UTRECHT UNIVERSITY

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MASTER'S THESIS

BPS rotating black holes in $\mathcal{N} = 1$ $D = 4$ anti-de Sitter supergravity

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June 2016

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Introduction

Motivation and goals

The study of black holes in anti-de Sitter (AdS) space has been of particular interest since Maldacena conjectured the celebrated AdS/CFT correspondence [1], a duality between quantum gravity in anti-de Sitter space and gauge theories on its conformal boundary. The weak version of the correspondence relates the strong coupling limit of a gravity theory to the weak coupling limit of a gauge theory. This means we can use the AdS/CFT correspondence to translate a hard problem in the strongly coupled system in one theory to an easier problem in the weakly coupled system in the dual theory. The duality is not yet fully understood and work is still to be carried out to build a complete dictionary that takes us from one description to the other.

Besides the motivation from the AdS/CFT correspondence, black holes in AdS are interesting in their own right because of their stability against radiation. It is known that a black hole cannot be in stable thermal equilibrium with an indefinitely large reservoir of energy [2]. By putting it in a finite box, as is the case in AdS space, we can make the black hole come to a thermal equilibrium where statistical canonical ensemble techniques can be employed to study macroscopic behavior of the black hole.

The main goal of the research is to determine whether or not there are BPS rotating black holes in $\mathcal{N} = 1$, $D = 4$ AdS supergravity. There are hints that the $\mathcal{N} = 1$ supersymmetry algebra for AdS in four dimensions permits $\frac{1}{2}$-BPS states at the bound $E = |J|$ [3]. Recent literature [4] suggests that for $\mathcal{N} = 2$ with zero magnetic and electric charge, BPS states occurs at the same bound as [3], which correspond to the ultra spinning limit. As a result, we aim to determine the explicit preserved supersymmetries of rotating black holes in the ultra spinning limit. The approach taken to attain the goal is outlined in the next section.

Outline

We will first review some background on black holes, in particular the Kerr solution in both flat and AdS space. This will be done in chapter one.

In chapter two a brief introduction to supersymmetry, in particular the minimal ($\mathcal{N} = 1$) case will be given. We will discuss what supersymmetric and BPS states are. We will see that the superalgebra in Minkowski space for $\mathcal{N} = 1$ does not allow for massive BPS states. The $\mathcal{N} = 1$ AdS superalgebra and its BPS bound are also discussed.

Chapter three is reserved for introducing $\mathcal{N} = 1$ supergravity. Supergravity in both flat and AdS space will be examined. We will clarify the connection between the BPS bound which comes from the superalgebra and the Killing spinor equation. The relationship between Killing vectors and Killing spinors will also be examined.

Chapter five presents the work in progress towards solving for the Killing spinor equation. We will first examine the $\mathcal{N} = 1$ Kerr black hole in Minkowski space to confirm the non-existence of BPS Kerr black holes in flat space. This is in agreement with the fact that the superalgebra in flat space does not allow for massive BPS states. We then investigate Kerr black holes in AdS and present our current working in solving for the complete set of Killing spinors.

Chapter 1

Review of black holes and the Kerr solution

In this chapter a brief introduction to black hole, in particular the Kerr solution, will be given to pave the way to our discussion in the next chapter. Treatment will first be carried out in flat Minkowski spacetime and then in anti-de Sitter space.

1.1 The Einstein equation

Black holes emerged as possible solutions to the Einstein equation for general relativity. The basic idea of general relativity is that the geometry of spacetime is dynamical: spacetime geometry is determined by the distribution of matter and conversely, the motion of matter is influenced by the geometry of spacetime. Let us quickly review how one arrives at the Einstein equation.

Our spacetime is the usual pseudo-Riemannian manifold with dynamical metric $g_{\mu\nu}$ and the $(-+++)$ signature. All curvature properties of the metric are encoded in the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$, which is expressed in terms of second order derivatives of $g_{\mu\nu}$. It will vanish if and only if the metric is perfectly flat. If we restrict the action of general relativity to be at most quadratic in derivatives then the simplest possible choice for a gravitational action is the Einstein-Hilbert action

$$S_{EH} = \int \sqrt{-g}(R - 2\Lambda)\, d^4x, \tag{1.1}$$

where Λ is the cosmological constant. The Einstein-Hilbert action, together with the matter action S_M, gives the complete action that describes both gravitational fields and matter. Upon defining the energy-momentum tensor of matter to be

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.2}$$

and varying the combined action $S_{EH} + S_M$ with respect to the metric, we arrive at the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda) = T_{\mu\nu}. \tag{1.3}$$

In vacuum, the energy-momentum tensor vanishes. Upon taking the trace (i.e. contracting both sides of equation (1.3) with $g^{\mu\nu}$) the Einstein equation takes the form

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{1.4}$$

1.2 Black holes in flat spacetime

The Schwarzschild black hole was the first exact solution to the vacuum Einstein equation with vanishing cosmological constant, and hence $R_{\mu\nu} = 0$. The Schwarzschild metric takes the form

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2. \tag{1.5}$$

The event horizons are located at the zeros of g^{rr}. In the case of the Schwarzschild metric there is only one event horizon, located at $r = 2M$.

While the Schwarzschild solution is simple, it is not particularly interesting due to its spherical symmetry. Black holes in nature are expected to form due to stellar collapse and since all stars have angular momentum, we expect that the gravitational endstate of such collapse would instead be a rotating black hole. The Kerr metric is the unique axisymmetric solution to the Einstein equation in vacuum. The Kerr metric in the Boyer-Lindquist coordinates is given by [5] and takes the form

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big(adt - (r^2 + a^2)d\phi\big)^2, \tag{1.6}$$

where

$$\Delta = \Delta(r) = r^2 - 2Mr + a^2 \qquad \text{and} \qquad \rho^2 = \rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{1.7}$$

The two constants, M and a, parameterise the possible solutions. The mass, M, is the Komar energy of the black hole and a is the angular momentum per unit mass, $a = J/M$, where J is the Komar angular momentum.

Kerr black holes have two distinct event horizons, located at the zeros of Δ, i.e., at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

An important concept which will be relevant to later chapters is the concept of global timelike Killing vectors. Killing vectors give the direction along which the metric is invariant [6]. Since the Kerr metric is independent of both t and ϕ we can immediately deduce the existence of two Killing vector fields in Kerr geometry: the time translation Killing vector ∂_t and the azimuthal Killing vector ∂_ϕ. In Boyer-Lindquist coordinates they correspond to

$$K^\mu = (1,0,0,0) \qquad \text{and} \qquad R^\mu = (0,0,0,1). \tag{1.8}$$

The Komar energy and Komar angular momentum mentioned earlier are the conserved quantities associated with the time translation and azimuthal Killing vector respectively.

It can be shown that K^μ and R^μ are the only two linearly independent Killing vectors admitted by the Kerr metric [7], hence any Killing vector of the Kerr metric can be expressed as a linear combination of them. The norm of a generic Killing vector $\xi^\mu = K^\mu + \alpha R^\mu$ for constant α is given by

$$\xi^2 = g_{\mu\nu}\xi^\mu\xi^\nu = g_{tt} + 2\alpha\, g_{t\phi} + \alpha^2 g_{\phi\phi}. \tag{1.9}$$

Since $g_{\phi\phi} \to +\infty$ and $g_{tt} \to -1$ at spatial infinity, ξ^μ is spacelike at spatial infinity unless $\alpha = 0$. As we are interested in timelike Killing vectors, we set $\alpha = 0$ and recover the Killing vector associated with time translation $\xi^\mu = K^\mu$.

The time translation Killing vector K^μ has norm

$$K^2 = g_{\mu\nu}K^\mu K^\nu = g_{tt} = -\Big(1 - \frac{2Mr}{\rho^2}\Big) = -\frac{r^2 - 2Mr + a^2\cos^2\theta}{\rho^2}. \tag{1.10}$$

This quantity is timelike at spatial infinity however as we travel towards the black hole past $r = M + \sqrt{M^2 - a^2\cos^2\theta} > r_+$ it becomes spacelike. The region of spacetime outside the outer event horizon where ∂_t is spacelike is called an ergoregion and the boundary at which ∂_t is lightlike is called an ergosphere. More explicitly, the ergoregion is found to be

$$r_+ \le r \le M + \sqrt{M^2 - a^2\cos^2\theta}, \tag{1.11}$$

that is, it extends from the outer horizon to the surface of $g_{tt} = 0$.

As a result, there is no global timelike Killing vector for Kerr black holes in flat space. This is an important result which will be referred back to later in our discussion of BPS black holes.

