Differentiable stacks: stratifications, measures and deformations

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Differentiable stacks: stratifications, measures and deformations

Differentieerbare Stacks : stratificaties, maten en deformaties

(met een samenvatting in het Nederlands)

Proefschrift

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Aos meus pais e Miguel

Contents

Introduction

The core of this thesis consists of two studies on the differential geometry of Lie groupoids and of the singular (i.e., not smooth) spaces modelled by them, called differentiable stacks. The first deals with measures on differentiable stacks (Chapter 4) and the second is about deformations of Lie groupoids and deformation cohomology (Chapter 5). These two pieces of work are complemented by some expository material focused on a special kind of differentiable stacks - those arising as orbit spaces of proper Lie groupoids (Chapters 2 and 3).

Lie groupoids

Lie groups model global symmetries of geometric objects. We can represent a symmetry of an object as an arrow from that object to itself. Lie groupoids are a generalization of Lie groups where we consider several objects at once and symmetries, i.e., arrows between them. They have smooth structure maps similar to those of Lie groups, such as a multiplication m (but only defined for pairs of arrows that "match"), inversion, etc., satisfying group-like axioms.

More precisely, a **Lie groupoid** $\mathcal{G} \rightrightarrows M$ consists of two smooth manifolds, $\mathcal{G}$ and M, called the space of arrows and the space of objects respectively, together with the following smooth structural maps:

- $s, t : \mathcal{G} \to M$ are submersions, called the **source** and **target** respectively;
- $u : M \to \mathcal{G}$, called the **unit**;
- $i : \mathcal{G} \to \mathcal{G}$, called the **inversion**;
- $m : \mathcal{G}^{(2)} \to \mathcal{G}$, called the **multiplication** or **composition**, where

  $$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \mid s(g) = t(h)\}$$

  is the space of composable arrows

and which are required to satisfy group-like axioms. One can spell these axioms in a short form by saying that $\mathcal{G}$ is required to be a category, with all arrows invertible, where the structure maps of the category are precisely s, t, m, u, i.

Many geometric structures can be encoded by Lie groupoids. For example: Lie groups (Lie groupoids with a single object), manifolds (Lie groupoids which only have identity arrows), Lie group actions on manifolds, submersions, principal bundles, foliations, etc. As such, Lie groupoids provide a unified framework for the study of several geometric structures. For example, one can relate deformation theories of all the examples mentioned to the deformation theory of Lie groupoids, as discussed in Chapter 5.

Differentiable stacks

Lie groupoids can serve as "atlases" for singular (i.e., not smooth) spaces. Almost by definition, the differential geometry of a singular space X can be difficult to handle, given its non-smooth nature. The advantage of describing X by means of a Lie groupoid $\mathcal{G}$ is that we can use classical differential-geometric techniques on $\mathcal{G}$ in order to understand the differential geometry of X. As an example, one can in this way make sense of measures and integration on X, as done in Chapter 4.

The way in which a singular space X is modelled by means of a Lie groupoid $\mathcal{G}$ is by realizing X as the orbit space of $\mathcal{G}$: any Lie groupoid $\mathcal{G} \rightrightarrows M$ defines an equivalence relation on M:

$$x \sim y \Leftrightarrow \text{there is an arrow } g \in \mathcal{G} \text{ such that } g : x \to y.$$

The equivalence classes are called the **orbits** of the groupoid; the quotient of M by this relation, endowed with the quotient topology, is called the **orbit space** of $\mathcal{G}$ and is denoted by $M/\mathcal{G}$. The orbit space of an action of a Lie group on a manifold, or the leaf space of a foliation are examples of spaces that can be modelled in this way.

One should keep in mind that the actual quotient *topological space* $M/\mathcal{G}$ may be very pathological and uninteresting; for that reason, when one refers to the "orbit *space*" one often has in mind much more than just the topological space itself. One way to make sense of the "orbit space" in a satisfactory way is to study it as a differentiable stack.

Differentiable stacks have recently received increased attention [5, 8, 11, 14, 33, 48, 51, 57, 68, 100, 103, 110]. While they are inspired by the notion of stack from algebraic geometry, differentiable stacks are better behaved than abstract stacks - they are those stacks in the differentiable setting which are "geometric", meaning that they are presentable by Lie groupoids (i.e., realized as orbit spaces of Lie groupoids):

A **differentiable stack atlas** on a topological space X is given by a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $f : M/\mathcal{G} \longrightarrow X$. A **differentiable stack** is a topological space equipped with an equivalence class of differentiable stack atlases. We use the notation $M//\mathcal{G}$ to denote the stack presented by $\mathcal{G}$.

The idea is that the geometry of a differentiable stack is encoded in the "transverse geometry" of any groupoid presenting it (which intuitively should correspond to the geometry of its orbit space); groupoids having "the same transverse geometry" give rise to equivalent atlases.

The appropriate equivalence relation among groupoids which recognizes when two groupoids have the same transverse geometry is that of Morita equivalence. As such, differentiable stacks are often studied as Morita equivalence classes of Lie groupoids; this point of view has problems if one wants to talk about morphisms between differentiable stacks; nonetheless, it is enough for the scope of this thesis, where we focus our attention on the geometry of a fixed differentiable stack at a time and it lets us avoid unnecessary technicalities.

A simple class of differentiable stacks is that of **orbifolds**. These are spaces that are locally the quotient of Euclidean space by a *finite* group action. Any orbifold can be modelled by a particular kind of Lie groupoid "atlas" and this is a very convenient way to model them, first proposed by Moerdijk and Pronk [73].

Another important class of differentiable stacks, which generalizes orbifolds, are **orbispaces**. These are spaces that are locally a quotient of Euclidean space by actions of *compact* Lie groups [85]. It is not known whether any classical orbispace (i.e., given by charts) can be modelled by a Lie groupoid. In the following we focus our attention on those that can, or in other words, on those orbispaces which are differentiable stacks.

Orbispaces

Those orbispaces which are differentiable stacks are presented by proper Lie groupoids. Properness is a compactness condition: $\mathcal{G} \rightrightarrows M$ is proper if for any compact subsets K, L of M, the subset of $\mathcal{G}$ consisting of arrows with source in K and target in L is compact as well. It generalizes the notions of compactness of a Lie group and properness of a Lie group action.

Proper Lie groupoids admit normal form theorems [25, 28, 32, 109, 112] describing neighbourhoods of the orbits, similar to the Tube theorem in the theory of proper Lie group actions [37]. This is the central tool that permits studying the rich structure of orbispaces.

If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid presenting a differentiable stack $M//\mathcal{G}$, one can make sense of smooth the ring $C^\infty(M//\mathcal{G})$ of smooth functions on $M//\mathcal{G}$: it is identified with the ring of smooth functions on M which are constant along the orbits, via the pullback by the projection map $\pi : M \longrightarrow M/\mathcal{G}$. When $\mathcal{G}$ is a proper Lie groupoid, $C^\infty(M//\mathcal{G})$ has a richer structure than in the general case. For example, the topological space $M/\mathcal{G}$ can be reconstructed from its real spectrum.

Another feature of orbispaces is that although they are in general not manifolds, they do admit a canonical stratification - a decomposition into manifolds which fit together nicely.

Transverse measures

The measures we work with are Radon measures on locally compact Hausdorff spaces X, interpreted as positive linear functionals on the space $C_c(X)$ of compactly supported continuous functions, or $C_c^\infty(X)$ if X is a manifold. Therefore, for a Lie groupoid $\mathcal{G} \rightrightarrows M$ (representing the stack $M//\mathcal{G}$), the main question is how to define the space $C_c^\infty(M//\mathcal{G})$. To make sure that the resulting notion of measure (called transverse measure for $\mathcal{G}$) is intrinsic to the differentiable stack, we have to prove that the notion of transverse measure for Lie groupoids is invariant under Morita equivalences.

On manifolds X, one usually works with "geometric measures", i.e., measures induced by densities $\rho \in \mathcal{D}(X)$, i.e., sections of the density bundle $\mathcal{D}_X$ (so that the resulting integration is based on the standard Lebesgue integration). We will explain that the same discussion applies to differentiable stacks.

The motivation for this work comes from the study of Poisson manifolds of compact type [21, 22, 23] and from Weinstein's work on volumes of differentiable stacks [110].

Deformations and cohomology

A central problem in geometry is that of understanding the behaviour of geometric structures under deformations. Typically, the final goal is to describe the moduli space of such structures up to an appropriate equivalence relation. Each class of geometric structures comes with its deformation theory, which generally includes a cohomology theory that controls such deformations (cf. [9, 49, 56, 62, 63, 80, 82, 83]).

