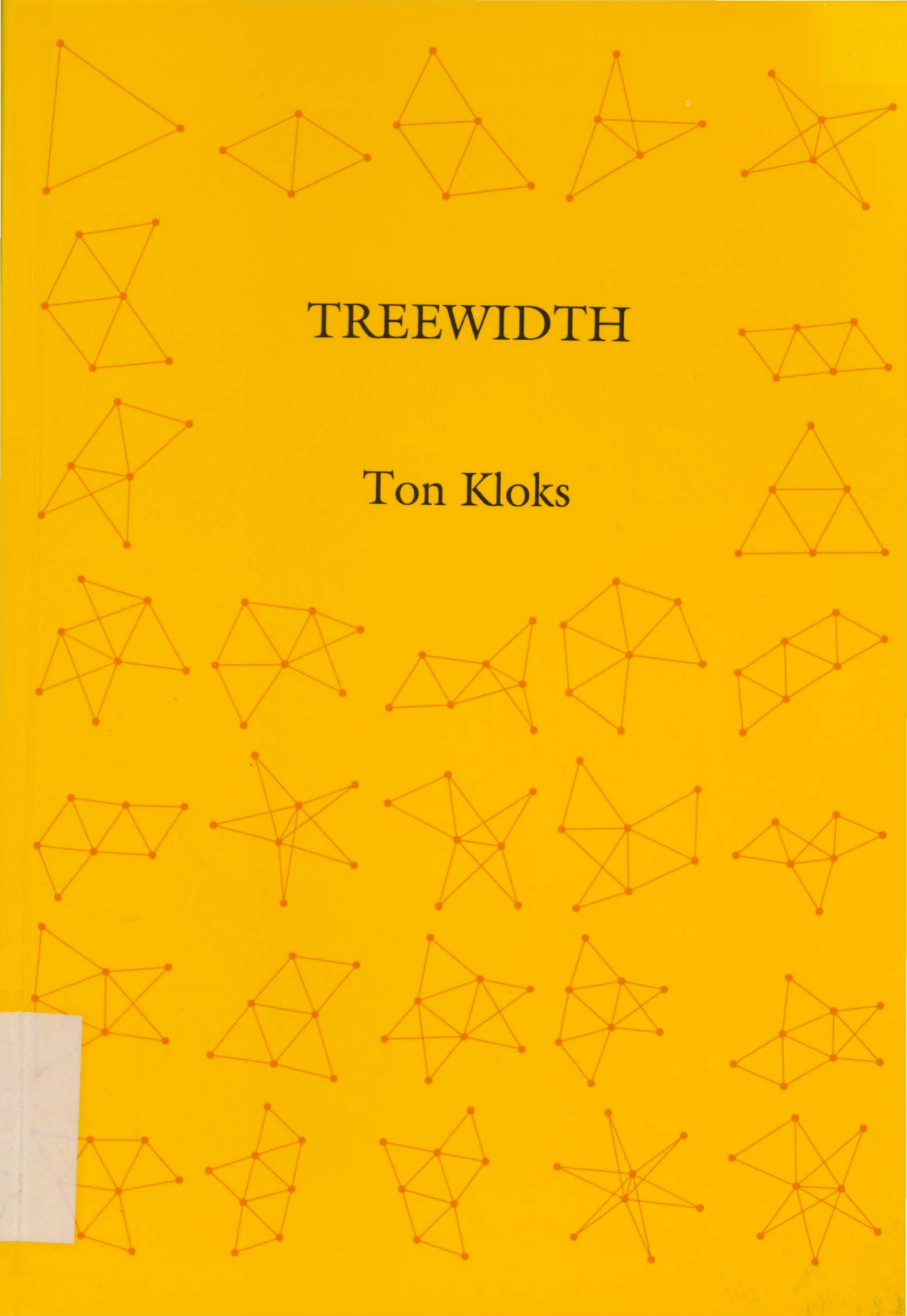

TREEWIDTH

Ton Kloks

Treewidth

Boombreedte

(met een samenvatting in het Nederlands)

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Rijksuniversiteit te Utrecht
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ingevolge het besluit van het College van Dekanen
in het openbaar te verdedigen
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te Tilburg

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Cover: 2-trees up through eight vertices. The drawings were obtained using the algorithm described in: T. Kamada and S. Kawai, An algorithm for drawing general undirected graphs, *Information Processing Letters* **31**, (1989), pp. 7–15.

Inhoudsopgave

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Chapter 1

Introduction

In this thesis we consider a number of problems related to treewidth and pathwidth of graphs. The main goal is to obtain good bounds on the complexity of determining the treewidth and pathwidth for various interesting classes of graphs. In this section we review some basic concepts and notations and outline the main results of the thesis.

We assume that the reader is familiar with the most well-known notions from graph theory (see, e.g., [64]). Unless stated otherwise we assume that all graphs are finite, undirected and without loops or multiple edges.

As a standard notation we use $G = (V, E)$, where G is a graph with vertex set V and edge set E. We refer to elements of V as vertices and to elements of E as edges. Each edge is an unordered pair of vertices. Two different vertices are called adjacent if they are both incident with the same edge. For each vertex x we denote by $N(x)$ the neighborhood of x, the set of vertices which are adjacent to x.

Let $G = (V, E)$ be a graph. A subgraph of G is a graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. If $W \subseteq V$ is a subset of vertices then $G[W]$ denotes the subgraph *induced* induced by W, i.e., $G[W]$ is the subgraph with vertex set W in which two vertices are adjacent whenever they are adjacent in G. If $W = \{v_1, \ldots, v_s\}$ then we also use the notation $G[v_1, \ldots, v_s]$ for $G[W]$. If W is a subset of vertices such that every pair of vertices is adjacent then W is called a clique in G. the clique size of a clique W in G is the number of vertices of W. The maximum clique size of a clique in G, $\omega(G)$, is called the clique number of G.

Let $G = (V, E)$ be a graph. A vertex coloring is a mapping $f : V \to S$ from the vertex set into a set S of colors such that no two adjacent vertices have the same color, i.e., $f(x) \neq f(y)$ whenever x and y are adjacent. The minimum number of colors for which there exists a vertex coloring of a graph G is called the chromatic number, $\chi(G)$.

A walk is an alternating sequence of vertices and edges, starting and ending with a vertex, in which every edge is incident with the two vertices immediately preceding and following it. If all vertices of a walk are distinct (and hence also all edges in the walk are distinct) the walk is called a path. If the walk is closed,

i.e., the starting and ending vertex are the same, and if all vertices in the walk are distinct, then the walk is called a cycle. If two vertices of a cycle in G are adjacent if and only if the incident edge is also in the cycle, then the cycle is called chordless. A chordless cycle with three vertices is called a triangle and a chordless cycle with four vertices is called a square.

A tree is a special type of graph, which is connected (i.e., every two vertices are connected by a path) and does not contain any cycle. The vertices in a tree usually are called nodes or points. A natural generalization of trees are the so-called k-trees. A k-tree is a graph which can be recursively defined as follows. A clique with $k+1$ vertices is a k-tree and a k-tree with $n+1$ vertices can be obtained from a k-tree with n vertices by making a new vertex adjacent to exactly all vertices of a k-clique. Subgraphs of k-trees are called partial k-trees. If a partial k-tree G is a subgraph of a k-tree H, we call H a k-tree embedding for G.

The minimum value k for which a graph is a subgraph of a k-tree is called the treewidth of a graph. It is not difficult to see that k-trees are partial ℓ-trees for every $\ell \geq k$. Hence every graph with treewidth k is a partial ℓ-tree for every $\ell \geq k$, i.e., the class of partial k-trees is exactly the class of graphs with treewidth at most k.

Because of the underlying tree-like structure, it turns out that for many difficult (e.g., NP-hard) problems, there exist algorithms of which the running time is polynomial in the number of vertices but exponential in the treewidth k of a graph G, if a k-tree embedding of G is given. It follows that for each value k these problems are solvable in polynomial time when restricted to the class of graphs with treewidth at most k, assuming a k-tree embedding of the graph is part of the input. It is an NP-complete problem to decide whether or not for a given graph and given integer k, the graph has treewidth at most k [4]. So finding a k-tree embedding of the input graph for which the value k is as small as possible is a hard problem. Fortunately, in many important special cases this problem can be solved efficiently. Using results of chapter 11 and chapter 12 (see also [88]), it is shown in [18] that for each value k there is a linear time algorithm to decide whether a given graph G has treewidth k. Furthermore, if the graph has indeed treewidth at most k, the algorithm can be implemented such that it outputs a k-tree of which the graph is a subgraph. As a consequence, we may conclude that for each k there exist fast algorithms for many NP-hard problems when restricted to the class of partial k-trees.

The existence of polynomial time algorithms for treewidth and pathwidth for graphs of bounded treewidth and pathwidth respectively, can also be shown by using the theory of graph minors. Although we do not use much of the theory of graph minors, we include a short list of the most important results. If G is a graph and (u,v) is an edge of G, the graph obtained by *contracting* the edge (u,v) is the graph obtained from $G[V \setminus \{u,v\}]$, by adding a new vertex z and adding edges (z,w) for all $w \in (N(u) \cup N(v)) \setminus \{u,v\}$. A graph H obtained from G by a series of vertex deletions, edge deletions and contractions is called a *minor* of G. A class

of graphs $\mathcal{F}$ is called minor-closed if for every graph G in $\mathcal{F}$, every minor of G is also a member of $\mathcal{F}$. For a class of graphs $\mathcal{F}$, a finite obstruction set S is a finite set of minors such that a graph is a member of $\mathcal{F}$ if and only if it does not contain an element of S as a minor. Some of the most important results and conjectures from the theory of graph minors are listed below.

1. Kuratowski showed that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

2. Hadwiger's conjecture states that if a graph does not have K_s as a minor, then the chromatic number is less than s. Wagner showed that in the case of $s = 5$, this is implied by the 4-Color-Conjecture [129].

3. Wagner's conjecture, stating that every minor-closed class of graphs has a finite obstruction set, was proved by Robertson and Seymour [110].

4. If X is a graph, and $\mathcal{G}[HX]$ the class of graphs with no minor isomorphic to X, then there is a constant c such that all graphs of $\mathcal{G}[HX]$ have treewidth $\leq c$ if and only if X is planar. This was proved by Robertson and Seymour [112].

5. For every graph H, there is an $O(n^3)$ algorithm to test whether H is a minor of a given graph G with n vertices. It follows that the graphs in every minor-closed class of graphs are recognizable in $O(n^3)$ time [110].

6. For every graph H and every constant k, there is an $O(n)$ algorithm to test whether H is a minor of a given graph G with n vertices and treewidth at most k. It follows that every minor-closed class of graphs which does not contain all planar graphs is recognizable in $O(n)$ time (see [18] and chapters 11 and 12).

Notice that for each constant k, the class of partial k-trees is minor-closed. For $k = 2, 3$ the obstruction sets have been determined [6]. For larger values of k there does not seem to be a practical way to do this.

In chapter 2, we start with some basic terminology and tools which are useful when dealing with k-trees and chordal graphs in general. We introduce tree-decompositions of width k of a partial k-tree G as an important alternative and equivalent way to describe k-tree embeddings of G. A special type of tree-decompositions are path-decompositions, and the related embeddings in triangulated graphs are interval graphs.

Chapters 3, 4, 5 and chapter 6 deal with problems related to the structure of partial k-trees. In chapter 3 a complete characterization is given for the structure of partial 2-trees. With this characterization we are able to solve the following problem in linear time and using a linear amount of memory: Given a graph of which each vertex is colored with one of three colors such that adjacent vertices have a different color, find, if it exists, an embedding of the graph in a triangulated

graph which is also properly colored. If such an embedding exists then there also exists an embedding in a 2-tree. The general problem, to find a k-tree embedding which is properly colored for a graph colored with k colors, stems from molecular biology where it is known as the 'perfect phylogeny problem'.

In chapter 4 we use the characterization of partial 2-trees, in order to solve the enumeration problem of biconnected partial 2-trees. It turns out that biconnected partial 2-trees generalize clusters, which is related to the well-known and often studied structures called 'animals'. The result we present is of interest also because the enumeration of these animals in general is, to our knowledge, an unsolved problem.

In chapter 5 we show that a random graph with a number of edges which is only linear in the number of vertices, has, with high probabibity, a large treewidth (proportional to the number of vertices). Perhaps this illustrates the importance to restrict the search for graph with bounded treewidth to graphs which already have some underlying structure. Research in this direction is reported in chapter 7, 8 and chapter 9. In these chapters we restrict ourselves to special classes of perfect graphs.

In chapter 6 we show for each k there exists a linear time algorithm to test superperfection of k-trees. The fact that the class of superperfect graphs is indeed very large is illustrated by the fact that all comparability graphs and hence, for example, all bipartite graphs, are superperfect. Strangely enough, there are only few other classes of graphs for which superperfection can be tested in polynomial time. Indeed we do not even know if the problem is in NP. We show that for each k, there is a finite list of structures, which can be computed in $O(1)$ time, such that a k-tree is superperfect if and only if it does not contain a structure from this list. It is then easy to see that for each k testing superperfection of k-trees can be done in linear time.

Since the algorithms to compute a tree-decomposition of a graph with treewidth at most k are of practical use only for small values of k at the moment, it is of great importance to devise fast algorithms for determining the treewidth for classes of graphs which are of special practical importance. For some classes of graphs the treewidth and pathwidth can be computed in polynomial time. For example this is the case for cographs [22]. The treewidth can also be computed efficiently for circular arc graphs [125].

In chapter 7, we show that for many important classes of perfect graphs, there exist very fast algorithms which approximate treewidth within a constant factor. For example we give algorithms to approximate treewidth within a constant factor for cotriangulated graphs, split graphs, convex graphs and permutation graphs. For the large class of cocomparability graphs, we give a fast approximation algorithm to compute a tree-decomposition of width at most k^2 where k is the treewidth of the graph. As an important, and perhaps surprising side-effect, we see that for all these graphs the approximate tree-decomposition we construct is in fact a path-decomposition, which shows a strong relationship between the

pathwidth and treewidth of these graphs.

In chapter 8 we give a polynomial time algorithm to compute the treewidth of chordal bipartite graphs. This is an important subclass of the class of bipartite graphs, containing many other interesting classes of bipartite graphs. Our result is even more of interest since the problem of determining the treewidth of bipartite graphs in general is NP-hard [4].

In chapter 9 we give a fast algorithm to compute the treewidth and pathwidth of a permutation graph. We show that this generalizes in the following way. For each d there is a polynomial time algorithm to compute the treewidth of a cocomparability graph of which the partial order dimension of the complement is bounded by d. This is of special interest since treewidth is NP-hard for cocomparability graphs in general.

It turns out that the treewidth and pathwidth of a permutation graph are equal, and in fact this is true for all cocomparability graphs. This was generalized recently by M. Habib and R. Möhring [61] who showed that for all AT-free graphs the treewidth and pathwidth are equal.

We like to mention here that recently we found that the results of chapter 8 and chapter 9 can be generalized in the following way. Consider a class of graphs $\mathcal{G}$ having a polynomial time algorithm computing the set of all minimal separators for every graph in $\mathcal{G}$. Then there exists a polynomial time algorithm for TREEWIDTH and the related problem MINIMUM FILL-IN when restricted to the class $\mathcal{G}$. This shows that TREEWIDTH and MINIMUM FILL-IN can be computed in polynomial time for many interesting classes like permutation graphs, circular permutation graphs, cocomparability graphs of bounded dimension, trapezoid graphs, cotriangulated graphs, circle graphs, circular arc graphs, distance hereditary graphs, chordal bipartite graphs etc.

In chapter 10 we look at the problem to find approximate tree-decompositions and path-decompositions for graphs in general. We show that the problem is closely related to that of finding balanced vertex separators in graphs. We use two of these algorithms. The first is an algorithm of Alon et al. [1] that finds a good vertex separator with approximation ratio $O(\sqrt{kn})$. Using this algorithm as a basic tool we show there exists a fast algorithm that finds a path-decomposition of width $O(k^{3/2}\sqrt{n})$, where k is the treewidth of the graph.

An approximation for vertex separators within a factor $O(\log n)$ due to Leighton and Rao was reported in [74]. We use this to develop an algorithm which finds a tree-decomposition of width within a factor $O(\log n)$ off the optimum.

In chapter 11 and chapter 12 we show that for each k the decision problems 'treewidth $\leq k$' and 'pathwidth $\leq k$' can be solved in $O(n \log(n))$ time. We use a result of [107] stating that for every constant k there exists an $O(n \log(n))$ algorithm that given a graph with n vertices either outputs that the treewidth is larger than k, or outputs a tree-decomposition of width at most $3k + 2$ (see also [17]). Using this approximate tree-decomposition we show that the exact pathwidth and treewidth can be computed in linear time. Furthermore, a path-

or tree-decomposition of optimal width can be found within the same time bound.

Recently, using the results of chapter 11 and chapter 12, it was shown in [18] that for each constant k it can be decidecd in linear time whether a graph has treewidth or pathwidth at most k.

Chapter 2

Preliminaries

In this chapter we review some basic terminology and aspects of triangulated graphs. We define the concepts treewidth and pathwidth of a graph in terms of subgraphs of triangulated graphs. We then show the equivalence between the treewidth and pathwidth of a graph and the minimum width of a tree-decomposition and path-decomposition. We end this section with some notes on perfect graphs.

2.1 Triangulations

In this section we review some important aspects of triangulated graphs and of triangulations of graphs. The various names given to triangulated graphs, such as chordal graphs, rigid circuit graphs, perfect elimination graphs, monotone transitive graphs and so on, illustrate the importance of this class of graphs. We shall mainly use the term triangulated graphs. For a good overview of the different aspects of triangulated graphs the reader is referred to Golumbic's book [58].

Definition 2.1.1 *A graph is triangulated if it contains no chordless cycle of length greater than three.*

This is equivalent to saying that the graph does not contain an induced subgraph isomorphic to C_n (i.e., a cycle of length n) for $n > 3$. Notice that being triangulated is a hereditary property, i.e., if a graph G is triangulated then every induced subgraph of G also is triangulated.

There are many ways to characterize triangulated graphs. Characterizations have been appearing from 1960 [43] until very recently [10]. Although many of these characterizations are interesting and useful, it suffices for our purposes to list only some of them.

One of the most important tools that can be used when working with triangulated graphs is the concept of a perfect elimination scheme.

Definition 2.1.2 *Let* $G = (V, E)$ *be a graph. A* simplicial vertex *of* G *is a vertex of which the neighborhood induces a clique. An ordering of the vertices* $\sigma = [v_1, \ldots, v_n]$ *is called a* perfect elimination scheme *if for every* $1 \leq i \leq n$, v_i *is a simplicial vertex in* $G[v_i, \ldots, v_n]$.

Fulkerson and Gross, in a paper in 1965 [53], characterized triangulated graphs by means of a perfect elimination scheme.

Lemma 2.1.1 *A graph* G *is triangulated if and only if there exists a perfect elimination scheme for* G. *Moreover, if a graph is triangulated, any simplicial vertex can start a perfect elimination scheme for it.*

In 1975 Rose and Tarjan [115] gave a linear time algorithm for recognizing triangulated graphs, by finding a perfect elimination scheme if it exists. Many NP-hard problems are solvable in linear time when restricted to triangulated graphs like CLIQUE, INDEPENDENT SET, CLIQUE COVER etc. Lemma 2.1.1 has many useful consequences. For example, every triangulated graph has at least one simplicial vertex. In fact, Dirac showed that if a graph is not a clique, then it has at least two nonadjacent simplicial vertices. This can be used to obtain yet another characterization of triangulated graphs: a graph is triangulated if and only if every induced subgraph is either a clique or has two nonadjacent simplicial vertices. We conclude that if G is triangulated and C is the vertex set of a maximal clique in G, then there exists a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$ such that $C = \{v_{n-t+1}, \ldots, v_n\}$ (where $t = |C|$).

Notice that if x is a simplicial vertex in a triangulated graph, then $\{x\} \cup N(x)$ is a maximal clique, and in fact this is the only maximal clique containing x. It follows that there are at most n maximal cliques, with equality holding if and only if the graph has no edges. This was first pointed out in [53]. Notice that if $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme, then all maximal cliques are of the form $\{v_i\} \cup (N(v_i) \cap \{v_i, \ldots, v_n\})$.

The last immediate consequence of Lemma 2.1.1 that we mention here is that a triangulated graph is perfect (a graph is called perfect if the maximum clique size is equal to the chromatic number for every induced subgraph; see section 2.3). This can be seen as follows. First, since every induced subgraph of a triangulated graph is again triangulated, we only have to show that the chromatic number $\chi(G)$ and maximum clique size $\omega(G)$ are equal for any triangulated graph G. To prove it, let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme. For $i = n, n-1, \ldots, 1$, color v_i with a color which is not used in $N(v_i) \cap \{v_{i+1}, \ldots, v_n\}$. This gives a correct coloring, and since each $\{v_i\} \cup (N(v_i) \cap \{v_{i+1}, \ldots, v_n\})$ is a clique, this coloring can be done using no more than $\omega(G)$ colors.

Another characterization of triangulated graphs was given by Dirac in 1961 [43], by means of minimal vertex separators.

Definition 2.1.3 *Given a graph* G *with vertex set* V, *a subset* $S \subset V$ *is called a* vertex separator *for nonadjacent vertices* a *and* b *in* $V \setminus S$ *if* a *and* b *are in*

different connected components of $G[V \setminus S]$. *If* S *is a vertex separator for* a *and* b *but no proper subset of* S *separates* a *and* b *in this way, then* S *is called a* minimal vertex separator for a and b. *A subset* $S \subseteq V$ *is called a minimal vertex separator if there exists a pair of nonadjacent vertices for which* S *is a minimal vertex separator.*

We like to stress that one minimal vertex separator might well be properly contained in another minimal vertex separator.

Lemma 2.1.2 *A graph* G *is triangulated if and only if every minimal vertex separator induces a complete subgraph of* G.

The following lemma, which must have been rediscovered many times, appears as an exercise in [58].

Lemma 2.1.3 *Let* S *be a minimal vertex separator for nonadjacent vertices* a *and* b, *and let* C_a *and* C_b *be the connected components of* $G[V \setminus S]$ *containing* a *and* b, *respectively. Then every vertex of* S *has a neighbor in* C_a *and a neighbor in* C_b.

Proof. Assume $w \in S$ has no neighbor in C_a. We claim that $S' = S \setminus \{w\}$ is also an a, b-separator. Suppose there is a path between a and b in $G[V \setminus S']$. Then the path contains no vertex of S', hence the path passes through w. This leads to a contradiction. □

Especially useful is the fact that a triangulated graph has *balanced* separators. The following lemma is a generalization of probably the oldest separator theorem of which one version is due to C. Jordan in 1869 [71, 84]. For a similar result see e.g. [111].

Lemma 2.1.4 *Let* $G = (V, E)$ *be a triangulated graph with* n *vertices. There is a clique* C, *such that every component of* $G[V \setminus C]$ *has at most* $\lceil \frac{1}{2}(n - |C|) \rceil$ *vertices.*

As mentioned earlier, there are many more characterizations of triangulated graphs. For example, a graph is triangulated if and only if every induced connected subgraph with $p \geq 2$ vertices contains at most $p - 1$ maximal cliques (see for example [14] page 84). Another interesting characterization appears in [10] and uses the so-called stability function of a graph. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. With each vertex we can associate a 0/1-variable x_i. If x is some vector in $\{0,1\}^n$ then the subgraph of G induced by x is defined as the subgraph induced by exactly those vertices v_i for which the corresponding variable x_i is set to one. The stability function $\alpha_G : \{0,1\}^n \to \mathbf{N}$ is defined as follows. For each $x \in \{0,1\}^n$, $\alpha_G(x)$ is the stablity number (the maximum cardinality of an independent set) of the subgraph induced by x. It can be shown that this function

can be expressed uniquely in the form $\sum_{t\in\Delta} a_t \prod_{i\in t} x_i$, where Δ is a collection of subsets of $\{1,\ldots,n\}$ and the a_t's are real coefficients. Now the following statement holds. A graph is triangulated if and only if the polynomial expression of the stability function has all its coefficients in $\{0,-1,1\}$.

We conclude with one more characterization, showing that triangulated graphs are a class of intersection graphs. This was shown for example in [132].

Lemma 2.1.5 *A graph* G *is triangulated if and only if* G *is the intersection graph of a family of subtrees of a tree.*

The kind of intersection graph referred to in the lemma is defined as follows. Given a family of subtrees of a tree, a graph is constructed in the following way. The vertices of the graph are the subtrees and two vertices are adjacent if the corresponding subtrees have at least one node in common. Notice that a family of subtrees of a tree satisfies the Helly property, which is the following (see for example [58]).

Definition 2.1.4 *A family* $\{S_i\}_{i\in I}$ *of sets satisfies the* Helly property *if for all* $J \subset I$*:* $S_i \cap S_j \neq \emptyset$ *for all* $i,j \in J$ *implies that* $\cap_{j\in J} S_j \neq \emptyset$.

Notice that this implies the following. Assume we have a set of connected subgraphs of a triangulated graph such that for every pair the subgraphs have at least one vertex in common. Then there is at least one vertex present in each of the subgraphs.

An important subclass of the triangulated graphs are the interval graphs (which were first mentioned by Hajös [62]).

Definition 2.1.5 *An* interval graph *is a graph for which one can associate with each vertex an interval on the real line such that two vertices are adjacent if and only if their corresponding intervals have a nonempty intersection.*

Definition 2.1.6 *An ordering* $(X_1,\ldots,X_t)$ *of the maximal cliques of a graph* G *is called an* interval ordering *if, for every vertex, the maximal cliques containing it occur consecutively in the ordering.*

The following characterization of interval graphs was found by Gilmore and Hoffman in 1964 [55].

Lemma 2.1.6 *The following statements are equivalent.*

1. G *is an interval graph.*
2. G *is triangulated and its complement* $\overline{G}$ *is a comparability graph.*
3. *There is an interval ordering of the maximal cliques of* G.

A comparability graph is a graph which admits a transitive orientation of its edges [58].

Interval graphs have many practical applications in various fields like archeology, geology, criminology, genetics etc., see for example [36, 58]. For a much more extensive bibliographical list of applications we refer to page 90 of the overview paper of Duchet which appeared in [14].

As an interval graph is triangulated, the number of maximal cliques is bounded by the number of vertices. By Lemma 2.1.6 the maximal clique-versus-vertex incidence matrix has the consecutive ones property. In [30] Booth and Lueker give a fast algorithm to test for the consecutive ones property using PQ-trees, which implies the following result:

Lemma 2.1.7 *A graph* $G = (V, E)$ *can be tested for being an interval graph in* $O(n + e)$ *steps, where* n *is the number of vertices and* e *is the number of edges of* G.

Recently a simpler algorithm was discovered by Hsu [70] which does not use the maximal cliques and places the intervals directly. As with triangulated graphs, there are many more characterizations for interval graphs. For example, Lekkerkerker and Boland in 1962, found a list of graphs such that a graph is an interval graph if and only if it does not contain an induced subgraph isomorphic to a graph in this list [92]. Since the complements of interval graphs are comparability graphs, such a list is also easily obtained from the complete list of forbidden induced subgraphs for comparability graphs, which was found by Gallai [54] in 1967 (this list appears also in [14] on page 78). Lekkerkerker and Boland also proved another important characterization, using astroidal triples.

Definition 2.1.7 *Three vertices in a graph* G *are called an* astroidal triple *if any two of them is connected by a path which avoids the neighborhood of the third.*

For example, in a 3-sun (see figure 6.1 on page 66) the three vertices of the independent set are an astroidal triple.

Lemma 2.1.8 *A graph is an interval graph if and only if* G *is triangulated and does not contain an astroidal triple.*

A second important subclass of triangulated graphs is the class of k-trees. The first mention we found of 2-trees is in a paper of Harary and Palmer [65] and dates back to 1968 (in this paper they are introduced as being simple connected, acyclic 2-plexes). The related concept of a (acyclic) pure k-complex is defined in a paper of Harary in 1955 [63]. When restricted to 2-trees another related concept, called a two-terminal series-parallel network, can even be traced back to Macmahon in 1892 [95], who described a method to determine the number of distinct series-parallel two-terminal networks. This is mentioned in a paper of

R. M. Foster [51] (see also [52]). Foster counts the number of different series-parallel networks without specification of terminals. (He mentions that in [126] the enumeration through $n = 7$ has been performed by Tellegen.)

In a paper from 1969 Beineke and Pippert enumerate labeled k-dimensional trees (after enumerating 2-dimensional trees in 1968) which they call k-trees for short [9]. See chapter 4 for an overview of these results in enumerations.

Usually a clique with k vertices is also considered to be a k-tree. For convenience we prefer to define k-trees to have at least $k+1$ vertices. With this definition, the treewidth of a k-tree is k (the treewidth of a k-clique is $k-1$), and the maximum clique size of a k-tree is $k+1$.

Definition 2.1.8 k-Trees *are defined recursively as follows: A clique with* $k+1$ *vertices is a* k*-tree; given a* k*-tree* T_n *with* n *vertices, a* k*-tree with* $n+1$ *vertices is constructed by taking* T_n *and creating a new vertex* x_{n+1} *which is made adjacent to a* k*-clique of* T_n *and nonadjacent to the* $n-k$ *other vertices of* T_n.

The first characterization shows that k-trees are indeed triangulated.

Lemma 2.1.9 *A graph* G *with* n *vertices is a* k*-tree if and only if* $n > k$ *and there exists a perfect elimination scheme* $\sigma = [v_1, \ldots, v_n]$ *such that for all* $i \leq n-k$, v_i *is adjacent to a* k*-clique in the subgraph* $G[v_i, \ldots, v_n]$.

Proof. Assume $G = (V, E)$ is a k-tree with n vertices. If $n = k+1$ then G is a $(k+1)$-clique and any ordering of the vertices yields a perfect elimination scheme satisfying the desired property. If $n > k+1$, by definition there is a vertex x such that $G[V \setminus \{x\}]$ is a k-tree with $n-1$ vertices, and x is adjacent to a k-clique. By induction we may assume that there is a perfect elimination scheme $\sigma' = [v_2, \ldots, v_n]$ for $G[V \setminus \{x\}]$ such that v_i is adjacent to a k-clique in $G[\{v_i, \ldots, v_n\}]$, for all $2 \leq i \leq n-k$. Define $v_1 = x$. Then $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme as mentioned for G.

Now assume $n > k$ and $\sigma = [v_1, \ldots, v_n]$ is a perfect elimination scheme such that for all $i \leq n-k$, v_i is adjacent to a k-clique in $G[v_i, \ldots, v_n]$. If $n = k+1$ it follows that the graph must be a $(k+1)$-clique and hence a k-tree. Assume $n > k+1$. Then $\sigma' = [v_2, \ldots, v_n]$ is a perfect elimination scheme for $G[V \setminus \{v_1\}]$ with the stated property guaranteeing by induction that $G[V \setminus \{v_1\}]$ is a k-tree. Since v_1 is adjacent to a k-clique it follows by definition of a k-tree that G is a k-tree. □

It follows that all maximal cliques in a k-tree have $k+1$ vertices. Lemma 2.1.9 makes it very easy to calculate the number of certain subgraphs in a k-tree. We give one example.

Lemma 2.1.10 *For* $1 \leq t \leq k+1$, *the number of* t*-cliques in a* k*-tree is*

$$\binom{k}{t} + (n-k)\binom{k}{t-1}$$

The following characterization of k-trees appears in [114].

Lemma 2.1.11 *A graph* G *is a* k*-tree if and only if*

1. G *is connected,*
2. $\omega(G) = k+1$,
3. *every minimal vertex separator of* G *is a* k*-clique.*

The following lemma shows that k-trees are triangulated graphs with maximum clique size $k+1$ which have a maximum number of edges (see also [114]). In Lemma 2.1.13 we will show that, conversely, every triangulated graph with maximum clique size $k+1$ and with a maximal number of edges is a k-tree.

Lemma 2.1.12 *A graph* G *with* n *vertices is a* k*-tree if and only if*

1. G *is triangulated,*
2. $\omega(G) = k+1$,
3. *the number of edges is at least* $nk - \frac{1}{2}k(k+1)$.

Proof. First assume G is a k-tree. By Lemma 2.1.9 G is triangulated and has maximum clique size $k+1$. By Lemma 2.1.10 the number of edges in a k-tree is $nk - \frac{1}{2}k(k+1)$. Conversely, assume that a graph G satisfies the three conditions. Since G is triangulated there is a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$. Since $\omega(G) = k+1$, it follows that for each i, $|N(v_i) \cap \{v_i, \ldots, v_n\}| \leq k$. Hence we find for the number of edges in G:

$$\sum_{i=1}^{n} |N(v_i) \cap \{v_i, \ldots, v_n\}| \leq \sum_{i=1}^{n-k} k + \frac{1}{2}k(k-1) = nk - \frac{1}{2}k(k+1)$$

We can have equality only if for each $i \leq n-k$ v_i has k neighbors in $\{v_i, \ldots, v_n\}$. By Lemma 2.1.9 G is a k-tree. □

Definition 2.1.9 *A* partial k-tree *is any subgraph of a* k*-tree.*

There exist linear time algorithms for many NP-complete problems when these problems are restricted to the class of partial k-trees for some *constant* k and when an embedding in a k-tree (i.e., a k-tree of which the graph is a subgraph) is given [3, 15, 7, 5]. There are also results stating that large *classes* of problems can be solved in linear time for the class of partial k-trees with k bounded by some constant [37, 38, 39].

In this thesis we shall primarily be concerned with the problem of finding nice embeddings of a graph in triangulated graphs and in interval graphs. We call such an embedding a triangulation of the graph.

Definition 2.1.10 *A triangulation of a graph* G *is a graph* H *with the same set of vertices such that* G *is a subgraph of* H *and such that* H *is triangulated. We say that* G *is* triangulated into H.

Clearly, every graph can be triangulated (into a clique). There are two problems which have drawn much attention because of the large number of applications. The first problem is to triangulate a graph such that the number of added edges is minimum and the second one is to triangulate a graph such that the maximum clique size in the triangulated graph is minimum. The first one is called the MINIMUM FILL-IN problem and the second one is the TREEWIDTH problem. These problems are both NP-hard [4, 133]. Our concern, as one might guess by the title of this thesis, lies in finding embeddings which minimize the maximum clique.

Remark. The fact that these problems are indeed different can be seen by the following example [20]. Let $\delta > 3$ be an integer. Let C be a cycle with $2\delta + 3$ vertices, I an independent set with δ vertices and K a clique with 2δ vertices. Consider the graph $G = C + (I \cup K)$. There are two possible triangulations with a minimal number of edges. In the first edges are added such that $I \cup K$ becomes a clique and such that C becomes triangulated. In this case the maximum clique size becomes $3\delta + 3$ and the number of added edges is $\frac{1}{2}\delta(5\delta + 3)$. The second possible triangulation with a minimal number of edges, adds edges such that C becomes a clique. In this way the maximum clique has $4\delta + 3$ vertices and the number of added edges is $\delta(2\delta + 3)$. Hence the first triangulation minimizes the clique size and the second minimizes the number of edges.

Another result obtained in [20] is the following. An elimination scheme for a graph G is simply an ordering of the vertices $\sigma = [v_1, \ldots, v_n]$. Given an elimination ordering σ let $G' = G(\sigma)$ be the graph obtained from G by adding the minimum number of edges such that σ is a perfect elimination scheme for G'. Let a graph G have property $\mathcal{P}$ if for *every* elimination ordering σ the triangulation $G(\sigma)$ realizes the minimum fill-in. In [20] it is shown that the only graphs having property $\mathcal{P}$ are graph for which every connected component is either a clique, a cycle or a cocktailparty graph (i.e. the complement of $K_2 \cup K_2 \cup \ldots \cup K_2$). For these graphs every $G(\sigma)$ also minimizes the clique size. As far as we know, it is an open problem to characterize those graphs for which every triangulation with a minimal number of edges is a triangulation which minimizes the clique size.

Lemma 2.1.13 *If* G *is a triangulated graph with at least* $k + 1$ *vertices and has maximum clique size at most* $k + 1$*, then* G *can be triangulated into a* k*-tree.*

Proof. Let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme for G. We make an embedding of G in a k-tree recursively as follows. First we add edges such that the subgraph induced by $\{v_{n-k}, v_{n-k+1}, \ldots, v_n\}$ becomes a $k + 1$-clique. For the induction step, assume the subgraph with vertices $\{v_{i+1}, \ldots, v_n\}$ has been

triangulated into a k-tree T_{i+1}. In G vertex v_i is adjacent to a clique C with at most k vertices in $\{v_{i+1}, \ldots, v_n\}$. In the k-tree T_{i+1} C is contained in a k-clique C'. We make v_i adjacent to all vertices of C' and we obtain a k-tree T_i which is a triangulation of G. □

Lemma 2.1.14 *Every partial* k*-tree with at least* $k+1$ *vertices can be triangulated into a* k*-tree.*

Proof. Let the partial k-tree $G = (V, E)$ be a subgraph of the k-tree $H = (W, F)$. Then $H[V]$ is triangulated and has maximum clique size at most $k+1$. By Lemma 2.1.13, H[V] can be triangulated into a k-tree. Clearly this is a triangulation of G. □

We have seen that k-trees are triangulations with a maximal number of edges. We now show some properties of triangulations with a minimal number of edges.

Definition 2.1.11 *A* minimal triangulation *of a graph* G *with treewidth* k *is a triangulation of* G *into a triangulated graph* H *such that the following three conditions are satisfied:*

1. $\omega(H) = k+1$,
2. *If* a *and* b *are nonadjacent vertices in* H *then every minimal* a, b*-separator of* H *is also a minimal* a, b*-separator in* G,
3. *If* S *is a minimal separator in* H *and* C *is the vertex set of a connected component of* $H[V \setminus S]$, *then* C *induces also a connected component in* $G[V \setminus S]$.

Theorem 2.1.1 *Every graph* G *has a minimal triangulation.*

Proof. Consider a triangulation H of G with a minimum number of edges. Suppose H has a minimal vertex separator C for nonadjacent vertices a and b, such that either C induces no minimal vertex separator for a and b in G, or the vertex sets of the connected components of $H[V \setminus C]$ are different from those of $G[V \setminus C]$. Let $S \subseteq C$ be a minimal a, b-separator in G. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus S]$. Make a chordal graph H' as follows. For each $C_i \cup S$ take the chordal subgraph of H induced by these vertices. Since S is a clique in H, this gives a chordal subgraph H' of H. Notice that the vertex sets of the connected components of $H'[V \setminus S]$ are the same as those of $G[V \setminus S]$. We claim that the number of edges of H' is less than the number of edges of H, which is a contradiction. Clearly, the number of edges of H' does not exceed the number of edges of H. First assume that $S \neq C$, and let $x \in C \setminus S$. By Lemma 2.1.3, in H, x has a neighbor in the component containing a and a neighbor in the

component containing b. Not both these edges can be present in H'. Thus we may assume $S = C$. Then the vertex sets of the connected components of $H[V \setminus C]$ are different from those of $H'[V \setminus C]$. Since H' is a subgraph of H, every connected component $H'[V \setminus C]$ is contained in some connected component of $H[V \setminus C]$. It follows that there must be a connected component in $H[V \setminus C]$ containing two different connected components of $H'[V \setminus C]$. This can only be the case if there is some edge between these components in $H[V \setminus C]$ (which is not there in $H'[V \setminus C]$). This proves the theorem. □

A somewhat stronger result was recently obtained in [82].

Let $G = (V, E)$ be a graph and let C be a minimal vertex separator. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus C]$. Denote by $\overline{C}_i$ $(i = 1, \ldots, t)$ the graph obtained as follows. Take the induced subgraph $G[C \cup C_i]$, and add all possible edges between vertices of C such that the subgraph induced by C is complete. The following lemma easily follows from Theorem 2.1.1 (a similar result appears in [4]).

Lemma 2.1.15 *A graph G with at least $k+2$ vertices is a partial k-tree if and only if there exists a minimal vertex separator with at most k vertices such that all graphs $\overline{C}_i$ are partial k-trees.*

Proof. Assume G is a partial k-tree. Let H be a minimal triangulation. Then $\omega(H) \leq k + 1$. Thus, because H has more than $k + 1$ vertices, there must exist nonadjacent vertices in H. Let C be a minimal vertex separator in H. Then this is also a minimal vertex separator in G. Since H is triangulated, C is a clique in H. Let $C_1, \ldots, C_t$ be the vertex sets of the connected components of $H[V \setminus C]$. These are also the vertex sets of the connected components of $G[V \setminus C]$. The graphs $H[C_i \cup C]$ are triangulations of $\overline{C}_i$, hence these are partial k-trees.

Conversely, let C be a minimal vertex separator and let $C_1, \ldots, C_t$ be the vertex sets of the connected components of $G[V \setminus C]$. Assume all $\overline{C}_i$ are partial k-trees. Make triangulations H_i of each of these graphs and identify the vertices of C. We claim that the result is a triangulation H of G. This can be seen for example as follows. Let the vertices of C be $c_1, \ldots, c_w$. For each triangulation H_i take a perfect elimination scheme $\sigma_i = [v_1^i, \ldots, v_{n_i}^i, c_1, \ldots, c_w]$. We can construct a perfect elimination scheme for H of the following form: $\sigma = [v_1^1, \ldots, v_{n_1}^1, \ldots, v_1^t, \ldots, v_{n_t}^t, c_1, \ldots, c_w]$. □

2.2 Treewidth and pathwidth

In this section we give a brief introduction to some important notions related to partial k-trees: treewidth, pathwidth, tree-decomposition and path-decomposition.

We define the treewidth of a graph in two ways and show the equivalence of the two definitions (see also [89, 17]). The first definition, which we have seen already in the previous section, is by means of partial k-trees and the second is by means of tree-decompositions.

Definition 2.2.1 *The* treewidth *of a graph* G *is the minimum value* k *for which* G *is a partial* k*-tree.*

Determining the treewidth or the pathwidth of a graph is NP-hard [4]. However, for *constant* k, graphs with treewidth $\leq k$ are recognizable in linear time, see chapter 11 and chapter 12.

Notice that a k-clique has treewidth $k-1$ (since it is a $(k-1)$-tree). The following result may serve as an example, and is needed in later chapters (see also, e.g., [22]).

Lemma 2.2.1 *The complete bipartite graph* $G = K(m,n)$ *has treewidth equal to* $\min(m,n)$.

Proof. First notice that in *any* triangulation of G at least one of the color classes is a clique. Otherwise there are nonadjacent vertices x and y in one colorclass and nonadjacent vertices p and q in the other colorclass, and $H[x,y,p,q]$ would be a chordless cycle. This shows that the treewidth is at least $\min(m,n)$.

For a bipartite graph $G = (X,Y,E)$ the graph $\mathrm{split}(G) = (X,Y,\hat{E})$ is defined as the graph with $\hat{E} = E \cup \{(x,x') \mid x,x' \in X\}$. A *split graph* is a graph for which there exists a partition of the vertex set into a clique and a stable set. Assume without loss of generality that $|X| \leq |Y|$. If G is complete bipartite then $\mathrm{split}(G)$ is a k-tree with $k = |X|$ consisting of a maximal clique with $|X|+1$ vertices and a set of $|Y|-1$ simplicial vertices. This proves that the treewidth is at most $\min(m,n)$. □

We now come to the second way to define the treewidth of a graph, namely by a concept called the tree-decomposition of a graph.

Definition 2.2.2 *A* tree-decomposition *of a graph* $G = (V,E)$ *is a pair* $D = (S,T)$ *with* $S = \{X_i \mid i \in I\}$ *a collection of subsets of vertices of* G *and* $T = (I,F)$ *a tree, with one node for each subset of* S*, such that the following three conditions are satisfied:*

1. $\bigcup_{i \in I} X_i = V$,
2. *for all edges* $(v,w) \in E$ *there is a subset* $X_i \in S$ *such that both* v *and* w *are contained in* X_i,
3. *for each vertex* x *the set of nodes* $\{i \mid x \in X_i\}$ *forms a subtree of* T.

A path-decomposition *of a graph* G *is a tree-decomposition* (S, T) *such that* T *is a path. A path-decomposition is also denoted as* $(X_1, X_2, \ldots, X_\ell)$.

An alternative way to formulate the third condition is the following: for all $i, j, k \in I$: if j is on the path from i to k in T then $X_i \cap X_k \subset X_j$. Notice that if D is a tree-decomposition for a graph G and H is a subgraph of G with the same vertex set, then D is also a tree-decomposition for H. Also, if H is a subgraph of G and $D = (S, T)$ is a tree-decomposition for G, we can obtain a tree-decomposition D' for H by taking the restriction of every subset in S to the vertex set of H (the three conditions in Definition 2.2.2 are trivially satisfied).

Definition 2.2.3 *Let* $D = (S, T)$ *be a tree- or path-decomposition for a graph* G *and let* H *be any subgraph of* G. *The tree- or path-decomposition for* H *obtained from* D *by taking the restriction of every subset in* S *to the vertex set of* H *is called the* subdecomposition of D for H.

Definition 2.2.4 *The* width *of a tree-decomposition* $(\{X_i \mid i \in I\}, T = (I, F))$ *is* $\max_{i \in I}(|X_i| - 1)$.

To show the equivalence between the treewidth of a graph and the minimum width over all tree-decompositions, we make use of the following lemma [22], which is a direct consequence of the Helly property satisfied by a set of subtrees of a tree (for alternative proofs see [15, 118]).

Lemma 2.2.2 *Let* $(S = \{X_i \mid i \in I\}, T = (I, F))$ *be a tree-decomposition for* G. *For every clique* C *in* G *there exists a subset* $X_i \in S$ *such that* $C \subset X_i$.

Proof. Let C be a clique in G. For every pair of vertices x and y of C there exists a subset X_i containing both x and y. Also, for every vertex x of C the set of subsets X_i containing x forms a subtree of T. The lemma follows since a family of subtrees of a tree satisfies the Helly property (Definition 2.1.4). □

Given a tree-decomposition D of a graph G of width k, we can triangulate G into a triangulated graph H with maximal clique size $k + 1$ as follows.

Definition 2.2.5 *Let* $D = (S, T)$ *be a tree-decomposition for* $G = (V, E)$. *Define* $H = \mathcal{H}(G, D)$ *as the graph with the same vertex set as* G, *with two vertices in* H *adjacent if and only if they appear in some common subset* $X \in S$. *We call* H *the triangulation of* G *implied by* D.

Lemma 2.2.3 *Let* $D = (S, T)$ *be a tree-decomposition for* $G = (V, E)$ *of width* k. *Then* $H = \mathcal{H}(G, D)$ *is a triangulation of* G *and has maximum clique size* $k + 1$. *The tree-decomposition* D *is also a tree-decomposition for* H.

Proof. First we show that G is a subgraph of H. If (v, w) is an edge in G, then v and w appear in some common subset $X \in S$. By definition v and w are thus also adjacent in H. The fact that D is also a tree-decomposition for H follows immediately from Definition 2.2.2. By Lemma 2.2.2 the maximum clique size of H is $k + 1$.

It remains to show that H is triangulated. For each vertex $x \in V$, consider the subtree of T consisting of those nodes i for which $x \in X_i$. Two vertices x and y are adjacent in H if and only if the corresponding subtrees intersect. Hence, H is the intersection graph of a family of subtrees of a tree, and by Lemma 2.1.5 H is triangulated. □

An alternative way to prove Lemma 2.2.3 is to show that H has a perfect elimination scheme. This can be seen as follows. First, if T has only one node, H is a clique and thus H is triangulated. If T has more than one node, let i be a leaf and let j be the neighbor of i. Assume that for every vertex $x \in X_i$ there exists another subset X_k $(k \neq i)$ that also contains x. Then, by definition, x must also be contained in X_j and thus $X_i \subset X_j$. Now we can make a new tree-decomposition D′ for H by removing X_i from the set of subsets and by removing i from the tree. Otherwise, let x be a vertex in X_i which is contained only in X_i. Then clearly all neighbors of x are contained in X_i and thus x is simplicial. In this way we obtain a perfect elimination scheme for H, showing that H is triangulated.

Corollary 2.2.1 *If* G *has a tree-decomposition of width* k*, then the treewidth of* G *is at most* k.

The next lemma shows the equivalence between the two concepts.

Lemma 2.2.4 *The treewidth of a graph* G *equals the minimum width over all tree-decompositions of* G.

Proof. Let the treewidth of G be k. We show how to construct a tree-decomposition of width k for G. Since the treewidth of G is k we know there exists a k-tree H such that G can be triangulated into H. First notice that a tree-decomposition for H is also a tree-decomposition for G, since G is a subgraph of H with the same set of vertices. Hence it is sufficient to show there exists a tree-decomposition for H of width k.

Since H is triangulated, it is the intersection graph of a family of subtrees of a tree $T = (I, F)$. For each node i in this tree, define a subset X_i consisting of those vertices for which the corresponding subtree contains i. Let $S = \{X_i \mid i \in I\}$. We claim that $D = (S, T)$ is a tree-decomposition of width at most k. It is easy to check that D is indeed a tree-decomposition for H. Furthermore, each subset corresponds with a clique in H and hence has cardinality at most $k + 1$. This shows that the width of D is at most k.

The converse is stated in Corollary 2.2.1. □

Lemma 2.2.5 *For any graph with* n *vertices and treewidth* k*, there exists a tree-decomposition* $D = (S, T)$ *with* $|S| \leq n - k$ *such that every subset* $X \in S$ *contains exactly* $k + 1$ *vertices.*

Proof. Let $G = (V, E)$ be a graph with n vertices and treewidth k. Let H be a triangulation of G into a k-tree. We show there exists a tree-decomposition as claimed for H. If $n = k+1$, then we take a tree with one node and a corresponding subset containing all nodes. Otherwise let x be a simplicial vertex. By induction there exists a tree-decomposition for $H[V \setminus \{x\}]$ with $n - k - 1$ nodes and such that every subset in the tree-decomposition has $k + 1$ elements. Since $N(x)$ is a clique, there is a subset X_i in this tree-decomposition such that $C \subseteq N(x)$. Take a new node i' and make this adjacent to i. Let the corresponding subset be $X_{i'} = \{x\} \cup N(x)$. □

For a related result see also Lemma 7.3.1 on page 80.

We now show some related results for the path-decomposition of a graph. In the same manner in which triangulated graphs are related to tree-decompositions, interval graphs are related to path-decompositions.

Definition 2.2.6 *A* k-path *is a* k*-tree which is an interval graph. A* partial k-path *is a subgraph of a* k*-path. The* pathwidth *of a graph* G *is the minimum value* k *for which* G *is a partial* k*-path.*

Analogous to Lemma 2.1.13 and Lemma 2.1.14 we can obtain the following results.

Lemma 2.2.6 *An interval graph* G *with at least* $k + 1$ *vertices and with* $\omega(G) \leq k + 1$ *can be triangulated into a* k*-path.*

Proof. Let $G = (V, E)$ be an interval graph with $n \geq k + 1$ vertices and with $\omega(G) \leq k + 1$. Let $(X_1, \ldots, X_t)$ be an interval ordering of the maximal cliques of G. We prove there exists a triangulation H of G into a k-path such that there is an interval ordering $(Y_1, \ldots, Y_{n-k})$ of the maximal cliques of H, with $X_1 \subseteq Y_1$. This can be seen as follows. If G has $k+1$ vertices then we can take for H a clique with all vertices and let $Y_1 = V$. Otherwise G can not be a clique and there must exist a vertex $x \in X_1 \setminus X_2$. Then x is a simplical vertex in G and $G[V \setminus \{x\}]$ is an interval graph with at least $k + 1$ vertices and with maximum clique size at most $k + 1$. There are two cases to consider.

