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Shrinking random β-transformation

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Abstract

For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

In this paper we define the shrinking random β-transformation K and investigate natural invariant measures for K, and the induced transformation of K on a special subset of the domain. We prove that both transformations have a unique measure of maximal entropy. However, the measure induced from the intrinsically ergodic measure for K is not the intrinsically ergodic measure for the induced system.

Keywords: Random β-transformation; Unique measure of maximal entropy; Invariant measure

1. Introduction

Let $\beta \in (1, 2)$ and $x \in \mathcal{A}_\beta = [0, (\beta - 1)^{-1}]$, we call a sequence $(a_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ a β-expansion of x if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Renyi [11] introduced the greedy map, and showed that the greedy expansion $(a_i)_{i=1}^{\infty}$ of $x \in [0, 1)$ can be generated by defining $T(x) = \beta x \mod 1$ and letting $a_i = k$ whenever

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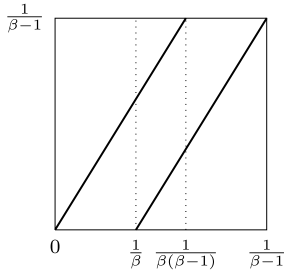

Fig. 1. The dynamical system for $\{T_0(x) = \beta x,\ T_1(x) = \beta x - 1\}$.

$T^{i-1}(x) \in [k\beta^{-1}, (k+1)\beta^{-1})$. Since then, many papers were dedicated to the dynamical properties of this map, see for example [12,5,8,10,4,9] and references therein. However, Renyi's greedy map is not the unique dynamical approach to generate β-expansions. In [6] (see also [3,4]) a new transformation was introduced, the random β-transformation, that generates all possible β-expansions, see Fig. 1. This transformation makes random choices between the maps $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$ whenever the orbit falls into $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, which we refer to as the switch region.

Although, all possible β-expansions can be generated via the random β-transformation, nevertheless, for some practical problems one would want to make choices only on a subset of the switch region $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, for instance, in A/D (analog-to-digit) conversion [7]. This motivates our study of the shrinking random β-transformation described below.

Let $1 < \beta < 2^{-1}(1+\sqrt{5})$, $\Omega = \{0, 1\}^{\mathbb{N}}$, and $E = [0, (\beta-1)^{-1}]$. Set $a = (\beta^2-1)^{-1}$, $b = \beta(\beta^2-1)^{-1}$, i.e. $T_0(a) = b$, $T_1(b) = a$. The shrinking random β-transformation K is defined in the following way (see Fig. 2).

Definition 1.1. $K : \Omega \times E \to \Omega \times E$ is defined by

$$K(\omega, x) = \begin{cases} (\omega, \beta x) & x \in [0, a) \\ (\sigma(\omega), \beta x - \omega_1) & x \in [a, b] \\ (\omega, \beta x - 1) & x \in (b, (\beta-1)^{-1}]. \end{cases}$$

Given $(\omega, x) \in \Omega \times [a, b]$, the first return time is defined by

$$\tau(\omega, x) = \min\{n \geq 1 : K^i(\omega, x) \notin \Omega \times [a, b],\ 1 \leq i \leq n-1,\ K^n(\omega, x) \in \Omega \times [a, b]\}.$$

Define $K_{\Omega\times[a,b]}(\omega, x) = K^{\tau(\omega,x)}(\omega, x)$, and denote it for simplicity by I.

We now consider a special family of algebraic bases defined as follows. For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

The following lemma is clear.

Lemma 1.2. *For any $n \geq 3$, let $\beta_n > 1$ be the largest positive real root of the equation*

$$x^n = x^{n-2} + x^{n-3} + \cdots + x + 1. \tag{1}$$

Then (β_n) is an increasing sequence which converges to $2^{-1}(1+\sqrt{5})$.