Morally, the cohomology represents the tangent space to the moduli space under consideration. Therefore, vanishing of cohomology is usually called infinitesimal rigidity and one of the main questions is the guiding principle

$$\textit{Infinitesimal rigidity} \Longrightarrow \textit{rigidity}. \tag{1}$$

We study deformations of a Lie groupoid $\mathcal{G}$ and the resulting deformation cohomology $H^*_{\mathrm{def}}(\mathcal{G})$, which can be used to prove several rigidity results. Rigidity of a geometric structure σ means that any other structure σ' which is a small deformation of σ is in the same equivalence class as σ, or in other words, that σ is an isolated point in the moduli space.

In general, a family $\{M_\varepsilon \mid \varepsilon \in I\}$ of manifolds smoothly parameterized by a real parameter ε can be understood as a manifold $\tilde{M}$ together with a submersion $\tilde{\pi} : \tilde{M} \longrightarrow I$, so that M_ε is just the fibre of π above ε; similarly for families of groupoids $\mathcal{G}_\varepsilon$. We can see such a family as a deformation of the groupoid $\mathcal{G}_0$.

The deformation cohomology is built in such a way that deformations of a groupoid $\mathcal{G}$ give rise to 2-cocycles inducing elements in $H^2_{\mathrm{def}}(\mathcal{G})$, by considering the variation of the structure maps of the groupoid ($\frac{d}{d\varepsilon}m_\varepsilon$, etc); 1-cochains that transgress (integrate) these 2-cocycles (when they exist) are then used to produce flows that allow us to prove rigidity results, as expected from (1).

One of the main sources of inspiration for this work is the theory of Kodaira and Spencer on deformations of complex manifolds [56]. Other sources are several related deformation theories (for example those of foliations [9, 49] and equivariant geometry) that fit our framework, such as:

- Palais' theory of deformations of actions of Lie groups [82], [83].
- Nijenhuis and Richardson's work on deformations of Lie groups [80].
- Rigidity in Poisson geometry by Crainic, Loja Fernandes and Mărcuț [19, 62, 63].

Outline of the chapters

Let us give a brief description of what is done in each chapter. The short intro at the beginning of each chapter and the small descriptions at the start of each such subsection give a more comprehensive account of the content of each chapter.

Chapter 1 This chapter recalls some background on Lie groupoids, including Morita equivalences, some basics on orbifolds and orbispaces, local structure of proper Lie groupoids and smooth functions on orbit spaces.

Chapter 2 In this chapter we recall three approaches to studying smooth structures on singular spaces; most of the material of this chapter is known - we review some of the basic concepts, compare the three approaches and look at how the orbit space of a proper Lie groupoid fits into each of the pictures.

Chapter 3 We discuss the canonical orbit type stratification associated to proper Lie group actions and the similar canonical stratification for proper Lie groupoids; most of the material is standard, but we try to clarify some points that are not clear in the literature. We also relate the stratifications on the orbit space of a proper Lie groupoid with the smooth structures of Chapter 2.

Chapter 4 This chapter deals with measures and densities on differentiable stacks. We introduce the notions of transverse measure and transverse density for any Lie groupoid, generalizing Haefliger's approach for foliations [45]. We then prove some fundamental results on transverse measures and densities:

For general Lie groupoids we prove Morita invariance (Theorem 4.19), which lets us interpret transverse measures as objects associated to stacks; a Stokes formula (Proposition 4.42) which provides reinterpretations in terms of (Ruelle-Sullivan type) algebroid currents and a Van Est isomorphism (Theorem 4.49).

In the proper case we reduce the theory to classical (Radon) measures on the underlying space (Theorem 4.30), we provide explicit (Weyl-type) formulas (Corollaries 4.32, 4.33) that shed light on Weinstein's notion of volumes of differentiable stacks; in particular, in the symplectic groupoid case, we prove the conjecture left open in [110] (Corollary 4.41). We also revisit the notion of modular class and of Haar systems.

This chapter is based on the paper [24].

Chapter 5 This chapter deals with deformations of Lie groupoids. We introduce the deformation cohomology of a Lie groupoid (Definition 5.11), which provides an intrinsic model for the cohomology of a Lie groupoid with values in its adjoint representation (Lemma 5.53). We then prove several fundamental properties of the deformation cohomology including Morita invariance (Theorem 5.58), a Van Est theorem (Theorem 5.55), and a vanishing result in the proper case (Theorem 5.41). Finally, we use the deformation cohomology in order to prove rigidity and normal form results (e.g. Theorems 5.50 and 5.51). This chapter is based on the paper [25].

Chapter 1

Background

1.1 Lie groupoids

A (Lie) groupoid is a generalization of the concept of a (Lie) group where we consider not the symmetries, or transformations, of an object, but of possibly many objects, taking into account both their internal symmetry and the symmetries between different objects. If we think of a symmetry between two objects as an arrow between them, we are led to the following definition.

Definition 1.1. A **Lie groupoid** consists of two smooth manifolds, $\mathcal{G}$ and M, called the space of arrows and the space of objects respectively, together with the following smooth structural maps:

- $s, t : \mathcal{G} \to M$ are submersions, called the **source** and **target** respectively;
- $u : M \to \mathcal{G}$, called the **unit**;
- $i : \mathcal{G} \to \mathcal{G}$, called the **inversion**;
- $m : \mathcal{G}^{(2)} \to \mathcal{G}$, called the **multiplication** or **composition**, where

  $$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \mid s(g) = t(h)\}$$

  is the **space of composable arrows** (which is a manifold because s and t are submersions).

These maps are also often denoted in a more suggestive way by $u(x) = 1_x$, $i(g) = g^{-1}$ and $m(g, h) = gh$. Additionally, these structural maps are required to satisfy the following group-like (or categorical) axioms:

- $s(1_x) = t(1_x) = x$;
- $1_{t(g)}g = g1_{s(g)} = g$;
- $s(g^{-1}) = t(g)$ and $t(g^{-1}) = s(g)$;
- $gg^{-1} = 1_{t(g)}$ and $g^{-1}g = 1_{s(g)}$;

- $s(gh) = s(h)$ and $t(gh) = t(g)$, whenever $(g, h) \in \mathcal{G}^{(2)}$;
- associativity: $g(hk) = (gh)k$, whenever $(g, h), (h, k) \in \mathcal{G}^{(2)}$,

for any $g, h, k \in \mathcal{G}$ and $x \in M$. An arrow g with source x and target y is sometimes denoted more graphically by $g : x \to y$, $x \xrightarrow{g} y$ or $y \xleftarrow{g} x$ and we commonly denote the groupoid $\mathcal{G}$ over M by $\mathcal{G} \rightrightarrows M$.

A more concise way to define a groupoid is to say that a groupoid is a small category in which all arrows are invertible. It is a Lie groupoid if the space of arrows and the space of objects are both smooth manifolds, all structural maps are smooth, and the source and target maps are submersions.

Some references with introductions to the theory of Lie groupoids are [13, 20, 59, 72].

Remark 1. The space of arrows $\mathcal{G}$ is not required to be Hausdorff, but the space of objects M and the fibres of the source map $s : \mathcal{G} \to M$ are. This is done in order to accommodate several natural examples of groupoids for which the space of arrows may fail to be Hausdorff. A typical source of such examples is foliation theory.

Remark 2. From the definition of a Lie groupoid $\mathcal{G} \rightrightarrows M$ we can conclude that the inversion map is a diffeomorphism of $\mathcal{G}$ and that the unit map is an embedding $u : M \hookrightarrow \mathcal{G}$. We often identify the base of a groupoid with its image by the unit embedding.

Example 1.2.

