Chapter 6
Digital Technology and Mathematics Education: Core Ideas and Key Dimensions of Michèle Artigue's Theoretical Work on Digital Tools and Its Impact on Mathematics Education Research

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6.1 Introduction

In 2002 Michèle Artigue published an article entitled *Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work*. That paper reflects a fundamental contribution to theory on the teaching and learning of mathematics in technological environments, and to instrumentation theory in particular. Clearly, Michèle's work[1] did not end with her 2002 paper; rather, the article presents important threads that she has continued to develop, and that have inspired other researchers in the field. As such, the paper has had an important influence on the international research agenda in the domain of technology-enhanced learning, as well as a considerable impact on recent research. This chapter, therefore, has two goals. The first goal is to revisit the central themes elaborated in that paper. The second is to follow the evolutionary paths of the paper's main themes and to outline some new directions that have emerged from them.

To achieve these goals, we distinguish the threads that are *general key dimensions*, which run through the body of Michèle's work, from the threads that are *core*

[1]Michèle would be the first to insist that the contributions we describe in this chapter were not hers alone, nor just those of her DIDIREM team in Paris, but were also based on collaboration that included a team in Rennes piloted by Jean-Baptiste Lagrange, and another in Montpellier piloted by Dominique Guin and Luc Trouche.

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theoretical ideas, which are interwoven into and provide specific perspectives on the key dimensions. The key dimensions are generic in nature; they include the mathematics, the teacher, the learner, and the tool—dimensions that are in fact touched upon in much of the research on the use of digital technology in mathematics classes. The cross-wise threads of core theoretical ideas are those particular notions that underpin and elaborate the ways in which the general dimensions are considered and without which the dimensional terms would be devoid of specific interpretation. In collaboration with others, Michèle has contributed uniquely to the generation of core theoretical ideas that have profoundly impacted the way in which we think about some of mathematics education's basic dimensions. We also believe that the metaphor of interweaving, which permeates this chapter, fits well with the kind of 'tinkering' that we all try to do in our work, and at which Michèle excelled.

6.2 The Importance of Theoretical Foundations

6.2.1 Towards a New Theoretical Framework

The first theme we identify in Artigue's (2002) IJCML article concerns the importance of theoretical foundations. In one of the first sections, entitled *A theoretical framework for thinking about learning issues in CAS environments*, Artigue emphasises the need that had been felt by her research group for a framework other than the ones that were then in use, in particular a framework that would avoid the traditional "technical-conceptual cut":

> In the mid-nineties, we thus became increasingly aware of the fact that we needed other frameworks in order to overcome some research traps that we were more and more sensitive to, the first one being what we called the "technical-conceptual cut" (Artigue 2002, p. 247).

In the search for such frameworks, she and her collaborators turned toward the anthropological theory of the didactic (ATD, or TAD within the French community) with its socio-cultural and institutional basis (Chevallard 1999) and the cognitive ergonomic approach with its tools for thinking about instrumentation processes (Rabardel 1995; Vérillon and Rabardel 1995). Together, these two theories formed the foundational principles for a new theoretical framework, the *instrumental approach to tool use*—a framework that was supported by the earlier research carried out by Artigue and her collaborators (e.g., Artigue et al. 1998; Guin and Trouche 2002; Lagrange 1999, 2000; Trouche 1997). This theoretical work is testimony to the importance Artigue attributed to what we consider an overall characteristic of her research, that of *theoretical frameworks* in the area of technology-enhanced learning. An important feature of this framework is the underlying process of combining, integrating, and adapting the two theoretical orientations for the specific purpose of investigating the opportunities and constraints of the use of digital tools in mathematics education.

It is noted that the combining of Chevallard's anthropological theory of the didactic (with its institutional aspects that impact upon the generic dimensions of teacher, learner, and mathematics) with the cognitive ergonomic approach of Vérillon and Rabardel (with its tool and learner dimensions) into the instrumental approach could be viewed as an early attempt at networking two theories before the term came into vogue—a notion that Artigue addressed in her plenary talk at CERME-5 in 2007. She remarked that this combining had been productive, even if at times it had yielded tensions:

> The difference [between the two frames] reflects in the evident tension existing between on the one hand the language of praxeologies and techniques used in the TAD, and on the other hand the language of schemes used by Rabardel. This tension between schemes and techniques, … between the institutional and the individual, has been extensively discussed in recent years … but up to now has not been overcome. … For me, this is a good illustration of the difficulties that one necessarily meets when trying to integrate two different logics, to build something starting from two different coherences. It shows the difficulties raised by the connection of theoretical frames (Artigue 2007, p. 75).