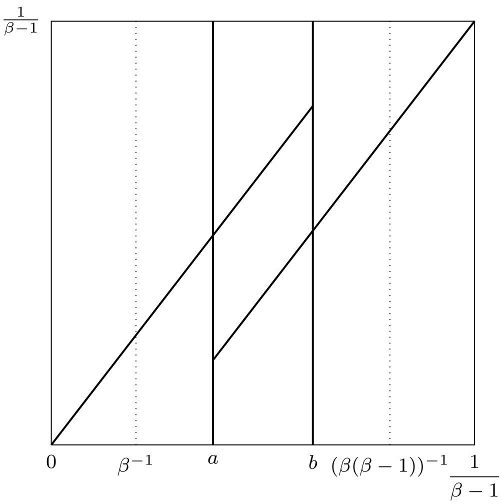

Fig. 2. Shrinking random β-transformation.

Throughout the paper we will assume $\beta = \beta_n$ for some $n \geq 3$. In Section 2, we will show that I can be identified with a full left shift. As a result it will be easy to find I-invariant measures, and to show that I is intrinsically ergodic (i.e. has a unique measure of maximal entropy). In the last section, we identify the dynamics of K with a topological Markov chain, and then use Parry's recipe to prove the following result.

Theorem 1.3. *For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation*

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

Then the shrinking random β-transformation K and the induced transformation $I = K_{\Omega\times[a,b]}$ have intrinsically ergodic measures. Moreover, the measure induced from the intrinsically ergodic measure of K on $\Omega \times [a, b]$, does not yield the unique measure of maximal entropy for I.

2. Invariant measures for $I = K_{\Omega\times[a,b]}$

As above, β satisfies $\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1$, $n \geq 3$. It is easy to check that for $(\omega, x) \in \Omega \times [a, b]$, the first return time $\tau(\omega, x) \in \{2, 3, \ldots, n\}$. We give a simple proof for this statement. Since $a = T_1(b)$, $b = T_0(a)$ and the fact that β satisfies the equation $\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1$, it follows that the largest return time of a is n. Similarly, we can show that those points, which are very close to a or b, have the return time 2. What we want to emphasize here is that we delete these points with return time 1. As there are only countable such points, we can delete these points without affecting our result.

Consider the space $\Omega \times \{2, 3, \ldots, n\}^{\mathbb{N}}$ equipped with the product σ-algebra, and the left shift σ'. Define the map $\phi : (\Omega \times [a, b]) \setminus (\cup_{i=0}^{\infty} K^{-i}(\Omega \times \{a\} \cup \Omega \times \{b\})) \to \Omega \times \{2, 3, \ldots, n\}^{\mathbb{N}}$ by

$$\phi(\omega, x) = (\omega, (n_1, n_2, \ldots, n_k, \ldots)),$$

where n_i is the ith return time of (ω, x) to $(\Omega \times [a, b]) \setminus (\cup_{i=0}^{\infty} K^{-i}(\Omega \times \{a\} \cup \Omega \times \{b\}))$, i.e. $n_i = \tau(I^{i-1}(\omega, x))$.

Given $(a_n) \in \{0, 1\}^{\mathbb{N}}$, we denote the value of the sequence (a_n) by $(a_n)_\beta = \sum_{n=1}^{\infty} a_n \beta^{-n}$.

Lemma 2.1. *The sequences*

$$(01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots$$

and

$$(10)^{j_1}(0\underbrace{1\cdots1}_{n-1})^{j_2}(10)^{j_3}(0\underbrace{1\cdots1}_{n-1})^{j_4}\cdots$$

are the possible β-expansions of a and b generated by the map K respectively, where $j_1 \geq 1$ and $j_k \geq 0$ for $k \geq 2$.

Proof. The proof follows from the fact that $a = T_1(b) = T_0^{n-1}T_1(a)$ and $b = T_0(a) = T_1^{n-1}T_0(b)$. $\square$

Lemma 2.2. *ϕ is a measurable bijection.*

Proof. Firstly we prove ϕ is one-to-one. Let $\phi(\omega, x_1) = \phi(\tau, x_2)$. Then we have $\omega = \tau$, and the first return time functions coincide. We denote the values of this function by $(n_i)_{i=1}^{\infty}$. Since $x_1 \in [a, b]$ and $\omega = (\omega_1\omega_2\omega_3\cdots)$, by the definition of K, we choose ω_1 for the first digit of x_1. Then the orbit of x_1 jumps out of $[a, b]$, and in the region $[0, a) \cup (b, (\beta - 1)^{-1}]$, we can only choose digits $\overline{\omega_1}$ for $n_1 - 1$ times. After n_1 times later, the orbit of x_1 goes back to $[a, b]$. Therefore we can implement similar algorithm again. Using this idea, one can easily check that