- (Lie groups) Any Lie group G can be seen as a Lie groupoid over a point $G \rightrightarrows \{*\}$.
- (Bundles of Lie groups; vector bundles) A bundle of Lie groups parameterized by a manifold M is the same as a Lie groupoid $\mathcal{G}$ for which the source map coincides with the target map. As a particular case, any vector bundle gives rise to a Lie groupoid in this way, with source and target equal to the projection of the vector bundle, and fibrewise addition as the composition.
- (Unit groupoids) Any smooth manifold M can be seen as a Lie groupoid $M \rightrightarrows M$, with only the identity arrows, called the unit groupoid of M.
- (Pair groupoids) Associated to any manifold M we can also construct the pair groupoid $M \times M \rightrightarrows M$, with structure maps $s(x, y) = y$, $t(x, y) = x$, $(x, y)^{-1} = (y, x)$, $1_x = (x, x)$ and $(x, y)(y, z) = (x, z)$.
- (Submersion groupoids) Given any submersion $\pi : M \to B$ there is a groupoid $M \times_\pi M \rightrightarrows M$, for which the arrows are the pairs (x, y) such that $\pi(x) = \pi(y)$, and the structure maps are defined in the same way as with the pair groupoid. This is called the submersion groupoid of π and it is sometimes denoted by $\mathcal{G}(\pi)$. In the particular case of π being the identity map of M we recover the unit groupoid; when B is a point, we recover the pair groupoid of M.
- (Action groupoids) Let G be a Lie group acting smoothly on a manifold M. Then we can form the action Lie groupoid $G \ltimes M \rightrightarrows M$. The objects are the points of M, and the arrows are pairs $(g, x) \in G \times M$. The structure maps are defined by $s(g, x) = x$, $t(g, x) = g \cdot x$, $1_x = (e, x)$, $(g, x)^{-1} = (g^{-1}, g \cdot x)$ and $(g, h \cdot x)(h, x) = (gh, x)$.

- (Gauge groupoids) Let $P \to M$ be a principal bundle with structure group G. Then we can take the quotient $(P \times P)/G$ of the pair groupoid of P by the diagonal action of G, to obtain a Lie groupoid over M called the gauge groupoid of P and denoted by $Gauge(P) \rightrightarrows M$.

- (Tangent groupoids) For any Lie groupoid $\mathcal{G} \rightrightarrows M$, applying the tangent functor gives us a groupoid $T\mathcal{G} \rightrightarrows TM$, which we call the tangent groupoid of $\mathcal{G}$, for which the structural maps are the differential of the structural maps of $\mathcal{G}$.

Structure of Lie groupoids

Definition 1.3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $x \in M$. The subsets $s^{-1}(x)$ and $t^{-1}(x)$ of $\mathcal{G}$ are called the **source-fibre** of x and the **target-fibre** of x respectively (or s-fibre and t-fibre). The subset $\mathcal{G}_x := \{g \in \mathcal{G} \mid s(g) = t(g) = x\} \subset \mathcal{G}$ is called the **isotropy group** of x.

Definition 1.4. Any Lie groupoid $\mathcal{G} \rightrightarrows M$ defines an equivalence relation on M such that two objects x and y are related if and only if there is an arrow $g \in \mathcal{G}$ such that $s(g) = x$ and $t(g) = y$. The equivalence classes are called the **orbits** of the groupoid and the orbit of a point $x \in M$ is denoted by $\mathcal{O}_x$. A subset of M is said to be **invariant** if it is a union of orbits. Given a subset U of M, the **saturation** of U, denoted by $\langle U \rangle$, is the smallest invariant subset of M containing U.

The quotient of M by this relation, endowed with the quotient topology, is called the **orbit space** of $\mathcal{G}$ and is denoted by $M/\mathcal{G}$.

The following result gives more information about these pieces of a groupoid (cf. e.g. [72]).

Proposition 1.5 (Fundamental structure of Lie groupoids)**.** *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $x, y \in M$. Then:*

1. *the set of arrows from x to y, $s^{-1}(x) \cap t^{-1}(y)$ is a Hausdorff submanifold of $\mathcal{G}$;*
2. *the isotropy group $\mathcal{G}_x$ is a Lie group;*
3. *the orbit $\mathcal{O}_x$ through x is an immersed submanifold of M;*
4. *the s-fibre of x is a principal $\mathcal{G}_x$-bundle over $\mathcal{O}_x$, with projection the target map t.*

The partition of the manifolds into connected components of the orbits forms a foliation, which is possibly singular, in the sense that different leaves might have different dimension. To give an idea of some different kinds of singular foliations that might occur let us look at two very simple examples coming from group actions.

Example 1.6. Let the circle S^1 act on the plane $\mathbb{R}^2$ by rotations. Then the leaves of the singular foliation on the plane corresponding to the associated action groupoid are the orbits, i.e., the origin and the concentric circles centred on it.

Let now $(\mathbb{R}_+, \times)$ act on the plane $\mathbb{R}^2$ by scalar multiplication. The leaves of the corresponding singular foliation are the origin and the radial open half-lines.

Note that the first example has a Hausdorff orbit space; in the second example, on the other hand, there is a point in the orbit space which is dense, defined by the orbit composed of the origin.

Note that the inversion map induces diffeomorphisms between the source and target-fibres at each point. As for translations by arrows, we can apply a right-translation by $y \xleftarrow{g} x$ only to arrows that have source equal to y, so that right-translation by g is a diffeomorphism

$$R_g : s^{-1}(y) \to s^{-1}(x), \quad h \mapsto hg.$$

Similarly, left-translation by g is a diffeomorphism

$$L_g : t^{-1}(x) \to t^{-1}(y), \quad k \mapsto gk.$$

These maps are related by

$$L_g \circ R_k = R_k \circ L_g, \quad i \circ R_g = L_{g^{-1}} \circ i,$$

whenever the expressions make sense.

Morphisms of Lie groupoids

Definition 1.7. A **Lie groupoid morphism** between $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is a smooth functor, i.e., a pair of smooth maps $\Phi : \mathcal{G} \to \mathcal{H}$ and $\phi : M \to N$ commuting with all the structure maps. An **isomorphism** is an invertible Lie groupoid morphism.

Definition 1.8. An **isomorphism** $\alpha : \Phi \cong \Psi$ between two Lie groupoid morphisms $\Phi, \Psi : (\mathcal{G} \rightrightarrows M) \to (\mathcal{H} \rightrightarrows N)$ is a smooth natural transformation, i.e., a smooth map $\alpha : M \to \mathcal{H}$ such that for each $x \in M$ we have an arrow $\alpha(x) : \phi(x) \longrightarrow \psi(x)$ of $\mathcal{H}$, satisfying

$$\alpha(y)\Phi(g) = \Psi(g)\alpha(x),$$

for all $g : x \longrightarrow y$.

Definition 1.9. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **Lie subgroupoid** of $\mathcal{G}$ is a pair $(\mathcal{H}, j)$ consisting of a Lie groupoid $\mathcal{H}$ and an injective immersive groupoid morphism $j : \mathcal{H} \to \mathcal{G}$.

The Lie algebroid of a Lie groupoid

Any Lie group can be differentiated at the unit in order to obtain its infinitesimal counterpart, a Lie algebra. The corresponding notion for a Lie groupoid $\mathcal{G} \rightrightarrows M$ is that of a Lie algebroid.

Definition 1.10. A **Lie algebroid** over a manifold M is a vector bundle $A \to M$, together with a Lie bracket on its space of sections $[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and a vector bundle map $\sharp : A \to TM$ called the **anchor**, satisfying the Leibniz rule

$$[\alpha, f\beta] = f[\alpha, \beta] + (\mathcal{L}_{\sharp(\alpha)} f)\beta,$$

for all $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$.

Remark 3. From the definition of a Lie algebroid we get as a consequence that the anchor induces a Lie algebra map $\sharp : \Gamma(A) \to \mathfrak{X}(M)$.

We now construct for a Lie groupoid $\mathcal{G} \rightrightarrows M$ the Lie algebroid $A = \mathrm{Lie}(\mathcal{G})$ associated to it. Unlike the case of Lie groups, we may have more than one unit and it only makes sense to talk about right-invariance for vector fields tangent to the source-fibres (because right-translations are defined only on the s-fibres).

Having this in mind, as a vector bundle, A is equal to the pullback by the unit map of the sub-bundle $T^s\mathcal{G} := \mathrm{Ker}\,(ds) \subset T\mathcal{G}$, consisting of s-vertical tangent vectors,

$$A = u^*(T^s\mathcal{G}).$$

In other words, the fibres of A are the tangent spaces to the s-fibres at the units of $\mathcal{G}$. The anchor is simply the restriction to A of the differential of the target map, $dt_{|A} : A \to TM$.

We define the bracket in terms of right-invariant vector fields. An s-vertical vector field $X \in \Gamma(\mathrm{Ker}\,(ds))$ is called **right-invariant** if

$$dR_g(X_h) = X_{gh}$$

for any $(g, h) \in \mathcal{G}^{(2)}$. Any section α of A gives rise to a right-invariant vector field $\overrightarrow{\alpha}$ by letting $\overrightarrow{\alpha}_g = dR_g(\alpha_{t(g)})$. The map $\alpha \mapsto \overrightarrow{\alpha}$ gives an isomorphism between the space of sections of A and the space of right-invariant vector fields, with inverse the evaluation at the units. The space of right-invariant vector fields is closed under the Lie bracket, so the bracket on $\Gamma(A)$ is determined by

$$\overrightarrow{[\alpha, \beta]} = [\overrightarrow{\alpha}, \overrightarrow{\beta}].$$

For any Lie algebroid $A \to M$, the image of the anchor, $\sharp(A) \subset TM$ defines a smooth generalized distribution in the sense of Sussmann [98], which is integrable, so it defines a singular foliation on M (cf. [102, Ch. 2]); when A is the Lie algebroid of a Lie groupoid $\mathcal{G}$, this singular foliation coincides with the partition of M given by the connected components of the orbits of $\mathcal{G}$.