1.3 Anti-de Sitter spacetime

Let us begin with a review of AdS space before discussing Kerr-AdS black holes. AdS space is a maximally symmetric solution of the Einstein equation with negative cosmological constant

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \quad , \quad \Lambda = -\frac{3}{l^2} < 0. \tag{1.12}$$

AdS space in d dimensions can be thought of as a submanifold of $R^{2,d-1}$. Let us recall that $R^{2,d-1}$ is endowed with the following metric

$$ds^2 = -(dx^0)^2 - (dx^d)^2 + \sum_{i=1}^{d-1}(dx^i)^2. \tag{1.13}$$

The d-dimensional AdS space, AdS_d, is then defined as the set of points $(x^0, ..., x^d)$ that satisfy

$$-(x^0)^2 - (x^d)^2 + \sum_{i=1}^{d-1}(x^i)^2 = -l^2. \tag{1.14}$$

There are many ways to write down the metric for AdS, we will work in particular with AdS_4 in static coordinates (t, r, θ, ϕ). The AdS_4 metric is given by [8] and takes the form

$$ds^2 = -\Big(1 + \frac{r^2}{l^2}\Big)dt^2 + \Big(1 + \frac{r^2}{l^2}\Big)^{-1}dr^2 + r^2 d\Omega^2, \tag{1.15}$$

where $l = \sqrt{-3/\Lambda}$ is the AdS radius, a cosmological length associated with the cosmological constant Λ. Any metric that asymptotically approaches the metric in equation (1.15) is called asymptotically AdS.

1.4 Kerr black holes in anti-de Sitter space

How might a black hole analogous to the Kerr black hole look like in AdS space? The Kerr-AdS metric in Boyer-Lindquist coordinates [9] takes the form

$$ds^2 = -\frac{\Delta_r}{\rho^2}\Big(dt - \frac{a}{\Sigma}\sin^2\theta\, d\phi\Big)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}dt^2 + \frac{\Delta_\theta}{\rho^2}\sin^2\theta\Big(adt - \frac{r^2 + a^2}{\Sigma}d\phi\Big)^2, \tag{1.16}$$

where

$$\begin{aligned} \Delta_r &= (r^2 + a^2)\big(1 + \frac{r^2}{l^2}\big) - 2Mr \quad , \quad \Sigma = 1 - \frac{a^2}{l^2}, \\ \Delta_\theta &= 1 - \frac{a^2}{l^2}\cos^2\theta \quad , \quad \rho^2 = r^2 + a^2\cos^2\theta. \end{aligned} \tag{1.17}$$

The parameter M can no longer be interpreted as the Komar energy. Rather, the Komar energy is determined to be M/Σ^2 [8]. Similarly, the Komar angular momentum is $J = aM/\Sigma^2$. Also note that the rotation parameter a is constrained by the AdS radius, $a^2 < l^2$. This is because the metric is singular when $a^2 = l^2$ and is inconsistent with the initial pseudo-Riemannian signature when $a^2 > l^2$. Although we skip the derivation of this metric, it can be checked that in the limit $l \to \infty$ we recover the Kerr metric in flat space and in the limit $a \to 0$, $M \to 0$ we recover empty AdS space.

Upon examining the zeros of the horizon function Δ_r, we can determine (see section 1.6) the critical mass parameter M_{extr}

$$M_{\text{extr}}(a) = \frac{l}{3\sqrt{6}}\left(\sqrt{\left(1+\frac{a^2}{l^2}\right)^2+\frac{12a^2}{l^2}}+\frac{2a^2}{l^2}+2\right)\times\left(\sqrt{\left(1+\frac{a^2}{l^2}\right)^2+\frac{12a^2}{l^2}}-\frac{a^2}{l^2}-1\right)^{\frac{1}{2}}, \quad (1.18)$$

such that the metric exhibits a naked singularity for $M < M_{\text{extr}}$. The metric represents a black hole with two distinct event horizons for $M > M_{\text{extr}}$. For $M = M_{\text{extr}}$ the horizon function Δ_r has a double zero and the black hole is extremal. The value for the critical mass parameter we determined here agrees with result from earlier work by Caldarelli and Klemm [10]. We will focus on the case $M \geq M_{\text{extr}}$, which guarantees the existence of an event horizon at $r = r_+$ when solving for the largest zero of Δ_r.

By checking the coefficients of the metric, we see that the two Killing vectors of this space are again the Killing vectors associated with time translation ∂_t and axial symmetry ∂_ϕ.

As with the case of asymptotically flat Kerr, we would like to determine if asymptotically AdS Kerr black holes also possess an ergoregion. This can be done by repeating exactly what we did for flat Kerr, i.e. solving explicitly for $g_{tt} = 0$, which gives the location of the ergosphere. There is, however, a quicker way to confirm the existence of an ergoregion without the need to solve for $g_{tt} = 0$, which is a quartic equation in r in Kerr-AdS.

This can be done by noticing that

$$g_{tt} = -\frac{\Delta_r}{\rho^2} + \frac{\Delta_\theta}{\rho^2} a^2 \sin^2\theta = -\frac{1}{\rho^2}(\Delta_r - \Delta_\theta\, a^2 \sin^2\theta). \quad (1.19)$$

Solving for $g_{tt} = 0$ is thus equivalent to solving for

$$\Delta_r - \Delta_\theta\, a^2 \sin^2\theta = 0. \quad (1.20)$$

Now we recall that $\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta$, which is positive for all values of θ since $a^2 < l^2$. This implies that $\Delta_\theta\, a^2\sin^2\theta \geq 0$ for all θ and equality occurs only at $\theta = 0, \pi/2$.

The fact that $\Delta_\theta\, a^2\sin^2\theta \geq 0$ has the effect of shifting the graph of $\Delta_r - \Delta_\theta\, a^2\sin^2\theta = 0$ down the vertical axis as compared to the graph of $\Delta_r = 0$, as is illustrated in figure 1.1. This will open a gap in the r-axis between the largest zero of $\Delta_r = 0$ and the corresponding largest zero of $\Delta_r - \Delta_\theta\, a^2\sin^2\theta = 0$. As a result, the existence of the largest event horizon r_+, as guaranteed by $M \geq M_{\text{extr}}$, ensures the existence of r_{erg}, the largest zero of equation (1.20) and that $r_{\text{erg}} \geq r_+$ with equality occurs at $\theta = 0, \pi/2$. This confirms the existence of an ergoregion for Kerr black holes in AdS space. The ergoregion is given by

$$r_+ \leq r \leq r_{\text{erg}}. \quad (1.21)$$

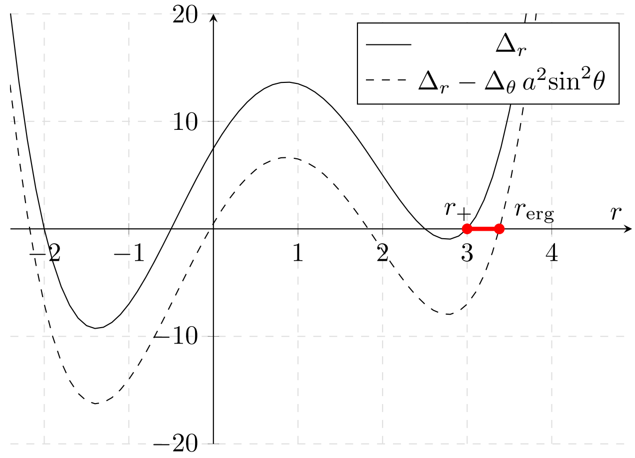

Figure 1.1: Schematic illustration of the horizon function Δ_r and $\Delta_r - \Delta_\theta\, a^2\sin^2\theta$. The effect of shifting the graph of Δ_r down by $\Delta_\theta\, a^2\sin^2\theta$ is the movement of r_+ to $r_{\rm erg} \geq r_+$. The gap between r_+ and $r_{\rm erg}$ (coloured in red) represents the ergoregion.

It should be noted that for Kerr-AdS $g_{\phi\phi} \to +\infty$ and $g_{tt} \to -\infty$ at spatial infinity. This means we cannot set $\alpha = 0$ in $\xi^\mu = K^\mu + \alpha R^\mu$ when searching for global timelike Killing vectors like we did for flat Kerr. As a result, in contrast to the case of asymptotically flat Kerr, the existence of an ergoregion does not preclude the existence of a global timelike Killing vector. It was determined that when the horizon radius is larger than the AdS radius, i.e. $r_+ > l$, the black hole admits a global Killing vector. This result is worked out in [11].

1.5 Ultra-spinning black holes

Since we suspect that the BPS bound is satisfied by the ultra-spinning limit $a \to l$, we should be working in a coordinate system where the metric is well-defined in that limit. The Kerr-AdS metric in Boyer-Lindquist coordinates given in equation (1.16) is clearly singular in the limit $a \to l$. Fortunately, recent works by Caldarelli et al. [12, 13] show that one can do a coordinate transformation of the form

$$\sin\theta = \sqrt{\Sigma}\,\sinh^2\tfrac{\sigma}{2}, \tag{1.22}$$

on the metric (1.16) then the resulting metric has a regular $a \to l$ limit

$$ds^2 = \frac{dr^2}{V(r)} + \frac{l^2+r^2}{4}\left(d\sigma^2 + \sinh^2\sigma d\phi^2\right) - V(r)\Big(dt - l\,\sinh^2\tfrac{\sigma}{2}d\phi\Big)^2, \tag{1.23}$$

where

$$V(r) = 1 + \frac{r^2}{l^2} - \frac{2Mr}{l^2+r^2}. \tag{1.24}$$

As a sanity check, we compute the critical mass parameter for the new metric. It was found to be $M_{\text{extr}} = \frac{8}{3\sqrt{3}} l$, which agrees with the limit $a \to l$ of the result in equation (1.18).