$$x_1 = (\omega_1\overline{\omega_1}^{n_1-1}\omega_2\overline{\omega_2}^{n_2-1}\cdots)_\beta,$$

and

$$x_2 = (\tau_1\overline{\tau_1}^{n_1-1}\tau_2\overline{\tau_2}^{n_2-1}\cdots)_\beta$$

where $\overline{\omega_i} = 1 - \omega_i$, and $(\omega_i)^k$ means k consecutive ω_i.

As such we have $x_1 = x_2$. Now we prove ϕ is also a surjection. Given any $(\omega, (n_1, n_2, \ldots, n_k, \ldots))$, it is sufficient to show that

$$x = (\omega_1\overline{\omega_1}^{n_1-1}\omega_2\overline{\omega_2}^{n_2-1}\cdots)_\beta \in [a, b].$$

We decompose the sequence $(\omega_1\overline{\omega_1}^{n_1-1}\omega_2\overline{\omega_2}^{n_2-1}\cdots)$ into the blocks $\omega_i\overline{\omega_i}^{n_i-1}$. Note that for any $i \geq 1$, the value of the block can be classified in the following way:
If $\omega_i = 0$ and $n_i = 2$, then

$$(\omega_i\overline{\omega_i}^{n_i-1})_\beta = (01)_\beta.$$

If $\omega_i = 0$ and $n_i \geq 3$, then

$$(\omega_i\overline{\omega_i}^{n_i-1})_\beta \geq (1\underbrace{0\cdots0}_{n-1})_\beta.$$

If $\omega_i = 1$, then

$$(\omega_i\overline{\omega_i}^{n_i-1})_\beta \geq (1\underbrace{0\cdots0}_{n-1})_\beta.$$

Here we use the fact $1 < \beta < \frac{\sqrt{5}+1}{2}$, see Lemma 1.2. Hence, we have

$$x = (\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)_\beta \geq ((01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots)_\beta = a$$

or

$$x = (\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)_\beta \geq ((1\underbrace{0\cdots0}_{n-1})^{j_1}(01)^{j_2}(1\underbrace{0\cdots0}_{n-1})^{j_3}(01)^{j_4}\cdots)_\beta = a.$$

Similarly, we prove by symmetry that

$$\begin{aligned}\bar{x} &= (\beta-1)^{-1} - x = (\overline{\omega_1}\omega_1^{n_1-1}\overline{\omega_2}\omega_2^{n_2-1}\cdots)_\beta \\ &\geq ((01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots)_\beta = a\end{aligned}$$

or

$$\begin{aligned}\bar{x} &= (\beta-1)^{-1} - x = (\overline{\omega_1}\omega_1^{n_1-1}\overline{\omega_2}\omega_2^{n_2-1}\cdots)_\beta \\ &\geq ((1\underbrace{0\cdots0}_{n-1})^{j_1}(01)^{j_2}(1\underbrace{0\cdots0}_{n-1})^{j_3}(01)^{j_4}\cdots)_\beta = a.\end{aligned}$$

Since $b = (\beta-1)^{-1} - a$, we have $a \leq x \leq b$ and ϕ is surjective. It remains to show that ϕ is measurable. For any cylinders $C = \{\omega \in \Omega : \omega_1 = i_1, \ldots, \omega_m = i_m\}$ and $D = \{y \in \{2, 3, \ldots, n\}^{\mathbb{N}} : y_1 = n_1, \ldots, y_m = n_m\}$, we have

$$\begin{aligned}&\phi^{-1}(C \times D) \\ &\quad = \{(\omega, x) \in \Omega \times [a, b] : \tau(\omega, x) = n_1, \tau(I(\omega, x)) = n_2 \cdots, \tau(I^{m-1}(\omega, x)) = n_m\}\end{aligned}$$

which is a measurable set, since τ and I are measurable. □

Lemma 2.3. *Let μ be any $\sigma \times \sigma'$-invariant measure on $\Omega \times \{2, 3, \ldots, n\}^{\mathbb{N}}$. Then, the measure $\mu \circ \phi$ is I-invariant, and the dynamical systems $(\Omega \times [a, b], I, \mu \circ \phi)$, and $(\Omega \times \{2, 3, \ldots, n\}^{\mathbb{N}}, \sigma \times \sigma', \mu)$ are isomorphic.*