6.2.2 *Further Developments and Impact: Networking of Theories*

In order to follow the evolutionary paths of the 2002 paper's main themes and to outline some new directions that have emerged from them, we now address some further developments concerning the combination and confrontation of different theoretical frameworks. While the instrumental approach to tool use continued to develop in France and elsewhere during the years following the turn of the millennium, researchers who were conducting research on the use of digital tools in mathematical learning and teaching were adapting frames involving several other constructs, such as activity theory and social semiotics. The field was becoming marked by fragmentation with respect to the theoretical frameworks used in designing technological tools and in conducting research with these tools (Lagrange et al. 2003). This was making difficult not only productive collaboration among researchers but also the transporting of tools to educational contexts different from those for which they had initially been designed.

To overcome this theoretical fragmentation, the European project Technology Enhanced Learning in Mathematics (TELMA) was created, with Artigue one of the main collaborators. Project participants explored possibilities for connecting and integrating theoretical frames. According to Artigue et al. (2009, p. 218), "very soon, we became convinced that integration could not mean for us the building of a unified theory that would encompass the main theories we were relying on; the number and diversity of theories at stake made such an effort totally unrealistic." Artigue and her collaborators realised that in order to develop an integrated approach to research they needed a shared research practice so as to look at theories in operational terms. Such a practice also needed an appropriate methodology and

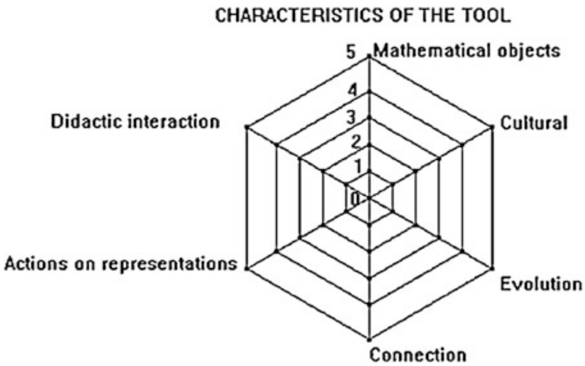

Fig. 6.1 Tool characteristic radar chart within the Integrative Theoretical Framework (ReMath Deliverable 1 2006)

instruments. Radar charts, for example, were used to help position the tools used in different studies (see Fig. 6.1).

Developing this shared research practice led to the constructs of didactical functionality and shared concerns, where tool characteristics, modalities of tool use, and educational goals were central. Tool characteristics included concerns related to ergonomics, semiotic representations, and institutional/cultural distances. Modalities of tool use included concerns related to the interaction with paper-and-pencil work, the social organisation and roles of the different actors, and the functions given over to the tool. Educational goals included concerns related to epistemological, cognitive, social, and institutional considerations. The several cross-experimentation[2] studies carried out by the various TELMA teams revealed that the concerns related to tool, tool use, and goals do indeed drive the entire experimentation process. The development of these concerns constitutes a major contribution by Artigue and her collaborators with respect to the theoretical elaboration of the *tool dimension* in research on technology-enhanced learning of mathematics. The work of the TELMA researchers in developing methodological and conceptual tools was to evolve further when the TELMA teams engaged in another project in continuity with their previous research: the ReMath project[3] (Representing Mathematics with Digital Media).

[2]The TELMA cross-experimentation studies involved pairs of teams coming from different theoretical cultures, but both using the same digital technology—a technology that was well known to one of the teams but alien to the other.

[3]The ReMath project relied on the TELMA meta-language of didactic functionalities and concerns, as well as the system of cross-experiments, but had somewhat different aims. It focused more specifically on representations and issues related to the design of digital artefacts and extended the TELMA methodology to include cross-case-study analyses. For further elaboration of the ways in which the ReMath project developed, modified, and extended the ideas initiated in the TELMA project, see the recently published Artigue and Mariotti (2014) paper, which appeared after this chapter was written.

One of Artigue's early initiatives within the ReMath project was the formulation of a first version of an integrative theoretical frame (ITF), a document that—we note with interest—began to use the language of *networking of theories*:

> The first version of the ITF is neither a theory, nor a meta-structure integrating the seven main theoretical frames used in ReMath into a unified whole. It is more a meta-language allowing the communication between these, a better understanding of the specific coherence underlying each theoretical framework, pointing out overlapping or complementary interests as well as possible conflicts, connecting constructs which, in different frameworks are asked to play similar or close roles or functions. … What has been achieved in TELMA … tends to show that the metaphor of *networking* is, as regards the idea of integrative perspective, better adapted than the metaphor of *unification*, but it only suggests some hints as regards the strategies we could engage for making this networking productive. (ReMath Deliverable 1 2006, p. 31, italics in the original document).