It should be noted that the result of taking the limit is not unique and depends crucially on how the limit is performed. The procedure described by [12] is the only known procedure in four dimensions. The limit is taken while keeping the horizon radius r_+ fixed and simultaneously zooming into the poles of $\frac{1}{a^2 - l^2}$.

1.6 Derivations

For section 1.4

Derivation of the critical mass parameter M_{extr} in equation (1.18):

We would like to find the critical value M_{extr} at which the horizon function Δ_r, which is quartic in r, has a double zero.

The zeros of Δ_r are also the zeros of $\dfrac{\Delta_r}{l^2}$ and therefore, we can solve for the zeros of $\dfrac{\Delta_r}{l^2}$ since it can be put in a more suggestive form

$$\frac{\Delta_r}{l^2} = \tilde{r}^4 + \tilde{r}^2(1 + \tilde{a}^2) - 2\tilde{M}\tilde{r} + \tilde{a}^2, \tag{1.25}$$

where

$$\tilde{r} = \frac{r}{l} \quad , \quad \tilde{a} = \frac{a}{l} \quad , \quad \tilde{M} = \frac{M}{l}. \tag{1.26}$$

Demanding that $\dfrac{\Delta_r}{l^2}$ has a double root means equating (1.25) with a generic quartic which has a double root.

$$\begin{aligned} \tilde{r}^4 + \tilde{r}^2(1 + \tilde{a}^2) - 2\tilde{M}\tilde{r} + \tilde{a}^2 &= (\tilde{r} - A)^2(\tilde{r}^2 + B\tilde{r} + C) \\ &= \tilde{r}^4 + \tilde{r}^3(B - 2A) + \tilde{r}^2(A^2 - 2AB + C) + \tilde{r}(A^2B - 2AC) + A^2C. \end{aligned}$$

This leads to

$$\begin{cases} B - 2A = 0 \\ A^2 - 2AB + C = 1 + \tilde{a}^2 \\ A^2B - 2AC = -2\tilde{M} \\ A^2C = \tilde{a}^2 \end{cases} \Rightarrow \begin{cases} B = 2A \\ C - 3A^2 = A^2C + 1 \\ \tilde{M} = A(C - A^2) \\ A^2C = \tilde{a}^2 \end{cases} \Rightarrow \begin{cases} B = 2A \\ C^2 - C(1 + \tilde{a}^2) - 3\tilde{a}^2 = 0 \\ \tilde{M} = A(C - A^2) \\ 3A^4 + A^2(1 + \tilde{a}^2) - \tilde{a}^2 = 0. \end{cases}$$

Upon solving the two quadratic equations for A and C under the conditions that $A, C \geq 0$ (since $\tilde{a}^2, \tilde{M} \geq 0$) we get unique solutions for A and C

$$C = \frac{1}{2}\Big((1 + \tilde{a}^2) + \sqrt{(1 + \tilde{a}^2)^2 + 12\tilde{a}^2}\Big) \tag{1.27}$$

$$A = \frac{1}{6}\Big(-(1 + \tilde{a}^2) + \sqrt{(1 + \tilde{a}^2)^2 + 12\tilde{a}^2}\Big) \tag{1.28}$$

Then $\tilde{M}$ at which $\dfrac{\Delta_r}{l^2}$ has double zeros is

$$\tilde{M}_{\rm extr} = \frac{1}{3\sqrt{6}}\Big(2(1+\tilde{a}^2) + \sqrt{(1+\tilde{a}^2)^2 + 12\tilde{a}^2}\Big) \times \Big(- (1+\tilde{a}^2) + \sqrt{(1+\tilde{a}^2)^2 + 12\tilde{a}^2}\Big)^{\frac{1}{2}} \quad (1.29)$$

It is then straight forward to calculate $M_{\rm extr}$ from $\tilde{M}_{\rm extr}$ using equation (1.26).

Chapter 2

Introduction to supersymmetry and BPS states

Supersymmetry, in very simple terms, is a spacetime symmetry between particles of integer spin (bosons) and particles of half integer spin (fermions). Since it is a spacetime symmetry (relating particles of different spins) we begin by reviewing some basics about spacetime symmetries.

2.1 The Poincaré algebra

The isometries, mappings that preserve the metric, of our flat Minkowski spacetime are elements of the Poincaré group, which includes Lorentz transformations and translations. For Minkowski spacetime we take the convention of a mostly plus signature

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1).$$

Lorentz transformations and translations act on the coordinates as

$$x^{\mu} \ \rightarrow \ \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu},$$

where the matrix $\Lambda^{\mu}{}_{\nu}$ is an element of the Lorentz group $SO(1,3)$ specifying a Lorentz transformation and a^{μ} is a vector parameterising a translation.

The Lorentz group has six generators, which consist of three spatial rotations J_i and three boosts K_i, $i = 1, 2, 3$, which satisfy the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad , \quad [J_i, K_j] = i\epsilon_{ijk}K_k \quad , \quad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.1}$$

For later purposes let us introduce a four-vector notation for the generators. Since there are six generators for the Lorentz group, we can express them in terms of an anti-symmetric tensor $M_{\mu\nu}$ where $\mu = 0, 1, 2, 3$ as follows

$$M_{0i} = K_i \quad , \quad M_{ij} = \epsilon_{ijk} J_k \quad , \quad M_{\mu\nu} = -M_{\nu\mu}. \tag{2.2}$$

The matrices $M_{\mu\nu}$ are generators of the Lorentz group in the sense that any element of the group can be expressed uniquely as

$$\Lambda^{\kappa}{}_{\sigma} = \exp\Big(-\frac{i}{2}\omega^{\mu\nu}(M_{\mu\nu})^{\kappa}{}_{\sigma}\Big), \tag{2.3}$$

where $\omega^{\mu\nu}$ is a real antisymmetric matrix which contains the rotation angles and boost parameters, and $M_{\mu\nu}$ are 4×4 matrices satisfying

$$(M^{\mu\nu})_{\kappa\sigma} = i(\delta^{\mu}{}_{\kappa}\delta^{\nu}{}_{\sigma} - \delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\kappa}). \tag{2.4}$$

There is a distinction to be made between elements of a group and those of its associated algebra. For any group G, some group elements $g \in G$ can be written as $g = \exp(i\xi^a t_a) = \mathbb{1} + i\xi^a t_a + \mathcal{O}(\xi^2)$ where the t_a matrices are the *generators* of the group. The generators are in an algebra, because we can add and multiply them, whereas the group elements are in a group, because we can only multiply them. The Lorentz group belongs to a class of groups called Lie groups which have an infinite number of elements but a finite number of generators.

The commutation relations in (2.1) can then be written in a more compact way

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}. \tag{2.5}$$

This is the commutation relation that defines the Lorentz algebra. The symmetry algebra that describes Minkowski spacetime is the Poincaré algebra which is simply the Lorentz algebra together with the spacetime translation generators P_μ. The Poincaré algebra is defined as follows

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}, \\ [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu. \end{aligned} \tag{2.6}$$

2.2 The superalgebra and BPS states in flat space

So far we have only discussed spacetime symmetry, another type of symmetry that is allowed by the S-matrix is internal symmetry. An example of internal symmetry is the $SU(3)$ gauge symmetry of the strong interactions which transforms quark states of different colour into each other. Note that internal symmetries do not change the Lorentz index, that is, an $SU(3)_c$ rotation changes the colour of a quark, not its spin. There have been attempts to combine all known symmetries into a single group, however the idea was abandoned after a series of no-go results that led to the Coleman-Mandula theorem, which states that in a generic quantum field theory, the most general symmetry enjoyed by the S-matrix is

$$G = G_{\text{Poincaré}} \times G_{\text{internal}}. \tag{2.7}$$

The Coleman-Mandula theorem can be evaded by weakening one of its assumptions. One such assumption is that the symmetry algebra only involves *commutators* of *bosonic* generators. If we allow for *fermionic* generators, which satisfy anti-commutation relations, then we can enlarge the group of allowed symmetries. Haag, Lopuskanski and Sohnius later showed that the most general symmetry of the S-matrix is

$$G = G_{\text{superPoincaré}} \times G_{\text{internal}}, \tag{2.8}$$

where the superPoincaré algebra is, as we will see, a specific extension of the Poincaré algebra which includes transformations that turn bosons into fermions and vice versa.

We will include $\mathcal{N}$ such fermionic generators $Q^I_\alpha,\ I = 1, \ldots, \mathcal{N}$ together with their Hermitian conjugates $\bar{Q}^I_{\dot{\alpha}}$ into the algebra. Note that here α and $\dot{\alpha}$ are both spinor indices and they transform correspondingly in the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group.

A supersymmetry algebra with one fermionic generator Q_α is called $\mathcal{N} = 1$ supersymmetric, with two fermionic generators Q^1_α, Q^2_α called $\mathcal{N} = 2$ supersymmetric, etc. Since a Weyl spinor in $D = 4$ has four real components, a supersymmetric theory in $D = 4$ with $\mathcal{N} = 1$ supersymmetry has four real supercharges.