Special classes of groupoids

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.

- (Regular and transitive groupoids) $\mathcal{G}$ is called **regular** if all the orbits have the same dimension. It is called **transitive** if it has only one orbit.

Example 1.11. The gauge groupoid $Gauge(P)$ of a principal bundle P is transitive. Conversely, if $\mathcal{G} \rightrightarrows M$ is a transitive groupoid, then $\mathcal{G}$ is isomorphic to $Gauge(s^{-1}(x))$, the gauge groupoid of the $\mathcal{G}_x$-principal bundle $s^{-1}(x) \xrightarrow{t} M$, for any object $x \in M$.

- (Étale groupoids) $\mathcal{G}$ is called **étale** if the source map $s : \mathcal{G} \to M$ is a local diffeomorphism.

Example 1.12. An action groupoid $G \ltimes M \rightrightarrows M$ is étale if and only if the group G is discrete.

- (Proper groupoids) $\mathcal{G}$ is called **proper** if it is Hausdorff and $(s, t) : \mathcal{G} \to M \times M$ is a proper map.

Example 1.13. For several of the examples of Lie groupoids described before the condition of properness becomes some sort of familiar compactness condition.

- A Lie group G is proper when seen as a Lie groupoid if and only if it is compact.
- The submersion groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : M \to B$ is always proper.
- An action groupoid is proper if and only if it is associated to a proper Lie group action.
- The gauge groupoid of a principal G-bundle is proper if and only if G is compact.
- If $\mathcal{G} \rightrightarrows M$ is a proper Lie groupoid and $S \subset M$ a submanifold such that the restriction $\mathcal{G}_S \rightrightarrows S$ is a Lie groupoid, then $\mathcal{G}_S$ is proper as well.

Actions and representations

A groupoid can act on a space fibred over its base, with an arrow $g : x \to y$ mapping the fibre over x onto the fibre over y.

Definition 1.14. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider a surjective smooth map $\mu : P \to M$. A **(left) action** of $\mathcal{G}$ on P along the map μ, which is called the **moment map**, is a smooth map

$$\mathcal{G} \times_M P = \{(g, p) \in \mathcal{G} \times P \mid s(g) = \mu(p)\} \to P,$$

denoted by $(g, p) \mapsto g \cdot p = gp$, such that $\mu(gp) = t(g)$, and satisfying the usual action axioms $(gh)p = g(hp)$ and $1_{\mu(p)}p = p$. We then say that P is a **left $\mathcal{G}$-space**.

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll, "\mu"] \\ M & & \end{tikzcd}$$

Example 1.15. Any Lie groupoid $\mathcal{G} \rightrightarrows M$ acts canonically on its base, with moment map the identity on M, by letting $g : x \to y$ act by $gx = y$; it also acts on $\mathcal{G}$ itself by left translations, with the target map $t : \mathcal{G} \to M$ as moment map, and action $g \cdot h = gh$.

Definition 1.16. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **representation** of $\mathcal{G}$ is a vector bundle E over M, together with a linear action of $\mathcal{G}$ on E, meaning that for each arrow $g : x \to y$, the induced map $g : E_x \to E_y$ is a linear isomorphism.

In general, given a groupoid $\mathcal{G}$, there might not be many interesting representations, so let us focus on some particular classes that have natural examples.

Example 1.17 (Representations of regular groupoids)**.** Let $\mathcal{G} \rightrightarrows M$ be a regular Lie groupoid, with algebroid A. Then $\mathcal{G}$ has natural representations on the kernel of the anchor map of A, denoted by $\mathfrak{i}$, and on the normal bundle to the orbits (which is the cokernel of the anchor), denoted by ν. An arrow $g \in \mathcal{G}$ acts on $\alpha \in \mathfrak{i}_{s(g)}$ by conjugation,

$$g \cdot \alpha = dR_{g^{-1}} \circ dL_g \alpha$$

and it acts on $[v] \in \nu_{s(g)}$ by the so called **normal representation**: if $g(\epsilon)$ is a curve on $\mathcal{G}$ with $g(0) = g$ such that $[v] = \left[\frac{d}{d\epsilon}_{|\epsilon=0} s(g(\epsilon))\right]$, then

$$g \cdot [v] = \left[\frac{d}{d\epsilon}_{|\epsilon=0} t(g(\epsilon))\right].$$

In other words, $g \cdot [v]$ can be defined as $[dt(X)]$, where $X \in T_g\mathcal{G}$ is any s-lift of v, meaning that $ds(X) = v$.

Example 1.18 (Restriction to an orbit)**.** If $\mathcal{G} \rightrightarrows M$ is any Lie groupoid, not necessarily regular, then the normal spaces to the orbits may no longer form a vector bundle. Nonetheless, we can still get a representation of an appropriate restriction of $\mathcal{G}$ on some appropriate normal bundle. To be precise, if $\mathcal{O}$ is an orbit of $\mathcal{G}$, then the restriction

$$\mathcal{G}_{\mathcal{O}} = \{g \in \mathcal{G} \mid s(g), t(g) \in \mathcal{O}\}$$

is a Lie groupoid over $\mathcal{O}$ and it has a natural representation on $\mathcal{N}\mathcal{O}$, the normal bundle to the orbit inside of M, defined in the following way. Denote by $\mathcal{N}_x := T_xM/T_x\mathcal{O}_x$ the fibre of $\mathcal{N}\mathcal{O}$ at x. Let $g \in \mathcal{G}_{\mathcal{O}}$ and $[v] \in \mathcal{N}_{s(g)}$. Then, just as in the regular case, we define $g \cdot [v]$ to be $[dt(X)]$, for any s-lift $X \in T_g\mathcal{G}$ of v. Furthermore given any point x in the orbit $\mathcal{O}$ we can restrict this representation to a representation of the isotropy Lie group $\mathcal{G}_x$ on the normal space $\mathcal{N}_x$, also called the **normal representation** (or isotropy representation) of $\mathcal{G}_x$.

Differentiable Lie groupoid cohomology

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and denote by $\mathcal{G}^{(k)}$ the space of strings of k composable arrows, meaning

$$\mathcal{G}^{(k)} = \{(g_1, \ldots, g_k) \in \mathcal{G}^k \mid s(g_i) = t(g_{i+1}),\ i = 1, \ldots, k-1\}.$$

Let E be a representation of $\mathcal{G}$. A **differentiable** k**-cochain** on $\mathcal{G}$ with values in E is a smooth map $c : \mathcal{G}^{(k)} \longrightarrow E$ such that $c(g_1, \ldots, g_k) \in E_{t(g_1)}$. The space of differentiable k-cochains is denoted by $C^k_{\text{diff}}(\mathcal{G}, E)$, or simply by $C^k(\mathcal{G}, E)$, and is equipped with a **differential** $d : C^k(\mathcal{G}, E) \to C^{k+1}(\mathcal{G}, E)$ defined by

$$\begin{aligned}(dc)(g_1, \ldots, g_{k+1}) = g_1 \cdot c(g_2, \ldots, g_{k+1}) + \sum_{i=1}^{k} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) \\ + (-1)^{k+1} c(g_1, \ldots, g_k).\end{aligned}$$

The **differentiable cohomology of** $\mathcal{G}$ **with coefficients in** E is the cohomology of the resulting cochain complex $(C^*(\mathcal{G}, E), d)$ and it is denoted by $H^*_{\text{diff}}(\mathcal{G}, E)$, or simply by $H^*(\mathcal{G}, E)$. For the particular case of the trivial representation of $\mathcal{G}$ on the trivial line bundle we use the notation $C^*(\mathcal{G})$ and $H^*(\mathcal{G})$, and we call this cohomology simply the differentiable cohomology of $\mathcal{G}$.

1.2 Morita equivalences

Recall that a morphism between Lie groupoids is a smooth functor and that two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if there are morphisms $\phi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{H} \to \mathcal{G}$ such that $\phi\psi = \mathrm{Id}_{\mathcal{H}}$ and $\psi\phi = \mathrm{Id}_{\mathcal{G}}$.