Artigue was not the only one to elaborate on this core idea of *networking of theories*; it received considerable attention at the 2005 Fourth Congress of the European Society for Research in Mathematics Education (CERME 4), as well as at successive ERME congresses (see also Bikner-Ahsbahs and Prediger 2006; Prediger et al. 2008). Some of the strategies proposed for networking theories included comparing, contrasting, coordinating or combining—in fact, strategies that bear a certain relationship to the approaches that were part of the ongoing discourse of researchers from the TELMA and ReMath projects. The interactions among the various researchers participating in the Theory Working Group at the ERME congresses, as well as the reflections of the networking group set up by Angelika Bikner-Ahsbahs and Susanne Prediger at CERME 5 to work between the ERME congresses, have not only advanced researchers' thinking about this emerging area (e.g., Artigue et al. 2005; Cerulli et al. 2005; Kidron et al. 2008; Artigue et al. 2010; Bikner-Ahsbahs et al. 2010) but have also served to stimulate an increase in the very activity of theorising within the field (e.g., Monaghan 2010, 2011; Drijvers et al. 2013a; Godino et al. 2013; Lagrange and Psycharis 2013).

More recently, Artigue et al. (2011) have proposed a broadening of the discussion on networking of theories to include the construct of research praxeologies. Artigue and her co-authors argue that talking about "theories," as in "networking theories," indicates only the theoretical part of research practice. They have therefore extended Chevallard's ATD notion of praxeology to elaborate the pivotal notion of *research praxeology:* It comprises the practice of research (with its task-technique block) along with its technological-theoretical discourse. Artigue et al. stress that research praxeologies are dynamic entities whereby changes in the practical block lead to evolution of the technological-theoretical block and vice versa (i.e., the technical-theoretical dialectic)—changes that involve considering the notion of research phenomena. They maintain that "networking between theoretical frameworks must be situated in a wider perspective than that consisting of the search for connections between the objects and relationships structuring these. … Our reflection tends to show that an approach in terms of research praxeologies can be productive for networking between theories, especially because it helps address the essential issue of the functionality of theoretical frameworks, by inserting these in systems of practices" (Artigue et al. 2011, p. 9).

In sum, our above review of recent literature shows that Artigue's (2002) article describing the interwoven roots of the instrumental approach to tool use was central to the later theoretical work of combining and integrating theoretical frameworks that grew into the networking of theories approach to research in mathematics education.

6.2.3 *Further Work and Impact: Ongoing Developments of ATD*

The above-mentioned Artigue et al. (2011) paper also reflects a second direction of follow-up work in the field of theoretical frameworks, in this case concerning ATD. In particular, researchers around the world have been inspired by Artigue's and her research group's insistence on avoiding the technical-conceptual cut. Her group's development of the idea that the technical has a strong conceptual element, especially during the period of the initial learning of a technique (Lagrange 2000), has been taken up not just in ensuing research involving digital tools (e.g., Nicaud et al. 2004; Boon and Drijvers 2005; Haspekian 2005; Martinez 2013) but also in the theorising of mathematical learning at large (Kieran 2013). As an example of the former, we refer to a research project on the interaction between the technical and the conceptual in the learning of algebra with CAS tools (Kieran et al. 2006), which was framed within the instrumental approach's task-technique-theory (TTT) adaptation of Chevallard's ATD. Within that project, Hitt and Kieran (2009) investigated in detail at close range the task-based activity of a pair of 10th grade students and documented, with the aid of a specially-developed notation (see Fig. 6.2), the ways in which students' emerging theories were systematically being revised as they engaged with CAS tools in concept-building actions within technique-oriented algebraic activity.

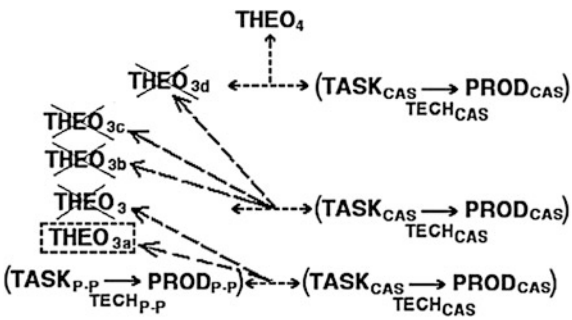

Fig. 6.2 Students' revisions of their theoretical explanations to account for task-based phenomena in a learning environment involving the use of CAS techniques (Hitt and Kieran 2009)

This core idea of the *technical-conceptual connection* (also referred to as the technical-theoretical connection), which was explored in the research of Artigue and her group (Artigue et al. 1998; Lagrange 1999, 2000) and further developed in the above more recent research, has provided a vital new theoretical tool for reflecting on the learning of mathematics. As such, it has led to a different way of thinking about the *learner dimension* within school mathematics, especially in the area of algebra. In this area, where the technical has for decades held sway and conceptual understanding considered all but an oxymoron, the work of Artigue and her colleagues in changing the relationship between technical skills and conceptual understanding has been truly ground breaking. We will come back to this technical-conceptual connection in Sect. 6.4.

