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Hypernormal form of the Hopf singularity

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Introduction

The basic properties of dynamical systems can be derived by a classification of the eigenvalues of the linearisation given that the eigenvalues have nonzero real part. When this condition is not met, higher order terms determine the dynamical behaviour. The Hopf singularity arises when the eigenvalues are purely imaginary. This thesis focusses on the Hopf singularity and its generalisation. Classical normal form theory aims to study those system by removing all terms that do not change the dynamical behaviour locally around the equilibrium. Let the Taylor expansion of the Hopf singularity be given by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \dots \tag{1}$$

where the terms v_k are the homogeneous polynomials of degree $k+1$. It can be shown that using near identity transformations the system can be formally transformed to the following classical normal form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sum_{k=1}^{\infty} (x^2+y^2)^k \left[\alpha_k \begin{pmatrix} x \\ y \end{pmatrix} + \beta_k \begin{pmatrix} -y \\ x \end{pmatrix} \right] \qquad \alpha_k, \beta_k \in \mathbb{R}. \tag{2}$$

In polar coordinates it is given by:

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=1}^{\infty} \begin{pmatrix} \alpha_k r^{2k+1} \\ \beta_k r^{2k} \end{pmatrix}, \qquad \alpha_k, \beta_k \in \mathbb{R}. \tag{3}$$

In Chapter 2 it will be shown that this normal form is in fact not unique. The problem is that there exist different transformations that produce different normal forms. Given this non-uniqueness, the investigation in which one is correct will lead to further simplifications in Chapter 2 and in fact uniqueness. The classical normal form in most cases also has an infinite number of terms and usually does not converge. After further simplification, only finitely many terms remain up to any order and this is a quite strong result.
To show the result for the Hopf singularity, consider it's classical normal form. When $\alpha_1 \neq 0$ and $\beta_1 \neq 0$ the system can be transformed to it's hypernormal form up to any order:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \alpha_1(x^2+y^2) \begin{pmatrix} x \\ y \end{pmatrix} + \beta_1(x^2+y^2) \begin{pmatrix} -y \\ x \end{pmatrix} + \alpha_2(x^2+y^2)^2 \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \alpha_1, \alpha_2, \beta_1 \in \mathbb{R} \tag{4}$$

Since α_1 and β_1 are nonzero, the system can be classified according to these two coefficients.

In the normal form theory there are two problems that need to be tackled. The first one is the problem of describing what the normal form actually looks like. As will be explained in the next chapter, normal form theory is the search of the minimal or simplest representative under the action of near identity transformations. This simplest representative usually implies that as many coefficients are set zero as possible that have no effect on the dynamical behaviour. On the other hand, the computation problem is the problem that concerns itself with the best way of generating the transformations that put the system into normal form. This

assumes that the such a simplest system is already known and concerns with the general transformations that transforms one system into another.
So the problem is clear: the description problem assumes that computation problem is solved and vice versa. To solve the normal form problem both problems need to be solved sort of simultaneously. The assumption made is that the description is already solved and the system is given in terms of a general system. Then the computation problem is solved. When this is done an analysis is performed as to how many coefficients we are able to set zero and then the total problem is solved.

In [2] a subcase of the general theorem is proven using the techniques presented there, only the specific subcase above is proven. The total result is proven in [4] but using very abstract methods. Here the proof is redone in full generality using the easiest possible methods and this is not found elsewhere. Not many textbooks exist on the subject of hypernormal forms. In [3] the topic is touched and some examples are given for the Hopf singularity. At the time of writing, James Murdock and Jan Sander had the idea of using spectral sequences for the hypernormalisation. This theory is rather involved, is definitely not entry level and requires extensive knowledge on the subject of homological algebra. It resulted in [7] treating the subject, but still had a very steep learning curve. In 2004 James Murdock wrote [2] and that aimed to be a self contained treatment of hypernormal forms. To be able to read it it still require a deep understanding of classical normal forms and it did not provide with a full proof for the Hopf singularity. It is exactly this gap that the thesis aims to fill. To give a full treatment of the Hopf singularity, developing the theory starting from the point of view of classical normal forms, and working towards the hypernormal form. The result in proven using the techniques presented there, but without the need of spectral sequences.

Before reading this thesis some comments have to be made with regards to the general approach of the problem. The hypernormalisation of the Hopf singularity can be done completely without the need of any advanced techniques. The main reason the abstract theory is still developed is because for the other singularities this abstractness is absolutely needed. The Hopf singularity also provides enough simplicity to shift between abstract reasoning and the effect of these ideas on calculation level. This greatly helps to understand the mechanics on both the abstract level as well as the concrete level. One disadvantage of this minimal theory approach is that some topics tend to skimmed over and done on ad hoc basis regardless of the fact that a lot of theory is needed to prove that the results hold. A perfect example is the kernel of the homological operator, it takes a great deal of machinery to prove that the basis that is chosen is the simplest.

First an introduction into the classical norm form theory is given and the basic results are stated and proven using ad hoc methods, specialising on the Hopf singularity. Since no theory is developed, the goal of normal form theory will be to remove as many terms as possible. Whether this is possible and whether a simplest normal form exists is something that has to wait until Chapter 3. Hypernormal form theory will be developed first using ad hoc methods and transformations and in the last chapter the work is redone using abstract theory. All theory will be developed along the way.

Chapter 1

The classical normal form

1.1 Introduction

Usually vector fields are studied through their Taylor series. In the case of polynomial vector fields this does not make any difference, but when a dynamical system is studied through it's Taylor expansions the normal forms should be taken modulo any flat function. A flat functions is a function such that it's Taylor expansion is identically zero. The fact that normal form calculation is always done modulo these functions is always implicitly assumed.
It is always assumed that the equilibrium being studied is at the origin and hence that the vector field does not have any constant terms. In normal form theory it is custom to work with the formal Taylor series. The terminology "formal power series" is due to the fact that the convergence or divergence is not taken into account for all transformations.

The next general assumption is that the linear term of the vector field has been put into Jordan normal form. Note that this coincides with diagonalisation when the linear part is diagonalisable. Suppose that the linear part has matrix $A = P^{-1}JP$, then the transformation $x = Py$ transforms the system into one that has linear part equal to J. It has the same dynamical behaviour because it is a local diffeomorphism. Let the different terms $v_k(x)$ belong to the finite dimensional vector spaces of homogeneous degree $k+1$ and denote them by V_k. It can be concluded that the starting point is:

$$\dot{x} = Jx + v_1 + v_2 + \ldots, \quad v_k \in V_k$$

1.2 The transformation

The first part of normal form theory is to choose a near identity transformation that will simplify the terms of a certain order. The transformations that will be used in this chapter only[1] are transformations of the form $x = y + h_k(y)$. Here the $h_k(y)$ are homogeneous polynomials of degree k. This will then perform a transformation to the vector field such that the terms with degree lower then k are not modified and the terms of degree k are modified in a special way. The general principle is to apply a series of normalising transformations such that each time the next order is simplified.

Definition 1 (Homogeneous terms)**.** *Denote the space of homogeneous polynomials of degree by H_k. A general homogeneous polynomial is denoted by $h_k \in H_k$*

Note that there are two definitions for the homogeneous space of polynomials, namely H_k and V_k. They are equal, but H_k is used for transformations and V_k is used for elements of the original or transformed vector field. The space of homogeneous polynomials forms a vector space over the real numbers. The following theorem shows the effect of a near identity transformation on the vector field.

[1]These are used for now because no extra theory is needed. There are better ways to generate the transformations and they are shown in the next chapter.

Theorem 1. *The coordinate change $x = y + h_k(y)$ will take the system*

$$x = v_0(x) + v_1(x) + v_2(x) + \dots$$

into the following system:

$$y = v_0(y) + v_1(y) + v_2(y) + \cdots + v_{k-1}(y) + b_k(y) + \dots.$$

Here

$$b_k(y) = v_k(y) + Dv_0(x)h_k(y) - Dh_k(y)v_k(y)$$

Proof. The proof consists of a simple calculation. The general is approach is that first it is proven that the transformation provides a smooth diffeomorphism close to the origin, which is true by the Inverse Function Theorem. Then the inverse is expanded as a power series and truncated. After multiplication the result follows directly and the calculation can be found on page 166 of [3]. □

The expression $Dv_0(x)h_k(y) - Dh_k(y)v_k(y)$ is called the homological expression. If there exists an h_k such that this expression becomes equal to $-v_k$, the term b_k vanishes. In order to find such an h_k, the expression should be further studied. For each degree the expression will change, because the basis of the polynomials on which it acts changes. This expression allows for a generalisation that will be discussed at the end of chapter.

The mechanism by which normal form theory works is now clear. By applying a series of transformations starting at the quadratic terms, it is possible to normalise one specific degree at a time. When moving to the next degree the near identity transformation does not change the terms already in normal form. The most beneficial outcome is when there exists a transformation such that the terms b_k vanish for each degree. The homological expression can be seen as an operator mapping H_k into H_k when the term v_0 is kept fixed.

Definition 2 (Homological operator)**.** *Define the homological operator $\mathscr{L}_{v_0}^k : H_k \to H_k$ as the map:*

$$h_k \mapsto Dv_0(x)h_k(y) - Dh_k(y)v_k(y)$$

A calculation shows that the operator is linear. Because it is a linear operator acting on a vector space, a matrix representation of this operator exists. The isomorphism between H_k and a $\mathbb{R}^m$ for a suitable m will be made explicit below.

Given this matrix representation of the homological operator, if the matrix is invertible there exists a transformation that sets the coefficients of grade k to zero. In the case that the matrix is not invertible, a simplification can still be accomplished by removing all terms in the image of the homological operator and leaving only those terms that are not removable by these transformations. As it turns out, by the theorems below, the kernel of the operator is left. Then any basis suffices as a complement to the image of the operator. When such a basis is found, the calculations can be repeated for the next order.

Theorem 2. *Let $L : V \to V$ be a linear operator acting on the vector space V, then a complement to the image of L is given by the kernel of the adjoint operator. In other words:*

$$V = im\ L \oplus ker\ L^T$$

if the standard inner product is used.