In this section we recall a coarser equivalence relation between Lie groupoids: that of Morita equivalence - which is crucial for most of the following chapters. Some references for this material are [31, 72, 77].

Description in terms of weak equivalences

Definition 1.19. Given two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, a Lie groupoid morphism $\phi : \mathcal{G} \to \mathcal{H}$ (where we denote the map on objects by $\phi_0 : M \to N$) is called a **weak equivalence** if it satisfies the following two conditions.

- It is **full and faithful**: the following diagram is a pullback

$$\begin{tikzcd} \mathcal{G} \arrow[r, "\phi"] \arrow[d, "(s,t)_{\mathcal{G}}"'] & \mathcal{H} \arrow[d, "(s,t)_{\mathcal{H}}"] \\ M \times M \arrow[r, "(\phi_0,\phi_0)"] & N \times N \end{tikzcd}$$

- It is **essentially surjective**: the map

$$t \circ pr_1 : \mathcal{H} \times_N M \to N$$

  is a surjective submersion, where the fibred product is taken over the source map of $\mathcal{H}$ and $\phi_0 : M \to N$.

Remark 4 (Reformulating essential surjectivity). Essential surjectivity of ϕ has the following equivalent and more geometric formulation.

- The morphism ϕ is essentially surjective if and only if $\phi_0 : M \to N$ is transverse to the orbits of $\mathcal{H}$, and $\phi_0(M)$ meets every orbit of $\mathcal{H}$.

Indeed, note first that in order for $t \circ pr_1 : \mathcal{H} \times_N M \to N$ to be surjective, $\phi_0(M)$ must have non-empty intersection with every orbit of $\mathcal{H}$. Let $x \in M$ and $h \in \mathcal{H}$ such that $\phi_0(x) = s(h)$, and let $y = t(h)$. Recall that if A is the Lie algebroid of $\mathcal{H}$, then the image of the anchor of A forms a smooth distribution that coincides with the tangent spaces to the orbits of $\mathcal{H}$, meaning that by definition $T_y\mathcal{O}_y = dt(A_y)$. So, for any $V = dt(\alpha) \in T_y\mathcal{O}_y$, with $\alpha \in A_y$, we have that $(dR_h(\alpha), 0) \in T_{(h,x)}(\mathcal{H} \times_N M)$, so that at least $T_y\mathcal{O}_y = dt(A_y)$ is always contained in the image of $d(t \circ pr_1)_h$.

Note that for any pair $(X, V) \in T_{(h,x)}(\mathcal{H} \times_N M)$ we have that X is an s-lift of $d\phi_0(V)$. We can now observe that $d(t \circ pr_1)_h(X, V) = dt(X)$ is by definition a representative for $h \cdot [d\phi_0(V)] \in \mathcal{N}_y$, where h is acting by the normal representation (as defined in Example 1.18). From this we can see that the image of $d(t \circ pr_1)_h$ is the whole of T_yN if and only if any normal vector $[v] \in \mathcal{N}_{\phi_0(x)}$ has a representative of the form $[d\phi_0(V)]$ (since we already knew that it contains $T_y\mathcal{O}_y$ for sure). This is equivalent to requiring that ϕ_0 is transverse to $\mathcal{O}_{\phi_0(x)}$ at $\phi_0(x)$.

Definition 1.20. Two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are said to be **Morita equivalent** if there is a third Lie groupoid $\mathcal{K}$ and weak equivalences $\mathcal{K} \to \mathcal{G}$ and $\mathcal{K} \to \mathcal{H}$.

Note that Morita equivalence is the smallest equivalence relation among Lie groupoids such that two Lie groupoids are equivalent if there exists a weak equivalence between them.

Example 1.21. Here are some examples of weak equivalences and Morita equivalences. Some more are given in the next section.

1. A weak equivalence between Lie groups is the same as an isomorphism.

2. For any transitive Lie groupoid, the inclusion of the isotropy group of any point in the base is a weak equivalence.

3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, let $N \subset M$ be a submanifold that intersects transversely every orbit it meets and let $\langle N \rangle$ denote the saturation of N. Then the inclusion of $\mathcal{G}_N \rightrightarrows N$ into $\mathcal{G}_{\langle N \rangle} \rightrightarrows \langle N \rangle$ is a weak equivalence. As a particular case, we can take N to be any open subset of M.

4. The groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : M \to N$ is Morita equivalent to $\pi(M)$.

Definition 1.22. A morphism of Lie groupoids that is both surjective and fully faithful is called a **surjective equivalence**.

Remark 5. If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent groupoids, then it is possible to realize the equivalence by means of two surjective equivalences $\mathcal{K} \to \mathcal{G}$ and $\mathcal{K} \to \mathcal{H}$ which are submersive at the level of objects (cf. e.g. [31, 72]).

Description in terms of bibundles

Definition 1.23. A **left $\mathcal{G}$-bundle** is a left $\mathcal{G}$-space P together with a $\mathcal{G}$-invariant surjective submersion $\pi : P \to B$. A left $\mathcal{G}$-bundle is called **principal** if the map $\mathcal{G} \times_M P \to P \times_\pi P$, $(g, p) \mapsto (gp, p)$ is a diffeomorphism. So for a principal $\mathcal{G}$-bundle, each fibre of π is an orbit of the $\mathcal{G}$-action and all the stabilizers of the action are trivial.

$$\begin{tikzcd}
\mathcal{G} \arrow[d, shift left] \arrow[d, shift right] & \curvearrowright & P \arrow[dll] \arrow[d, "\pi"] \\
M & & B
\end{tikzcd}$$

The notions of right action and right principal $\mathcal{G}$-bundle are defined in an analogous way.

Definition 1.24. A **Morita equivalence** between two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is given by a **principal $\mathcal{G} - \mathcal{H}$-bibundle**, i.e., a manifold P together with moment maps $\alpha : P \to M$ and $\beta : P \to N$, such that $\beta : P \to N$ is a left principal $\mathcal{G}$-bundle, $\alpha : P \to M$

is a right principal $\mathcal{H}$-bundle and the two actions commute: $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for any $g \in \mathcal{G}$, $p \in P$ and $h \in \mathcal{H}$. We say that P is a bibundle realising the Morita equivalence.

$$\begin{tikzcd}
\mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\curvearrowright"] & P \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[r, phantom, "\curvearrowleft"] & \mathcal{H} \arrow[d, shift left] \arrow[d, shift right] \\
M & & N
\end{tikzcd}$$

The relation with the previous definition of Morita equivalence using weak equivalences is explained in Proposition 1.31.

Example 1.25 (Isomorphisms)**.** If $f : \mathcal{G} \to \mathcal{H}$ is an isomorphism of Lie groupoids, then $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent. A bibundle can be given by the graph $Graph(f) \subset \mathcal{G} \times \mathcal{H}$, with moment maps $t \circ pr_1$ and $s \circ pr_2$, and the natural actions induced by the multiplications of $\mathcal{G}$ and $\mathcal{H}$.

Example 1.26 (Pullback groupoids)**.** Let $\mathcal{G}$ be a Lie groupoid over M and let $\alpha : P \to M$ be a surjective submersion. Then we can form the *pullback groupoid* $\alpha^*\mathcal{G} \rightrightarrows P$, that has as space of arrows $P \times_M \mathcal{G} \times_M P$, meaning that arrows are triples (p, g, q) with $\alpha(p) = t(g)$ and $s(g) = \alpha(q)$. The structure maps are determined by $s(p, g, q) = q$, $t(p, g, q) = p$ and $(p, g_1, q)(q, g_2, r) = (p, g_1 g_2, r)$.

The groupoids $\mathcal{G}$ and $\alpha^*\mathcal{G}$ are Morita equivalent, a bibundle being given by $\mathcal{G} \times_M P$. The left action of $\mathcal{G}$ has moment map $t \circ pr_1 : \mathcal{G} \times_M P \to M$ is given by $g \cdot (h, p) = (gh, p)$ and the right action of $\alpha^*\mathcal{G}$ has moment map $pr_2 : \mathcal{G} \times_M P \to P$ and is given by $(h, p) \cdot (p, k, q) = (hk, q)$.

Example 1.27 (Čech groupoids)**.** The following special case of the example above arises in Mayer-Vietoris type arguments concerning Lie groupoids (for example in Proposition 5.66 and the proof of Theorem 5.58).