6.2.4 Core Theoretical Ideas and Key Dimensions

To summarise Sect. 6.2, which has focused on Artigue's passion for theory, a main theme that has been highlighted is the importance of and need for theoretical foundations of research and development in the field of mathematics education. Two of the key dimensions that we have identified as being central to the theoretical advances that have been made are the *tool* and the *learner* dimensions. The theoretical threads that have been woven into, and have provided texture to, these dimensions include the core idea of the *instrumental approach to tool use* frame, with its concomitant core idea of the *technical-conceptual connection*—the latter yielding novel theoretical perspectives particularly with regard to the *learner dimension* in school mathematics. The *tool dimension* was significantly elaborated by the theorising initiated within the TELMA project and further developed within the ReMath project. Artigue's emphasis on theoretical foundations and the fact that these foundations can arise by a process of 'tinkering', integrating and adapting existing theoretical frameworks within the domain of study, or from outside, is another core idea of Artigue's work—a core idea that may be seen as *networking of theories* 'avant la lettre'.

6.3 Instrumental Approaches and Instrumental Genesis

6.3.1 The Complexity of Instrumental Genesis

In the previous section we drew attention to the emergence of the instrumental approach to tool use, based on principles from ATD and cognitive ergonomics. In our opinion, this instrumental approach was the first fundamental theoretical lens for studying the use of digital tools in mathematics education, and CAS in particular. It proved to be a major contribution to the field (Hoyles and Lagrange 2010) and underlines the importance of tools in use, which through their opportunities and constraints shape and are shaped by student knowledge. Instrumental approaches—we use the plural here because of the different variations that now exist for the theory—acknowledge the impact tools have on the ways in which students do and think about mathematics: "Tools matter: they stand between the user and the phenomenon to be modelled, and shape activity structures" (Hoyles and Noss 2003, p. 341).

In line with Rabardel's (1995) distinction between an artefact and an instrument, Artigue in her 2002 IJCML article points out that an instrument is a mixed entity that is part artefact and part cognitive schemes (see also Guin and Trouche 1999). We can summarise this in a 'formula': instrument = artefact + scheme. The process by which an artefact becomes an instrument is referred to as *instrumental genesis*—another core theoretical idea. This genetic process works in two ways: in one, the process is directed from the user toward the artefact in that the artefact becomes loaded with potentialities—the instrumentalisation of the artefact; in the other, the process is directed from the artefact toward the user in that the user develops schemes of instrumented action that permit an effective response to given tasks—the instrumentation of the user. An important contribution to our knowledge of using digital technology in mathematics education, now, is the notion that the use of cognitive tools such as advanced calculators or computers is neither self-evident nor trivial, and that the instrumental genesis needed is a complex and time-consuming process.

The research on instrumental genesis emanating from Artigue's collaborative research group included doctoral theses that illustrated, for example, the diversity of the instrumental relationships that students studying the concept of limit develop with the digital technology of graphical and symbolic calculators (Trouche 1997, whose doctoral thesis was directed by Dominique Guin). Students' conceptions and ways of interacting with the digital tools led Trouche to characterise five different student profiles: theorist, rationalist, scholastic, tinkerer, and experimentalist. Another thesis (Defouad 2000), which focused on the study of functional variation over the course of the school year and involved Grade 11 students equipped with the TI-92 CAS calculator, pointed to the complexity and fragility of the process of instrumental genesis. For Defouad's students, instrumental genesis was found to progress slowly through various stages, beginning with the graphical application being used for exploration and solving, and evolving through to the symbolic application for the computation of exact values, at which point the graphical was being used primarily for anticipation and control. A key dimension of research with digital tools that is highlighted in both of these studies is that of the *learner* and the way in which his/her characteristics interact with those of the tool.

6.3.2 Further Developments and Impact: Instrumental Orchestration

The notion of instrumental genesis was followed up in several studies that identified instrumentation schemes and that documented the difficult process of building these up in students (e.g., see Fig. 6.3). However, it was not long before research related

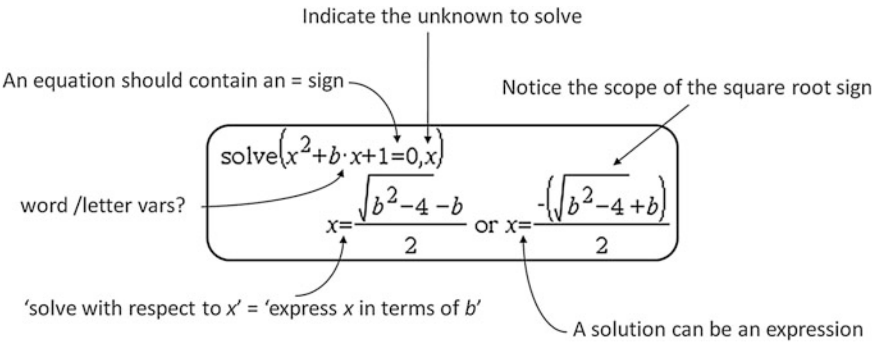

Fig. 6.3 Elements of an instrumentation scheme for solving equations in a CAS environment (Drijvers et al. 2013a)

to the core theoretical idea of instrumental genesis was to focus on the *teacher* dimension, both from the point of view of his/her role within the digitally enhanced learning environment and from the perspective of his/her own instrumental genesis.