Proof. It is easy to check that $(\sigma \times \sigma') \circ \phi = \phi \circ I$. Since ϕ is a measurable bijection, $\mu \circ \phi$ is I-invariant and the result follows. □

Corollary 2.4. *Let m_p be the $(p, 1-p)$ product measure on Ω, and μ_π the product measure on $\{2, 3, \ldots, n\}^{\mathbb{N}}$ induced by the probability vector $\pi = (\pi_2, \ldots, \pi_n)$, i.e. $\mu_\pi(\{(a_n) \in \{2, 3, \ldots, n\}^{\mathbb{N}} : a_1 = i_i, \ldots, a_m = i_m\}) = \pi_{i_1} \ldots \pi_{i_m}$. Then, $(m_p \times \mu_\pi) \circ \phi$ is an I-invariant ergodic measure on $\Omega \times [a, b]$.*

Proof. Note that μ_π is σ'-invariant, and since σ is weakly mixing, we have that $(m_p \times \mu_\pi)$ is $\sigma \times \sigma'$-invariant ergodic measure. By Lemma 2.3, it follows that $(m_p \times \mu_\pi) \circ \phi$ is an I-invariant ergodic measure on $\Omega \times [a, b]$. □

Note that for different probability vectors $\pi^{(1)}$ and $\pi^{(2)}$, the corresponding measures $(m_p \times \mu_{\pi^{(1)}}) \circ \phi$ and $(m_p \times \mu_{\pi^{(2)}}) \circ \phi$ are singular with respect to each other. It is natural to ask the following question: when do we have $(m_p \times \mu_\pi) \circ \phi = m_p \times \lambda$, where λ is the normalized Lebesgue measure on $[a, b]$?

To answer this question, we need to find an explicit expression for the induced transformation $K_{\Omega \times [a,b]}$ in terms of the first return time. We begin by partitioning $[a, b]$ using the greedy orbits, i.e. when $x \in [a, b]$ we implement T_1 on x. Define the greedy map $L_1(x) = \beta^{n-i+1}x - \beta^{n-i}$, where $x \in [c_i, c_{i+1})$, $c_1 = a$, $c_n = b$, $c_i = \beta^{i-1}a - \beta^{i-2} + \beta^{-1}$, $2 \le i \le n-1$. Similarly, we can define the lazy map $L_0(x)$ (we choose T_0 if the orbits fall into $[a, b]$) by

$$L_0(x) = \beta^{n-i+1}x - \beta^{n-i-1} - \beta^{n-i-2} - \beta^{n-i-3} - \cdots - \beta - 1,$$

if $x \in (d_i, d_{i+1}]$, where $d_1 = a$, $d_n = b$, $d_i = \beta^{n-i}b - \beta^{n-i-2} - \cdots \beta - 1 - \beta^{-1}$, $2 \le i \le n-1$. It is easy to see that L_1 and L_0 are Generalized Lüroth Series (GLS) maps [2]. Hence, the induced transformation $I = K_{\Omega\times[a,b]}$ is given by $I(\omega, x) = (\sigma(\omega), L_{\omega_1}(x))$. We now answer the question posed above.

Theorem 2.5. *Let* $P = \left(\frac{p_1}{b-a}, \frac{p_2}{b-a}, \dots, \frac{p_{n-1}}{b-a}\right)$, *where* $p_i = \beta^i a - \beta^{i-1}a - (\beta^{i-1} - \beta^{i-2})$, $1 \le i \le n-2$, $p_{n-1} = b - \beta^{n-2}a - \beta^{n-3}a + \beta^{-1}$. *Then,* $m_p \times \lambda$ *is an* I*-invariant ergodic measure and* $(m_p \times P) \circ \phi = m_p \times \lambda$.