The $\mathcal{N} = 1$ supersymmetry algebra, besides the commutators of (2.6), contains the following (anti)-commutation relations

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \\ \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0 \\ [P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot{\alpha}}] &= 0 \\ [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \end{aligned} \tag{2.9}$$

The supersymmetry algebra can be extended to have more fermionic generators. If there are $\mathcal{N}$ generators, the algebra is enlarged to

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\} &= 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \\ \{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ} \end{aligned} \tag{2.10}$$

where Z is the *central charge* of the algebra, in the sense that it commutes with all other generators.

From the anti-commutation relation in (2.10) we can deduce that Z^{IJ} is anti-symmetric. This means that Z^{IJ} only exists in algebras with extended supersymmetry, i.e. when $\mathcal{N} \geq 2$.

Two important concepts in a supersymmetric theory that will be relevant to later discussion are the concepts of supersymmetric and BPS states. Let us first discuss supersymmetric states.

Consider an arbitrary state $|\psi\rangle$. Using the relevant anti-commutation relation we get

$$2\sigma^{\mu}_{\alpha\dot{\alpha}}\langle\psi|P_\mu|\psi\rangle = \langle\psi|\{Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}\}|\psi\rangle = ||Q^I_\alpha|\psi\rangle||^2 + ||\bar{Q^I}_{\dot{\alpha}}|\psi\rangle||^2 \geq 0. \tag{2.11}$$

Summing over $\alpha, \dot{\alpha}$, noticing that σ^0 is the only Pauli matrix with non-vanishing trace, gives

$$4\langle\psi|P_0|\psi\rangle = \sum_\alpha ||Q_\alpha|\psi\rangle||^2 + \sum_{\dot{\alpha}} ||\bar{Q}_{\dot{\alpha}}|\psi\rangle||^2 \geq 0. \tag{2.12}$$

If the state is massless we can find the lightcone frame where $P_\mu = (E, 0, 0, E)$. If it is massive we can find its rest frame where $P^\mu = (m, 0, 0, 0)$. P_0, therefore, gives the energy of the state. Equation (2.12) implies that

$$E \geq 0 \tag{2.13}$$

for any state $|\psi\rangle$ in a supersymmetric theory. Furthermore,

$$E = 0 \iff Q_\alpha|\psi\rangle = \bar{Q}_{\dot{\alpha}}|\psi\rangle = 0 \quad \forall\ \alpha, \dot{\alpha}. \tag{2.14}$$

This means a zero energy state must be invariant under all supersymmetry generators and this is precisely what we mean by supersymmetric state.

Let us move on and discuss what it means to be BPS states. Notice that in the discussion of supersymmetric states we have implicitly taken $I = J$, that is, the case of trivial central charge Z^{IJ}. If we now take the central charge into account, let us consider the simplest case of such extension, the case $\mathcal{N} = 2$. Consider the expectation value of a linear combination of the supercharges as follow

$$\langle\psi|\big\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger, (Q^1_\alpha + \epsilon_{\alpha\beta}\big(Q^2_\beta)^\dagger\big)^\dagger\big\}|\psi\rangle = ||\big(Q^2_\beta)^\dagger\big)|\psi\rangle||^2 + ||\big(Q^2_\beta)^\dagger\big)^\dagger|\psi\rangle||^2 \geq 0 \tag{2.15}$$

The left hand side, however, can be expressed as

$$\begin{aligned}
&\langle\psi|\big\{Q^1_\alpha + \epsilon_{\alpha\beta}(Q^2_\beta)^\dagger, (Q^1_\alpha + \epsilon_{\alpha\beta}\big(Q^2_\beta)^\dagger\big)^\dagger\big\}|\psi\rangle \\
&= \langle\psi|\{Q^1_\alpha, (Q^1_\alpha)^\dagger\} + \epsilon_{\alpha\beta}\{Q^1_\alpha, Q^2_\alpha\} + \epsilon_{\alpha\beta}\{(Q^1_\alpha)^\dagger, (Q^2_\alpha)^\dagger\} + \{Q^2_\alpha, (Q^2_\alpha)^\dagger\}|\psi\rangle \\
&= 8\langle\psi|P_0|\psi\rangle + 4\langle\psi|Z^{12}|\psi\rangle
\end{aligned} \tag{2.16}$$

The right hand sides of equations (2.15) and (2.16) give

$$E \geq \frac{1}{2} Z^{12} \tag{2.17}$$

Generalisation to extended supersymmetry gives

$$E \geq \frac{1}{2}|Z_r| \quad , \quad \forall r = 1, ..., \tfrac{\mathcal{N}}{2}. \tag{2.18}$$

where we have relabeled the central charges as $Z^{12} = Z_1, \ Z^{34} = Z_2, \dots$

States that saturate this bound, i.e. $E = \frac{1}{2}|Z_r| \ \forall r = 1, ..., \frac{\mathcal{N}}{2}$ are called $\frac{1}{2}$BPS.

In this sense supersymmetric states ($E = 0 \iff Q_\alpha|\psi\rangle = \bar{Q}_{\dot{\alpha}}|\psi\rangle = 0$) can be seen as complete BPS. For $\mathcal{N} = 1$, since the central charge is trivially realised, $Z^{IJ} = 0$, the $\frac{1}{2}$-BPS bound reduces to $E \geq 0$.

2.3 The superalgebra and BPS states in anti-de Sitter space

The relevant anti-commutation relation of the $\mathcal{N} = 1$ superalgebra in AdS_4 is given by [3] in terms of a Majorana spinor supercharge Q, the four-vector P_μ that generates translation and the Lorentz generators $M_{\mu\nu}$ as follows

$$\{Q_\alpha, Q_\beta\} = (C\gamma^\mu)_{\alpha\beta} P_\mu + \frac{1}{2}(C\gamma^{\mu\nu})_{\alpha\beta} M_{\mu\nu} \tag{2.19}$$

where C is the charge conjugation matrix. Note that instead of using the two-component Weyl spinor formalism as in the earlier section we have adopted the four-component Majorana formalism to be consistent with the authors of [3].

The anti-commutation relation can be written more compactly as

$$\{Q_\alpha, Q_\beta\} = \frac{1}{2} M_{AB} (C\Gamma^{AB})_{\alpha\beta} \tag{2.20}$$

where

$$\Gamma^A = (\gamma^\mu, \gamma_5) \qquad , \qquad M_{AB} = -M_{BA} \qquad , \qquad M_{\mu 5} = P_\mu \tag{2.21}$$

and $A, B = 0, 1, 2, 3, 5$. The convention we follow is

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \tag{2.22}$$

where η is the flat metric on $\mathbb{E}^{(2,4)}$ with signature $(-+++-)$. One important difference between the Poincaré supersymmetry algebra and the AdS supersymmetry algebra is that P^2 commutes with all elements of the former and is therefore a Casimir operator. The Casimir operators for the AdS superalgebra is given by

$$c_2 = \frac{1}{2} M_{AB} M^{AB} \tag{2.23}$$

$$c_4 = M^A{}_B M^B{}_C M^C{}_D M^D{}_A \tag{2.24}$$

Similar to the energy condition $E \geq 0$ for physical states in the Poincaré superalgebra that we derived in equation (2.13), the condition for physical states in AdS superalgebra is

$$c_4 - c_2^2 \geq 0 \tag{2.25}$$

A complete treatment can be found in [3], however here we only quote the results.

Complete BPS states are those that saturate the bound in equation (2.25), that is, $c_4 - c_2^2 = 0$. This corresponds to states with $E = 0$ where E is the eigenvalue of $M_{05} = P_0$.

$\frac{1}{2}$-BPS states are those for which $E \geq |J|$ where J is the eigenvalue of M_{23}, which gives the angular momentum around the 1-axis. This suggests that, unlike the case in flat space, the superalgebra in AdS permits $\frac{1}{2}$-supersymmetric states with non-vanishing angular momentum J. This bound coincides with the bound $a \leq l$, thus $\frac{1}{2}$-BPS states, if in existence, will occur in the ultra spinning limit $a \to l$.

The angular momentum J arises from the form of the anti-commutation relation in the AdS superalgebra, which involves both P_μ and $M_{\mu\nu}$ as given in equation (2.19). In contrast, the anti-commutation relation in the Poincaré supersymmetry algebra involves only P_μ, as given in equation (2.9).

Chapter 3

$\mathcal{N}=1, \;\; D=4$ supergravity and BPS states

After having introduced the ideas of global supersymmetry in chapter two, we will now begin to study how to combine global supersymmetry, Lorentz invariance and gauge principles to formulate supergravity, a *local* theory of supersymmetry.

We will start by constructing a free spin-$\frac{3}{2}$ field and then couple it to gravity to obtain the pure $\mathcal{N}=1$, $D=4$ supergravity consisting of the graviton and one Majorana spinor gravitino. In doing this many important features of the theory, in particular the transformation rules for individual fields, emerge. A brief discussion of supergravity in anti-de Sitter space is also presented. Finally, we will examine BPS states in the context of supergravity.