The potential synergy between the instrumental approach and the role of the teacher led to Trouche's (2004) elaboration of the construct of instrumental orchestration: "the necessity (for a given institution—a teacher in her/his class, for example) of external steering of students' instrumental genesis" (p. 296). According to Trouche, an instrumental orchestration is defined by didactic configurations and their exploitation modes, the latter of which are aimed at providing students with the means to reflect on their own instrumented activity. In pointing to the instructional role involved in managing and fine-tuning an entire classroom of individualised instruments so as to bring out their collective aspects, Trouche integrates the individual concerns of the ergonomic frame with the institutional concerns of the ATD. Further research on teachers' instrumental orchestrations is reported in, for example, Drijvers and Trouche (2008) and Drijvers et al. (2010), and has resulted in some categorisations (see Fig. 6.4).

Teachers' instrumental genesis has also been an area of study that has evolved from the theoretical frame of the instrumental approach. Bueno-Ravel and Gueudet (2007), who participated in the GUPTEN (Genesis of Professional Uses of Technologies by Teachers) project spearheaded by Jean-Baptiste Lagrange, focused specifically on e-exercises and the way in which these artefactual resources become instruments for the teacher through a process of instrumental genesis. Artigue and Bardini (2010) studied teachers' instrumental geneses in a project involving the use of a new tool, the TI-Nspire CAS. In particular, they addressed the issue of the relationships between the development of mathematical knowledge and instrumental genesis and noted the impact of new kinds of instrumental distance (see Haspekian and Artigue 2007) and closeness that shape teachers' activities.

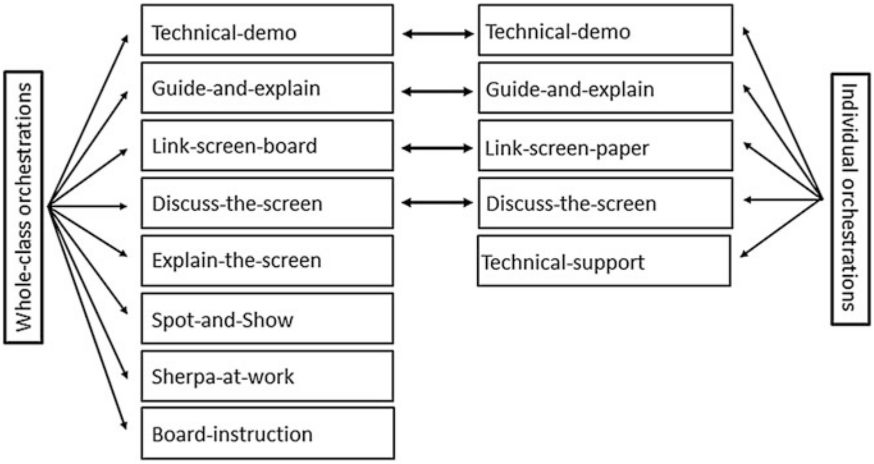

Fig. 6.4 A first inventory of teachers' orchestrations (Drijvers et al. 2013b)

6.3.3 Further Developments and Impact: The Documentational Approach

A further evolution of the research on teachers' instrumental geneses has been the theoretical transformation of this focus into a new frame that is referred to as the documentational approach of didactics (Gueudet and Trouche 2009). In this theoretical frame for studying teachers' documentation work, the artefact-instrument dialectic within instrumental genesis has been recrafted as the resource-document dialectic within the process of documentational genesis. The new 'formula' thus becomes: document = resource + scheme. This theoretical frame, which places documentation work at the core of teachers' professional growth, has been further developed in Gueudet and Trouche (2010) and Gueudet et al. (2012). As an elaboration, Sabra (2011) sketches the 'fabric' of a resource system for one particular teacher (see Fig. 6.5). Even more recently, this approach has evolved to take into account the way in which documentation work is also central to the professional activity of design researchers (Kieran et al. 2013).

6.3.4 Core Theoretical Ideas and Key Dimensions

To summarise, in Sect. 6.3 we have focused on the complexity of the use of digital tools and the corresponding instrumental genesis, and on the ways in which this construct had been applied and developed by Artigue's collaborators and by other researchers outside France. The dimensional threads that have been theoretically elaborated in that research include: the *tool* (and its use), the *learner,* and the