Proof. By [2, Theorems 1], the GLS maps L_0 and L_1 preserve the normalized Lebesgue measure. Since the induced transformation I is a skew product, it follows that $m_p \times \lambda$ is an I-invariant measure. To show $(m_p \times P) \circ \phi = m_p \times \lambda$, it is enough to show that $(m_p \times P) = (m_p \times \lambda) \circ \phi^{-1}$. Let $C = \{\omega \in \Omega : \omega_1 = i_1, \dots, \omega_m = i_m\}$ and $D = \{y \in \{2, 3, \dots, n\}^{\mathbb{N}} : y_1 = n_1, \dots, y_m = n_m\}$. Then,

$$\begin{aligned}\phi^{-1}(C \times D) &= \{(\omega, x) \in \Omega \times [a, b] : \tau(\omega, x) = n_1, \dots, \tau(I^{m-1}(\omega, x)) = n_m\}\\ &= C \times J,\end{aligned}$$

where

$$J = D_{n_1} \cap L_{i_1}^{-1}(D_{n_2}) \cap (L_{i_2} \circ L_{i_1})^{-1}(D_{n_3}) \cap \cdots \cap (L_{i_{m-1}} \circ \cdots \circ L_{i_1})^{-1}(D_{n_m})$$

with $D_{n_j} = [c_{n-n_j+1}, c_{n-n_j+2})$ if $i_j = 1$ and $D_{n_j} = (d_{n_j-1}, d_{n_j}]$ if $i_j = 0$. Since the maps L_0 and L_1 are piecewise linear and surjective, an easy calculation shows that J is an interval of length $\frac{p_{n_1}\cdots p_{n_m}}{(b-a)^m} = P(D)$, see [2, Theorem 1]. Thus,

$$(m_p \times \lambda)\Big(\phi^{-1}(C \times D)\Big) = m_p(C)P(D) = (m_p \times P)(C \times D). \quad \square$$

Now, we turn our attention in finding the intrinsically ergodic measure for $I = K_{\Omega\times[a,b]}$, i.e. the unique measure of maximal entropy. For this, we will identify the dynamics of I with a full left shift. Consider the space

$$\Lambda = \{(0, 2), (0, 3), \dots, (0, n), (1, 2), (1, 3), \dots, (1, n)\}^{\mathbb{N}}.$$

Here the first coordinate denotes the outcome of the coin toss (heads $= 0$ or tails $= 1$), and the second denotes the return time to $\Omega \times [a, b]$. Let S be the left shift on Λ, i.e. $S((i, j)_n) = (i', j')_n$, where $((i', j')_n) = (i, j)_{n+1}$. We define the following map

$$\rho : \Omega \times \{2, 3, \dots, n\}^{\mathbb{N}} \to \Lambda$$

by

$$\rho((\omega, (n_1, n_2, \dots))) = ((\omega_1, n_1), (\omega_2, n_2), (\omega_3, n_3), \dots).$$

Evidently, ρ is a bijection and $\rho \circ (\sigma \times \sigma') = S \circ \rho$. This leads to the following theorem.

Theorem 2.6. *The induced transformation I is intrinsically ergodic with maximal entropy* $\log(2n-2)$.

Proof. Let m be the product $\left(\frac{1}{2n-2}, \frac{1}{2n-2}, \dots, \frac{1}{2n-2}\right)$ measure on Λ. Note that m is shift invariant, and is intrinsically ergodic. Since ρ is a commuting bijection, the measure $m \circ \rho$ is $\sigma \times \sigma'$-invariant and is intrinsically ergodic. By Lemma 2.3, $m \circ \rho \circ \phi$ is the unique measure of maximal entropy for I. Since entropy is preserved under an isomorphism, the maximal entropy is $\log(2n-2)$. □

3. Invariant measures for K

It is a classical fact that if ν is an I-invariant probability measure on $\Omega \times [a,b]$, then the probability measure μ defined on $\Omega \times [T_1(a), T_0(b)]$ by

$$\mu(E) = \frac{1}{\int \tau \, d\nu} \sum_{n \geq 0} \nu(\{(\omega, x) \in \Omega \times [a,b] : \tau(\omega, x) > n\} \cap K^{-n}(E))$$

is a K-invariant probability measure. So for any measure ν as defined in Lemma 2.3 of the previous section.

