep}_{1,n}^k(\Omega G))$ and the category of positive energy representations of the free loop group at level k (Definition 2), provided one removes the condition that the energy operator has finite-dimensional eigenspaces.

Proof: Conjecture 4 implies $\mu(\mathcal{A}_{G,k}) < \infty$ because of the equivalence 8 and of Huang's theorem (59, 67), according to which the representation categories of the relevant VOAs are modular. \square

By Lurie's classification of extended topological field theories (ref. 69, theorem 1.4.9), the map $Z \mapsto Z(pt)$ which sends a TQFT Z to its value on the point provides a bijection between extended n -dimensional TQFTs and fully dualizable objects in the given target n -category. From the Remark at the end of the previous section, it might look like T_A is not fully dualizable, which would imply that there is no TQFT whose value on a point is T_A . We believe that it is possible to restore the full dualizability of T_A by viewing it as an object not of the 3-category of tensor categories, but of a yet to be constructed 3-category of bicommutant categories.

Let R be a hyperfinite III_1 factor, and let $\text{Bim}(R)$ denote its bimodule category, equipped with Connes' relative tensor product. The latter comes with antilinear involutions at the level of objects (the contragredient of a bimodule) and at the level of morphisms (the adjoint of a linear map).

Definition 8 (ref. 81, section 3). A bicommutant category is a tensor category T equipped with two involutions, as above, and a tensor functor $T \rightarrow \text{Bim}(R)$, compatible with the two involutions, so that the natural map $T \rightarrow T''$ of the category to its bicommutant is an equivalence.

Here, we write T' for the commutant of the tensor category T . It is the category whose objects are pairs (Y, e) with $Y \in \text{Bim}(R)$ and $e = \{e_X : X \boxtimes Y \rightarrow Y \boxtimes X\}_{X \in T}$ a half-braiding with all of the elements of T [which we abusively identify with their image in $\text{Bim}(R)$]. The bicommutant $T'' := (T')'$ is equipped with a natural "inclusion" functor $T \rightarrow T''$.

Theorem 9. If \mathcal{A} is a conformal net with finite μ -index, then T_A is a bicommutant category.

The proof of this theorem can be found in our companion paper (70).

Remark: In our earlier paper (82), we suggested using the 3-category of conformal nets, constructed in refs. 42, 83, and 84, as a target category for extended 3D TQFTs, and to have $\mathcal{A}_{G,k}$ be the value of Chern–Simons theory on a point. We conjecture that the construction $\mathcal{A} \mapsto T_A$ extends to a fully faithful but maybe not essentially surjective 3-functor $T : \{\text{conformal nets}\} \rightarrow \{\text{bicommutant categories}\}$. Such a functor would make our current proposal 7 for the value of Chern–Simons theory on a point "backward compatible" with respect to our earlier proposal (82).

5. CS(pt) for Disconnected Groups

In sections 2–4, the gauge group was always connected. Let now G be an arbitrary compact Lie group, and let $k \in H^4(BG, \mathbb{Z})$ be a positive level. We propose a general answer to the question of what Chern–Simons theory assigns to a point that simultaneously generalizes the previous answers (Eqs. 3 and 7) in the cases of finite gauge group and connected gauge group, respectively.

Let $\text{Bun}_G(S^1; *)$ be the moduli space of G bundles over S^1 trivialized to infinite order at the base point $* = 1 \in S^1$. This stack has finitely many points (the isomorphism classes of G bundles over S^1) classified by their monodromy in $\pi_0(G)$, and each point has an infinite-dimensional isotropy group (the automorphism group of the G bundle) which is isomorphic to ΩG . For ease of notation, we fix for every $[g] \in \pi_0(G)$ a principal bundle $P_{[g]} \in \text{Bun}_G(S^1; *)$ with monodromy $[g]$. A vector bundle V over $\text{Bun}_G(S^1; *)$ is then equivalent to a collection of representations $V_{[g]}$ of the groups $\text{Aut}(P_{[g]})$. The level $k \in H^4(BG, \mathbb{Z})$ induces a gerbe over $\text{Bun}_G(S^1; *)$, which can equivalently be thought of as a collection of central extensions of the above isotropy groups. We say that a k -twisted vector bundle over $\text{Bun}_G(S^1; *)$ is locally normal if its restriction to each $P_{[g]} \in \text{Bun}_G(S^1; *)$ yields a locally normal representation of $\text{Aut}(P_{[g]}) \cong \Omega G$, in the sense we introduced. We propose