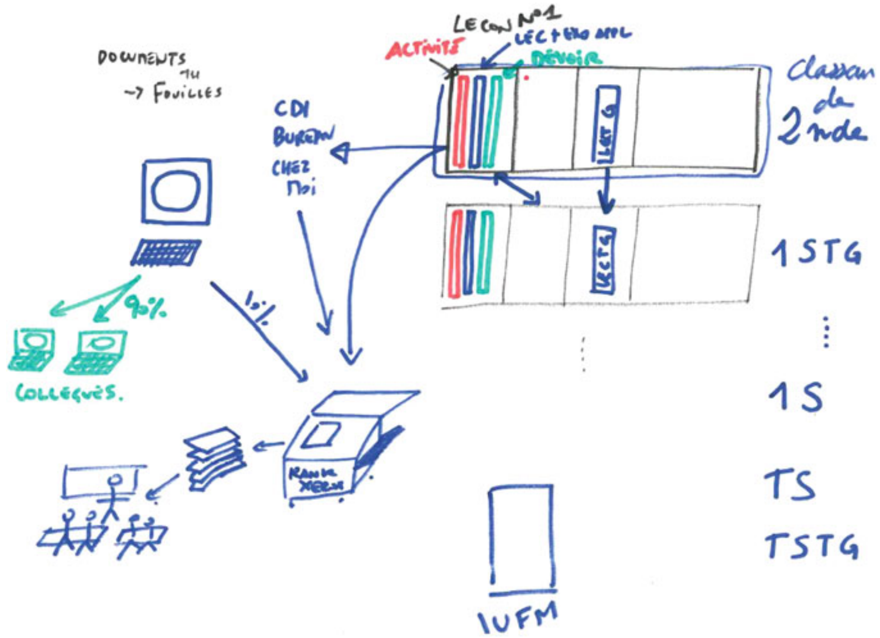

Fig. 6.5 An inventory of one teacher's resource system (Sabra 2011)

teacher. The core theoretical idea that has been interwoven through, and that has given a particular theoretical sense to, these dimensional threads has been the construct of *instrumental genesis*.

6.4 The Pragmatic-Epistemic Duality

6.4.1 The Pragmatic and Epistemic Value of Techniques

In Sect. 6.2, the avoidance of the technical-conceptual cut was mentioned as a hallmark of research on the use of digital tools in mathematics education—one that has been inherited from the instrumental approach to tool use. The technical aspects of using digital tools clearly incorporate a strong conceptual element and reconciling these two can be seen as an important component of instrumental genesis. Thus, while the conceptual is intricately interwoven with the technical within the core idea of the *technical-conceptual connection,* the role of technique in contributing to the development of the conceptual is central—and this brings us to the pragmatic-epistemic duality.

An important contribution of Artigue's (2002) article is the distinction she draws between the pragmatic and epistemic values of techniques. Within the instrumental approach, the *pragmatic* value of techniques refers to their "productive potential" (Artigue 2002, p. 248), while their *epistemic* value refers to "their contribution to the understanding of the objects they involve", particularly during their period of learning when they constitute a source of questions about mathematical knowledge (see also Lagrange 2000). In her CERME-5 plenary lecture, Artigue (2007, p. 72) clarified an important point about this duality within the instrumental approach to tool use: "While technique is a fundamental object of the ATD, the ATD does not distinguish between the epistemic and pragmatic values of techniques; these terms come from cognitive ergonomy, but there they are linked to schemes and not to techniques."

Taking the pragmatic-epistemic notion of the ergonomic approach and connecting it with the objects of the ATD was an astute move on Artigue's part. Having already linked techniques to schemes by having the former designate the visible part of the latter, Artigue could then refer to the epistemic and pragmatic values of techniques. However, the appropriation of the pragmatic-epistemic duality within the instrumental approach allowed for much more than this. It provided for considering the 'mathematical needs of instrumentation' (a phrase that combined the mathematical underpinnings of the ATD with the instrumentational aspects of the ergonomic approach) and for these mathematical needs to be interpreted in terms of the epistemic value of instrumented techniques. In addition, it supported a pragmatic-epistemic perspective on the two ATD objects of technique and theory and highlighted the relationship between the two. As well, it opened up a discourse for comparing and contrasting the pragmatic and epistemic values of "official" mathematics with the pragmatic and epistemic values of instrumented mathematics. The multiple ways in which the notion of *pragmatic-epistemic duality* allowed for aligning the contributions of the ATD and of the ergonomic approach within the instrumental frame, as well as for operationalising their interactions, render it a truly core theoretical idea of Artigue's work.

Three elements of Artigue's research that are intertwined with the pragmatic-epistemic duality, but which can also be considered central notions in their own right, are the following: the institutional aspect, the task design component, and the mathematical dimension. The first element, the institutional aspect, refers to the educational, social and institutional contexts of techniques. In line with ATD, Artigue (2002) describes how teachers in French mathematics classes during the first year of a study were observed to have difficulty in giving adequate status to instrumented techniques. In contrast to the standard way in which paper-and-pencil techniques were explored, routinised, and institutionalised, the several digital techniques that were introduced suffered from ad hoc treatments that prevented them from becoming efficient and productive. The theoretical discourse accompanying the use of such techniques remained fragmentary and underdeveloped. Artigue points out that, while the "kinds of discourse which can be developed are well known for official paper and pencil techniques, … a discourse has to be constructed for instrumented techniques … a discourse that will call up knowledge which goes beyond the standard mathematics culture" (Artigue 2002, p. 261). The institutional roots of this difficulty are emphasised: "The institutional negotiation of the specific mathematical needs required by instrumentation [is] a negotiation which today is not an easy one" (p. 268). This *institutional aspect*, which was central to the ATD, remained a core theoretical idea that was threaded through all of Artigue's research (see Artigue 2012).