Now we consider the intrinsically ergodic measure of K. It can be found via Parry's work, see [13]. For the sake of convenience, we give a brief introduction to Parry's result. Given any one-dimensional subshift of finite type with irreducibility condition, the Parry measure given by a probability vector $(p_0, p_1, \dots, p_{k-1})$ and stochastic matrix (p_{ij}) is constructed as follows. If λ is the largest positive eigenvalue of A ($A = (a_{ij})$ is the adjacency matrix of the subshift of finite type) and $(u_0, u_1, \dots, u_{k-1})$ is a strictly positive left eigenvector and $(v_0, v_1, \dots, v_{k-1})$ is a strictly positive right eigenvector with $\sum_{i=0}^{k-1} u_i v_i = 1$, then $p_i = u_i v_i$ and $p_{ij} = \frac{a_{ij} v_j}{\lambda v_i}$. We state the following classical result.

Theorem 3.1. *Given any one-dimensional subshift of finite type with irreducibility condition, then the Parry measure is the intrinsically ergodic measure for this subshift of finite type. The maximal entropy is* $\log \lambda$.

Recall the definition of β, given $n \geq 3$, let β be the largest positive root of the following equation:

$$\beta^n = \sum_{i=0}^{n-2} \beta^i.$$

We can partition $[T_1(a), T_0(b)]$ in terms of the image of $[a,b]$ under K. More precisely, let

$$\{[T_0^k T_1(a), T_0^{k+1} T_1(a)], 0 \leq k \leq n-2, [a,b], [T_1^i T_0(b), T_1^{i+1} T_0(b)], 1 \leq i \leq n-1\}$$

be a Markov partition of $[T_1(a), T_0(b)]$, where $T_j^0 = id$, $j = 0, 1$. It is easy to see that the image of each set of the Markov partition is the union of some sets of this partition. For instance, when $n = 3$, let

$$A = [T_1(a), T_0 T_1(a)], \qquad B = [T_0 T_1(a), a], \qquad C = [a,b], \qquad D = [b, T_1 T_0(b)],$$
$$E = [T_1 T_0(b), T_0(b)].$$

Evidently,

$$T_0(A) = B, \qquad T_0(B) = C, \qquad T_0(C) = D \cup E, \qquad T_1(C) = A \cup B,$$
$$T_1(D) = C, \qquad T_1(E) = D.$$

Hence the associated adjacency matrix for this Markov partition is

$$S_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix can generate a subshift of finite type, denoted by Σ_3, i.e.

$$\Sigma_3 = \{(i_n) \in \{1, 2, 3, 4, 5\}^{\mathbb{N}} : S_{3_{i_n, i_{n+1}}} = 1\}.$$

Similarly, for general n, we can find the adjacency matrix S_n and its corresponding subshift of finite type Σ_n. It is easy to see that the matrix S_n is irreducible. Hence, we can make use of Parry's idea to find the unique measure of maximal entropy.

Denote $a_n = \det(\lambda E - S_n)$. The following lemma is doing some trivial calculation in linear algebra.

Lemma 3.2. $a_{n+1} = \lambda^2 a_n - 2\lambda^n$ *for any* $n \geq 3$, *and* $a_3 = \lambda^2(\lambda^3 - 2\lambda - 2)$. *By induction, we have*

$$a_n = \lambda^{n-1}(\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2})).$$

The right eigenvector of S_n *is*

$$\vec{v} = (v_0, v_1, \ldots, v_{2n-2}) = (c, \lambda c, \lambda^2 c, \ldots, \lambda^{n-2} c, \lambda^{n-1} c, \lambda^{n-2} c, \lambda^{n-3} c, \ldots, \lambda c, c)$$

where $c > 0$.