Let $\mathcal{U} = \{U_i\}_{i \in J}$ be an open cover of M and let $\pi : \amalg U_i \to M$ denote the obvious surjective submersion. The **Čech groupoid of $\mathcal{G}$ w.r.t $\mathcal{U}$** is defined to be the groupoid

$$\check{\mathcal{G}}(\mathcal{U}) = \pi^*\mathcal{G} \rightrightarrows \amalg U_i,$$

and it is Morita equivalent to $\mathcal{G}$. Note that an arrow (x, g, y) of $\check{\mathcal{G}}(\mathcal{U})$ can be unambiguously denoted by (i, g, j) where $s(i, g, j) = y \in U_j$, and $t(i, g, j) = x \in U_i$.

Remark 6 (Decomposing Morita equivalences). Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalent, with bibundle P as above. Using that P is a principal bibundle, it is easy to check that

$$\alpha^*\mathcal{G} = P \times_M \mathcal{G} \times_M P \cong P \times_M P \times_N P \cong P \times_N \mathcal{H} \times_N P = \beta^*\mathcal{H},$$

as Lie groupoids over P.

This means that we can break a Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$, using a bibundle P, into a chain of simpler Morita equivalences: $\mathcal{G}$ is Morita equivalent to $\alpha^*\mathcal{G} \cong \beta^*\mathcal{H}$, which is Morita equivalent to $\mathcal{H}$. Therefore, in order to check invariance under Morita equivalences of a property or construction associated to a Lie groupoid, it is enough to check invariance under isomorphisms and under Morita equivalences between a Lie groupoid and its pullback by a surjective submersion. Of course, after we make the connection with the definition of Morita equivalence in terms of weak equivalences (Proposition 1.31), this corresponds to the idea that in order to check Morita invariance of a property, it is enough to check invariance under surjective equivalences which are submersive at the level of objects (as in Remark 5).

Lemma 1.28 (Morita equivalences preserve transverse geometry)**.** *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Morita equivalent Lie groupoids and let P be a bibundle realising the equivalence. Then P induces:*

1. *A homeomorphism between the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$,*

$$\Phi : M/\mathcal{G} \longrightarrow N/\mathcal{H};$$

2. *isomorphisms $\phi : \mathcal{G}_x \longrightarrow \mathcal{H}_y$ between the isotropy groups at any points $x \in M$ and $y \in N$ whose orbits are related by Φ, i.e., for which $\Phi(\mathcal{O}_x) = \mathcal{O}_y$;*

3. *isomorphisms $\tilde{\phi} : \mathcal{N}_x \longrightarrow \mathcal{N}_y$ between the normal representations at any points x and y in the same conditions as in point 2, which are compatible with the isomorphism $\phi : \mathcal{G}_x \longrightarrow \mathcal{H}_y$.*

Proof. First, let us define the map Φ: fix a point x in the base of $\mathcal{G}$. Then for any point x' in the orbit of x, the fibre $\alpha^{-1}(x')$ is a single orbit for the $\mathcal{H}$-action on P; it projects via β to a unique orbit of $\mathcal{H}$, which we define to be $\Phi(\mathcal{O}_x)$. Invariance of β under the action of $\mathcal{G}$ implies that $\Phi(\mathcal{O}_x)$ does not depend on the choice of x', so Φ is well defined. In order to see that Φ is a homeomorphism, note that bi-invariant open sets on P correspond to invariant opens on $\mathcal{G}$ and $\mathcal{H}$.

Let $x \in M$ and $y \in N$ be points such that their orbits are related by Φ. Then there is a $p \in P$ such that $\alpha(p) = x$ and $\beta(p) = y$. Any such p induces an isomorphism $\phi_p : \mathcal{G}_x \to \mathcal{H}_y$ between the isotropy groups at x and y, uniquely determined by the condition $gp = p\phi_p(g)$.

Moreover, p induces an isomorphism $\tilde{\phi}_p : \mathcal{N}_x \to \mathcal{N}_y$ between the normal representations at x and y, uniquely determined by

$$\tilde{\phi}([(d\alpha)_p(X_p)]) = [(d\beta)_p(X_p)], \quad \text{for all } X_p \in T_pP. \qquad \square$$

Comparison of descriptions using weak equivalences and bibundles

Finally, let us recall how the descriptions of Morita equivalences in terms of weak equivalences (as in Definition 1.20) and in terms of bibundles (as in Definition 1.24) relate to each other. See for example [31] or [77] for more details.

In order to state the relation, we need the notions of isomorphism between Morita equivalences and between bibundles.

Definition 1.29. An **isomorphism between two Morita equivalences** given by weak equivalences $\mathcal{G} \longleftarrow \mathcal{K} \longrightarrow \mathcal{H}$ and $\mathcal{G} \longleftarrow \mathcal{K}' \longrightarrow \mathcal{H}$ is given by a third Morita equivalence

$\mathcal{G} \longleftarrow \mathcal{K}'' \longrightarrow \mathcal{H}$, fitting into a diagram that commutes up to isomorphism (see Definition 1.8):

$$\begin{tikzcd}
\mathcal{G} \arrow[d, equal] & \mathcal{K} \arrow[l] \arrow[r] & \mathcal{H} \arrow[d, equal] \\
\mathcal{G} \arrow[d, equal] & \mathcal{K}'' \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathcal{H} \arrow[d, equal] \\
\mathcal{G} & \mathcal{K}' \arrow[l] \arrow[r] & \mathcal{H}
\end{tikzcd}$$

Definition 1.30. An **isomorphism of bibundles** between two $\mathcal{G} - \mathcal{H}$-bibundles P and P' is a smooth map $\phi : P \longrightarrow P'$ which is compatible with the actions of $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ and with the submersions $\alpha, \alpha' : P \longrightarrow M$ and $\beta, \beta' : P \longrightarrow M$.

Proposition 1.31. *There is a $1-1$ correspondence between isomorphism classes of Morita equivalences (given by weak equivalences) between Lie groupoids $\mathcal{G}$ and $\mathcal{H}$, and isomorphism classes of $\mathcal{G} - \mathcal{H}$-bibundles.*

Sketch of the proof. We briefly recall the proof from [31]. If we start with a Morita equivalence between $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ given by a principal bibundle P, then the pullback groupoid $\alpha^*\mathcal{G}$ (which is isomorphic to $\beta^*\mathcal{H}$) comes together with the obvious maps into $\mathcal{G}$ and $\mathcal{H}$, which are weak equivalences.

Conversely, let $\mathcal{K} \rightrightarrows P$ be a Lie groupoid, together with weak equivalences $\phi : \mathcal{K} \to \mathcal{G}$ and $\psi : \mathcal{K} \to \mathcal{H}$. The manifold $\mathcal{G} \times_M P \times_N \mathcal{H}$ is endowed with an action of $\mathcal{K}$ given by

$$k \cdot (g, p, h) = (g\phi(k)^{-1}, t(k), \psi(k)h).$$

The fact that ϕ is fully faithful implies that this action is free and proper; $\mathcal{G}$ and $\mathcal{H}$ act on the quotient manifold $(\mathcal{G} \times_M P \times_N \mathcal{H})/\mathcal{K}$ by multiplication on the first and last factor respectively. These actions define a principal bibundle because ϕ and ψ are weak equivalences.

These two constructions are inverses of each other up to isomorphism - we refer to [31] for the proof of this fact. □

1.3 Structure of proper Lie groupoids

Proper Lie groupoids generalize compact Lie groups and proper Lie group actions - for example, a Lie group, seen as a Lie groupoid, is proper if and only if it is compact; the action groupoid of a Lie group action is proper if and only if the action is proper.

Similarly to the theory of proper Lie group actions (cf. [37]), one can obtain slices and a normal form result around orbits, which pushes a bit further the analogy by telling us that in fact, proper Lie groupoids are locally (on a neighbourhood of each orbit) Morita equivalent to linear action groupoids associated to the action of compact groups.

Basic structure of proper Lie groupoids

Proposition 1.32. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then the orbit space $M/\mathcal{G}$ is Hausdorff and the isotropy group $\mathcal{G}_x$ is compact for every $x \in M$.*

Proof. Since the map $(s,t) : \mathcal{G} \to M \times M$ is proper, it is closed, and has compact fibres. This automatically implies that the isotropy groups are compact and since the orbit space is the quotient of M by a closed relation $(s,t)(\mathcal{G}) \subset M \times M$, it is Hausdorff. □

Proposition 1.33. *Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalent Lie groupoids. If one of them is proper, then the other one is proper as well.*

Proof. As mentioned in Remark 5, in order to prove invariance of a property, we may assume that we have a surjective equivalence $\phi : \mathcal{G} \to \mathcal{H}$. But then since ϕ is full and faithful, we have a pullback diagram relating the maps $(s,t) : \mathcal{G} \longrightarrow M \times M$ and $(s',t') : \mathcal{H} \longrightarrow N \times N$. The result follows from stability of proper maps (with Hausdorff domain) under pullback. □

Before we describe the normal form of a proper Lie groupoid around an orbit, let us recall, for motivation and for use in the rest of the text, the analogous normal form theorem for proper Lie group actions.