First assume that $(X_1 \setminus \{x\}, X_2, \ldots, X_t)$ is an interval ordering of the maximal cliques of $G[V \setminus \{x\}]$. By induction there is a k-path H', which is a triangulation of $G[V \setminus \{x\}]$, and there is an interval ordering $(Y_1, \ldots, Y_{n-k-1})$ with $X_1 \setminus \{x\} \subseteq Y_1$. Notice there must be an element $z \in Y_1 \setminus X_1$ (otherwise X_1 contains more than $k + 1$ vertices). Define $Y_0 = \{x\} \cup (Y_1 \setminus \{z\})$. Let H be the graph obtained from H' by making x adjacent to all vertices of $Y_0 \setminus \{x\}$. Then H is a k-tree, and $(Y_0, Y_1, \ldots, Y_{n-k-1})$ is an interval ordering of the maximal cliques of H (hence H is a k-path) with $X_1 \subseteq Y_0$.

Now assume $X_1 \setminus \{x\}$ is not a maximal clique in $G[V \setminus \{x\}]$. Then $X_1 \setminus \{x\} \subset X_2$. Clearly $(X_2, \ldots, X_t)$ is an interval ordering of the maximal cliques of $G[V \setminus \{x\}]$. Let H' be a triangulation of $G[V \setminus \{x\}]$ into a k-path, and let $(Y_2, \ldots, Y_{n-k-1})$ be an interval ordering of the maximal cliques of H with $X_2 \subseteq Y_2$. Let $z \in Y_2 \setminus X_1$ (this vertex must exist since $X_2 \setminus X_1 \neq \emptyset$). Let $Y_1 = \{x\} \cup (Y_2 \setminus \{z\})$. Let H be the graph obtained from H' by making x adjacent to all vertices of $Y_1 \setminus \{x\}$. Then H is a k-tree and $(Y_1, \ldots, Y_{n-k})$ is an interval ordering of the maximal cliques of H with $X_1 \subseteq Y_1$. □

Lemma 2.2.7 *Every partial* k*-path with at least* $k+1$ *vertices can be triangulated into a* k*-path.*

Proof. Let $G = (V, E)$ be a partial k-path with at least $k+1$ vertices. There is a k-path H such that G is a subgraph of H. Notice that $H[V]$ is an interval graph with maximal clique size at most $k+1$, such that G is a subgraph of $H[V]$. By Lemma 2.2.6, $H[V]$ can be triangulated into a k-path. □

Lemma 2.2.8 *The pathwidth of a graph* G *equals the minimum width over all path-decompositions of* G.

Proof. Assume the pathwidth of G is k. Hence G is the subgraph of an interval graph H which is simultaneously a k-tree. Since H is a k-tree all maximal cliques of H have size $k+1$. We show there exists a path-decomposition for H of width k. By Lemma 2.1.6 there is an interval ordering $(X_1, X_2, \ldots, X_\ell)$ of the maximal cliques of H. It is easy to check that $(X_1, X_2, \ldots, X_\ell)$ is a path-decomposition for H of width k.

Conversely let $D = (X_1, X_2, \ldots, X_\ell)$ be a path-decomposition for G of width k. Let H be the triangulation of G implied by D. Each maximal clique of H is contained in some subset X_i and each subset contains at most one maximal clique. Hence there exists an ordering of the maximal cliques of H such that for every vertex the maximal cliques containing it are consecutive. Hence H is an interval graph with maximum clique size $k+1$. It follows that H can be triangulated into a k-path. □

We show that the treewidth and pathwidth of a graph differ at most by a factor $\log n$. We need the following lemma. For a slightly more general result see also [111]. See also Lemma 2.1.4.

Lemma 2.2.9 *Every* k*-tree* $G = (V, E)$ *contains a clique* C *with* $k+1$ *vertices such that every connected component of* $G[V \setminus C]$ *has at most* $\frac{1}{2}(n-k)$ *vertices.*

Proof. Consider the following algorithm. Start with any $k+1$-clique S_0. Assume there is a connected component C in $G[V \setminus S_0]$ which has more than $\frac{1}{2}(n-k)$ vertices. Notice that the other components together have less than $\frac{1}{2}(n-k)-1$ vertices. There exists a vertex x in C which has k neighbors in S_0. Let $y \in S_0 \setminus N(x)$. Define $S_1 = \{x\} \cup (N(x) \cap S_0)$. Notice that S_1 also has $k+1$ vertices. The algorithm continues with S_1.

To show that this algorithm terminates, we prove that in each step of the algorithm the number of vertices in the largest component decreases. Notice that $G[V \setminus S_1]$ has two types of components. One type consists only of vertices of $C \setminus \{x\}$. If the largest component of $G[V \setminus S_1]$ is among these, the number of vertices has clearly decreased. The other type of components consists only of vertices of $\{y\} \cup V \setminus (C \cup S_0)$. By the remark above, the total number of vertices in this set is less than $\frac{1}{2}(n-k)$. As the largest component of $G[V \setminus S_0]$ has more than this number of vertices, this shows that the number of vertices in the largest component of $G[V \setminus S_1]$ is at least one less than the number of vertices in the largest component of $G[V \setminus S_0]$. This shows that the algorithm terminates. □

Corollary 2.2.2 *Let G be a graph with n vertices and treewidth k. There exists a set S with $k+1$ vertices such that every connected component of $G[V \setminus S]$ has at most $\frac{1}{2}(n-k)$ vertices.*

Lemma 2.2.10 *Let $G = (V, E)$ be a graph with n vertices and treewidth $k \geq 1$. Then the pathwidth of G is at most $(k+1)\log n - 1$.*

Proof. Lemma 2.2.9 shows there is a clique X with $k+1$ vertices such that every connected component of $G[V \setminus X]$ has at most $\frac{1}{2}(n-k)$ vertices.

If $n = k+1$ the upperbound on the pathwidth clearly holds. We proceed by induction. Let $n > k+1$, and let X be a balanced separator as mentioned above. Let $C_1, \ldots, C_t$ be the connected components of $G[V \setminus X]$. By induction there are path-decompositions P_i for the induced subgraphs $G[C_i]$ with pathwidth $\leq (k+1)\log|C_i| - 1$. Add X to every subset of every path-decomposition, and let P_i' $(i = 1, \ldots, t)$ be the new path-decompositions so obtained. Let P be the concatenation of the P_i''s, i.e., the path-decomposition obtained by putting all P_i''s after each other: $P = P_1' +\!+ \ldots +\!+ P_t'$. Clearly, the width of P is at most $k + (k+1)\log\frac{n-k}{2} \leq (k+1)\log n - 1$. □

2.3 Perfect graphs

Perfect graphs were introduced by Claude Berge around 1960 (see [12]). A graph $G = (V, E)$ is called *perfect* if the following two conditions are both satisfied: First the *clique number* and the *chromatic number* must be equal for all induced

subgraphs, (i.e. $\omega(G[A]) = \chi(G[A])$ for all $A \subseteq V$) and second, the *stability number* must equal the *clique cover number* for all induced subgraphs of G (i.e. $\alpha(G[A]) = k(G[A])$ for all $A \subseteq V$). Notice that it is quite a natural question to ask which graphs are perfect, since all graphs satisfy $\omega(G) \leq \chi(G)$ and $\alpha(G) \leq k(G)$. Notice also that the two conditions are dual in the sense that a graph satisfies the first condition if and only if its complement satisfies the second. The remarkable fact that a graph satisfies the first equality if and only if it satisfies the second equality, was conjectured by Berge [11] and proven by Lovász [93]. This has become known as the *perfect graph theorem*. Lovász also proved that the two conditions are equivalent to a third condition, namely: $\omega(G[A])\alpha(G[A]) \geq |A|$ for all $A \subseteq V$.

A graph is called *minimal imperfect* (or p-*critical*) if it is not perfect but every proper induced subgraph of it is. The *strong perfect graph conjecture* made by Berge (see [13]) states that the only minimal imperfect graphs are the chordless odd cycles of length at least five and their complements. This is equivalent to saying that a graph G is perfect if and only if in G and in $\overline{G}$ every chordless odd cycle of length at least five has a chord. The chordless odd cycles of length at least five and their complements are often referred to as the odd holes and the odd antiholes, respectively. Until now, the strong perfect graph conjecture is unsettled.

It is interesting to notice that for every constant k there is a polynomial time algorithm to test whether a given partial k-tree satisfies the strong perfect graph conjecture. This can be seen as follows. Let G be a partial k-tree. We may assume that a tree-decomposition of G of width bounded by some constant is given. Using standard techniques it can be tested in linear time whether or not the graph is perfect (for example, it can be stated in monadic second order logic [38] or, it can be tested using dynamic programming). Also, testing whether the graph has an odd hole can be done in linear time using one of these techniques. The only thing left to test is whether the graph has an odd antihole. Now notice that an antihole $\overline{C_{2t+1}}$ has a clique of size at least t. Since a partial k-tree can only have cliques of size at most $k + 1$ it follows that we can restrict the search for odd antiholes to those with at most $2k + 3$ vertices. It follows that we can perform this test also in linear time. In chapter 12 we show a linear time algorithm to obtain an optimal tree-decomposition for the graph. This proves that for each k there is a linear time algorithm to test whether a given partial k-tree satisfies the strong perfect graph conjecture. Using results of [37] we can even obtain a much stronger result; for each k there is an algorithm to decide whether or not for *every* partial k-tree the strong perfect graph conjecture holds. We do not claim that this is a very practical algorithm. This is even more so since for partial 3-trees the strong perfect graph conjecture is already known to hold. This can be seen as follows. Notice that every graph with at most three vertices is either bipartite or a clique. Consider a partial k-tree for $k \leq 3$ which is minimal imperfect. Then there is a vertex u with at most three neighbors. Hence the neighborhood of this vertex

induces a bipartite graph or a clique (which is multipartite). In [69] it is shown that every minimal imperfect graph which has a vertex of which the neighborhood induces a bipartite or multipartite graph, must be an odd hole or an odd antihole.

Since the discovery of perfect graphs in 1960, much research has been devoted to special classes of perfect graphs. Among the most well-known classes of perfect graphs are the comparability graphs and the triangulated graphs. The class of triangulated graphs contains graph classes such as interval graphs, split graphs, k-trees, and indifference graphs. The class of comparability graphs contains complements of interval graphs, permutation graphs, threshold graphs and P_4-free graphs (or cographs). Much work has been done to characterize these graph classes and to find relationships between them. Interest has only increased since Lovász settled the perfect graph conjecture. From an algorithmic point of view, perfect graphs have become of great interest since Grötschel, Lovász and Schrijver discovered polynomial time algorithms for NP-hard problems like CLIQUE, STABLE SET and CHROMATIC NUMBER for perfect graphs [14]. Special classes of perfect graphs have proven their importance by the large number of applications (see for example [58, 14, 109] for applications in general and [36] for an overview of applications of interval graphs).

Chapter 3

Triangulating 3-colored graphs

In this chapter we consider the problem of determining whether a given colored graph can be triangulated such that no edges between vertices of the same color are added. This problem originated from the Perfect Phylogeny problem from molecular biology, and is strongly related with the problem of recognizing partial k-trees. In this chapter we give a simple linear time algorithm that solves the problem when there are three colors. We do this by using the structural characterization of the class of partial 2-trees.

Consider a graph of which the vertices are colored such that no two adjacent vertices have the same color. In this chapter we consider the problem to determine whether we can triangulate the graph such that in the triangulated graph no two adjacent vertices have the same color. The problem of triangulating a colored graph such that no two adjacent vertices have the same color is polynomially equivalent to the *Perfect Phylogeny problem*, see e.g. [72]. This problem, which is concerned with the inference of evolutionary history from DNA sequences, is of importance to molecular biologists. Recently, this problem was proven to be NP-complete [19]. In [100] it was shown by Morris, Warnow and Wimer that the problem is solvable in $O(n^{k+1})$ time for k-colored graphs. Another special case was solved in [72] by Kannan and Warnow. In this chapter we consider the problem, for the case in which there are at most three colors. In [73] Kannan and Warnow solved this problem by an $O(n\alpha(n))$ time algorithm. Recently, Kannan and Warnow improved on their algorithm, and found a variant that uses linear time but more than linear space. In this chapter we give another, in our opinion much simpler, linear time algorithm which uses only linear space. This work was done more or less simultaneously with, and independent from this recent result of Kannan and Warnow.

3.1 A characterization of partial 2-trees

In this section we first address the problem of characterizing partial 2-trees. A graph is a partial 2-tree if and only if all its biconnected components are partial 2-trees, so when characterizing partial 2-trees we can restrict ourselves to biconnected partial 2-trees.

Lemma 3.1.1 *Let* G *be a biconnected partial* 2*-tree. Let* $S = \{x, y\}$ *be a vertex separator such that* $G[V \setminus S]$ *has at least three connected components. Then in any triangulation of* G *into a a* 2*-tree* (x, y) *is an edge.*

Proof. Let x and y be vertices such that $G[V \setminus \{x, y\}]$ has at least three connected components. Since the graph is biconnected, both x and y have at least one neighbor in every connected component and so there are paths between x and y with internal vertices in every connected component. Assume that G can be triangulated into a 2-tree H such that (x, y) is not an edge in H. In a 2-tree every minimal vertex separator is an edge (Lemma 2.1.11). Every minimal vertex separator for x and y contains at least one vertex of every connected component of $G[V \setminus \{x, y\}]$ (otherwise there would be a path between x and y). So the minimal separator contains at least three vertices, which is a contradiction. □

Definition 3.1.1 *The* cell-completion *of* G *is the graph* $\overline{G}$*, obtained from* G *by adding an edge between every pair* $\{x, y\}$ *for which* $G[V \setminus \{x, y\}]$ *has at least three connected components. A* cell *of* G *is a set of vertices which form a chordless cycle in the cell-completion.*

Note that by Lemma 3.1.1 the cell-completion $\overline{G}$ of a partial 2-tree G is a subgraph of every triangulation of G into a 2-tree.

Definition 3.1.2 *A* tree of cycles *is any member of the class of graphs* $\mathcal{C}_T$ *defined recursively as follows:*

1. *Every chordless cycle is an element of* $\mathcal{C}_T$.
2. *Given an element* T *of* $\mathcal{C}_T$ *and a chordless cycle* C*, the graph obtained by identifying an edge and its two end-vertices of* C *with an edge and its two end-vertices of* T *is also in* $\mathcal{C}_T$.

Call a tree of cycles T, a tree of k-cycles if all chordless cycles in T have length k. Then for example a 2-tree is a tree of 3-cycles (triangles). We show in the next theorem that a biconnected graph is a partial 2-tree if and only if its cell-completion is a tree of cycles. The following lemma will be useful.

Lemma 3.1.2 *Let* G *be a biconnected partial* 2*-tree and let* G *be triangulated into a* 2*-tree* H*. For any edge* $e = (x, y)$ *in* H*, the number of common neighbors of* x *and* y *in* H *is at least three if and only if the number of components in* $G[V \setminus \{x, y\}]$ *is at least three.*

Proof. Let $e = (x, y)$ be an edge in H such that x and y have at least three common neighbors in H. Two of these common neighbors cannot be adjacent in H otherwise there would be a 4-clique. Also, two of these common neighbors cannot be in the same connected component of $H[V \setminus \{x, y\}]$, since e must be a minimal vertex separator for them. Hence the number of components in $G[V \setminus \{x, y\}]$ is at least three. To prove the converse, let the number of components of $G[V \setminus \{x, y\}]$ be at least three. Let C be a component of $G[V \setminus \{x, y\}]$. G is biconnected hence there is a path between x and y with internal vertices in C. Since (x, y) is an edge in H this implies that, in H, x and y must have a common neighbor in C. This shows that the number of common neighbors of x and y in H is at least three. □

Throughout this chapter, the phrase 'triangulating a chordless cycle with m vertices' will mean the addition of $m - 3$ edges (i.e., the *minimum* number) to the cycle such that the new graph that is obtained is triangulated.

Theorem 3.1.1 *A biconnected graph* G *is a partial 2-tree if and only if its cell-completion* $\overline{G}$ *is a tree of cycles. By triangulating each chordless cycle of* $\overline{G}$ *in all possible ways we obtain all possible triangulations of* G *into a 2-tree.*

Proof. The 'only if'-part can be seen as follows. We must show that $\overline{G}$ is a tree of cycles. Consider a triangulation of $\overline{G}$ into a 2-tree H. Consider the edges in H that are not in $\overline{G}$. Any such edge of H must be incident with at least two triangles since $\overline{G}$ is biconnected. If there are *three* triangles incident with an edge of H, the edge is also present in $\overline{G}$ (Lemma 3.1.2). Since H is a tree of triangles and the only edges that are not in $\overline{G}$ are edges that are incident with two triangles, it follows that $\overline{G}$ must be a tree of cycles.

Conversely, let $\overline{G}$ be a tree of cycles. Then by triangulating each cycle into a 2-tree, the resulting graph is a tree of triangles, and hence a 2-tree. □

3.2 c-Triangulating 3-colored graphs

The problem we consider in this chapter is the following.

Definition 3.2.1 *Let* t *be an integer. A* t*-colored graph is a graph* $G = (V, E)$ *together with a vertex coloring which is a mapping* $f : V \to S$ *such that*

1. *each vertex is colored with one of the colors such that no two adjacent vertices have the same color (i.e.,* $f(x) \neq f(y)$ *whenever* x *and* y *are adjacent),*

2. $|S| = t$ *and each color is used at least once (i.e.,* f *is surjective).*

The problem is to triangulate G (if possible) without introducing edges between vertices which have the same color. If such a triangulation exists we call it a *c-triangulation* (c stands for colored).

Note that a graph can be c-triangulated if and only if all the biconnected components can be c-triangulated, and a c-triangulation of a graph G can be obtained by c-triangulating all biconnected components. Hence, in the remainder we assume that G is a biconnected graph. We can also observe that the maximum clique size in a c-triangulation can be at most the number of different colors, since otherwise there would be an edge between vertices of the same color. The following lemma states an even stronger result.

Lemma 3.2.1 *Let* $G = (V, E)$ *be a* $(k+1)$*-colored graph. Then* G *can be* c*-triangulated if and only if* G *can be* c*-triangulated into a* k*-tree.*

Proof. Let G be a $(k+1)$-colored graph and assume that G can be c-triangulated. We show that G can also be c-triangulated into a k-tree. The converse is trivial: if G can be c-triangulated into a k-tree then, a fortiori, G can be c-triangulated.

We show, by induction on the number of vertices, that every $(k+1)$-colored graph which is triangulated can be c-triangulated into a k-tree. This will prove the lemma since G can be c-triangulated into a triangulated graph H (which is $(k+1)$-colored) and, by the statement above, H can be c-triangulated into a k-tree.

Notice that, since H is $k+1$-colored, $n \geq k+1$. If $n = k+1$, then the lemma is obviously true. Assume $n > k+1$. Since H is triangulated, it has a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$. Consider the graph obtained from H by removing the simplicial vertex v_1. Let C be the set of neighbors of v_1 in H. Hence C is a clique and $|C| \leq k$. By induction, $H[V \setminus \{v_1\}]$ can be triangulated into a k-tree H'. Since H' is a k-tree and C is a clique in H', C is contained in a $k+1$-clique C'. Since C' is a clique with $k+1$ vertices it contains exactly one vertex x with the same color as v_1. Clearly x cannot be contained in C. By making v_1 adjacent to every vertex of $C' \setminus \{x\}$ we obtain a c-triangulation of H into a k-tree. □

In particular, it follows that a $(k+1)$-colored graph can be c-triangulated only if it is a partial k-tree. Consider a biconnected 3-colored graph $G = (V, E)$. In order that G can be c-triangulated, G must be a biconnected partial 2-tree. The following results give a precise characterization of partial 2-trees that can be c-triangulated.

Theorem 3.2.1 *Let* G *be a biconnected 3-colored partial 2-tree, and let* $\overline{G}$ *be the cell completion of* G. G *can be* c*-triangulated if and only if the following two conditions hold:*

1. *no two adjacent vertices of* $\overline{G}$ *have the same color, and*
2. *every cell of* $\overline{G}$ *has at least three vertices with different colors.*

Proof. From Theorem 3.1.1 it follows that it is sufficient to show that a chordless cycle can be c-triangulated if and only if it contains three vertices of different colors. This fact was proved in [73]. For reasons of completeness we also present a proof here.

If a cycle has only two colors, then it is easy to see that it cannot be c-triangulated, since cycles of length 3 cannot be made. Suppose a cycle S has three colors. Notice that S must have a vertex such that its two neighbors have different colors.

If S is a triangle there is nothing to prove. If S has more than three vertices we can add a chord to S such that the two new cycles made by this chord each have three colors. To find such a chord, we consider two cases. In case that there is a color that appears only once, then take any chord containing this color at one of its end-vertices. If every color appears at least twice in S, then take any vertex v such that its two neighbors have different colors. Then take the chord connecting the two neighbors of v.

This proves that any chordless cycle which is colored by at least three colors can be c-triangulated. □

Remark. Notice that the theorem is not true for t-colored partial 2-trees with $t > 3$. As a counterexample, take $G = K(2,3)$, which is a partial 2-tree. Let A be the color class with two vertices and B be the color class with three vertices. In the cell-completion of G, A is made an edge. Now color the two vertices of A with the same color a, and the three vertices of B with different colors b_1, b_2 and b_3 with $b_i \neq a$ for $i = 1,2,3$. Then the cell-completion is not correctly colored. But G can be c-triangulated by making a clique of B.

A slightly different characterization is obtained in the following theorem.

Theorem 3.2.2 *Let G be a biconnected 3-colored partial 2-tree, and let $\overline{G}$ be the cell completion of G. G can be c-triangulated, if and only if:*

1. *no two adjacent vertices of $\overline{G}$ have the same color, and*
2. *every cycle in $\overline{G}$ contains at least three vertices with different colors.*

Proof. Use Theorem 3.2.1. Clearly, if each cycle in $\overline{G}$ contains at least three different colors, then each cell in the cell completion does so too. This shows the 'if'-part. The 'only if'-part follows from the fact that, if $\overline{G}$ contains a cycle with only two colors, then $\overline{G}$ contains a subgraph that cannot be triangulated, hence G cannot be triangulated. □

It follows that a 3-colored partial 2-tree can be c-triangulated if and only if $\overline{G}$ does not contain an edge between vertices with the same color, and for every pair of colors, the subgraph of $\overline{G}$ induced by the vertices with these colors is cycle-free, i.e., is a partial 1-tree. This partly generalizes:

Lemma 3.2.2 *Let* H *be a* k*-tree colored with* $k+1$ *colors such that no two adjacent vertices have the same color. Let* S *be a subset of the colors and let* $W = W(S)$ *be the set of vertices with a color in* S*. Then the induced subgraph* $H[W]$ *is a* $(|S|-1)$*-tree.*

Proof. Let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme for H. Remove from σ all vertices with a color that is not in S and let σ' be the ordering of the vertices of $H[W]$ obtained in this way. We show that:

1. $|\{v_{n-k}, \ldots, v_n\} \cap W| = |S|$,
2. for all $i \leq n-k$: $|N(v_i) \cap \{v_i, \ldots, v_n\} \cap W| = \begin{cases} |S|-1 & \text{if } v_i \in W \\ |S| & \text{if } v_i \notin W \end{cases}$

The first item follows from the fact that $\{v_{n-k}, \ldots, v_n\}$ is a $(k+1)$-clique, and hence each color of S occurs exactly once. The second item follows by a similar argument. It follows that σ' is a perfect elimination scheme for $H[W]$ such that all $v_i \in W$ are adjacent to a clique with $|S|-1$ vertices in $N(v_i) \cap \{v_i, \ldots, v_n\} \cap W$. This proves the lemma. □

Corollary 3.2.1 *If a* t*-colored graph* G *can be* c*-triangulated, then for every subset of* $s \leq t$ *colors, the subgraph of* G *induced by the vertices with a color in this set is a partial* $(s-1)$*-tree.*

Remark. This necessary condition is unfortunately not sufficient, not even for $t = 3$. Construct a graph G as follows. Take $K(2,3)$. Let A be the color class with two vertices and let B be the color class with three vertices. On every edge put an extra vertex. Let C be the set of these extra vertices. This completes the construction of G. Clearly G is a partial 2-tree since the cell-completion, obtained by creating an edge between the two vertices of A, is a tree of cycles. On the other hand, the induced subgraphs $G[A \cup B]$, $G[A \cup C]$ and $G[B \cup C]$ are all acyclic.

The following result follows directly from Lemma 3.2.1 and Theorem 3.2.2.

Corollary 3.2.2 *Let* G *be a biconnected 3-colored graph and let* $\overline{G}$ *be the cell completion of* G*.* G *can be* c*-triangulated if and only if:*

1. G *is a partial 2-tree,*
2. *no two adjacent vertices of* $\overline{G}$ *have the same color,*
3. *every cycle in* $\overline{G}$ *contains at least three vertices with different colors.*

3.3 Algorithm for 3-colored graphs

In this section we describe an algorithm to c-triangulate a 3-colored graph whenever such a triangulation is possible. In section 3.4 we give an easier variant that only tests whether it is possible to c-triangulate the graph, without actually yielding a c-triangulation.

Let G be a biconnected 3-colored graph. Recall that a c-triangulation (if it exists) can be obtained by triangulating each chordless cycle of the cell-completion (Theorem 3.1.1). Our algorithm to c-triangulate G has the following structure:

1. Make *any* triangulation of G into a 2-tree H. If this step fails (i.e., G is not a partial 2-tree) then STOP; the graph cannot be c-triangulated.
2. Given H we then can make the cell-completion $\overline{G}$ of G using Lemma 3.1.2. Check if no two adjacent vertices in $\overline{G}$ have the same color. If there are adjacent vertices in $\overline{G}$ which have the same color then STOP; the graph cannot be c-triangulated.
3. Make a list of all cells.
4. Check if all cells have three different colors. If there is a cell with only two different colors then STOP; the graph cannot be c-triangulated. Otherwise, c-Triangulate each cell.

Notice that when step 1, 2 or 4 fails, the graph can not be c-triangulated. Otherwise the algorithm outputs a correct c-triangulation. The correctness of the algorithm follows from Theorem 3.2.1.

Below we describe each step of this algorithm in more detail. As each step has a linear time implementation, we get the following result.

Theorem 3.3.1 *There exists a linear time algorithm that, given a 3-colored graph* G, *tests whether* G *can be c-triangulated and, if so, outputs a c-triangulation of* G.

3.3.1 Triangulating G into a 2-tree H

In this subsection we consider the problem to determine whether there exists a 2-tree H that contains a given graph G as a subgraph and to output such a 2-tree when it exists. It is well-known that this problem can be solved in linear time, see e.g. [96]. For completeness we describe the algorithm also here.

The following lemma is well-known, see for example [131].

Lemma 3.3.1 *If* G *is a biconnected partial 2-tree it can be triangulated into a 2-tree* H *by repeatedly choosing a vertex of degree two, making the neighbors of this vertex adjacent and removing the vertex from the graph. The order in which the vertices are removed form a perfect elimination scheme for* H.

We can implement this as follows. Assume that G is represented by means of adjacency lists. Assume that with each edge (x, y) in the adjacency list of x, there is a pointer to the edge (y, x) in the adjacency list of y. Keep a list of vertices of degree 2.

Step 1 Initialize H to G (i.e., make a copy of the adjacency lists).

Step 2 Take a vertex x of the list of vertices of degree 2. Let y and z be the neighbors of x. Add z to the adjacency list of y and y to the adjacency list of z. It is possible, that a duplicate edge (y, z) s created in this way; in that case y appears twice in the adjacency list of z, and vice versa.

Step 3 Now test, whether y or z has degree 2 (and hence whether they must be put on the list of vertices of degree 2).

Step 4 Scan the adjacency list of y until either we have encountered *three different* neighbors of y, or the list becomes empty. When a duplicate edge is encountered while scanning the list, say (y, x'), remove the second copy of it from the adjacency list of y and remove its counterpart from the list of x'. If y has only two different neighbors put y in the list of vertices of degree two. We do the same for z.

Step 5 Iterate Step 2, 3 and 4 until there are no vertices of degree 2 left. If the graph now has more than two vertices, then STOP; G is not a partial 2-tree. Otherwise, H is a 2-tree, containing G as a subgraph. The order in which the vertices are removed, is a perfect elimination scheme for the 2-tree H.

The correctness of the algorithm follows from [131]. To see that this algorithm runs in linear time notice the following. When scanning the adjacency lists, in every step we either encounter a duplicate edge (which is then removed) or we find a new neighbor. The total number of duplicate edges is at most n (the number of vertices of G), since every time a vertex is removed at most one duplicate edge is created.

We remark here that the description given in [131] is insufficient to show the linear time and linear space bound of the algorithm. In particular, in [131] the test '*given vertices* x *and* y*, test whether* $(x, y) \in E$' is assumed to take constant time. However, when the edges are given as adjacency lists, this test can take more time (when using a standard model of computation).

3.3.2 Making the cell-completion $\overline{G}$

Suppose that we now have a triangulation into a 2-tree H with a perfect elimination scheme $\sigma = [v_1, \ldots, v_n]$ for H. Recall that $\overline{G}$ is a subgraph of H, so to find $\overline{G}$, we only have to test for each edge in H whether it belongs to $\overline{G}$ or not. We use Lemma 3.1.2 to make the cell-completion. The algorithm is as follows. Make a

copy of G in $\overline{G}$. If vertices x and y have at least three common neighbors in H, add the edge (x, y) to $\overline{G}$. This property can be tested as follows.

Step 1 From the adjacency list of a vertex v_i in H, remove the vertices v_j for all $j < i$. Do the same for $\overline{G}$, using the ordering of vertices of σ. Each adjacency list now has at most two elements, since σ is a perfect elimination scheme of a 2-tree.

Step 2 Number the edges of H in any order $1, \ldots, 2n-3$, and let a pointer point from the edges in the adjacency lists to its number and vice versa. Initialize an array $cn(1 \ldots 2n-3)$ to zero.

Step 3 Start with the triangle $\{v_{n-2}, v_{n-1}, v_n\}$. For each edge of this triangle, look up its number and increase the value in cn at this position by 1.

Step 4 Consider the other vertices one by one, in the reversed order of the perfect elimination scheme, i.e., in the order $v_{n-3}, v_{n-4}, \ldots, v_1$. For each vertex v_i $(i < n-2)$ do the following. Suppose v_i is adjacent to vertices v_j and v_k (with $j > i$ and $k > i$). For the edges in this triangle, increase the value in cn by one.

Step 5 If a final value in cn is at least three, look up the edge in the adjacency list of H and add this edge to $\overline{G}$ if it was not already present.

Clearly, this procedure uses linear time. It is straightforward to see (by induction) that in Step 4, after vertex v_i has been encountered, for each edge in the induced subgraph $H[\{v_i, v_{i+1}, \ldots, v_n\}]$ the number of common neighbors of its end-vertices is given by the value in cn. We use here that each triangle v_i, v_j, v_k is considered exactly once, namely when considering the lowest numbered vertex in the triangle. Correctness follows now from Lemma 3.1.2.

3.3.3 Making a list of the cells

Notice that the number of cells in $\overline{G}$ is at most $n-2$ (with equality if and only if $\overline{G}$ is a 2-tree). For each cell initialize an empty list. During the algorithm keep, for each edge in H that has been encountered, pointers to the number of the last cell the edge is contained in and to the positions of its end vertices in this cell.

Again let $\sigma = [v_1, \ldots, v_n]$ be a perfect elimination scheme for H. Recall that for each vertex v_i, all vertices v_j with $j < i$ are removed from its adjacency list.

Step 1 Put the vertices v_n, v_{n-1}, v_{n-2} in the first cell and for each edge in this triangle make pointers to the number of this cell and to the positions of its end vertices in this cell.

Step 2 Consider the other vertices one by one, in the reversed order of the perfect elimination scheme. Suppose v_i has neighbors v_j and v_k in H with $j > i$ and $k > i$. Let $j < k$. Look in the adjacency list of v_j in $\overline{G}$ (which has at most two elements) whether the vertices v_j and v_k are adjacent.

If v_j and v_k are adjacent in $\overline{G}$, make a new cell containing the three vertices v_i, v_j and v_k and for each edge of this triangle update the number of the cell it is contained in.

If v_j and v_k are not adjacent in $\overline{G}$, then they can be contained in at most one cell, since otherwise v_j and v_k would have at least three different neighbors in H (v_i included) and hence would be adjacent in $\overline{G}$. There is a pointer to this cell and to the position of v_j and v_k in this cell. Add the vertex v_i to this cell in the place between vertices v_j and v_k and update the cell number for the edges (v_i, v_j) and (v_i, v_k).

It is straightforward to see by induction that at each step each cell contains the vertices of the cell of $\overline{G}$, restricted to the vertices $v_i, v_{i+1}, \ldots, v_n$. So in this way we obtain a list of all cells in $\overline{G}$ in linear time.

3.3.4 Triangulating each cell

Thus far for each cell a list of vertices is made such that consecutive vertices in this list are adjacent in $\overline{G}$. For each cell do the following.

Step 1 If there are two adjacent vertices in the cell with the same color then STOP; G cannot be c-triangulated.

Step 2 For each of the three colors make a list of vertices in the cell that are of this color, and for every vertex make a pointer to its position in the list containing the vertex.

Step 3 If there is a color with an empty list, the cell can not be triangulated; STOP.

Step 4 Assume every color occurs in the cell.

If there is a color such that there is only one vertex of that color, make this vertex adjacent to all other vertices.

Otherwise, by going around the cycle clockwise, find a vertex x such that the two neighbors are of a different color. Add the edge between the two neighbors and delete x from the cell and from the color list. Move counterclockwise to the neighbor of x, and start again.

It is easy to see that this algorithm runs in linear time.

3.4 Some variants

A simple variant of the algorithm is the following.

Lemma 3.4.1 *There exists a linear time algorithm that, given a* t*-colored graph* G*, tests whether* G *can be* c*-triangulated into a 2-tree.*

Proof. If $t < 3$, then the graph can not be c-triangulated into a 2-tree. For $t \geq 3$ the same algorithm as used in section 3.3 will do. □

We now describe an algorithm that, given a 3-colored graph G, decides whether G can be c-triangulated. It does not produce a c-triangulation of G. However, it is slightly easier than the algorithm in section 3.3. The algorithm has the following structure:

1. Make *any* triangulation of G into a 2-tree H.
2. Given H make the cell-completion of G, $\overline{G}$. Check that no two adjacent vertices in $\overline{G}$ have the same color.
3. For each pair of colors c_1, c_2, take the subgraph of G induced by vertices of color c_1, or c_2, and check whether this graph does not contain a cycle.

G can be c-triangulated if and only if step 1 succeeds (G is a partial 2-tree), in step 2 no pair of adjacent vertices with the same color is found, and in step 3 all three considered induced subgraphs are cycle-free. The correctness of this procedure follows from Corollary 3.2.2. Steps 1 and 2 can be implemented as in section 3.3.1 and section 3.3.2. It is easy to see, that step 3 has a linear time implementation.

Chapter 4

Enumerations of partial k-trees

In this chapter we concentrate on the enumeration problem of (partial) k-trees. The enumeration problem of k-trees has by now a considerable history. It started with the enumeration of labeled 2-trees [103, 8] and continued with the enumeration of labeled k-trees [9, 99, 50, 108]. In section 4.1 we show a simple method to derive a formula for the number of labeled k-trees. The method we describe is similar to the method described in [99]. The main difference is that we explicitly show a one-to-one correspondence between rooted k-trees and certain labeled rooted trees, where some of the vertices have a color attached to them.

In section 4.2 we consider the enumeration of unlabeled biconnected 2-trees. The unlabeled 2-trees were first enumerated by Harary and Palmer [65]. A special type of (unlabeled) k-trees, equivalent with stack polytopes or simplicial clusters, in which every k-clique is contained in at most two $(k+1)$-cliques, is enumerated in [68]. We generalize this to the enumeration of biconnected partial 2-trees in section 4.2. This method also generalizes the results of Harary, Palmer and Read [66] for the cell-growth problem (enumeration of *clusters*) and the enumeration of the *tree-like polyhexes* in [67], and is related to the (unsolved) combinatorial problem of the enumeration of *animals*, such as *polyominoes*. The difficulty with animals seems to be the restriction that newly added cells must remain in the plane. With the clusters a newly added cell has only an edge in common with one already existing cell, and it may be put 'above' other cells. A typical example of a cluster is an *outerplanar* 2-tree, in which case the cells are triangles.

In a certain sense, biconnected partial 2-trees are a generalization of the polygonal clusters of [66]. The main differences are:

1. cells do not have to be fixed polygons,

2. there can be more than two cells incident with one cell-side, and

3. cell-sides may be missing, provided that there are at least three cells incident with this cell-side.

The methods we use are roughly the same as those in [66]. The main contribution of this chapter, is the exhibition of a *dissimilarity characteristic* which holds true for all biconnected partial 2-trees. This characteristic establishes a relation between the different cells and cell-sides of a partial 2-tree. This resembles the result of Otter for trees [102], and is a generalization of the dissimilarity characteristic theorem derived in [65] and of the related equation for clusters derived in [66].

Our methods can easily be adapted for ordinary 2-trees and related structures such as clusters. The enumeration methods presently known do not seem strong enough for solving the generalized problem of enumerating the unlabeled (partial) *k-trees*. Another open problem is the enumeration of the *labeled* partial 2-trees.

4.1 Labeled k-trees

In this section we want to derive a formula for the number of different labeled k-trees with n vertices. Most results also appeared in [99]. Only the proof of Lemma 4.1.3 is different. We establish a one-to-one correspondence between rooted k-trees and rooted trees with a color from a set of k colors attached to each vertex except the root and the neighbors of the root.

Definition 4.1.1 *Let* $T(n,d)$ *be the number of rooted trees with* n *nodes in which the root has degree* d.

Lemma 4.1.1 *Let* $d > 1$. *Then:*

$$T(n,d) = \frac{n-d}{(n-1)(d-1)} T(n,d-1)$$

Proof. Consider a rooted tree T such that the root r has degree $d-1$. Let $i \neq r$ be a vertex which is not adjacent to r, and let j be the father. Construct a new rooted tree T', such that the root has degree d as follows. Cut the edge (i,j) in T, and add the edge (r,i). We count the number of such tuples (T,i,j,T') in two different ways. Starting with T there are $n-d$ possible choices for i in T and, after i is chosen, the node j and tree T' are uniquely determined.

Alternatively, we can start with a rooted tree T' such that the root has degree d, and change this into a tree T such that r has degree $d-1$ as follows. First choose some vertex $j \neq r$ in T'. Next choose a neighbor i of r, which is *not* the root of the subtree containing j. Then cut the edge (r,i) and make i adjacent to j instead. There are $n-1$ possible choices for a vertex j, and, when j is chosen, there are $d-1$ choices for a neighbor i of r which is not the root of the subtree containing j. In this way we find:

$$T(n,d-1)(n-d) = T(n,d)(n-1)(d-1)$$

□

Lemma 4.1.2

$$T(n,d) = \binom{n-2}{d-1}(n-1)^{n-d-1}$$

Proof. From Lemma 4.1.1 it follows that for $d > 1$:

$$T(n,d) = \frac{1}{(n-1)^{(d-1)}}\binom{n-2}{d-1}T(n,1)$$

We can find $T(n,1)$ as follows. First we take a labeled unrooted tree with $n-1$ vertices, and then we make the root r adjacent to any vertex in this tree. From Cayley's formula we know that there are $(n-1)^{n-3}$ labeled trees with $n-1$ vertices. It follows that $T(n,1) = (n-1)^{n-2}$. The lemma follows. □

Definition 4.1.2 *Consider k-trees with vertices labeled* $1,\ldots,n$. *A* d-rooted k-tree *is a labeled k-tree in which the vertices* $n, n-1, \ldots, n-k+1$ *form a clique with* k *vertices and in which there are exactly* d *other vertices which are adjacent to every vertex of this clique. Let* $C_k(n,d)$ *be the number of these* d*-rooted* k*-trees.*

Lemma 4.1.3

$$C_k(n,d) = \{k(n-k)\}^{n-d-k}\binom{n-k-1}{d-1}$$

Proof. Let $\mathcal{T}$ be a d-rooted k-tree, and let C be the clique in $\mathcal{T}$ with vertices $n,\ldots,n-k+1$. Consider a perfect elimination scheme $\sigma[x_1,\ldots,x_n]$ such that $x_i = i$ for $i = n-k+1,\ldots,n$. We construct a rooted tree T with $n-k+1$ vertices as follows. Start with a root r. For $i = n-k,\ldots,1$, if i is adjacent to every vertex of C, then make i adjacent to r in T. Otherwise, there is a vertex j in T, which is a neighbor of i in $\mathcal{T}$, such that all neighbors of i which are in T are either in C or are on the path from j to the root r. Make i adjacent to this vertex j.

Consider a vertex x in T which is not the root or a neighbor of the root. Let y be the father of x. Notice that the number of neighbors of x in C plus the number of neighbors of x on the path from x to the root r is k. It follows that there is exactly one vertex on the path from y to the root which is a neighbor of y but not of x. We can give x a color from a set with k colors indicating which neighbor of y this is. In this way we construct a rooted tree T such that the root r has d neighbors, and in which each vertex except the root or the neighbors of the root has a color from a set with k colors attached to it. Such a tree uniquely determines a k-tree such that $\{n,\ldots,n-k+1\}$ is a clique. Hence $C_k(n,d) = k^{n-d-k}T(n-k+1,d)$. □

Corollary 4.1.1 *There is a one-to-one correspondence between* d*-rooted* k*-trees rooted trees in which the root has degree* d *and with a color from a set of* k *colors attached to every vertex except the root and the neighbors of the root.*

Theorem 4.1.1 *Let* $T_k(n)$ *be the number of labeled* k*-trees with* n *vertices. Then*

$$T_k(n) = \binom{n}{k}(1 + k(n-k))^{n-k-1}$$

Proof. Let $R_k(n)$ be the number of k-trees, such that vertices $n, n-1, \ldots, n-k+1$ are a clique. Notice that:

$$R_k(n) = \sum_{d=1}^{n-k} C_k(n,d) = (1 + k(n-k))^{n-k-1}$$

Count the number of pairs $(\mathcal{T}, C)$, where $\mathcal{T}$ is a k-tree, and C is a clique with k vertices in $\mathcal{T}$. If we start by choosing the clique C, we find $\binom{n}{k} R_k(n)$ for the number of these pairs. On the other hand, if we first choose the k-tree, we find $T_k(n)(1 + k(n-k))$, since the number of cliques with k vertices in a k-tree is $1 + k(n-k)$. The result follows. □

4.2 The unlabeled case. An introduction

Harary and Palmer [65] gave a method to enumerate unlabeled 2-trees. In the next sections we describe a method to enumerate the biconnected partial 2-trees. We use the term *2-partials* for short.

Definition 4.2.1 *A* 2-partial *is a biconnected partial 2-tree.*

Recall that a 2-tree is a tree of triangles (Definition 3.1.2, page 26). So a 2-tree can be obtained by gluing together a number of triangles at their sides (one at a time). Recall that a *cell* in a 2-partial is a t-*gon* for some t, of which zero or more edges are missing (Definition 3.1.1, page 26). The sides of the *n-gon* are called *cell-sides*; i.e., the cell-sides are simply the edges of the cell-completion. If a cell-side of the n-gon is not an edge of the 2-partial then there must be at least three components of the 2-partial incident with this cell-side.

In the next section we shall obtain a *dissimilarity characteristic* for 2-partials which will enable us to enumerate them.

4.3 A dissimilarity characteristic

Lemma 4.3.1 *Consider a 2-partial* $T = (V, E)$. *Let* p *be the number of cells of* T *and let* q *be the number of cell-sides. Denote the set of cell-sides of* T *by* E. *For a cell-side* e *let* $c(e)$ *be the number of components in* $T[V \setminus e]$. *Then we have:*

$$p - \sum_{e \in E} (c(e) - 1) = 1$$

Proof. By induction on the number of cells. If $p = 1$ then it is trivial because $c(e) = 1$ for all the sides. For any newly added cell in a 2-partial there is exactly one cell-side (on which it is glued) of which the number of components increases by one. □

Definition 4.3.1 *Let* T *be a 2-partial. Two cells of* T *are called* similar *if there is an automorphism of* T *which maps one cell onto the other. Two cell-sides are called similar if there is an automorphism which maps one cell-side onto the other. We denote by* p^* *the number of dissimilar cells, and by* q^* *the number of dissimilar cell-sides. Let* E^* *be a set of dissimilar cell-sides, containing one cell-side for every equivalence class.*

Notice that, of course, two cell-sides can only be similar if they are both edges or both are not edges in the 2-partial.

Definition 4.3.2 *A cell-side is called* symmetric *if there is an automorphism which maps the cell-side onto itself and interchanges the two end vertices of the cell-side. Let* $S \subseteq E^*$ *be a maximal set of dissimilar symmetric cell-sides.*

Let $T = (V, E)$ be a 2-partial, and let e be a cell-side. We call a connected component of $T[V \setminus e]$, A component of e.

Definition 4.3.3 *Consider a cell-side* e *of a 2-partial* T. *Two components of* e *are* equivalent *if there is an automorphism of* T *which maps one component onto the other and leaves* e *invariant. For a cell-side* e, *let* $c^*(e)$ *be the number of equivalence classes of the components incident with this cell-side.*

Definition 4.3.4 *A component incident with a symmetric cell-side* e *is called* symmetric *if there is an automorphism which maps the component onto itself but interchanges the end vertices of* e. *For a symmetric cell-side* e, *let* $\theta(e)$ *be the number of* non-equivalent and non-symmetric *components incident with* e.

Theorem 4.3.1 (Dissimilarity Characteristic) *For any 2-partial we have:*

$$p^* - \sum_{e \in E^*} (c^*(e) - 1) + \sum_{e \in S} \frac{\theta(e)}{2} = 1$$

Proof. Let G be a 2-partial. First notice the following. Consider the components incident with one cell-side. If two of these components are equivalent, remove one of them. The new structure G′ does not have to be a 2-partial, since cell-sides that are not edges can be incident with fewer than three components. But as long as they are identified as cell-sides, Lemma 4.3.1 remains valid, and if the dissimilarity characteristic holds for these somewhat more general structures, it also holds for 2-partials. Notice that G′ has at least one cell of each equivalence class of cells in G and at least one cell-side of each equivalence class of cell-sides in G. Also, two cells or cell-sides are similar in G′ if and only if they are similar in G. It follows that if the dissimilarity characteristic holds for G′, it also holds for G. Notice that $c^*(e) = c(e)$ for all $e \in E^*$ in G′.

Notice that if the automorphism group of G′ consists only of the trivial automorphism, the theorem clearly is true. Hence we may assume there is at least one non-trivial automorphism.

We consider the different kinds of symmetry the structure G′ can have. To analyze this, construct a tree T as follows. The nodes of T are of two types: one type consists of the inner cell-sides (i.e., the cell-sides incident with at least two cells), and the other type consists of the cells. Two nodes in T are adjacent if and only if they are of different type and incident. According to a well-known theorem of König (see for example [64]), the *center* of T consists either of one point or of two adjacent points. Clearly, any automorphism of G′ maps the center onto itself. We consider two different cases.

case 1 The center contains a cell-side s. Clearly there can be only one cell-side in the center. Since any automorphism maps this cell-side onto itself, it easily follows that the only non-trivial automorphism can be a reflection. Hence, s must be a symmetric cell-side. For each symmetric cell-side, the non-symmetric components incident with with this cell-side, must appear in pairs of conjugates. For each pair of conjugates, choose one component, and remove this. Consider the symmetric components incident with s. Consider a cell D, incident with s, which is in a symmetric component. then the cell-sides of D, except s and one other if the number is even, appear in pairs of conjugates (which are mapped onto each other by the reflection). For each pair, choose one, and remove all incident components except the one component containing vertices of D. Let G^*, be the new structure. It is easy to see that the number of cells of G^* is p^*, and for all equivalence classes of cell-sides in G′, there is exactly one cell-side e, in G^*, such that the number of components incident with e in G^* equals $c^*(e)$. For all other cell-sides, similar with e, the number of components is 1. Now according to Lemma 4.3.1,

$$1 = p^* - \sum_{e \in E^*} (c(e) - 1)$$

$$= p^* - \sum_{e\in E^*\setminus S}(c^*(e)-1) - \sum_{e\in S}(c(e)-1)$$

$$= p^* - \sum_{e\in E^*\setminus S}(c^*(e)-1) - \sum_{e\in S}(c^*(e) - \frac{\theta(e)}{2} - 1)$$

$$= p^* - \sum_{e\in E^*}(c^*(e)-1) + \sum_{e\in S}\frac{\theta(e)}{2}$$

case 2 The center consists of one point which corresponds with a cell C. In this case the automorphism group can contain both reflections and rotations. For each equivalence class of cell-sides of C, choose one representative, and remove all components incident with the other cell-sides of the equivalence class, except the component containing C. For all symmetric cell-sides, apply the trick described above. Application of Lemma 4.3.1 gives the result.

□

If we sum the equation from Theorem 4.3.1 over the different 2-partials, the righthandside simply becomes the number of different 2-partials. The summation over p^* becomes the number of *cell-rooted* 2-partials (i.e., 2-partials in which one cell is distinguished from all other cells). In section 4.5.2 we consider the problem of enumerating these cell-rooted 2-partials. This turns out to be easy if mirror images are regarded as distinct. In section 4.6.1 we sum the second term of the dissimilarity characteristic. This means that we obtain a method to determine the coefficients of $E(x,y) = \sum_n \sum_k e_{n,k}x^n y^k$, where $e_{n,k}$ is the number of 2-partials rooted at a cell-side with n vertices and k non-equivalent components incident with this cell-side. In section 4.6.2 we sum the third term of the dissimilarity characteristic. Let $\theta(x,y) = \sum_n \sum_k \theta_{n,k}x^n y^k$ be the counting series for the symmetric cell-side rooted 2-partials with n vertices and k different non-symmetric components. We give a method to determine the coefficients $\theta_{n,k}$. At the basis of most computations are the counting series of the 2-partials which are rooted at a labeled edge. In the next section we give a method to obtain these counting series.

4.4 2-partials rooted at a labeled edge

Definition 4.4.1 *Let $\mathcal{B}$ be the set of 2-partials with at least three vertices rooted at a labeled edge (i.e., one edge, with end vertices different, distinguished from all other edges). Let $F(x) = \sum_{n=3}^{\infty} f_n x^n$ be the counting series of the 2-partials in $\mathcal{B}$, where f_n is the number of elements of $\mathcal{B}$ with n vertices.*

Let $\mathcal{B}^*$ be those 2-partials of $\mathcal{B}$ in which there is exactly one *component incident with the root edge. Let $G(x)$ be the counting series of the elements of $\mathcal{B}^*$.*

The following theorem illustrates how the edge-rooted 2-partials can be enumerated.