The left eigenvector of S_n, *denoted by* $\vec{u} = (u_0, u_1, u_2, \ldots, u_{2n-2})$, *is*

$$\left(d, \frac{1+\lambda}{\lambda} d, \frac{1+\lambda+\lambda^2}{\lambda^2} d, \ldots, \frac{1+\lambda+\cdots+\lambda^{n-2}}{\lambda^{n-2}} d, \lambda d, \right.$$
$$\left. \frac{1+\lambda+\cdots+\lambda^{n-2}}{\lambda^{n-2}} d, \ldots, \frac{1+\lambda}{\lambda} d, d \right)$$

where $d > 0$. *By the construction of the Parry measure, we assume* $\vec{u} \cdot \vec{v} = 1$, *which implies that* c *and* d *have the following relation*

$$\frac{1}{cd} = \frac{2}{\lambda - 1}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) + \lambda^n.$$

Now we can find the Parry measure as follows, given any $(a_1 a_2 \cdots a_k) \in \{1, \ldots 2n-2\}^k$, the Parry measure defined on the cylinder $[a_1 a_2 \cdots a_k]$ is

$$\mu([a_1 a_2 \cdots a_k]) = p_{a_1} p_{a_1 a_2} \cdots p_{a_{k-1} a_k}.$$

Let ν be the induced measure of μ on $\Omega \times [a, b]$, that is

$$\nu(E) = \frac{\mu(E)}{\mu(\Omega \times [a, b])},$$

for E a measurable subset of $\Omega \cap [a, b]$. By Abramov formula,

$$h(K, \mu) = h(I, \nu) \times \mu(\Omega \times [a, b]),$$

where h denotes the entropy of the underlying system, and $I = K_{\Omega\times[a,b]}$. By the construction of the Parry measure,

$$h(I, \nu) = \frac{\log\lambda}{u_n v_n} = \frac{\log\lambda}{cd\lambda^n}.$$

To prove the remaining part of Theorem 1.3, we need to compare $h(I, \nu) = \frac{\log\lambda}{cd\lambda^n}$ with $\log(2n-2)$, the maximal entropy of I.

Lemma 3.3. *For any $n \geq 3$,*

$$\log(2n-2) > \frac{\log\lambda}{cd\lambda^n}.$$

Proof. For $n = 3$, we can show

$$\log 6 > \frac{\log\lambda}{cd\lambda^3} = \frac{\log\lambda}{\lambda^3}\left(\frac{2}{\lambda-1}\left(\lambda^2 - 3 + \frac{\lambda^2}{2}\right)\right).$$

This is trivial as we can find the exact value of λ in terms of some characteristic polynomial. Similarly, for $n = 4$ the lemma is still correct. Hence, it suffices to prove this lemma when $n \geq 5$. Note that λ is the largest positive root of the following equation

$$\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2}) = 0.$$

Since S_n is irreducible, it follows by Perron–Frobenius Theorem that such a λ exists, and furthermore $1 < \lambda < 2$. By the construction of the Parry measure, it follows that

$$\frac{1}{cd} = \frac{2}{\lambda-1}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) + \lambda^n.$$

Hence, in order to prove

$$\frac{\log\lambda}{cd\lambda^n} = \left[\frac{2}{\lambda-1}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) + \lambda^n\right]\frac{\log\lambda}{\lambda^n} < \log(2n-2),$$

it suffices to prove that

$$\lambda^{\frac{2}{(\lambda-1)\lambda^n}\left(\lambda^{n-1}-n+\frac{\lambda^n}{2}\right)} < \frac{2n-2}{\lambda}.$$

Since $n \geq 5$ and $1 < \lambda < 2$, it follows that $\frac{2n-2}{\lambda} \geq \frac{8}{\lambda} \geq \lambda^2$. Hence it remains to show that

$$\frac{2}{(\lambda-1)\lambda^n}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) < 2.$$

However, this inequality immediately follows from

$$\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2}) = 0$$

and $1 < \lambda < 2$. $\square$

Proof of Theorem 1.3. By Lemma 3.3, Theorems 2.6 and 3.1, we finish the proof of Theorem 1.3. $\square$

4. Some remarks

The shrinking random β-transformation we defined is very special. For a general sub switch region, i.e. $(a, b) \subset [\beta^{-1}, \beta^{-1}(\beta - 1)^{-1}]$, does the intrinsically ergodic measure exist? For general $1 < \beta < 2^{-1}(1 + \sqrt{5})$, how can we find an invariant measure (or intrinsically ergodic measure) for the shrinking random β-transformation? In the setting of classical random beta transformation, similar questions can be considered, see [1].

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