Whenever a Lie group G acts on a manifold M, we can differentiate the action to get an induced action of G on TM, the **tangent action** of G, defined by

$$g \cdot X = \frac{d}{d\epsilon}_{|\epsilon=0}(g \cdot x(\epsilon)),$$

where $X \in T_xM$ and $x(\epsilon)$ is a curve representing X.

For any point $x \in M$, if we restrict this action to an action of the isotropy group G_x, then we obtain a representation of G_x on T_xM. Since the action of G_x leaves the tangent space to the orbit through x invariant, we obtain an induced representation on the quotient $\mathcal{N}_x = T_xM/T_xO_x$, called the **isotropy representation** at x. This representation is used in the normal form around an orbit for a proper action of a Lie group, described by the Slice theorem, also called Tube theorem [37, p. 109].

Theorem 1.34 (Slice theorem for proper actions)**.** *Let a Lie group G act properly on a manifold M and let $x \in M$. Then there is an invariant open neighbourhood of x (called a tube for the action) which is equivariantly diffeomorphic to $G \times_{G_x} B$, where B is a G_x-invariant open neighbourhood of 0 in $\mathcal{N}_x$ (the isotropy representation).*

Let us return to the case of a proper Lie groupoid. First, let us recall that there is a pointwise version of properness.

Definition 1.35. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $x \in M$. The groupoid $\mathcal{G}$ is **proper at** x if every sequence $(g_n) \in \mathcal{G}$ such that $(s,t)(g_n) \to (x,x)$ has a converging subsequence.

The following result relates the notions of properness and of properness at a point.

Lemma 1.36. *([31]) A Lie groupoid is proper if and only if it is proper at every point and its orbit space is Hausdorff.*

Definition 1.37. Let $\mathcal{G}$ be a Lie groupoid over M and $x \in M$. A **slice** at x is an embedded submanifold $\Sigma \subset M$ of dimension complementary to $\mathcal{O}_x$ such that it is transverse to every orbit it meets and $\Sigma \cap \mathcal{O}_x = \{x\}$.

The following result gives us some information about the longitudinal (along the orbits) and the transverse structure of a groupoid $\mathcal{G}$, at a point x at which $\mathcal{G}$ is proper. For a proof we refer to [28].

Proposition 1.38. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid which is proper at $x \in M$. Then*

1. *The orbit $\mathcal{O}_x$ is an embedded closed submanifold of M;*
2. *there is a slice Σ at x.*

Local models and linearization

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\mathcal{O}$ an orbit of $\mathcal{G}$. We recall that the restriction

$$\mathcal{G}_{\mathcal{O}} = \{g \in \mathcal{G} \mid s(g), t(g) \in \mathcal{O}\}$$

is a Lie groupoid over $\mathcal{O}$. The normal bundle of $\mathcal{G}_{\mathcal{O}}$ in $\mathcal{G}$ is naturally a Lie groupoid over the normal bundle of $\mathcal{O}$ in M:

$$\mathcal{N}(\mathcal{G}_{\mathcal{O}}) := T\mathcal{G}/T\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{N}\mathcal{O} := TM/T\mathcal{O},$$

with the groupoid structure induced from that of $T\mathcal{G} \rightrightarrows TM$. The groupoid $\mathcal{N}(\mathcal{G}_{\mathcal{O}})$ is called the **local model**, or the linearization, of $\mathcal{G}$ at $\mathcal{O}$.

Recall also that the restricted groupoid $\mathcal{G}_O$ has a natural representation on the normal bundle to the orbit, called the normal representation, defined by $g \cdot [v] = [dt(X)]$, for any $v \in T_{s(g)}M$ and any s-lift $X \in T_g\mathcal{G}$ of v.

This representation can be restricted, for any $x \in M$, to a representation of the isotropy group $\mathcal{G}_x$ on $\mathcal{N}_x$, also called the **normal representation**, or isotropy representation, at x.

Using this representation, there is a more explicit description of the local model using the isotropy bundle by choosing a point x of $\mathcal{O}$. Let P_x denote the s-fibre of x and recall that it is a principal $\mathcal{G}_x$-bundle over $\mathcal{O}_x$ (Proposition 1.5). Then the normal bundle to $\mathcal{O}$ is isomorphic to the associated vector bundle

$$\mathcal{N}_{\mathcal{O}} \cong P_x \times_{\mathcal{G}_x} \mathcal{N}_x,$$

and the local model is

$$\mathcal{N}(\mathcal{G}_{\mathcal{O}}) \cong (P_x \times P_x) \times_{\mathcal{G}_x} \mathcal{N}_x.$$

The structure on the local model is given by

$$s([p, q, v]) = [q, v],\ t([p, q, v]) = [p, v],\ [p, q, v] \cdot [q, r, v] = [p, r, v].$$

Remark 7. Since $\mathcal{G}_{\mathcal{O}}$ is transitive, it is Morita equivalent to $\mathcal{G}_x$. Moreover, the linearization $\mathcal{N}(\mathcal{G}_{\mathcal{O}})$ is Morita equivalent to $\mathcal{G}_x \ltimes \mathcal{N}_x$.

The following linearization result is essential to most of the results in this text.

Theorem 1.39 (Linearization theorem for proper groupoids). *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\mathcal{O}$ be the orbit through $x \in M$. If $\mathcal{G}$ is proper at x, then there are neighbourhoods U and V of $\mathcal{O}$ such that $\mathcal{G}_U \cong \mathcal{N}(\mathcal{G}_{\mathcal{O}})_V$.*

The proof of the linearization result around a fixed point (an orbit consisting of a single point) was first completed by Zung [112]; together with previous results of Weinstein [109], it gave rise to a similar result to the one we present here.

The final version of the Linearization theorem that we present here has appeared in [28], where issues regarding which were the correct neighbourhoods of the orbits to be taken were solved; the geometric proof in *loc. cit.* served as motivation for the work of Chapter 5. In particular, we prove a generalization of the Linearization theorem around invariant submanifolds (Theorem 5.50).

Remark 8. Combining the Linearization theorem with the previous remarks on Morita equivalence, we conclude that any orbit $\mathcal{O}_x$ of a proper groupoid $\mathcal{G}$ has an invariant neighbourhood such that the restriction of $\mathcal{G}$ to it is Morita equivalent to $\mathcal{G}_x \ltimes \mathcal{N}_x$. For this we use also that $\mathcal{N}_x$ admits arbitrarily small $\mathcal{G}_x$-invariant open neighbourhoods of the origin which are equivariantly diffeomorphic to $\mathcal{N}_x$.

When we are interested in local properties of a groupoid, it is often enough to have an open around a point in the base, not necessarily containing the whole orbit, and the restriction of the groupoid to it. In this case it is possible to give a simpler model for the restricted groupoid [86, Cor. 3.11]:

Proposition 1.40 (Local model around a point). *Let $x \in M$ be a point in the base of a proper groupoid $\mathcal{G}$. There is a neighbourhood U of x in M, diffeomorphic to $O \times W$, where O is an open ball in $\mathcal{O}_x$ centred at x and W is a $\mathcal{G}_x$-invariant open ball in $\mathcal{N}_x$ centred at the origin, such that under this diffeomorphism the restricted groupoid $\mathcal{G}_U$ is isomorphic to the product of the pair groupoid $O \times O \rightrightarrows O$ with the action groupoid $\mathcal{G}_x \times W \rightrightarrows W$.*

Haar systems and averaging

One of the main features of proper groupoids is that it is possible to take averages of several objects (functions, Riemannian metrics, ...) to produce invariant versions of the same objects. This is done using a Haar system, in an analogous way to how one uses a Haar measure on a compact Lie group.

Definition 1.41. Given a Lie groupoid $\mathcal{G}$ over M, a (right) **Haar system** μ is a family of smooth measures $\{\ \mu^x \mid x \in M\}$ with each μ^x supported on the s-fibre of x, satisfying the properties

1. (Smoothness) For any $f \in C_c^\infty(\mathcal{G})$, the formula

$$I_\mu(f)(x) := \int_{s^{-1}(x)} f(g)\ d\mu^x(g)$$

defines a smooth function $I_\mu(\phi)$ on M.