3.1 A local theory of the free spin-$\frac{3}{2}$ field

Supergravity is the gauge theory of global supersymmetry: the parameter of transformation of global supersymmetry is a constant spinor ϵ_α, in supergravity it becomes a general function of spacetime $\epsilon_\alpha(x)$. The associated gauge field is a spin-$\frac{3}{2}$ vector-spinor quantity $\psi_{\mu\alpha}(x)$ called a gravitino.

Motivated by a general gauge transformation of the form $A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$ and a global supersymmetry transformation $\theta_\alpha \to \theta_\alpha + \epsilon_\alpha$ we demand that the vector-spinor quantity $\psi_{\mu\alpha}(x)$ transforms as

$$\psi_{\mu,\alpha}(x) \to \psi_{\mu,\alpha}(x) + \partial_\mu \epsilon_\alpha(x). \tag{3.1}$$

We wish to construct an action which is (i) invariant under this gauge transformation, (ii) Lorentz invariant, (iii) Hermitian and (iv) contains term with first order derivative. It is straight forward to see that the action

$$S_{\frac{3}{2}} = -\int d^D x \;\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho, \tag{3.2}$$

satisfies these requirements. The equation of motion is

$$\gamma^{\mu\nu\rho} \partial_\nu \psi_\rho = 0. \tag{3.3}$$

Note that in both equation (3.2) and (3.3) we have suppressed the spinor index α on the gravitino. It is important to keep this in mind, particularly since we will soon review the vielbein formalism to couple a fermionic quantity to curved space.

3.2 Vielbein formalism

One key issue we need to address is how to couple fermions to gravity. Normally for bosonic fields the effects of gravity are taken into account by the simple replacements $\{\partial_\mu, \eta_{\mu\nu}\} \to \{\nabla_\mu, g_{\mu\nu}\}$ in the matter Lagrangian. The same replacements, however, do not work in the case of fermionic fields. Since we know how to work with fermions in flat Minkowski spacetime, we can try to formulate our gravity theory using locally flat coordinates. More explicitly, we will rely the existence of a locally inertial frame, one for which all physical laws become those known from Minkowski spacetime, to build our theory. Such formalism is called the vielbein formalism and fortunately for our case, a pseudo-Riemannian metric in four dimensions, the existence of such coordinates (often known as *local inertial coordinates*) is guaranteed [6].

The conventional approach to gravity uses partial derivatives as a natural basis for the tangent space T_p at a point p.

$$\hat{e}_{(\mu)} = \partial_{(\mu)}. \tag{3.4}$$

A four-vector V in the tangent space can be written as

$$V = V^\mu \hat{e}_{(\mu)} = (V_0, V_1, V_2, V_3). \tag{3.5}$$

Similarly, the cotangent space, denoted by T_p^*, is spanned by the differential elements

$$\hat{e}^{(\mu)} = dx^{(\mu)}. \tag{3.6}$$

Since we wish to couple fermions to gravity, let us construct a new set of orthonormal basis vectors $\hat{e}^a$ that is locally flat, i.e. in these coordinates the metric is given by

$$ds^2 = \eta_{ab}\, de^a\, de^b, \tag{3.7}$$

where $\eta = \text{diag}(-1, +1, +1, +1)$. We now introduce the vielbein fields (or frame fields) $e^a_\mu(x)$ by

$$de^a = \frac{\partial e^a}{\partial x^\mu} dx^\mu \equiv e^a_\mu(x) dx^\mu. \tag{3.8}$$

We can thus view the vielbeins $e^a_\mu(x)$ as the transformation matrix between the usual coordinates x^μ and the inertial coordinates e^a or equivalently as a set of orthonormal vector in T^*_p. Note that the new basis is indexed by Latin letters rather than Greek because it is not derived from any coordinate system.

The inverse vielbein e^μ_a can be define analogously as

$$dx^\mu = \frac{\partial x^\mu}{\partial e^a} de^a \equiv e^\mu_a(x) de^a. \tag{3.9}$$

If we check explicitly,

$$de^a = e^a_\mu dx^\mu = e^a_\mu e^\mu_b de^b, \tag{3.10}$$

that is, $e^a_\mu e^\mu_b = \delta^a_b$. Thus, the vielbein and inverse vielbein are defined consistently. Vielbeins are often considered to be the square root of the metric tensor due to

$$g_{\mu\nu}(x)\, dx^\mu dx^\nu = ds^2 = \eta_{ab}\, de^a de^b = \eta_{ab}\, e^a_\mu(x) e^b_\nu(x)\, dx^\mu dx^\nu. \tag{3.11}$$

Hence

$$g_{\mu\nu}(x) = e^a_\mu(x)\, \eta_{ab}\, e^b_\nu(x). \tag{3.12}$$

Equation (3.12) gives the general relation between the metric and a frame field. In general, for a given metric tensor $g_{\mu\nu}(x)$, the frame field $e^a_\mu(x)$ is not unique. Let us check this explicitly.

Given any matrix $\Lambda^a{}_b(x)$ which leaves η_{ab} invariant, that is, given a *local* Lorentz transformation, we can construct another frame field that also satisfies (3.12), namely

$$e'^a_\mu(x) = \Lambda^{-1a}{}_b(x) e^b_\mu(x). \tag{3.13}$$

As a result, all choices of frame fields that are related by local Lorentz transformations are equivalent. We must therefore insist that the frame field and all geometrical quantities derived from it transform covariantly with respect to local Lorentz transformations. It is this requirement that will allow us to derive the correct covariant derivative acting on a fermionic quantity. We will now derive this explicitly.

Since a one-form vielbein field transforms as

$$e^a \;\rightarrow\; \Lambda^{-1a}{}_b e^b, \tag{3.14}$$

the differential two-form de^a should transform as

$$de^a \;\rightarrow\; d(\Lambda^{-1a}{}_b\, e^b) = (d\Lambda^{-1})^a{}_b e^b + (\Lambda^{-1})^a{}_b de^b. \tag{3.15}$$

The first term, $(d\Lambda^{-1})^a{}_b e^b$, spoils the covariance we wish to keep. To compensate for the term induced by spacetime dependence of $\Lambda^{-1}(x)$, in a manner completely analogous to general relativity, we have to introduce a new term with transformation properties that will make the sum transform covariantly. It can be verified that

$$de^a + \omega^a_b \wedge e^b \equiv T^a, \tag{3.16}$$

transforms as a vector under local Lorentz transformation if the new term $\omega^a{}_b$ transforms as

$$\omega^a{}_b \;\rightarrow\; (\Lambda^{-1})^a{}_c \, d\Lambda^c{}_b + (\Lambda^{-1})^a{}_c \, \omega^c{}_d \, \Lambda^d{}_b. \tag{3.17}$$

Since a spinor transforms under the local Lorentz transformation as

$$\psi_\alpha(x) \;\rightarrow\; \exp\Big(-\frac{1}{4}\lambda^{ab}(x)\gamma_{ab}\Big)\psi_\alpha(x), \tag{3.18}$$

the Lorentz covariant derivative can be found to be

$$D_\mu\psi_\alpha(x) = \Big(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu(x)\gamma_{ab}\Big)\psi_\alpha(x). \tag{3.19}$$

For quantities with mixed indices, those that need to transform covariantly under both local Lorentz and general coordinate transformations, the covariant derivative is given by

$$\nabla_\mu\psi_{\nu\alpha} = D_\mu\psi_{\nu\alpha} - \Gamma^\rho_{\mu\nu}\psi_{\rho\alpha}. \tag{3.20}$$

Comparing to (3.19) we can see that covariant indices are contracted with the usual affine connection $\Gamma^\rho_{\mu\nu}$ whereas vielbein indices are contracted with $\omega^a_\mu{}_b$. ∇_μ is covariant with both local Lorentz and general coordinate transformations.

3.3 $\mathcal{N} = 1$ supergravity in flat space

We have gathered the machinery needed to couple the free spin-$\frac{3}{2}$ to curved space. Covariantising the equation of motion with respect to both the local Lorentz transformation and general coordinate transformation turns the equation of motion (3.3) into

$$\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho = 0, \tag{3.21}$$

where the partial derivative ∂_ν has been replaced by the general covariant derivative ∇_ν, which contains the spin connection ω^{ab}_μ and the affine connection $\Gamma^\rho_{\mu\nu}$. Let us first assume there is no torsion, therefore the affine connection becomes symmetric in the two lower indices $\mu\nu$. When contracting with the antisymmetric $\gamma^{\mu\nu\rho}$ the affine connection will vanish. The general covariant derivative will then reduce to a Lorentz covariant derivative and the equation of motion becomes

$$\gamma^{\mu\nu\rho}D_\nu\psi_\rho = 0, \tag{3.22}$$

which gives the action for spin-$\frac{3}{2}$ in curved space as

$$S_{\frac{3}{2}} = -\frac{1}{2k^2}\int d^D x e\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho. \tag{3.23}$$

The gauge transformation of the gravitino is now

$$\delta\psi_\mu = D_\mu \varepsilon(x) = \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon. \tag{3.24}$$

It can be shown [14] that the variation of the spin-$\frac{3}{2}$ action will cancel against the variation of the Einstein-Hilbert action to linear order in ψ_μ

$$S_2 = \frac{1}{2k^2}\int d^D x \ \sqrt{-g}R(g) = \frac{1}{2k^2}\int d^D x e e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega), \tag{3.25}$$

provided that the vielbein transforms as

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu. \tag{3.26}$$

The complete action is

$$S = S_{\frac{3}{2}} + S_2. \tag{3.27}$$

where $S_{\frac{3}{2}}$ and S_2 are given in equation (3.23) and (3.25).