Theorem 4.4.1 *The counting series* F *and* G *are determined by:*

$$F(x) = x^2 \exp\left(\sum_{t=1}^{\infty} \frac{G(x^t)}{t x^{2t}}\right) - x^2$$

$$G(x) = \frac{(x^2 + 2F(x))^2}{x^2 + x + 2F(x)}$$

Proof. The first formula can be obtained by taking a number of 2-partials of $\mathcal{B}^*$ and identifying the roots. In this way we obtain:

$$F(x) = x^2 \prod_{b \in \mathcal{B}^*} (1 + x^{\nu(b)-2} + x^{2(\nu(b)-2)} + \ldots) - x^2$$

where $\nu(b)$ denotes the number of vertices for a 2-partial b. We subtract x^2 because we only consider 2-partials with at least three vertices. This leads as follows to the first formula:

$$\begin{aligned} F(x) &= x^2 \prod_{b \in \mathcal{B}^*} (1 + x^{\nu(b)-2} + x^{2(\nu(b)-2)} + \ldots) - x^2 \\ &= x^2 \exp\left(-\sum_{b \in \mathcal{B}^*} \ln\left(1 - x^{\nu(b)-2}\right)\right) - x^2 \\ &= x^2 \exp\left(\sum_{t=1}^{\infty} \frac{1}{t x^{2t}} \sum_{b \in \mathcal{B}^*} x^{t\nu(b)}\right) - x^2 = x^2 \exp\left(\sum_{t=1}^{\infty} \frac{G(x^t)}{t x^{2t}}\right) - x^2 \end{aligned}$$

Consider a 2-partial of $\mathcal{B}^*$ with root edge (r_1, r_2). There is exactly one cell C incident with this edge. Let z be the "other" neighbor of r_1 in C. This vertex exists in any element of $\mathcal{B}^*$ and is unique. We can built an arbitrary element of $\mathcal{B}^*$ as follows. Start with a triangle (r_1, r_2, z). Put edge rooted partials at the sides (r_1, z) and (r_2, z). We put an arbitrary element of $\mathcal{B}$ at (r_2, z) and for *all choices* we allow (r_2, z) to be an edge or not. In this way arbitrary cells C are created. At (r_1, z), if this is not an edge, we only allow elements of $\mathcal{B} \setminus \mathcal{B}^*$, i.e., edge-rooted 2-partials with at least two components incident with the root (thus ensuring the unique role of z). Considering the four different cases we obtain:

$$\begin{aligned} xG(x) &= (x^2 + F(x))^2 + F(x)(x^2 + F(x)) + \\ &\quad + (x^2 + F(x))(F(x) - G(x)) + F(x)(F(x) - G(x)) \end{aligned}$$

The first term corresponds to the case in which there is an edge at (r_1, z) and at (r_2, z), the second term to the case with an edge at (r_1, z) and no edge at (r_2, z), the third term to the case with an edge at (r_2, z) and no edge at (r_1, z), and finally,

the fourth term corresponds to the case with no edge at (r_1, z) and (r_2, z). This proves the theorem. □

With these formulae we can determine $F(x)$ and $G(x)$. For $F(x)$ the first few terms are: $F(x) = x^3 + 4x^4 + 19x^5 + 104x^6 + \ldots$.

We now enumerate the 2-partials rooted at an edge, which are *symmetric* at this edge.

Definition 4.4.2 *Let $F_s(x)$ be the counting series for the 2-partials rooted at an unlabeled symmetric edge and $G_s(x)$ be the counting series for the 2-partials rooted at an unlabeled symmetric edge which is incident with* one *component. Let*

$$G_{ns}(x) = \frac{G(x) - G_s(x)}{2} \quad \textit{and} \quad F_{ns}(x) = \frac{F(x) - F_s(x)}{2}$$

Remark. F_{ns} and G_{ns} are the counting series for the *non-symmetric* edge-rooted 2-partials (with *unlabeled* root edge).

Theorem 4.4.2 *The counting series F_s and G_s are determined by the following functional equations.*

$$\begin{aligned} G_s(x) &= \frac{(x + x^2 + 2F_s(x))(x^4 + 2F(x^2))}{x^2 + x^4 + 2F(x^2)} \\ F_s(x) &= x^2 \exp\left(\sum_{t=1}^{\infty} \frac{G_s(x^t)}{t x^{2t}} + \frac{G_{ns}(x^{2t})}{t x^{4t}} \right) - x^2 \end{aligned}$$

Proof. Consider a 2-partial rooted at a symmetric edge (r_1, r_2) which is incident with exactly one component. Since there is *one* component incident with (r_1, r_2), there is exactly one cell incident with (r_1, r_2). There are two cases to consider, namely the case in which this cell contains an odd number of vertices and the case in which this cell contains an even number of vertices. Let $G_{s,odd}(x)$ be the counting series for the 2-partials of the first type, and $G_{s,even}(x)$ be the counting series for the 2-partials of the second type. Then, clearly $G_s(x) = G_{s,odd}(x) + G_{s,even}(x)$. First consider the case in which the cell incident with (r_1, r_2) has an odd number of vertices, say $2t+1$ besides r_1 and r_2, i.e., $2t+3$ in total ($t \geq 0$). We put 2-partials rooted at cell-sides at the sides of the cell in the following way. We choose $t+1$ 2-partials and put these on the left side of the cell. The 2-partials at the right side must be the mirror images. For each of the $t+1$ 2-partials we can choose one rooted at an edge or one rooted at a non-edge with at least two components. In this way we find:

$$G_{s,odd}(x) = \sum_{t=0}^{\infty} \sum_{\ell=0}^{t+1} \binom{t+1}{\ell} \left(\frac{x^4 + F(x^2)}{x^4} \right)^{\ell} \left(\frac{F(x^2) - G(x^2)}{x^4} \right)^{t+1-\ell} x^{2t+3}$$

This easily leads to:

$$G_{s,odd}(x) = \frac{x(x^4 + 2F(x^2) - G(x^2))}{x^2 - x^4 - 2F(x^2) + G(x^2)}$$

Using Theorem 4.4.1 we obtain:

$$G_{s,odd}(x) = \frac{x^4 + 2F(x^2)}{x}$$

In a similar manner we find:

$$G_{s,even}(x) = (x^2 + 2F_s(x) - G_s(x))\frac{x^4 + 2F(x^2)}{x^2}$$

Some calculations then give the stated result.

The formula for $F_s(x)$ can be found as follows. Let $\mathcal{B}_s^*$ be the set of 2-partials rooted at a symmetric edge and with *one* component incident with this edge. A symmetric 2-partial rooted at an edge can be obtained by identifying the root edges of a number of elements of $\mathcal{B}_s^*$, and a number of elements of $(\mathcal{B} \setminus \mathcal{B}_s^*)_r$ which is a set of representatives of $\mathcal{B} \setminus \mathcal{B}_s^*$, these last elements appearing twice at the root edge, because the resulting 2-partial must be symmetric. Hence we find:

$$\begin{aligned} x^2 + F_s(x) &= x^2 \prod_{b \in \mathcal{B}_s^*} (1 + x^{v(b)-2} + x^{2(v(b)-2)} + \ldots) \\ &\quad \prod_{b \in (\mathcal{B} \setminus \mathcal{B}_s^*)_r} (1 + x^{2(v(b)-2)} + x^{4(v(b)-2)} + \ldots) \end{aligned}$$

This gives the stated result. □

4.5 Cell-rooted 2-partials

Definition 4.5.1 *A* cell-rooted 2*-partial is a 2-partial with one cell distinguished from all the other cells.*

In this section we enumerate the cell-rooted 2-partials. The method we use is similar to the method used in [66] for the cell-growth problem. First we consider the problem in which the mirror images are considered as being different.

4.5.1 Mirror images different

Definition 4.5.2 *Let* $U_t(x)$ *be the counting series for the cell-rooted 2-partials with mirror images different, and with a root-cell consisting of* t *vertices.*

At the cell-sides of the root we either have an edge, a 2-partial rooted at an edge, or a 2-partial rooted at a non-edge with at least two components. Thus the *figure counting series* [64] is $f(x) = x^2 + F(x) + (F(x) - G(x))$. The configuration group of permutations that is required, is the cyclic group C_t of order t. The *cycle index* is [64]:

$$Z(C_t; a_1, a_2, \ldots, a_d) = \frac{1}{t} \sum_{k|t} \phi(k) a_k^{t/k}$$

where $\phi(k)$ is the number of positive integers less than k and relatively prime to k and $\phi(1) = 1$. By Pólya's theorem [105, 106, 64] we can find the counting series for the cell-rooted 2-partials with mirror images different by substituting the figure counting series into the cycle index according to:

$$Z(C_t, f(x)) = Z(C_t; f(x), f(x^2), f(x^3), \ldots, f(x^d))$$

Since each vertex of the cell appears in two "figures", we divide by x^t. Hence we obtain:

Theorem 4.5.1 *The counting series $U_t(x)$ for the cell-rooted partials with mirror images different and with a root-cell consisting of t vertices is:*

$$U_t(x) = \frac{1}{x^t} Z(C_t, f(x))$$

where Z is the cycle index of the configuration group, $f(x) = x^2 + 2F(x) - G(x)$ where $F(x)$ and $G(x)$ are determined by the functional equations of Theorem 4.4.1.

For example, we find: $U_3(x) = x^3 + x^4 + 6x^5 + 34x^6 + \ldots$, and $U_4(x) = x^4 + x^5 + 7x^6 + \ldots$.

4.5.2 Mirror images not different

In this section we discuss the general case in which also reflections are taken into account. The difficulty in this case is that the figures (the 2-partials rooted at the cell-sides of the root-cell) are replaced by their mirror images (see also [66]). Hence we cannot use Pólya's theorem directly. Instead we count the cell-rooted 2-partials which are invariant under a reflection. Then we apply Burnside's lemma.

Definition 4.5.3 *Let $V_t(x)$ be the counting series for the cell-rooted 2-partials in which the root-cell has t vertices.*

Theorem 4.5.2 *Let $Y(x) = x^2 + 2F_s(x) - G_s(x)$ where F_s and G_s are determined by the functional equation in Theorem 4.4.2 and $K(x) = x^2 + 2F(x) - G(x)$ with F and G as in Theorem 4.4.1. Then*

$$V_t(x) = \frac{1}{2} U_t(x) + \begin{cases} \frac{1}{2x^t} Y(x) K(x^2)^m & \text{if } t = 2m+1 \\ \frac{1}{4x^t} (K(x^2)^m + K(x^2)^{m-1} Y(x)^2) & \text{if } t = 2m \end{cases}$$

Proof. The cycle index of the dihedral group can be written as [64]:

$$Z(D_t) = \frac{1}{2}Z(C_t) + \begin{cases} \frac{1}{2}a_1 a_2^m & \text{if } t = 2m+1 \\ \frac{1}{4}(a_2^m + a_1^2 a_2^{m-1}) & \text{if } t = 2m \end{cases}$$

First consider the case in which there is an odd number of vertices in the root-cell, say $t = 2m+1$. We enumerate the number of cell-rooted 2-partials which are invariant under a reflection. Put 2-partials rooted at a cell-side at the sides of the root-cell. We need *one* symmetric 2-partial rooted at a cell-side to put at the cell-side which is cut by the axis of the reflection. The figure counting series for the symmetric 2-partials rooted at a cell-side is given by $Y(x)$ (see also Theorem 4.5.1). The other ones we can choose freely, so their figure counting series is given by $K(x)$. Hence the number of cell-rooted 2-partials which are invariant under a reflection is $\frac{1}{x^t}Y(x)K(x^2)^m$. According to Burnside's lemma [34, 64] the number of cell-rooted 2-partials is given by $V_{2m+1}(x)$.

Now consider the case in which the root-cell has an even number of vertices, say $t = 2m$. In this case there are two kinds of reflections. The axis can go through two cell-sides or it can go through no cell-side at all. In the first case we need *two* symmetric 2-partials rooted at a cell-side. The number of invariant cell-rooted 2-partials in this case is $\frac{1}{x^t}K(x^2)^{m-1}Y(x)^2$. In the other case it is $\frac{1}{x^t}K(x^2)^m$. This proves the theorem. □

For example we find $V_3(x) = x^3+x^4+5x^5+21x^6+\ldots$ and $V_4(x) = x^4+x^5+6x^6+\ldots$

4.6 Cell-side rooted 2-partials

In this section we sum the second and the third term of the dissimilarity characteristic. For the second term we need the counting series for the cell-side rooted 2-partials with the additional information of the number of non-equivalent components. For the last term of the dissimilarity characteristic we need the number of different non-symmetrical components of the cell-side rooted 2-partials.

4.6.1 Nonequivalent components

In this section we sum the second term of the dissimilarity characteristic over the 2-partials. In this case we need the counting series for the cell-side rooted 2-partials with a given number of non-equivalent components.

Definition 4.6.1 *Let* $E(x,y) = \sum_{n=3}^{\infty}\sum_{k=1}^{\infty} e_{n,k}x^n y^k$, *where* $e_{n,k}$ *is the number of 2-partials rooted at a cell-side with* n *vertices and with* k *non-equivalent components incident with this cell-side. We define* E_L *as the corresponding series for the case the cell-side is* labeled *and* $\tilde{E}_L$ *is the series for the special case the labeled cell-side is an edge.*

Then, with this definition, the summation over the different 2-partials of $\sum_{e\in E}(c^*(e)-1)$ becomes $\frac{d}{dy}E(x,y)_{y=1} - E(x,1)$.

Lemma 4.6.1 *The counting series* E_L *and* $\tilde{E}_L$ *satisfy the following equations.*

$$\begin{aligned}\tilde{E}_L(x,y) &= x^2\exp\left(\sum_{k=1}^{\infty}\frac{1-(1-y)^k}{kx^{2k}}G(x^k)\right) - x^2\\ E_L(x,y) &= 2\tilde{E}_L(x,y) - y\left(G(x)+\frac{G(x^2)}{x^2}\right) - \frac{1}{2}y^2\frac{G(x)^2-G(x^2)}{x^2}\end{aligned}$$

Remark. Notice that $F(x) = \tilde{E}_L(x,1)$.

Proof. Again, we can take a number of elements of $\mathcal{B}^*$ and identify their roots. We obtain:

$$\tilde{E}_L(x,y) = x^2\prod_{b\in\mathcal{B}^*}(1+y(x^{\nu(b)-2}+x^{2(\nu(b)-2)}+\ldots)) - x^2$$

After some manipulations this leads to the desired result.

To obtain the formula for E_L, notice the following. A cell-side can be a non-edge only if there are at least three components incident with it (by definition). The counting series of the 2-partials with *one* component is given by $yG(x)$. The counting series for the 2-partials with *two* components is given by $\frac{1}{x^2}(\frac{1}{2}y^2(G(x)^2-G(x^2))+yG(x^2))$, where the first term corresponds with the case the two components are different and the second term corresponds with two equivalent components.

Subtracting these from $2\tilde{E}_L(x,y)$ gives the counting series $E_L(x,y)$. □

To get the counting series $E(x,y)$ from the labeled one $E_L(x,y)$ we need the counting series for the symmetric case first.

Definition 4.6.2 *Let* $E_s(x,y) = \sum_{n=3}^{\infty}\sum_{k=1}^{\infty} e_{s,n,k}x^n y^k$, *where* $e_{s,n,k}$ *is the number of 2-partials rooted at a* symmetric *cell-side with* n *vertices and* k *different components incident with this cell-side. Let* $\tilde{E}_s$ *be the corresponding series in case the cell-side is an edge.*

Lemma 4.6.2

$$\begin{aligned}\tilde{E}_s(x,y) &= x^2\exp\left(\sum_{k=1}^{\infty}\frac{1-(1-y)^k}{kx^{2k}}G_s(x^k) + \frac{1-(1-y^2)^k}{kx^{4k}}G_{ns}(x^{2k})\right) - x^2\\ E_s(x,y) &= 2\tilde{E}_s(x,y) - yG_s(x) +\\ &\quad -\frac{1}{x^2}\left(\frac{1}{2}y^2(G_s(x)^2-G_s(x^2)) + yG_s(x^2) + y^2G_{ns}(x^2)\right)\end{aligned}$$

Remark. Notice that $F_s(x) = E_s(x, 1)$.

Proof.

$$x^2 + \tilde{E}_s(x,y) = x^2 \prod_{b \in \mathcal{B}_s^*} (1 + y(x^{\nu(b)-2} + x^{2(\nu(b)-2)} + \ldots)) \prod_{b \in (\mathcal{B} \setminus \mathcal{B}_s^*)_r} (1 + y^2(x^{2(\nu(b)-2)} + x^{4(\nu(b)-2)} + \ldots))$$

where $(\mathcal{B} \setminus \mathcal{B}_s^*)_r$ is a set of representatives of $\mathcal{B} \setminus \mathcal{B}_s^*$. Some manipulations lead to the stated result.

To obtain a formula for $E_s(x, y)$, again notice that a cell-side can be a non-edge, only if there are at least three components incident with this cell-side. The number with *one* component is given by $yG_s(x)$ and the number with *two* components by:

$$\frac{1}{x^2}\left(\frac{1}{2}y^2(G_s(x)^2 - G_s(x^2)) + yG_s(x^2) + y^2G_{ns}(x^2)\right)$$

The first term corresponds with the case in which the two components are symmetric and different. The second term corresponds with the case in which the two components are symmetric and equivalent. The third term corresponds with a non-symmetric component and its mirror image.

Subtracting these from $2\tilde{E}_s$ gives the formula for E_s. □

Finally we obtain a formula for $E(x, y)$:

Theorem 4.6.1

$$E(x,y) = \frac{E_L(x,y) + E_s(x,y)}{2}$$

where E_L *and* E_s *as defined by the functional equations in Lemma 4.6.1 and Lemma 4.6.2.*

4.6.2 Nonsymmetric components

In this section we sum the last part $(\sum_{s \in S} \frac{\theta(s)}{2})$ of the dissimilarity characteristic over the different 2-partials. We need a counting series which contains the information of the different non-symmetric components of the 2-partials rooted at a symmetric cell-side.

Definition 4.6.3 *Let* $\theta(x,y) = \sum_{n=3}^{\infty} \sum_{k=1}^{\infty} \theta_{n,k} x^n y^k$, *with* $\theta_{n,k}$ *the number of 2-partials rooted at a symmetric cell-side with* n *vertices and* k *different unsymmetric components. Let* $\tilde{\theta}(x,y)$ *be the corresponding series for the case in which the cell-side is an edge.*

The summation over the 2-partials of $\sum_{s\in S}\frac{\theta(s)}{2}$ is then given by $\frac{1}{2}\frac{d}{dy}\theta(x,y)_{y=1}$.

Theorem 4.6.2

$$\begin{aligned}\tilde{\theta}(x,y) &= x^2\exp\left(\sum_{k=1}^{\infty}\frac{G_s(x^k)}{kx^{2k}}+\frac{1-(1-y^2)^k}{kx^{4k}}G_{ns}(x^{2k})\right)-x^2\\ \theta(x,y) &= 2\tilde{\theta}(x,y)-G_s(x)-\frac{1}{x^2}\left(\frac{1}{2}(G_s(x)^2+G_s(x^2))+y^2G_{ns}(x^2)\right)\end{aligned}$$

Proof.

$$\begin{aligned}x^2+\tilde{\theta}(x,y) &= x^2\prod_{b\in\mathcal{B}_s^*}(1+x^{\nu(b)-2}+x^{2(\nu(b)-2)}+\ldots)\\ &\quad\prod_{b\in(\mathcal{B}\setminus\mathcal{B}_s^*)_r}(1+y^2(x^{2(\nu(b)-2)}+x^{4(\nu(b)-2)}+\ldots))\end{aligned}$$

To obtain a formula for $\theta(x,y)$ we subtract the ones with one or two components. This proves the theorem. □

4.7 The counting series for 2-partials

Theorem 4.7.1 *The counting series for the (unlabeled) biconnected partial 2-trees with at least three vertices is*

$$\sum_{t=3}^{\infty}V_t(x)-\frac{d}{dy}E(x,y)_{y=1}+E(x,1)+\frac{1}{2}\frac{d}{dy}\theta(x,y)_{y=1}$$

where V_t is given in Theorem 4.5.2, E is given in Theorem 4.6.1 and θ in Theorem 4.6.2.

Chapter 5

Only few graphs have bounded treewidth

In this chapter we consider the treewidth of random graphs. To be more precise, for random graphs with n vertices and $m \geq \delta n$ edges, we obtain asymptotic lower bounds for the treewidth. If $\delta \geq 1.18$, then almost every (a.e.) graph with $m \geq \delta n$ edges has treewidth $\Theta(n)$. Our results also show the following. Let $\mathcal{G}$ be a class of graphs which is closed under taking minors. Let $\delta \geq 1.18$. Then almost every graph with n vertices and m edges, where $m \geq \delta n$, is *not* an element of $\mathcal{G}$. Part of our results, concerning clique minors of graphs, extend earlier results of Bollobás, Catlin and Erdös [29] and is related to results of Kostochka [85] and Thomason [127].

First we mention some earlier, related results. The most important method we use in this chapter is similar to that used in [128] in relation to the *bandwidth* of random graphs. In [128] de la Véga proves that almost all graphs on n vertices with cn edges have bandwidth $\geq b_c n$ where b_c is strictly positive for $c > 1$. We show that if $\delta \geq 1.18$, almost all graphs with n vertices and δn edges have treewidth $\geq b_\delta n$, where b_δ is strictly positive.

Bollobás et al. [29] obtain the following result. Let $0 < p < 1$ be fixed. For a.e. graph $G_{n,p}$ the maximum value s such that $G_{n,p}$ has a minor K_s is $(1 + o(1))\frac{n}{\sqrt{\log_d n}}$, where $d = \frac{1}{q}$. For a graph G, let $e(G)$ be the number of edges and let $|G|$ the number of vertices. For graphs G and H let $G > H$ denote the "G has H as a minor" relation. Let $c(s) = \inf\{c \mid e(G) \geq c|G| \Rightarrow G > K_s\}$. The result of [29] shows that $c(s) \geq 0.265s\sqrt{\log_2 s}$ for large values of s (see [127]). Subsequently, Kostochka [85] showed that $s\sqrt{\log s}$ is the correct order for $c(s)$. Thomason [127] shows the best upper bound as far as we know: $c(s) \leq 2.68s\sqrt{\log_2 s}\,(1 + o(1))$, for large s. For the treewidth problem this bound is of no interest; although a graph with K_s as a minor has treewidth at least $s - 1$, a graph with treewidth at most s, can have at most $ns - \frac{1}{2}s(s+1)$ edges. We extend the result of [29] as follows. Let $\delta > 1.18$. Then there exists a positive constant c_δ such that a.e. graph $G_{n,m}$

with $m \geq \delta n$ has a minor K_s with $s \geq \lfloor c_\delta^{2/3} n^{1/3} \rfloor$.

Finally we like to mention two more related results. Cohen et al. [36] give the exact asymptotic probability that a graph is an interval graph and that a graph is a circular arc graph (this paper also contains numerous applications of interval graphs). In the common random graph model, interval graphs play only a minor role. (If $m/n^{5/6} \to \infty$, where m is the number of edges and n is the number of vertices of a random graph, then the probability that the random graph is an interval graph goes to zero, if n tends to infinity.) For this reason, some work has been done to find separate models for random interval graphs, see e.g. [119]. This paper of Scheinerman also contains results on maximum degree, Hamiltonicity, chromatic number etc., for random interval graphs. In this chapter we only look at the common random graph model, i.e., all graphs with a certain number of vertices and a certain number of edges are equi-probable.

In [45], results are obtained concerning the size of a triangulated *subgraph*, given a graph with n vertices and m edges. For example, every graph with n vertices and $m = \frac{n^2}{4} + 1$ edges has a triangulated subgraph with $\frac{3n}{2} - 1$ edges.

We can summarize the results of this chapter as follows:

1. In section 5.1 we start with restating a result of Bollobás: almost all graphs with $< \frac{1}{2}n$ edges have treewidth ≤ 2.

2. In section 5.2 we show that for all $\delta > 1$ and for all $0 < \epsilon < (\delta - 1)/(\delta + 1)$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ has treewidth $\geq n^\epsilon$.

3. In section 5.3, we show that:

   (a) For all $0 < b < 1$ there exists a constant δ such that if $m \geq \delta n$, then a.e. graph $G_{n,m}$ has treewidth $\geq bn$.

   (b) For all $\delta \geq 1.18$, if $m \geq \delta n$, then a.e. graph $G_{n,m}$ has treewidth $\Theta(n)$.

4. In section 5.4 we prove the following related results:

   (a) Let $\delta \geq 1.18$. Then a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a clique minor K_s with $s = \Omega(n^{1/3})$.

   (b) Let $\mathcal{G}$ be a minor-closed class of graphs. For all $\delta \geq 1.18$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ is *not* in $\mathcal{G}$.

We do not know whether these results are optimal. Indeed, the smallest constant c such that almost every graph with cn edges has treewidth bn, for some $b > 0$, remains an open problem. Also, we do not have any results on random graphs of which the number of edges is in the range $(\frac{1}{2}n, n)$.

5.1 Preliminaries

For random graphs in general, the reader is referred to [28]. Let $P(Q)$ be the probability that a random graph with n vertices and m edges has a certain property Q. In most cases m is a function of n. We say that almost every (a.e.) graph has property Q if $P(Q) \to 1$ as $n \to \infty$. Throughout this chapter we use $N = \binom{n}{2}$, where n is the number of vertices of a graph. (N is the number of edges in the complete graph K_n with n vertices.) We use $G_{n,p} \in \mathcal{G}(n,p)$ to denote a random graph with edge probability p, and $G_{n,m} \in \mathcal{G}(n,m)$ for a random graph with n vertices and m edges. In [28] it is shown that if m is close to $pN = p\binom{n}{2}$, the two models $\mathcal{G}(n,m)$ and $\mathcal{G}(n,p)$ are practically interchangeable. In fact, since treewidth $\geq k$ is a *monotone increasing* property, it follows that, if $pqN \to \infty$ and x is some fixed constant, almost every graph in $\mathcal{G}(n,p)$ has treewidth $\geq k$ if and only if a.e. graph in $\mathcal{G}(n,m)$ has treewidth $\geq k$, where $m = \lfloor pN + x(pqN)^{\frac{1}{2}} \rfloor$, ([28], page 35).

As a first result on treewidth we can restate a result of Bollobás, ([28], page 99). Define a connected unicyclic graph with t vertices to be any connected graph with t vertices and t edges. Notice that a unicyclic graph has treewidth at most 2.

Lemma 5.1.1 *Suppose* $p = \frac{c}{n}$, $0 < c < 1$. *Then a.e.* $G_{n,p}$ *is such that every connected component is a tree or a unicyclic graph.*

Corollary 5.1.1 *If* $m < \frac{1}{2}n$, *then a.e. graph* $G_{n,m}$ *has treewidth at most two.*

5.2 A bound on the number of subgraphs of k-trees

In this section we show that almost all graphs with δn edges, have treewidth $\geq n^{\epsilon}$, for all fixed $\epsilon < \frac{\delta-1}{\delta+1}$. We do this by proving an upper bound for the number of subgraphs of k-trees.

Recall that the number of k-trees is given by the following formula, which was shown in Theorem 4.1.1 on page 40 and in different manners in a number of papers [9, 50, 99, 108].

$$T_k(n) = \binom{n}{k}(1 + k(n-k))^{n-k-2}$$

Lemma 5.2.1 *Let* $0 < \epsilon < 1$, *and let* $k \leq n^{\epsilon}$. *Then for* $n \to \infty$:

$$T_k(n) = o\left(n^{(1+\epsilon)(n-2)}\right)$$

Proof.

$$\begin{aligned} T_k(n) &= \binom{n}{k}(1+k(n-k))^{n-k-2} \\ &\leq n^k(nk)^{n-k-2} \\ &\leq n^{n-2}n^{\epsilon(n-k-2)} \\ &= n^{(1+\epsilon)(n-2)}n^{-\epsilon k} = o\left(n^{(1+\epsilon)(n-2)}\right) \end{aligned}$$

This proves the lemma. □

Lemma 5.2.2 *For any integer* k*, the number of partial* k*-trees with* m *edges is at most* $T_k(n)\binom{nk}{m}$.

Proof. The number of edges in a k-tree is $nk - \frac{1}{2}k(k+1)$ (see Lemma 2.1.10 on page 12 or e.g. [9]). It follows that the number of partial k-trees with m edges is at most

$$T_k(n)\binom{nk - \frac{1}{2}k(k+1)}{m} \leq T_k(n)\binom{nk}{m}$$

□

Theorem 5.2.1 *Let* $\delta > 1$ *and* $0 < \epsilon < \frac{\delta-1}{\delta+1}$*. Then for all* $m \geq \delta n$*, a.e.* $G_{n,m}$ *has treewidth at least* n^ϵ.

Proof. Let k be any integer with $k \leq n^\epsilon$. The total number of graphs with m edges is $\binom{N}{m}$, where $N = \binom{n}{2}$. Let F be the fraction of all graphs with m edges that have treewidth $\leq k$. We show that $F \to 0$. If $m \geq nk$, then clearly $F = 0$. Assume henceforth that $m < nk$. We have, if n is sufficiently large:

$$F \leq T_k(n)\frac{\binom{nk}{m}}{\binom{N}{m}} \leq T_k(n)\left(\frac{nk}{N-m}\right)^m \leq T_k(n)\left(3\frac{n^{1+\epsilon}}{n^2}\right)^{\delta n}$$

since $m < nk \leq n^{1+\epsilon}$ and $N - m \geq \frac{1}{3}n^2$ if n is large enough. We also used $m \geq \delta n$ and $k \leq n^\epsilon$. It follows that for large enough n (using Lemma 5.2.1):

$$F \leq T_k(n)\left(3n^{\epsilon-1}\right)^{\delta n} \leq 3^{\delta n}n^{n(\epsilon(\delta+1)-(\delta-1))} \to 0$$

since $\epsilon(\delta+1) - (\delta-1) < 0$. □

In the next section we show that, if δ is somewhat larger, almost every graph with n vertices and at least δn edges has a treewidth which is linear in n.

5.3 The separator method

In this section we show the following. Let $\delta \geq 1.18$. There exists a *positive* number b_δ, such that for $m \geq \delta n$, a.e. graph $G_{n,m}$ does not have a balanced separator of size $\leq b_\delta n$. It is well-known that a partial k-tree has a balanced separator with at most $k+1$ vertices. We show that a random graph with at least δn edges does not have such a separator for small k.

In this section a balanced separator is a set C with $k+1$ vertices such that every connected component has at most $\frac{1}{2}(n-k)$ vertices. Recall Lemma 2.2.9 on page 21 which shows there exist such separators in partial k-trees:

> If G is a k-tree then there is a clique C in G with $k+1$ vertices such that every connected component of $G[V \setminus C]$ has at most $\frac{1}{2}(n-k)$ vertices.

Corollary 5.3.1 *Let $G = (V, E)$ be a graph with at least $k+1$ vertices and with treewidth $\leq k$. Then there exists a set S of $k+1$ vertices such that every component of $G[V-S]$ has at most $\frac{1}{2}(n-k)$ vertices.*

Let $G = (V, E)$ be a graph. To ease the later computations somewhat, we partition the vertices of G in three sets.

Definition 5.3.1 *Let $G = (V, E)$ be a graph with n vertices. A partition (S, A, B) of the vertices is a* balanced k-partition *if the following three conditions are satisfied:*

1. $|S| = k+1$
2. $\frac{1}{3}(n-k-1) \leq |A|, |B| \leq \frac{2}{3}(n-k-1)$
3. *S separates A and B, i.e., there are no edges between vertices of A and vertices of B.*

Lemma 5.3.1 *Let $G = (V, E)$ be a partial k-tree with n vertices such that $n \geq k+4$. Then G has a balanced k-partition.*

Proof. Let S be a balanced separator in G, which exists by Corollary 5.3.1. Let $C_1, \ldots, C_t$ be the connected components of $G[V-S]$. Hence, $|C_i| \leq \frac{1}{2}(n-k)$. We consider two cases:
Case 1: There exists a component C_i such that $|C_i| \geq \frac{1}{3}(n-k-1)$.
In this case let $A = C_i$ and B the union of the other components. Then clearly, since $n-k \geq 4$, and $|C_i| \leq \frac{1}{2}(n-k)$ it follows that $|A| \leq \frac{2}{3}(n-k-1)$.
Case 2: All components have less than $\frac{1}{3}(n-k-1)$ vertices.
In this case, choose s such that:

$$|C_1| + \ldots + |C_s| \leq \frac{2}{3}(n-k-1) < |C_1| + \ldots + |C_{s+1}|$$

It follows that:

$$|C_{s+2}| + \ldots + |C_t| < \frac{1}{3}(n-k-1)$$

Let $A = C_{s+1} \cup \ldots \cup C_t$ and $B = C_1 \cup \ldots \cup C_s$. Then:

$$|A| < |C_{s+1}| + \frac{1}{3}(n-k-1) \leq \frac{2}{3}(n-k-1)$$

Clearly, also:

$$|A| = n-k-1-|B| \geq \frac{1}{3}(n-k-1)$$

□

We show that the fraction of all graphs that have a balanced k-partition is negligible if the number of edges is not too small.

Lemma 5.3.2 *Let $L_k(n,m)$ be the number of graphs with m edges that have a balanced k-partition. Then:*

$$L_k(n,m) \leq \frac{1}{2} \sum_{\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)} \binom{n}{k+1} \binom{n-k-1}{a} \binom{N-a(n-k-1-a)}{m}$$

where $N = \binom{n}{2}$.

Proof. First choose a separator S with $k+1$ vertices, and a set A with a vertices, where $\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)$. Finally, choose m edges. Since no edges between A and B are allowed, these m edges must be chosen from a set of $N - a(n-k-1-a)$ available edges. Since A and B are interchangeable, we can divide by 2 to find an upper bound for the number of graphs which have a balanced k-partition. □

Lemma 5.3.3

$$L_k(n,m) \leq \binom{n}{k+1} \cdot 2^{n-k-2} \cdot \binom{N - \frac{2}{9}(n-k-1)^2}{m}$$

Proof. Consider the bound on $L_k(n,m)$ in Lemma 5.3.2. Notice that:

$$\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1) \Rightarrow N - a(n-k-1-a) \leq N - \frac{2}{9}(n-k-1)^2$$

and hence

$$\binom{N-a(n-k-1-a)}{m} \leq \binom{N-\frac{2}{9}(n-k-1)^2}{m}$$

Also notice that

$$\sum_{\frac{1}{3}(n-k-1) \leq a \leq \frac{2}{3}(n-k-1)} \binom{n-k-1}{a} \leq 2^{n-k-1}$$

Using these upperbounds, the lemma follows. □

Definition 5.3.2 *Let* $F_k(n,m)$ *be the fraction of all graphs with* n *vertices and* m *edges that have a balanced* k*-partition, i.e.,*

$$F_k(n,m) = \frac{L_k(n,m)}{\binom{N}{m}}$$

where $N = \binom{n}{2}$.

Lemma 5.3.4

$$F_k(n,m) \leq 2^{n-k-2} \cdot \binom{n}{k+1} \cdot \left(1 - \frac{\frac{4}{9}(n-k-1)^2}{n^2}\right)^m$$

Proof. Let $t = \frac{2}{9}(n-k-1)^2$. Then

$$\begin{aligned}
\frac{\binom{N-t}{m}}{\binom{N}{m}} &= \frac{N-t}{N} \cdot \frac{N-t-1}{N-1} \cdot \ldots \cdot \frac{N-t-m+1}{N-m+1} \\
&= \left(1 - \frac{t}{N}\right)\left(1 - \frac{t}{N-1}\right) \ldots \left(1 - \frac{t}{N-m+1}\right) \\
&\leq \left(1 - \frac{t}{N}\right)^m \leq \left(1 - \frac{2t}{n^2}\right)^m
\end{aligned}$$

Using this the lemma follows from Lemma 5.3.3. □

Definition 5.3.3 *For* $0 < b < 1$ *and* $\delta > 0$, *let:*

$$\varphi(b,\delta) = 2^{1-b} \cdot \frac{\left(1 - \frac{4}{9}(1-b)^2\right)^\delta}{b^b(1-b)^{1-b}}$$

Theorem 5.3.1 *Let* $0 < b < 1$ *and* δ *be fixed. Let* $m \geq \delta n$ *and let* $k+1 = \lceil bn \rceil$. *Then*

$$F_k(n,m) = o\left(\varphi(b,\delta)^n\right)$$

Proof. Since $k+1 \to \infty$ and $n-k-1 \to \infty$, we find with the aid of Stirling's formula:

$$\binom{n}{k+1} \sim \left(\frac{n}{k+1}\right)^{k+1} \cdot \left(\frac{n}{n-k-1}\right)^{n-k-1} \cdot \frac{1}{\sqrt{2\pi(k+1)(1-\frac{k+1}{n})}}$$

Since $bn \leq k+1 \leq bn+1$, it follows that $\left(\frac{n}{k+1}\right)^{k+1} \leq \left(\frac{1}{b}\right)^{bn+1}$. And also

$$\begin{aligned}
\left(\frac{n}{n-k-1}\right)^{n-k-1} &\leq \left(\frac{n}{n-bn-1}\right)^{n-bn} \\
&= \left(\frac{1}{1-b}\right)^{(1-b)n} \cdot \frac{1}{\left(1 - \frac{1}{n(1-b)}\right)^{n(1-b)}} \sim e \cdot \left(\frac{1}{1-b}\right)^{(1-b)n}
\end{aligned}$$

Since $m \geq \delta n$ we have

$$\begin{aligned}
\left(1 - \frac{\frac{4}{9}(n-k-1)^2}{n^2}\right)^m &\leq \left(1 - \frac{\frac{4}{9}(n-bn-1)^2}{n^2}\right)^{\delta n} \\
&\leq \left(1 - \frac{4}{9}(1-b)^2 + \frac{8}{9n}(1-b)\right)^{\delta n} \\
&\leq \left(1 - \frac{4}{9}(1-b)^2\right)^{\delta n} \cdot \left(1 + \frac{8(1-b)}{9n(1-\frac{4}{9}(1-b)^2)}\right)^{\delta n} \\
&\sim \left(1 - \frac{4}{9}(1-b)^2\right)^{\delta n} \cdot \exp\left(\frac{8\delta(1-b)}{9-4(1-b)^2}\right)
\end{aligned}$$

The result now follows from the fact that $\sqrt{2\pi(k+1)(1-\frac{k+1}{n})} \to \infty$ and Lemma 5.3.4. □

Theorem 5.3.2

1. *For all* $0 < b < 1$ *there exists a* δ_b *such that if* $m \geq \delta_b n$ *a.e. graph* $G_{n,m}$ *has treewidth* $\geq bn$.

2. *Let* $\delta \geq 1.18$. *There exists a positive constant* b_δ *such that if* $m \geq \delta n$ *and* $k+1 = \lceil b_\delta n \rceil$, *then* $F_k(n,m) \to 0$.

3. *Let* $\delta \geq 1.18$. *Then a.e. graph* $G_{n,m}$ *with* $m \geq \delta n$ *has treewidth* $\Theta(n)$.

Proof. Notice that $\lim_{\delta\to\infty} \varphi(b,\delta) = 0$. The first statement now follows from Theorem 5.3.1.

To prove the second statement, notice that

$$\lim_{b \downarrow 0} \left(\frac{1}{b}\right)^b \cdot \left(\frac{2}{1-b}\right)^{1-b} = 2$$

Hence $\varphi(0,\delta) = 2(\frac{5}{9})^\delta < 1$ if $\delta \geq 1.18$. Hence, from the fact that φ is a continuous function for b in $(0,1)$ it follows that for $\delta \geq 1.18$, there exists a *positive number* b_δ such that $\varphi(b_\delta,\delta) < 1$. The second statement now follows immediately Theorem 5.3.1.

The third statement is an immediate consequence of the second one. □

5.4 Further results

In this section, we show some results in the theory of graph minors. To be more specific, we show that, for any minor-closed class of graphs, a.e. graph with at least a certain amount of edges is *not* in the class. Apart from Theorem 5.3.2, the following theorem is the main ingredient. In [1] Alon et al. proved the following theorem.

Theorem 5.4.1 *Let $h \geq 1$ be an integer and let $G = (V, E)$ be a graph with n vertices and no K_h minor. There exists a subset $X \subseteq V$, with $|X| \leq h^{3/2}n^{1/2}$ such that every connected component of $G[V - X]$ has at most $\frac{1}{2}n$ vertices.*

Together with our results this proves the following theorem.

Theorem 5.4.2 *Let $\delta \geq 1.18$. There exists a positive constant c_δ, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ edges has a clique K_h with $h \geq \lfloor c_\delta n^{1/3} \rfloor$ as a minor.*

Proof. By Theorem 5.3.2, there is a positive number b_δ such that a.e. graph $G_{n,m}$ with $m \geq \delta n$, does not have a balanced k-partition for $k \leq b_\delta n$. Let $c_\delta = (\frac{1}{3}b_\delta)^{2/3}$ and let $h = \lfloor c_\delta n^{1/3} \rfloor$. By Theorem 5.4.1, if a graph does not have K_h as a minor, then it has a separator X with $|X| \leq c_\delta^{3/2} n \leq \frac{1}{3}b_\delta n$, such that every component has at most $\frac{1}{2}n$ elements. By a similar argument as in Lemma 5.3.1 there is a partition of the vertices (X, A, B) such that $|A|, |B| \leq \frac{2}{3}n$. Assume this is not a balanced k-partition. Then without loss of generality we may assume that $|A| > \frac{2}{3}(n - |X|)$. Let $t = 3|A| - 2(n - |X|)$. Take t vertices out of A and move them into X. Call the new sets X' and A'. Clearly, X' separates A' and B, and $|A'| = \frac{2}{3}(n - |X'|)$. Since $|A| \leq \frac{2}{3}n$:

$$|X'| = |X| + t = |X| + 3|A| - 2(n - |X|) \leq 3|X|$$

It follows that (X', A', B) is a balanced k-partition with $k + 1 = |X'| \leq b_\delta n$. □

Let $\mathcal{G}$ be a minor-closed class of graphs (e.g. the class of planar graphs). Robertson and Seymour [110] proved Wagner's conjecture; there is a *finite* set of forbidden minors. From this result the following lemma easily follows.

Lemma 5.4.1 *Let $\mathcal{G}$ be a minor-closed class of graphs. Then there exists an integer h such that K_h is not a minor of any graph $G \in \mathcal{G}$.*

Proof. Take a finite set S of forbidden minors. Let h be the minimum number of vertices of any element of S. □

Theorem 5.4.3 *Let $\mathcal{G}$ be a minor-closed class of graphs. For all $\delta \geq 1.18$, a.e. graph $G_{n,m}$ with $m \geq \delta n$ is* not *in $\mathcal{G}$.*

Proof. Take h such that no graph in $\mathcal{G}$ has K_h as a minor. From Theorem 5.4.2, there exists a positive number c_δ, such that a.e. graph $G_{n,m}$ with $m \geq \delta n$ has a K_t minor with $t \geq \lfloor c_\delta n^{1/3} \rfloor$. It immediately follows that a.e. graph $G_{n,m}$ is not in $\mathcal{G}$. □

Chapter 6

Testing superperfection of k-trees

Much work has been done in recognizing classes of perfect graphs in polynomial time [123, 30, 58, 104, 55]. An exception seems to be the class of superperfect graphs.

The results presented in this chapter can be summarized as follows. First we give in section 6.2 a complete characterization, by means of forbidden induced subgraphs, of 2-trees that are superperfect. We also characterize those 2-trees that are comparability graphs and those that are permutation graphs. Secondly, in section 6.3 we give *for each constant* k an O(1) time algorithm which produces a complete characterization of superperfect k-trees, by means of forbidden configurations. With the aid of this characterization we find, for each constant k, a linear-time algorithm to test superperfection of k-trees.

Until now we have not been able to find a polynomial algorithm to test superperfection on partial k-trees (for general k). Since by definition a graph is superperfect if for each assignment of non-negative weights to the vertices the interval chromatic number is equal to the maximum weight clique, the following observation is of interest. Determining the interval chromatic number of a weighted interval graph with weights one and two is NP-hard. When restricted to weighted partial k-trees, for some constant k, it can be seen that the interval chromatic number can be determined in linear time.

The class of superperfect graphs contains the class of comparability graphs (hence also the class of bipartite graphs), but these classes are not equal. This has been pointed out by Golumbic [58] who showed the existence of an infinite class $\mathcal{S}$ of superperfect graphs which are not comparability graphs. However all graphs in $\mathcal{S}$ are neither triangulated nor co-triangulated, and therefore Golumbic [58] raises the question whether for triangulated graphs the classes of superperfect and comparability graphs coincide. For split graphs this equivalence has been shown. Our results show it is not the case in general. We show the existence of triangulated graphs which are superperfect but are comparability graphs.

6.1 Preliminaries

We start with some definitions and easy lemmas. Most definitions and results in this section are taken from [58]. For further information on superperfect graphs the reader is referred to this book.

Definition 6.1.1 *An undirected graph* $G = (V, E)$ *is called a comparability graph, or a transitively orientable graph, if there exists an orientation of the edges such that the resulting oriented graph* (V, F) *satisfies the following conditions.*

$$F \cap F^{-1} = \emptyset \text{ and } F + F^{-1} = E \text{ and } F^2 \subseteq F$$

where $F^2 = \{(a,c) \mid \exists_{b \in V} (a,b) \in F \wedge (b,c) \in F\}$. *An orientation* F *of the edges satisfying the conditions above is called a* transitive *orientation.*

So, if F is a transitive orientation, then $(a,b) \in F$ and $(b,c) \in F$ imply $\{a,c\}$ is an edge with orientation $(a,c) \in F$. There exists a somewhat weaker equivalent condition (which we do not use): a graph is a comparability graph if and only if it admits an orientation of its edges that represents a pseudo-order relation (see [14], page 76).

If a graph G is a comparability graph, then this also holds for every induced subgraph of G. In [58] it is shown that comparability graphs are perfect, and can be recognized in polynomial time (see also [123]).

A *weighted* graph is a pair (G, w), where G is a graph and w a weight function which associates to every vertex x a non-negative weight $w(x)$. For a subset S of the vertices we define the weight of S, denoted by $w(S)$, as the sum of the weights of the vertices in S.

Definition 6.1.2 *An* interval coloring *of a weighted graph* (G, w) *maps each vertex* x *to an open interval* I_x *on the real line, of width* $w(x)$, *such that adjacent vertices are mapped to disjoint intervals. The* total width *of an interval coloring is defined to be* $|\bigcup_x I_x|$. *The* interval chromatic number $\chi(G, w)$ *is the least total width needed to color the vertices with intervals.*

Determining whether $\chi(G, w) \leq r$ is an NP-complete problem, even if w is restricted to values one and two and G is an interval graph. (This has been shown by L. Stockmeyer as reported in [58].) In this paper we shall only use the following alternative characterization of the interval chromatic number (see [58]).

Theorem 6.1.1 *If* (G, w) *is a weighted undirected graph, then*

$$\chi(G, w) = \min_F \left(\max_\mu w(\mu) \right)$$

where F *is an acyclic orientation of* G *and* μ *is a path in* F.

If w is a weight function and F is an acyclic orientation, then we say that F is a *superperfect orientation* with respect to w if the weight of the heaviest path in F does not exceed the weight of the heaviest clique.

Definition 6.1.3 *The* clique number $\Omega(G, w)$ *of a weighted graph* (G, w) *is defined as the maximum weight of a clique in* G.

In this chapter we use the capital Ω to denote the (weighted) clique number rather then ω, to avoid confusion with the weight function w. It is easy to see that $\Omega(G, w) \leq \chi(G, w)$ holds for all weighted graphs, since for any acyclic orientation and for every clique there exists a path in the orientation which contains all vertices of the clique.

Definition 6.1.4 *A graph* G *is called* superperfect *if for* every non-negative weight function w, $\Omega(G, w) = \chi(G, w)$.

Notice that each induced subgraph of a superperfect graph is itself superperfect, and also that every superperfect graph is perfect. If G is a comparability graph, then there exists an orientation such that every path is contained in a clique. This proves the following theorem (see also [58]).

Theorem 6.1.2 *Every comparability graph is superperfect.*

The converse of this theorem is not true. In [58] an infinite class of superperfect graphs is given that are not comparability graphs. However, none of these graphs is triangulated. In [58] (page 214) the question is raised whether the converse of the theorem holds for triangulated graphs; is it true or false that, for *triangulated* graphs, G is a comparability graphs if and only if G is superperfect? In the next section we answer this question in the negative, and we give a complete characterization of superperfect 2-trees.

6.2 2-trees and superperfection

In this section we give a characterization of the 2-trees that are superperfect by means of forbidden subgraphs. In 1967 Gallai [54] published a list of all minimal forbidden subgraphs of the comparability graphs (see also [14] page 78). Extracting from this list the triangulated graphs which are subgraphs of 2-trees (or: have treewidth at most two), we find a characterization of the 2-trees which are comparability graphs. We find two types of forbidden induced subgraphs, which we call the *3-sun* and the *odd wing*. They are illustrated in figure 6.1. Notice that a 3-sun and a wing are 2-trees, and that a wing has at least seven vertices. We call a wing odd (even) if the total number of vertices is odd (even). The following lemma is easy to check.

Lemma 6.2.1 *A wing is a comparability graph if and only if it is even.*

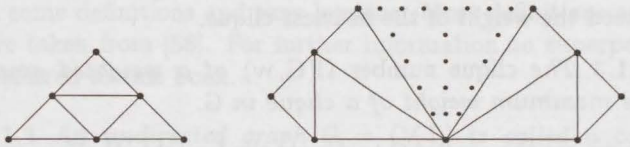

Figure 6.1: 3-sun (left) and wing (right)

We thus find the following characterization of 2-trees that are comparability graphs.

Theorem 6.2.1 *A 2-tree is a comparability graph if and only if it does not contain a 3-sun or an odd wing.*

It is interesting to notice that we can get a characterization of the 2-trees that are permutation graphs. Recall that a graph is an interval graph if and only if it is triangulated and a cocomparability graph [55]. Also, a graph is a permutation graph if and only if the graph and its complement are comparability graphs [104]. It follows that a 2-tree is a permutation graph if and only if it is an interval graph without an induced odd wing.

The next theorem shows that the smallest odd wing, with seven vertices, (which is not a comparability graph) is superperfect. As we shall see later, this is in fact the only odd wing that is superperfect. Notice that in [58] (page 212, figure 9.9) this graph is mistakenly placed in the position of a non-superperfect graph. See also [97] and [49]; the result of [97] is wrong: A wing is an interval graph.

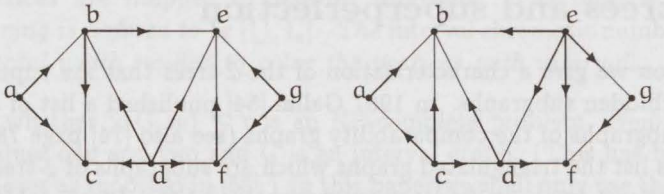

Figure 6.2: two orientations of the wing with seven vertices

Theorem 6.2.2 *The odd wing with seven vertices is superperfect and hence there exists a triangulated graph which is superperfect but not a comparability graph.*

Proof. Label the vertices of the graph as in figure 6.2. We consider two orientations of this wing as illustrated in figure 6.2 and we show that for every weight function one of these orientations is superperfect. Notice that both orientations are such that there is exactly one path not contained in a triangle. In the first orientation this is the path $\{a, b, e\}$ and in the second orientation the path $\{c, d, f\}$. Consider a non-negative weight function w of the vertices. Suppose the orientation of the first type is *not* superperfect with respect to w. Then the path $\{a, b, e\}$ must be heavier then every triangle. Since $\{a, b, c\}$ is a triangle, this implies that $w(e) > w(c)$. But then $w(\{c, d, f\}) < w(\{e, d, f\})$, and since $\{e, d, f\}$ is a triangle, the second orientation is superperfect with respect to w. □

In the last part of this section we give a complete characterization of the superperfect 2-trees. In figure 6.3 we give a list of forbidden induced subgraphs. Notice that the fourth subgraph starts an infinite series. The following lemma can be easily checked.

Lemma 6.2.2 *The graphs shown in figure 6.3 are not superperfect. For each of the graphs the weight function that is shown is such that for any acyclic orientation, there exists a path which is heavier than the heaviest clique.*

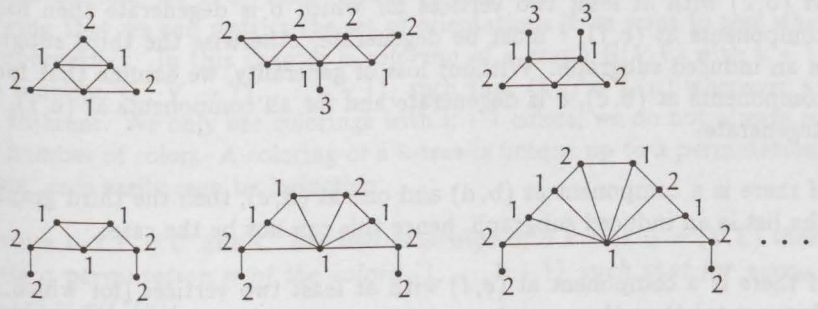

Figure 6.3: critical non-superperfect graphs

Theorem 6.2.3 *A 2-tree is superperfect if and only if it does not contain an induced subgraph isomorphic to one of the graphs shown in figure 6.3.*

Proof. Assume the 2-tree G has no induced subgraph from this list. Then the graph cannot have an induced odd wing with nine or more vertices. We may assume the graph is not a comparability graph, hence it contains an odd wing with seven vertices. Consider the labeled wing of figure 6.2. Let H be the subgraph obtained from this wing by removing the vertices a and g. Let (x, y) be an edge of

H. We say that C is a component at (x, y) if C is a maximal connected subgraph of $G[V \setminus \{x, y\}]$ *containing no vertices of* H. If C is a component at (x, y), we say that x is a degenerate vertex of this component if x is adjacent to all vertices of C. Notice that there are only four edges of H at which there can be components, namely (b, c), (b, d), (d, e) and (e, f), otherwise the 3-sun would be an induced subgraph. The following remarks restrict the possible components.