Proof. Found in any standard linear algebra textbook. □

Remark 1. *For a semisimple linear part, the decomposition $V = im\ L \oplus ker\ L$ can be used.*

Proof. The result holds in general when the operator is diagonalisable over the complex numbers. The proof follows from linear algebra. □

1.3 Normal form to third order

The calculation will be done explicitly as an example for the quadratic and third order terms and will demonstrate the various mechanics described above. The example is the Hopf singularity.
The starting point of the discussion is $\dot{x} = f(x)$ with formal power series

$$\begin{aligned}
&\dot{x} = Ax + v_1(x) + v_2(x) + v_3(x) + \dots \\
&A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and} \\
&v_k \in H_k.
\end{aligned}$$

Usually the period is different from 1, but time can always be rescaled to 1 and hence no generality is lost. Since the first order terms are already in normal form, the first step is to normalise the quadratic terms. This consists of the following steps:

1. Calculate a basis for the homogeneous terms.
2. Set up a matrix for the homological operator.
3. Calculate a complement space to the image of the homological operator. In this case, calculate the kernel of the homological operator.
4. When a basis of the kernel is found, these terms with coefficients are left in the classical normal form.
5. Proceed to the next order and repeat the steps.

The coefficients and vectors will be denoted by a three digit subscript: α_{102} could be such a subscript. To read to which vector it belongs: the last two digits denote the powers of the variable. So in the example the first variable gets a power of 0 and the second variable gets a power of 2. So the entry is $x^0y^2 = y^2$. If the first digit is 1, then the entry is multiplied with $e_1 \in \mathbb{R}^2$ and if it is a 2 then it is multiplied with $e_2 \in \mathbb{R}^2$. So in the coefficient belongs to the vector:

$$\begin{pmatrix} y^2 \\ 0 \end{pmatrix}$$

The general quadratic terms of the original vector are denoted by the same span but in terms of x and y, with the coefficients denoted by a_{***}, with the same notation as the transformation. So the general transformation is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = +c_{102}v_{102} + c_{111}v_{111} + c_{120}v_{120} + c_{202}v_{202} + c_{211}v_{211} + c_{220}v_{220}$$

The matrix can be calculated by plugging in all the different vectors. The result is:

$$\begin{aligned}
\mathscr{L}_A v_{120} &= 2v_{111} + v_{220} \\
\mathscr{L}_A v_{111} &= -v_{120} + v_{102} + v_{211} \\
\mathscr{L}_A v_{102} &= -2v_{111} + v_{202} \\
\mathscr{L}_A v_{220} &= -v_{120} + 2v_{211} \\
\mathscr{L}_A v_{211} &= -v_{111} - v_{220} + v_{202} \\
\mathscr{L}_A v_{202} &= -v_{102} - 2v_{211}
\end{aligned}$$

These calculations can be verified by hand. To set up the matrix, each vector v_{***} is associated with an element of the basis of (in this case) $\mathbb{R}^6$. The result is a 6×6 matrix and the representation of $\mathscr{L}_A^2$ in $\mathbb{R}^6$ is denoted L_A^2:

$$L_A^2 = \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

A simple calculation shows that this matrix is invertible. In order to conclude the discussion we assume that the normal form has the following representation:

$$\begin{pmatrix} b_{120} \\ b_{111} \\ b_{102} \\ b_{220} \\ b_{211} \\ b_{202} \end{pmatrix}.$$

Then the equation that needs to be solved is the following:

$$\begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_{120} \\ c_{111} \\ c_{102} \\ c_{220} \\ c_{211} \\ c_{202} \end{pmatrix} = \begin{pmatrix} a_{120} - b_{120} \\ a_{111} - b_{111} \\ a_{102} - b_{102} \\ a_{220} - b_{220} \\ a_{211} - b_{211} \\ a_{202} - b_{202} \end{pmatrix}.$$

Now the crux of the homological operator is clear: because it is invertible there exists a transformation such that when multiplied by the matrix the result is exactly the coefficients a_{***}. When this happens, the result is that the equation is satisfied when all b_{***} are set zero. To set as many terms to zero without changing the behaviour of the dynamical system. So the conclusion is that the quadratics can be completely removed in the Hopf singularity.

For the third order terms the analysis proceeds in a similar way. First a basis is written for a general third order transformation, then the effect of the homological operator is calculated on these vectors and the matrix representation is found. The result is:

$$L_A^3 = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This matrix is not invertible. By the theory developed so far the only thing we can say is that those terms that belong to the image of L_A^3 can be removed. Those elements that belong to the kernel of the homological operator can not be removed and stay. A basis for the kernel of the operator is given by the following two terms:

$$u_1 = (x^2 + y^2) \begin{pmatrix} x \\ y \end{pmatrix} \qquad w_1 = (x^2 + y^2) \begin{pmatrix} -y \\ x \end{pmatrix}$$

These two terms can be found by first finding a basis for the kernel of the matrix above and then finding the polynomials.

By this calculation the normal form up to third order is given by:

$$\dot{x} = Ax + \alpha_1 u_1 + \beta_1 w_1 + h.o.t.$$

From the calculation done it can be concluded that this way of calculating the normal form is inefficient and does not provide with a description of all terms. In the next section improvements are made.

1.4 The normal form to all orders

The homological operator had risen in the form

$$\mathscr{L}_{v_0}^k : H_k \to H_k$$

as an operator acting on two finite dimensional vector spaces. For every degree a new operator was defined. To improve on the previous section we view the homological operator as an operator acting on the infinite dimensional vector space of power series. To this end, define the vector $m \in \mathbb{Z}^n$ and define $x^m = x_1^{m_1} \dots x_n^{m_n}$. Now define the vector space of formal power series as

$$V = \prod_{k=0}^{\infty} V_k.$$

Here the product is used instead of the sum, because we want to allow an infinite number of element to be nonzero. Now define the new homological operator as:

$$\mathscr{L}_{v_0} : V \to V, \quad w \mapsto Dv_0 w - Dw v_0.$$

Here $w \in V$ is a formal power series and a basis of the space is given by the homogeneous polynomials x^m. A calculation shows that this basis consists of the eigenvectors of the space V. In order to prove this statement, the definition of a semisimple operator is needed:

Definition 3 (Semisimple Operator)**.** *Let $A : V \to V$ be a linear map on a vector space V. Then the operator A is said to be semisimple if the matrix A associated with the operator is diagonalisable over the complex numbers.*

The next result shows how the homological operator can be diagonalised:

Theorem 3. *When the linear part of the vector field is diagonalisable over the complex numbers, the homological operator is semisimple. If moreover the linear part of the vector field is diagonal, the eigenvalues are given by:*

$$\mathscr{L} x^m e_i = (\langle m, \lambda \rangle - \lambda_i) x^m e_i$$

Proof. See Corollary 4.5.3 of [3]. □

For the normal form to all orders it is best to remove the restriction and let m vary over all possible values. Then easy relations can be set up and this yields the normal form. This is best shown in the example. The restriction $|m| = m_1 + \cdots + m_n = k+1$ yields the eigenvalues of the the map $\mathscr{L}_{v_0}^k$. So the kernel of the homological operator in the previous section is found by setting $|m| = k+1$ and finding the solutions to the equations $m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_i = 0$ for $1 \le i \le n$.

1.4.1 Classical normal form for the Hopf-singularity

The theorem above allows for a general description of the normal form of the Hopf singularity to all orders. Non-uniqueness is still an issue, but will be addressed in the next chapter.

$$\begin{aligned}
&\dot{x} = Ax + v_1(x) + v_2(x) + v_3(x) + \dots \\
&A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and} \\
&v_k \in V_k.
\end{aligned}$$

One of the restrictions of Theorem 3 is that linear part of the vector field should be in diagonal form. This is done below with the transformation:

$$x = Tz, \qquad T = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

It can be seen that $z_2 = \bar{z}_1$ and result of the transformation is a complex system:

$$\begin{aligned} \dot{z} &= Az + v_1(z) + v_2(z) + v_3(z) + \dots \\ A &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \text{and} \\ v_k &\in V_k. \end{aligned}$$

Later this can be transformed back to the real system. A monomial is of the form $x^m = x_1^k x_2^l$ and $k+l$ equal to the various degrees. From Theorem 3, two equations need to be satisfied: $\langle m, \lambda \rangle - \lambda_1 = 0$ and $\langle m, \lambda \rangle - \lambda_2 = 0$. The first corresponds to e_1, the first basis vector for $\mathbb{R}^2$ and the second equation corresponds to e_2.
The first yields the equation:

$$ik - li - i = 0$$

and it follows that $k = l + 1$. Hence the top vectors of the monomials are of the form:

$$\begin{pmatrix} z_1^{l+1} z_2^l \\ 0 \end{pmatrix}.$$

The second equation yields:

$$ik - li + i = 0$$

This results in the following set of equations:

$$\begin{pmatrix} 0 \\ z_1^k z_2^{k+1} \end{pmatrix}.$$

It follows that the normal form up to any degree is given by the following equation:

$$\dot{x} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \sum_{k=0}^{\infty} (z_1 z_2)^k \left[\alpha_k \begin{pmatrix} 0 \\ z_2 \end{pmatrix} + \beta_k \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \right]$$

The α_k and β_k are real coefficients. This system is in complex coordinates and coincides with the calculations done in the first chapter. The linear change of variables does transform it back to real variables. Let:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This transforms the system back into the system:

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sum_{k=0}^{\infty} (x_1^2 + x_2^2)^k \left[\alpha_k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta_k \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right]$$

And this is exactly the result we got using the ad hoc methods, but to all orders.

Remark 2. *Until now the basis of the kernel was just chosen. If a different basis would have been chosen, a different normal form would have been obtained. In appendix A of [3] a treatment of the subject is given and the result is that this is the best possible basis of the kernel.*

Chapter 2

Further simplifications

2.1 Non-uniqueness of the normal form

So far a description for the classical normal form was derived to all degrees and it turned out that this could be done when the linear part of the vector field is diagonalisable over the real numbers. In this chapter we will follow [1] and his approach to the hypernormalisation. When the vector field is in classical normal form the system has the following form:

$$v = w_0 + \sum_{i=1}^{\infty}(\alpha_i u_i + \beta_i w_i)$$

with real coefficients $\alpha_i,\ \beta_i$ and u_i and w_i defined as below.

$$u_i = (x^2+y^2)^i \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textit{and} \qquad w_i = (x^2+y^2)^i \begin{pmatrix} -y \\ x \end{pmatrix}.$$

In the introduction non-uniqueness of the normal form was mentioned. The different ways that the normal form can not be unique have been suppressed. In the calculation of the first order normal form the step was to find a basis for the kernel of the normal form. Choosing a different basis for the kernel will result in a different normal form. For the other planar cases, especially the Bogdanov-Takens case, this is essential to deriving the first order normal form. A natural question is what the best choice of basis is and to answer that question a whole new chapter should be devoted to the answer. This will not be done here and it is best to assume that the choice of kernel is just a convention. The interested reader can read Appendix A of [3]

The second way that normal forms are not unique are worse in the sense that even if a standard basis for the kernel is used, a different transformation can lead to a different normal form, in the sense that the coefficients do not match up. For instance, [1] gives a good example of two different transformations that yield two different normal forms for the same systems. The transformations used can be found there and are rather complicated, but a summary of his results will be given.