3.4 $\mathcal{N} = 1$ supergravity in anti-de Sitter spaces

In this section we will carry out our discussion in AdS space. There are two crucial steps in obtaining the AdS supergravity action from the complete action in flat space. The first step is to add a cosmological term to (3.27), this is equivalent to replacing R with $R - 2\Lambda$. The second step is to replace the Lorentz covariant derivative D_μ with a modified covariant derivative which is appropriate for the de Sitter group [15] of the form

$$\hat{D}_\mu \equiv D_\mu - \frac{1}{2l}\gamma_\mu. \tag{3.28}$$

The supersymmetric action for AdS gravity is given by [14] and takes the form

$$S = \frac{1}{2\kappa^2}\int d^D x \ e\Big(R - \bar{\psi}_\mu \gamma^{\mu\nu\rho}\hat{D}_\nu \psi_\rho + \frac{(D-1)(D-2)}{l^2}\Big). \tag{3.29}$$

3.5 BPS states in $\mathcal{N} = 1$, $D = 4$ supergravity

To motivate the concept of BPS states in supergravity, we will borrow some techniques from the superspace formalism. Although supergravity is not formulated in superspace, going to superspace will help us understand how the operator formalism for supersymmetry that we discussed in chapter two relates to supersymmetry variations on a field.

The basic idea of the $\mathcal{N}=1$ superspace is to enlarge ordinary spacetime coordinates x^μ, which are associated with the generators P_μ, by adding $2+2$ anti-commuting coordinates $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$, which are associated with the supersymmetry generators $Q_\alpha, \bar{Q}_{\dot{\alpha}}$, to get an eight coordinate superspace $(x^\mu, \theta, \bar{\theta})$.

Before discussing supersymmetry transformations in superspace, let us first recall translations in ordinary spacetime. Consider an infinitesimal translation of a scalar field $\phi(x)$ by a parameter a^μ

$$\phi(x+a) = e^{-ia\mathcal{P}}\phi(x)e^{ia\mathcal{P}} = \phi(x) - ia^\mu[\mathcal{P}_\mu, \phi(x)] + \dots \tag{3.30}$$

Here $\mathcal{P}_\mu$ is the generator of translations in ordinary spacetime. The left hand side of equation (3.30) can also be expanded as

$$\phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x) + \dots \tag{3.31}$$

Equating equation (3.30) and (3.31) gives

$$[\phi(x), \mathcal{P}_\mu] = -i\partial_\mu \phi(x) = P_\mu \phi(x) \tag{3.32}$$

where P_μ is the differential operator that represents the translation operator $\mathcal{P}_\mu$ in field space. As a result, a translation of a field by a parameter a^μ can be expressed as

$$\delta_a \phi = \phi(x+a) - \phi(x) = ia^\mu P_\mu \phi \tag{3.33}$$

The same treatment can be applied to a superfield, this was done in [16] and we will quote the result here. A translation in superspace on a field $Y(x,\theta,\bar{\theta})$ by an infinitesimal parameter $(\epsilon_\alpha, \bar{\epsilon}_{\dot{\alpha}})$ is given by

$$\delta_{\epsilon,\bar{\epsilon}} Y = (i\epsilon Q + i\bar{\epsilon}\bar{Q})Y \tag{3.34}$$

where $Q, \bar{Q}$ are the supersymmetry generators. The Weyl spinors $(\epsilon_\alpha, \bar{\epsilon}_{\dot{\alpha}})$ are the parameters of the supersymmetry transformation.

Recall from chapter two that a BPS state is invariant under some of the supersymmetry generators $Q, \bar{Q}$, that is,

$$Q|\psi\rangle = \bar{Q}|\psi\rangle = 0. \tag{3.35}$$

Equation (3.34) tells us that if a state, or equivalently, a field configuration is BPS with respect to a certain number of supersymmetry operators then its supersymmetric variation with respect to a certain number of spinors ϵ_α will vanish, i.e.

$$\delta_{\epsilon,\bar{\epsilon}} Y = (i\epsilon Q + i\bar{\epsilon}\bar{Q})Y = 0 \tag{3.36}$$

Since the field $Y(x, \theta, \bar{\theta})$ contains separate field components describing each physical particle in the theory, the supersymmetric variation of each field component also vanishes.

Thus a field component $\psi(x)$ is BPS if there exists some spinor $\epsilon(x)$ such that

$$\delta_{\epsilon(x)}\psi = 0. \tag{3.37}$$

Let us return to the $\mathcal{N} = 1$ theory of a graviton and a gravitino we discussed earlier. The BPS condition in equation (3.37) can be expressed explicitly in terms of each of the field as

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu = 0, \tag{3.38}$$

$$\delta\psi_\mu = D_\mu\epsilon(x) = 0. \tag{3.39}$$

As a result, to determine if a state is BPS we need to solve the equations given in (3.38) and (3.39). Very often the fermionic field ψ is set to zero to simplify the calculation. The first condition is automatically satisfied thus we only need to solve the second condition, which is also called the Killing spinor equation.

Spinors which satisfy equation (3.39) are called Killing spinors. Intuitively they are the parameters of preserved supersymmetry of a state. If the Killing spinors are characterized by n_Q constants, the state has n_Q preserved supersymmetries [14].

Provided that we have the explicit form of the Killing spinors, Killing vector can then be constructed as bilinears of Killing spinors as follows

$$K^\nu = \bar{\epsilon'}\gamma^\nu\epsilon. \tag{3.40}$$

3.6 Derivations

For section 3.5

Show that Killing vectors can always be constructed as γ_ν bilinears of Killing spinors:

Let ϵ_α be a Killing spinor, this implies that $\hat{\nabla}_\mu\epsilon_\alpha = \hat{D}_\mu\epsilon_\alpha = 0$. The first equality comes from the fact that ϵ_α is a scalar with respect to the μ index, therefore the affine connection contained within the general covariant derivative $\hat{\nabla}_\mu$ vanishes and the action of $\hat{\nabla}_\mu$ reduces to that of the modified covariant derivative $\hat{D}_\mu$. The second equality is a straight forward application of the definition of a Killing spinor.

We construct a vector K_ν from the γ_ν bilinear of two Killing spinor ϵ_α and ϵ'_α

$$K_\nu = \bar{\epsilon}'\gamma_\nu\epsilon = \bar{\epsilon}'_{\dot{\alpha}}\gamma_\nu^{\dot{\alpha}\alpha}\epsilon_\alpha, \tag{3.41}$$

then

$$\hat{\nabla}_\mu K_\nu = \hat{\nabla}_\mu(\bar{\epsilon}'\gamma_\nu\epsilon) = \bar{\epsilon}'(\hat{\nabla}_\mu\gamma_\nu)\epsilon. \tag{3.42}$$

Let us consider

$$\hat{\nabla}_\mu\gamma_\nu = \partial_\mu\gamma_\nu + \frac{1}{4}\omega_\mu^{ab}[\gamma_{ab},\gamma_\nu] - \Gamma^\rho_{\mu\nu}\gamma_\rho - \frac{1}{2l}[\gamma_\mu,\gamma_\nu] = -\frac{1}{l}\gamma_{\mu\nu}. \tag{3.43}$$

The first three terms vanish when acting on γ_ν due to the vielbein postulate

$$\hat{\nabla}_\mu K_\nu = -\frac{1}{l}\bar{\epsilon}'\gamma_{\mu\nu}\epsilon. \tag{3.44}$$

Since $\nabla_\mu K_\nu$ is antisymmetric in $\mu\nu$, the Killing vector equation, which is symmetric in $\mu\nu$, is satisfied

$$\hat{\nabla}_{(\mu}K_{\nu)} = -\frac{1}{l}\bar{\epsilon}'(\gamma_{\mu\nu} - \gamma_{\nu\mu})\epsilon = 0. \tag{3.45}$$

Hence K_ν is a Killing vector.

Chapter 4

BPS Kerr black holes

We will examine BPS states for both asymptotically flat and AdS Kerr black holes in the context of $\mathcal{N} = 1$ supergravity in four dimensions. In flat space, the answer is already known from the superalgebra, there are no BPS rotating black holes. We will confirm this by directly manipulating the Killing spinor equation to show that the equation admits no massive solution. In AdS space, there are hints from the AdS superalgebra that there might exist BPS black holes with non-vanishing angular momentum. The limit in which these black holes can exist coincides with the ultra-spinning limit $a \to l$ [4]. We will present our work in progress towards solving for the complete set of Killing spinors of this case.