1. It can be checked that if C is a component at (x, y), then either x or y must be degenerate.

2. For components at (b, d), b must be degenerate and for components at (d, e), e must be degenerate, otherwise the second graph in the list is an induced subgraph.

3. Consider components at (b, c) with at least two vertices. Either b or c is degenerate for *all* these components, otherwise the second graph from list 6.3 is an induced subgraph.

4. If there is a component at (b, c) with at least two vertices for which c is degenerate, then for all components at (e, f), e must be degenerate, otherwise the fourth graph in the list is an induced subgraph. If there is a component at (b, c) with at least two vertices for which b is degenerate then for all components at (e, f), f must be degenerate, otherwise the third subgraph is an induced subgraph. Without loss of generality, we assume that for all components at (b, c), c is degenerate and for all components at (e, f), e is degenerate.

5. If there is a component at (b, d) and one at (d, e), then the third graph in the list is an induced subgraph, hence this can not be the case.

6. If there is a component at (e, f) with at least two vertices (for which e is degenerate) then there can be no component at (b, d), otherwise the third graph from the list is an induced subgraph.

7. If there is a component at (b, c) with at least two vertices, then all components at (b, d) can have only one vertex, otherwise the second graph of the list is an induced subgraph.

8. Suppose there is a component at (b, d). Then all components at (b, c) can have at most two vertices. Furthermore, if there is a component at (b, c) with two vertices, then it is the only component at (b, c). Otherwise, the third subgraph of the list is an induced subgraph.

It follows that only one of two cases can occur.

- There is a component at (b, d). Then there is no component at (d, e). All components at (b, d) and at (e, f) have one vertex. Either all components at (b, c) have one vertex or there is only one component at (b, c), in which case it can have at most two vertices and c is degenerate.

- There are only components at (b, c), (d, e) and (e, f). For all components at (b, c), c is degenerate. For all components at (d, e) and at (e, f), e is degenerate.

It is easily checked (e.g. by methods described in the next section) that both types are superperfect. □

Notice that the list of forbidden induced subgraphs is infinite. In the next section we show that for each k there is a finite characterization of the class of superperfect k-trees, by means of forbidden configurations. Furthermore we give for each k a $O(1)$ time algorithm to find this characterization. As a consequence we find a linear time algorithm to check whether a k-tree is superperfect.

6.3 k-trees and superperfection

In this section, let k be some constant, and let G be a k-tree. We start by showing that we can restrict the set of orientations if we want to test whether G is superperfect. In this chapter a *coloring* of a graph $G(V, E)$ with $k+1$ colors is a function $C : V \to \{1, \ldots, k+1\}$, such that $C(x) \neq C(y)$ whenever x and y are adjacent. We only use colorings with $k+1$ colors; we do not always mention the number of colors. A coloring of a k-tree is unique up to a permutation of the colors, as is easily seen by induction.

Lemma 6.3.1 *If C and C' are two colorings of a k-tree $G = (V, E)$ then there exists a permutation π of the colors $\{1, \ldots, k+1\}$ such that for every vertex x, $C(x) = \pi(C'(x))$.*

By this fact, the following set of orientations is uniquely defined for each k-tree.

Definition 6.3.1 *Let G be a graph and let C be a coloring of G with $k+1$ colors. For each permutation π of the colors we define an orientation F_π as follows. Direct the edge (x, y) from x to y if $\pi(C(x)) < \pi(C(y))$. Let $\mathcal{F}_c(G)$ be the set of orientations obtained in this way.*

Lemma 6.3.2 *If G is a k-tree then:*

1. $|\mathcal{F}_c| = (k+1)!$.
2. *Each $F \in \mathcal{F}_c$ is acyclic.*

3. If $F \in \mathcal{F}_c$ then each path μ in F has at most $k+1$ vertices.

Definition 6.3.2 *Let G be a k-tree. Let $\mathcal{F}^*(G)$ be the set of those acyclic orientations of G in which every path contains at most $k+1$ vertices.*

Notice that $\mathcal{F}_c \subseteq \mathcal{F}^*$.

Lemma 6.3.3 $\mathcal{F}_c = \mathcal{F}^*$.

Proof. We prove that $|\mathcal{F}^*| = (k+1)!$ which implies the result. Let $F \in \mathcal{F}^*$. Let S be a k-clique in G, and let x and y be two vertices which are adjacent to all vertices of S. Since S is a clique and F is acyclic, there is a unique ordering of the vertices of S, say $s_1, s_2, \ldots, s_k$, such that there is an arc from s_i to s_j ($s_i \to s_j$) if and only if $i > j$. Since x is adjacent to all vertices of S, there exists an index $0 \le t_x \le k$ such that $x \to s_i$ for all $1 \le i \le t_x$ and $s_j \to x$ for all $t_x < j \le k$. The same holds for y with index t_y. Consider the case $t_x < t_y$. Then F has a path of length $k+2$:

$$(s_k, s_{k-1}, \ldots, s_{t_y+1}, y, s_{t_y}, \ldots, s_{t_x+1}, x, s_{t_x}, \ldots, s_1)$$

Since $F \in \mathcal{F}^*$, we find that $t_x = t_y$. Now consider the inductive construction of G as a k-tree. Start with an acyclic orientation of a $(k+1)$-clique. This can be done in $(k+1)!$ manners. If we add a new vertex v and make it adjacent to a k-clique, by the argument above, the orientations of the edges incident with v are determined. Hence $|\mathcal{F}^*| = (k+1)!$. □

Definition 6.3.3 *Let F be an acyclic orientation. A path μ in F is* contained *in a path μ', if all vertices of μ are also vertices of μ'.*

Lemma 6.3.4 *Let $F \in \mathcal{F}_c$. Then any path μ in F is contained in a path μ' with $k+1$ vertices.*

Proof. Let C be a coloring and let $F = F_\pi$ for some permutation π. The colors in the path μ must appear in the same order as in the permutation. Assume there is a gap between adjacent colors c_1 and c_2 in the path (i.e., there is a color in the permutation between c_1 and c_2). Since the edge of the path with colors c_1 and c_2 is contained in a $(k+1)$-clique, the missing colors can be put between c_1 and c_2. Thus we can make a longer path μ' containing μ. □

Theorem 6.3.1 *Let G be a k-tree. Then G is superperfect if and only if for all weight functions w*

$$\min_{F \in \mathcal{F}_c} \max_{\mu} w(\mu) = \Omega(G, w)$$

Proof. Assume G is superperfect. Let w be a non-negative weight function. There is an orientation F such that $\max_\mu w(\mu) = \Omega(G, w)$. If every path in F has at most $k+1$ vertices, we are done. Assume F has a path with more than $k+1$ vertices. Now increase all weights with some constant $L > \Omega(G, w)$. Let w' be this new weight function. Notice that $\Omega(G, w') = \Omega(G, w) + (k+1)L$. Since G is superperfect, there must be an orientation F' for this new weight function w'. Suppose F' also has a path μ with more than $k+1$ vertices. Then (with $|\mu|$ the number of vertices of μ):

$$\begin{aligned} \Omega(G, w') = \Omega(G, w) + (k+1)L &\geq w'(\mu) \\ &= w(\mu) + |\mu|L \\ &\geq w(\mu) + (k+2)L \\ &\geq (k+2)L \end{aligned}$$

Since $L > \Omega(G, w)$, this is a contradiction. We may conclude that $F' \in \mathcal{F}^* = \mathcal{F}_c$. We show that F' is also a good orientation for the weight function w. Let ν be a path in F'. By Lemma 6.3.4, ν is contained in a path ν^* with $k+1$ vertices. Hence

$$w(\nu) \leq w(\nu^*) = w'(\nu^*) - (k+1)L \leq \Omega(G, w') - (k+1)L = \Omega(G, w)$$

The converse is trivial. □

Definition 6.3.4 *Let* G *be a triangulated graph and let* C *be a coloring of* G *with* $k+1$ *colors. For each permutation* π *of the colors, let* $\mathcal{P}(\pi)$ *be the set of paths in* F_π, *which are* not *contained in a clique and which have* $k+1$ *vertices. If* $\mathcal{Q}$ *is a set of paths in* G, *we say that* $\mathcal{Q}$ *is a* cover *if, for every permutation* π, *there is a path* $\mu \in \mathcal{Q}$ *which can be oriented such that it is in* $\mathcal{P}(\pi)$. *A cover is called* minimal *if it contains* $\frac{1}{2}(k+1)!$ *paths.*

Lemma 6.3.5 *Let* G *be a* k*-tree. If for some permutation* π, $\mathcal{P}(\pi) = \emptyset$, *then* G *is a comparability graph (hence superperfect).*

Proof. Suppose $\mathcal{P}(\pi) = \emptyset$. Consider the orientation F_π. If there is a path in F_π which is not contained in a clique then, by Lemma 6.3.4, $\mathcal{P}(\pi)$ cannot be empty. Hence, every path in F_π is contained in a clique. Since F_π is acyclic, the lemma follows. □

Definition 6.3.5 *Let* G *be a* k*-tree and let* C *be a coloring of* G. *Let* S *be a maximal clique of* G, *and let* $\mathcal{Q}$ *be a minimal cover. Define* $LP(G, S, \mathcal{Q})$ *as the following set of inequalities:*

1. *For each vertex* x: $w(x) \geq 0$.

2. *For each maximal clique* $S' \neq S$: $w(S') \leq w(S)$.

3. *For each path* μ *of* $\mathcal{Q}$: $w(\mu) > w(S)$.

We call the second type of inequalities, the clique inequalities. *The inequalities of the third type are called the* path inequalities.

Lemma 6.3.6 *There are* $n - k - 1$ *clique inequalities and* $\frac{1}{2}(k+1)!$ *path inequalities.*

Proof. Notice that a k-tree has $n - k$ cliques with $k + 1$ vertices. □

Theorem 6.3.2 *Let* G *be a* k*-tree with a coloring* C. G *is* not *superperfect if and only if there is a maximal clique* S *and a minimal cover* $\mathcal{Q}$ *such that* $LP(G, S, \mathcal{Q})$ *has a solution.*

Proof. Suppose $LP(G, S, \mathcal{Q})$ has a solution. Take this solution as a weight function. Then, clearly, for any orientation F_π there is a path (in $\mathcal{Q}$ and in $\mathcal{P}(\pi)$) which is heavier than the heaviest clique S. By Theorem 6.3.1 G is not superperfect. On the other hand, if G is not superperfect, there exists a weight function w such that for every orientation F_π there is a path which is heavier than the heaviest clique (hence it can not be contained in a clique). By Lemma 6.3.4 we may assume this path has $k + 1$ vertices, hence it is in $\mathcal{P}(\pi)$. Take S to be the heaviest clique and let $\mathcal{Q}$ be a minimal cover for these paths. □

Notice that we could use Theorem 6.3.2 for a polynomial time algorithm to test superperfection on k-trees. There are at most n^{k+1} different paths of length k, hence the number of minimal covers is at most $(n^{k+1})^{\frac{1}{2}(k+1)!}$. Since the number of maximal cliques of G is at most $n - k$, we only have to check for a polynomial number of sets of inequalities if it has a solution. This checking can be done in polynomial time, e.g. by the ellipsoid method. We now show that there also exists a linear time algorithm.

Consider a set of inequalities $LP(G, S, \mathcal{Q})$ which has a solution. Notice that if some vertex y does *not* appear in the path inequalities then we can set the weight $w(y) = 0$. This new weight function is also a solution. Hence we can transform the set of inequalities as follows:

Definition 6.3.6 *Let* G *be a* k*-tree and let* C *be a coloring of* G. *Let* S *be a maximal clique, and let* $\mathcal{Q}$ *be a cover. Let* H *be the subgraph of* G *induced by the vertices of* S *and of all paths in* $\mathcal{Q}$. *Define* $LP'(H, S, \mathcal{Q})$ *as the following set of inequalities:*

1. *For each vertex* x *of* H: $w(x) \geq 0$.

2. *For each maximal clique* $S' \neq S$ *of* H: $w(S') \leq w(S)$.

3. *For each path* μ *in* $\mathcal{Q}$: $w(\mu) > w(S)$.

The following lemma follows directly from Definitions 6.3.5 and 6.3.6.

Lemma 6.3.7 $LP(G, S, \mathcal{Q})$ *has a solution if and only if* $LP'(H, S, \mathcal{Q})$ *has a solution.*

Lemma 6.3.8 *The number of inequalities of* $LP'(H, S, \mathcal{Q})$ *is bounded by a constant.*

Proof. There are $\frac{1}{2}(k+1)!$ paths in $\mathcal{Q}$, each involving $k+1$ variables. The clique S has also $k+1$ vertices, hence it follows that the subgraph H, has at most $(\frac{1}{2}(k+1)!+1)(k+1)$ vertices. Since H is triangulated, the number of maximal cliques in H is bounded by the number of vertices. Hence the number of clique inequalities is bounded by $(\frac{1}{2}(k+1)!+1)(k+1)$. □

We now have the following algorithm to test superperfection of k-trees.

Algorithm to test superperfection of G

Step 1 Generate a list of all $(k+1)$-colored triangulated graphs H, with at most $(1+\frac{1}{2}(k+1)!)(k+1)$ vertices, for which there exists:

1. A maximal clique S with $k+1$ vertices
2. A set $\mathcal{Q}$ of $\frac{1}{2}(k+1)!$ paths, which is a minimal cover,

such that $LP'(H, S, \mathcal{Q})$ has a solution.

Step 2 Make a coloring of G (with $k+1$ colors).

Step 3 Check whether a graph H from the list is an induced subgraph of G (preserving colors). If G has a subgraph from the list then G is not superperfect, otherwise it is.

Notice that generating the list takes $O(1)$ time (if k is a constant). Since the subgraphs have constant size, we can check whether such a subgraph is an induced subgraph of G in linear time, using standard techniques for (partial) k-trees (see [5]).

Theorem 6.3.3 *The algorithm correctly determines whether* G *is superperfect, and does so in linear time.*

Proof. Assume G is not superperfect. Let C be a coloring of G. By Theorem 6.3.2, $LP(G, S, \mathcal{Q})$ has a solution for some maximal clique S and some minimal cover $\mathcal{Q}$. Take H the colored subgraph induced by vertices of S and of paths in $\mathcal{Q}$. By Lemma 6.3.7, $LP'(H, S, \mathcal{Q})$ has a solution, so the subgraph H is in the list.

Conversely, suppose the colored graph G has a colored induced subgraph H from the list. Then H has a clique S with $k+1$ vertices and a minimal cover $\mathcal{Q}$ such that $LP'(H, S, \mathcal{Q})$ has a solution. Since H is an induced subgraph of G preserving colors, the clique S is also a maximal clique in G and the cover $\mathcal{Q}$ is also a cover for G. Since $LP(G, S, \mathcal{Q})$ has a solution, G can not be superperfect. □

Notice that the list only has to contain those subgraphs H, of which every vertex is either in the maximal clique S or on some path in $\mathcal{Q}$ (with S and $\mathcal{Q}$ as defined in the algorithm). For reasons of simplicity, we left this detail out of the algorithm.

Chapter 7

Approximating treewidth and pathwidth for some classes of perfect graphs

For some *special classes* of graphs, it has been shown that the treewidth can be computed efficiently. In this chapter we discuss the problem of finding *approximate* tree- and path-decompositions for cotriangulated graphs, convex graphs, permutation graphs and for cocomparability graphs. We also show that for these graphs, if the treewidth is at most k, then the pathwidth is bounded by some polynomial in k. Our results show that it is often very easy to find good approximations for tree- and pathwidth.

The problem of determining the pathwidth of a triangulated graph is NP-hard [60]. It is unknown whether there exist good approximations, which can be computed efficiently, for the pathwidth of a triangulated graph. Surprisingly, for *cotriangulated* graphs, the pathwidth and treewidth are related in a linear fashion, and there is a very simple algorithm that computes an approximate path-decomposition. We show this in section 7.2. Perhaps even more surprising, the exact pathwidth and treewidth of cotriangulated graphs can be computed in polynomial time. We show this in subsection 7.2.1

It is easy to see that the problem of determining the treewidth and pathwidth for bipartite graphs is NP-hard. Indeed, finding an approximation algorithm with a certain performance ratio for bipartite graphs is at least as difficult as finding an approximation with the same performance for graphs in general. This can be seen as follows. Given a graph G, let S(G) be the subdivision graph (see [64, pages 79–80]). Clearly, S(G) is bipartite. The treewidth of S(G) is equal to the treewidth of G. Given a tree-decomposition for S(G), it can easily be transformed into a tree-decomposition for G with the same width. This shows that finding an approximation for the treewidth of bipartite graphs (within a constant factor) is as hard as finding approximations for the treewidth in general.

In section 7.3 we show that, if a graph is bipartite graph and *convex*, and

has treewidth k, then the pathwidth of the graph is at most $2k + 1$ and it is almost straightforward to find a path-decomposition of width $2k + 1$ in $O(nk)$ time. In chapter 8, we show that the treewidth of chordal bipartite graphs can be computed polynomial time. Chordal bipartite graphs (or graphs that are weakly chordal and bipartite) are bipartite graphs such that every induced cycle of length at least six has a chord. There is a strong connection between chordal bipartite graphs, strongly chordal graphs and totally balanced matrices. Indeed, a graph is chordal bipartite if and only if the adjacency matrix is totally balanced [47]. As the class of convex graphs is properly contained in the class of chordal bipartite graphs, this shows that the exact treewidth of convex graphs can be computed in polynomial time. However, the running time of this algorithm is not very good ($O(e^3)$, where e is the number of edges) and we think that, especially for practical applications, the algorithm described section 7.3 for the convex graphs, is of importance. Furthermore, as far as we know, computing the exact pathwidth is still an open problem.

One of the most well-known and well-studied classes of perfect graphs is the class of *permutation* graphs. Permutation graphs are exactly the comparability graphs of posets of dimension at most two. They can also be characterized as the graphs which are both a comparability and cocomparability graph. We show in section 7.4 that if the treewidth of a permutation graph is k, then the pathwidth is at most $2k$ and there is a linear time algorithm which produces a path-decomposition with this width. For more general information on permutation graphs and cocomparability graphs the reader is referred to chapter 9, or to Golumbic's book [58].

7.1 Preliminaries

We start with some definitions and easy lemmas. For more information on the special perfect graph classes treated in this section, the reader is referred to [31, 58]. An example of a class of perfect graphs is the class of convex graphs. (In fact all bipartite graphs are perfect.)

Definition 7.1.1 *Let* $G = (X, Y, E)$ *be a bipartite graph. An ordering of* X *has the* adjacency property *if for each* $y \in Y$ *the neighbors of* y *in* X *are consecutive in the ordering of* X.

Definition 7.1.2 *A bipartite graph* $G = (X, Y, E)$ *is called* convex *if there is an ordering of* X *or of* Y *that has the adjacency property.*

A bipartite graph $G = (X, Y, E)$ is *biconvex* if there is an ordering of X *and* Y with the adjacency property. Convex graphs contain the bipartite permutation graphs. Notice that convex graphs can be recognized in linear time, using for example the PQ-tree algorithms of Booth and Lueker [30] (see also [31] and [122]).

Recall the Definition 6.1.1, on page 64, of a comparability graph $G = (V, E)$. We know that if G is a comparability graph, then this holds for every induced subgraph of G. There exists a complete list of critical non-comparability graphs [54]. Comparability graphs can be recognized in $O(n^\alpha)$ time, which is the time needed to square the 0/1-adjacency matrix [123]. By the perfect graph theorem, also cocomparability graphs, which are complements of comparability graphs, are perfect. For our algorithm we shall need the concept of a height function defined in [58]. Let F be an acyclic orientation of an undirected graph $G = (V, E)$. A *height function* h assigns a non-negative integer to each vertex as follows: $h(v) = 0$ if v is a sink; otherwise, $h(v) = 1 + \max\{h(w) \mid (v, w) \in F\}$. In other words, $h(v)$ is the maximal length of a path from v to a sink. A height function can be assigned in linear time [58], and represents a proper vertex coloring of G. If F is a transitive orientation, the coloring by h is optimal (i.e., uses the least possible number of colors) [58].

We also need a coloring of the cocomparability graph. In [58] a method is described to find an optimal coloring of a cocomparability graph by using a minimum flow algorithm. It follows that this coloring can be found in $O(n^3)$ time. Recently, this result was improved upon in [122]. Here an $O(\sqrt{n}m)$ algorithm is given to find a clique partition of a comparability graph.

Cocomparability graphs are intersection graphs [117].

Lemma 7.1.1 *A graph G with n vertices is a cocomparability graph if and only if G is the intersection graph of n continuous functions $F_i : (0, 1) \to \mathbf{R}$.*

We only use this lemma implicitly.

We think of a permutation π of the numbers $1, \ldots, n$ as the sequence $\pi = [\pi_1, \ldots, \pi_n]$. We use the notation π_i^{-1} for the position of the number i in this sequence.

Definition 7.1.3 *If π is a permutation of the numbers $1, \ldots, n$, we can construct an undirected graph $G[\pi] = (V, E)$ with vertex set $V = \{1, \ldots, n\}$, and edge set E:*

$$(i, j) \in E \Leftrightarrow (i - j)(\pi_i^{-1} - \pi_j^{-1}) < 0$$

An undirected graph is called a permutation graph *if there exists a permutation π such that $G \cong G[\pi]$.*

$G[\pi]$ is sometimes called the *inversion graph* of π. Notice that we can obtain the complement of $G[\pi]$, by reversing the sequence π. Hence the complement of a permutation graph is also a permutation graph. It is also easy to see that a permutation graph is a comparability graph. Pnueli, Lempel and Even [104] showed that a graph G is a permutation graph if and only if both G and $\overline{G}$ are comparability graphs. It follows that permutation graphs are *perfect*. They can be recognized in time $O(n^2)$ (see [123]). There exist fast algorithms for many

NP-hard problems like CLIQUE, INDEPENDENT SET, FEEDBACK VERTEX SET and DOMINATING SET when restricted to permutation graphs [58, 48, 33, 32].

In this chapter we assume that the permutation π is given, and we show some results on the pathwidth and treewidth of $G[\pi]$. If the permutation π is *not* given, transitive orientations of G and $\overline{G}$ can be computed in $O(n^2)$ time [123]. Given these orientations, a permutation can be computed in $O(n^2)$ time [58].

Every permutation graph $G[\pi]$ is an intersection graph, which is illustrated by the *matching diagram* of π [58].

Definition 7.1.4 *Let π be a permutation of $1,\ldots,n$. Write the number $1,\ldots,n$ horizontally from left to right. Underneath, write the numbers $\pi_1,\ldots,\pi_n$, also horizontally from left to right. Draw n straight line segments joining the two 1's, the two 2's, etc. The resulting diagram is called the* matching diagram *of π.*

An example of a matching diagram of a permutation graph $G[\pi]$ is shown in figure 7.1. Notice that two vertices i and j of $G[\pi]$ are adjacent if and only if the

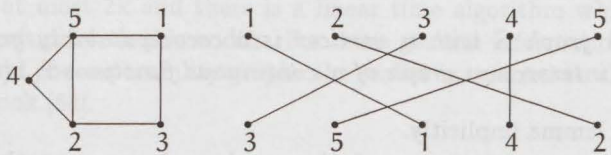

Figure 7.1: permutation graph and matching diagram

corresponding line segments in the matching diagram of π intersect. Matching diagrams are often useful in visualizing certain concepts.

7.2 Splitgraphs and cotriangulated graphs

Recall that for triangulated graphs the treewidth is equal to the maximum clique size minus one. It follows that, for triangulated graphs, the treewidth can be computed in linear time. The problem of determining the pathwidth of a triangulated graph is NP-hard, as shown in [60].

We show in this section that the treewidth and pathwidth of cotriangulated graphs can be computed efficiently. We start by showing how good approximations for the treewidth and pathwidth of cotriangulated graphs can be found. Let G be a cotriangulated graph with treewidth at most k. We show that the pathwidth

of G is at most $3k+4$ and that there exists an $O(n^2)$ algorithm to compute a path-decomposition with this width. According to Lemma 2.1.4, if $H = (V, E)$ is a triangulated graph with n vertices then there is a clique C, such that every component of $H[V - C]$ has at most $\lceil \frac{1}{2}(n - |C|) \rceil$ vertices. Also Lemma 2.2.1 will be useful: for the complete bipartite graph $K(m, n)$, the treewidth is $\min(m, n)$.

Let C be a clique in $\overline{G}$ as mentioned in Lemma 2.1.4. The vertices of $V \setminus C$ can be partitioned in two set A and B such that no vertex of A is adjacent to a vertex of B, and both A and B have at most $\lceil \frac{2}{3}(n - |C|) \rceil$ vertices. Notice that the subgraph induced by A and B has a complete bipartite subgraph in G, since every vertex of A is adjacent to every vertex of B. As the treewidth of G is at most k we have by Lemma 2.2.1:

$$\lfloor \frac{1}{3}(n - |C|) \rfloor \leq \min(|A|, |B|) \leq k$$

Hence $|A \cup B| \leq 3(k+1)$. We can triangulate G by adding edges such that $A \cup B$ becomes a clique. Since C is a stable set in G, the result is a split graph. For a split graph we have the following result.

Lemma 7.2.1 *Let $H = (V, E)$ be a split graph with maximum size clique C. Then the treewidth of H is $|C| - 1$. The pathwidth of H is either $|C| - 1$ or $|C|$. The pathwidth of H is equal to the treewidth if and only if there are vertices x and y in C (possibly equal) such that $N(x) \cap N(y) \subseteq C$.*

Proof. Since H is triangulated the treewidth is equal to the maximum clique size minus one. We can construct a path-decomposition of width $|C|$ as follows. Let $y_1, y_2, \ldots, y_s$ be the vertices of $V \setminus C$. Construct a path-decomposition $(X_1, X_2, \ldots X_s)$ with $X_i = C \cup \{y_i\}$ for $i = 1, \ldots, s$. It is easy to check that this is indeed a path-decomposition.

Assume there exists a path-decomposition $P = (X_1, \ldots, X_t)$ of width $|C| - 1$. By Lemma 2.2.2 there exists a subset X_i such that $C \subseteq X_i$. Since the width of P is $|C| - 1$, we have $X_i = C$. We can assume that $X_{i+1} \neq X_i$ and that $X_{i-1} \neq X_i$ (take $X_0 = \emptyset$ and $X_{t+1} = \emptyset$). Let $x \in X_i \setminus X_{i+1}$ and $y \in X_i \setminus X_{i-1}$. Let $z \in N(x) \cap N(y)$. Then z must be an element of a subset X_p which also contains x and of a subset X_q which also contains y. Since $p \leq i$ and $q \geq i$ we must have $z \in X_i$. This shows that $N(x) \cap N(y) \subseteq C$.

Finally, assume there are vertices x and y in C with $N(x) \cap N(y) \subseteq C$. Let $z_1, \ldots z_r$ be the vertices of $N(x) \setminus C$ and let $z_{r+2} \ldots z_t$ be the vertices of $V \setminus (C \cup \{z_1 \ldots, z_r\})$. Define subsets X_i for $i = 1, \ldots, t$ as follows:

$$X_i = \begin{cases} \{z_i\} \cup (C \setminus \{y\}) & \text{for } i = 1, \ldots, r \\ C & \text{if } i = r+1 \\ \{z_i\} \cup (C \setminus \{x\}) & \text{for } i = r+2, \ldots, t \end{cases}$$

It is easy to check that this gives a correct path-decomposition of width $|C| - 1$. □

As an immediate consequence we have the following result.

Theorem 7.2.1 *If G is a cotriangulated graph with treewidth at most k, then the pathwidth is at most $3k+4$ and there exists an $O(n^2)$ algorithm which produces a path-decomposition of this width.*

In the next section we show how to find the exact treewidth and pathwidth of a cotriangulated graph. The following result will be useful.

Lemma 7.2.2 *Let $H=(V,E)$ be a split graph. There is an algorithm to find a path-decomposition of width at most $\omega(H)$. This algorithm can be implemented to run in time $O(\omega(H)n)$ (where n is the number of vertices of H). The exact pathwidth of H can be found in time $O(n^\alpha)$, which is the time needed to square the $0/1$-adjacency matrix of H (currently $\alpha = 2.37\ldots$).*

Proof. Since H is triangulated, a maximum clique C can be found in $O(|V|+|E|)$ time [58]. As is shown in the proof of Lemma 7.2.1, a path-decomposition of width at most $|C|$ can be found in time $O(\omega(H)n)$. Using fast matrix multiplication techniques [89] and using the characterization of Lemma 7.2.1, the exact pathwidth can be found in $O(n^\alpha)$. □

Remark. In [60] a $O(n^3)$ algorithm is given to determine the pathwidth of a split graph. In fact a somewhat more general class of triangulated graphs is treated there.

7.2.1 Exact algorithms for the treewidth and pathwidth of cotriangulated graphs

In this subsection we give an algorithm to compute the treewidth of a cotriangulated graph. We start by showing that there exists a special tree-decomposition for triangulated graphs.

Definition 7.2.1 *Let $G=(V,E)$ be a triangulated graph. A* clique-tree-decomposition *for G is a tree-decomposition $D=(S,T)$ with $T=(I,F)$ and $S=\{X_i \mid i \in I\}$ such that for every $i \in I$, X_i is a maximal clique in G and such that for every $i,j \in I$, if $i \neq j$ then $X_i \neq X_j$.*

Lemma 7.2.3 *Let $G=(V,E)$ be triangulated. There is a clique-tree-decomposition for G.*

Proof. By induction. If G is a clique, then make a tree T with one node and a corresponding subset containing all vertices of G. Otherwise, let x be a simplicial vertex of G, and let C be the set of neighbors of x. By induction there is a clique-tree-decomposition $D'=(S',T')$ for $G[V \setminus \{x\}]$. Suppose there is a subset Y_i in S'

with $Y_i = C$. Add x to this subset. The new tree-decomposition is a clique-tree-decomposition for G. Suppose there is no subset in S' equal to C. There must exist a subset Y_i such that $C \subset Y_i$. Take a new node i_0 and make it adjacent to i in T'. Make a corresponding subset $Y_{i_0} = \{x\} \cup C$. It is easy to check that the new tree-decomposition is a clique-tree-decomposition for G. □

We use the following notation. Let $G = (V, E)$ be triangulated, and let $D = (S, T)$ be a clique-tree-decomposition for G with $T = (I, F)$ and $S = \{X_i | i \in I\}$. Let $(i, j) \in F$. We write $T^i_{(i,j)}$ for the maximal subtree containing i obtained by removing the edge (i, j). Let $I^i_{(i,j)}$ be the set of nodes of $T^i_{(i,j)}$. Also we write $V^i_{(i,j)}$ for the following subset of vertices:

$$V^i_{(i,j)} = \{x \mid x \in X_t \setminus X_j \wedge t \in I^i_{(i,j)}\}$$

Lemma 7.2.4 *Let $G = (V, E)$ be triangulated and let $D = (S, T)$ be a clique-tree-decomposition. Consider a node i in T. For each connected component C of $G[V \setminus X_i]$ there is exactly one node j adjacent to i such that $C \subseteq V^j_{(i,j)}$.*

Proof. Since C is connected, the nodes of T containing vertices of C must form a connected subtree of T. Since X_i contains no vertices of C, the lemma follows. □

Lemma 7.2.5 *Let $G = (V, E)$ be triangulated, and let H be a triangulation of the complement $\overline{G}$. Let C be a clique in G. There is at most one connected component of $G[V \setminus C]$ with vertex set Y such that $H[Y]$ is not a clique.*

Proof. Clearly, if x and y are vertices in different connected components of $G[V \setminus C]$, then x and y are adjacent in H since they are already adjacent in $\overline{G}$. Assume there are two different connected components with vertex sets C_i and C_j such that $H[C_i]$ and $H[C_j]$ are not cliques. Then there are vertices p and q in C_i which are not adjacent in H and vertices r and s in C_j which are not adjacent in H. But then H cannot be triangulated since $H[\{p, q, r, s\}]$ is a cycle of length four. □

Theorem 7.2.2 *Let $G = (V, E)$ be triangulated. Let H be a triangulation of $\overline{G}$. There is a maximal clique with vertex set C in G such that $H[V \setminus C]$ is a clique.*

Proof. We show that there is a maximal clique C in G such that the vertex sets of all connected components of $G[V \setminus C]$ induce cliques in H. Assume this is not the case. Then, by Lemma 7.2.5, for each maximal clique C in G there is exactly one connected component which is not a clique in H. Call $\mathcal{C}(C)$ the vertex set of this connected component.

Let $D = (S, T)$ be a clique-tree-decomposition for G with $T = (I, F)$ and $S = \{X_i \mid i \in I\}$. Make a digraph with vertex set I as follows. Consider a node

$i \in I$. By Lemma 7.2.4 there exists exactly one neighbor j of i in T such that all vertices of $\mathcal{C}(X_i)$ are contained in $V^j_{(i,j)}$. Direct an arc from i to j in this case. By assumption every node i gets one outgoing arc in this way. Since there are only arcs between nodes which are adjacent in T and since T is a tree, there must exist nodes i and j which are adjacent in T and such that there is an arc from i to j and an arc from j to i. Let $A = V^i_{(i,j)} \setminus X_i$ and let $B = V^j_{(i,j)} \setminus X_j$. Then $H[A \cup B]$ is a clique.

Let $X^*_i = X_i \setminus X_j$ and $X^*_j = X_j \setminus X_i$. We claim that if $x \in X^*_i$ and $y \in X^*_j$ then x and y are adjacent in H. This can be seen as follows. Assume x and y are adjacent in G. Then there must be a subset X_t of S containing both x and y. But if $t \in I^i_{(i,j)}$ then by definition of a tree-decomposition, y must also be in X_i which is a contradiction. Similarly t cannot be in $I^j_{(i,j)}$ and the statement follows. We also claim that each vertex of X^*_i is adjacent to each vertex of B in H and that each vertex of X^*_j is adjacent to each vertex of A. This follows by a similar argument.

We know that $H[V^j_{(i,j)}]$ is not a clique, since there is an arc from i to j. Let x and y be two non adjacent vertices in $H[V^j_{(i,j)}]$. Since B is a clique, either both x and y are elements of X^*_j or one is in B and the other in X^*_j. In both cases all vertices of A and all vertices of X^*_i are common neighbors of x and y in H. Since H is triangulated a minimal separator for x and y in H is a clique and any minimal separator must contain all common neighbors. It follows that $H[A \cup X^*_i]$ is a clique and hence also $H[A \cup B \cup X^*_i]$ is a clique. Finally notice that $A \cup B \cup X^*_i = V \setminus X_j$ and hence $H[V \setminus X_j]$ is a clique. This proves the theorem. □

Definition 7.2.2 *Let G be triangulated. Let C be a maximal clique in G. The split graph $\mathcal{S}(C)$ is obtained from $\overline{G}$ by adding edges to $\overline{G}$ such that $\mathcal{S}(C)[V \setminus C]$ becomes a clique.*

Notice that $\mathcal{S}(C)$ is a triangulation of $\overline{G}$.

Theorem 7.2.3 *Let $G = (V, E)$ be triangulated. If there is a maximum clique C (with $\omega(G)$ vertices) such that all vertices x in C have degree larger than $\omega(G) - 1$ in G, then the treewidth of $\overline{G}$ is $|V| - \omega(G) - 1$. Otherwise, the treewidth of G is $|V| - \omega(G)$.*

Proof. Let H be a triangulation of $\overline{G}$. By Theorem 7.2.2 there is a maximal clique C in G such that $H[V \setminus C]$ is a clique. Notice that $\mathcal{S}(C)$ is a subgraph of H. It follows that we can restrict ourselves to split graphs $\mathcal{S}(C)$ with C a maximal clique in G. The treewidth of $\mathcal{S}(C)$ is either $|V| - |C| - 1$ or $|V| - |C|$. It follows that we can restrict ourselves to maximum cliques in G (of size $\omega(G)$).

Let C be a maximum clique in G. Then the treewidth of $\mathcal{S}(C)$ is $|V| - |C| - 1$ if every vertex x in C is adjacent to some vertex of $V \setminus C$ in G (in that case $V \setminus C$ is a maximum clique in $\mathcal{S}(C)$). Otherwise the treewidth of $\mathcal{S}(C)$ is $|V| - |C|$.

Hence we we may conclude that the treewidth of $\overline{G}$ is $|V| - \omega(G) - 1$ if there is a maximum clique C such that all vertices of C are adjacent to some vertex of $V \setminus C$ in G. Otherwise the treewidth of $\overline{G}$ is $|V| - \omega(G)$. □

Theorem 7.2.4 *Let $\overline{G}$ be a cotriangulated graph. There is an $O(n^2)$ algorithm to find a tree-decomposition with width equal to the treewidth of $\overline{G}$ (where n is the number of vertices of $\overline{G}$).*

Proof. We can construct the complement G of $\overline{G}$ in $O(n^2)$ time. For each vertex compute the degree in G. In [58] it is shown that a list of maximum cliques can be constructed in $O(n^2)$ time. By Theorem 7.2.3 we can compute in linear time the treewidth of the split graph $\mathcal{S}(C)$. Since there are at most n maximum cliques, this proves the theorem. □

We now show an algorithm to compute the pathwidth of a cotriangulated graph. Let G be triangulated and let H be an interval graph embedding of $\overline{G}$. Hence H is triangulated and by Theorem 7.2.2 we know there exists a maximal clique C in G such that $\mathcal{S}(C)$ is a subgraph of H. Hence we can find the pathwidth of G by computing the pathwidth of $\mathcal{S}(C)$ for all maximal cliques C of G and taking the minimum of those. We can make a list of all maximal cliques in time $O(n^2)$ time. Using Lemma 7.2.2 we can compute the pathwidth of $\mathcal{S}(C)$ in $O(n^\alpha)$ time, for each maximal clique C. Since there are at most n maximal cliques, the computation of the pathwidth of $\overline{G}$ can be done in $O(n^{\alpha+1})$. The algorithm can be adapted such that it produces an optimal path-decomposition within the same time bound. This proves the following theorem.

Theorem 7.2.5 *There exists a polynomial time algorithm which, given a cotriangulated graph $\overline{G}$, computes a path-decomposition of width equal to the pathwidth of $\overline{G}$.*

7.3 Convex graphs

Let $G = (X, Y, E)$ be a convex graph, with $|X| = m$, and assume the vertices of X have been ordered $1, 2, \ldots, m$ such that this ordering fulfills the adjacency property (i.e., for each $y \in Y$ the neighbors in X are consecutive). Let k be some integer. In this section we describe an $O(nk)$ algorithm which produces a path-decomposition of width at most $2k + 1$ or shows that the treewidth of G is larger than k.

Consider the case where $k \geq m$. If we add edges such that X becomes a clique, then the result is a split graph with maximum clique size at most $k + 1$. Hence, according to Lemma 7.2.1, we find a path-decomposition of width at most $k + 1$. Hence we may assume without loss of generality that $k \leq m - 1$. Let Y' be the

set of vertices of Y with degree at least $k+1$. Consider the subgraph G', induced by vertices of X and the vertices of Y'. We construct a path-decomposition for G' as follows. For $i = 1, \ldots, m-k$ let Z_i be the subset with:

$$\begin{aligned} Z_i \cap X &= \{i, i+1, \ldots, i+k\} \\ Z_i \cap Y &= \{y \in Y \mid Z_i \cap X \subseteq N(y)\} \end{aligned}$$

where $N(y)$ is the neighborhood of $y \in Y$. Notice that $Z_i \cup Y \subseteq Y'$.

Lemma 7.3.1 *Assume* $k \leq m-1$. *If the treewidth of* G' *is* k, *then the path-width of* G' *is at most* $2k$ *and* $(Z_1, \ldots, Z_{m-k})$ *is a path-decomposition for* G' *of width at most* $2k$.

Proof. Notice that each Z_i has $k+1$ vertices of X. Each vertex $y \in Y'$ which is in Z_i is adjacent to all vertices of $Z_i \cap X$. This shows that there can be at most k vertices of Y in Z_i, otherwise G' has a complete bipartite subgraph $K(k+1, k+1)$, which is forbidden by Lemma 2.2.1. Obviously, $(Z_1, \ldots, Z_{m-k})$ is indeed a path-decomposition. □

We now show how to extend this path-decomposition to a path-decomposition for G. Consider a vertex $y \in Y$, with at most k neighbors. Notice that there exists a subset Z_i containing $N(y)$. Take such a subset Z_i containing $N(y)$ with at most $2k+1$ elements. Make a new subset $Z_{i'} = Z_i \cup \{y\}$, and make a new path-decomposition $(Z_1, \ldots, Z_i, Z_{i'}, Z_{i+1}, \ldots, Z_{m-k})$. This clearly is a path-decomposition for the subgraph of G induced by the vertices $X \cup Y' \cup \{y\}$. By induction the following lemma follows.

Lemma 7.3.2 *Let* $G = (X, Y, E)$ *be convex with treewidth* k. *Then the path-width of* G *is at most* $2k+1$.

We now show that the algorithm described above can be implemented to run in $O(nk)$ time. We assume that the vertices of X are ordered $1, \ldots, m$ such that this ordering has the adjacency property. Assume $Y = \{y_1, \ldots, y_t\}$.

Step 1 If $k \geq m$ then make a subset $Z_y = X \cup \{y\}$ for each vertex $y \in Y$. The sequence $(Z_{y_1}, Z_{y_2}, \ldots, Z_{y_t})$, (for *any* ordering $y_1, \ldots, y_t$ of the vertices of Y) is a correct path-decomposition. Stop.
Otherwise, if $k \leq m-1$, check if the number of edges does not exceed $nk - \frac{1}{2}k(k+1)$. If it does, then stop; the treewidth of G is larger than k.
Otherwise, if the number of edges does not exceed $nk - \frac{1}{2}k(k+1)$, proceed with step 2.

Step 2 Determine for each $y \in Y$ the maximum and minimum neighbor in X, say $\min(y)$ and $\max(y)$. If a vertex y has degree 0 then we set $\min(y) = 0$ and $\max(y) = -1$. The result of this step is that the set of neighbors of $y \in Y$, with degree at least one, is $\{x \in X \mid \min(y) \leq x \leq \max(y)\}$.

Step 3 Calculate for each vertex $y \in Y$ the degree, $\max(y) - \min(y) + 1$, and make a list Y' of vertices of degree at least $k+1$.

Step 4 Initialize subsets $Z_i = \{i, \ldots, \min(m, i+k)\}$, for $i = 1, \ldots, m$. For isolated vertices of Y, we initialize $Z_0 = \emptyset$.

Step 5 For each $y \in Y'$, put y in the subsets Z_i, for $i = \min(y), \ldots, \max(y) - k$. If one of these subsets, say Z_i, gets more than $2k+1$ vertices, then stop; the treewidth of G exceeds k. The vertices of Z_i induce a $K(k+1, k+1)$.

Step 6 For each $y \in Y \setminus Y'$, make a subset $Z'_y = Z_{\min(y)} \cup \{y\}$.

Step 7 For $i = 0, \ldots, m$, let $y_1^i, y_2^i, \ldots$ be the vertices $y \in Y \setminus Y'$ with $\min(y) = i$. Then the path-decomposition is $(Z_0, Z'_{y_1^0}, Z'_{y_2^0}, \ldots, Z_1, Z'_{y_1^1}, Z'_{y_2^1}, \ldots, Z_2, \ldots)$.

This proves the following theorem.

Theorem 7.3.1 *Let G be a convex graph and let k be an integer. There exists an $O(nk)$ algorithm which either determines that the treewidth exceeds k, or produces a path-decomposition of G of width at most $2k+1$.*

7.4 Permutation graphs

In this section, let $G[\pi]$ be a permutation graph with n vertices and with treewidth k. We show there exists a path-decomposition of width at most $2k$, and we give a linear time algorithm to compute this. The algorithm outputs a set X_i for $1 \leq i \leq n$. A vertex j is put in all sets X_k with $\pi_j^{-1} \leq k < j$ or $j \leq k < \pi_j^{-1}$. The precise algorithm is given below.

```
procedure Pathdec (input π; output X);
begin
 for i ← 1 to n do X_i ← ∅;
 for j ← 1 to n do
     begin
      if π_j^{-1} = j then X_j ← X_j ∪ {j};
      if π_j^{-1} > j then
        for k ← j to π_j^{-1} − 1 do X_k ← X_k ∪ {j};
      if π_j^{-1} < j then
        for k ← π_j^{-1} to j − 1 do X_k ← X_k ∪ {j}
     end
end
```

For example, for the graph of figure 7.1, the computed sets are: $X_1 = \{1,3\}$, $X_2 = \{1,2,3,5\}$, $X_3 = \{2,5\}$, $X_4 = \{4,2,5\}$ and $X_5 = \emptyset$. Notice that this

path-decomposition is not optimal since the pathwidth of this graph is 2 and the computed path-decomposition has width 3. The next lemma shows that the constructed sets form indeed a path-decomposition.

Lemma 7.4.1 *Let* $S = \{X_i \mid 1 \leq i \leq n\}$ *be the subsets of vertices constructed by the algorithm. Let* $P = (1, \ldots, n)$ *be the path with* n *vertices. Then* (S, P) *is a path-decomposition for the permutation graph* $G[\pi]$.

Proof. We first show that each vertex is in at least one subset of S. Consider a vertex i. If $\pi_i^{-1} \geq i$ then i is in the subset X_i. If $\pi_i^{-1} < i$ then i is in the subset X_{i-1}. Notice that the subsets containing i are consecutive. The only thing left to show is that every edge is in at least one subset. Consider again a vertex i and let j be a neighbor of i. Assume without loss of generality that $i < j$. In the matching diagram, the line segment corresponding with j must intersect the line segment of i. Since $i < j$, this implies that $\pi_j^{-1} < \pi_i^{-1}$. We consider the different orderings of i, j, π_i^{-1} and π_j^{-1}. If $i < j \leq \pi_j^{-1} < \pi_i^{-1}$, then both i and j are contained in the subset X_j. If $i \leq \pi_j^{-1} \leq j \leq \pi_i^{-1}$ then both are contained in $X_{\pi_j^{-1}}$. If $i \leq \pi_j^{-1} < \pi_i^{-1} \leq j$, then both are contained in $X_{\pi_j^{-1}}$. If $\pi_j^{-1} \leq i \leq \pi_i^{-1} \leq j$, then both are contained in X_i. Finally, if $\pi_j^{-1} < \pi_i^{-1} \leq i < j$, then both i and j must be in $X_{\pi_i^{-1}}$. □

We now show that the width of this path-decomposition is at most 2k.

Lemma 7.4.2 *Each subset produced by the algorithm has at most* $2k + 1$ *elements.*

Proof. Consider a subset X_i. Notice that $X_i \subset S_1 \cup S_2 \cup \{i\}$ where S_1 and S_2 are defined by: $S_1 = \{j \mid j \leq i < \pi_j^{-1}\}$ and $S_2 = \{j \mid \pi_j^{-1} \leq i < j\}$. Note that, as π is a permutation, there must be as many lines in the matching diagram with their upper point left of i and their lower point right of i, as lines with their upper point right of i and their lower point left of i. Hence $|S_1| = |S_2|$. Every vertex in S_1 is adjacent to every vertex in S_2, hence the subgraph induced by $S_1 \cup S_2$ contains a complete bipartite subgraph $K(m, m)$, with $m = |S_1|$. By Lemma 2.2.1, this implies that $k \geq m$. Hence $|X_i| \leq |S_1| + |S_2| + 1 \leq 2k + 1$. □

Notice that the algorithm can be implemented to run in $O(nk)$ time, since at each step one new element is put into a subset. Hence we have proved the following theorem:

Theorem 7.4.1 *If* $G[\pi]$ *is a permutation graph with treewidth at most* k, *then the pathwidth of* $G[\pi]$ *is at most* 2k, *and the algorithm Pathdec produces a path-decomposition of width at most* 2k *in time* $O(nk)$.

It follows that we have $O(nk)$ approximation algorithms for pathwidth and treewidth with performance ratio 2. In chapter 9 we show that the pathwidth and treewidth are in fact equal, and can be computed exactly in $O(nk)$ time. Because of its simplicity and also as an introduction for the next section, we decided to include this approximation algorithm.

7.5 Cocomparability graphs

In this section, let G be a cocomparability graph with treewidth at most k. Notice that computing the exact treewidth of a cocomparability graph is NP-hard. In fact, it is shown in [4] that treewidth is NP-hard for complements of bipartite graphs.

We start with an informal discussion of the approximation algorithm. Recall Lemma 7.1.1. There exist n continuous functions $F_i : (0,1) \to \mathbf{R}$, such that two vertices are adjacent if and only if the corresponding functions intersect. First make a vertex coloring of G, such that no two adjacent vertices have the same color. Since the treewidth of G is at most k, this can be done by using at most $k+1$ colors. Fix any position L at the line $x = 0$. This partitions the vertices of G into two sets: one set of vertices for which the corresponding functions start below L, and one set for which the corresponding functions start at position at least L. For each of these positions L we make a subset X_L as follows. For each color class C_t, take the top most $k+1$ functions which start below L, say $\overline{C_t}$ (take all if there are less than $k+1$). Notice that the functions corresponding with a color class do not intersect, hence the 'topmost functions' are well defined. For functions starting at position at least L, take those which are adjacent to $k+1$ vertices of $\overline{C_t}$, say $\mathcal{F}_t$. Notice that $\mathcal{F}_t$ is empty if $|\overline{C_t}| \leq k$. Also notice that if $|\overline{C_t}| = k+1$, then each vertex of $\mathcal{F}_t$ is adjacent to all vertices of $\overline{C_t}$. In this last case $|\mathcal{F}_t| \leq k$, otherwise there is a $K(k+1, k+1)$ subgraph. It follows that X_L has at most $(k+1)(2k+1)$ vertices, since there are at most $k+1$ color classes and for each color class C_t we have $|\overline{C_t}| + |\mathcal{F}_t| \leq 2k+1$. Notice that we only used the ordering of the functions at position $x = 0$. We can find a suitable ordering using the height function of G.

We now give the formal description of the algorithm and prove the correctness, without using the function model. We assume that a height function h of the complement $\overline{G}$ with transitive orientation F, and a coloring of G are given. Let $C_1, \ldots, C_s$ be the color classes of G, where $s = \chi(G)$ is the chromatic number of G. Since a partial k-tree can always be colored with $k+1$ colors, we may assume that the number of color classes $s \leq k+1$.

The first step of the algorithm is to renumber the vertices.

Definition 7.5.1 *Let $G = (V, E)$ be a cocomparability graph with n vertices and let h be a height function of the transitively oriented complement $\overline{G}$. A* height labeling *of G is a bijection $L : V \to \{1, \ldots, n\}$, such that $h(x) > h(y)$ implies that $L(x) > L(y)$.*

A height labeling clearly can be computed in $O(n)$ time, if the height function h is given. This height labeling is sometimes called a cocomparability labeling (see, e.g., [86]).