Consider the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y + x^2 + xy \\ -x + y^2 \end{pmatrix} \tag{2.1}$$

Then a classical normal form is obtained using Maple in [1]:

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{8}r^3 - \frac{1}{144}r^5 - \frac{63973}{663552}r^7 + O(9) \\ 1 - \frac{1}{6}r^2 + \frac{451}{6912}r^4 + \frac{74699}{829440}r^6 + O(9) \end{pmatrix}. \tag{2.2}$$

However a different transformation yields the normal form:

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{8}r^3 - \frac{1}{144}r^5 - \frac{55525}{663552}r^7 + O(9) \\ 1 - \frac{1}{6}r^2 + \frac{451}{6912}r^4 + \frac{27901}{829440}r^6 + O(9) \end{pmatrix}. \tag{2.3}$$

Both transformations can be found in Appendix A of [1]. Loosely speaking, in the kernel of each homological operator, there is "room" left for an arbitrary choice of some of the coefficients in the transformation. A different choice of coefficients will result in a different normal form. This "freedom" can be used to our advantage by choosing the coefficients in such a way that more terms become zero. The statements are admittedly rather handwavy, but it will be clear after reading the next section and in particular the next chapter.

It should also be noted that in both equations the coefficients up to order 5 match. In the next section we will show that the classical normal form for this system is uniquely determined up to order 5 and any term with higher degree can be completely removed. The hypernormal form up to any order will be calculated in the example below.

2.2 The hypernormal form for the Hopf singularity

The last section hinted that further simplifications could be made. In the following section the suggestion is further investigated. This will be done using a brute force method assuming nothing and applying the most general transformations possible. The advantage is that it requires no abstract framework and gives good insight into the mechanics of normal form theory. The disadvantage is that the calculations are tedious and there is no generality and this brings the advantage of the abstract framework: it allows statements for general Lie algebra's which allows for normalisation of a great deal more examples.

Remark 3. *At this stage a general transformation is used to transform the system. In the next chapter, further refinement of the theory will result in a simplification of the transformation. The main reason is that the first order normal form forms a Lie algebra. The theory will be developed for general Lie algebra's and this allows us to work within this smaller Lie algebra.*

The third order transformation has the following form:

$$\begin{aligned} x &= \xi + c_{130}\xi^3 + c_{121}\xi^2\eta + c_{112}\xi\eta^2 + c_{103}\eta^3 \\ y &= \eta + c_{230}\xi^3 + c_{221}\xi^2\eta + c_{212}\xi\eta^2 + c_{203}\eta^3 \end{aligned}$$

In order to make the calculation more visible write it in the form:

$$x = h_1(\xi, \eta), \qquad y = h_2(\xi, \eta)$$

, then:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} D_x h_1(\xi,\eta) & D_y h_1(\xi,\eta) \\ D_x h_2(\xi,\eta) & D_y h_2(\xi,\eta) \end{pmatrix}^{-1} \begin{pmatrix} \dot{x}(\xi,\eta) \\ \dot{y}(\xi,\eta) \end{pmatrix}$$

To make sure no confusion can arise: the α_1 and β_1 are from the original system. Then the a_1 and b_1 are the coefficients from the new system and our hope is that both can be set to zero. This is done in the following way: if the matrix is invertible, the transformation variables can be chosen in such a way that after matrix multiplication the result is equal to the old variables. It follows that all the new variables should be zero. Let the general transformed system have the following form:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \end{pmatrix} = w_0 + a_1 u_1 + b_1 w_1 + \dots$$

After working out the straightforward substitution the result is the following set of 8 equations:

$$\begin{aligned}
-c_{121} - c_{230} &= \alpha_1 - a_1 \\
3c_{130} - 2c_{112} - c_{221} &= \beta_1 - b_1 \\
2c_{121} - 3c_{103} - c_{212} &= \alpha_1 - a_1 \\
c_{112} - c_{203} &= \beta_1 - b_1 \\
c_{221} - c_{130} &= \beta_1 - b_1 \\
3c_{230} - 2c_{212} + c_{121} &= \alpha_1 - a_1 \\
-2c_{221} + 3c_{203} - c_{112} &= \beta_1 - b_1 \\
c_{212} + c_{103} &= \alpha_1 + a_1
\end{aligned}$$

The above equations are linear and hence can be put in matrix form. Notice that this is exactly the same matrix as before, completely as expected. The matrix is:

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 & 0 & 0 & 0 \\ 3 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{130} \\ c_{112} \\ c_{221} \\ c_{203} \\ c_{230} \\ c_{212} \\ c_{121} \\ c_{103} \end{pmatrix} = \begin{pmatrix} \alpha_1 - a_1 \\ \alpha_1 - a_1 \\ \alpha_1 - a_1 \\ \alpha_1 - a_1 \\ \beta_1 - b_1 \\ \beta_1 - b_1 \\ \beta_1 - b_1 \\ \beta_1 - b_1 \end{pmatrix} \tag{2.4}$$

After a quick calculation it turns out that the matrix is not invertible and the hope of setting all third order terms equal to zero vanishes. The matrix has rank 6 and hence 6 transformation variables can be chosen independently. The kernel is spanned by two vectors and also two variables can be choses arbitrarily. They will have effect on the fifth order terms and will contribute to the removal of those terms. The two variables chosen to be arbitrary are c_{203} and c_{103}. Then the transformation has the following form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_{103}u_1 + c_{203}w_1.$$

Remark 4. *The result above could have been derived in two ways that do not require any calculations. First, by the calculation of the first order normal form the kernel was to be determined to be exactly the set of this form. The next transformation should lie in should be a linear combination of elements that lie in the kernel of the first homological operator. If not, a third round of transformations could be applied and would result in the removal of any term not in the kernel. Using this argument, it follows that the transformation should have the above form.*

A second way of deriving this result is by using the structure of the ring of formal vector fields, which forms a Lie algebra under the bracket operation. When the linear part of the vector field is semisimple the resulting kernel of the homological forms a Lie subalgebra under the Lie bracket. This implies that the calculations can be done in this smaller Lie algebra and then the result is that the coefficients should have the above form. Unfortunately an approach using Lie algebra's is bound to wait until the next chapter.

Since the third order terms can not be removed, the focus shifts to the fifth order terms. In the previous calculation two variables remained free to choose, so their effect on the fifth order terms is calculated. This is the main difference between the transformations of the classical normal form and the calculations done now. The general transformation of the fifth order terms is of the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} + c_{203}u_1 + c_{103}w_1 + \sum_{k=0}^{5} \begin{pmatrix} c_{1(5-k)(k)}\xi^{5-k}\eta^k \\ c_{2(5-k)(k)}\xi^{5-k}\eta^k \end{pmatrix}.$$

A calculation similar to the calculation done with the third order terms yields the following twelve by twelve matrix, when put in the reduced row echelon form:

$$\begin{pmatrix} 1&0&0&0&0&-1&0&0&0&0&0&0\\ 0&1&0&0&0&-2&0&0&0&0&0&0\\ 0&0&1&0&0&-1&0&0&0&0&0&0\\ 0&0&0&1&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&-2&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&0&0&0&2\\ 0&0&0&0&0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&-1\\ 0&0&0&0&0&0&0&0&0&0&1&-2\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} c_{150}\\ c_{132}\\ c_{114}\\ c_{241}\\ c_{223}\\ c_{205}\\ c_{250}\\ c_{232}\\ c_{214}\\ c_{141}\\ c_{123}\\ c_{105} \end{pmatrix} = \begin{pmatrix} -8C_1\\ -31C_1\\ -33C_1\\ 7C_1\\ 9C_1\\ 24C_1\\ -8C_2\\ -31C_2\\ -33C_2\\ -7C_2\\ -9C_2\\ -24C_2 \end{pmatrix} \tag{2.5}$$

with:

$$C_1 = \frac{1}{15}(2\alpha_1 c_{203} - 2\beta_1 c_{103} + \alpha_2 - a_2) \tag{2.6}$$

$$C_2 = \frac{1}{15}(\beta_2 - b_2). \tag{2.7}$$

The difficult part is to interpret the equation. The coefficients c_{***} should lie in the kernel of the homological operator. If not, extra terms will be added instead of removed. The rank of the matrix is ten and two variables are free to be chosen, just as was done for the third order. The difference is that in the right hand side of the equation two more variable can chosen independently, c_{203} and c_{105}. If they can be chosen such that the right hand side is moved to the image of the homological operator, then there exists a right inverse for the equation. The matrix is already put in reduced row echelon form and it can be seen that the right hand side is in the image if $C_1 = C_2 = 0$. From $C_1 = 0$ it follows that a_1 can be set zero if $\alpha_1 \neq 0$ or if $\beta_1 \neq 0$. From $C_2 = 0$ it follows that $\beta_2 = b_2$. Hence it follows that $a_2 = 0$.

As a last observation: from the matrix it should be clear that the fifth order terms can not be removed and the following relations hold.

$$c_{150} = \frac{1}{2}c_{132} = c_{114} = c_2 41 = \frac{1}{2}c_{223}$$

and

$$-c_{250} = -\frac{1}{2}c_{232} = -c_{214} = c_{141} = \frac{1}{2}c_{123} = c_{105}$$

The pattern that appears is that the transformations should be having the same form as the first order normal form. This will be proven in the next chapter, but follows from Remark 4. It will, without the need of matrices, yield the two equations that need to be set zero. They also can be derived by the method described above.

The result in matrix form is:

$$\begin{pmatrix} 2\alpha_1 & 0\\ -2\beta_1 & 4\alpha_1 \end{pmatrix} \begin{pmatrix} c_{205}\\ c_{105} \end{pmatrix} = \begin{pmatrix} \alpha_3 - a_3 - \frac{1}{2}\beta_2 c_{103}\\ \beta_3 - b_3 + \beta_1 c_{103}^2 + \alpha_2 c_{103} \end{pmatrix} \tag{2.8}$$

From these equations it is easy to see that the appropriate choice of c_{105} and c_{205} both b_3 and a_3 can be set zero. The matrix equation is not a coincidence and for each degree the same equation will arise but with a different right hand side which depends on the known variables. If they are denoted by $P_1(...)$ and $P_2(...)$ the result will be always invertible if $\alpha_1 \neq 0$

The next theorem can be found in [1].