4.1 Non-existence of BPS Kerr black holes in flat space

As explained in chapter three, the supersymmetry admitted by a Kerr solution in $\mathcal{N} = 1$ supergravity is characterised by the solutions to the Killing spinor equation

$$D_\mu \epsilon_\alpha = 0, \tag{4.1}$$

where D_μ is the Lorentz covariant derivative which acts on spinors as

$$D_\mu \epsilon_\alpha = \Big(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\Big)\epsilon_\alpha.$$

To solve for the spinor $\epsilon(x)$ we first need to determine the spin connection ω^{ab} for Kerr black holes embedded in Minkowski space.

The defining equation for the spin connection in the language of differential form is given by

$$de^a + \omega^a{}_b(e) \wedge e^b = 0. \tag{4.2}$$

The Kerr metric in flat space was given in (1.6) but let us rewrite it here for the completeness of the calculation

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\big(adt - (r^2+a^2)d\phi\big)^2, \tag{4.3}$$

where

$$\Delta = \Delta(r) = r^2 - 2Mr + a^2 \qquad , \qquad \rho^2 = \rho^2(r,\theta) = r^2 + a^2\cos^2\theta.$$

If we choose the vielbeins to be

$$\begin{aligned} e^0 &= \frac{\sqrt{\Delta}}{\rho}(dt - a\sin^2\theta d\phi), & e^1 &= \frac{\rho}{\sqrt{\Delta}}dr, \\ e^3 &= \frac{\sin\theta}{\rho}\big(adt - (r^2+a^2)d\phi\big), & e^2 &= \rho d\theta. \end{aligned} \tag{4.4}$$

we can then work out the non-zero components of the spin connection:

$$\begin{aligned} \omega_t^{01} &= \frac{(r-M)\rho^2 - r\Delta + ra^2\sin^2\theta}{\rho^4} & \omega_\phi^{01} &= \frac{a\sin^2\theta}{\rho^4}\Big(r\big(\Delta - (r^2+a^2)\big) - \rho^2(r-M)\Big) \\ \omega_\phi^{02} &= -\frac{\sqrt{\Delta}a\sin\theta\cos\theta}{\rho^2} \\ \omega_r^{03} &= \frac{r a\sin\theta}{\sqrt{\Delta}\rho^2} & \omega_\theta^{03} &= \frac{\sqrt{\Delta}a\cos\theta}{\rho^2} \\ \omega_r^{12} &= -\frac{a^2\cos\theta\sin\theta}{\rho^2\sqrt{\Delta}} & \omega_\theta^{12} &= \frac{r\sqrt{\Delta}}{\rho^2} \\ \omega_\phi^{13} &= \frac{r\sqrt{\Delta}\sin\theta}{\rho^2} \\ \omega_t^{23} &= -\frac{a\cos\theta((r^2+a^2)-\Delta)}{\rho^4} & \omega_\phi^{23} &= \frac{\cos\theta}{\rho^4}\big(-a^2\sin^2\theta\Delta + (r^2+a^2)^2\big). \end{aligned} \tag{4.5}$$

Notice that all components of the spin connection are functions of only r and θ, not of t and ϕ. It is therefore sensible to first examine the $\mu = t$ and $\mu = \phi$ components of the Killing spinor equation (4.1).

For $\mu = t$, the Killing spinor equation becomes

$$\partial_t \epsilon_\alpha = -\frac{1}{2}\Big(\omega_t^{01}\gamma_{01} + \omega_t^{23}\gamma_{23}\Big)\epsilon_\alpha. \tag{4.6}$$

Since ω_t^{01} and ω_t^{23} are constant with respect to t, we can solve this first order differential equation by choosing a basis for the γ-matrices and using the eigenvalue method. In the real basis for the γ-matrices, equation (4.6) becomes

$$\partial_t \epsilon_\alpha = -\frac{1}{2}\begin{bmatrix} & i\omega_t^{23} & & -\omega_t^{01} \\ i\omega_t^{23} & & -\omega_t^{01} & \\ & -\omega_t^{01} & & i\omega_t^{23} \\ -\omega_t^{01} & & i\omega_t^{23} & \end{bmatrix}\epsilon_\alpha. \tag{4.7}$$

The eigenvalues and corresponding eigenvectors for this system are

$$\lambda_1 = \tfrac{1}{2}(-\omega_t^{01} + i\omega_t^{23}) \quad , \quad \lambda_2 = -\lambda_1 \quad , \quad \lambda_3 = \tfrac{1}{2}(-\omega_t^{01} - i\omega_t^{23}) \quad , \quad \lambda_4 = -\lambda_3 \tag{4.8}$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad , \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad , \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}. \tag{4.9}$$

As a result, the general solution for the real spinor ϵ is

$$\epsilon_\alpha(t,r,\theta,\phi) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 e^{\lambda_1 t} \\ \alpha_2 e^{\lambda_2 t} \\ \alpha_3 e^{\lambda_3 t} \\ \alpha_4 e^{\lambda_4 t} \end{bmatrix}, \tag{4.10}$$

where $\alpha_i = \alpha_i(r,\theta,\phi)$ for $i = 1,2,3,4$. We have separated the t-dependence of ϵ into the exponents $e^{\lambda_i t}$ and the r,θ,ϕ-dependence is contained within the coefficients α_i. This is the best we can do with $\mu = t$.

For $\mu = \phi$ the Killing spinor equation gives

$$\partial_\phi \epsilon_\alpha = \frac{1}{2}\Big(\omega_\phi^{01}\gamma_{01} + \omega_\phi^{02}\gamma_{02} + \omega_\phi^{13}\gamma_{13} + \omega_\phi^{23}\gamma_{23}\Big)\epsilon_\alpha. \tag{4.11}$$

In the real basis for the γ-matrices equation (4.11) becomes

$$\partial_\phi \epsilon_\alpha = \frac{1}{2}\begin{bmatrix} & -\omega_\phi^{13} + i\omega_\phi^{23} & & -\omega_\phi^{01} + i\omega_\phi^{02} \\ \omega_\phi^{13} + i\omega_\phi^{23} & & -\omega_\phi^{01} + i\omega_\phi^{02} & \\ & -\omega_\phi^{01} + i\omega_\phi^{02} & & -\omega_\phi^{13} + i\omega_\phi^{23} \\ -\omega_\phi^{01} + i\omega_\phi^{02} & & \omega_\phi^{13} + i\omega_\phi^{23} & \end{bmatrix}\epsilon_\alpha. \tag{4.12}$$

We could try to solve for the eigenvalues and eigenvectors of this system, in a similar manner to what we did earlier for $\mu = t$. However, a faster way to solve this is to notice that we can write $\epsilon(t,r,\theta,\phi)$ as

$$\epsilon_\alpha(t,r,\theta,\phi) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 e^{\lambda_1 t} \\ \alpha_2 e^{\lambda_2 t} \\ \alpha_3 e^{\lambda_3 t} \\ \alpha_4 e^{\lambda_4 t} \end{bmatrix} = A\xi_\alpha, \tag{4.13}$$

where A is the 4×4 matrix and ξ_α is the column vector. Similarly we can shorten equation (4.12) as $\partial_\phi\epsilon = B\epsilon$ where B is the 4×4 matrix in equation (4.12).

As a result, we have the relations

$$\begin{cases}\partial_\phi\epsilon_\alpha = B\epsilon_\alpha = BA\xi_\alpha\\ \partial_\phi\epsilon_\alpha = \partial_\phi(A\xi_\alpha) = A\partial_\phi\xi_\alpha.\end{cases} \tag{4.14}$$

therefore

$$\partial_\phi\xi_\alpha = A^{-1}BA\xi_\alpha. \tag{4.15}$$

Since we already chose to work in a particular basis (in our case, the real basis) of the γ-matrices, we can easily compute and examine $A^{-1}BA$. Equation (4.15) then becomes

$$\partial_\phi\begin{bmatrix}\alpha_1 e^{\lambda_1 t}\\ \alpha_2 e^{\lambda_2 t}\\ \alpha_3 e^{\lambda_3 t}\\ \alpha_4 e^{\lambda_4 t}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-\omega_\phi^{01}+i(\omega_\phi^{02}+\omega_\phi^{23}) & \omega_\phi^{13} & & \\ -\omega_\phi^{13} & \omega_\phi^{01}-i(\omega_\phi^{02}+\omega_\phi^{23}) & & \\ & & -\omega_\phi^{01}+i(\omega_\phi^{02}+\omega_\phi^{23}) & -\omega_\phi^{13}\\ & & \omega_\phi^{13} & \omega_\phi^{01}-i(\omega_\phi^{02}+\omega_\phi^{23})\end{bmatrix}\begin{bmatrix}\alpha_1 e^{\lambda_1 t}\\ \alpha_2 e^{\lambda_2 t}\\ \alpha_3 e^{\lambda_3 t}\\ \alpha_4 e^{\lambda_4 t}\end{bmatrix}$$