Definition 7.5.2 *For $i = 1, \ldots, n$ and for $t = 1, \ldots, s$, let $C_t(i)$ be the set of vertices in color class C_t with label at most i: $C_t(i) = C_t \cap \{x \mid L(x) \leq i\}$.*

Write $C_t(i) = \{x_1, \ldots, x_m\}$, *with* $i \geq L(x_1) > L(x_2) \ldots > L(x_m)$. *We define for* $i = 1, \ldots, n$ *and* $t = 1, \ldots, s$:

$$\overline{C_t}(i) = \begin{cases} C_t(i) & \text{if } m \leq k+1 \\ \{x_1, \ldots, x_{k+1}\} & \text{otherwise} \end{cases}$$

Notice that, since C_t is a clique in $\overline{G}$, all heights of vertices in C_t are different. It follows that $\overline{C_t}(i)$ is uniquely determined as the set of $k+1$ vertices of $C_t(i)$ with the largest heights.

Definition 7.5.3 *For* $i = 1, \ldots, n$ *and* $t = 1, \ldots, s$, *define*

$$\mathcal{F}_t(i) = \{x \in V \mid L(x) > i \wedge |N(x) \cap C_t(i)| \geq k+1\}$$

where $N(x)$ *is the set of neighbors of* x.

We can now define the subsets of the path-decomposition:

Definition 7.5.4 *For* $i = 1, \ldots, n$, *let* X_i *be the following set of vertices:*

$$X_i = \bigcup_{1 \leq t \leq s} (\mathcal{F}_t(i) \cup \overline{C_t}(i))$$

In the rest of this section we prove that each subset X_i has at most $(2k+1)\chi(G)$ elements and we show that $(X_1, \ldots, X_n)$ forms indeed a path-decomposition. The following lemma is crucial.

Lemma 7.5.1 *Each* $y \in \mathcal{F}_t(i)$ *is adjacent to all vertices of* $\overline{C_t}(i)$.

Proof. Suppose not. Let $y \in \mathcal{F}_t(i)$ be not adjacent to every vertex in $\overline{C_t}(i)$. Let $C_t(i) = \{x_1, \ldots, x_m\}$ with $h(x_1) > h(x_2) > \ldots > h(x_m)$. If $m < k+1$, then by definition $\mathcal{F}_t(i) = \emptyset$. Hence we may assume $m \geq k+1$ and $\overline{C_t}(i) = \{x_1, \ldots, x_{k+1}\}$. Let $x_w \in \overline{C_t}(i)$ (with $1 \leq w \leq k+1$) be not adjacent to y. Consider the complement $\overline{G}$ with the transitive orientation F. $C_t(i)$ is a clique in $\overline{G}$ and $(x_p, x_q) \in F$ for all x_p and x_q in $C_t(i)$ with $p < q$. Since y is adjacent to x_w in $\overline{G}$, we must have $h(y) \neq h(x_w)$. Since $L(y) > L(x_w)$, it follows that $h(y) > h(x_w)$ and hence $(y, x_w) \in F$. Since F is transitive, we find that y is adjacent in $\overline{G}$ to all vertices x_p with $w \leq p \leq m$ in $\overline{G}$. So y can have at most $w-1$ neighbors in $C_t(i)$ in G, hence it cannot be in $\mathcal{F}_t(i)$, contradiction. □

Corollary 7.5.1 *A vertex* y *with* $L(y) > i$ *is in* $\mathcal{F}_t(i)$ *if and only if it is adjacent to the vertex in* $\overline{C_t}(i)$ *with the smallest label.*

Theorem 7.5.1 *If* G *is a cocomparability graph with treewidth at most* k, *then* $|\overline{C_t}(i)| \leq k+1$ *and* $|\mathcal{F}_t(i)| < k+1$.

Proof. Notice that $|\overline{C_t}(i)| \leq k+1$, by definition. If $|\overline{C_t}(i)| < k+1$, then $C_t(i)$ contains less than $k+1$ elements, and then, by definition $\mathcal{F}_t(i) = \emptyset$.

Now assume that $C_t(i)$ contains at least $k+1$ vertices. Then $|\overline{C_t}(i)| = k+1$. By Lemma 7.5.1 all vertices of $\mathcal{F}_t(i)$ are adjacent to all vertices of $\overline{C_t}(i)$.

The treewidth of G is at most k. Hence G cannot have a complete bipartite subgraph $K(k+1, k+1)$. This implies that $|\mathcal{F}_t(i)| < k+1$. □

Corollary 7.5.2 *For all* i, $|X_i| \leq (2k+1)\chi(G)$.

Corollary 7.5.2 shows that, if $(X_1, \ldots, X_n)$ is a path-decomposition of G, then the width of this path-decomposition is at most $2k^2 + 3k$. The next three lemmas show that $(X_1, \ldots, X_n)$ is indeed a path-decomposition.

Lemma 7.5.2 *For every vertex* x *of* G: $x \in X_{L(x)}$.

Proof. Let x be in the color class C_p. By definition, $x \in \overline{C_p}(L(x)) \subseteq X_{L(x)}$. □

Lemma 7.5.3 *Let* $\{x, y\} \in E$ *be an edge of* G. *Then there is a subset* X_i *such that* x *and* y *are both in* X_i.

Proof. Assume $L(x) < L(y)$. Notice that x and y are not in the same color class, since they are adjacent. Consider the color class of x, say $C_p = \{x_1, \ldots, x_m\}$, and let $L(x_1) > L(x_2) > \ldots > L(x_m)$. Let $x = x_j$ for some $1 \leq j \leq m$. Clearly, if $j \leq k+1$, then x and y are both contained in $X_{L(y)}$ since $x \in \overline{C_p}(L(y)) \subseteq X_{L(y)}$. Now assume that $j > k+1$. Consider $z = x_{j-k}$. If $L(y) < L(z)$ then x and y are both contained in $X_{L(y)}$, since $x \in \overline{C_p}(L(y))$. If $L(y) > L(z)$, then $x \in \overline{C_p}(L(z))$ and $y \in \mathcal{F}_p(L(z))$. □

Lemma 7.5.4 *The subsets* X_i *containing a given vertex* x, *are a consecutive subsequence of* $(X_1, \ldots, X_n)$.

Proof. Assume $L_1 < L_2$ and $x \in X_{L_1} \cap X_{L_2}$. Let $L_1 < L < L_2$. We prove that $x \in X_L$. We consider three cases:

case 1 $L(x) \geq L_2$. Since $x \in X_{L_1}$, x must be adjacent to at least $k+1$ vertices of some color class which are in X_{L_1}. Clearly, this also holds for every $L_1 \leq L < L(x)$. Hence $x \in X_L$.

case 2 $L(x) \leq L_1$. Since $x \in X_{L_2}$, x must be among the vertices in its color class with the $k+1$ largest heights which are in X_{L_2}. But, then clearly this must hold for every $L(x) \leq L \leq L_2$.

case 3 $L_1 < L(x) < L_2$. The argument of the first case shows that $x \in X_L$ for all $L_1 \leq L < L(x)$. In the second case it is shown that $x \in X_L$ for all $L(x) \leq L \leq L_2$.

□

By Lemmas 7.5.2, 7.5.3, and 7.5.4 the sequence of subsets $(X_1, \ldots, X_n)$ is a path-decomposition for the cocomparability graph G. According to Corollary 7.5.2, the width of this path-decomposition is at most $(2k+1)\chi(G) - 1$. In the following theorem we summarize these results.

Theorem 7.5.2 *Let* G *be a cocomparability graph. The sequence* $(X_1, \ldots, X_n)$ *with* X_i *defined in Definition 7.5.4, is a path-decomposition for* G. *If the treewidth of* G *is at most* k, *then the width of this path-decomposition is at most* $(2k+1)\chi(G) - 1 \leq 2k^2 + 3k$.

Consider the time it takes to compute the sets X_i. We can sort all the color classes according to increasing labels in $O(nk)$ time. Then each set $\overline{C_t}(i)$ can be computed in $O(k)$ time. Now notice that $\mathcal{F}_t(i) \subseteq \mathcal{F}_t(i+1) \cup \{x \mid L(x) = i+1\}$. If we use an adjacency matrix to represent G, we can compute each set $\mathcal{F}_t(i)$ in time $O(k)$: An element y in $\mathcal{F}_t(i+1) \cup \{x \mid L(x) = i+1\}$ is in $\mathcal{F}_t(i)$ if and only if y is adjacent to the vertex with the smallest label in $\overline{C_t}(i)$ (Corollary 7.5.1). The sets $\overline{C_t}(i)$ and $\mathcal{F}_t(i)$ have at most $k+1$ elements and there are at most $k+1$ of each for each i. Hence we can easily compute each set X_i in time $O(k^2)$.

Corollary 7.5.3 *Let* G *be a cocomparability graph with treewidth* k. *Assume a vertex coloring of* G *with at most* $k+1$ *colors, and a transitive orientation* F *of the complement* $\overline{G}$ *are given. If an adjacency matrix is used to represent* G, *then a path-decomposition of* G *with width at most* $2k^2 + 3k$ *can be computed in time* $O(nk^2)$.

Chapter 8

Treewidth of chordal bipartite graphs

The chordal bipartite graphs form a large class of perfect graphs containing, for example, not only the convex and biconvex bipartite graphs, but also the bipartite permutation graphs and the bipartite distance-hereditary graphs (or $(6,2)$-chordal bipartite graphs). Many NP-hard problems remain NP-hard when restricted to the class of chordal bipartite graphs. For example HAMILTONIAN CYCLE, HAMILTONIAN PATH, DOMINATING SET, CONNECTED DOMINATING SET, INDEPENDENT DOMINATING SET, and STEINER TREE [101, 41]. The recognition problem for chordal bipartite graphs can be solved in $O(\min(n^2))$ [124, 94]. Since so many NP-hard problems remain NP-hard when restricted to chordal bipartite graphs, it is of importance to be able to use the partial k-tree algorithms for these problems. In this chapter we give a polynomial time algorithm to find the treewidth and a tree-decomposition of optimal width for a chordal bipartite graph.

We do not claim that our algorithm for the treewidth of chordal bipartite graphs is a very practical one, but we feel that it is one of the first non-trivial polynomial time algorithms for computing the treewidth of a relatively large class of graphs. Note that it considerably narrows the gap between classes where treewidth is computable in polynomial time and the classes for which the corresponding decision problem is NP-complete.

8.1 Preliminaries

We start with some definitions and easy lemmas. For more information the reader is referred to [58] or [31].

Definition 8.1.1 *A graph is called* chordal bipartite *(or weakly chordal bipartite) if it is bipartite and each cycle of length at least six has a chord.*

A chord (x,y) in a cycle C of even length is *odd* if the distance between x and y in the cycle is odd.

Definition 8.1.2 *A graph is called* strongly chordal *if it is chordal and each cycle of even length at least six has an odd chord.*

Definition 8.1.3 *For a bipartite graph* $G = (X, Y, E)$ *let* $\text{split}(G) = (X, Y, \hat{E})$ *with* $\hat{E} = E \cup \{(x, x') \mid x, x' \in X \wedge x \neq x'\}$.

A split graph is a graph for which there is a partition of the vertices in two sets such that one set induces an independent set and the other set induces a clique. These are exactly the graph which are both triangulated and cotriangulated. The following characterization of chordal bipartite graphs appeared in [40].

Lemma 8.1.1 $G = (X, Y, E)$ *is chordal bipartite if and only if* $\text{split}(G)$ *is strongly chordal.*

If x is a vertex of a graph $G = (V, E)$, we denote by $N[x]$ the *closed neighborhood* of x, i.e. $N[x] = N(x) \cup \{x\} = \{y \mid y = x \text{ or } (x, y) \in E\}$.

Definition 8.1.4 *A vertex* v *is* simple *if for all* $x, y \in N[v]$, $N[x] \subseteq N[y]$ *or* $N[y] \subseteq N[x]$.

Notice that a simple vertex is simplicial (i.e., the neighborhood is complete). We shall use the following property of strongly chordal graphs [47].

Lemma 8.1.2 *A graph* G *is strongly chordal if and only if every induced subgraph has a simple vertex.*

Definition 8.1.5 *Let* $G = (X, Y, E)$ *be a bipartite graph. Then* $(u, v) \in E$ *is called a* bisimplicial edge *if* $N(u) \cup N(v)$ *induces a complete bipartite subgraph of* G.

Definition 8.1.6 *Let* $G = (X, Y, E)$ *be a bipartite graph. Let* $(e_1, \ldots, e_k)$ *be an ordering of the edges of* G. *For* $i = 0, \ldots, k$ *define the subgraph* $G_i = (X_i, Y_i, E_i)$ *as follows:* $G_0 = G$, *and for* $i \geq 1$ G_i *is the subgraph of* G_{i-1} *with* $X_i = X_{i-1}$, $Y_i = Y_{i-1}$ *and* $E_i = E_{i-1} \setminus \{e_i\}$ *(i.e., the edge* e_i *is removed but not the end vertices). The ordering* $(e_1, \ldots, e_k)$ *is a* perfect edge-without-vertex elimination ordering *for* G *if each edge* e_i *is bisimplicial in* G_i, *and* G_k *has no edge.*

The following lemma appears for example in [31].

Lemma 8.1.3 G *is chordal bipartite if and only if there is a perfect edge-without-vertex elimination ordering.*

The following lemma implies that we can start a perfect edge-without-vertex elimination ordering with *any* bisimplicial edge.

Lemma 8.1.4 *Let* G *be chordal bipartite. Let* e *be a bisimplicial edge in* G. *Let* G' *be the graph obtained from* G *by deleting the edge* e *but not the end vertices of* e. *Then* G' *is chordal bipartite.*

Proof. Assume G' has a chordless cycle C of length ≥ 6. Let $e = (x, y)$. Then, clearly, x and y must be elements of C. The neighbors of x and y in the cycle form a square. This shows that C cannot be chordless in G'. □

In [57] it is shown that a bisimplicial edge in a chordal bipartite graph with n vertices can be found in $O(n^2)$ time.

Corollary 8.1.1 *A perfect edge-without-vertex elimination scheme in a chordal bipartite graph can be determined in time $O(n^2m)$, where n is the number of vertices and m is the number of edges of the graph.*

8.2 Triangulations of chordal bipartite graphs

In this section, let $G = (X, Y, E)$ be chordal bipartite. We give a method to triangulate G and prove the correctness. We denote complete bipartite subgraphs as $M = (A, B)$, i.e., the vertex set of this graph M is $A \cup B$ and the edge set $E = \{(a, b) \mid a \in A \wedge b \in B\}$. In this chapter we require by definition that a complete bipartite graph (A, B) is such that $|A| \geq 2$ and $|B| \geq 2$. If $G = (X, Y, E)$ is a bipartite graph, then we call the sets X and Y the color classes of G.

Lemma 8.2.1 *If $G = (X, Y, E)$ is chordal bipartite, then it contains at most $|E|$ maximal complete bipartite subgraphs.*

Proof. G is chordal bipartite, hence there is a perfect edge-without-vertex elimination ordering $(e_1, \ldots, e_k)$. Consider a maximal complete bipartite subgraph, (A, B). Let e_i be the first edge in the ordering which is an edge of (A, B). Let $e_i = (x, y)$ with $x \in A$ and $y \in B$. Since e_i is bisimplicial and (A, B) is maximal we have $A = N(y)$ and $B = N(x)$. Thus the maximal complete bipartite subgraph (A, B) is completely and uniquely determined by e_i. This proves the lemma. □

Remark. It is not difficult to see that there exist chordal bipartite graphs for which the number of maximal complete bipartite subgraphs is $\Omega(n^2)$.

If (A, B) is a complete bipartite graph and H is a triangulation, then either $H[A]$ or $H[B]$ is a complete subgraph of H (otherwise there would be a chordless square). Now let G be chordal bipartite, and let $\mathcal{M}$ be the set of maximal complete bipartite subgraphs (A, B) of G with $|A| \geq 2$ and $|B| \geq 2$. If H is a triangulation of G, then for each $(A, B) \in \mathcal{M}$, either $H[A]$ or $H[B]$ is a complete subgraph of H. Consider the following process. For *each* $(A, B) \in \mathcal{M}$, choose one color class $C \in \{A, B\}$, and add all edges between vertices of C. We say the color class C is completed. The following example shows, that the resulting graph need not be chordal.

Example. Take the $K(4, 4)$ with color classes $A = \{a, b, c, d\}$ and $B = \{p, q, r, s\}$

and delete the edges (p, d) and (a, s). Call this graph G. It is easy to see that G is chordal bipartite (it is even bipartite permutation). For the maximal complete bipartite subgraph $(\{a, b, c, d\}, \{q, r\})$ we choose the color class $\{a, b, c, d\}$ and change this into a clique. For the maximal complete bipartite subgraph $(\{b, c\}, \{p, q, r, s\})$ we choose the color class $\{p, q, r, s\}$ and make this complete. The resulting graph is not triangulated, because there is a chordless square induced by $\{a, d, p, s\}$.

Definition 8.2.1 *Let* $M_1 = (A_1, B_1)$ *and* $M_2 = (A_2, B_2)$ *be two maximal complete bipartite subgraphs of a graph* G. *We say that* M_1 *and* M_2 cross *if either* $A_2 \subset A_1$ *and* $B_1 \subset B_2$, *or* $A_1 \subset A_2$ *and* $B_2 \subset B_1$.

In the example above, the maximal complete bipartite subgraphs $(\{b, c\}, \{p, q, r, s\})$ and $(\{a, b, c, d\}, \{q, r\})$ cross.

Definition 8.2.2 *For each* $M \in \mathcal{M}$ *let* $C(M)$ *be one of the color classes. The set* $\mathcal{C} = \{C(M) \mid M \in \mathcal{M}\}$ *is called* feasible, *if for each pair* $(A_1, B_1), (A_2, B_2) \in \mathcal{M}$ *that cross with* $A_2 \subseteq A_1$ *and* $B_1 \subseteq B_2$, not both A_1 *and* B_2 *are in* $\mathcal{C}$.

We want to prove in this section that if $\mathcal{C}$ is feasible and we complete each $C \in \mathcal{C}$, then the resulting graph is triangulated. Notice that there exists a feasible set of color classes: simply complete for each $M \in \mathcal{M}$ the smallest color class. The following example shows that this needs not be optimal. This (con-

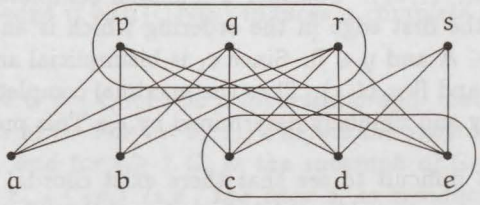

Figure 8.1: feasible set need not be optimal

vex) graph has two maximal complete bipartite subgraphs, $(\{p, q, r, s\}, \{c, d, e\})$ and $(\{p, q, r\}, \{a, b, c, d, e\})$. If we complete for both the smaller color class we obtain a triangulated graph with a maximal clique $\{p, q, r, c, d, e\}$ of six vertices. Another possibility is that we complete $\{p, q, r, s\}$. In this case the largest clique has only five vertices.

Theorem 8.2.1 *Let* $\mathcal{C}$ *be a feasible set of color classes of a chordal bipartite graph* G. *Let* H *be the graph obtained by making each* $C \in \mathcal{C}$ *complete. Then* H *is triangulated.*

Proof. Assume by way of contradiction that G is a *minimal* counterexample. Thus for every induced subgraph the theorem is true. By assumption H is not chordal. Let S be a chordless cycle of length greater than three in H. Let $G = (X, Y, E)$. We call the vertices of X red and the vertices of Y black. An edge of S is called red (black), if *both* its end vertices are red (black). The red and black edges of S are called colored edges. Notice that there must exist at least one colored edge, otherwise the cycle would also be a chordless cycle in G, and since G is chordal bipartite, S must be a square. But a square is contained in some maximal complete bipartite subgraph. Hence at least one of the color classes must have been completed. Let $\mathcal{R}$ be the set of all red edges of S and $\mathcal{B}$ be the set of all black edges of S. Without loss of generality we may assume that $\mathcal{R} \neq \emptyset$.

Consider a red vertex $x \notin S$, and let $e \in \mathcal{B}$ be a black edge of S. We say that x *creates* e, if there is a maximal complete bipartite subgraph (A, B) with $e \subseteq B$, $x \in A$, and $B \in \mathcal{C}$. A red vertex r is called *redundant*, if $r \notin S$ and it does not create any black edge. A black vertex $b \notin S$ is called redundant, if it does not create a red edge.

Claim *There are no redundant vertices.*

Proof of claim. Suppose there is a redundant red vertex x. Consider the induced subgraph $G' = G[V \setminus \{x\}]$. Clearly, also G' is chordal bipartite. We create a feasible set of color classes $\mathcal{C}'$ as follows. Consider a maximal complete bipartite subgraph $M = (A, B)$ of G'. If M is also maximal in G, we put $C(M)$ in $\mathcal{C}'$. If M is not maximal in G then $N = (A \cup \{x\}, B)$ must be a maximal complete bipartite subgraph of G. In this case we put $C(N) \setminus \{x\}$ in $\mathcal{C}'$. It is easy to see that $\mathcal{C}'$ is a feasible set of color classes for G'.

Let H' be the graph obtained from G' by completing each set of $\mathcal{C}'$. We claim that S is also a chordless cycle in H'.

Let e be a colored red edge of S. There is a maximal complete bipartite subgraph (A, B), with $e \subseteq A$ and $A \in \mathcal{C}$. If $x \notin A$, then (A, B) is a maximal complete bipartite subgraph of G'. Hence $A \in \mathcal{C}'$. Now assume $x \in A$. Then $|A| \geq 3$. Take B' such that $B \subseteq B'$ and $(A \setminus \{x\}, B')$ is a maximal complete bipartite subgraph of G' (i.e. B' is the set of common neighbors of $A \setminus \{x\}$). If (A, B') is maximal in G, then $B' = B$ and $A \setminus \{x\} \in \mathcal{C}'$. Otherwise $(A \setminus \{x\}, B')$ is maximal in G. Notice that (A, B) and $(A \setminus \{x\}, B')$ cross. Hence $A \setminus \{x\}$ is in $\mathcal{C}$ and hence also in $\mathcal{C}'$.

Now let e be a black edge, and let (A, B) be a maximal complete bipartite subgraph with $e \subseteq B$ and $B \in \mathcal{C}$. Notice that since x is redundant, $x \notin A$. Hence (A, B) is also maximal in G' and $B \in \mathcal{C}'$. This shows that S is also a cycle in G'.

Since G was a minimal counterexample, there must be a chord in G' between two vertices p and q of S. Without loss of generality assume p and q are both red (the case where they are both black is similar). There exists a maximal complete bipartite subgraph (A, B) in G' such that $p, q \in A$ and $A \in \mathcal{C}'$. By definition

either $A \cup \{x\} \in \mathcal{C}$ or $A \in \mathcal{C}$. Hence (p, q) is also an edge in G. This proves that there are no redundant red vertices. A similar argument show that there are no redundant black vertices either. □

Consider the graph $G^* = \text{split}(G)$ obtained by making Y complete. Since G is chordal bipartite, we know that G^* is strongly chordal. Hence we know that G^* has a simple vertex. Let s be the simple vertex. The next lemma shows that s must be a red vertex.

Claim *A simple vertex* s *of* G^* *is red.*

Proof of claim. Assume $s \in Y$ is simple. If s is not an element of the cycle, it must be in some maximal complete bipartite subgraph (A, B) creating some edge, otherwise s would be redundant. But then it has at least two nonadjacent neighbors, which is a contradiction. Assume s is a black vertex of the cycle S. Clearly s can not have two red neighbors, because red vertices are not adjacent in G. Hence s is incident with at least one edge of $\mathcal{B}$. Consider a maximal path of S, which contains only black vertices, and which contains s. Let p and q, be the end vertices of this path. We know that $p \neq q$. Since s is simple, and p and q are neighbors of s in G^*, either $N[p] \subseteq N[q]$ or $N[q] \subseteq N[p]$. Since p and q are both incident with a red vertex in S, it follows that S contains only one red vertex. This is a contradiction, because we assumed that $\mathcal{R} \neq \emptyset$. □

Claim *A simple vertex* s *is an element of* S.

Proof of claim. We know s is red. Assume s is not in S. Then we know that s creates some black edge $e = (p, q)$. Without loss of generality assume $N[p] \subseteq N[q]$. If p has a red neighbor in S, then this would also be a neighbor of q, and there would be a chord in S. Hence p has another black neighbor r in S. Consider the maximal complete bipartite subgraph (K, L) with $p, r \in L$ and $L \in \mathcal{C}$. Then p is adjacent to every vertex of K. Hence also q is adjacent to every vertex of K. It follows that $q \in L$. But $L \in \mathcal{C}$. Hence q and r are adjacent in H. This is a chord in S. □

We have to consider three more cases. First we show that s is incident with at least one edge of $\mathcal{R}$.

Claim *A simple vertex* s *is incident with at least one red edge of* S.

Proof of claim. We know s is a red vertex of S. Assume it has two black neighbors in S, x and y. We may assume $N[x] \subseteq N[y]$. If in the cycle x has two red neighbors, then also y is adjacent to both red neighbors, which is a contradiction. Hence x is adjacent to a black vertex z in S. Consider a maximal complete bipartite subgraph (K, L) creating the edge (x, z). Then x is adjacent to every vertex of K, and hence

this also holds for y. But then $y \in L$. $L \in \mathcal{C}$ hence x and y are adjacent in H. This is a chord in S. □

Claim *A simple vertex s is incident with exactly one red edge of S.*

Proof of claim. We know that $s \in S$ and s has at least one red neighbor in S. Assume s has *two* red neighbors x and y in S. Consider the maximal complete bipartite subgraphs (K_x, L_x) creating the edge (s, x) and (K_y, L_y) creating (s, y). Now there must exist $p \in L_x \setminus L_y$ which is not adjacent to y, otherwise $(K_x \cup \{y\}, L_x)$ is complete bipartite. Also there must exist $q \in L_y \setminus L_x$ which is not adjacent to x. Without loss of generality we may assume $N[p] \subseteq N[q]$. But then we have a contradiction since $x \in N[p] \subseteq N[q]$. □

Hence we know that a simple vertex s is a red vertex of the cycle S with one red neighbor x and one black neighbor y. Let (K_x, L_x) be the maximal complete bipartite subgraph that creates (s, x). Since s is simple and y is not adjacent to x, we must have $N[y] \cap X \subseteq K_x$. Now assume that in the cycle y is adjacent to another red vertex u. Then $u \in N[y] \cap X \subseteq K_x$. But then u, x and s would form a triangle, which is a contradiction. Hence y is adjacent to a black vertex v in the cycle. Let (K_v, L_v) be the maximal complete bipartite subgraph creating (y, v). Notice that $K_v \subseteq N[y] \cap X \subseteq K_x$. It follows that $L_x \subseteq L_v$, and hence (K_x, L_x) and (K_v, L_v) cross. But L_v and K_x are completed, hence $\mathcal{C}$ is not feasible. This completes the proof of Theorem 8.2.1. □

In the next section we show that there exists a feasible set of color classes of G such that the triangulated graph H is optimal, i.e. has the smallest possible maximum clique size.

8.3 Optimal triangulations of chordal bipartite graphs

Let $G = (X, Y, E)$ be a chordal bipartite graph. If $\mathcal{C}$ is a feasible set of color classes, we denote by $H_{\mathcal{C}}$ the chordal graph obtained from G by completing each $C \in \mathcal{C}$. In this section we show the following. There is a feasible set of color classes $\mathcal{C}$, such that the triangulation $H_{\mathcal{C}}$ minimizes the clique size.

Theorem 8.3.1 *Let G be a chordal bipartite graph with treewidth $\leq k$. Then there exists a feasible set of color classes $\mathcal{C}$ such that $H_{\mathcal{C}}$ has clique size $\leq k+1$.*

Proof. Let H be any triangulation of G. We show that there is a feasible set of color classes $\mathcal{C}$ such that $H_{\mathcal{C}}$ is a subgraph of H. Let $\mathcal{M}$ be the set of all maximal complete bipartite subgraphs of G. Suppose we list the elements of $\mathcal{M}$ one by

one, creating a feasible set of color classes $\mathcal{C}$ as follows. Start with $\mathcal{C} = \emptyset$. For each $M \in \mathcal{M}$ at least one of the color classes is a complete subgraph in H. Let $M = (A, B)$ and assume A is complete in H. Check the list $\mathcal{C}$ created thus far, if there is a maximal complete subgraph (C, D) which crosses with (A, B), such that $C \subseteq A$ and $D \in \mathcal{C}$. If not, then put A in $\mathcal{C}$. Otherwise put B in $\mathcal{C}$. Notice that there cannot be a maximal complete bipartite subgraph (K, L) crossing with (A, B), such that $L \subseteq B$ and $K \in \mathcal{C}$. Otherwise (K, L) and (C, D) would also cross with $C \subseteq K$ and $K, D \in \mathcal{C}$. This shows that in this way we create a feasible set of color classes. Now notice that $H_{\mathcal{C}}$ is a subgraph of H, hence the clique size of $H_{\mathcal{C}}$ does not exceed the clique size of H. □

8.4 Cliques in optimal triangulations of chordal bipartite graphs

Let $G = (X, Y, E)$ be a chordal bipartite graph and let $\mathcal{C}$ be a feasible set of color classes. In this section we analyze the structure of the maximal cliques in $H_{\mathcal{C}}$. We first look at a clique B with vertices only in one color class.

Definition 8.4.1 *Let B be a clique with all vertices in Y. We say that a set $\mathcal{A}$ of maximal complete bipartite subgraphs is a* cover *for B if the following two conditions are satisfied:*

1. *for each $(A', B') \in \mathcal{A}$: $B' \in \mathcal{C}$, and*
2. *for every pair $x, y \in B$ there is a maximal complete bipartite subgraph $(A', B') \in \mathcal{A}$ such that $x, y \in B'$.*

Theorem 8.4.1 *Let B be a clique of $H_{\mathcal{C}}$ with all vertices in Y. Assume $|B| \geq 2$. A minimal cover for B has only one element.*

Proof. Let $\mathcal{A} = \{(A_1, B_1), \ldots (A_t, B_t)\}$ be a minimal cover for B. Assume $t \geq 2$. Let $A = \bigcup_i A_i$.

Claim *Every vertex of A has at least two neighbors in B.*

Proof of claim. Assume there is a vertex $a \in A$ with only one neighbor b in B. There is a complete bipartite subgraph (A_i, B_i) containing a. We claim that B is covered by $\mathcal{A} \setminus \{(A_i, B_i)\}$. For every other vertex $b' \in B$ the edge (b, b') must be in some (A_j, B_j) with $j \neq i$. This proves the lemma. □

Claim *Every vertex of B has at least two neighbors in A.*

Proof of claim. Let $b \in B$. Take another vertex $b' \in B$. The edge (b, b') must be in some maximal complete bipartite subgraph (A_i, B_i). □

Take the subgraph H of G induced by $A \cup B$. Let $W = \text{split}(H)$ obtained by making A a complete graph. W is strongly chordal. Each vertex of A has two nonadjacent neighbors, hence A cannot contain a simple vertex. Let $b \in B$ be a simple vertex. Let $a_1, \ldots, a_k$ be the neighbors of b in A, and let $N[a_1] \subseteq N[a_2] \subseteq \ldots \subseteq N[a_k]$. Notice that $k \geq 2$. Since A is a clique in W, we have that $N[a_1] \cap B \subseteq \ldots \subseteq N[a_k] \cap B$. We claim that not all a_i's can be contained in one color class A_i.

Claim *None of the color classes* A_i *contains* all *neighbors* $a_1, \ldots, a_k$ *of* b *in* A.

Proof of claim. Assume that $\{a_1, \ldots, a_k\} \subseteq A_i$ for some color class A_i. Take $b' \notin B_i$. There exists a maximal complete bipartite subgraph (A_j, B_j) such that $\{b, b'\} \subseteq B_j$. Clearly $A_j \subseteq \{a_1, \ldots, a_k\} \subseteq A_i$. This implies $B_i \subseteq B_j$. This is a contradiction since $\mathcal{A}$ is minimal. □

Now we can finish the proof of Theorem 8.4.1. Let (A_1, B_1) be a maximal complete bipartite subgraph containing a_1. Let $i > 1$. Notice $B_1 \subseteq N[a_1] \cap B \subseteq N[a_i] \cap B$. Hence $(A_1 \cup \{a_i\}, B_1)$ is a complete bipartite subgraph. This implies that $a_i \in A_1$ for all $1 \leq i \leq k$, which is a contradiction. This proves Theorem 8.4.1. □

The following theorem shows that the structure of the maximal cliques in H_C is very simple.

Theorem 8.4.2 *Let* K *be a maximal clique in* H_C *with* $|K| > 2$. *Let* $K_x = K \cap X$ *and* $K_y = K \cap Y$. *Assume* $|K_x| \geq 2$. *Then one of the following two cases holds:*

1. $|K_y| = 1$ *and there exists a maximal complete bipartite subgraph* (A, B) *such that* $K_x = A$, $y \in B$ *and* $A \in \mathcal{C}$, *or*

2. $|K_y| > 1$ *and there exist maximal complete bipartite subgraphs* (A_1, B_1) *and* (A_2, B_2), *with* $A_1 \in \mathcal{C}$ *and* $B_2 \in \mathcal{C}$ *such that* $K_x \subseteq A_1$ *and* $K_y \subseteq B_2$.

Proof. By Theorem 8.4.1 there exists a maximal complete bipartite subgraph (A_1, B_1) such that $K_x \subseteq A_1$ and $A_1 \in \mathcal{C}$. Assume $K_y = \emptyset$. Then clearly, K cannot be maximal, since for any vertex $y \in B_1$, $A_1 \cup \{y\}$ is a clique. Assume $|K_y| = 1$. Let $K_y = \{y\}$. Notice that $(K_x, \{y\})$ is contained in a complete bipartite subgraph $(K_x, B_1 \cup \{y\})$ in G. Hence $(K_x, B_1 \cup \{y\})$ is contained in a maximal complete bipartite subgraph (A_2, B_2) in G. Since $B_1 \subseteq B_2$, we have $A_2 \subseteq A_1$. Since $\mathcal{C}$ is feasible, and $A_1 \in \mathcal{C}$, we must have $A_2 \in \mathcal{C}$. Hence the first case holds. Now assume $|K_y| \geq 2$. By Theorem 8.4.1 there exists a maximal complete bipartite subgraph (A_2, B_2) with $K_y \subseteq B_2$ and $B_2 \in \mathcal{C}$. □

8.5 An algorithm for the treewidth of chordal bipartite graphs

Let $G = (X, Y, E)$ be a chordal bipartite graph. In this section we show a polynomial time algorithm which outputs either a valid statement that the treewidth of G exceeds k, or a triangulation of G such that the maximum clique size does not exceed $k+1$. We first describe the algorithm. In the method we use, we construct a digraph with vertex set $\mathcal{M}$. We direct an edge from $M_1 = (A_1, B_1)$ to $M_2 = (A_2, B_2)$, with $A_1, A_2 \in X$, if $A_1 \in \mathcal{C}$ then necessarily also $A_2 \in \mathcal{C}$. Next we color some of the vertices of this graph red or black. A vertex (A, B) is colored red if necessarily $A \in \mathcal{C}$ and black if necessarily $B \in \mathcal{C}$. In the next step we try to extend this coloring to the whole graph.

step 1 If $k \leq 3$ then use the algorithm described in [96] to either decide that the treewidth of G exceeds k or, to find a suitable triangulation. If $k > 3$ perform the following steps.

step 2 First make a list $\mathcal{M}$ of all maximal complete bipartite subgraphs of G.

step 3 Make a directed graph W with vertex set $\mathcal{M}$ as follows. Let $M_1 = (A_1, B_1)$ and $M_2 = (A_2, B_2)$ be two different elements of $\mathcal{M}$. We direct an edge from M_1 to M_2 if one of the following cases holds:

- Compute the maximal number of vertices of a maximal complete bipartite subgraph in the induced subgraph $G[A_1 \cup B_2]$. (M_1, M_2) is a directed edge if this size exceeds $k+1$.
- If M_1 and M_2 cross, with $A_2 \subseteq A_1$ then (M_1, M_2) is a directed edge.

step 4 Color some of the vertices of W as follows. If $M = (A, B)$ is a maximal complete bipartite subgraph with $|A| > k$ and $|B| > k$ then output that the treewidth of G exceeds k. Otherwise, if $|B| > k$, then we color M red and if $|A| > k$, then we color M black. In case $|A| \leq k$ and $|B| \leq k$, then we do not color M yet.

step 5 While there is some arc (M_1, M_2) with M_1 colored red and M_2 not colored red then consider the following cases:

- If M_2 is black then output that the treewidth of G exceeds k
- If M_2 is not colored, then color M_2 red.

step 6 All vertices which do not yet have a color, are colored black.

step 7 For all elements $M = (A, B)$ of $\mathcal{M}$: If the color of M is red, then complete A, and if the color of M is black then complete B.

Theorem 8.5.1 *Let G be chordal bipartite and let k be an integer. The treewidth of $G \leq k$ if and only if the algorithm produces a triangulation with maximum clique size at most $k+1$.*

Proof. Clearly, the theorem holds when $k \leq 3$. Assume $k > 3$. Clearly, if the algorithm does not produces a triangulation, then some necessary condition is not satisfied and hence the treewidth is more than k. Assume each $M \in \mathcal{M}$ is colored. Consider the set $\mathcal{C}$ of color classes, defined as follows. If $M = (A, B)$ is red, then $A \in \mathcal{C}$, otherwise $B \in \mathcal{C}$. It is easily checked that $\mathcal{C}$ is a feasible set of color classes. It follows from Theorem 8.2.1 that the algorithm produces a triangulation H. Assume H has a maximal clique K with more than $k+1$ vertices. By Theorem 8.4.2 there are two cases to consider. First consider the case $|K_y| = 1$, let $K_y = \{y\}$. Then $|K_x| > k$. There is a maximal complete bipartite subgraph $M = (A, B)$, with $K_x \subseteq A \in \mathcal{C}$, and $y \in B$. However, the algorithm can not color M red (step 4) hence A cannot be in $\mathcal{C}$. A similar argument shows that $|K_x| = 1$ is also not possible. Now assume $|K_x| \geq 2$ and $|K_y| \geq 2$. By Theorem 8.4.2, there exist complete maximal bipartite subgraphs, $M_1 = (A_1, B_1)$ and $M_2 = (A_2, B_2)$, such that $K_x \subseteq A_1$ and $K_y \subseteq B_2$, with $A_1 \in \mathcal{C}$ and $B_2 \in \mathcal{C}$. This means that M_1 is colored red and M_2 is colored black. But (K_x, K_y) is a complete bipartite subgraph in the induced subgraph $G[A_1 \cup B_2]$, hence there is an arc (M_1, M_2) (step 3). This is a contradiction. □

In the last part of this section we analyze the running time of the algorithm and we show it is polynomial. Consider the time it takes to find all maximal complete bipartite subgraphs. Corollary 8.1.1 shows we can find a perfect edge-without-vertex elimination ordering in time $O(n^2m)$. Each bisimplicial edge gives a complete bipartite subgraph. It follows that we can find a list with at most m complete bipartite subgraph in time $O(m^2)$ from the scheme. From this list we can get the maximal complete bipartite subgraphs in time $O(m^2n)$. It follows that step 2 can be performed in $O(m^2n)$. Now clearly, the time complexity is dominated by step 3 of the algorithm. For each pair $M_1 = (A_1, B_1)$ and $M_2 = (A_2, B_2)$ we have to find the maximum cardinality of a complete bipartite subgraph in $G[A_1 \cup B_2]$. We can do this as follows. First we compute for each pair (A_1, B_1) and (A_2, B_2) the number of vertices in $A_1 \cap A_2$ and in $B_1 \cap B_2$. Then for each $M \in \mathcal{M}$ we can compute the number of vertices in $A_1 \cup B_2$ in time $O(n \log n)$. It follows that for each pair M_1, M_2 we can decide if there is an arc (M_1, M_2) in time $O(m)$ (checking whether they cross can be done in $O(n \log n)$). It follows that step 3 of the algorithm can be performed in $O(m^3)$. This is also the total time complexity of the algorithm which can be easily checked. This proves the following theorem.

Theorem 8.5.2 *Let G be chordal bipartite and let k be an integer. Then there is a polynomial time algorithm to decide whether the treewidth of G is at most k. If so, the algorithm returns a triangulation of G with clique size at most $k+1$.*

Chapter 9

Treewidth and pathwidth of permutation graphs

We showed in section 7.4 that the pathwidth of a permutation graph is at most two times the treewidth of that graph, and we gave a linear time algorithm which produces a path-decomposition of width within this bound. In this chapter, we show that for permutation graphs, the treewidth and pathwidth are in fact equal, and we give an algorithm which computes the (exact) treewidth. Our algorithm to decide whether the treewidth (pathwidth) is at most some given integer k, can be implemented to run in $O(nk)$ time, when the matching diagram is given. We show that this algorithm can easily be adapted to compute the treewidth of a permutation graph in $O(nk)$ time, where k is the actual treewidth.

Permutation graphs have many applications in scheduling problems. See for example [46] where permutation graphs are used to describe the memory requirements of a number of programs at a certain time (see also [58]). Permutation graphs also arise in a very natural way in the problem of sorting a permutation, using queues in parallel. In [58] it is shown that this problem is closely related to the coloring problem of permutation graphs. Other applications occur for example in VLSI layout (see e.g. [120]). There is a long list of papers, which mainly appeared in the last ten years, studying the algorithmic complexity of NP-hard graph problems when restricted to permutation graphs. Indeed, there exist fast algorithms for many NP-hard problems like MAX WEIGHT CLIQUE, MAX WEIGHT INDEPENDENT SET, FEEDBACK VERTEX SET and DOMINATING SET when restricted to permutation graphs [58, 48, 32, 33, 35]. However some problems remain NP-hard, like COCHROMATIC NUMBER [130, 56], and ACHROMATIC NUMBER [16].

Some of the results of this chapter for permutation graphs can be generalized to cocomparability graphs. We showed in chapter 7.5 that, if the treewidth of a cocomparability graph G is at most k, then the pathwidth is at most $O(k^2)$, and we gave a simple algorithm which finds a path-decomposition with this width. The results for permutation graphs presented in this chapter, i.e., the algorithm to compute the treewidth of a permutation graph and the proof that the treewidth

and pathwidth are equal for permutation graphs, can be generalized to cocomparability graphs as follows.

1. The treewidth and pathwidth of *any* cocomparability graph are equal. Independently, this was also remarked in [61], and only very recently, R. H. Möhring reported to us that for graphs containing no astroidal triple, the treewidth and pathwidth are equal as well.

2. Using the results of [59], the treewidth can be computed in polynomial time for cocomparability graphs of which the partial order dimension of the complement is bounded by some constant.

3. In [61] it is shown that the results of this chapter can be extended to give an approximation for the treewidth of cocomparability graphs, of which the performance ratio depends on the partial order dimension of the complement.

Finally, we want to remark that the problem to determine the treewidth of cocomparability graphs in general, is NP-hard (this is also mentioned in [61]). In fact, it follows from [4], that finding the treewidth of complements of bipartite graphs is NP-hard.

9.1 Preliminaries

In this section we start with some definitions and easy lemmas. For more information on classes of perfect graphs the reader is referred to [58, 31, 14]. Recall the definition of an interval graph. There are many ways to characterize interval graphs. We restate the characterization of Lekkerkerker and Boland given in Lemma 2.1.8 (page 11).

Lemma 9.1.1 *An undirected graph is an interval graph if and only if the following two conditions are satisfied:*

1. G *is triangulated, and*

2. *every three vertices of* G *can be ordered in such a way that every path from the first to the third vertex passes through a neighbor of the second vertex.*

Three vertices which do not satisfy the second condition are called an *astroidal triple* (see Definition 2.1.7). These vertices are pairwise non-adjacent and for any pair of them there is a path that avoids the neighborhood of the remaining vertex. Let k be an integer. Recall that a graph G has pathwidth $\leq k$ if and only if there is a triangulation H of G such that H is an interval graph with $\omega(H) \leq k+1$.

In this chapter we show that the treewidth and pathwidth of a permutation graph can be computed in polynomial time. Recall Definition 7.1.3 on page 77

of a permutation graph. In this chapter we assume that the permutation is given and we identify the permutation graph with the inversion graph. Recall that a permutation graph is an intersection graph, which is illustrated by the matching diagram (Figure 7.1, on page 78).

9.2 Separators obtained from scanlines

In this section we show that every minimal separator in a permutation graph can be obtained by using a *scanline*. Recall the definition of the matching diagram (Definition 7.1.4, page 78). (We call a matching diagram a diagram for short). It consists of two horizontal lines, one above the other, and a number of straight-line segments, one for each vertex, such that each line segment has one end vertex on each horizontal line. Two vertices are adjacent, if the corresponding line segments intersect. We say that two line segments *cross* if they have a nonempty intersection.

Definition 9.2.1 *A* scanline *in the diagram is any line segment with one end vertex on each horizontal line. A scanline* s *is* between *two non crossing line segments* x *and* y *if the top point of* s *is in the open interval bordered by the top points of* x *and* y *and the bottom point of* s *is in the open interval bordered by the bottom points of* x *and* y.

If a scanline s is between line segments x and y then the intersection of each pair of the three line segments is empty. Consider two nonadjacent vertices x and y. The line segments in the diagram corresponding to x and y do not cross in the diagram. Hence we can find a scanline s between the lines x and y. Take out all the lines that cross the scanline s. Clearly this corresponds with an x, y-separator in the graph. The next lemma shows that we can find all minimal x, y-separators in this way.

Lemma 9.2.1 *Let* G *be a permutation graph, and let* x *and* y *be nonadjacent vertices in* G. *Every minimal* x, y*-separator consists of all line segments crossing a scanline which lies between the line segments of* x *and* y.

Proof. Let S be a minimal x, y-separator. Consider the connected components of $G[V - S]$. Let C_x be the component containing x and C_y be the component containing y. Clearly these must also be 'connected' parts in the diagram, and we may assume without loss of generality that the component containing x is completely to the left of the component containing y. Every vertex of S is adjacent to some vertex in C_x and to some vertex in C_y (Lemma 2.1.3, page 9). Notice that we can choose a scanline s crossing no line segment of $G[V - S]$, and which is between x and y. Then all lines crossing the scanline must be elements of S. But for all elements of S the corresponding line segment must cross s, since it is

intersecting with a line segment of C_x, which is to the left of s, and with a line segment of C_y, which is to the right of s. □

If s is a scanline, then we denote by S the set of vertices of which the corresponding line segments cross s. In the rest of this chapter we consider only scanlines of which the end points do not coincide with end points of other line segments.

Definition 9.2.2 *Two scanlines* s_1 *and* s_2 *are* equivalent, *denoted as* $s_1 \equiv s_2$, *if they have the same position in the diagram relative to every line segment; i.e. the set of line segments with the top (or bottom) end point to the left of the top (or bottom) end point of the scanline is the same for* s_1 *and* s_2.

Notice that $S_1 = S_2$ does not necessarily imply that $s_1 \equiv s_2$ (the converse implication of course holds).

Corollary 9.2.1 *There are* $O(n^2)$ *minimal separators in a permutation graph with* n *vertices.*

We are only interested in scanlines which do not cross too many line segments, since these correspond with suitable separators.

Definition 9.2.3 *A scanline* s *is* k-small *if it crosses with at most* $k+1$ *line segments.*

Lemma 9.2.2 *There are* $O(nk)$ *pairwise non-equivalent* k*-small scanlines.*

Proof. Consider the matching diagram, with numbers $1, \ldots, n$ written from left to right and underneath written $\pi_1, \ldots, \pi_n$. Consider a scanline t and assume the top end point is between i and $i+1$ and the bottom end point is between π_j and π_{j+1}. Assume that α line segments are such that the top end point is to the left of the top of t and the bottom end point to the left of the bottom of t. Then the number of line segments crossing t is $i + j - 2\alpha$. Since $\alpha \leq i$ and $\alpha \leq j$, it follows that $i - k - 1 \leq j \leq i + k + 1$ must hold. This proves the lemma. □

9.3 Treewidth and pathwidth of permutation graphs are equal

In this section we show that a permutation graph can be triangulated optimally such that the result is an interval graph. Recall the definition of a minimal triangulation of a graph (Definition 2.1.11, page 15).

Theorem 9.3.1 *Let* G *be a permutation graph, and let* H *be a minimal triangulation of* G. *Then* H *is an interval graph.*

Proof. Assume H has an astroidal triple x, y, z. Since x, y and z are pairwise nonadjacent, the corresponding line segments in the matching diagram pairwise do not cross. We may assume without loss of generality that the line segment of y is between those of x and z. Take a path p between x and z which avoids the neighborhood of y. Then each line of the path lies totally to the left or totally to the right of y. It follows that there are x' to the left of y and z' to the right of y such that x' and z' are adjacent in H, but neither x' or z' is a neighbor of y in H. Let S be a minimal x', y-separator in H. Since H is a minimal triangulation, S is also a minimal x', y-separator in G. By Lemma 9.2.1 S consists of all lines crossing some scanline s between x' and y. Clearly, the connected component of $G[V - S]$ containing x' lies totally to the left of s and the connected component containing z' in $G[V - S]$ lies totally to the right of s (notice $z' \notin S$ since z' lies totally to the right of y). It follows that x' and z' must be in different components of $G[V - S]$. Since H is minimal, by Theorem 2.1.1 they must also be in different components of $H[V - S]$. But then x' and z' can not be adjacent in H. It follows that there can not be an astroidal triple, and by the characterization of Lekkerkerker and Boland ([92], stated in Lemma 9.1.1), H is an interval graph. □

Corollary 9.3.1 *For any permutation graph* G, *the pathwidth of* G *is equal to the treewidth of* G.

9.4 Candidate components

A graph with at least $k + 2$ vertices has treewidth at most k if and only if there is a minimal vertex separator with at most k vertices such that all components with S added as a clique have treewidth at most k. We can find a minimal separator using a scanline. The main result of this section is that we can change the diagram of a component in such a way that that the separator is changed into a clique.

Consider the matching diagram of G.

Definition 9.4.1 *Let* s_1 *and* s_2 *be two scanlines of which the intersection is either empty or one of the end points of* s_1 *and* s_2. *A* candidate component $C = \mathcal{C}(s_1, s_2)$ *is a subgraph of* G *induced by the following sets of lines:*

- *all lines that are* between *the scanlines (in case the scanlines have a common end point, this set is empty), and*
- *all lines crossing at least one of the scanlines.*

We identify the candidate component $C = \mathcal{C}(s_1, s_2)$ with the diagram containing s_1, s_2 and the set of lines corresponding with vertices of C.

Definition 9.4.2 *Let* k *be an integer. A candidate component* $C = \mathcal{C}(s_1, s_2)$ *is* k-feasible *if there is a triangulation* H *of* C *such that* $\omega(H) \leq k+1$ *and such that for each scanline* s_i $(i = 1, 2)$ *the set of lines crossing this scanline forms a clique in* H.

Notice that, if a candidate component has at most $k+1$ vertices, then it is k-feasible.

Definition 9.4.3 *Let* $C = \mathcal{C}(s_1, s_2)$ *be a candidate component. We define the* realizer $R(C)$ *as the graph obtained from* C *by adding all edges between vertices of* S_1 *and between vertices of* S_2 *(i.e., the two subgraphs of* $R(C)$ *induced by* S_1 *and by* S_2 *are cliques).*

A candidate component $C = \mathcal{C}(s_1, s_2)$ is k-feasible if and only if the realizer $R(C)$ has treewidth at most k.