Theorem 4. *When $\alpha_1 \neq 0$, the normal form up to all orders is given by:*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = w_0 + \alpha_1 u_1 + \beta_1 w_1 + \alpha_2 u_2 \tag{2.9}$$

Proof. So far, none of the terms could be eliminated. A similar calculation for the seventh order terms will show that the they are completely removable and the fifth order terms make sure the seventh order terms can be removed. To prove that from seventh order terms everything can be eliminated, an induction argument is used. The pattern of transformations suggested by the third and fifth order is one of the form:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} + c_{203}u_1 + c_{103}w_1 + c_{205}u_2 + c_{105}w_2 + \cdots + c_{20(2k+1)}u_k + c_{10(2k+1)}w_k + \ldots \tag{2.10}$$

At each degree, the kernel of the homological operator has dimension two and hence two variables are free. Take the seventh order terms to be the first terms that can be completely removed as the start of the mathematical induction. Now we assume that it worked for degree $(k-1)$ with k even and show that the $(k+1)$ order terms can be fully removed. To calculate the coefficients $c_{20(2k+1)}$ and $c_{10(2k+1)}$, we substitute the transformation and collect all terms of degree $2k+1$. The result is:

$$c_{20(2k+1)} = \frac{1}{2(k-1)\alpha_1}(\alpha_k - P_1(...)) \tag{2.11}$$

$$c_{10(2k+1)} = \frac{1}{2k\alpha_1}(\beta_k + 2\beta_1 c_{20(2k+1)} - P_2(...)) \tag{2.12}$$

$$\tag{2.13}$$

Then rewriting this in matrix form yields:

$$\begin{pmatrix} 2(k-1)\alpha_1 & 0 \\ -2\beta_1 & 2k\alpha_1 \end{pmatrix} \begin{pmatrix} c_{20(2k+1)} \\ c_{10(2k+1)} \end{pmatrix} = \begin{pmatrix} P_1(...) \\ P_2(...) \end{pmatrix} \tag{2.14}$$

□

Since α_1 is nonzero, the matrix is invertible and a solution exists.

2.3 Concluding remarks

In the next chapter some of the details that can be verified by hand, will be proven using a little extra machinery because it gives more insight into the calculations. The proof given here is for the Hopf singularity, but will also generalised for the Generalised Hopf singularity.

Chapter 3

General Problem

3.1 Introduction

In the previous section the classical normal form was derived. Historically seen questions of uniqueness were asked. Moreover, after the transformation removed certain terms, are these the transformations that remove the maximal amount of "uninteresting" terms? The answer to this question is generally no, this is not true. The reason we did not find the simplest normal form is that the transformation was not "general" enough. In order to simplify it even further, a second round of transformations needed to be applied. This is also the big picture of the next section. In [1], Yu derives the simplest normal form by a substantial amount of calculation and an induction argument. For someone that has seen normal form theory in terms of its calculations this provides a great introduction to the theory of hypernormal forms. However it is not the most efficient approach. A constant factor is the need to simplify calculations since they always seem to find a way to get out of hand rather quickly. Abstract theory is needed to provide with the framework to manage the transformations and prove quite strong results. The main example is the Generalised Hopf singularity. The other planar singularities are even more dependent on this abstract framework. For the Hopf singularity we are able to prove that the whole system can be reduced to a system with only finitely many terms. So far, terms like "simplest" and "remove as many terms" are used without ever giving any hint as to whether a simplest normal form even existed in all cases. Neither was it explained what this simplest normal form even looked like. In this chapter this will be shown and proven.

3.2 Generating the transformations

Suppose a linear differential equation $\dot{x} = Ax$ is studied and it's normal form is sought. The idea of normal form theory is to apply a sequence of diffeomorphisms that transforms the system in such a way that as many variables are set zero. If $A = X^{-1}BX$, then A and B have the same eigenvalues and hence the same dynamical behaviour. Now the whole purpose of normal form theory becomes clear. The set of matrices with the same eigenvalues form an equivalence class under the similarity operator. Now normal form theory is all about finding simplest representative of this equivalence class. In the linear case it is obvious that the Jordan canonical form is this simplest representative of the equivalence class and it is unique.
Now return to the original question, that of normalising a nonlinear vector field with Hopf singularity. To do this we have to find the representative of the equivalence class of vector fields that the vector field under investigation belongs to. Baider proved in [4] that when the vector field has a diagonalisable linear part, there exist exactly four equivalence classes and exactly one simplest representative of each class. For the rest of the section it is assumed that such a simplest representative is already found. The remaining question is to prove that the original vector field can be transformed to the new simpler vector field and to show how this is done.
Suppose a map $f : M \to M$ is given together with a diffeomorphism $g : M \to N$ which can be seen as a change of coordinates. To express the map f in terms of the new coordinates $y = g(x)$ first the inverse needs to be applied, $x = g^{-1}$. Then the map f. Afterwards the map g is applied to carry the result over to the

new system. The result is $g \circ f \circ g^{-1} : N \to N$. One recognizes the conjugation operator in this and this is exactly how it should be viewed.
Since normal form theory works with the formal expansion of the original system, it is natural also to work with the formal expansion of the diffeomorphisms as well. The problem with doing this is that it works one way. Each diffeomorphism defines a smooth vectorfield, but every smooth vectorfield only generates a local flow. Hence it follows that the normal form calculations are only valid in a neighbourhood of the origin.

Definition 4 (Conjugation operator)**.** *Given two flows φ^t and ψ^s, the conjugation operator is given by:*

$$\mathtt{C}_\psi \varphi = (\psi^s)^{-1} \circ \varphi^t \circ \psi^s$$

The way to look at this operator is as a map that takes one vector field and calculates the coordinate change directly. The problem with the conjugation is that usually the flow is not at all known and only the similarity operator is used w.r.t. a to be chosen vector field that simplifies the old system.

Definition 5 (Similarity operator)**.** *The flow generated by the conjugation operator is defined by:*

$$b(x) = (\mathtt{S}_\psi a)(x) = \frac{d}{dt}(\mathtt{C}_\psi \varphi^t)(x)\bigg|_{t=0} = D\psi(x)^{-1} a(\psi(x))$$

The above formula needs a proof:

Proof. For the second equality the proof of [3] page 180 is followed. Note that $\psi^{-1}(\psi(x)) = x$ implies that $D(\psi^{-1})(\psi(x)) = (D\psi)^{-1}(x)$
Now

$$\frac{d}{dt}\psi^{-1}(\varphi^t(\psi(x))) = D\psi^{-1}(\varphi^t(\psi(x)))\frac{d}{dt}\varphi^t(\psi(x))$$

and setting $t = 0$ yields the required results. □

Definition 6 (Lie operator)**.** *Define the Lie operator as:*

$$(\mathtt{L}_b a)(x) = \frac{d}{ds}(D\psi^s)^{-1} a(\psi^s)\bigg|_{s=0} = Da(x) \cdot b(x) - Db(x) \cdot a(x) = [a, b]$$

Proof. The proof is similar to the proof of the similarity operator and can be found in [3] on page 181 (Lemma 4.4.7). □

Then the final result is that the vector field $b(x)$ is generated by an exponential expressed in terms of Lie brackets:

Theorem 5 (Lie series for vector fields)**.** *The power series expansion of the similarity operator is given by the formal power series:*

$$(D\psi^s)^{-1} a(\psi_s(x)) \sim e^{s\mathtt{L}_b} a(x) = a(x) + s\mathtt{L}_b + s^2 \mathtt{L}_b^2 a(x) + \cdots \tag{3.1}$$

$$= a(x) + [b(x), a(x)] + \frac{1}{2}[b(x), [b(x), a(x)]] + \frac{1}{6}[b(x), [b(x), [b(x), a(x)]]] + \ldots \tag{3.2}$$

Proof. See [3] on page 184 proof that proves the theorem by a calculation. □

3.3 Lie theory

The first thing that needs attentions is the connection between the theory of Lie algebra's and the normal form theory. Someone who has read [3] might wonder why in chapter 3 he starts with an analysis of the linear vector fields, develops theory for it then just brings it over to the nonlinear case. The reason is that both form a Lie algebra and since the normal form is always with respect to a Lie algebra, it works.

Definition 7 (Lie algebra)**.** *A Lie algebra is a vector space V together with a bilinear operator $[\cdot,\cdot] : V \times V \to V$ satisfying:*

- *Bilinearity*
- *Skew-symmetry*
- *The Jacobi-identity*

The set of formal polynomial power series forms a Lie algebra under the Lie bracket defined as:

$$[a(x), b(x)] = Da(x) \cdot b(x) - Db(x) \cdot a(x)$$

The proof is rather simple and can be done by hand. In the previous sections we always worked with the homogeneous terms of degree k. This has a natural abstraction and is called a grading. It partitions the space of vector fields into a countable sum of finite dimensional vector fields. It also allows for a natural filtration, taking the set of vector fields starting with grade k and higher order terms. Both of these definitions will arise in the rest of the chapter and are rather important.

The normal form works on any Lie algebra with a grading defined on it. Two criteria should be satisfied, the first being a grading and the second that each grade is a finite dimensional vector space. So, given a sequence of finite dimensional vector spaces, V_k, the graded Lie algebra is defined as

$$V = \prod_{k=0}^{\infty} V_k.$$

The Lie bracket has a particularly interesting structure defined on it:

$$[V_n, V_m] \subset V_{m+n}$$

This seems rather abstract, but a lot of examples exist for these vector spaces exist. The simplest example is the set of power series, the finite dimensional vector spaces are defined as the homogeneous terms H_k in the first chapter and it can be verified that all properties are full-filled. The second example already encountered in the previous example was the linear differential equation depending on one parameter was encountered in the second chapter. In this case the Lie bracket is defined as the commutator bracket and maps within the various degrees of ε.

Definition 8 (Set of generators)**.** *Let $g \in V$ be of the form*

$$g = g_1 + g_2 + \cdots, \qquad g_k \in V_k$$

then g is called a generator of the transformations. The collection

$$g \in G := \prod_{k \geq 1} V_k \subset V$$

is called the set of generators.

In other words, all vector fields starting with quadratic terms are in this set. These generators generate the set of transformation that can put a vector field into its normal form. The goal of the rest of the chapter is to break up these generators into the smallest possible steps, so that the calculation becomes doable. So suppose that the first transformation is given by $g \in G$ and denote the original vector field by v^0. One of the goals of normal form theory is to break the transformations down in such a way that each step only normalises a part of the original problem. In doing this the calculations become manageable, but in the process of classical normal form theory the transformations are broken down in the wrong way, implying that not all transformations are used up and hence more terms can be removed. Hypernormal form theory aims to find the remaining transformations in order leverage out as many terms as possible. In the previous chapter this happened exactly when we applied the second round of transformations.