$$CS_{G,k}(pt) = \left\{ \begin{array}{l} k\text{-twisted locally normal vector} \\ \text{bundles over } \text{Bun}_G(S^1; *) \end{array} \right\} \quad [9]$$

and conjecture that it is a bicommutant category. This is equivalent (noncanonically) to the category of $\pi_0(G)$ -tuples of locally normal representations of ΩG at level k .

The tensor structure on Eq. 9 is obtained by thinking about G bundles over \bigcirc^* , trivialized at the base point $*$. Given two locally normal vector bundles V and W over $\text{Bun}_G(S^1; *)$ and given a G -bundle P over S^1 , the value of $V \boxtimes W$ at P is computed as follows. Consider the finite set [indexed by $\pi_0(G)$] of isomorphism classes of extensions of P over the above theta graph, and let Q_i be representatives of the isomorphism classes. Let Q_i^+ and Q_i^- be the restrictions of Q_i to the upper and lower halves of the theta graph. We identify those two halves \bigcirc^* and \bigcirc^* with S^1 , to be able to view Q_i^+ and Q_i^- as elements of $\text{Bun}_G(S^1; *)$.

Let $G_0 \subset G$ be the connected component of the identity. Using a trivialization of Q_i over the middle edge of the theta graph, we get left and right actions of the algebra $\mathcal{A}_{G_0,k}(\rightarrow)$ on the spaces $W(Q_i^+)$ and $V(Q_i^-)$ [the right action uses the identification $\mathcal{A}_{G_0,k}(\rightarrow)^{\text{op}} \cong \mathcal{A}_{G_0,k}(\leftarrow)$]. The monoidal structure on the category of locally normal vector bundles over $\text{Bun}_G(S^1; *)$ is then given by

$$(V \boxtimes W)(P) := \bigoplus_i V(Q_i^-) \boxtimes_{\mathcal{A}_{G_0,k}(\rightarrow)} W(Q_i^+).$$

The proposal 9 for $CS(pt)$ unifies and generalizes the previous ones: Eqs. 3 and 7. It fits into the following commutative diagram,

$$\begin{array}{ccccc} \left\{ \begin{array}{l} \text{finite group } G \\ + \text{ level} \\ k \in H^4(BG, \mathbb{Z}) \end{array} \right\} & \leftrightarrow & \left\{ \begin{array}{l} \text{compact Lie group} \\ G + \text{ level} \\ k \in H_+^4(BG, \mathbb{Z}) \end{array} \right\} & \leftarrow & \left\{ \begin{array}{l} \text{compact connected} \\ \text{Lie group } G + \text{ level} \\ k \in H_+^4(BG, \mathbb{Z}) \end{array} \right\} \\ 1 \downarrow & & \downarrow 5 & & \downarrow 3 \\ \left\{ \begin{array}{l} \text{Unitary fusion} \\ \text{categories} \end{array} \right\} & \xrightarrow{2} & \left\{ \begin{array}{l} \text{Bicommutant} \\ \text{categories} \end{array} \right\} & \xleftarrow{4} & \left\{ \begin{array}{l} \text{Conformal} \\ \text{nets} \end{array} \right\} \end{array}$$

where $H_+^4(BG, \mathbb{Z})$ denotes the set of positive levels (Definition 1). The arrow labeled 1 is the well-known isomorphism $H^4(BG, \mathbb{Z}) \cong H^3(G, U(1))$. The arrow labeled 2 is constructed in ref. 81. The arrow labeled 3 is constructed in ref. 61, section 8. The arrow labeled 4 is constructed in ref. 70 under the assumption that the conformal net has finite μ -index. The arrow labeled 5 is our proposal Eq. 9; the fact that it produces a bicommutant category is presently just a conjecture.

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