Lemma 9.4.1 *If* $C = \mathcal{C}(s_1, s_2)$ *is a candidate component, then the realizer* $R(C)$ *is a permutation graph.*

Proof. Consider the matching diagram. Assume s_1 is to the left of s_2. First consider lines that cross only s_1 and with top end point to the right of the top end point of s_1. Let $(a_1, b_1), \ldots, (a_r, b_r)$ be these line segments with top end points $a_1, \ldots, a_r$. Assume $a_1 < a_2 < \ldots < a_r$. Change the order of $b_1, \ldots, b_r$ such that $b_1 > b_2 > \ldots > b_r$. This is illustrated in figure 9.1. Now consider the line segments crossing s_1 of which the top end point is to the left of the top end point of s_1. Reorder in the same way the *top end points* of these line segments. The lines crossing s_2 are handled similarly. The resulting diagram is a matching diagram for $R(C)$. □

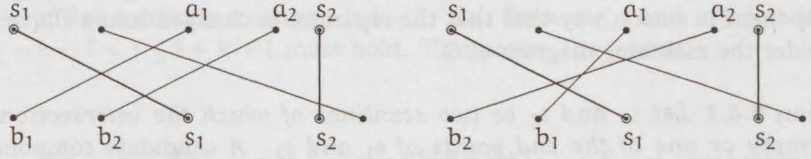

Figure 9.1: diagrams of candidate component and realizer

Remark. Let $C = \mathcal{C}(s_1, s_2)$ be a candidate component, such that the scanlines s_1 and s_2 have an end point in common. In this case the graph $R(C)$ is chordal and therefore, there is an efficient algorithm to check whether the candidate component is k-feasible.

Let $C = \mathcal{C}(s_1, s_2)$ be a candidate component such that C has at least $k+2$ vertices. The realizer R(C) has treewidth at most k if and only if there is a minimal vertex separator S with at most k vertices such that all components, with S added as a clique, have treewidth at most k (see Lemma 2.1.15 on page 16). Consider the diagram of R(C), obtained from the diagram of C by the method described in the proof of Lemma 9.4.1. By Lemma 9.2.1 a minimal separator can be found by a scanline. Let H be a minimal triangulation of R(C) and let the scanline s represent a minimal vertex separator in H for non-adjacent vertices a and b. The separator consists exactly of the lines crossing this scanline s.

Definition 9.4.4 *Let* $C = \mathcal{C}(s_1, s_2)$ *be a candidate component with realizer* R(C). *A scanline* t *is* nice *if the top point of* t *is in the closed interval between the top points of* s_1 *and* s_2, *and the bottom point of* t *is in the closed interval between the bottom points of* s_1 *and* s_2.

Lemma 9.4.2 *There is a scanline* $s^* \equiv s$ *such that* s^* *is nice.*

Proof. Consider the diagram of R(C) with the scanlines s_1, s_2 and s. Without loss of generality, we assume s_1 is to the left of s_2. s separates non adjacent vertices a and b. Let the line segment of a be to the left of the line segment of b. The scanline s lies between the line segments of a and b. Assume s is not nice. Without loss of generality assume it crosses s_1. Then $a \in S_1$ and $b \notin S_1$ (since a and b are not adjacent). (see the left diagram in figure 9.2). Let s^* be the line segment with top point the top of s_1 and bottom point the bottom of s. We want to proof that $s \equiv s^*$. This clearly is the case if there is no line segment of which the top point is between the top points of s and s_1. Assume there is such a vertex p (see the right diagram in figure 9.2). Notice that, since p and a both cross s_1 they are adjacent.

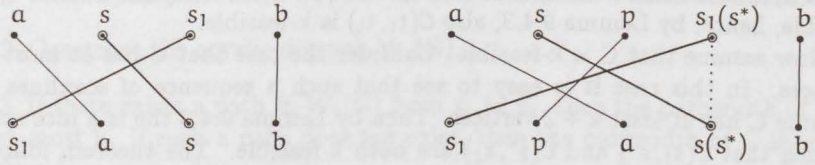

Figure 9.2: there is an equivalent nice scanline

We claim that $S^* \subset S$. Let x be a line segment crossing s^*. If the bottom end of x is to the left of the bottom end of s^*, then the segment x clearly also crosses s. Assume the bottom vertex of x is to the right of the bottom vertex of s^*. Then the line segment also crosses s_1. But then x and a are adjacent, hence the top vertex of x must be to the left of the top vertex of a. This implies that x also crosses s. Clearly, S^* is an a, b-separator in R(C). Since $p \in S \setminus S^*$, S cannot be a minimal

a, b-separator in $R(C)$, and since H is a minimal triangulation of $R(C)$, it cannot be a minimal a, b-separator in H (Theorem 2.1.1). This is a contradiction. □

This proves that all lines crossing the scanline s are in C, and the next lemma follows.

Lemma 9.4.3 *Let $C = \mathcal{C}(s_1, s_2)$ be a candidate component with at least $k+2$ vertices. Then C is k-feasible if and only if there is a nice scanline s such that the two candidate components $C_1 = \mathcal{C}(s_1, s)$ and $C_2 = \mathcal{C}(s, s_2)$ are both smaller than C and are both k-feasible.*

Definition 9.4.5 *Two scanlines s_1 and s_2 are* neighbors *if they have one endpoint in common and in the interval determined by the other endpoints, lies exactly one endpoint of a line segment.*

We are now ready to proof our main theorem.

Theorem 9.4.1 *Let $C = \mathcal{C}(s_1, s_2)$ be a candidate components with s_2 to the right of s_1. Then C is k-feasible if and only if there exists a sequence of scanlines $s_1 = t_1, t_2, \ldots, t_r = s_2$ such that the following conditions are satisfied:*

- *the scanlines t_i and t_{i+1} are neighbors for $i = 1, \ldots, r-1$, and one end point of t_{i+1} lies to the right of the end point of t_i.*
- *Each $\mathcal{C}(t_i, t_{i+1})$ has at most $k+1$ vertices ($i = 1, \ldots, r-1$).*

Proof. First assume such a sequence exists. Let $t_1, \ldots t_r$ be the sequence of scanlines. If C has at most $k+1$ vertices, then C is k-feasible. Hence we may assume that C has at least $k+2$ vertices. Then $r \geq 3$. By induction we show that $\mathcal{C}(t_1, t_i)$ is k-feasible for $i = 2, \ldots r$. If $i = 2$ then $\mathcal{C}(t_1, t_2)$ has at most $k+1$ vertices and hence it is k-feasible. Assume $\mathcal{C}(t_1, t_{i-1})$ is k-feasible and $\mathcal{C}(t_1, t_i) \neq \mathcal{C}(t_1, t_{i-1})$. Then t_{i-1} is a nice scanline in $\mathcal{C}(t_1, t_i)$. $\mathcal{C}(t_1, t_{i-1})$ and $\mathcal{C}(t_{i-1}, t_i)$ are both k-feasible, hence, by Lemma 9.4.3, also $\mathcal{C}(t_1, t_i)$ is k-feasible.

Now assume that C is k-feasible. Consider the case that C has at most $k+1$ vertices. In this case it is easy to see that such a sequence of scanlines exist. Assume C has at least $k+2$ vertices. Then by Lemma 9.4.3 the is a nice scanline s^* such that $\mathcal{C}(s_1, s^*)$ and $\mathcal{C}(s^*, s_2)$ are both k-feasible. The theorem follows by induction on the number of vertices of C. □

9.5 Algorithm for the treewidth of a permutation graph

In this section we show how to compute the treewidth (and pathwidth) of a permutation graph G. Let k be an integer. The algorithm we present checks whether the treewidth of the permutation graph does not exceed k.

We use a directed acyclic graph $W_k(G)$ as follows. The vertices of the graph are the pairwise non-equivalent k-small scanlines. Direct an arc from scanline s to t if the following two conditions hold:

- The scanlines s and t are neighbors such that one end point of t is to the right of the correponding end point of s, and
- the candidate component $\mathcal{C}(s,t)$ has at most $k+1$ vertices.

We call this graph the *scanline graph*.

Lemma 9.5.1 *Let s_L be the scanline which lies totally to the left of all line segments and let s_R be the scanline which lies totally to the right of all line segments. G has treewidth at most k if and only if there is a directed path in the scanline graph from s_L to s_R.*

Proof. Clearly s_L and s_R are k-small. The result follows immediately from Theorem 9.4.1. □

Lemma 9.5.2 *The scanline graph has $O(nk)$ vertices and each vertex is incident with at most 4 arcs.*

Proof. The bound on the number of vertices is proved in Lemma 9.2.2. For each scanline s there are at most 4 scanlines t such that s and t are neighbors. □

We now describe the algorithm which determines if the treewidth of G is at most k.

Step 1 Make a maximal list $\mathcal{L}$ of pairwise non-equivalent k-small scanlines.

Step 2 Construct the acyclic digraph $W_k(G)$.

Step 3 If there exists a path in $W_k(G)$ from s_L to s_R, then the pathwidth of G is at most k. If such a path does not exist, then the pathwidth of G is larger than k.

We now discuss the running time of the algorithm in more detail.

Lemma 9.5.3 *The algorithm can be implemented to run in $O(nk)$ time.*

Proof. By Lemma 9.2.2 each k-small scanline can be characterized by two indices i and θ with $0 \leq i \leq n$ and $-(k+1) \leq \theta \leq k+1$. The scanline with these indices has top end point between i and $i+1$ and bottom end point between $\pi_{i+\theta}$ and $\pi_{i+\theta+1}$ (with obvious boundary restrictions). Let $A(i,\theta)$ be the number of line segments of which the top end point is to the left of the top endpoint of this

scanline and which cross the scanline. Notice that $A(0,\theta) = 0$ for $\theta = 0,\ldots,k+1$. The rest of the table follows from:

$$A(i,\theta) = \begin{cases} A(i,\theta-1) & \text{if } \pi_{i+\theta} > i \\ A(i,\theta-1)-1 & \text{if } \pi_{i+\theta} \leq i \end{cases} \quad \text{for } \theta > -\min(k+1,i), \text{ and}$$

$$A(i,\theta) = \begin{cases} A(i-1,\theta+1)+1 & \text{if } \pi_i^{-1} > i+\theta \\ A(i-1,\theta+1) & \text{if } \pi_i^{-1} \leq i+\theta \end{cases} \quad \text{for } i \geq 1 \wedge \theta = -\min(k+1,i).$$

Notice that the number of line segments crossing the scanline with indices i and θ is $2A(i,\theta)+\theta$. It follows that the list $\mathcal{L}$ of k-small scanlines can be made in $O(nk)$ time.

We now show that the scanline graph $W_k(G)$ can be constructed in $O(nk)$ time. Consider two k-small scanlines s and t that are neighbors and which have a top end point in common, say between i and $i+1$. Assume the bottom end point of s is between π_{j-1} and π_j and the bottom end point of t is between π_j and π_{j+1}. Now notice that the number of vertices of $\mathcal{C}(s,t)$ is $j-i+A(i,j-i)+A(i,j-i-1)$. This shows that the adjacency list for each k-small scanline can be computed in $O(k)$ time. Computing a path in W from s_L to s_R, if it exists, clearly also takes $O(nk)$ time. Hence the total algorithm can be implemented to run in $O(nk)$ time. □

Theorem 9.5.1 *Let G be a permutation graph. Then the pathwidth and treewidth of G are the same, and there is an $O(nk)$ algorithm that determines whether the treewidth of G is at most k.*

We end this section by noting that the algorithm can be adapted such that it computes, within the same time bound, the treewidth (resp. pathwidth) of a permutation graph (assuming that the matching diagram is given). This can be seen as follows. Let the treewidth of G be k. First compute a number L such that $L < k \leq 2L$. This can be done, using the algorithm described above $O(\log k)$ times, in time $O(nk)$ (execute the algorithm described above for $L = 1,2,4,\ldots$). In fact, this step can also be performed using the algorithm described in section 7.4. Now construct the scanline graph $W_{2L}(G)$, and put weights on the arcs, saying how many vertices are in the corresponding candidate component. Then search for a path from s_L to s_R, such that the maximum over weights of arcs in the path is minimized. This maximum weight minus one gives the exact treewidth k of G.

Chapter 10

Approximating treewidth and pathwidth of a graph

In this chapter we look at the problem of efficiently approximating treewidth and pathwidth for graphs in general. Since computing the treewidth or pathwidth of a graph is NP-hard, it is of interest to have a good approximation algorithm. In this chapter, we show first that there exists a fast approximation algorithm for the treewidth of a graph with performance ratio $O(\sqrt{kn})$, where n is the number of vertices of the graph and k is the treewidth. We also give a polynomial time approximation algorithm with performance ratio $O(\log n)$.

In fact, the first algorithm we give computes a *path-decomposition* with width at most $O(k^{\frac{3}{2}}\sqrt{n})$ where k is the *treewidth* and n the number of vertices of the graph. We give an upper bound for the *exact* performance ratio, which shows that the constant is quite reasonable. Our main tool is the algorithm of Alon et al. [1], for finding good vertex separators. We use this algorithm to find vertex separators, because of its simplicity, and nice behavior (i.e. there are no large hidden constants involved). Our methods however show that basically *any* approximation algorithm for balanced vertex separators can be used to find approximations for the treewidth and pathwidth.

Recently, it was reported in [74] that Leighton and Rao obtain balanced vertex separators with performance ratio $O(\log n)$, using an approximate max-flow min-cut theorem [91]. However, we do not know how fast the algorithm is and neither do we know much about the constant in this approximation. However, we do show that such an algorithm can be used to give an approximation algorithm for the treewidth of a graph with performance ratio $O(\log n)$. With a simple trick we can then find a path-decomposition of which the width is at most factor $O(\log^2 n)$ off the optimal width.

To our knowledge it is open whether the problem of approximating the treewidth within a *constant* factor is NP-hard or not. In fact, we feel this is one of the biggest open problems in the research dealing with treewidth and pathwidth at the moment. If the fast algorithms for solving NP-hard problems for

graphs with bounded treewidth are ever to become of practical importance, it is undoubtedly of importance to find good tree-decompositions for these graphs (having small width). Since the $O(\log n)$ approximations do not seem to make many of these algorithms practical, it is of great interest to know whether approximations within a small constant exist. We show in this chapter that finding approximations within an *additive* constant (absolute approximation) is NP-hard.

Recently, Seymour and Thomas [121] have obtained a polynomial algorithm for the related notion of *branchwidth*, restricted to planar graphs. As the branchwidth and treewidth of a graph differ by at most a factor of $\frac{3}{2}$ [113], this gives a polynomial time approximation algorithm for treewidth of planar graphs with performance ratio $\frac{3}{2}$, and polynomial time approximation algorithms for pathwidth of planar graphs with performance ratio $O(\log n)$. Interesting questions are whether a polynomial time algorithm exists that determines treewidth exactly for planar graphs, and whether the pathwidth of a planar graph can be approximated with a constant performance ratio.

10.1 Approximating pathwidth

In this section we give a polynomial algorithm to approximate the pathwidth of a graph. We use the algorithm of Alon et al. [1] to find balanced vertex separators. For convenience we recall their result (using our own terminology). As usual, if $G = (V, E)$ is a graph and $w : V \to \mathbf{R}^+$ is a weight function which maps every vertex onto a non-negative real number, then for a subset Q of the vertices, we denote by $w(Q)$ the sum of the weights of the vertices in Q.

> There is an algorithm with running time $O(m\sqrt{hn})$, which takes as input an integer $h \geq 1$, a graph $G = (V, E)$ (with n vertices and $m = |V| + |E|$), and a weight function $w : V \to \mathbf{R}^+$ (the set of non-negative real numbers), and that outputs either:
>
> - a K_h-minor of G, or
> - a subset X of V, with $|X| \leq h\sqrt{hn}$ such that $w(F) \leq \frac{1}{2}w(V)$ for every connected component F of $G[V \setminus X]$.

We need only a restricted version of the theorem of Alon et al. where the weights of the vertices are restricted to values of zero and one.

Definition 10.1.1 *Let $G = (V, E)$ be a graph, $W \subseteq V$ a subset of the vertices and α any real number between 0 and 1. An* α-vertex-separator of W *in* G *is a set of vertices $S \subseteq V$ such that every connected component of $G[V \setminus S]$ has at most $\alpha|W|$ vertices of W.*

The following lemma is an immediate consequence of the theorem of Alon et al.

Lemma 10.1.1 *There is an algorithm with running time* $O((kn)^{\frac{3}{2}})$, *which takes as input an integer* k, *a graph* $G = (V, E)$ *(with* $|V| = n$*), and a subset of the vertices* Q *and that outputs either:*

- *a valid statement that the treewidth of* G *is larger than* k, *or*
- *a* $\frac{1}{2}$*-vertex-separator* $X \subseteq V$ *of* Q *in* G *with* $|X| \leq (k+2)^{\frac{3}{2}}\sqrt{n}$.

Proof. First check if the number of edges in G does not exceed $\frac{1}{2}k(2n-k-1)$. If it does, the graph cannot have treewidth $\leq k$. Otherwise use the theorem of Alon et al., with $h = k+2$, and a 0/1 valued weight function which has value 1 precisely for the vertices of Q. Notice that if K_h is a minor of G, then the treewidth is at least $h-1$. □

Definition 10.1.2 *Let* $G = (V, E)$ *be a graph with* n *vertices. A* partition *of* G *is a partition* (X, A, B) *of the set of vertices into three subsets such that:*

- X *separates* A *and* B, *and*
- $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$.

Lemma 10.1.2 *There is an algorithm with running time* $O((nk)^{\frac{3}{2}})$ *which takes as input an integer* k *and a graph* $G = (V, E)$. *It outputs either:*

- *a correct statement that the treewidth of* G *is larger than* k, *or*
- *a partition* (X, A, B) *with* $|X| \leq (k+2)^{\frac{3}{2}}\sqrt{n}$.

Proof. Use Lemma 10.1.1 to either determine that the treewidth of G exceeds k or find an $\frac{1}{2}$-vertex-separator X of V, such that $|X| \leq (k+2)^{\frac{3}{2}}\sqrt{n}$. Let $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$. Notice that X is an α-vertex-separator with $|X| \leq (k+2)^{\frac{3}{2}}\sqrt{n}$. Let $C_1, C_2, \ldots, C_t$ be the vertex sets of the connected components of $G[V \setminus X]$. We group these sets together into two sets A and B as follows.

If there is a set C_i with $|C_i| \geq \frac{1}{3}n$ then take $A = C_i$ and $B = V \setminus (X \cup A)$.

If $|V \setminus X| \leq \frac{2}{3}n$, then take $A = V \setminus X$ and $B = \emptyset$.

In all other cases take ℓ such that $|C_1| + \ldots + |C_{\ell-1}| < \frac{1}{3}n$ and $|C_1| + \ldots + |C_\ell| \geq \frac{1}{3}n$. Take $A = C_1 \cup \ldots \cup C_\ell$ and $B = V \setminus (X \cup A)$. This partitions the vertices of $V \setminus X$ into two sets A and B with $|A|, |B| \leq \frac{2}{3}n$, and such that X separates A and B. □

In figure 10.1 (page 116) we present the algorithm to obtain a path-decomposition. First we show that the algorithm is correct.

Lemma 10.1.3 *Let* $G = (V, E)$ *be a graph with* n *vertices, and let* k *be an integer.* Pathdec(G, k, P) *either outputs a correct statement saying that the* treewidth *of* G *exceeds* k *or returns a path-decomposition* P *of* G.

```
procedure Pathdec(G, k, P);
comment {Input: Graph G = (V, E) with n vertices and integer k.
         Output: A path-decomposition P of G or
          a correct statement that the treewidth of G is larger than k}.
begin
 β ← (1/(1 − √(2/3)))²;
 if n ≤ β(k + 2)³ then
   let P consist of one subset containing all vertices of G
 else
   begin
    determine whether there exists a partition (X, A, B) in G with
      |X| ≤ (k + 2)^(3/2) √n (Lemma 10.1.2);
    if such a partition is not found then output("the treewidth exceeds k")
    else
     begin
      call pathdec(G[A], k, P_A);
      call pathdec(G[B], k, P_B);
      add X to all subsets of P_A and to all subsets of P_B;
      let P'_A and P'_B be the new path-decompositions;
      P ← P'_A ++P'_B
     end
   end
end
```

Figure 10.1: Procedure to find a path-decomposition.

Proof. Let $\beta = (1/(1-\sqrt{\frac{2}{3}}))^2$. Clearly, if the algorithm outputs that the treewidth exceeds k, then this is indeed the case by Lemma 10.1.2. Assume this does not occur. We show that the algorithm outputs a path-decomposition P. Clearly this is the case if $n \leq \beta(k+2)^3$. Let $n > \beta(k+2)^3$. Assume that the recursive calls $pathdec(G[A], k, P_A)$ and $pathdec(G[B], k, P_B)$ return path-decompositions P_A of $G[A]$ and P_B of $G[B]$ (if not, the treewidth of G exceeds k). In the next step all vertices of X are added to all subsets of the path-decompositions P_A and P_B, thus creating P'_A and P'_B. Clearly P'_A and P'_B are correct path-decompositions for $G[A \cup X]$ and $G[B \cup X]$. Finally, it easily follows that $P = P'_A +\!+ P'_B$ is a path-decomposition of G. □

Lemma 10.1.4 *If G is a graph with n vertices and with treewidth k, then* $\mathrm{pathdec}(G, k, P)$ *returns a path-decomposition P of G of width at most*

$$\frac{1}{1-\sqrt{\frac{2}{3}}}(k+2)^{\frac{3}{2}}\sqrt{n}-1.$$

Proof. Let $\beta = (1/(1-\sqrt{\frac{2}{3}}))^2$. If $n \leq \beta(k+2)^3$, the result follows directly. Thus we assume without loss of generality that $n > \beta(k+2)^3$.

Let $\alpha = \frac{1}{1-\sqrt{\frac{2}{3}}}(k+2)^{\frac{3}{2}}$. By induction, the recursive calls $\mathrm{pathdec}(G[A], k, P_A)$ and $\mathrm{pathdec}(G[B], k, P_B)$ return path-decompositions P_A of $G[A]$ and P_B of $G[B]$ of width at most $\alpha\sqrt{|A|}-1$ and $\alpha\sqrt{|B|}-1$, respectively. Hence we find for the width of P, since $|A| \leq \frac{2}{3}n$ and $|B| \leq \frac{2}{3}n$:

$$\begin{aligned} \mathit{width}(P) &\leq (k+2)^{\frac{3}{2}}\sqrt{n} + \alpha\max(\sqrt{|A|}, \sqrt{|B|}) - 1 \\ &\leq (k+2)^{\frac{3}{2}}\sqrt{n} + \alpha\sqrt{\frac{2}{3}n} - 1 = \alpha\sqrt{n} - 1. \end{aligned}$$

□

We can now state the main result.

Theorem 10.1.1 *There is an algorithm with running time $O((nk)^{\frac{3}{2}})$ which takes as input an integer k and a graph $G = (V, E)$ with n vertices and that outputs either:*

- *a correct statement that the treewidth of G exceeds k, or*
- *a path-decomposition of G of width at most*

$$\frac{1}{1-\sqrt{\frac{2}{3}}}(k+2)^{\frac{3}{2}}\sqrt{n}-1.$$

Proof. By Lemma 10.1.3 and Lemma 10.1.4 it remains to verify the running time of the algorithm. First we show that the number of subsets returned in the path-decomposition for a graph with n vertices is at most n. This is clearly the case when $n \leq \beta(k+2)^3$. Otherwise the number of subsets is the number of subsets in P_A plus the number of subsets in P_B, which is at most $|A|+|B| \leq n$ (by induction).

Let $T_k(n)$ be the running time of $\mathrm{pathdec}(G, k, P)$. We show that there is a constant C such that $T_k(n) \leq C(k+2)^{\frac{3}{2}}n^{\frac{3}{2}}$. This is clearly so when $n \leq \beta(k+2)^3$. Otherwise, finding the partition takes time $O((nk)^{\frac{3}{2}})$, by Lemma 10.1.2. By induction, the recursive calls can be done in $C(k+2)^{\frac{3}{2}}(|A|^{\frac{3}{2}} + |B|^{\frac{3}{2}})$ time. Finally adding all vertices of X to the subsets of P_A and P_B also takes time $O((nk)^{\frac{3}{2}})$,

since $|X| \leq (k+2)^{\frac{3}{2}}\sqrt{n}$ and the total number of subsets is at most n. Thus we find for some constant γ:

$$T_k(n) \leq C(k+2)^{\frac{3}{2}}\left(|A|^{\frac{3}{2}} + |B|^{\frac{3}{2}}\right) + \gamma(k+2)^{\frac{3}{2}}n^{\frac{3}{2}}$$

Notice that, since $n > \beta(k+2)^3$, we have that $|X| \leq (1-\sqrt{\frac{2}{3}})n$. Since $|A|, |B| \leq \frac{2}{3}n$, we find:

$$\begin{aligned} |A|^{\frac{3}{2}} + |B|^{\frac{3}{2}} &\leq \left(\frac{2}{3}n\right)^{\frac{3}{2}} + \left(\frac{1}{3}n - |X|\right)^{\frac{3}{2}} \\ &\leq \left(\frac{2}{3}n\right)^{\frac{3}{2}} + \left(n\sqrt{\frac{2}{3}} - \frac{2}{3}n\right)^{\frac{3}{2}} \leq \left(\frac{2}{3}\right)^{\frac{3}{4}} n^{\frac{3}{2}} \end{aligned}$$

By chosing $C \geq \gamma/(1-(2/3)^{\frac{3}{4}})$, the result follows. □

Notice that this also proves that if $G = (V, E)$ is a graph with n vertices and of treewidth k, then the pathwidth is at most $O(k^{\frac{3}{2}}\sqrt{n})$. Lemma 2.2.10 on page 22 shows that in fact a better upper bound holds; the pathwidth of a graph is at most $\log n$ times the treewidth of a graph.

10.2 Approximating treewidth

In this section we present a polynomial time algorithm to find a $\log n$ approximation for the treewidth of a graph. As reported in [74], Leighton et al. [90] obtain the following theorem.

> There exists a polynomial time algorithm that, given a graph $G = (V, E)$ and a set $W \subseteq V$ of vertices, finds a $\frac{2}{3}$-vertex-separator W in G of size $O(k \log n)$, where k is the minimum size of a $\frac{1}{2}$-vertex-separator of W in G.

Recall that if G is a graph with treewidth $\leq k$ and W is a subset of vertices, then there is a $\frac{1}{2}$-vertex-separator of W in G with size at most $k+1$. We conclude the following:

Theorem 10.2.1 *There exists a polynomial time algorithm and a constant Δ that, given a graph $G = (V, E)$ and a set $W \subseteq V$, finds a $\frac{2}{3}$-vertex-separator S of W in G of size $|S| \leq \Delta k \log(n+1)$, where $n = |V|$ and k is the treewidth of G.*

In the remainder of this section, Δ is assumed to be the constant mentioned in this theorem.

Definition 10.2.1 *Let $G = (V, E)$ be a graph. Let $W \subseteq V$. A* W-partition *is a partition $(X, (S_i)_{i=1,\ldots,4})$ of the vertices in five sets, such that:*

1. *for $i \neq j$, X separates S_i and S_j,*
2. *for $i = 1, \ldots, 4$: $|S_i| \leq \frac{2}{3}n$, and*
3. *for $i = 1, \ldots, 4$: $|S_i \cap W| \leq \frac{2}{3}|W|$.*

Lemma 10.2.1 *There is a polynomial time algorithm $\mathcal{A}$, which takes as input an integer k, a graph $G = (V, E)$ (with $|V| = n$), and a subset of the vertices W and that outputs either:*

- *a correct statement that the treewidth of G is larger than k, or*
- *a W-partition $(S, (S_i)_{i=1,\ldots,4})$ with $|S| \leq 3\Delta k \log(n+1)$.*

Proof. If the treewidth of G is not larger than k then the algorithm of Theorem 10.2.1 finds a $\frac{2}{3}$ vertex separator X of V with $|X| \leq \Delta k \log(n+1)$. If this separator is not found, then the treewidth of G exceeds k. Otherwise, find a partition (X, A, B) of V in G (see the proof of Lemma 10.1.2). Take a $\frac{2}{3}$ vertex separator U_1 of W in $G[X \cup A]$ and a $\frac{2}{3}$ vertex separator U_2 of W in $G[X \cup B]$ using Theorem 10.2.1. If $|U_1|$ or $|U_2|$ is larger than $\Delta k \log n$ then the treewidth of G exceeds k. Otherwise, in the same manner as described in the proof of Lemma 10.1.2, we group the the components of $G[(X \cup A) \setminus U_1]$ in two sets T_1 and T_2, such that $|T_i \cap W| \leq \frac{2}{3}|W|$. Similar, we group the components of $G[(X \cup B) \setminus U_2]$ in two sets T_3 and T_4. Let $S = X \cup U_1 \cup U_2$ and let $S_i = T_i \setminus X$ $(i = 1, \ldots, 4)$. Clearly $(X, (S_i)_{i=1,\ldots,4})$ is a W-partition with $|S| \leq 3\Delta k \log(n+1)$. □

In figure 10.2 (page 120) we describe an algorithm that, given a graph $G = (V, E)$ and a subset of the vertices W, triangulates G such that W is a clique and such that the maximum clique size is at most $\max(\frac{4}{3}|W|, 12k\Delta \log(n+1))$, or outputs that the treewidth of this graph is larger than k. Hence for $W = \emptyset$ the algorithm triangulates G such that the maximum clique size is not larger than $12k\Delta \log(n+1)$ or outputs that the treewidth is larger than k.

Lemma 10.2.2 *If $G = (V, E)$ is a graph with n vertices, k an integer and W a subset of V, then* triang(G, W, k) *triangulates G such that the vertices of W form a clique and such that the maximum clique size is at most*

$$\max\left(\frac{4}{3}|W|, 12k\Delta \log(n+1)\right)$$

or correctly outputs that the treewidth of G is larger than k.

```
procedure triang(G, W, k);
comment {Input: Graph G = (V, E) with |V| = n,
         subset of vertices W and integer k.
         Output: A triangulation of G such that W is a clique and
         such that ω ≤ max(4/3|W|, 12kΔ log(n + 1)) or,
         a valid statement that the treewidth of G is larger than k}.
begin
 β ← 12kΔ;
 if n ≤ β log(n + 1) then
   make a clique of G
 else
   begin
    find a W-partition (X, (S_i)_{i=1,...,4}) with |X| ≤ 3kΔ log(n + 1) in G;
    if such a W-partition is not found then ouput("the treewidth exceeds k");
    else
     begin
      for i ← 1 to 4 do
          begin
           W_i ← S_i ∩ W;
           call triang(G[S_i ∪ X], W_i ∪ X, k)
          end;
       add edges between vertices of W ∪ X, making a clique of G[W ∪ X]
     end
   end
end
```

Figure 10.2: Procedure to triangulate G such that W is a clique.

Proof. If the algorithm outputs that the treewidth of G is larger than k, then this is a correct statement by Lemma 10.2.1. Assume the algorithm does not output that the treewidth exceeds k. First we show that the algorithm terminates. Let $\beta = 12k\Delta$. If $n > \beta \log(n+1)$, then $|X \cup S_i| \leq 3k\Delta \log(n+1) + \frac{2}{3}n < \frac{11}{12}n$. This guarantees termination.

Now we show that the algorithm returns a triangulated graph. We prove this by induction using the recursive structure of the algorithm. Clearly the claim is true if $n \leq \beta \log(n+1)$. Assume $n > \beta \log(n+1)$. By induction the recursive calls $\text{triang}(G[S_i \cup X], W_i \cup X, k)$ return triangulations of $G[S_i \cup X]$, such that $W_i \cup X$ is a clique. Finally the algorithm makes a clique of $W \cup X$. Notice that, for $i = 1, \ldots, 4$, the graphs $G[S_i \cup W \cup X]$ are triangulated. Since the intersection of these triangulated graphs is a clique, the union is triangulated.

We show that the maximum clique size of this triangulation is at most $\max(\frac{4}{3}|W|, \beta \log(n+1))$. This is clearly true if $n \leq \beta \log(n+1)$. Hence assume $n > \beta \log(n+1)$. Let M be a maximal clique. If M contains no vertex of $S_i \setminus W_i$, for $i = 1, \ldots, 4$, then M contains only vertices of $W \cup X$. By Lemma 10.2.1 we have $|W \cup X| \leq |W| + 3k\Delta \log(n+1)$. If $|W| \leq (\beta - 3k\Delta)\log(n+1)$ then $|W \cup X| \leq \beta \log(n+1)$. Otherwise, $|W \cup X| \leq \frac{4}{3}|W|$.

Now assume M contains a vertex of $S_i \setminus W_i$. Then clearly, it contains no vertex of S_j for $j \neq i$. Hence M is a clique in the triangulation of $G[S_i \cup X]$. By induction we know

$$|M| \leq \max\left(\frac{4}{3}|W_i \cup X|, \beta \log(1 + |S_i \cup X|)\right)$$

Notice that:

$$\beta \log(1 + |S_i \cup X|) \leq \beta \log(n+1)$$

and:

$$\frac{4}{3}|W_i \cup X| \leq \frac{8}{9}|W| + 4k\Delta \log(n+1).$$

If $\frac{8}{9}|W| \leq (\beta - 4k\Delta)\log(n+1)$ then $\frac{4}{3}|W_i \cup X| \leq \beta \log(n+1)$, and otherwise $\frac{4}{3}|W_i \cup X| \leq (\frac{8}{9} + \frac{4}{9})|W| = \frac{4}{3}|W|$. Hence $|M| \leq \max(\frac{4}{3}|W|, \beta \log(n+1))$. This proves the lemma. □

Theorem 10.2.2 *There exists a polynomial time algorithm $\mathcal{B}$ which takes as input an integer k and a graph $G = (V, E)$ with n vertices and that outputs either:*

- *a correct statement that the treewidth of G exceeds k, or*
- *a triangulation of G with maximum clique size at most $12k\Delta \log(n+1)$.*

If algorithm $\mathcal{A}$ can be implemented to run in time $O((nk)^c)$ for some constant $c > 1$, then also $\mathcal{B}$ can be implemented to run in time $O((nk)^c)$.

Proof. The algorithm $\mathcal{B}$ consists of calling $\mathtt{triang}(G, \emptyset, k)$. By Lemma 10.2.2 the algorithm either outputs that the treewidth exceeds k, or triangulates G with a maximum clique size not larger than $12k\Delta \log(n+1)$. Hence we only have to bound the running time. Assume that $\mathcal{A}$ runs in $O((nk)^c)$ for some $c > 1$.

Let $T_k(n, w)$ be the running time for $\mathtt{triang}(G, W, k)$, if G is a graph with n vertices and W a set with w vertices. We show there exist constants α and β such that:

$$T_k(n, w) \leq \alpha(nk)^c + \beta nw$$

Taking $w = 0$ proves the theorem. Let A be some constant larger than 12 (we fix A later). We start by showing that if $n \leq Ak\Delta \log(n+1)$ then the formula holds. Consider the *recursion tree*, which is defined recursively as follows. If no recursive

call is made, the recursion tree consists of one node. Otherwise, take a new root node corresponding to the calling program, and make this adjacent to the roots of the recursion trees corresponding to the recursive calls. As we have shown before (in the proof of Lemma 10.2.2), each recursive call decreases the number of vertices by at least a factor $\frac{11}{12}$. We show that the fraction $n/(k\log(n+1))$ decreases by at least a constant factor, thus showing that the depth of the recursion tree is bounded by some constant. Clearly if $n \leq 8$ the depth of the recursion tree is bounded by a constant. Hence we may assume $\ln(n+1) \geq 2$.

$$\begin{aligned}\frac{11n/12}{k\log(11n/12+1)} \cdot \frac{k\log(n+1)}{n} &\leq \frac{11}{12} \cdot \frac{\log(n+1)}{\log(11/12)+\log(n+12/11)} \\ &\leq \frac{11}{12} \cdot \frac{\log(n+1)}{\log\frac{11}{12}+\log(n+1)} \leq \frac{11}{12} \cdot \frac{1}{1-\frac{\log(12/11)}{\log(n+1)}} \\ &\leq \frac{11}{12} \cdot \frac{1}{1-\frac{1}{2}\ln(12/11)} \leq \frac{11}{12} \cdot \frac{1}{1-\frac{1}{22}} < 1.\end{aligned}$$

This shows that the depth of the recursion tree is bounded by some constant D. Each node in the recursion tree has at most four children, hence the recursion tree contains at most a number of nodes proportional to 4^D. We show that the extra amount of time needed for each node is $O((nk)^c)$. Finding the W-partition takes $O((nk)^c)$ time. Apart from the recursive calls and from finding the W-partition, the time needed is $O(n^2)$. If $n < k$ then $O(n^2) = O((nk)^c)$, and otherwise since $n \leq Ak\Delta\log(n+1)$, there is a constant C such that:

$$n^{1/c} \leq \frac{Cn}{\log(n+1)} \leq AC k\Delta \Rightarrow n^2 \leq (AC\Delta)^{2c}k^{2c} \leq (AC\Delta)^{2c}(nk)^c.$$

This shows that the running time is $O((nk)^c)$ in case $n \leq Ak\Delta\log(n+1)$.

Let $n > 12k\Delta\log(n+1)$. We show a recurrence relation for $T(n,w)$. We assumed that finding the W-partition can be done in $O((nk)^c)$ time. The only other non-trivial part is the construction of the clique $W \cup X$, after the recursive calls are made. The recursive calls $\mathrm{triang}(G[S_i \cup X], W_i \cup X, k)$ make cliques of $W_i \cup X$. Hence, when making a clique of $W \cup X$, we only have to introduce edges between vertices x and y which are in different sets W_i and W_j. But these vertices were not adjacent before. This shows that introducing such an edge can be done in $O(1)$ time. The number of edges in the triangulated graph is at most $n \cdot \max(\frac{4}{3}|W|, 12k\Delta\log(n+1))$. This shows there exists constants γ and δ such that:

$$T(n,w) \leq \sum_{i=1}^{4} T(n_i+|X|, w_i+|X|) + \gamma(kn)^c + \delta nw$$

where $n_i = |S_i|$ and $w_i = |W_i|$. By induction:

$$T(n,w) \leq \sum_{i=1}^{4} \{\alpha k^c(n_i+|X|)^c + \beta(n_i+|X|)(w_i+|X|)\} + \gamma(kn)^c + \delta nw$$

Notice that since $w_i \leq \frac{2}{3}w$, $\sum_{i=1}^{4} n_i = n - |X|$ and $|X| \leq 3k\Delta \log(n+1)$:

$$\sum_{i=1}^{4} \beta(n_i + |X|)(w_i + |X|) \leq \beta\left(\frac{2}{3}w + 3k\Delta \log(n+1)\right)(n + 9k\Delta \log(n+1))$$

We can choose A large enough such that $9k\Delta \log(n+1) \leq \frac{1}{5}n$ for all $n > Ak\Delta \log(n+1)$. Then, if $\beta \geq 5\delta$:

$$\frac{2}{3}\beta w(n + 9k\Delta \log(n+1)) + \delta nw \leq \frac{4}{5}\beta nw + \delta nw \leq \beta nw$$

Clearly, we can choose a new constant γ' such that:

$$\beta \cdot 3k \log(n+1)(n + 9k\Delta \log(n+1)) + \gamma(kn)^c \leq \gamma'(nk)^c$$

Hence we obtain, for $n > Ak\Delta \log(n+1)$:

$$T(n, w) \leq \alpha k^c \sum_{i=1}^{4} (n_i + |X|)^c + \beta nw + \gamma'(nk)^c$$

Without loss of generality, we may assume that $n_1 \geq \frac{1}{4}(n - |X|)$. Then it is easily checked that:

$$\begin{aligned}
\sum_{i=1}^{4} (n_i + |X|)^c &\leq (n_1 + |X|)^c + \left(\sum_{i=2}^{4} (n_i + |X|)\right)^c \\
&= (n_1 + |X|)^c + (n - n_1 + 2|X|)^c \\
&\leq \left(\frac{1}{4}n + \frac{3}{4}|X|\right)^c + \left(\frac{3}{4}n + \frac{9}{4}|X|\right)^c \\
&\leq \left(\left(\frac{1}{4}\right)^c + \left(\frac{3}{4}\right)^c\right)\left(1 + 9k\Delta \frac{\log(n+1)}{n}\right)^c n^c
\end{aligned}$$

It follows that there is a constant $\kappa < 1$ such that if we choose A large enough then for all $n > Ak\Delta \log(n+1)$:

$$\sum_{i=1}^{4} (n_i + |X|)^c \leq \kappa n^c$$

This proves the theorem because, if we choose $\alpha \geq \frac{\gamma'}{1-\kappa}$:

$$T(n, w) \leq \kappa\alpha(nk)^c + \beta nw + \gamma'(nk)^c \leq \alpha(nk)^c + \beta nw$$

□

We end this section by showing an approximation algorithm for pathwidth with performance ratio $O(\log^2 n)$. Recall Lemma 2.2.10. It is easily seen that the proof of this lemma can be made constructive in case the graph is triangulated:

There is a polynomial time algorithm that given a triangulated graph H with n vertices and treewidth k, finds a path-decomposition for H with width at most $(k+1)\log(n) - 1$.

This proves the following theorem.

Theorem 10.2.3 *There exists a polynomial algorithm that, given a graph* G *with* n *vertices and with treewidth* k*, finds a path-decomposition of* G *with pathwidth* $O(k \log^2 n)$.

10.3 Absolute approximations

In this section we show that if $P \neq NP$, then no absolute approximation algorithms exist for treewidth and for pathwidth.

Given an approximation algorithm $\mathcal{A}$ for a minimization problem, we can distinguish between three kinds of performance guarantees. First, in an *absolute approximation*, the approximate solution $\mathcal{A}(I)$ is within an additive constant of the optimal solution $\mathcal{OPT}(I)$. Second, the approximate solution can be within a multiplicative constant of the optimal one. Finally, the ratio between the optimal and approximate solutions can grow with the size of the problem. The algorithms we have presented above all have performance guarantees of the latter kind, with ratios of $O(\sqrt{kn})$, $O(\log n)$ or $O(\log^2 n)$. The hardest of these bounds to achieve is the absolute bound; few NP-hard problems have absolute approximation algorithms.

Recall Lemma 2.2.2 (page 18). Given a tree-decomposition (S, T) for a graph $G = (V, E)$, and let $W \subseteq V$ induce a clique in G. Then there is a subset $X \in S$, such that $W \subseteq X$.

Theorem 10.3.1 *If* $P \neq NP$*, then no polynomial time approximation algorithm* $\mathcal{A}$ *for the treewidth problem can guarantee* $\mathcal{A}(G) - \mathcal{OPT}(G) \leq C$ *for a fixed constant* C.

Proof. Assume we have a polynomial time algorithm $\mathcal{A}$ that, given a graph $G = (V, E)$, finds a tree-decomposition of G with treewidth at most C larger than the treewidth of G. Let a graph $G = (V, E)$ be given. Let $G' = (V', E')$ be the graph obtained by replacing every vertex of G by a clique of $C + 1$ vertices, and adding edges between every pair corresponding to adjacent vertices in G. Thus

$$\begin{aligned} V' &= \{v_i \mid v \in V, 1 \leq i \leq C+1\} \text{ and} \\ E' &= \{(v_i, w_j) \mid (v = w \wedge i \neq j) \vee (v, w) \in E\} \end{aligned}$$

We examine the relationship between the treewidth of G and the treewidth of G'.

Suppose we have a tree-decomposition (X, T), with $X = \{X_i | i \in I\}$ and $T = (I, F)$ of G with treewidth L. One easily checks that

$$(\{Y_i \mid i \in I\}, T = (I, F)) \text{ with } Y_i = \{v_j \mid v \in X_i, 1 \leq j \leq C + 1\}$$

is a tree-decomposition of G' with treewidth $(L + 1)(C + 1) - 1$. It follows that:

$$treewidth(G') \leq (treewidth(G) + 1) \cdot (C + 1) - 1$$

Next suppose we have a tree-decomposition (Y, T) with $Y = \{Y_i | i \in I\}$ and $T = (I, F)$ of G' with treewidth M. Let $X_i = \{v \in V \mid \{v_1, v_2, \cdots, v_{C+1}\} \subseteq Y_i\}$. We claim that (X, T) with $X = \{X_i | i \in I\}$ is a tree decomposition of G with treewidth $(M + 1)/(C + 1) - 1$. Let $(v, w) \in E$. Note that $v_1, v_2, \cdots, v_{C+1}, w_1, w_2, \cdots, w_{C+1}$ form a clique in G'. Hence, by Lemma 2.2.2 there exists an $i \in I$ with $\{v_1, \cdots, v_{C+1}, w_1, \cdots w_{C+1}\} \subseteq Y_i$, and thus $v, w \in X_i$. Let $j \in I$ be on the path in T from $i \in I$ to $k \in I$. If $v \in X_i \cap X_k$, then $\{v_1, \cdots, v_{C+1}\} \subseteq Y_i \cap Y_k$, and hence by definition of a tree-decomposition $\{v_1, \cdots, v_{C+1}\} \subseteq Y_j$, so $v \in X_j$. Clearly, $\max_{i \in I} |X_i| \cdot (C + 1) \leq \max_{i \in I} |Y_i|$. This finishes the proof of our claim. It follows that $treewidth(G') \geq (treewidth(G) + 1) \cdot (C + 1) - 1$ and hence:

$$treewidth(G') = (treewidth(G) + 1) \cdot (C + 1) - 1$$

Now consider the following algorithm for the treewidth problem. Let G be the input graph. Form G', and apply algorithm $\mathcal{A}$ to G'. Apply the construction described above to form a tree-decomposition of G. This must be a tree-decomposition with minimum treewidth: if the treewidth of G is k, then the treewidth of G' is $(k + 1)(C + 1) - 1$. Hence, $\mathcal{A}$ outputs a tree-decomposition of G' with treewidth at most $(k + 1)(C + 1) - 1 + C$, and the algorithm described above outputs a tree-decomposition of G with treewidth at most $\lfloor ((k + 1)(C + 1) + C)/(C + 1) - 1 \rfloor = k$. Thus we would have a polynomial time algorithm for treewidth contradicting the NP-completeness of the problem. □

In the same way we can prove the following theorem. This result was also proved (with different terminology) by Deo et al. [42].

Theorem 10.3.2 *If* $P \neq NP$, *then no polynomial time approximation algorithm* $\mathcal{A}$ *for the pathwidth problem can guarantee* $\mathcal{A}(G) - \mathcal{OPT}(G) \leq C$ *for a fixed constant* C.

Chapter 11

An algorithm to determine the pathwidth of a graph

The problem of determining the treewidth or pathwidth of a graph, and finding tree-decompositions or path-decompositions with small (optimal) width are NP-hard, but if k is a fixed constant the problems are solvable in polynomial time. The first algorithms solving these problems for fixed k are based on dynamic programming and use $O(n^{k+2})$ [4] and $O(n^{2k^2+4k+8})$ [44] time, respectively. With help of graph minor theory, Robertson and Seymour showed the existence of $O(n^2)$ algorithms. However, these algorithms are non-constructive in two ways: first, only existence of the algorithms is proved and secondly the claimed algorithm gives an output *yes* or *no*, but no tree- or path-decomposition. Another drawback is that the constant factors in the estimated running times of these algorithms make them infeasible. It should be remarked that for $k = 1, 2$ and 3 fast linear time algorithms exist.

In this chapter we show for each k there exists an $O(n \log n)$ algorithm that, given a graph $G = (V, E)$, decides whether the pathwidth of G is at most k and, if it is, constructs a path-decomposition of width at most k. In fact we show a slightly stronger result: Given constants A and B, we exhibit a linear time algorithm which, given a tree-decomposition of width at most A, outputs a path-decomposition of width at most B, or correctly states that such a path-decomposition does not exist. Using the approximation algorithm of B. Reed [107], which finds an approximate tree-decomposition of bounded width in $O(n \log n)$ time, we obtain the stated result. Recently these results have been improved by H. Bodlaender [18]. Using results from this and the next chapter he shows that for each k there exists a linear time algorithm to decide whether or not the treewidth of a given graph is at most k.

Our algorithm does not use non constructive arguments or graph minors but is given directly and needs only constructive combinatorial arguments to prove its correctness. The constant factor, although still growing fast with k is only singly exponential in k.

11.1 Preliminaries

In this section we introduce some extra terminology related to tree-decompositions.

Definition 11.1.1 *A* rooted *tree-decomposition is a tree-decomposition* $D = (S, T)$ *in which* T *is a rooted tree.*

Definition 11.1.2 *Let* $D = (S, T)$ *be a rooted tree-decomposition for a graph* G. *For each node* i *of* T, *let* T_i *be the subtree of* T, *rooted at node* i. *Define:* $V_i = \bigcup_{f \in T_i} X_f$ *and let* $G_i = G[V_i]$ *(so if* r *is the root of* T, $G_r = G$*). We call* G_i *the subgraph of* G rooted at i.

We can obtain a rooted tree-decomposition $D_i = (S_i, T_i)$ for G_i from D:

Definition 11.1.3 *Let* $D = (S, T)$ *be a rooted tree-decomposition for a graph* G. *Let* i *be a node of* T. *Let* $D_i = (S_i, T_i)$, *where* T_i *is the subtree of* T *rooted at* i, *and* $S_i = \{X_f \mid f \in T_i\}$. *We call* D_i *the rooted tree-decomposition of* G_i rooted at node i.

Lemma 11.1.1 *For each node* i, D_i *is a tree-decomposition for* G_i.

Proof. Clearly, every vertex of G_i is in at least one subset of S_i. To check the second condition for tree-decompositions, let (v, w) be an edge in G_i. There exists a subset X_k in S which contains both end vertices. Assume $X_k \notin S_i$, i.e., k is not a descendant of i in T. But v also is contained in a subset X_p with p a descendant of i. But then, by the third condition of tree-decompositions, v must also be contained in X_i and the same holds for w. Hence the second condition is satisfied. Finally, if T' is a subtree of T, then the restriction of T' to the vertices of T_i is a subtree of T_i, hence a fortiori, also the third condition is satisfied. □

In order to describe our algorithms more easily, we introduce a special type of rooted tree-decompositions.

Definition 11.1.4 *A rooted tree-decomposition* $D = (S, T)$ *with* $S = \{X_i \mid i \in I\}$ *and* $T = (I, F)$) *is called a* nice *tree-decomposition, if the following conditions are satisfied:*

1. *every node of* T *has at most two children,*
2. *if a node* i *has two children* j *and* k, *then* $X_i = X_j = X_k$
3. *if a node* i *has one child* j, *then either* $|X_i| = |X_j| + 1$ *and* $X_j \subset X_i$ *or* $|X_i| = |X_j| - 1$ *and* $X_i \subset X_j$.

We show that every partial k-tree has a nice tree-decomposition of width k.

Lemma 11.1.2 *Every graph* G *with treewidth* k *has a nice tree-decomposition of width* k. *Furthermore, if* n *is the number of vertices of* G *then there exists a nice tree-decomposition with at most* $4n$ *nodes.*

Proof. Since G has treewidth k, G has at least $k+1$ vertices and there is a triangulation of G into a k-tree H. We show that there is a nice tree-decomposition for H of width k. Let n be the number of vertices of H. If $n = k+1$, we can take the trivial tree-decomposition which has one node and the corresponding subset containing all vertices. Assume $n > k+1$. Then H has a simplicial vertex x. Let H' be the k-tree obtained from H by removing x. By induction there is a nice tree-decomposition $(\{X_i | i \in I\}, T = (I, F))$ for H' with at most $4(n-1)$ nodes. Let C be the set of neighbors of x. Since C is a clique (with k vertices) there exists a subset X_i in the nice tree-decomposition containing C. Take such a node i. We consider three cases.

case 1 First assume i has two children j and k. Then $X_i = X_j = X_k$. We then go down in the tree to either one of the children. We continue until we end in a node p with at most one child.

case 2 If p is a leaf, we add one or two vertices below p in the following manner. If $X_p = C$ we take a new node $a \notin I$ and a subset $X_a = C \cup \{x\}$ and make a a child of p. Otherwise, if $X_p \neq C$, let $z \in X_p \setminus C$. Then we give p a new child a and we create a subset $X_a = X_p \setminus \{z\}$. Notice that since $C \subset X_p$ and $k = |C| \leq |X_p| \leq k+1$, $X_a = C$. We take another new node b and a subset $X_b = C \cup \{x\}$ and make b a child of a. Hence the number of new vertices introduced when p is a leaf is at most two.

case 3 Finally assume p has one child q. Remove the edge (p, q) from the tree. Take a new node a and corresponding subset $X_a = X_p$. Make a a child of p and q a child of a. Take another new node b, make a new subset $X_b = X_p$ and make b a child of p in the tree (thus p has now two children a and b). We now go down in the tree to b which is a leaf, and continue as described in case 2. Notice that in this case we introduced at most 4 new nodes.