For any $v \in V$ and $g \in G$, the $\bullet$ describes the Lie exponential:

$$g \bullet v = exp(ad\ g)v = v + [g, v] + \frac{1}{2}[g, [g, v]]] + \frac{1}{6}[g, [g, [g, v]]] + \dots$$

Usually when a transformation is applies it should modify terms from a certain order and modify the higher order terms is an uncontrollable way. It is then said that g *targets* grade k in v and write $g \bullet v = v + \mathscr{O}_k$. In simple words it means that g can be used to transform v_j without modifying any lower order terms.

Theorem 6 (Campbell-Baker-Hausdorff theorem)**.** *Suppose the above action of the the set G is given on the Lie algebra V. Then an operation $*$ exists such that the set G becomes a group and the Lie exponential is the action of G on V.*

Proof. The proof of this theorem is complicated and comes from general Lie theory. The proof nor the explicit formula of $*$ give more insight into the theory presented here and are therefore omitted. □

The operator define in the above theorem is the Campbell–Baker–Hausdorff formula and it is rather complicated. For the use of normal forms it is rather unnecessary to know what is looks like, but rather the fact that it is a group and its action are used. The following two subsets are those that encode the necessary information.

$$G_{jk} = \{g \in G | g = \mathscr{O}_j \ and \ g \bullet v = v + \mathscr{O}_k\}$$

and the second is a subspace of this and is defined as

$$\Gamma_{jk} = \{g \in G | g = g_j + \dots + g_k \ and \ g \bullet v = \mathscr{O}_k\}$$

For both sometimes a different notation is used and it is written as: $G_k^l = G_{(k-l)k}$ and similarly for Γ. The upper number of the is called the lag and determines the jet that we normalise to. So in the case that $l = 0$, the normalisation is done with respect to the zero-jet, which is nothing more then the linear part of the vector field. By increasing the jet to a higher number, the amount of "room" in the set of transformations increases, but the calculations increase dramatically. The golden question is how we can minimalise the lag while still removing as many terms as possible.
The G_{ij} provide the space of all transformations that target a specific grade and for the hypernormal form this is of particular interest since they do not modify lower order terms. This is also one of things that normal form calculations try to accomplish: to break down the immense calculation to smaller or even minimal calculation that can be done with a computer. To find the minimal set that is needed to simplify grade k the theorem below is needed. It shows the effect of G_{ij} on v. This theorem shows that in order to simplify order k only the k-jet of $g \in G_{ij}$ is needed. This set is then defined as Γ_{ij}. The next theorem shows what the action looks like in terms of the commutation relation for the target grade.

Theorem 7. *Suppose $g \bullet v = v + \mathscr{O}_k$, then*

$$g \bullet v = v + \sum_{i=1}^{k} [g_i, v_{k-i}] + \mathscr{O}_{k+1}$$

Proof. The proof of this theorem will be a straightforward calculation that shows the desired identity. By definition: $g \bullet v - v = \frac{1}{2}[g, [g, v]]] + \frac{1}{6}[g, [g, [g, v]]] + \dots$. Since $[\mathscr{O}^k, \mathscr{O}^m] \subset \mathscr{O}^{k+m}$ any double bracket will belong to $\mathscr{O}^{k+1}$. The result follows from a straightforward calculation and the fact that some of the terms should not modify lower order terms which in terms means that some brackets should be zero.

$$\begin{aligned}
[g, v] &= [g_1 + g_2 + g_3 + \dots, v_0 + v_1 + v_2 + \dots] \\
&= [g_1, v_0 + v_1 + v_2 + \dots] + [g_2, v_0 + v_1 + v_2 + \dots] + [g_3, v_0 + v_1 + v_2 + \dots] + \dots \\
&= \sum_{i=1}^{k} [g_i, v_{k-i}] + \mathscr{O}_{k+1}
\end{aligned}$$

□

Theorem 8. *A generator $g \in G$ belongs to G_{jk} when:*

$$\begin{aligned} [g_j, v_0] &= 0 \\ [g_j, v_1] + [g_{j+1}, v_1] &= 0 \\ &\vdots \\ [g_j, v_{k-j-1}] + \cdots + [g_{k-1}, v_0] &= 0 \end{aligned}$$

Proof. The proof is a repeated application of the previous theorem. Equivalently the result can be obtained by calculating the bracket of a general $g = \mathscr{O}_j$. The first effect of g is on the terms of grade j, calculating the bracket it yields $[g_j, v_0]$. If g should belong to G_{jk}, it should have no effect on the j-th order terms and hence the bracket should be zero. Also any higher order bracket with then becomes automatically zero. Working all the way up yields the result. □

Definition 9. *Define the partial action $\odot_v : G \times V_k \to V_k$ as the effect of the action $\bullet$ on the vector space V_k.*

$$g \odot_v w = w + [g_j, v_{j-k}] + \cdots + [g_k, v_0] \qquad w \in V_k$$

Define the projection π_{jk} as:

Definition 10 (The projection)**.** *For any $v \in V$, projection $\pi_{jk} v$ is defined as*

$$\pi_{jk} v = v_j + .. + v_k$$

Theorem 9. *The G_{jk} are a subgroup of G, the subgroup $G_{j(k+1)}$ is a normal subgroup of G_{jk} and the quotient is abelian. The quotient is isomorphic to:*

$$G_{jk}(v)/G_{j(k+1)}(v) \cong \Gamma_{jk}(v)/\pi_{jk}\Gamma_{j(k+1)}(v)$$

Proof. The proof can be found in [2], Lemma 6. □

In order to make the set of generators as small as possible we look at the quotient. The information that is divided out is exactly those terms that are needed for the following terms and not the current terms.

Theorem 10. *$G_{(j+1)k}$ is a normal subgroup of G_{jk} and the same quotient can be formed as in the previous theorem.*

$$G_{jk}(v)/G_{(j+1)k}(v) \cong \Gamma_{jk}(v)/\pi_{jk}\Gamma_{(j+1)k}(v)$$

Proof. This is Lemma 7 of [2] and the proof can be found there. □

For any transformation $g \in \Gamma_{ij}$ the effect is a sum of brackets. Those brackets form a linear subspace in the space V_k. For any term of degree k+1 to removed, there should be a transformation that can serve as a partial inverse. This is summarised in the following definition:

Definition 11 (Removable subspace)**.** *Let*

$$R_{ij}(v) = \{[g_j, v_{k-j}] + \cdots + [g_k, v_0] | g \in G_{jk}\}$$

In terms of the classical normal form theory the removable subspace is the image of the homological operator. The difference is that in hypernormal forms, multiple Lie brackets arise instead of one.

Theorem 11. *For any removable subspace the following holds*

$$R_{jk}(g \bullet v) = R_{jk}(v)$$

Proof. The proof comes from [2]. Let $u \in R_{jk}(v)$, then there must exist a n $h \in \mathscr{O}_j$ such that the leading term of u coincides with the leading terms of $[h, v]$. Let $\tilde{h} = g \bullet h$. Then obviously by the definitions or by working out the brackets, it is found that $\tilde{h} = \mathscr{O}_j$. Also $[\tilde{h}, g \bullet h] = g \bullet [h, v]$ with leading term u. It can be concluded that $u \in R_{jk}(g \bullet v)$ and thus that $R_{jk}(v) \subset R_{jk}(g \bullet v)$. By a similar argument the reverse can be shown. □

Definition 12. *When the removable subspaces are divided out of the V_k, the complementary spaces remain. They are denoted by C_{jk}*

So in order to calculate the hypernormal form the idea is to apply a series of transformations $g^r \in G_{1,r}(v^{r-1})$ so that the result of applying this to v^r will become $v^r = g^r \bullet v^{r-1}$. In the approach that Baider took in his paper [4], each generator started at degree one for each target grade.
Because the removable subspaces are invariant under the transformations of g, see theorem 11, the removable subspace at the r-th stage of the normalisation process is the same as the removable subspace at the first stage. Written out: $R_{jk}(v^r) = R_{jk}(v^0)$. The next step is to define a complement C^l_k and hence a decomposition of the target grade. Then $V_k = R^l_k \oplus C^l_k$. Since not all terms of v^{r-1}_k are in the image of the differential, the projection is taken.
Set $v = v^0$ and $g^r \in G^{r-1}_r$. For the first round so to say, $v^1 = g^1 \bullet v^0$ and $g^1 \in G^{11}$. The space G_{11} is described as the space of vector fields starting at degree 1, the lowest term is the linear part, and targeting the linear part.
Using the projection $P^l_k : V_k \to C^l_k$ we get:

$$g^1 \odot_0 v^0_1 = P^0_1 v^0_1.$$

The projection is made so that the transformation g can be chosen to simplify the terms as much as possible. To form the second stage the generator is chosen in the set $g \in \Gamma_{12}$ so that $g^1 = g^1_1 + g^1_2$.
Then the second round produces a second homological operator that is not the same as the first homological operator. This can be carried further to the higher order terms.

3.3.1 Classical Normal forms

Now enough abstract theory has been developed in order to reformulate the classical normal form in the new abstract language. In the first chapter quadratic transformations were used in order to simplify the quadratic terms and in the next stage the cubic transformations were used to simplify the cubic terms. In other words, for target grade k only terms of the same grade where used. The definition of G is the same and the action of G on V is also the same. Since only terms of grade k are used to simplify grade k, the subspaces G_{jk} has to have $j = k$. Only then the transformation start at degree k and simplify degree degree k. Since any terms with degree higher then k will do nothing to terms of degree k, it is best to leave them out completely. Then we arrive at the space Γ_{kk} and this is the space used in the first chapter to simplify the Hopf singularity.
Writing it out in formula's yields:

$$g^k_k \in \Gamma_{kk} = H_k.$$

So Γ_{kk} is the space of homogeneous polynomials of degree k. Let's work out the calculations for the first few terms. The vector field is given by

$$v = v_0 + v_1 + \dots$$

and the dependence on the different variables is suppressed. The linear term should be known by now for the anharmonic oscillator and the first transformation is g^1 and lies in Γ_{11} and is defined by the quadratic terms.
Using the theory of the second chapter on Lie series the partial action $\odot_v$ is found by calculating the first few terms of the Lie series and collecting everything that is of quadratic order.
Written in terms of abstract language it looks like:

$$g \odot_{v_0} v^0_1 = v^1_1$$

and in terms of the language of the first chapter it boils down to the homological operator:

$$[g_1^1, v_0] + v_0 = v_1^1.$$

Obviously the third order terms and higher order terms are done in a similar way.
The big problem is that the lower order terms have an effect on the higher order terms. When a specific grade k is removable (the homological operator is invertible) all variables are needed to leverage that specific term out of the equation. Hence no variables are left free to choose to effect the higher order terms. In the case that the homological operator is not invertible, not all variables are used in the simplification of that specific grade k. If the effect is calculated on the higher order terms they might be used for further simplification.