Since the matrix $A^{-1}BA$ is block-diagonal, the first and second component of ξ are coupled together, similarly for the third and fourth component. Let us examine the first two components

$$\begin{cases}\partial_\phi(\alpha_1 e^{\lambda_1 t}) = (\partial_\phi\alpha_1)e^{\lambda_1 t} = \big[-\omega_\phi^{01}+i(\omega_\phi^{02}+\omega_\phi^{23})\big]\alpha_1 e^{\lambda_1 t} + \omega_\phi^{13}\,\alpha_2 e^{\lambda_2 t}\\ \partial_\phi(\alpha_2 e^{\lambda_2 t}) = (\partial_\phi\alpha_2)e^{\lambda_2 t} = -\omega_\phi^{13}\,\alpha_1 e^{\lambda_1 t} + \big[\omega_\phi^{01}-i(\omega_\phi^{02}+\omega_\phi^{23})\big]\alpha_2 e^{\lambda_2 t}.\end{cases} \tag{4.16}$$

Since the t-dependence has been put into the exponentials, we can remove the exponentials and examine what remains

$$\begin{cases}\partial_\phi\alpha_1 = \big[-\omega_\phi^{01}+i(\omega_\phi^{02}+\omega_\phi^{23})\big]\alpha_1 + \omega_\phi^{13}\,\alpha_2 e^{(\lambda_2-\lambda_1)t}\\ \partial_\phi\alpha_2 = -\omega_\phi^{13}\,\alpha_1 e^{(\lambda_1-\lambda_2)t} + \big[\omega_\phi^{01}-i(\omega_\phi^{02}+\omega_\phi^{23})\big]\alpha_2.\end{cases} \tag{4.17}$$

The left hand sides are no longer dependent on t while the right hand sides are still functions of t. As a result, in order for the Killing spinor equation to be self-consistent, we need to require $\lambda_1 = \lambda_2$. Since we defined $\lambda_1 = -\lambda_2$ earlier, it must be that $\lambda_1 = \lambda_2 = 0$. The same can be deduced about λ_3 and λ_4. Therefore

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0. \tag{4.18}$$

Upon recalling what the eigenvalues λ_i are in terms of the spin connection, we get the condition

$$\omega_t^{01} = \omega_t^{23} = 0. \tag{4.19}$$

From the coefficients of the spin connection (4.5) this happens if and only if either $a = 0$ or $M = 0$.

Since a is the rotation parameter and M is the mass parameter in our Kerr metric, the condition for self-consistency of the Killing spinor equation in flat space implies that there is no massive spinning supersymmetric black hole in flat space.

This is consistent with the result from the supersymmetry algebra, that the $\mathcal{N} = 1$ superalgebra does not admit massive BPS states. This is also consistent with the fact that there is no global timelike Killing vector for asymptotically flat Kerr black hole.

4.2 BPS Kerr black holes in anti-de Sitter space

The Killing spinor equation for AdS space is slightly different from the one in flat space. The existence of a non-zero cosmological constant implies a constant curvature of spacetime independent of matter. This leads to the need for a modified Lorentz covariant derivative suitable for the de Sitter group as mentioned in earlier chapter.

The Killing spinor equation takes the form

$$\hat{D}_\mu \epsilon_\alpha = \Big(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \frac{1}{2l}\gamma_\mu\Big)\epsilon_\alpha = 0. \tag{4.20}$$

Since we suspect that the BPS bound is satisfied by the ultra-spinning limit, we will be working in limit $a \to l$. The metric in this limit is given in (1.23), which we will rewrite here

$$ds^2 = \frac{dr^2}{V(r)} + \frac{l^2 + r^2}{4}\big(d\sigma^2 + \sinh^2\sigma d\phi^2\big) - V(r)\Big(dt - l\sinh^2\tfrac{\sigma}{2}d\phi\Big)^2, \tag{4.21}$$

with

$$V(r) = 1 + \frac{r^2}{l^2} - \frac{2Mr}{l^2 + r^2}. \tag{4.22}$$

Similar to the flat case, we can choose a set of vielbeins as follows

$$\begin{aligned} e^0 &= \sqrt{V(r)}\big(dt - l\sinh^2\tfrac{\sigma}{2}d\phi\big) & e^1 &= \frac{1}{\sqrt{V(r)}}dr \\ e^3 &= \frac{\sqrt{l^2 + r^2}}{2}\sinh\sigma d\phi & e^2 &= \frac{\sqrt{l^2 + r^2}}{2}d\sigma \end{aligned} \tag{4.23}$$

Solving for the spin connection gives

$$
\begin{aligned}
\omega_t^{01} &= \frac{1}{2}V'(r) \qquad\qquad \omega_\phi^{01} = -\frac{1}{2}lV'(r)\,\sinh^2\tfrac{\sigma}{2} \\
\omega_\phi^{02} &= -\frac{l\sqrt{V(r)}\sinh\sigma}{2\sqrt{l^2+r^2}} \\
\omega_\sigma^{03} &= \frac{l\sqrt{V(r)}}{2\sqrt{l^2+r^2}} \\
\omega_\sigma^{12} &= -\frac{r\sqrt{V(r)}}{2\sqrt{l^2+r^2}} \\
\omega_\phi^{13} &= -\frac{r\sqrt{V(r)}\sinh\sigma}{2\sqrt{l^2+r^2}} \\
\omega_t^{23} &= -\frac{l\,V(r)}{l^2+r^2} \qquad\qquad \omega_\phi^{23} = -\cosh\sigma + \frac{l^2\,V(r)\,\sinh^2\frac{\sigma}{2}}{l^2+r^2}
\end{aligned}
\tag{4.24}
$$

Notice that all of the coefficients are functions of only r and σ, not of t and ϕ, and that all the $\mu = r$ components vanish. Hence, the $\mu = r$ component of the Killing spinor equation reads

$$\hat{D}_r\epsilon_\alpha = 0. \tag{4.25}$$

This gives

$$\partial_r\epsilon_\alpha = \frac{1}{2l\sqrt{V}}\gamma_1\epsilon_\alpha. \tag{4.26}$$

In the real basis for the γ-matrices we can derive

$$
\begin{cases}
\epsilon_1 = c_1\ \exp\left(\int_0^r \frac{1}{2l\sqrt{V(r')}}dr'\right) + d_1 \\
\epsilon_2 = c_2\ \exp\left(\int_0^r \frac{1}{2l\sqrt{V(r')}}dr'\right) + d_2 \\
\epsilon_3 = c_3\ \exp\left(-\int_0^r \frac{1}{2l\sqrt{V(r')}}dr'\right) + d_3 \\
\epsilon_4 = c_4\ \exp\left(-\int_0^r \frac{1}{2l\sqrt{V(r')}}dr'\right) + d_4
\end{cases}
\tag{4.27}
$$

where the r-dependence has been isolated to the exponential, leaving c_i and d_i dependent on only (t, σ, ϕ).

The $\mu = t$ component of equation (4.20) gives

$$
\begin{aligned}
\partial_t\epsilon_\alpha &= -\frac{1}{2}\Big(\omega_t^{01}\gamma_{01} + \omega_t^{23}\gamma_{23} - \frac{1}{l}e_t^0\gamma_0\Big)\epsilon_\alpha \\
&= -\frac{1}{2}\Big(\omega_t^{01}\gamma_{01} + \omega_t^{23}\gamma_{23} - \frac{\sqrt{V}}{l}\gamma_0\Big)\epsilon_\alpha.
\end{aligned}
\tag{4.28}
$$

Work is still to be done to solve equation (4.28) and the $\mu = \sigma, \phi$ component of the Killing spinor equation given the form of the Killing spinors we obtained in equation (4.27).

Chapter 5

Conclusion and outlook

We began by reviewing Kerr black holes in both flat and AdS space. As we have seen, asymptotically flat Kerr black holes have no global timelike Killing vectors. Kerr black holes in AdS were shown to possess an ergoregion, however this does not preclude them from admitting timelike Killing vectors.

We examined the $\mathcal{N} = 1$ Poincaré and AdS supersymmetry algebras and showed that the former does not allow massive BPS states, while the later suggests that $\frac{1}{2}$-BPS states occur at $E = |J|$, which is the ultra spinning limit of the black hole.

We also gave a brief introduction to $\mathcal{N} = 1$ supergravity in both flat and AdS space in four dimensions and saw how the Killing spinor equation emerges from the BPS bound in supergravity. As a result, searching for BPS black holes means solving the Killing spinor equation. For asymptotically flat Kerr black holes, self consistency of the Killing spinor equation implies that there are no massive, rotating BPS solutions. This is consistent with the result from the supersymmetry algebra, that there is no massive BPS state for $\mathcal{N} = 1$. This is also consistent with the fact that asymptotically flat Kerr admits no globally timelike Killings vectors.

For asymptotically AdS Kerr the fact that a globally timelike Killing vector exists on this geometry, together with the hints from the superalgebra that BPS states occur at $E = |J|$, suggests that we might be able to find the complete form of the preserved supersymmetries for this space. We presented our solution to the Killing spinor equation as it currently stands and work is still needed to determine the complete set of Killing spinor equations.

One puzzle that we still need to understand is the ultra spinning limit. The resulting metric after taking the limit is not unique, and depends on how the limit is taken. We would like to understand if and how the procedure affects the number of preserved supersymmetries of the original asymptotically flat Kerr black hole.

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