Since the tree-decomposition for H' has at most $4(n-1)$ nodes, the tree-decomposition for G has at most $4n$ nodes. □

Lemma 11.1.3 *For* constant k, *given a tree-decomposition of a graph* G *of width* k *and* $O(n)$ *nodes, where* n *is the number of vertices of* G, *one can find a* nice *tree-decomposition of* G *of width* k *and with at most* $4n$ *nodes in* $O(n)$ *time.*

Proof. Constructing the triangulated graph implied by the tree-decomposition clearly takes linear time. Given a triangulated graph, one can find in linear time

a perfect elimination scheme (see Lemma 2.1.1, page 8). The lemma now easily follows from the constructive proof of Lemma 11.1.2. □

Definition 11.1.5 *In a nice tree-decomposition* $(\{X_i | i \in I\}, T = (I, F))$ *every node is of one of four possible types. We name the types as follows.*

"**Start**" *If a node is a leaf, it is called a* start node.

"**Join**" *If a node has two children, it is called a* join node.

"**Forget**" *If a node* i *has one child* j *and if* $|X_i| < |X_j|$, *node* i *is called a* forget node.

"**Introduce**" *If a node* i *has one child* j *and if* $|X_i| > |X_j|$, *node* i *is called an* introduce node.

Notice that every node in the nice tree-decomposition must have one of the four mentioned labels.

Our algorithm for determining the pathwidth of graphs with small treewidth roughly work as follows. First we make a (nice) tree-decomposition. Starting with the leaf nodes we work our way up in the tree, at each node computing a table of characteristics of path-decompositions for the subgraph thus far encountered. In each node we calculate the table of partial solutions from the tables at the children of this node. The main idea is to characterize partial solutions mainly by the vertices that are in the present node.

Notice that for this algorithm it is of crucial importance that we can compute a (nice) tree-decomposition for the graph with small treewidth. Recently B. A. Reed obtained the following result [107]:

Lemma 11.1.4 *For each* k *there exists an* $O(n \log n)$ *algorithm that, given a graph* $G = (V, E)$ *as input either finds a tree-decomposition of* G *with treewidth* $\leq 4k$ *or determines that the treewidth of* G *is larger than* k.

This improves an earlier result of Lagergren, who gave an $O(n \log^2 n)$ algorithm to find a tree-decomposition of width at most $6k + 5$ if the treewidth of G is at most k [87].

The first step of our algorithm is to change this approximate tree-decomposition found by the algorithm of B. A. Reed into a nice tree-decomposition, which can be done in linear time. In the next section we assume that we have a nice tree-decomposition and we show how to use this in order to find the pathwidth of the graph.

11.2 A decision algorithm for pathwidth

In this section we assume that we have a nice tree-decomposition $D = (S, T)$ with $S = \{X_i | i \in I\}$ and $T = (I, F)$, of width k of the input graph $G = (V, E)$. We give an algorithm to decide whether the pathwidth of G is at most ℓ. k and ℓ are considered to be constants for the algorithm. We use a somewhat more explicit notation for a path-decomposition of a graph, i.e., we use the notation $P = (Z_1, Z_2, \ldots, Z_m)$ (Definition 2.2.2, page 17). Recall Definition 11.1.3 for the tree-decomposition D_i rooted at node $i \in I$. We similarly define a partial path-decomposition rooted at node i.

Definition 11.2.1 *A* partial path-decomposition rooted at node $i \in I$ *is a path-decomposition for* G_i*, the subgraph of* G *rooted at* i.

Our algorithm roughly works as follows. For each node i we define a *full set of characteristics* for partial path-decompositions rooted at i of width at most ℓ. By this we mean:

> If there exists a partial path-decomposition of width at most ℓ, there also exists a partial path-decomposition of which the characteristic is in the full set of characteristics.

Assuming that we have a full set of characteristics for each of the children of a node i, we show how to compute the full set of characteristics for node i. We also show that a full set of characteristics is of *bounded* size, and that the full set of characteristics at node i can be computed in $O(1)$ time from the sets at the children of i. If we have computed the full set of characteristics for the root r of T, we can decide whether the pathwidth of G is at most ℓ: this is the case if and only if the full set of characteristics is not empty. In computing the full set of characteristics for node i we consider four different cases, namely corresponding to whether i is a **start**, **join**, **forget** or **introduce** node. Notice that, if k and ℓ are constants and the tree-decomposition D of width k and with a linear number of nodes is given, the algorithm to decide whether the pathwidth of G is at most ℓ takes only linear time.

11.3 The interval model

The characteristic of a partial path-decomposition is a pair of which the first element is called the *interval model* of the path-decomposition.

Definition 11.3.1 *Let* $Y = (Y_1, \ldots, Y_r)$ *be a partial path-decomposition rooted at node* i. *The* restriction of Y *is the sub-decomposition* Y^* *of* Y *for the subgraph induced by* X_i, *i.e.* $Y^* = (Y_1 \cap X_i, \ldots, Y_r \cap X_i)$.

In the restriction Y^* there can be many consecutive elements which are the same. If we remove these duplicates, we obtain the interval model for Y which is, of course, still a path-decomposition for the subgraph induced by X_i.

Definition 11.3.2 *Let* $Y^* = (Z_1, \ldots, Z_r)$ *be the restriction of a path-decomposition* Y *rooted at* i. *Let* $1 = t_1 < \ldots < t_{q+1} = r + 1$ *be defined by:*

$$\forall_{1 \leq i \leq q} \forall_{t_i \leq s < t_{i+1}} [Z_s = Z_{t_i}] \wedge \forall_{1 \leq i < q} [Z_{t_{i+1}} \neq Z_{t_i}]$$

The interval model *for* Y *at node* i *is the sequence* $(Z_{t_i})_{1 \leq i \leq q}$.

Notice that *not every* path-decomposition for a subgraph induced by X_p without repeating subsets is an interval model, since an interval model is defined by means of a partial path-decomposition rooted at p. We call a path-decomposition for the subgraph X_p without adjacent subsets that are the same, minimal:

Definition 11.3.3 *A path-decomposition* Z *for a graph* G *is called* minimal *if no two consecutive subsets in* Z *are the same.*

The next lemma shows that there are only $\Theta(1)$ different interval models at each node i.

Lemma 11.3.1 *For each node* i *the number of different interval models at* i *is bounded by* $(2k+3)^{2k+3}$. *The number of subsets in any interval model is at most* $2k+3$. *These bounds hold for the minimal path-decompositions for the subgraph induced by* X_i *as well.*

Proof. An interval model at node i is a path-decomposition $Z = (Z_1, \ldots, Z_r)$ for $G[X_i]$ which is minimal. We show the bounds hold for the minimal path-decompositions. Let $L(s)$ be the maximal number of subsets in a minimal path-decomposition of a graph with s vertices. We claim that $L(s) \leq 2s + 1$. Clearly, $L(1) = 3$. Now let $s > 1$, and let $Z = (Z_1, \ldots, Z_r)$ be a minimal path-decomposition for a graph $H = (V, E)$ with s vertices. Take any vertex x and let Z_a and Z_b be the first and the last subset of Z containing x. Now remove x from the graph and let $H' = H[V \setminus \{x\}]$. We can obtain a path-decomposition Z' for H' by removing vertex x from all subsets of Z. Notice that Z' can have at most two pairs of duplicate subsets, namely Z'_a can be the same as Z'_{a-1} and Z'_b can be the same as Z'_{b+1}. It follows that the number of subsets of Z is at most two more than the maximal number of subsets in a path-decomposition of a graph with $s - 1$ vertices. Hence $L(s) \leq L(s-1) + 2$. This proves our claim. Since $|X_i| \leq k + 1$, the number of subsets in a minimal path-decomposition of $G[X_i]$ is at most $2k + 3$.

We can find an upperbound for the number of interval models as follows. Notice that an interval model can be characterized by indicating for each vertex the first and last subset where it is contained in. Thus we find an upper bound of i^{2k+2} for the number of interval models with i subsets. Hence we find:

$$\text{number of interval models} \leq \sum_{i=1}^{2k+3} i^{2k+2} \leq (2k+3)^{2k+3}$$

This proves the lemma. □

11.4 Typical sequences

Consider a partial path-decomposition $Y = (Y_1, \ldots, Y_r)$ rooted at i of width at most ℓ. Let Z be the restriction of Y, i.e.,

$$\forall_{1 \le j \le r} [Z_j = Y_j \cap X_i]$$

In the previous section we defined the interval model of Y as a subsequence $(Z_{t_i})_{1 \le i \le q}$ of Z. For each *interval* $[t_i, t_{i+1})$ consider the integer sequence $(|Y_{t_i}|, |Y_{t_i+1}|, \ldots, |Y_{t_{i+1}-1}|)$. In this subsection we show how to characterize these integer sequences by *typical sequences*. We show that these typical sequences (which are also integer sequences) are bounded in length by $4\ell + 7$ and that the maximum element of these typical sequences is bounded by $\ell + 1$. In the next section we define the characteristic of the path-decomposition Y as the pair $((Z_{t_i})_{1 \le i \le q}, (\delta^{(i)})_{1 \le i \le q})$, where each $\delta^{(i)}$ is the typical sequence for $(|Y_{t_i}|, |Y_{t_i+1}|, \ldots, |Y_{t_{i+1}-1}|)$.

In this section we give some results on nonnegative integer sequences, of which the maximum element is bounded by some constant L (we use $L = l + 1$). When we say 'integer sequence' we shall always mean 'nonnegative integer sequence' of length at least one. We start with some notations.

- For any integer sequence $a(1 \ldots n)$, let $l(a) = n$ be the length and $max(a)$ be the maximum value: $max(a) = \max_{1 \le i \le n} a_i$.
- For two sequences a and b *of the same length* we define the sum $c = a + b$ as the sequence c with

$$\forall_{1 \le i \le l(a)} [c_i = a_i + b_i]$$

- For two sequences a and b *of the same length* we write $a \le b$ if $\forall_i a_i \le b_i$.
- For a *constant* A we write $a + A$ for the sequence with $\forall_i (a + A)_i = a_i + A$.

Definition 11.4.1 *For an integer sequence* $a(1 \ldots n)$ *we define the* typical sequence $\tau(a)$ *as the sequence obtained after iterating the following operations.*

- *Remove consecutive repetitions of the same element, i.e. if* $a_i = a_{i+1}$ *then the sequence* $a = (a_1, \ldots, a_n)$ *is replaced by* $(a_1, \ldots, a_i, a_{i+2}, \ldots, a_n)$.
- *If the sequence contains two elements* a_i *and* a_j *such that* $j - i > 2$ *and* $\forall_{i<k<j} [a_k \ge \max(a_i, a_j)]$, *then replace the subsequence* $a(i+1 \ldots j-1)$ *by one element equal to its maximum.*

The limit sequence is $\tau(a)$. *We refer to the second operation as the* typical operation.

If a typical operation is applied to a subsequence $a(i \ldots j)$ of $a = (a_1, \ldots, a_n)$, then a new sequence

$$a' = (a_1, \ldots, a_i, a_k, a_j, \ldots, a_n)$$

is constructed where a_k is the maximum element of the subsequence $a(i \ldots j)$. Notice that a' is at least one element shorter than a. By definition, $\tau(a)$ contains no repetitions and the typical operation is not applicable any further.

Lemma 11.4.1 *For every* a, *the typical sequence* $\tau(a)$ *is uniquely defined.*

Proof. Consider a sequence $a(1 \ldots n)$. An interval (i, j) is *maximal* if the typical operation is applicable to $a(i \ldots j)$, and the interval (i, j) is not contained in a larger interval to which the typical operation is applicable. We claim that the maximal intervals are disjoint (share at most an endpoint). Assume two maximal intervals (i_1, j_1) and (i_2, j_2) overlap, i.e., $i_1 < i_2 < j_1 < j_2$. Then notice that $a(i_2) \geq a(i_1)$ and $a(j_1) \geq a(j_2)$. Hence:

$$\forall_{i_2 < s < j_2} \left[a(s) \geq a(i_2) \geq a(i_1) \wedge a(s) \geq a(j_2) \right]$$

$$\forall_{i_1 < s < j_1} \left[a(s) \geq a(j_1) \geq a(j_2) \wedge a(s) \geq a(i_1) \right]$$

This shows that the typical operation is also applicable to (i_1, j_2), which is a contradiction. It follows that the order in which the typical operations are applied does not influence the resulting typical sequence. It is easy to see that the removal of repetitions may be applied at any moment. □

Lemma 11.4.2 *Let* $a(1 \ldots n)$ *be a sequence of nonnegative integers with* $\max(a) = L$. *Then* $\max(\tau(a)) = L$ *and* $l(\tau(a)) \leq 4L + 3$. *The number of different typical sequences of which the maximum element is bounded by* L *is at most* $(L + 1)^{4L+4}$.

Proof. The first part of the claim is trivial. We prove that the maximal length of any sequence for which the typical operation is not applicable is at most $4L + 3$ (with L the maximum value of the sequence).

Consider sequences that start with a 0 and that do not contain any other 0. Let $\tilde{T}(L)$ be the maximum length of such a sequence. Notice that $\tilde{T}(0) = 1$. The next minimum element (larger than zero) can occur only at the second or third position in the sequence since otherwise the typical operation is applicable. We find the following recurrence relation:

$$\tilde{T}(L) \leq \tilde{T}(L - 1) + 2$$

Hence $\tilde{T}(L) \leq 2L + 1$.

Now let $T(L)$ be the maximum length of a sequence with maximum value L and for which the typical operation is not applicable. Such a sequence can have at most three zeros of which the outermost two can be at distance at most two. Hence:

$$T(L) \leq 2\tilde{T}(L) + 1 \leq 4L + 3$$

The number of different typical sequences with length i is clearly bounded by $(L+1)^i$. Hence the total number of different typical sequence is at most

$$\text{number of typical sequences} \leq \sum_{i=0}^{4L+3} (L+1)^i \leq (L+1)^{4L+4}$$

□

Remark. The following sequence shows that the bound on the length of a typical sequence in Lemma 11.4.2 is optimal:

$$L, L, L-1, L, L-2, L, \ldots, 1, L, 0, L, 0, L, 1, L, 2, \ldots, L-2, L, L-1, L, L$$

Definition 11.4.2 *Let* $a(1..n)$ *be a sequence. We define* $E(a)$ *as the set of* extensions *of* a*:*

$$E(a) = \{a^* \mid \exists_{1=t_1<t_2<\ldots<t_{n+1}} \forall_{1\leq i\leq n} \forall_{t_i\leq k<t_{i+1}} [\, a^*(k) = a(i)\,]\}$$

Hence each element of $E(a)$ is of the form $(a_1, a_1, \ldots, a_2, a_2, \ldots, a_n, \ldots, a_n)$, where each a_i of the original sequence a appears at least once in the extension. For any interval $[\,\alpha, \beta\,]$ with $t_i \leq \alpha \leq \beta < t_{i+1}$ we say that $a(i)$ is *repeated* in this interval (in a^*).

Lemma 11.4.3 *If* $a^* \in E(a)$ *then* $\tau(a^*) = \tau(a)$.

Proof. In computing $\tau(a^*)$ we may start by removing all repetitions. □

Definition 11.4.3 *For two integer sequences* a *and* b *we write* $a \prec b$ *if there are* $a^* \in E(a)$ *and* $b^* \in E(b)$ of the same length *such that* $a^* \leq b^*$. *If both* $a \prec b$ *and* $b \prec a$ *hold we write* $a \equiv b$.

Lemma 11.4.4 *The relation* $\prec$ *is transitive.*

Proof. Let $a \prec b$ and $b \prec c$. First notice that:

$$b \prec c \wedge b^* \in E(b) \Rightarrow b^* \prec c$$

We show there exist extensions $a^* \in E(a)$ and $b^* \in E(b)$ such that $a^* \leq b^*$. By the remark above, $b^* \prec c$ and hence there are extensions $b^{**} \in E(b^*)$ and $c^* \in E(c)$ such that $b^{**} \leq c^*$. We make an extension $a^{**} \in E(a^*)$ as follows. If an element of b^* is repeated in b^{**} then we let the corresponding element of a^* repeat in a^{**}. Clearly,

$$a^{**} \leq b^{**} \leq c^*$$

and hence $a \prec c$. □

Corollary 11.4.1 *The relation $\equiv$ is an equivalence relation.*

Lemma 11.4.5 *If a sequence a' is obtained from a sequence a by a typical operation then $a' \equiv a$. Moreover, there exist extensions a'^* and a'^{**} both of a' such that $a'^* \leq a \leq a'^{**}$.*

Proof. We first show that $a \prec a'$. Let a' be the sequence obtained by applying the typical operation to the subsequence $a(i+1 \ldots j-1)$. So the subsequence $a(i+1 \ldots j-1)$ is replaced by its maximum element. We can make an extension of a', in which this maximum element is repeated $j-i-1$ times. This shows $a \prec a'$. To see that $a' \prec a$ also holds, consider again the typical operation applied to a subsequence $a(i+1 \ldots j-1)$. Let a_k be the maximum element of this subsequence. Now, in a', repeat the element a_i $k-i+1$ times, and element a_j $j-k$ times. This proves $a' \prec a$. Notice that in both cases we only used an extension of a', hence the 'moreover part' of the lemma follows. □

Lemma 11.4.6 *For any integer sequence a: $\tau(a) \equiv a$. Moreover, there exist extensions a' and a'' of $\tau(a)$ and both of the same length as a such that $a'_i \leq a_i$ and $a''_i \geq a_i$ for all i.*

Proof. First assume that a contains no two consecutive elements that are the same. Recall that, in this case, $\tau(a)$ is obtained from a by a series of typical operations. We prove the following: if a' is obtained from a by a series of typical operations then there exists an extension a'^* of a' such that $a'^* \leq a$. We prove this by induction on the number of typical operations applied. If this number is zero the statement follows immediately. Otherwise, consider the last typical operation applied. Let b be the sequence obtained from a before this last typical operation is applied. So a' is obtained from b by one typical operation. By Lemma 11.4.5 there is an extension c of a' such that $c \leq b$. By induction there is an extension d of b such that $d \leq a$. It immediately follows that there also exists an extension

a'^* of a' satisfying $a'^* \leq a$: if an element of b is repeated r times in d, repeat the corresponding element in c also r times.

In the same manner one can show that there is an extension a'^{**} of a' such that $a \leq a'^{**}$.

Finally, for any extension of a we can extend a' and a'' in the same manner which proves the lemma. □

From Lemma 11.4.6 and Lemma 11.4.4 it follows that:

Corollary 11.4.2 *If a and b are two sequences then $a \prec b$ if and only if $\tau(a) \prec \tau(b)$.*

Definition 11.4.4 *Let $a(1..n)$ and $b(1..m)$ be two integer sequences. The* ring-sum $a \oplus b$ *is defined as:*

$$a \oplus b = \{a^* + b^* \mid a^* \in E(a) \text{ and } b^* \in E(b) \text{ and } l(a^*) = l(b^*)\}$$

.

Lemma 11.4.7 *Let $c \in a \oplus b$, and let $a^* \in E(a)$ and $b^* \in E(b)$. Then there exists a sequence $c^* \in E(c)$ such that $c^* \in a^* \oplus b^*$.*

Proof. Let $c = a' + b'$ for some $a' \in E(a)$ and $b' \in E(b)$. Let a_i be repeated p'_i times in a' and p^*_i times in a^*. Let b_i be repeated q'_i times in b' and q^*_i times in b^*. Let $\lambda \geq 1$ be an integer. Make new extensions $a^\circ \in E(a)$ and $b^\circ \in E(b)$, by repeating a_i $\lambda p'_i$ times in a° and b_j $\lambda q'_j$ times in b°. Then a° and b° have the same length. If $c^* = a^\circ + b^\circ$, then $c^* \in E(c)$. If we take λ such that $\lambda p'_i \geq p^*_i$ for all i and $\lambda q'_j \geq q^*_j$ for all j then also $a^\circ \in E(a^*)$ and $b^\circ \in E(b^*)$. □

Next we show that if two sequences can be 'improved', then also the sum can be improved.

Lemma 11.4.8 *Let a and b be two integer sequences of the same length and let $y = a + b$. Let $a_0 \prec a$ and $b_0 \prec b$. Then there is a sequence $y_0 \in a_0 \oplus b_0$ such that $y_0 \prec y$.*

Proof. There are extensions a^*_0 of a_0 and a^* of a such that $a^*_0 \leq a^*$ and extensions b^*_0 of b_0 and b^* of b such that $b^*_0 \leq b^*$. Assume an element a_i is repeated p_i times in a^* and b_i is repeated q_i times in b^*. Now change the extensions a^* and a^*_0 into a^{**} and a^{**}_0 by repeating a_i $p_i q_i$ times in a^{**} and repeating each corresponding element in a^*_0 q_i times. We then have $a^{**}_0 \leq a^{**}$. In a similar way we obtain new extensions b^{**}_0 and b^{**}. Make an extension y^{**} of y by repeating each element y_i $p_i q_i$ times. Define $y_0 = a^{**}_0 + b^{**}_0$. We now have $y_0 = a^{**}_0 + b^{**}_0 \leq a^{**} + b^{**} = y^{**}$ and $y_0 \in a_0 \oplus b_0$. □

Lemma 11.4.9 *Let* a *and* b *be two integer sequences and let* $c \in a \oplus b$. *Then there exists an element* $c' \in \tau(a) \oplus \tau(b)$ *such that* $c' \prec c$.

Proof. This is an immediate consequence of Lemma 11.4.8. By Definition 11.4.4 $c = a^* + b^*$ for some extensions a^* of a and b^* of b. By Lemma 11.4.3 $\tau(a^*) = \tau(a)$ and $\tau(b^*) = \tau(b)$. By Lemma 11.4.6 $\tau(a^*) \prec a^*$ and $\tau(b^*) \prec b^*$. Hence, by Lemma 11.4.8, there is a $c^* \in \tau(a) \oplus \tau(b)$ such that $c^* \prec c$. □

Definition 11.4.5 *Let* $a(1 \ldots n)$ *and* $b(1 \ldots m)$ *be two integer sequences. The* concatenation *of* a *and* b, *is defined as the sequence:*

$$a +\!\!+ b = (a_1, \ldots, a_n, b_1, \ldots, b_m)$$

Lemma 11.4.10 *For two sequences* a *and* b: $\tau(a +\!\!+ b) = \tau(\tau(a) +\!\!+ \tau(b))$.

Proof. By Lemma 11.4.1 we can apply typical operations and removal of duplicates in any order to obtain $\tau(a +\!\!+ b)$. Start by applying the typical operations of a to the sequence $a +\!\!+ b$ and remove adjacent duplicates from this sublist. The result is the sequence $\tau(a) +\!\!+ b$. Next apply all typical operations and removal of duplicates to the sublist b. The result is $\tau(a) +\!\!+ \tau(b)$. This proves the lemma. □

Lemma 11.4.11 *If* $a^* \in E(a)$ *and* $b^* \in E(b)$ *then* $a^* +\!\!+ b^* \in E(a +\!\!+ b)$.

Proof. Obvious. □

Lemma 11.4.12 *If* $a' \prec a$ *and* $b' \prec b$, *then* $a' +\!\!+ b' \prec a +\!\!+ b$.

Proof. There are extensions $a'^* \in E(a')$, $a^* \in E(a)$, $b'^* \in E(b')$ and $b^* \in E(b)$ such that $a'^* \leq a^*$ and $b'^* \leq b^*$. Then clearly also: $(a'^* +\!\!+ b'^*) \leq (a^* +\!\!+ b^*)$. By Lemma 11.4.11: $a^* +\!\!+ b^* \in E(a +\!\!+ b)$ and $a'^* +\!\!+ b'^* \in E(a' +\!\!+ b')$. This proves the lemma. □

Definition 11.4.6 *Let* $a(1 \ldots n)$ *be an integer sequence* $(n > 0)$. *A* split *of* a *is a pair* (δ_1, δ_2) *of integer sequences of one of two types.*

1. *The* first type *split is such that there exists an index* $1 \leq f \leq n$ *with:* $\delta_1 = (a_1, \ldots, a_f)$ *and* $\delta_2 = (a_f, \ldots, a_n)$;
2. *The* second type *split is such that there is an index* $1 \leq f \leq n$ *with:* $\delta_1 = (a_1, \ldots, a_f)$ *and* $\delta_2 = (a_{f+1}, \ldots, a_n)$.

Notice that a_f occurs in both elements of the split of the first type. For an integer sequence of length one there can only be a split of the first type, since we assumed that integer sequences always have length at least one.

Lemma 11.4.13 *Let a be a nonempty sequence such that $a \in E(\tau(a))$. Let (δ_1, δ_2) be a split of $\tau(a)$ of any type. Let (a_1, a_2) be a split of a of the same type such that $a_1 \in E(\delta_1)$ and $a_2 \in E(\delta_2)$ (this split exists). Then $\tau(a_1) = \delta_1$ and $\tau(a_2) = \delta_2$.*

Proof. Write $\tau(a) = (\alpha_1, \ldots, \alpha_s)$ and let (δ_1, δ_2) be a split of the first type with $\delta_1 = (\alpha_1, \ldots, \alpha_f)$ and $\delta_2 = (\alpha_f, \ldots, \alpha_s)$. Make a split of a of the first type such that

$$a_1 = (\alpha_1, \ldots, \alpha_1, \ldots, \alpha_f, \ldots, \alpha_f) \wedge a_2 = (\alpha_f, \ldots, \alpha_f, \ldots, \alpha_s, \ldots, \alpha_s)$$

(with α_f appearing at least once in each a_i). This split clearly is possible since $a \in E(\tau(a))$. Since $a \in E(\tau(a))$, $\tau(a)$ is obtained from a by removing repetitions of elements in a. Clearly, δ_i contains no repetitions, and no typical operation is applicable to it. If the split (δ_1, δ_2) is of the second type, the proof is similar. Hence the lemma. □

We extend the results on integer sequences to *lists of integer sequences*. We use the notation $[a]$ to represent a list $(a^{(1)}, a^{(2)}, \ldots, a^{(n)})$ where each $a^{(i)}$ represents an integer sequence. For short, we call a list of integer sequences also a list. We start with some notations.

1. The length of a list is the number of integer sequences in the list.
2. For a list $[a] = (a^{(1)}, \ldots, a^{(n)})$ we define $max([a]) = max_{1 \leq i \leq n} max(a^{(i)})$.
3. For two lists $[a] = (a^{(1)}, \ldots, a^{(n)})$ and $[b] = (b^{(1)}, \ldots, b^{(n)})$ of the same length and such that $l(a^{(i)}) = l(b^{(i)})$ for all i, we say that $[a]$ and $[b]$ have the same length *in the strong sense*.
4. For two lists $[a]$ and $[b]$ with the same length in the strong sense we write $[a] \leq [b]$ if $a^{(i)} \leq b^{(i)}$ for each i.
5. For two such lists with the same length in the strong sense we use the notation $[a] + [b]$ for the list $(a^{(1)} + b^{(1)}, \ldots, a^{(n)} + b^{(n)})$.
6. Let $[a] = (a^{(1)}, \ldots, a^{(n)})$ be a list. The typical list $\tau[a]$ of $[a]$ is the list
$$\tau[a] = (\tau(a^{(1)}), \ldots, \tau(a^{(n)}))$$
7. Let $[a] = (a^{(1)}, \ldots, a^{(n)})$ be a list. The set of *extensions* of $[a]$ is defined as:
$$E[a] = \{[b] = (b^{(1)}, \ldots, b^{(n)}) \mid \forall_i\, b^{(i)} \in E(a^{(i)})\}$$

8. The *ringsum* of two lists $[a] = (a^{(1)}, \ldots, a^{(n)})$ and $[b] = (b^{(1)}, \ldots, b^{(n)})$ *of the same length* is defined as

$$[a] \oplus [b] = \{(c^{(1)}, \ldots, c^{(n)}) \mid \forall_i\, c^{(i)} \in a^{(i)} \oplus b^{(i)}\}$$

9. For two lists $[a]$ and $[b]$ *of the same length* we write $[a] \prec [b]$ if there exist extensions $[a^*] \in E[a]$ and $[b^*] \in E[b]$ such that $[a^*] \leq [b^*]$. If both $[a] \prec [b]$ and $[b] \prec [a]$ we write $[a] \equiv [b]$.

Most results on integer sequences trivially extend to lists of integer sequences. We summarize them in the following lemma.

Lemma 11.4.14

1. *The relation $\prec$ is transitive for lists and $\equiv$ is an equivalence relation for lists (Lemma 11.4.4 and Corollary 11.4.1).*

2. *If $[b] \in E[a]$ then $\tau[b] = \tau[a]$ (Lemma 11.4.3).*

3. *For two lists $[a]$ and $[b]$* of the same length*: $[a] \prec [b] \Leftrightarrow \tau[a] \prec \tau[b]$ (Corollary 11.4.2).*

4. *For any list $[a]$: $\tau[a] \equiv [a]$. Moreover there are extensions $[b'] \in E(\tau[a])$ and $[b''] \in E(\tau[a])$ such that $[b'] \leq [a] \leq [b'']$ (Lemma 11.4.6).*

5. *Let $[a]$ and $[b]$ be two lists of the same length and let $[c] \in [a] \oplus [b]$. Let $[a^*] \in E[a]$ and $[b^*] \in E[b]$. Then there exists a list $[c^*] \in E[c]$ such that $[c^*] \in [a^*] \oplus [b^*]$ (Lemma 11.4.7).*

6. *Let $[a]$ and $[b]$ be two list with the same length in the strong sense and let $[y] = [a] + [b]$. Let $[a_0] \prec [a]$ and $[b_0] \prec [b]$. Then there exists a list $[y_0] \in [a_0] \oplus [b_0]$ such that $[y_0] \prec [y]$ (Lemma 11.4.8).*

7. *Let $[a]$ and b be two lists* of the same length *and let $[c] \in [a] \oplus [b]$. There exists a list $[c'] \in \tau[a] \oplus \tau[b]$ such that $[c'] \prec [c]$ (Lemma 11.4.9).*

11.5 Characteristic path-decompositions

Let $D = (S, T)$ be a nice tree-decomposition with $S = \{X_i | i \in I\}$ and $T = (I, F)$ for the graph $G = (V, E)$ of width k. Consider partial path-decompositions rooted at some node $i \in I$. In this section we define a full set of characteristics of path-decompositions rooted at some node i. We start with defining the characteristic of a path-decomposition.

Definition 11.5.1 *Let Y be a partial path-decomposition rooted at a node i. Let $Z = (Z_{t_j})_{1 \le j \le q}$ be the interval model for Y. The* list representation *for Y is the pair $(Z, [Y])$, where Z is the interval model and $[Y] = (Y^{(1)}, Y^{(2)}, \ldots, Y^{(q)})$ is the sequence with $Y^{(m)} = (Y_{t_m}, Y_{t_m+1}, \ldots, Y_{t_{m+1}-1})$ for each $1 \le m \le q$.*

Definition 11.5.2 *Let $Y = (Y_1, Y_2, \ldots, Y_m)$ be a partial path-decomposition. The set of* extensions *of Y, $E(Y)$, is the set of path-decompositions*

$$Z = (Y_1, Y_1, \ldots, Y_1, Y_2, Y_2, \ldots, Y_2, \ldots, Y_m, \ldots, Y_m)$$

where each subset Y_i is repeated at least once.

Definition 11.5.3 *Let Y be a partial path-decomposition with list representation $(Z, [Y])$, with interval model $Z = (Z_{t_j})_{1 \le j \le q}$. Let $[y] = (y^{(1)}, y^{(2)}, \ldots, y^{(q)})$ be the list of integer sequences with $y^{(m)} = (|Y_{t_m}|, |Y_{t_m+1}|, \ldots, |Y_{t_{m+1}-1}|)$ for each interval $1 \le m \le q$. We call $[y]$ the list of Y and $\tau[y]$ the typical list of Y.*

Definition 11.5.4 *Let Y be a partial path-decomposition with list representation $(Z, [Y])$ and let $[y]$ be the list of Y. The* characteristic *of Y is the pair*

$$C(Y) = (Z, \tau[y])$$

Lemma 11.5.1 *The number of different characteristics is bounded by*

$$(2k+3)^{2k+3}(\ell+2)^{(4\ell+8)(4k+5)}$$

Proof. By Lemma 11.3.1 and by Lemma 11.4.2. For each subset in the interval model we choose a typical sequence. Since the number of subsets is bounded by $2k+3$, the number of typical lists with each interval model is at most $(\ell+2)^{(4\ell+8)(2k+3)}$. □

Definition 11.5.5 *For two partial path-decompositions Y and Z rooted at the same node i,* which have the same interval model, *we write $Y \prec Z$ if the corresponding lists satisfy $[y] \prec [z]$. If $Y \prec Z$ and $Z \prec Y$, we write $Y \equiv Z$.*

Definition 11.5.6 *A set of characteristics $FS(i)$ of partial path-decompositions rooted at some node i of width at most ℓ is called a* full set of characteristics *if for each partial path-decomposition Y rooted at i of width at most ℓ there is a path-decomposition $Y' \prec Y$ such that the characteristic of Y' is in $FS(i)$.*

Lemma 11.5.2 *If some full set of characteristics at a node i is nonempty, then* every *full set of characteristics at this node is nonempty. A full set of characteristics is nonempty if and only if the pathwidth of G_i is at most ℓ.*

Proof. By Definition 11.5.6. □

Notice that the pathwidth of G is at most ℓ if and only if any full set of characteristics at the root of the tree-decomposition is non-empty. In the next four subsections we show how to compute a full set of characteristics at a node p in $O(1)$ time, when a full set of characteristics of all the children of p is given.

11.6 A full set for a start node

In this subsection we consider the case in which p is a start node of the nice tree-decomposition $D = (S, T)$. Clearly, in this case G_p is the subgraph induced by the subset X_p.

Lemma 11.6.1 *If Y is partial path-decomposition for a* start *node* p *with list representation* $(Z, [Y])$, *then for each interval* $[t_m, t_{m+1})$ *the corresponding list of subsets is of the form* $Y^{(m)} = (Z_{t_m}, Z_{t_m}, \ldots, Z_{t_m})$ *and has length at least one.*

Proof. The restriction (Definition 11.3.1) of the path-decomposition Y rooted at p, is the path-decomposition Y itself, since p is a start node. The lemma follows by Definition 11.3.2. □

Lemma 11.6.2 *If* Y *is a partial path-decomposition for a* start *node* p, *with list representation* $(Z, [Y])$, *then the partial path-decomposition* $Y' = (Z_{t_1}, Z_{t_2}, \ldots, Z_{t_q})$ *satisfies* $Y' \prec Y$.

Proof. By Lemma 11.6.1 $Y \in E(Y')$. □

We can compute a full set of characteristics FS(p) for the start node p as follows. Make a list of all interval models of p (i.e. all minimal path-decompositions of G_p). By Lemma 11.3.1 the number of these is bounded by a constant and the different interval models can be computed in constant time. For each interval model $Z = (Z_{t_i})_{1 \le i \le q}$ let $[z] = ((|Z_{t_1}|), (|Z_{t_2}|), \ldots, (|Z_{t_q}|))$ be the list of the partial path-decomposition $(Z_{t_1}, \ldots, Z_{t_q})$. Notice that $\tau[z] = z$ (since each integer sequence has only one element). The full set of p consists of all pairs $(Z, \tau[z])$ such that each interval Z_{t_i} contains at most $\ell + 1$ elements (by Lemma 11.6.2 this *is* a full set).

11.7 A full set for a join node

Let p be a join node with children q and r. By definition $X_p = X_q = X_r$, since the tree-decomposition is nice. We now have $V_q \cap V_r = X_p$ (V_i is the vertex set of G_i) and G_p is obtained from G_q and G_r by identifying the vertices of G_q and G_r that are in X_p.

We first show how to compute a full set of characteristics of p when full sets of characteristics for q and r are given. Take $(Z, \tau[a]) \in FS(q)$ and $(Z, \tau[b]) \in FS(r)$ with the same interval model $Z = (Z_{t_i})_{1 \le i \le w}$ (thus $\tau[a]$ and $\tau[b]$ have the same length). Let $\tau[a] = (\tau(a^{(1)}), \ldots, \tau(a^{(w)}))$.

First compute the list $[a^*] = (\tau(a^{(1)}) - |Z_{t_1}|, \ldots, \tau(a^{(w)}) - |Z_{t_w}|)$ (we do this in order to prevent counting the elements of Z_{t_i} twice, as they are counted both in $\tau[a]$ and in $\tau[b]$). Compute the set

$$FS(p) = \{(Z, \tau[c]) \mid (Z, \tau[a]) \in FS(q) \wedge (Z, \tau[b]) \in FS(r) \wedge [c] \in [a^*] \oplus \tau[b] \wedge max([c]) \leq \ell + 1\}$$

We show that $FS(p)$ is a full set of characteristics for p. We first show that an element of $FS(p)$ is indeed a characteristic of a partial path-decomposition at node p. The following lemmas will be useful.

Lemma 11.7.1 *Let Y be a partial path-decomposition. Let* $[y]$ *be the list of* Y. *If* $[y^*] \in E[y]$, *then there exists a partial path-decomposition* $Y^* \in E(Y)$ *with list* $[y^*]$.

Proof. If an element of $y_i^{(u)}$ is repeated, we repeat the corresponding subset $Y_i^{(u)}$ the same number of times. □

Lemma 11.7.2 *Let Y be a partial path-decomposition with characteristic* $(Z, \tau[y])$. *Then there exists a partial path-decomposition* Y' with the same characteristic *such that the list* $[y']$ *of* Y' *satisfies* $[y'] \in E(\tau[y])$.

Proof. Assume that no integer sequence $y^{(u)}$ of $[y]$ has two consecutive elements that are the same. Recall the proof of Lemma 11.4.5. Consider a typical operation applied to an integer sequence $y^{(u)}$ of the list $[y]$ of Y. A subsequence $y^{(u)}(i+1 \ldots j-1)$ is replaced by its maximum element $y_k^{(u)}$. We write $Y_j^{(u)}$ for the set of the partial path-decomposition Y corresponding to $y_j^{(u)}$. Consider the path-decomposition Y^* obtained as follows. Initialize $Y_j^* = Y_j$ for all sets Y_j of the path-decomposition Y. Recursively for $j = k-1, \ldots, i+1$ add elements of $Y_{j+1}^{*(u)} \setminus Y_j^{*(u)}$ to the set $Y_j^{*(u)}$ until $Y_j^{*(u)}$ has the same number of elements as $Y_k^{(u)}$. For $j = k+1, \ldots, j-1$ recursively add elements of $Y_{j-1}^{*(u)} \setminus Y_j^{*(u)}$ to the set $Y_j^{*(u)}$ until $Y_j^{*(u)}$ has the same number of elements as $Y_k^{(u)}$. It is easy to see that Y^* is a partial path-decomposition with the same characteristic as Y. By induction on the number of typical operations applied the lemma follows. □

Lemma 11.7.3 *Let* p *be a* **join** *node with children* q *and* r. *Let* A *be a partial path-decomposition rooted at* q *and let* B *be a partial path-decomposition rooted at* r. *Let the restriction of* A *be* the same *as the restriction of* B. *For each* i *define* $C_i = A_i \cup B_i$. *Then* C *is a partial path-decomposition rooted at* p.

Proof. The restriction of A is a partial path-decomposition for the subgraph induced by X_q. Since p is a **join** node $X_p = X_q = X_r$. Notice that A and B contain the same number of subsets, hence C is well defined. The lemma now follows easily from the definition of a path-decomposition. □

Definition 11.7.1 *Let p be a* **join** *node with children q and r. Let A be a path-decomposition rooted at q and let B be a path-decomposition rooted at r, such that the restrictions of A and B are the same. Then we write $C = A \cup B$ for the path-decomposition rooted at p obtained by $C_i = A_i \cup B_i$ for all i.*

In the next three results we assume $FS(p)$ is computed from full sets of characteristics $FS(q)$ and $FS(r)$ as described in this section.

Theorem 11.7.1 *Let p be a* **join** *node with children q and r. For each $(Z, \tau[c]) \in FS(p)$ there is a partial path-decomposition rooted at p with this characteristic.*

Proof. Let A be a partial path-decomposition at q with characteristic $(Z, \tau[a]) \in FS(q)$ and let B be a partial path-decomposition at r with characteristic $(Z, \tau[b]) \in FS(r)$ with the same interval model Z. By Lemma 11.7.2 we may assume that the lists $[a]$ of A and $[b]$ of B satisfy $[a] \in E(\tau[a])$ and $[b] \in E(\tau[b])$. Define

$$[a'] = (a^{(1)} - |Z_{t_1}|, \ldots, a^{(w)} - |Z_{t_w}|) \wedge [a^*] = (\tau(a^{(1)}) - |Z_{t_1}|, \ldots, \tau(a^{(w)}) - |Z_{t_w}|)$$

Clearly $[a'] \in E[a^*]$ since $[a] \in E(\tau[a])$. Let $[c] \in [a^*] \oplus \tau[b]$ with $max([c]) \leq \ell + 1$. By Lemma 11.4.14.5 we may conclude that there is a list $[c^\circ] \in E[c]$ such that $[c^\circ] \in [a'] \oplus [b]$. Hence there are extensions $[a^\circ] \in E[a']$ and $[b^\circ] \in E[b]$ such that $[c^\circ] = [a^\circ] + [b^\circ]$. Notice that since $[c^\circ] \in E[c]$ also $max([c^\circ]) \leq \ell + 1$.

Now take extensions $A^\circ \in E(A)$, corresponding with the extension $[a^\circ]$, and $B^\circ \in E(B)$ corresponding with $[b^\circ]$. Define $C^\circ = A^\circ \cup B^\circ$ (since $[a^\circ]$ and $[b^\circ]$ have the same length in the strong sense C° is well defined). By Lemma 11.7.3 C° is a partial path-decomposition rooted at p. The list of C° is $[c^\circ]$ and hence C° has width at most ℓ. Finally, since $[c^\circ] \in E[c]$: $\tau[c^\circ] = \tau[c]$ (Lemma 11.4.14.2). Hence the characteristic of C° is $(Z, \tau[c]) \in FS(p)$. □

Theorem 11.7.2 *Let p be a* **join** *node with children q and r. If Y is a partial path-decomposition rooted at p of width at most l then there is a partial path-decomposition $Y' \prec Y$ such that $C(Y') \in FS(p)$.*

Proof. Let A be the subdecomposition of Y for G_q and let B be the subdecomposition of Y for G_r, so $Y = A \cup B$. Since $FS(q)$ and $FS(r)$ are full set of characteristics,

we know there exist path-decompositions $A_0 \prec A$ for G_q of which the characteristic is in $FS(q)$ and $B_0 \prec B$ for G_r of which the characteristic is in $FS(r)$. By Lemma 11.7.2 there exists also a partial path-decomposition A' with the same characteristic as A_0, such that $[a'] \in E(\tau[a'])$. Notice that

$$[a'] \equiv \tau[a'] = \tau[a_0] \equiv [a_0] \prec [a]$$

hence $A' \prec A$. In the same manner we find a partial path-decomposition $B' \prec B$ such that $[b'] \in E(\tau[b'])$. Notice that the interval model of all these path-decompositions is the same, say $(Z_{t_i})_{1 \leq i \leq w}$. Define the list

$$[y^*] = (y^{(1)} + |Z_{t_1}|, \ldots, y^{(w)} + |Z_{t_w}|)$$

(where $[y]$ is the list of Y). Then we have $[y^*] = [a] + [b]$. By Lemma 11.4.14.6 there exists a list $[y^\circ] \in [a'] \oplus [b']$ such that $[y^\circ] \prec [y^*]$. Hence there are extensions $[a^\circ] \in E[a']$ and $[b^\circ] \in E[b']$ such that $[y^\circ] = [a^\circ] + [b^\circ]$. By Lemma 11.7.1 there are path-decompositions $A^\circ \in E(A')$ with list $[a^\circ]$ and $B^\circ \in E(B')$ with list $[b^\circ]$. Define $Y^\dagger = A^\circ \cup B^\circ$. Notice that $Y^\dagger$ is a partial path-decomposition rooted at p with list

$$[y^\dagger] = (y^{\circ(1)} - |Z_{t_1}|, \ldots, y^{\circ(w)} - |Z_{t_w}|)$$

Notice that $[y^\dagger] \prec [y]$ (since $[y^\circ] \prec [y^*]$), hence $Y^\dagger \prec Y$. Since $[a'] \in E(\tau[a'])$ and $[a^\circ] \in E[a']$: $[a^\circ] \in E(\tau[a'])$. Also $[b^\circ] \in E(\tau[b'])$. Hence $[y^\circ] \in \tau[a'] \oplus \tau[b']$. If we define

$$[a'^*] = (\tau(a'^{(1)}) - |Z_{t_1}|, \ldots, \tau(a'^{(w)}) - |Z_{t_w}|)$$

we find $[y^\dagger] \in [a'^*] \oplus \tau[b']$, hence $C(Y^\dagger) \in FS(p)$. □

Corollary 11.7.1 $FS(p)$ *is a full set of characteristics for the* **join** *node* p.

11.8 A full set for a forget node

Let p be a **forget** node with child q. Then $G_p = G_q$ and by Definitions 11.1.4 and 11.1.5 $X_p \subset X_q$ and X_q contains exactly one vertex, say x, which is not in X_p. We call x the *forgotten* element of p. We first show how to compute the full set of characteristics $FS(p)$ from the full set of characteristics $FS(q)$.

Let $(Z, \tau[y])$ be a characteristic in $FS(q)$, with interval model $Z = (Z_{t_j})_{1 \leq j \leq w}$. Since Z is a path-decomposition for the subgraph induced by X_q there is a consecutive number of subsets in Z which contain the forgotten element x. We remove x from these sets, and remove consecutive subsets which are now the same. In this way we obtain an interval model Z' for p.

Lemma 11.8.1 Z' *is an interval model for* p.

Proof. Obvious. □

Let $i \leq j$ be such that Z_{t_i} is the first subset Z which contains x and Z_{t_j} is the last subset containing x. Notice that the number of subsets in Z is at most two more than the number of subsets of Z', namely Z_{t_i} can become the same as $Z_{t_{i-1}}$ after the removal of x and Z_{t_j} can become the same as $Z_{t_{j+1}}$. Consider the following four cases.

1. If the number of subsets of Z' is the same as the number of subsets in Z, then we put $(Z', \tau[y])$ in $FS(p)$.

2. If only the subset $Z_{t_i} \setminus \{x\}$ is the same as $Z_{t_{i-1}}$ then let

$$\tau^* = \tau(\tau(y^{(i-1)}) +\!\!+ \tau(y^{(i)}))$$

and change the typical list $\tau[y]$ into the list:

$$[y'] = (\tau(y^{(1)}), \ldots, \tau(y^{(i-2)}), \tau^*, \tau(y^{(i+1)}), \ldots, \tau(y^{(w)}))$$

i.e. we concatenate the typical sequences $\tau(y^{(i-1)})$ and $\tau(y^{(i)})$ and compute the typical sequence of the result. We put $(Z', [y'])$ in $FS(p)$.

3. If only $Z_{t_j} \setminus \{x\} = Z_{t_{j+1}}$ then compute

$$\tau(\tau(y^{(j)}) +\!\!+ \tau(y^{(j+1)}))$$

and change the typical list $\tau[y]$ into $[y'']$ as in the former case. Put $(Z', [y''])$ into $FS(p)$.

4. Finally, if both $Z_{t_{i-1}} = Z_{t_i} \setminus \{x\}$ and $Z_{t_j} \setminus \{x\} = Z_{t_{j+1}}$ then let

$$\tau_1 = \tau(\tau(y^{(i-1)}) +\!\!+ \tau(y^{(i)})) \wedge \tau_2 = \tau(\tau(y^{(j)}) +\!\!+ \tau(y^{(j+1)}))$$

and change the typical list $\tau[y]$ into the list:

$$[y^*] = (\tau(y^{(1)}), \ldots, \tau(y^{(i-2)}), \tau_1, \tau(y^{(i+1)}), \ldots$$
$$\ldots, \tau(y^{(j-1)}), \tau_2, \tau(y^{(j+2)}), \ldots, \tau(y^{(w)}))$$

We put $(Z', [y^*])$ in $FS(p)$.

Notice that if $i = j$ we compute in this last case the typical sequence of: $\tau(y^{(i-1)}) +\!\!+ \tau(y^{(i)}) +\!\!+ \tau(y^{(i+1)})$.

$FS(p)$ is obtained by carrying out the above for each element of $FS(q)$. Below we assume $FS(p)$ is computed in this way from $FS(q)$. To prove the correctness we first show that an element of $FS(p)$ is a characteristic of a partial path-decomposition rooted at p.

Theorem 11.8.1 *For each* $(Z', [c]) \in \mathrm{FS}(p)$ *there is a partial path-decomposition rooted at* p *with this characteristic.*

Proof. Let $(Z, \tau[y])$ be the corresponding characteristic in $\mathrm{FS}(q)$ (i.e. $(Z', [c])$ is computed from $(Z, \tau[y])$ by the algorithm described above). There exists a partial path-decomposition Y rooted at q, with this characteristic. Y is also a partial path-decomposition rooted at p. By Lemma 11.8.1 the interval model of Y at node p is Z'. We prove that the typical list is computed correctly. This is clearly the case when $Z' = Z$. Consider the second case: the number of subsets in Z' is one less and $Z_{t_{i-1}} = Z_{t_i} \setminus \{x\}$. Let $[y]$ be the list of Y. In this case the list of Y changes into

$$(y^{(1)}, \ldots, y^{(i-2)}, y^{(i-1)} +\!\!+ y^{(i)}, y^{(i+1)}, \ldots, y^{(w)})$$

By Lemma 11.4.10: $\tau(y^{(i-1)} +\!\!+ y^{(i)}) = \tau(\tau(y^{(i-1)}) +\!\!+ \tau(y^{(i)}))$, hence the typical list is computed correctly. The other cases are similar. □

The next theorem shows that $\mathrm{FS}(p)$ is indeed a full set of characteristics.

Theorem 11.8.2 *If* Y *is a partial path-decomposition rooted at* p *with width at most* ℓ*, then there is a partial path-decomposition* $Y' \prec Y$ *such that* $C(Y') \in \mathrm{FS}(p)$.

Proof. Y is also a partial path-decomposition rooted at q, since $G_q = G_p$. Hence there is a partial path-decomposition rooted at q, $Y' \prec Y$, of which the characteristic is in $\mathrm{FS}(q)$. In the proof of Theorem 11.8.1 it is shown that the characteristic of Y' is computed correctly for node p. We only have to show that $Y' \prec Y$ still holds for node p (recall Definition 11.5.5: the interval model may have changed!). This however is proved in Lemma 11.4.12. □

Corollary 11.8.1 $\mathrm{FS}(p)$ *is a full set of characteristics for the* **forget** *node* p.

11.9 A full set for an introduce node

In this last subsection we consider the case in which p is an **introduce** node with child q. In this case $V_p \setminus \{x\} = V_q$, for some vertex x, where V_i is the vertex set for subgraph G_i. We call the vertex x *introduced* at p. The neighbors of x are all contained in X_p. We first give an algorithm to compute a full set of characteristics $\mathrm{FS}(p)$ for p.

To simplify the presentation we compute $\mathrm{FS}(p)$ as follows. We first make a list of all feasible interval models for the for the node p (a minimal path-decomposition for the subgraph induced by X_p is a feasible interval model for p). We then check, for each feasible interval model, if there is a characteristic in $\mathrm{FS}(q)$ which can be 'extended' to a characteristic for X_p with this interval model. Clearly, this algorithm might not be the most efficient one to compute $\mathrm{FS}(p)$, since there could be many feasible interval models which are in fact not interval models.

Algorithm

Step 1 Make a list Q of all minimal path-decompositions for the subgraph induced by X_p (Definition 11.3.3).