3.3.2 Hypernormal form

In the section covering the basic intuition of the hypernormal form it is exactly this effect of the lower order terms that was used to simplify the higher order terms in a way that after the fifth order enough freedom was found that all terms could be completely removed. To prove the result, the theory written above in an abstract way should be understood a bit more, but in this subsection the idea presented in the previous section is restated in the abstract language.

The starting point of [1] is that a most general transformation of degree k and lower order terms is chosen without looking at the effect of those terms on the lower order terms. Then by a quite tedious calculation he restricts the transformations so that they do not modify the lower order term. This equivalent to the restriction that:

$$\begin{aligned}
[g_j, v_0] &= 0 \\
[g_j, v_1] + [g_{j+1}, v_1] &= 0 \\
[g_j, v_{k-j-1}] + \cdots + [g_{k-1}, v_0] &= 0.
\end{aligned}$$

But this exactly the restriction that $g \in G_{jk}$ with $j = 1$ and truncated at the k-th degree. Then the generator belongs exactly to Γ_{1k}.
In order to simplify the coefficients the following theorem is quite useful:

Theorem 12. *When the linear part of the vector field is semisimple, the first order normal form is a Lie subalgebra of the Lie algebra of formal vector fields.*

Proof. The statement can be verified by calculating the Lie brackets and showing that it is closed under the Lie bracket. □

Two remarks at place at this stage. The fist is that in general it is a bad idea to solve the whole description problem at each stage of simplification. The only exception is when the linear part is semisimple. By the theorem the result is a new Lie algebra and hence problem may be seen as new normalisation problem with coefficients chosen in this new Lie algebra. It is also the reason that the normal form is presented in terms of Lie algebra's instead of vector fields. Now we have the machinery at hand to normalise with respect to this new Lie algebra. Implicitly this was also used in [1]. He first chose the most general transformation, started calculating and at some derived that the transformation had to have the same form as the classical normal form. At some point the patterns was spotted and used in the succeeding terms, without proof.

3.4 Refinements of the methods

In the following section the method Baider originally used in his method is presented after which the theory is refined to a degree such that the anharmonic oscillator can be treated and the results be proven. Baider (see [4]) used maximum lag in his approach to normalise vector fields and hence certainly found the unique

normal form, but his method was quite computationally expensive. It was used in [1] to compute the simplest normal form to infinite order. In that case it worked and simplest normal form could be found, but a smaller set of generators can be used to find the simplest normals form.
To this end we are going to follow section 5 of [2] and define the extended partial normal form in the following way. Instead of following Baider and normalise the r-th order terms with respect to the (r-1)-jet, we now normalise with respect to a fixed jet. It is denoted by $\tilde{v}^r$ and in the best example of this extended partial normal form is the classical normal form. Now the 1-jet is taken to normalise the terms of any order. The classical normal form is denoted by $\tilde{v}^1$.

The second idea is also a deviation from the method Baider used. Instead of generating the transformation of v^r all at once, the calculations are broken down into smaller steps. To calculate v^r, v^r_r is first normalised with respect to the zero jet, then to the 1-jet and so forth so that in the end it is normalised with respect to the $(r-1)$-jet. Two main objections to this constructions: is the works done in the first step not redone in the second step and is this not more work then just generating it all at once? Both questions will be addressed and for both it turns out that it is not true.
The advantage, given the above is true, is that this way the minimal lag is used in order to remove as many terms with the minimal lag in order to reduce the amount of calculation needed.

Let us define $v_r^{r-1,l}$ to be the terms of grade r and soon to be normalised with respect the $(l-1)$-jet. This way we get a sequence of substages and it will end with v^r. It starts at $v_r^{r-1,0} = v_r^{r-1}$ and end with the terms $v_r^{r-1,r} = v_r^r$.
Now in order to give the concrete construction the following subspaces are defined:

$$F_r^l = F_{r-l,r} = \frac{\Gamma_{r-l,r}(v^{r-1})}{\Gamma_{r-l+1,r}(v^{r-1}) + \pi_{r-l,r}\Gamma_{r-l,r+1}} \tag{3.3}$$

First of all, these are the subspaces that will help us to prove the idea that had arisen in the previous chapter. The first term in the factor is $\Gamma_{r-l+1,r}(v^{r-1})$, this is the set of power series starting at degree $r-l+1$ targeting grade r. This is the next step in the process and is not needed right now and is divided out. The second term in the quotient is $\Gamma_{r-l,r+1}$ and is the set of power series that target grade $r+1$. These terms are also not necessary at the moment and also need to be divided out. The last space is is the space of transformations starting at grade $r-l$ and targeting grade r.

Given these spaces F_k^l it is possible to define $\bar{F}_k^l$ as the space obtained by removing the whole quotient. Given this construction it now is possible to break down Baider's method even further by the following decomposition:

$$\overline{\Gamma_{1k}(v)/\pi_{1k}\Gamma_{1,k+1}(v)} = \bigoplus_{j=1}^{k} \bar{F}_{jk}(v) \tag{3.4}$$

Using a similar decomposition we can define:

$$R_{1k}(v) = \bigoplus_{j=1}^{k} S_{jk} \tag{3.5}$$

$$S_{jk} = \{[g_j, v_{k-j}] + \cdots + [g_k, v_0] | g \in \bar{F}_{jk}(v)\} \tag{3.6}$$

3.5 The Hopf Singularity

3.5.1 Preliminary treatment and setup

The normal form theory developed in the previous sections is abstracted to the point that any graded Lie algebra can be brought into a unique normal form. The starting point for the hypernormalisation of the Hopf singularity is the classical normal form. When it is brought in this form the vector field forms a Lie algebra. This implies that the space of transformations is greatly reduced since we can take the transformations from this reduced space instead of the whole space.

In the case that the linear part of the original vector field is semisimple, the classical normal form to all degree's forms a Lie subalgebra that allows a new normalisation problem to be defined on this reduced problem. The generators that simplify this problem lie in this Lie subalgebra.
When the Hopf singularity is in classical normal form, we have the following system:

$$v = w_0 + \sum_{i=1}^{\infty}(\alpha_i u_i + \beta_i w_i)$$

with the α and β in the real coefficients and u_i and w_i defined as below.

$$u_i = (x^2 + y^2)\begin{pmatrix} x \\ y \end{pmatrix} \qquad \textit{and} \qquad w_i = (x^2 + y^2)\begin{pmatrix} -y \\ x \end{pmatrix}$$

The system defined by the above vector fields are in Cartesian coordinates and the system defined in the section on semisimple normal forms is in complex coordinates.

Theorem 13. *The vector fields spanned by u_i and w_i span a Lie algebra.*

Proof. □

Since the bracket relation between the vector field are crucial in the calculation of the normal form, they are calculated below. The general term for these relations are structure constants.

$$\begin{aligned} [u_i, u_j] &= 2(i-j)u_{i+j} \\ [u_i, w_j] &= -2jw_{i+j} \\ [w_i, w_j] &= 0 \end{aligned}$$

Proof. The proof is a straightforward calculation that proceeds as follows:

$$Du_k = \begin{pmatrix} 2kx^2(x^2+y^2)^{k-1} + (x^2+y^2)^k & 2kxy(x^2+y^2)^{k-1} \\ 2kxy(x^2+y^2)^{k-1} & 2ky^2(x^2+y^2)^{k-1} + (x^2+y^2)^k \end{pmatrix} \tag{3.7}$$

$$Dw_k = \begin{pmatrix} -2kxy(x^2+y^2)^{k-1} & -2ky^2(x^2+y^2)^{k-1} - (x^2+y^2)^k \\ 2kx^2(x^2+y^2)^{k-1} + (x^2+y^2)^k & 2kxy(x^2+y^2)^{k-1} \end{pmatrix} \tag{3.8}$$

Then by the definition:

$$[u_i, u_j] = Du_i \cdot u_j - Du_j \cdot u_i \tag{3.9}$$

$$[u_i, w_j] = Du_i \cdot w_j - Dw_j \cdot u_i \tag{3.10}$$

$$[w_i, w_j] = Dw_i \cdot w_j - Dw_j \cdot w_i \tag{3.11}$$

Working out these formula's yields the required relations

□

Since the normal form is given in terms of indeterminate coefficients, different cases need to be considered and in each case the simplest normal form has to be calculated. This is done using the theory developed in the previous section. In general there are 4 cases to consider.

Theorem 14. *When the Hopf singularity has been brought in classical normal form, it falls in exactly 4 cases:*

1. *All α_i and β_i are 0,*

2. *All $\alpha_i = 0$ and $\beta_\mu \neq 0$ is the first nonzero term,*

3. *For $\nu \in \mathbb{N}$, α_ν is the first nonzero term and $\beta_i = 0$ for $i \leq \nu$,*

4. *For $\nu \in \mathbb{N}$, α_ν is the first nonzero term and there is a first nonzero β_μ with $\mu \leq \nu$,*

For each of the cases, the system can be put in one of the corresponding forms:

1.
$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.12}$$

2.
$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + \beta_\mu r^{2\mu} \end{pmatrix} \tag{3.13}$$

3.
$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha_\nu r^{2\nu+1} + \alpha_2 r^{4\nu+1} \\ 1 \end{pmatrix} \tag{3.14}$$

4.
$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \alpha_\nu r^{2\nu+1} + \cdots + \alpha_{2\nu-\mu} r^{4\nu-2\mu+1} + \alpha_{2\nu} r^{4\nu+1} \\ 1 + \beta_\mu r^{2\mu} \end{pmatrix} \tag{3.15}$$

The rest of the chapter will consist of the proof of this theorem. The first case is trivial and needs no proof, the simplest normal form has already been obtained.

Case 4: The analysis of the general case breaks down into several cases that should be considered. The general tactic is that the various spaces are calculated for varying l until all terms can be removed, if this is the case. To do this, for each l, a parameter n needs to be calculated for values $0 \leq n \leq l$ and thus it is best first to perform an analysis on how this is done int general. Later the implications for each l are discussed. Central is the quotient (3.3) to the story.
The setting is that there is a smallest β_μ and α_ν with $\mu \leq \nu$ that are nonzero. The line of thought is that we calculate the subspaces F_k^l for $l = 0, \ldots$. The first two nonzero element are the ones that need to leverage out the rest of the terms in the system.
For the discussion the general bracket of the vectorfield with the various u's and w's is slightly modified in term of indexing, as the need will arise for letting n vary for the different l:

$$[u_{k-n}, w_0 + \beta_\mu w_\mu + \alpha_\nu u_\nu] = -2\beta_\mu w_{\mu+k-n} + 2(k-n-\nu)\alpha_\nu u_{k-n+\nu} \tag{3.16}$$
$$[w_{k-n}, w_0 + \beta_\mu w_\mu + \alpha_\nu u_\nu] = 2(k-n)\alpha_\nu w_{k-n+\nu} \tag{3.17}$$

The analysis starts by letting $n \geq 0$. The bracket is then given by (3.17) and following analysis holds true. For $n \leq \mu$ the w-term in the image of (3.16) has grade $k + \mu - n \geq k$, so then u_{k-n} targets grade k. Because the w terms has the lowest degree of the two. Then for $n \leq \mu - 1$, u_{k-n} also targets grade $k+1$. For the u term in the image a separate analysis should be made, as if the w term does not exist. For $n \leq \nu$, u_{k-n} targets grade k and for $n \leq \nu - 1$ grade $k+1$ through the $u_{k-n+\nu}$-term.