Second, in her discussion of the pragmatic-epistemic duality, Artigue relates the constructing of an adequate discourse for instrumented techniques to task design, that is, to the process of didactical engineering or ingénierie didactique. According to Artigue, developing appropriate situations and tasks for instrumental work was a challenge for the teachers involved in her research; they were unsure how to design tasks that make provision for developing the epistemic value of techniques. In this regard, Artigue (2002, p. 268) points out that "epistemic value is not something that can be defined in an absolute way; it depends on contexts, both cognitive and institutional; from the contextual [and mathematical] analysis of this potential to its effective realisation there is a long way, with situations to build, viability tests, and taking into account the connection and competition between paper and pencil and instrumented techniques." The latter remark highlights yet another core idea of her work: the *relationship between paper-and-pencil and digitally-instrumented techniques*. She notes that particular attention needs to be paid to the relationship between techniques for using digital tools and 'traditional' paper-and-pencil techniques: While both the pragmatic and the epistemic values are obvious for the case of "official" paper-and-pencil techniques in that "the epistemic value of a paper-and-pencil technique becomes evident through the details of its technical gestures" (Artigue 2002, p. 259), the epistemic value of instrumented techniques seems much less obvious.

Last but not least, a crucial step in the design of task sequences is a thorough analysis of the underlying mathematical domain. In commenting that more than the standard mathematics is called for when dealing with instrumented techniques, Artigue emphasises not only the mathematical needs of instrumentation but also the requirement for a deep a priori analysis of the mathematics embedded in the tool and its use. She thereby stresses the importance of elaborating the *mathematical dimension* within research studies—an emphasis that is shared by fellow researchers of the French *didactique* tradition (see also Brousseau 1997). In one of her examples, Artigue (2002) refers to the topic of equivalence of expressions and the problem of detecting equality for certain types of algebraic expressions in a CAS environment. She points out that the CAS tool can produce results—often quite surprising and unexpected—that go beyond what is usually faced in non-digital-technology-supported mathematics classrooms when algebraic expressions are to be simplified. In her ensuing discussion of the mathematical needs required for an efficient instrumentation, which she expresses in terms of the epistemic value of instrumented techniques, Artigue (2002, p. 260) suggests that the epistemic has to be provided for by constructing a mathematical discourse around it: "The epistemic value of instrumented gestures is something that must be thought about and reconstructed; in the teaching process, it has to be developed through an adequate set of situations and tasks".

6.4.2 Further Developments and Impact: The Institutional Aspect

The institutional/cultural aspect of the instrumental approach was highlighted in the work of the TELMA and ReMath projects, where institutional considerations figured into the three main theoretical developments of the two projects: tool characteristics, modalities of use, and educational goals. This aspect was also reflected in the practice of the participating research teams, as witnessed by their own institutional/cultural approaches to research. More recently, Artigue (2012) in her MERGA plenary presentation on multiculturalism in mathematics education research returned explicitly to the institutional aspect of Chevallard's ATD theory:

> Sensitivity to the cultural dependence of mathematics education must be supported by appropriate constructs and methodological tools for being productive. With the development of socio-cultural approaches, the field of mathematics education today offers a diversity of theoretical frameworks and constructs for such a purpose. As with many French colleagues, due to my cultural environment, I have found a support in the Anthropological Theory of Didactics (ATD). In this theory initiated by Chevallard, indeed, an initial postulate is that human knowledge emerges from practices which are institutionally situated thus a fortiori culturally situated (p. 6).

The attention paid to institutional conditions and constraints is also manifest in the documentational approach of didactics (Gueudet and Trouche 2009). As shown in Fig. 6.6, institutional influences may hinder or enhance teachers' documentational genesis to an important extent.

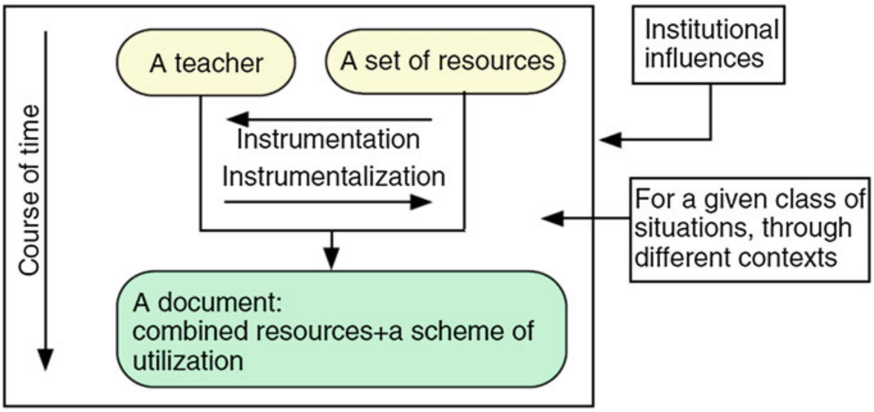

Fig. 6.6 The institutional aspect in documentational genesis (Gueudet and Trouche 2009)