Step 2 Make a new list Q^* as follows. For each minimal path-decomposition $Z \in Q$ compute Z': Remove the introduced vertex x from all subsets of Z and after that remove repetitions of subsets. Notice that Z' is a minimal path-decomposition for the subgraph induced by X_q. Put the pair (Z, Z') in Q^*.

Step 3 If for some pair $(Z, Z') \in Q^*$ there is *no* characteristic in the full set of characteristics for q, $FS(q)$, which has Z' as an interval model, then remove the pair (Z, Z') from the list Q^*.

Step 4 Initialize $FS(p) = \emptyset$. For each pair (Z, Z') in Q^* and for each characteristic $(Z', \tau[c]) \in FS(q)$ with Z' as an interval model do the following. Let $Z = (Z_{t_s})_{1 \le s \le w}$. Let $i \le j$ be such that Z_{t_i} is the first subset of Z containing the introduced vertex x and Z_{t_j} the last subset of Z containing x. Notice that the number of subsets of Z is at most two more than the number of subsets of Z'. Namely, after the removal of x Z_{t_i} can become the same as $Z_{t_{i-1}}$ and Z_{t_j} can become the same as $Z_{t_{j+1}}$. Hence one of the following four different cases is applicable:

1. If the number of subsets of Z' is the same as the number of subsets in Z, we change the typical list $\tau[c]$ into:

$$[c^\circ] = (\tau(c^{(1)}), \ldots, \tau(c^{(i-1)}), 1 + \tau(c^{(i)}), \ldots \\ \ldots, 1 + \tau(c^{(j)}), \tau(c^{(j+1)}), \ldots, \tau(c^{(w)}))$$

   i.e. we add one to all typical sequences $\tau(c^{(u)})$ with $i \le u \le j$. If $\max([c^\circ]) \le \ell + 1$, then we put $(Z, [c^\circ])$ in $FS(p)$.

2. If the number of subsets in Z' is one less than the number of subsets in Z, and $Z_{t_i} \setminus \{x\} = Z_{t_{i-1}}$: For convenience we write:

$$\tau[c] = (\tau(c^{(1)}), \ldots, \tau(c^{(i-2)}), \tau(c^{(i)}), \tau(c^{(i+1)}), \ldots, \tau(c^{(w)}))$$

   Consider *all splits of both types* (δ_1, δ_2) of $\tau(c^{(i)})$ (Definition 11.4.6). For each such split change the typical list $\tau[c]$ into:

$$[c'] = (\tau(c^{(1)}), \ldots, \tau(c^{(i-2)}), \delta_1, 1 + \delta_2, 1 + \tau(c^{(i+1)}), \ldots \\ \ldots, 1 + \tau(c^{(j)}), \tau(c^{(j+1)}), \ldots, \tau(c^{(w)}))$$

   If $\max([c']) \le \ell + 1$ then we put $(Z, [c'])$ in $FS(p)$.

3. If the number of subsets in Z' is one less than the number of subsets in Z, and $Z_{t_i} \setminus \{x\} = Z_{t_{j+1}}$: This case is similar to the second case. In this case we make all splits of $\tau(c^{(j)})$.

4. If the number of subsets in Z' is two less than the number of subsets in Z: Let

$$\tau[c] = (\tau(c^{(1)}), \ldots, \tau(c^{(i-2)}), \tau(c^{(i)}), \ldots, \tau(c^{(j)}), \tau(c^{(j+2)}), \ldots, \tau(c^{(w)}))$$

In this case consider all splits (α_1, α_2) of $\tau(c^{(i)})$ and all splits (β_1, β_2) of $\tau(c^{(j)})$. For each pair of such splits (α_1, α_2) and (β_1, β_2) change the typical list $\tau[c]$ into

$$[c^\dagger] = (\tau(c^{(1)}), \ldots, \tau(c^{(i-2)}), \alpha_1, 1 + \alpha_2, 1 + \tau(c^{(i+1)}), \ldots$$
$$\ldots, 1 + \beta_1, \beta_2, \tau(c^{(j+2)}), \ldots, \tau(c^{(w)}))$$

If $\max([c^\dagger]) \leq \ell + 1$ then put $(Z, [c^\dagger]$ in $FS(p)$.

Notice that in the last case, if $i = j$ then we split $\tau(c^{(i)})$ into *three* parts; i.e first split it into two parts and then split the second part again.

Step 5 Stop. The computation of $FS(p)$ is completed.

We prove that $FS(p)$ is a full set of characteristics for p in two stages. First we demonstrate that every element in $FS(p)$ is a characteristic of a partial path-decomposition rooted at p.

Theorem 11.9.1 *For each* $(Z, [d]) \in FS(p)$ *there is a partial path-decomposition rooted at* p *with this characteristic.*

Proof. Let $(Z', \tau[y])$ be the corresponding characteristic in $FS(q)$, i.e. $(Z, [d])$ is computed from this characteristic by the algorithm described above. There is a partial path-decomposition Y rooted at q, with $C(Y) = (Z', \tau[y])$. Let $[y]$ be the list for Y. By Lemma 11.7.2 we may assume that $[y] \in E(\tau[y])$. Let $(Z', [Y])$ be the list representation of Y.

First consider the case with $|Z'| = |Z|$ (the same number of subsets). If $i \leq j$ are the first and last subset of Z containing x, we change the path-decomposition Y into Y°, by adding x to all subsets of $Y^{(u)}$, for all $i \leq u \leq j$. Clearly, Y° is a partial path-decomposition for G_p. Since $\tau(y^{\circ(u)}) = \tau(1 + y^{(u)}) = 1 + \tau(y^{(u)})$ for all $i \leq u \leq j$, the typical list $\tau[y^\circ]$ for Y° equals the list $[d]$ as computed by the algorithmn. Hence Y° is a partial path-decomposition with characteristic $(Z, [d])$.

Now consider the case where the number of subsets in Z is one more than the number of subsets in Z' and $Z_{t_i} \setminus \{x\} = Z_{t_{i-1}}$ (the second case). In this case the typical sequence $\tau(y^{(i)})$ is split by the algorithm into, say, (δ_1, δ_2) in order to obtain $[d]$. The integer sequence $y^{(i)}$ is an extension of $\tau(y^{(i)})$. We make a split of $y^{(i)}$, say $(y_1^{(i)}, y_2^{(i)})$ such that $y_1^{(i)} \in E(\delta_1)$ and $y_2^{(i)} \in E(\delta_2)$. By Lemma 11.4.13

this split is always possible and $\tau(y_1^{(i)}) = \delta_1$ and $\tau(y_2^{(i)}) = \delta_2$. Take a similar 'split' of $Y^{(i)}$ into $(Y_1^{(i)}, Y_2^{(i)})$. Add the vertex x to all subsets of $Y_2^{(i)}$, and to all subsets of $Y^{(u)}$ with $i+1 \leq u \leq j$. Call the obtained path-decomposition Y° (notice that Y° is indeed a partial path-decomposition rooted at p of width at most ℓ). Since the typical sequence for $y_1^{(i)}$ is δ_1 and the typical sequence for $y_2^{(i)}$ is δ_2, the characteristic of Y° is indeed $(Z, [d])$, where $[d]$ is the typical list of Y° as computed by the algorithm.

The other two cases are similar. □

Theorem 11.9.2 *For every partial path-decomposition Y rooted at p of width at most ℓ there exists a partial path-decomposition $Y' \prec Y$, such that $C(Y') \in FS(p)$.*

Proof. Let Y° be the subdecomposition of Y for G_q. Since $FS(q)$ is a full set of characteristics, there exists a partial path-decomposition $Y_0 \prec Y^\circ$ of which the characteristic is in $FS(q)$. By Lemma 11.7.2 we know there exists a partial path-decomposition Y' with the same characteristic as Y_0, such that $[y'] \in E(\tau[y'])$ (where $[y']$ is the list of Y' and $\tau[y']$ is the typical list for Y'). Notice that $[y'] \equiv \tau[y'] = \tau[y_0] \equiv [y_0] \prec [y^\circ]$ (Lemma 11.4.14 and Definition 11.5.5) hence $Y' \prec Y^\circ$. So there are extensions $Y'^* \in E(Y')$ and $Y^{\circ *} \in E(Y^\circ)$ such that the respective lists satisfy $[y'^*] \leq [y^{\circ *}]$ (Lemma 11.7.1). Since Y and Y° have the same length we can take an extension $Y^* \in E(Y)$ corresponding with $Y^{\circ *}$ (i.e. if some subset Y_f° is repeated r times in $Y^{\circ *}$ then we repeat Y_f also r times in Y^*). Notice that now we have three partial path-decompositions of the same length Y^*, $Y^{\circ *}$ and Y'^* and that $Y^{\circ *}$ and Y'^* have the same interval model.

Make a partial path-decomposition $Y^\dagger$ rooted at p by changing Y'^* as follows. Add x to $Y_f'^*$ whenever $x \in Y_f^*$. *Notice that the interval model of $Y^\dagger$ is the same as the interval model of* Y and $[y^\dagger] \leq [y^*]$, hence $Y^\dagger \prec Y$.

We now show that the characteristic of $Y^\dagger$ is in $FS(p)$. Clearly, the characteristic of Y'^* is in $FS(q)$. Let $(Z', [Y'^*])$ be the list representation for Y'^* and let $(Z, [Y^\dagger]$ be the list representation for $Y^\dagger$. Write $Z = (Z_{t_s})_{1 \leq s \leq w}$. Let $i \leq j$ be the first and last interval of Z containing the vertex x.

Consider the case where the number of intervals of Z is the same as the number of intervals of Z'. Then x is added to *all* subsets of $Y'^{*(u)}$ for all $i \leq u \leq j$ (otherwise at least one interval of Z' would have been split). It follows that the characteristic of $Y^\dagger$ is computed in the first case of step 4 of the algorithm: $y^{\dagger(u)} = 1 + y'^{*(u)}$ for all $i \leq u \leq j$ hence $\tau(y^{\dagger(u)}) = 1 + \tau(y'^{*(u)})$.

Next consider the case where the number of intervals of Z' is one less than the number of intervals of Z and $Z_{t_i} \setminus \{x\} = Z_{t_{i-1}}$. For convenience we write:

$$[y'^*] = (y'^{*(1)}, \ldots, y'^{*(i-2)}, y'^{*(i)}, \ldots, y'^{*(j)}, y'^{*(j+1)}, \ldots, y'^{*(w)})$$

So $y'^{*(i)} = y^{\dagger(i-1)} +\!\!+ (y^{\dagger(i)} - 1)$. Now $[y'^*] \in E(\tau[y'])$. Write $\tau(y'^{(i)}) = (\alpha_1, \ldots, \alpha_s)$. Then it follows that either

$$y^{\dagger(i-1)} = (\alpha_1, \ldots, \alpha_1, \ldots, \alpha_f, \ldots, \alpha_f) \ \wedge \ y^{\dagger(i)} = 1 + (\alpha_f, \ldots, \alpha_f, \ldots, \alpha_s, \ldots, \alpha_s) \text{ or}$$
$$y^{\dagger(i-1)} = (\alpha_1, \ldots, \alpha_1, \ldots, \alpha_f, \ldots, \alpha_f) \ \wedge \ y^{\dagger(i)} = 1 + (\alpha_{f+1}, \ldots, \alpha_{f+1}, \ldots, \alpha_s, \ldots, \alpha_s)$$

It follows that the characteristic of $Y^\dagger$ is computed by the algorithm in the second case of step 4.

The other cases are similar. □

Corollary 11.9.1 *The set* FS(p) *computed by the algorithm is a full set of characteristics for the* **introduce** *node* p.

Chapter 12

An algorithm to determine the treewidth of a graph

In this chapter we give a linear time algorithm to determine the treewidth of a graph, if the graph has bounded treewidth. We assume an approximate tree-decomposition of width bounded by a constant is given. The algorithm is based on the results of the previous chapter. The method used to find the pathwidth of a graph can be extended to find the treewidth. We show that a tree-decomposition of width bounded by some constant can be characterized using a description of constant size. This description resembles the characteristic of a path-decomposition of the previous chapter. In fact, the characteristic tree-decomposition we define in this chapter, has, as one ingredient, characteristic path-decompositions of parts of the tree-decomposition. The tree itself is characterized by a concept we call the trunk. Recently, using our algorithm as a subroutine, H. Bodlaender succeeded in developing a linear time algorithm for treewidth if the treewidth is bounded by some constant. This algorithm improves many earlier results [44, 4, 107, 87].

Our methods and proofs are strictly of combinatorial nature. No non constructive arguments or graph minors are used. Although the constants involved are growing fast with the treewidth k, we think the algorithm can be of practical importance.

12.1 Preliminaries

We assume we have a nice tree-decomposition $NT = (\{X_i | i \in I\}, T = (I, F))$ of width k of the input graph $G = (V, E)$. In this section we give a linear time algorithm to decide whether the treewidth of G is at most ℓ (k and ℓ are assumed constant). We start by defining in this section the characteristic of a tree-decomposition. We may restrict ourselves to tree-decompositions which are minimal in some sense.

Definition 12.1.1 *Let* $D = (S, T)$ *be a tree-decomposition for a graph* G. D *is called* non-trivial *if for every pair of adjacent nodes* x *and* y *in* T *the corresponding subsets* S_x *and* S_y *are different.*

Clearly, if D is a tree-decomposition, it can be transformed to a tree-decomposition which is non-trivial.

Definition 12.1.2 *Let* $D = (S, T)$ *be a tree-decomposition. Let* x *be a leaf of* T *and let* S_x *be the corresponding subset. The leaf* x *is called* maximal *if* S_x *contains a vertex* v *which is not an element of any other subset of* S.

Notice that if x is a leaf and if y is the father of x, then x is exactly maximal if S_x is *not* a subset of S_y.

Definition 12.1.3 *Let* $D = (S, T)$ *be a tree-decomposition for a graph* G. D *is called* minimal *if the following two conditions are satisfied:*

1. D *is non-trivial, and*

2. *all leafs of* T *are maximal.*

Lemma 12.1.1 *Let* G *be a graph with treewidth* ℓ. *There exists a minimal tree-decomposition of* G *of width* ℓ.

Proof. Take any tree-decomposition $D = (S, T)$ of G of width ℓ. We transform D into a minimal tree-decomposition D' as follows. First, recursively remove leafs of T which are not maximal and remove the corresponding subsets from S. If $e = (x, y)$ is an edge of T such that $S_x = S_y$, then contract the edge in T and replace the subsets S_x and S_y by one new subset. It is easy to see that D' obtained in this way is a minimal tree-decomposition. □

We want to show an upper bound on the number of nodes in a minimal tree-decomposition. The following lemma will be useful.

Lemma 12.1.2 *Let* T *be a tree with* $p \geq 2$ *nodes* b *leafs. There are at most* $b - 2$ *nodes in* T *with degree at least three. Equality holds if and only if the maximum degree is at most three.*

Proof. For each vertex x of T let d_x be the degree. Count the number of edges in T:

$$\begin{aligned} 2(p-1) &= \sum_{x,d_x=1} 1 + 2\sum_{x,d_x=2} 1 + 3\sum_{x,d_x=3} 1 + 4\sum_{x,d_x=4} 1 + \ldots \\ &= 2p - b + \sum_{x,d_x=3} 1 + 2\sum_{x,d_x=4} 1 \ldots \\ &\geq 2p - b + \sum_{x,d_x\geq 3} 1 \end{aligned}$$

Hence $\sum_{x,d_x\geq 3} 1 \leq b - 2$. □

Lemma 12.1.3 *Let* G *be a graph with* n *vertices. Then the number of nodes in a minimal tree-decomposition is at most* $(2n-1)^2$.

Proof. Let $D = (S, T)$ be a minimal tree-decomposition for a graph G with n vertices. Let H be the chordal completion of G implied by D (Definition 2.2.5 on page 18). Since each leaf is maximal, the corresponding subsets are maximal cliques in H and all are different. Since H is triangulated, the number of maximal cliques in H is at most n (see for example [53]). Hence, T has at most n leafs, and by Lemma 12.1.2 the number of vertices of degree at least three is at most $n-1$ (we add one to account for the case $n = 1$). Since adjacent subsets are different, we can use Lemma 11.3.1 (page 132) to see that each path in T with all vertices of degree two can have length at most $2n-1$. It follows that T has at most $(2n-2)(2n-1)$ vertices of degree two. Hence, the total number of nodes in T is at most $(2n-1)^2$. □

Recall that we have a nice tree-decomposition $NT = (\{X_i | i \in I\}, T = (I, F))$. Recall Definition 11.1.2 (page 128) of a rooted subgraph.

Definition 12.1.4 *A* partial tree-decomposition *rooted at a node* $i \in I$ *is a tree-decomposition for* G_i, *i.e., the subgraph rooted at* i.

Definition 12.1.5 *Let* $Y = (SY, TY)$ *be a partial tree-decomposition rooted at a node* i. *The* restriction *of* Y *is the sub-decomposition* $Y^* = (SY^*, TY)$ *for the subgraph induced by* X_i, *i.e. if we define for each* $S \in SY$, $S' = S \cap X_i$, *then* $SY^* = \{S' \mid S \in SY\}$.

The characteristic of a partial tree-decomposition consists of three parts. We call the first part the trunk of the tree-decomposition.

Definition 12.1.6 *Let* $Y = (SY, TY)$ *be a partial tree-decomposition rooted at a node* i. *The* trunk *of* Y *is a tree* $\mathcal{T}$ *defined as follows. First take the restriction of* Y, *say* $Y^* = (SY^*, TY)$. *Next, recursively remove leafs of* TY *for which the corresponding subsets of* SY^* *are not maximal in* SY^*. *Finally, remove those vertices of the tree which have degree two and make the two neighbors adjacent.*

Lemma 12.1.4 *Let* Y *be a partial tree-decomposition rooted at a node* i. *The number of vertices of the trunk of* Y *is at most* 2k.

Proof. The number of vertices of X_i is at most $k+1$. Hence the number of maximal subsets of SY^* is at most $k+1$ (see the proof of Lemma 12.1.3). By Lemma 12.1.2, the number of nodes with degree at least three in the trunk is at most $k-1$. Since each node in the trunk is either a leaf or a node with degree at least three, the number of nodes is bounded by 2k. □

We now define the tree model of a partial tree-decomposition in analogue of the interval model defined in Definition 11.3.2. Let $Y = (SY, TY)$ be a partial tree-decomposition roote at a node i and let $\mathcal{T}$ be the trunk. For each edge e in the trunk, consider the corresponding path in TY with all internal vertices of degree 2. Let SY_e be the corresponding subsets of SY. We use the notation Y_e for the pair (SY_e, TY_e). Notice that Y_e is a path-decomposition for the subgraph induced by the vertices in $\cup_{S \in SY_e} S$.

Definition 12.1.7 *Let* $Y = (SY, TY)$ *be a partial tree-decomposition rooted at a node* i. *The* tree model *of* Y *is a pair*

$$(\mathcal{T}, (Z_e)_{e \in \mathcal{T}})$$

where $\mathcal{T}$ *is the trunk of* Y *and* Z_e *is the interval model of* Y_e *for each edge* e *of* $\mathcal{T}$.

Recall Definition 11.5.1 (page 141) of the list representation of of a path-decomposition.

Definition 12.1.8 *Let* Y *be a partial tree-decomposition rooted at a node* i. *The* trunk-representation *is:*

$$(\mathcal{T}, (Z_e)_{e \in \mathcal{T}}, ([Y_e])_{e \in \mathcal{T}})$$

where $(Z_e, [Y_e])$ *is the list representation for* Y_e *(for each edge* e *in the trunk).*

Recall Definition 11.5.3 (page 141) of a typical list of a path-decomposition.

Definition 12.1.9 *Let* Y *be a partial tree-decomposition rooted at a node* i *with tree model* $(\mathcal{T}, (Z_e)_{e \in \mathcal{T}})$. *The* characteristic *of* Y *is the triple:*

$$C(Y) = (\mathcal{T}, (Z_e)_{e \in \mathcal{T}}, (\tau[y_e])_{e \in \mathcal{T}})$$

where $\tau[y_e]$ *is the typical list of* Y_e.

Notice that the characteristic is of constant size by Lemma 12.1.4 and Lemma 11.5.1 (page 141).

Definition 12.1.10 *Let* Y *and* Z *be two partial tree-decompositions rooted at the same node* i, *which have* the same tree model *(i.e., the same trunk and for each edge of the trunk the same interval model). Then* $Y \prec Z$ *if for* each edge e in the trunk *the corresponding lists satisfy* $[y_e] \prec [z_e]$. *If* $Y \prec Z$ *and* $Z \prec Y$ *then we write* $Y \equiv Z$.

Recall Definition 11.5.6 (page 141) for the full set of characteristics. We define the full set of characteristics for partial tree-decompositions.

Definition 12.1.11
A set of characteristics FS(i) *of partial tree-decompositions rooted at some node* i *of width at most* ℓ *is called a* full set of characteristics *if for each partial tree-decomposition* Y *rooted at* i *and of width at most* ℓ *either its characteristic is in* FS(i) *or there is a partial tree-decomposition* Y′ *with* $Y' \prec Y$ *and the characteristic of* Y′ *is in* FS(i).

In the following sections we show how to compute a full set of characteristics for each node from the full sets of characteristics of the children of the node.

12.2 A full set for a start node

Let p be a start node. Then G_p is the subgraph induced by X_p. We show to compute the full set FS(p) of characteristics. Generate all minimal tree-decompositions $Y = (SY, TY)$ for the subgraph induced by X_p of width at most ℓ. This can be done in constant time by Lemma 12.1.3. The trunk $\mathcal{T}$ of Y in this case can be obtained by removing nodes of TY which have degree two. For each edge $e \in \mathcal{T}$, the interval model Z_e is simply the sequence $[Y_e]$ (since Y is minimal). The typical sequence for the ι^{th} interval satisfies $\tau(y_e^{(i)}) = (|Y_e^{(i)}|)$, i.e. consists of one element. In other words $\tau[y_e] = [y_e]$.

Lemma 12.2.1 *If* p *is a* start *node, then* FS(p) *is a full set of characteristics.*

Proof. Let Y be a partial tree-decomposition rooted at p, of width at most ℓ. Notice that if we remove leafs which are not maximal, this does not change the characteristic. Also, if two adjacent subsets are equal, we can contract the edge without changing the characteristic. This shows that for every partial tree-decomposition with width at most ℓ, the characteristic is in FS(p). □

12.3 A full set for a join node

Let p be a join node with children q and r. By definition $X_p = X_q = X_r$. First we show how to compute a full set of characteristics FS(p) from the sets FS(q) and FS(r). Take $(\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[a_e])_{e\in\mathcal{T}}) \in$ FS(q) and $(\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[b_e])_{e\in\mathcal{T}}) \in$ FS(r). Hence these characteristics have *the same tree model.* Compute, for each edge e of the trunk, the list

$$[a_e^*] = (\tau(a_e^{(1)}) - |Z_e^{(1)}|, \tau(a_e^{(2)}) - |Z_e^{(2)}|, \ldots) \tag{12.1}$$

Then compute *all* list $\tau[c_e]$, where $[c_e] \in [a_e^*] \oplus \tau[b_e]$ with $\max([c_e]) \leq \ell + 1$. Put

$$(\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[c_e])_{e\in\mathcal{T}})$$

in the set $FS(p)$ for all possible choices of $\tau[c_e]$.

In proving the correctness of the above construction, the following notion of the join of two partial tree-decompositions will be useful. Let p be a **join** node with children q and r. Let $A = (SA, TA)$ be a partial tree-decomposition rooted at q and let $B = (SB, TB)$ be a partial tree-decomposition rooted at r. Let $A' = (SA', TA)$ be the restriction of A and let $B' = (SB', TB)$ be the restriction of B. From the trees TA and TB recursively remove leafs which are not maximal in the restriction. Call these new trees TA^* and TB^*, and let SA^* and SB^* be those subsets of SA' and SB' corresponding with nodes in TA^* and TB^* respectively. Assume that:

$$(SA^*, TA^*) = (SB^*, TB^*)$$

We define a tree-decomposition C rooted at p as follows. Define a tree TC by taking the union of TA and TB and by *identifying* the nodes of TA^* and TB^*. For each node x of TC, define a subset SC_x as:

$$SC_x = \begin{cases} SA_x \cup SB_x & \text{if } x \in TA^* \\ SA_x & \text{if } x \in TA \setminus TA^* \\ SB_x & \text{if } x \in TB \setminus TB^* \end{cases}$$

If A and B are partial tree-decompositions rooted at q and r, satisfying the conditions, then we write $C = A \cup B$ for the tree-decomposition defined above.

Remark. Notice that the trunks of the three partial tree-decompositions A,B and C are the same.

Lemma 12.3.1 *$C = A \cup B$ is a partial tree-decomposition rooted at p.*

Proof. The conditions are easily verified. □

Theorem 12.3.1 *Let p be a* **join** *node with children q and r. Let*

$$C = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[y_e])_{e\in\mathcal{T}}) \in FS(p)$$

There exists a partial tree-decomposition of width at most ℓ and rooted at p with this characteristic.

Proof. Let

$$\begin{aligned} C(A) &= (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[a_e])_{e\in\mathcal{T}}) \in FS(q) \\ C(B) &= (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[b_e])_{e\in\mathcal{T}}) \in FS(r) \end{aligned}$$

Let C be obtained from $C(A)$ and $C(B)$ by the algorithm:

$$\forall_{e\in\mathcal{T}}\, [[y_e] \in [a_e^*] \oplus \tau[b_e] \wedge \max([y_e]) \leq \ell + 1]$$

where $[a_e^*]$ as defined in formula 12.1. It is easy to verify that the proof of Theorem 11.7.1 generalizes to obtain the following result. There exist partial tree-decompositions A° rooted at q and B° rooted at r with characteristics $C(A)$ and $C(B)$ respectively, such that $C^\circ = A^\circ \cup B^\circ$ is well defined and has characteristic C. □

Theorem 12.3.2 *Let* p *be a* **join** *node with children* q *and* r. *If* Y *is a partial tree-decomposition rooted at* p *of width at mist* ℓ *then there exists a partial tree-decomposition* Y' *such that* $Y' \prec Y$ *and such that* $C(Y') \in FS(p)$.

Proof. Let A be the subdecomposition of Y for G_q and let B be the subdecomposition for G_r. Notice that $\bar{Y} = A \cup B$ is well defined and satisfies $C(\bar{Y}) = C(Y)$. The result of Theorem 11.7.2 can easily be generalized and we obtain the following result. There exist partial tree-decompositions A° rooted at q and B° rooted at r, such that $C(A^\circ) \in FS(q)$, $C(B^\circ) \in FS(r)$ and $Y^\dagger = A^\circ \cup B^\circ$ is well defined and satisfies $Y^\dagger \prec \bar{Y}$. Moreover we may assume that for each edge of the trunk $[a_e^\circ] \in E(\tau[a_e^\circ])$ and $[b_e^\circ] \in E(\tau[b_e^\circ])$. Hence $[y_e^\dagger] \in [a_e^{\circ *}] \oplus \tau[b_e^\circ]$, with $[a_e^{\circ *}]$ computed from $\tau[a_e^\circ]$ as in formula 12.1. This proves the theorem. □

Corollary 12.3.1 $FS(p)$ *is a full set of characteristics for the* **join** *node* p.

12.4 A full set for a forget node

Let p be a forget node with child q, and let x be the forgotten element i.e. $X_p = X_q \setminus \{x\}$. We first show how to compute a full set of characteristics $FS(p)$ from a full set of characteristics $FS(q)$ for q.

Let $(\mathcal{T}, (Z_e)_{e \in \mathcal{T}}, (\tau[y_e])_{e \in \mathcal{T}})$ be a characteristic of a partial tree-decomposition Y rooted at q. Remove from all sets $Z_e^{(i)}$ the vertex x. Compute the new trunk $\mathcal{T}^*$. Remove repetitions from the interval models Z_e and for each edge $e \in \mathcal{T}^*$ let Z_e^* be the new interval model. Finally, for each edge $e \in \mathcal{T}^*$ change the typical list $\tau[y_e]$ into $\tau[y_e^*]$ as described in section 11.8. Put $(\mathcal{T}^*, (Z_e^*)_{e \in \mathcal{T}^*}, (\tau[y_e^*])_{e \in \mathcal{T}^*})$ in $FS(p)$.

Lemma 12.4.1 *The trunk* $\mathcal{T}^*$ *differs from* $\mathcal{T}$, *only if there is exactly one subset* $Z_e^{(i)}$ *containing the vertex* x. *Moreover, in that case at least one end vertex of* e *must be a leaf of* $\mathcal{T}$.

Proof. Let $Y^* = (SY^*, TY)$ be the restriction of Y at the node q. Recursively remove leafs from TY which are not maximal, and let TY^* be the new tree. Notice that the trunk $\mathcal{T}^*$ can only differ from $\mathcal{T}$ if some leaf node w of TY^* is not maximal any more, when x is removed from all subsets. Since the subset corresponding with w is maximal, it contains a vertex z which is not in the subset corresponding with

the neighbor of w. Since the only vertex which is removed is the forgotten vertex x, it follows that $z = x$. □

Lemma 12.4.2 *The tree model for* Y *at* p *is* $(\mathcal{T}^*, (Z_e)_{e\in\mathcal{T}^*})$.

Proof. See the proof of Lemma 11.8.1. □

Theorem 12.4.1 *For each* $(\mathcal{T}^*, (Z_e^*)_{e\in\mathcal{T}^*}, (\tau[y_e^*)_{e\in\mathcal{T}^*}) \in FS(p)$ *there is a partial tree-decomposition* Y *rooted at* p *with this characteristic.*

Proof. Let $(\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[y_e])_{e\in\mathcal{T}})$ be the corresponding characteristic in $FS(q)$. Then there is a partial tree-decomposition Y rooted at q with this characteristic. Then Y is also a partial tree-decomposition rooted at p since $G_p = G_q$. By Lemma 12.4.2, $(\mathcal{T}^*, (Z_e)_{e\in\mathcal{T}^*})$ is the tree model for Y at p. By Theorem 11.8.1 for each edge e in $\mathcal{T}^*$ the typical list $\tau[y_e^*]$ is computed correctly. □

Theorem 12.4.2 *If* Y *is a partial tree-decomposition rooted at* p *of width at most* ℓ *then there exists a partial tree-decomposition* $Y' \prec Y$ *such that* $C(Y') \in FS(p)$.

Proof. Y is also a partial tree-decomposition rooted at q. Since $FS(q)$ is a full set of characteristics, there is a partial tree-decomposition Y' with $Y' \prec Y$ such that $C(Y') \in FS(q)$. It is shown that the characteristic of Y' is computed correctly for node p. We have to show that $Y' \prec Y$ holds for node p. Since Y and Y' have the same tree model at q, they also have the same tree model at p (see Lemma 12.4.2; the tree model at p is computed from the tree model at q). Using Lemma 11.4.12 it follows that for each edge $e \in \mathcal{T}^*$ we have $[y'_e] \prec [y_e]$. Hence $Y' \prec Y$ for node p. □

Corollary 12.4.1 $FS(p)$ *is a full set of characteristics for the forget node* p.

12.5 A full set for an introduce node

Let p be an **introduce** node with child q. Let x be the vertex introduced at p, i.e. $X_p = X_q \cup \{x\}$. We start by giving the algorithm to compute the full set $FS(p)$. For reasons of simplicity we apply the same method as in section 11.9: First we compute all minimal tree-decompositions for X_p. Notice that by Lemma 12.1.3 this can be done in constant time. Remove the vertex x from all subsets, obtaining a tree-decomposition for X_q. Next compute the tree model of this tree-decomposition. Check if there is a characteristic in $FS(q)$ with this tree model, and change this into a new characteristic for $FS(p)$.

Algorithm

Step 1 Make a list Q of all tree models of minimal tree-decompositions for the graph induced by X_p of width at most ℓ.

Step 2 Make a new list Q′ as follows. For each element T^* of Q remove x from all subsets. Compute the tree model $T = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}})$ of the result. If there is a characteristic in FS(q) with this tree model, then put the pair (T^*, T) in the list Q′.

Step 3 Let $(T^*, T) \in Q'$, and let $C = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, ([\tau[y_e])_{e\in\mathcal{T}}) \in FS(q)$ such that $T = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}})$. Let $\mathcal{T}^*$ be the trunk of T^*. We show how to compute characteristics $C^* = (\mathcal{T}^*, (Z^*_e)_{e\in\mathcal{T}^*}, (\tau[y^*_e])_{e\in\mathcal{T}^*})$ for FS(p). We consider two cases.

1. The trunks $\mathcal{T}$ and $\mathcal{T}^*$ are the same. In this case we can proceed as described in section 11.9. For each edge of the trunk we change the typical list as described in step 4 of the algorithm given in that section (page 147).
2. The trunks are different. In that case, the trunk $\mathcal{T}^*$ contains a leaf say a, which is not a leaf of the trunk $\mathcal{T}$. Let b be the neighbor of a in $\mathcal{T}^*$. In general, when b is of degree three in $\mathcal{T}^*$, b is not a node in $\mathcal{T}$. In that case let c and d be the other neighbors of b in $\mathcal{T}^*$. Notice that c and d are adjacent in $\mathcal{T}$. Let Z_b be the interval corresponding with node b in T^*. Let

$$Z_e = (Z_e^{(c)}, \ldots, Z_e^{(b)}, \ldots, Z_e^{(d)})$$

be the interval model for $e = (c, d)$ in T. In T^* this interval model is split in two parts

$$Z^*_{e_1} = (Z_{e_1}^{(c)}, Z_{e_1}^{(c+1)}, \ldots, Z_{e_1}^{(b)}) \quad \text{for } e_1 = (c, b)$$
$$Z^*_{e_2} = (Z_{e_2}^{(b)}, Z_{e_2}^{(b+1)}, \ldots, Z_{e_2}^{(d)}) \quad \text{for } e_2 = (b, d)$$

Consider the typical list for e in T:

$$\tau[y_e] = (\tau(y_e^{(c)}), \ldots, \tau(y_e^{(b)}), \ldots, \tau(y_e^{(d)}))$$

The typical sequence

$$\tau(y_e^{(c)}) = (\alpha_1, \ldots, \alpha_s)$$

for the interval $Z_e^{(b)}$, is split into two parts *in all possible ways* (each split gives a characteristic for FS(p)):

$$\tau_1 = (\alpha_1, \ldots, \alpha_f) \wedge \tau_2 = (\alpha_f, \ldots, \alpha_s)$$

Notice that, by Lemma 11.4.13, τ_1 and τ_2 are typical sequences. The typical lists for the edges (c, b) and (b, d) in C^* are

$$\tau[y^*_{e_1}] = (\tau(y_e^{(c)}), \ldots, \tau(y_e^{(b-1)}), \tau_1) \qquad \text{for the edge } (c, b)$$
$$\tau[y^*_{e_2}] = (\tau_2, \tau(y_e^{(b+1)}), \ldots, \tau(y_e^{(d)})) \qquad \text{for the edge } (b, d)$$

When the node b is a node in $\mathcal{T}$, the interval model and typical list are not split.

Finally, we have to describe the typical sequence for the edge (a, b) in $f = (a, b)$. Consider the interval model $Z_f = (Z_f^{(1)}, \ldots, Z_f^{(r)})$ for this edge in T. The typical sequence $\tau(y_f^{*(i)}) = (|Z_f^{(i)}|)$ (i.e. consists of one element).

Step 5 Stop. The computation of FS(p) is completed.

In Step 3 of the algorithm, we claim that the trunks $\mathcal{T}$ and $\mathcal{T}^*$ differ only in some specified way. We start by proving this.

Lemma 12.5.1 *If the trunks $\mathcal{T}$ and $\mathcal{T}^*$ are different, then there is exactly one leaf a of $\mathcal{T}^*$ which is not a leaf of $\mathcal{T}$. Let b be the neighbor of a in $\mathcal{T}^*$. If b is of degree three in $\mathcal{T}^*$ then b is not a node in $\mathcal{T}$. In this case the two other neighbors of b are adjacent in $\mathcal{T}$.*

Proof. Let Y^* be a minimal tree-decomposition for the subgraph induced by X_p with tree-model $(\mathcal{T}^*, (Z_e)_{e\in\mathcal{T}})$. The trunk $\mathcal{T}$ is obtained by removing x from all subsets and then computing the trunk of the result. Assume the trunks are not the same. Then clearly, there must be a leaf in $\mathcal{T}^*$ which is not a node of $\mathcal{T}$. Hence the subset corresponding with a is not maximal in $\mathcal{T}$. It follows that x is contained only in this subset. □

Theorem 12.5.1 *Each element $C^* = (\mathcal{T}^*, (Z^*_e)_{e\in\mathcal{T}^*}, (\tau[y^*_e])_{e\in\mathcal{T}^*}) \in FS(p)$ is the characteristic of a partial tree-decomposition rooted at node p.*

Proof. Let $C = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[y_e])_{e\in\mathcal{T}})$ be the characteristic in FS(q) from which C^* is computed by the algorithm. Since FS(q) is a full set of characteristics for q, there exists a partial tree-decomposition $Y = (SY, TY)$ rooted at q with characteristic C. By Lemma 11.7.2 we may assume that $[y_e] \in E(\tau[y_e])$ for every edge $e \in \mathcal{T}$. We show how to compute a partial tree-decomposition rooted at p with characteristic C^*. If $\mathcal{T} = \mathcal{T}^*$ the result follows from Theorem 11.9.1. Hence assume the trunks are different. Let a be the leaf of $\mathcal{T}^*$ which is not in $\mathcal{T}$ and let b be the neighbor of a. Assume b is not an element of $\mathcal{T}$. Then b has two other neighbors c and d which are adjacent in $\mathcal{T}$. Let e be the edge (c, d) in $\mathcal{T}$. The algorithm splits the characteristic sequence $\tau(y_e^{(b)})$ into two parts:

$$\tau_1 = (\alpha_1, \ldots, \alpha_f) \wedge \tau_2 = (\alpha_f, \ldots, \alpha_s)$$

Since $[y_e] \in E(\tau[y_e])$ we know that $y_e^{(c)} \in E(\tau(y_e^{(c)}))$. Hence we can split the sequence $Y_e^{(c)}$ in two parts $Y_{e_1}^*$ and $Y_{e_2}^*$ such that τ_1 is the characteristic sequence of $y_{e_1}^*$ and τ_2 is the characteristic sequence of $y_{e_2}^*$. Let b_0 be the node in TY, corresponding with the split of $Y_e^{(c)}$. We make a new tree TY* by making a new path P adjacent to b_0. Let f be the edge (a, b) in $\mathcal{T}^*$. Each node i in P corresponds to a subset $Z_f^{*(i)}$. The corresponding subset in Y* is equal to $Z_f^{*(i)}$. It is easily checked that Y* is a partial tree-decomposition rooted at p with characteristic C*. □

Theorem 12.5.2 *Let Y be a partial tree-decomposition rooted at p of width at most ℓ. There exists a partial tree-decomposition Y' such that $Y' \prec Y$ and such that $C(Y') \in FS(p)$.*

Proof. We may assume Y is minimal. Let Y^0 be the subdecomposition of Y rooted at q. Since FS(q) is a full set of characteristics there exists a partial tree-decomposition $Y_0 \prec Y^0$ such that $C(Y_0) \in FS(q)$. Clearly we may assume that Y_0 is minimal. Let $C(Y_0) = (\mathcal{T}, (Z_e)_{e\in\mathcal{T}}, (\tau[y_e])_{e\in\mathcal{T}})$. We may assume that $[y_e] \in E(\tau[y_e])$ for all edges $e \in \mathcal{T}$. Since $Y_0 \prec Y^0$, there exist partial tree-decompositions Y_0^* and Y^{0*} such that $Y_{0e}^* \in E(Y_{0e})$, $Y_e^{0*} \in E(Y_e^0)$ and $[y_{0e}^*] \leq [y_e^{0*}]$ for every edge $e \in \mathcal{T}$. Since Y^0 is the restriction of Y, there exists a partial tree-decomposition Y* with the same characteristic as Y such that $[y_e^*] \in E([y_e])$ and such that $[y_e^*]$ and $[y_e^{0*}]$ have the same length in the strong sense for every edge $e \in \mathcal{T}$. Assume the trunk of Y and Y^0 are different. Since Y and Y_0 are minimal there is a simple path P in the tree TY of Y which is not present in the tree TY_0 of Y_0. Let i be the node in TY_0^* which is adjacent to a node of P in TY*. Make P adjacent to i in TY_0. □

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Index

Samenvatting

Dit proefschrift behandelt een aantal problemen die te maken hebben met de *boombreedte* en *padbreedte* van grafen.

Veel problemen blijken moeilijk oplosbaar voor grafen in het algemeen. Voor veel problemen is de boombreedte van een graaf een goede indicatie voor de snelheid waarmee een probleem is op te lossen: het probleem is snel oplosbaar voor grafen met een kleine boombreedte. Een voorwaarde is dat een *boomdecompositie* van de graaf gegeven is met zo klein mogelijke breedte. Een equivalente voorwaarde is dat een *triangulatie* van de graaf gegeven is zó dat het aantal punten in de maximale clique van de getrianguleerde graaf minimaal is. Een getrianguleerde graaf is een graaf waarin elke circuit ter lengte minstens vier een koorde heeft. De grafen met boombreedte hoogstens k, partiële k-bomen genoemd, zijn precies de subgrafen van getrianguleerde grafen waarvan de maximale clique hoogstens $k+1$ punten heeft.

In hoofdstuk 2 geven we de nodige definities. In de hoofdstukken 3, 4, 5 en 6 behandelen we problemen die te maken hebben met de structuur van de partiële k-bomen. Zo karakteriseren we in hoofdstuk 3 de structuur van partiële 2-bomen. Met behulp van deze karakterisering lossen we het volgende probleem (ontleend aan de mathematische biologie) in lineaire tijd op. Zij gegeven een graaf G waarvan elk punt gekleurd is met een van drie mogelijke kleuren zodat twee punten die verbonden zijn verschillend gekleurd zijn. Zoek een getrianguleerde graaf H met dezelfde puntverzameling als G en waarvan G een subgraaf is, zodat ook in H verbonden punten verschillend gekleurd zijn. We geven een lineaire tijd algoritme dat óf zo'n graaf H levert indien die bestaat óf meldt dat dit niet mogelijk is.

In hoofdstuk 4 gebruiken we de karakterisering van de partiële 2-bomen nogmaals om het enumeratie probleem van de tweevoudig samenhangende partiële 2-bomen op te lossen. Dit generaliseert eerdere resultaten met betrekking tot zogenaamde klusters en hangt samen met het tot nu toe onopgeloste enumeratie probleem van zogenaamde animals.

In hoofdstuk 5 bekijken we de boombreedte van random grafen. Verrassend is het feit dat random grafen met een aantal lijnen dat 'slechts' lineair is in het aantal punten, met grote kans al een boombreedte hebben die ook lineair is in het aantal punten. Dit is een reden om naar boomdecomposities te zoeken van grafen

met een zekere onderliggende structuur. We onderzoeken dergelijke klassen van grafen in de hoofdstukken 7, 8 en 9. We beperken ons hierbij tot verschillende klassen perfecte grafen.

In hoofdstuk 6 bewijzen we dat er voor elke k een lineaire tijd algoritme bestaat om superperfectie te testen voor k-bomen. De klasse superperfecte grafen is groot te noemen wat geïllustreerd wordt door het feit dat alle cocomparability grafen, en dus ook alle bipartite grafen superperfect zijn. Er zijn echter bijzonder weinig andere klassen van grafen bekend waarvoor superperfectie in polynomiale tijd te testen is. We laten zien dat er een eindige lijst 'verboden structuren' is, die in $O(1)$ tijd berekend kan worden, zodat een k-boom superperfect is dan en slechts dan als hij geen van deze structuren bevat.

Op dit moment zijn er nog geen snelle algoritmen bekend om optimale boomdecomposities te berekenen voor grafen in het algemeen. In de hoofdstukken 7, 8 en 9 bekijken we een aantal speciale klassen grafen.

In hoofdstuk 7 laten we zien dat er efficiënte algoritmen zijn om de boombreedte te *benaderen* voor grafen uit belangrijke klassen van perfecte grafen, zoals voor complementen van getrianguleerde grafen, splitgrafen, convexe grafen en permutatie grafen. Voor al deze klassen is er een algoritme dat een *pad*decompositie levert met een breedte die hoogstens een constante factor van de *boom*breedte afwijkt. Voor de klasse van cocomparability grafen geven we een algoritme dat een paddecompositie levert met breedte hoogstens k^2, waarbij k de boombreedte van de graaf is. Dit is een verrassend resultaat, temeer daar het beslissingsprobleem BOOMBREEDTE$\leq$ k NP-volledig is voor cobipartite grafen, en dus ook voor cocomparability grafen.

In hoofdstuk 8 geven we een polynomiaal algoritme dat de boombreedte bepaalt voor de klasse van chordale bipartite grafen. Dit resultaat is van belang ook omdat het berekenen van de boombreedte een NP-moeilijk probleem is voor bipartite grafen in het algemeen.

In hoofdtuk 9 geven we een lineaire tijd algoritme om de boombreedte van permuatie grafen te bepalen. Bovendien laten we zien dat er een generalisatie is naar cocomparability grafen waarvan de dimensie van het complement begrensd is door een constante. Een zijdelings resultaat is dat, voor cocomparability grafen in het algemeen, de padbreedte en boombreedte gelijk zijn.

We willen hier opmerken dat recentelijk gebleken is dat de resultaten van de hoofdstukken 8 en 9 op de volgende manier gegeneraliseerd kunnen worden. Beschouw een klasse $\mathcal{G}$ van grafen waarvoor er een polynomiaal algoritme bestaat dat voor elke graaf van $\mathcal{G}$ de verzameling van alle minimale separatoren levert. Dan is er een polynomiaal algoritme dat de boombreedte berekent voor elke graaf uit $\mathcal{G}$. Dit resultaat toont aan dat de boombreedte in polynomiale tijd berekend kan worden voor veel klassen van grafen zoals permutatie grafen, circulaire permutatie grafen, trapezoïde grafen, cirkelboog grafen, cirkel grafen, afstands-ervende grafen en chordale bipartite grafen.

In hoofdstuk 10 bekijken we het probleem van het vinden van benaderings

algoritmen voor de boombreedte van grafen in het algemeen. We geven twee algoritmen die gebruik maken van verschillende, al bestaande, algoritmen voor het vinden van zogenaamde gebalanceerde separatoren. Het eerste algoritme dat we beschrijven gebruikt resultaten van Alon et al. [1] en levert een paddecompositie met breedte $O(\sqrt{kn})$ voor grafen met boombreedte hoogstens k. We bewijzen dat dit algoritme geïmplementeerd kan worden zodanig dat het $O((nk)^{3/2})$ tijd vergt. Het tweede algoritme gebruikt als subroutine een algoritme van Leighton en Rao [74] wat een $O(\log n)$ benadering levert voor een gebalanceerde separator. Met behulp hiervan geven we een algoritme dat een boomdecompositie levert met een breedte die hoogstens een factor $O(\log n)$ keer te groot is.

In hoofdstuk 11 en 12 geven we expliciete constructies van algoritmen die voor elke k bepalen of de padbreedte respectievelijk de boombreedte van een graaf hoogstens k is en indien dit het geval is een pad- of boomdecompositie leveren met optimale breedte. We bewijzen dat deze algoritmen geïmplementeerd kunnen worden zodanig dat ze $O(n \log n)$ tijd vergen. We maken gebruik van een bestaand algoritme van B. Reed [107], dat voor elke k een algoritme geeft dat óf een boomdecompositie van de graaf levert met breedte hoogstens $3k + 2$, óf vaststelt dat de boombreedte van de graaf groter is dan k. We laten zien dat er lineaire tijd algoritmen zijn die, met behulp van zo'n benadering, de exacte padbreedte en boombreedte bepalen. In [18] is recentelijk aangetoond dat met behulp van deze resultaten voor elke k in lineaire tijd een pad- of boomdecompositie berekend kan worden met breedte hoogstens k, indien deze decompositie bestaat.

Curriculum Vitae

27 oktober 1957	Geboren te Tilburg.
1970-1977	Middelbare school: St. Janslyceum te 's Hertogenbosch. Atheneum B.
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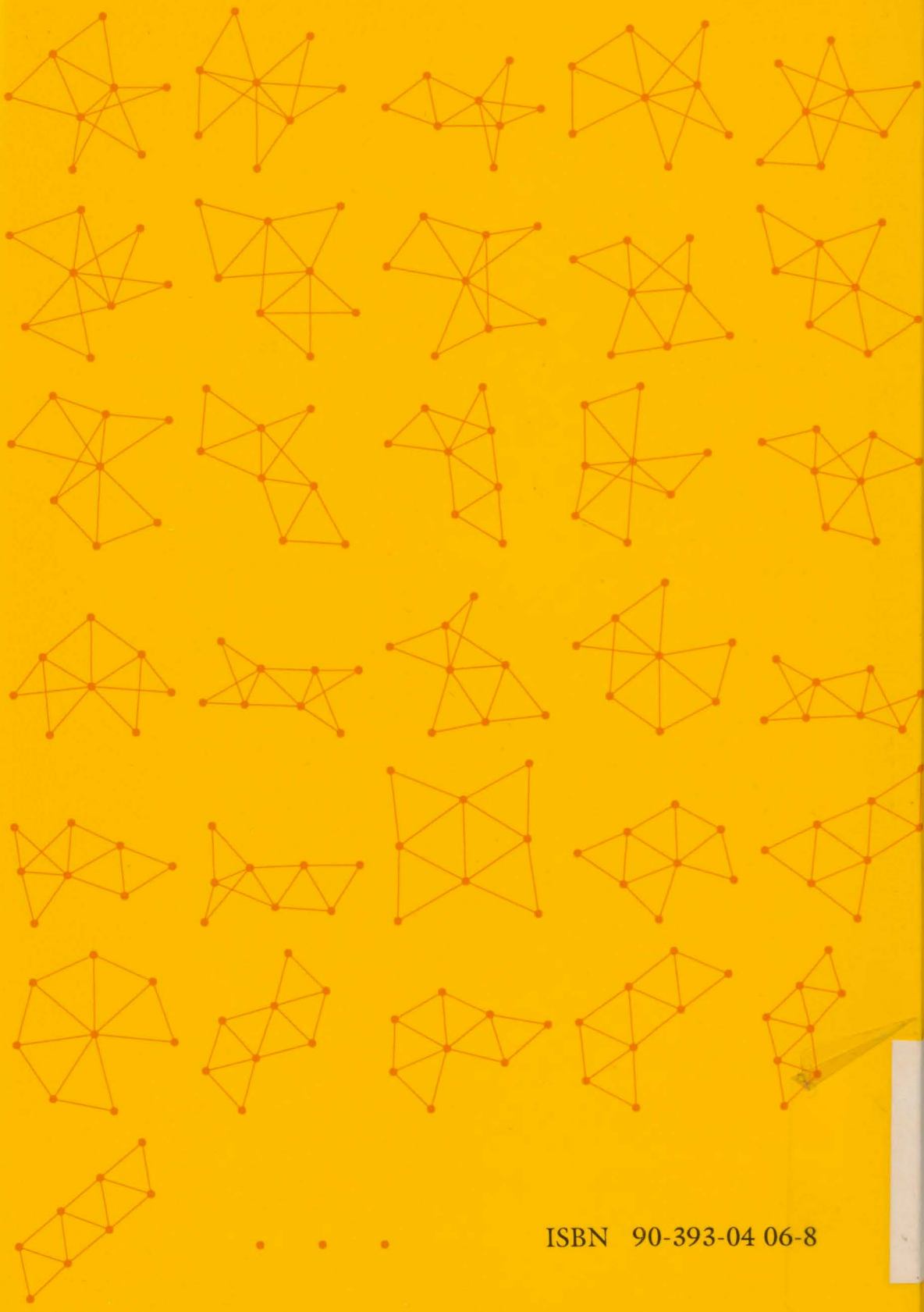

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