Now look at (3.17). For $n \leq \nu$ this targets grade k. The first part of the proof consists of a verification of the following result:

$$
\begin{aligned}
F_k^0 &= \{0\} \\
\vdots \quad & \quad \vdots \\
F_k^\mu &= \operatorname{span}\{w_{k-\mu} \ if \ k > \mu\} \\
\vdots \quad & \quad \vdots \\
F_k^{2\nu-\mu} &= \operatorname{span}\{u_k \ if \ k > 2\nu - \mu\}
\end{aligned}
$$

When this result is proven, the removable subspaces can be calculated and then it will be clear what the hypernormal form looks like. The proof will be done by a calculation of the various Γ's.

The space $\Gamma_{k-l,k}$ Given the previous discussion it follows that we need to vary $0 \leq n \leq l$ and collect all terms that target grade k. There are 3 distinct cases:

$l \leq \mu$ All u_{k-n} terms belong to $\Gamma_{k-l,k}$ also all w_{k-n} terms belong to $\Gamma_{k-l,k}$.

$\mu < l \leq \nu$ Now all w_{k-n} terms belong to $\Gamma_{k-l,k}$ and only those u_{k-n} term belong to $\Gamma_{k-l,k}$ that have $n \leq \mu$

$l > \nu$ Only those w_{k-n} terms belong to $\Gamma_{k-l,k}$ that have $n \leq \nu$ and only those u_{k-n} term belong to $\Gamma_{k-l,k}$ that have $n \leq \mu$

The space $\Gamma_{k-(l-1),k}$ The analysis proceeds the same way as the first space, but he difference is that now for each different l, $0 \leq n \leq l-1$. The only difference is that we stop looking at the l-th term and thus the indexing is a bit different.
Now the different cases are:

$l \leq \mu + 1$ All u_{k-n} terms belong to $\Gamma_{k-l+1,k}$ also all w_{k-n} terms belong to $\Gamma_{k-l+1,k}$.

$\mu + 1 < l \leq \nu + 1$ Now all w_{k-n} terms belong to $\Gamma_{k-l+1,k}$ and only those u_{k-n} term belong to $\Gamma_{k-l+1,k}$ that have $n \leq \mu - 1$

$l > \nu + 1$ Only those w_{k-n} terms belong to $\Gamma_{k-l+1,k}$ that have $n \leq \nu + 1$ and only those u_{k-n} term belong to $\Gamma_{k-l+1,k}$ that have $n \leq \mu + 1$

The space $\Gamma_{k-l,k+1}$ In the same manner, the indexing changes this time again a little bit:

$l \leq \mu - 1$ All u_{k-n} terms belong to $\Gamma_{k-l,k+1}$ also all w_{k-n} terms belong to $\Gamma_{k-l,k+1}$.

$\mu - 1 < l \leq \nu - 1$ Now all w_{k-n} terms belong to $\Gamma_{k-l,k+1}$ and only those u_{k-n} term belong to $\Gamma_{k-l,k+1}$ that have $n \leq \mu + 1$

$l > \nu - 1$ Only those w_{k-n} terms belong to $\Gamma_{k-l,k+1}$ that have $n \leq \nu - 1$ and only those u_{k-n} term belong to $\Gamma_{k-l,k+1}$ that have $n \leq \mu - 1$

In order to figure out what the quotient is for each different l, the union of the last two spaces should be taken. Since it is obvious that each space has different restrictions, a lot of cases should be considered. The tedious work is done term wise since complications will arise.

When $l \leq \mu - 1$ Then according to the above the spaces are given by the sets:

$$\Gamma_{k-l,k} = \text{span}\{u_{k-l}, w_{k-l}, \ldots, u_k, w_k\} \tag{3.18}$$
$$\Gamma_{k-(l-1),k} = \text{span}\{u_{k-(l-1)}, w_{k-(l-1)}, \ldots, u_k, w_k\} \tag{3.19}$$
$$\Gamma_{k-l,k+1} = \text{span}\{u_{k-l}, w_{k-l}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.20}$$

Then taking the projection $\pi_{k-l,k}$ is nothing more then removing all terms that have grade higher then k. Taking the direct sum yields:

$$\Gamma_{k-(l-1),k} + \pi_{k-l,k}\Gamma_{k-l,k+1} = \text{span}\{u_{k-l}, w_{k-l}, \ldots, u_k, w_k\} = \Gamma_{k-l,k} \tag{3.21}$$

It then immediately follows from the definition, see (3.3), that:

$$F_k^l = \{0\}, \qquad l \leq \mu - 1 \tag{3.22}$$

When $l = \mu$ Then according to the above the spaces are given by the sets:

$$\Gamma_{k-\mu,k} = \text{span}\{u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.23}$$
$$\Gamma_{k-(\mu-1),k} = \text{span}\{u_{k-(\mu-1)}, w_{k-(\mu-1)}, \ldots, u_k, w_k\} \tag{3.24}$$
$$\Gamma_{k-\mu,k+1} = \text{span}\{w_{k-\mu}, u_{k-\mu+1}, w_{k-\mu+1}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.25}$$

Notice that in the last term the term $u_{k-\mu}$ is missing. Taking the direct sum over the last two yields:

$$\Gamma_{k-(\mu-1),k} + \pi_{k-\mu,k}\Gamma_{k-l,k+1} = \text{span}\{w_{k-\mu}, u_{k-\mu+1}, w_{k-\mu+1}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.26}$$

Here again there is no $u_{k-\mu}$ term, but it is an element of $\Gamma_{k-\mu,k}$. So when the quotient is taken, this element remains:

$$F_k^\mu = \text{span}\{u_{k-\mu}\} \tag{3.27}$$

When $l = \mu + 1$ Then according to the above the spaces are given by the sets:

$$\Gamma_{k-\mu-1,k} = \text{span}\{w_{k-\mu-1}, u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.28}$$
$$\Gamma_{k-\mu,k} = \text{span}\{u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.29}$$
$$\Gamma_{k-\mu-1,k+1} = \text{span}\{w_{k-\mu-1}, w_{k-\mu}, u_{k-\mu+1}, w_{k-\mu+1}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.30}$$

Notice that in the last term the term u_k is missing. Taking the direct sum over the last two yields:

$$\Gamma_{k-\mu-1,k} + \pi_{k-\mu-1,k}\Gamma_{k-\mu-1,k+1} = \text{span}\{w_{k-\mu-1}, w_{k-\mu}, u_{k-\mu}, \ldots, u_k, w_k\} \tag{3.31}$$

Comparing the terms and factoring everything in (3.31) out, the result is that:

$$F_k^{\mu+1} = \{0\} \tag{3.32}$$

When $\mu + 1 < l \leq \nu - 1$ Then according to the above the spaces are given by the sets:

$$\Gamma_{k-l,k} = \text{span}\{w_{k-l}, ..., w_{k-\mu-1}, u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.33}$$
$$\Gamma_{k-l+1,k} = \text{span}\{w_{k-l+1}, ..., w_{k-\mu-1}, u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.34}$$
$$\Gamma_{k-l,k+1} = \text{span}\{w_{k-l}, \ldots, w_{k-l-1}, w_{k-l}, u_{k-l}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.35}$$

Taking the direct sum over the last two yields:

$$\Gamma_{k-l+1,k} + \pi_{k-l,k}\Gamma_{k-l,k+1} = \text{span}\{w_{k-l}, \ldots, w_{k-l-1}, w_{k-l}, u_{k-l}, \ldots, u_k, w_k\} = \Gamma_{k-l,k} \tag{3.36}$$

Comparing the terms and dividing everything in (3.31) out, the result is that:

$$F_k^{\mu+1} = \{0\} \tag{3.37}$$

When $l = \nu$ Then according to the above the spaces are given by the sets:

$$\Gamma_{k-\nu,k} = \text{span}\{w_{k-\nu}, ..., w_{k-\mu-1}, u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.38}$$
$$\Gamma_{k-\nu+1,k} = \text{span}\{w_{k-\nu+1}, ..., w_{k-\mu-1}, u_{k-\mu}, w_{k-\mu}, \ldots, u_k, w_k\} \tag{3.39}$$
$$\Gamma_{k-\nu,k+1} = \text{span}\{w_{k-\nu+1}, \ldots, w_{k-\mu-1}, w_{k-\mu}, u_{k-\mu}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.40}$$

Note that now $w_{k-\nu}$ is neither in $\Gamma_{k-\nu+1,k}$ or in $\Gamma_{k-\nu,k+1}$. So it nearly seems an element that remains. The problem is that this term also has image $w_{k-\nu}$, but all these terms where already divided out and it seems we are stuck.

$$\Gamma_{k-l+1,k} + \pi_{k-l,k}\Gamma_{k-l,k+1} = \text{span}\{w_{k-\nu+1}, \ldots, w_{k-\mu-1}, w_{k-\mu}, u_{k-\mu}, \ldots, u_{k+1}, w_{k+1}\} \tag{3.41}$$

Comparing the terms and dividing everything in 3.31 out, the result is that:

$$F_k^{\mu+1} = \{w_{k-\nu}\} \tag{3.42}$$

Now there is no need to analyse any higher orders for l because they do not belong to $\Gamma_{k-l,k}$.