6.4.3 Further Developments and Impact: Task Design and Mathematical Analysis

The potential interactions between, on the one hand, the pragmatic and epistemic values of techniques and, on the other hand, techniques instrumented digitally and paper-and-pencil techniques, served as a basis for designing tasks in a CAS study on equivalence reported by Kieran and Drijvers (2006). Task-sequences were designed that would invite both technical and theoretical development, as well as their co-emergence. One of the observations of the study was that most students wanted to be able to produce themselves, by means of paper and pencil, the results that were output by the CAS whenever the CAS results could not be explained by their existing technical and conceptual knowledge. That is, CAS and paper-and-pencil techniques were found to be interrelated epistemically and co-constitutive of students' theoretical development. However, it was also found that the a priori mathematical analysis of the notion of algebraic equivalence, which had guided the initial design of the study, did not go far enough. Data from student work indicated that the mathematical analysis by the task designers had to be further developed because it had not adequately taken into account the importance of domain considerations and transitivity in students' evolving conceptual understanding of equivalence (Kieran et al. 2013; also see Fig. 6.7). This led to a deeper theoretical elaboration of the dimensional thread related to the underlying mathematics and, at the same time, confirmed once again the importance of Artigue's insistence on the mathematical needs of instrumentation.

6.4.4 Core Theoretical Ideas and Key Dimensions

To summarise, Sect. 6.4 has highlighted the importance in Artigue's work of the dimensional thread related to the *mathematics*, that is, to the requirement for deep a priori mathematical analysis of the needs of instrumentation and for developing

We shall restrict ourselves to single-variable expressions. We just implicitly saw that there are two definitions for the equivalence of two expressions $f(x)$ and $g(x)$, equivalence for which we will use the usual notation $f(x) \equiv g(x)$:

- A *syntactic definition*: $f(x)$ and $g(x)$ are equivalent if and only if we can establish their equality by symbol manipulation, using rules recognized as true for the set $\mathcal{E}$.
- A *semantic definition*: $f(x)$ and $g(x)$ are equivalent if and only if for every element a in $\mathcal{E}$ we have an equality between $f(a)$ and $g(a)$ (we shall refer to this particular definition as Semantic Definition of Equivalence, Version 1).

Fig. 6.7 Extract from a mathematical analysis of the notion of algebraic equivalence in the Kieran et al. study (2013) adequate situations and tasks for instrumental work. Interwoven with the key mathematical dimension have been the three core theoretical ideas of the *pragmatic-epistemic duality*, the *relationship between paper-and-pencil and digitally-instrumented techniques*, and the *institutional* aspect.

6.5 Closing Remarks

In this chapter, we have revisited Michèle Artigue's classic 2002 IJCML article and have drawn out what we consider to be the core theoretical ideas and key dimensions of the body of work on tools and tool use that Michèle not only elaborated but also inspired others to further develop. We have traced the evolutionary path of these core ideas, noting the ways in which they theorised the four general key dimensions of learner, teacher, tool, and mathematics. Without claiming to be exhaustive in our selection, we have focused on seven core theoretical ideas that have been central to Michèle's work and that have impacted in various ways the research of others: the instrumental approach to tool use, instrumental genesis, the pragmatic-epistemic duality, the technical-conceptual connection, the paper-and-pencil versus digitally-instrumented-technique relationship, the institutional aspect, and the networking of theories.

We realise that we have discussed these core theoretical ideas as if they were separable, one from the other. Of course, they are all related, with each but the last being an intrinsic part of the frame of the instrumental approach to tool use. However, while the core idea that is the instrumental approach to tool use is an overarching one that subsumes most of the others, several of its component core ideas merited being singled out and discussed individually. Some have been further developed in various ways—sometimes without involving the use of digital tools—and have even taken on lives of their own. This was noted, for example, with the core theoretical idea of instrumental genesis, one strand of which has evolved into documentational genesis and the frame of the documentational approach. Another is the core theoretical idea of the technical-conceptual connection that has been applied more broadly in recent research on mathematical learning.

The dimension of tools and tool use has been at the heart of Michèle's work on instrumentation and thus has been central to her theoretical work. Nevertheless, her contributions extend beyond this dimension. Michèle's theoretical ideas have had a profound impact on the ways in which we think about some of the other basic dimensions of mathematics education, such as the learner, the teacher, and the mathematics. The further developments and impact of the core ideas and key dimensions that we have described in this chapter are clear testimony to the richness of Michèle Artigue's theoretical contributions, for which we have much to be thankful.

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