Linear Combinations The key to solving the problem and hence proving the result is by looking at linear combinations of the terms in order to get more terms out of the equation.
Recall that the brackets are given by:

$$[u_{k-n}, w_0 + \beta_\mu w_\mu + \alpha_\nu u_\nu] = -2\beta_\mu \underbrace{w_{\mu+k-n}}_{**} + 2(k-n-\nu)\alpha_\nu \underbrace{u_{k-n+\nu}}_{*} \tag{3.43}$$
$$[w_{k-n}, w_0 + \beta_\mu w_\mu + \alpha_\nu u_\nu] = 2(k-n)\alpha_\nu w_{k-n+\nu} \tag{3.44}$$

The goal of next the rest of this section is to find a way such that $(*)$ is removable in the sense that it can not be divided out. The problem now is that the term $(**)$is in the way, because it targets grades lower then k. The reasoning proceeds backward as to give insight in how the thought process goes. Eventually, the goal is to find a linear combination such that when plugged in, u_k comes out the the equation. This means that $n = \nu$. But then the $(**)$ term targets grades of degree $k + \mu - \nu < k$. To combat notice that the second equation also produces $(**)$ terms. So if they get the same degree after the bracket, suitable coefficients will make sure they cancel. Since the degree is $k - n + \nu$, it follows that if we choose $n = 2\nu - \mu$, it targets grade k. The terms should be of the form $Au_{k-\nu} + Bw_{k-2\nu+\mu}$ and by inspection it should be clear that $A = 2(k-2\nu+\mu)\alpha_\nu$

and that $B = 2\beta_\mu$. This element belongs to $\Gamma_{k-2\nu+\mu,k}$, simply because of the properties that we wanted. The last thing to notice is that for $k = 2\nu$ the coefficient vanishes and hence the term cannot be removed. To verify this, the bracket relation is: $[Au_{k-\nu} + Bw_{k-2\nu+\mu}, w_0 + \beta_\mu w_\mu + \alpha_\nu u_\nu] = 4(k - 2\nu)(k - 2\nu + \mu)$. This concludes the proof. The total result is that:

$$F_k^0 = \{0\} \tag{3.45}$$

$$\vdots \quad \vdots \tag{3.46}$$

$$F_k^\mu = \text{span}\{w_{k-\mu} \; if \; k > \mu\} \tag{3.47}$$

$$\vdots \quad \vdots \tag{3.48}$$

$$F_k^{2\nu-\mu} = \begin{cases} \text{span}\{u_k\} \text{ if } k \neq 2\nu \\ 0 \text{ if } k = 2\nu \end{cases} \tag{3.49}$$

Then the following holds:

$$R_{1k}(v) = \bigoplus_{j=1}^{k} S_{jk} \tag{3.50}$$

$$S_{jk} = \{[g_j, v_{k-j}] + \cdots + [g_k, v_0] | g \in \bar{F}_{jk}(v)\} \tag{3.51}$$

By definition:

$$\overline{\Gamma_{1k}(v)/\pi_{1k}\Gamma_{1,k+1}(v)} = \bigoplus_{j=1}^{k} \bar{F}_{jk}(v) \tag{3.52}$$

So we see that when $k > \mu$ $S_{jk} = \text{span}\{u_k\}$ from the definitions and it follows that there exists a transformation that removes the u-terms. A similar reasoning show that if $k > 2\nu - \mu$ and if $k \neq \nu$ all w-terms can be removed.

Now given that this was the hardest part, the second case is easily seen as an easier version of the fourth case. Just remove the last part of the discussion.

So going to case three, the approach is largely the same. The part that we normalise to is $w_0 + \alpha_\nu$ and the discussion proceeds as below. The approach is the same and for the spaces $l < \nu$ the results are the same as when $l \leq \mu$ in case 4 and all the subspaces can be set zero. To see why, consider $n \geq 0$. For the various terms the target grade is calculated. For 3.17, the terms $w_{k-n+\nu}$ target terms of grade k and higher when $n \leq \nu$ and hence belong to $\Gamma_{k-l+1,k}$ for $0 \leq l \leq \nu - 1$. For 3.16 the same analysis holds true and hence for $l < \nu$. Then for $l = \nu$, two things happen that make this case easier than case 4. The first being that $\nu \leq \mu$ which implies that when the grade gets higher, no linear combinations need to be sought to free a u term. The bracket of 3.16 gives the term $-2\beta_\mu w_{\mu+k-n} + 2(k - n - \nu)\alpha_\nu u_{k-n+\nu}$. Now it is the u term that targets grade k and cannot be divided out at the same time. On the other hand for 3.17, by a similar reasoning the w term in the image can not be divided out.
Note that the above reasoning is a bit sketchy in nature and the reader is invited to set up a reasoning similar to case 4, which is significantly more difficult. Setting up the relations yields the following subspaces:

$$F_k^0 = \{0\} \tag{3.53}$$

$$\vdots \quad \vdots \tag{3.54}$$

$$F_k^\nu = \text{span}\{w_{k-\nu}, u_{k-\nu}\} \; if \; k > \nu \tag{3.55}$$

Then the result follows.

Chapter 4

Example of the Hopf singularity

To conclude the thesis we include an example of the Hopf singularity. The example is chosen such that α_1 and β_1 are nonzero. The theory then tells us that everything after fifth order can be removed. Also from the classical normal form, the sixth order terms can be transformed away. In total, everything with a degree higher then 5 can be completely removed and it follows that all calculations can be done modulo terms of degree 6. This example is taken from [1] and is also found in the example of Chapter 2.
The system is given by:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} x_1 - x_1^2 + x_1 x_2 \\ -x_1 + x_1^2 \end{pmatrix}. \tag{4.1}$$

Then the following transformation transforms the system to it's fifth order formal normal form:

$$\begin{aligned}
x_1 &= y_1 + y_1^2 - \frac{2}{3}y_1y_2 + \frac{1}{8}y_1^3\frac{29}{32}y_1^2y_2 - \frac{3}{8}y_1y_2^2 + \frac{29}{96}y_2^3 - \frac{259}{720}y_1^4 + \frac{493}{2160}y_1^3y_2 + \frac{11}{180}y_1^2y_2^2 + \frac{1099}{720}y_1y_2^3 && (4.2)\\
&+ \frac{37}{240}y_2^4 - \frac{2423}{51840}y_1^5 + \frac{6479}{4320}y_4^1y_2 + \frac{5701}{10368}y_1^3y_2^2 + \frac{3733}{2160}y_1^2y_2^3 - \frac{1639}{5184}y_1y_2^4 - \frac{2789}{4320}y_2^5 && (4.3)\\
x_2 &= y2 - \frac{1}{3}y_1^2 + y_1y_2 - \frac{2}{3}y_2^2 - \frac{85}{96}y_1^3 + \frac{7}{24}y_1^2y^2 - \frac{145}{96}y_1y^2 - \frac{13}{24}y_2^3 - \frac{2681}{4320}y_1^4 - \frac{53}{240}y_1^3y_2 - \frac{4}{45}y_1^2y_2^2 && (4.4)\\
&- \frac{571}{720}y_1y_2^3 + \frac{641}{480}y_2^4 + \frac{5}{16}y_1^5 - \frac{16673}{51840}y_1^4y_2 + \frac{731}{240}y_1^3y_2^2 - \frac{45227}{51840}y_1^2y_2^3 + \frac{1939}{540}y_1y_2^4 - \frac{1649}{6480}y_2^5 && (4.5)
\end{aligned}$$

and the formal normal form up to order 5 is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = w_0 - \frac{1}{8}u_1 - \frac{1}{6}w_1 - \frac{1}{144}u_2 + \frac{451}{6912}w_2. \tag{4.6}$$

Then the coefficients $\alpha_1 = -\frac{1}{8} \neq 0$ and $\beta_1 = -\frac{1}{6} \neq 0$ and by our main result we are in case 4 with $\mu = \nu = 1$. This tells us that this is the hypernormal form up to any order. In polar coordinates the result is:

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} -\frac{1}{8}r^3 - \frac{1}{144}r^5 \\ 1 - \frac{1}{6}r^2 + \frac{451}{6912}r^4 \end{pmatrix}. \tag{4.7}$$

Concluding remarks and further reading

So far the normal form was treated only using a minimal theory approach. This has the advantage that the reader has a clean view on the the main concepts in the field of normal form theory, but some topics are rather skimmed over. To remedy this, here those area's will be pinpointed. Also this section aims to provide a number of references for a further study of the normal forms. Note that this list of suggestions will be far from complete.

The first thing to addres is the description of the homological operator. The theorem is derived only for a semisimple linear part. For a general linear part the theory is a lot more complicated compared to the semisimple case. A general matrix can be decomposed into three main categories: semisimple, nilpotent and a combination of both. The semisimple case was discussed here and in order to find a description when the linear part is nilpotent the structure of the kernel needs a deeper study. It turns out that the kernel has the structure of a module, even in the semisimple case and it is this structure together with sl(2) representation theory that help describe the module structure and hence the kernel of the nilpotent case. The best introduction to this topic is found in [3] in the starred sections, treating the sl(2) representations, and in section 4.6 and 4.7 treating the module structure. To read more on the hypernormal form of the nilpotent case, the Bogdanov-Takens singularity, the original papers of the Baider and Sanders are good starting points. See [5].

The next remark that should be made is about the computation problem. So far, apart from the example above, the theory has been focussed on describing what the normal form looks like and not on how to actually compute the normal form. This is a totally new problem and is also not the focus of this thesis. The computation could be done above using ad hoc methods, because the terms were of low order. In [3], some parts treat the computation problem and it would be good to read those sections for a basic understanding.

The last chapter follow the treatment given in [2]. However, it seems to be standard to approach the subject of hypernormalisation through spectral sequences. This higly abstract approach has originally been developped to treat the Bogdanov-Takens singularity. Moreover, in the treatment of parametric normal forms it seems to be the standard approach. However intresting, room did not permit a treatment of this new and even more abstract framework. For those first encountering the method are likely to be quite daunted by it. Not only by the abstractness, but also because different authors use different notations. To give references to papers that require minimal knowledge, the first is [2] as he gives the connection between the ideas of Jan Sanders and his method of normal form theory. As the treatment in [2] is more focussed on connecting the dots rather then to provide a thorough treatment, the next treatment is [6]. He gives a very thorough treatment that is both concrete and general. Moreover he only assumes only minimal knowledge of normal forms and also minimal knowledge on spectral sequences. With this as starting point, a description is given of the hypernormal form of a general lie algebra and gives several examples. It was not until I studied this paper that the larger perspective of the book by James Murdock became clear. In chapter 3 he describes the normal form for a simpler Lie algebra and transfers the mechanisms to the slightly more complicated general problem of normalising vector fields.

From this point onwards it is up to the reader where the focus will be. The reader interested in spectral

sequences may read the original papers by Sanders on the hypernormal form and the reader interested in parametric normal forms may first turn the series of papers by Yu on the parametric normal form.
Usually the dynamical system under study has bifurcation parameters. In order to study those systems and transform them to their normal form is a bit tricky. The problem is that in order to describe the bifurcation in the most general way, no division by parameters should be made. The extra restrictions hence pose extra difficulties. This results in the study of parametric normal forms.

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