2. (Right-invariance) For any $h \in \mathcal{G}$ with $h : x \to y$ and any $f \in C_c^\infty(s^{-1}(x))$ we have

$$\int_{s^{-1}(y)} f(gh)\ d\mu^y(g) = \int_{s^{-1}(x)} f(g)\ d\mu^x(g)$$

For such a Haar system, a **cut-off function** is a smooth function c on M satisfying

3. $s : supp(c \circ t) \to M$ is a proper map;
4. $\int_{s^{-1}(x)} c(t(g))\ d\mu^x(g) = 1$ for all $x \in M$.

Remark 9. Equivalently, condition 3 says that the intersection of the support of c with the saturation of any compact set is compact, or in other words, that the projection map onto the orbit space is a proper map when restricted to the support of c.

Proposition 1.42. *Any Lie groupoid admits a Haar system and cut-off functions exist for any proper Lie groupoid.*

Remark 10. One can also obtain a left-invariant Haar system, with each μ^x supported on the t-fibre of x and invariant under left-translations by elements of $\mathcal{G}$. To do so one may take the push forward of a right-invariant system by the inversion map. For proper groupoids, one can take the same cut-off function as for a right-invariant Haar system, which satisfies the appropriate versions of properties 3 and 4.

Haar systems are discussed in detail in the Appendix to Chapter 4. We will see in Subsection 4.9.2 how to construct Haar systems. The construction of a cut-off function for a proper groupoid is described in Proposition 4.66. For now let us illustrate how to use a Haar system in order to produce invariant objects with the following proposition.

Proposition 1.43. *$\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and $p : E \to M$ be a representation of $\mathcal{G}$. Then there exists an invariant inner product $\langle \cdot, \cdot \rangle$ on E, i.e., a family of inner products on the fibres of E such that*

$$\langle gv, gw \rangle = \langle v, w \rangle,$$

whenever v and w are in the same fibre and $g \in s^{-1}(p(v))$.

Proof. Start by choosing any inner product $\langle \cdot, \cdot \rangle'$ on E, a right-invariant Haar system on $\mathcal{G}$ and a cut-off function c. Define for any $v, w \in p^{-1}(x)$

$$\langle v, w \rangle = \int_{s^{-1}(x)} \langle gv, gw \rangle' c(t(g))\ d\mu(g).$$

This gives again an inner product on E, as it is smooth and positive, because of properties 1 and 4 of the Haar system and the cut-off function. It is invariant because of right-invariance of the Haar system. □

Another special feature of proper Lie groupoids that can be proved by an averaging argument is that, while in general it can be quite complicated to compute differentiable cohomology of a groupoid (defined in Section 1.1), for proper groupoids we have the following vanishing result.

Proposition 1.44 (Proposition 1 of [18])**.** *Let $\mathcal{G}$ be a proper Lie groupoid and E any representation of $\mathcal{G}$. Then*

$$H^k(\mathcal{G}, E) = 0, \quad \forall k \geq 1.$$

1.4 Orbifolds and proper étale groupoids

In this section we recall the definition of orbifolds and their relation with Lie groupoids; part of the reason for doing so is to motivate the notions of orbispace and of differentiable stacks, discussed in the next section.

Orbifolds are spaces which are locally modelled on quotients of Euclidean spaces by smooth actions of finite groups (and such that the actions are part of the structure). One way of trying to make this notion precise is defining orbifolds, similarly to manifolds, in terms of charts.

Definition 1.45. Let X be a topological space.

1. A **classical orbifold chart** of dimension n on X is a triple (U, G, ϕ), where $U \subset \mathbb{R}^n$ is a connected open subset, G is a finite group acting *effectively* by diffeomorphisms on U, and $\phi : U \to X$ is a G-invariant open map that induces a homeomorphism $\tilde{\phi} : U/G \longrightarrow \phi(U)$.

2. An **embedding** $\lambda : (U, G, \phi) \longrightarrow (V, H, \psi)$ between orbifold charts is a smooth embedding $\lambda : U \longrightarrow V$ such that $\psi \circ \lambda = \phi$.

3. A **classical orbifold atlas** on X is a family $\mathcal{U}$ of orbifold charts on X, which cover X, such that for any two orbifold charts (U, G, ϕ) and (V, H, ψ) in $\mathcal{U}$ and any point $x \in \phi(U) \cap \psi(V)$, there exists an open $\tilde{W} \subset \phi(U) \cap \psi(V)$ and an orbifold chart (W, K, χ) such that $\chi(W) = \tilde{W}$, and embeddings $\lambda_1 : (W, K, \chi) \longrightarrow (U, G, \phi)$ and $\lambda_2 : (W, K, \chi) \longrightarrow (V, H, \psi)$.

4. An atlas $\mathcal{U}$ is said to **refine** an atlas $\mathcal{V}$ is every chart of $\mathcal{U}$ has an embedding into a chart of $\mathcal{V}$.

5. Two atlases are said to be **equivalent** if they have a common refinement.

Remark 11. An important point to make is that the action of each group in the definition of an orbifold chart is *effective*. There is a different approach that allows for actions that are not necessarily effective, using groupoids (Definition 1.51). Provisionally let us consider the following definition of orbifolds.

Definition 1.46. A **(classical) orbifold** is a second-countable Hausdorff topological space together with an equivalence class of orbifold atlases.

Manifolds can be seen as orbifolds, as a smooth atlas induces an orbifold one. Global quotients of $\mathbb{R}^n$ by an effective action of a finite group of diffeomorphisms are also orbifolds, almost by definition. More generally, any foliated action of a compact connected Lie group G on a manifold M, i.e., such that the orbits of the action form a foliation) induces a canonical orbifold structure on the orbit space M/G (cf. Corollary 2.16 in [72]). The following result explains that these quotients encompass all examples.

Theorem 1.47 (Theorem 2.19 in [72])**.** *Any (classical) orbifold is isomorphic to the orbifold associated to the action of a compact connected Lie group with finite isotropy groups.*

We can also interpret this result as saying that any orbifold can be associated to the leaf space of a foliation, but one of a particular kind - given by the action of a compact connected Lie group. This brings us to the groupoid approach to the study of orbifolds, which originated in [46] and [73].

Leaf spaces and étale groupoids

The approach to studying foliations (and the transverse geometry of foliations) by means of Lie groupoids, as developed in Haefliger's work [46], is the following. First, to any foliation $(M, \mathcal{F})$ we can associate a holonomy groupoid $\mathrm{Hol}(M, \mathcal{F}) \rightrightarrows M$, whose arrows are determined by germs of holonomy transformations. The orbits of this groupoid are precisely the leaves of the foliation; hence the leaf space $M/\mathcal{F}$ is realized as the orbit space of $\mathrm{Hol}(M, \mathcal{F})$.

The groupoid $\mathrm{Hol}(M, \mathcal{F})$ can be replaced by a simpler, Morita equivalent, groupoid. Its restriction to a complete transversal $T \subset M$ of the foliation:

$$\mathrm{Hol}_T(M, \mathcal{F}) := \mathrm{Hol}(M, \mathcal{F})|_T$$

has "the same" orbit space as $\mathrm{Hol}(M, \mathcal{F})$ and has the extra property that it is *étale*.

Of course, we could choose a different complete transversal T' and we would like to end up with an equivalent model for the orbit space. The notion of Morita equivalence makes precise the idea of when two groupoids have "the same transverse geometry", or in other words, of when two groupoids "model the same orbit space" (see Lemma 1.28). Indeed, the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F})$ of a foliation is Morita equivalent to $\mathrm{Hol}_T(M, \mathcal{F})$, for any complete transversal T for $\mathcal{F}$ (cf. [46, Cor. 2.3.4]).

In summary, using this description one interprets leaf spaces (which are very singular in general) as Morita equivalence classes of étale groupoids; $\mathrm{Hol}_T(M, \mathcal{F})$ appear as "desingularizations" of the leaf spaces.

Orbifolds as groupoids

By Theorem 1.47, orbifolds arise as leaf spaces of a special kind. In the spirit of the approach to leaf spaces via Lie groupoids, orbifolds should then be modelled by étale Lie groupoids of a special kind. The precise relation between orbifolds and the groupoids modelling them was established by Moerdijk and Pronk in [73].

Definition 1.48. A Lie groupoid is called a **foliation groupoid** if all its isotropy groups are discrete (cf. [26]). An **orbifold groupoid** is a proper foliation groupoid (cf. [70]).

Note that any étale Lie groupoid is automatically a foliation groupoid. The following result explains the precise relation between the concepts.

Theorem 1.49 (Theorem 1 in [26])**.** *A Lie groupoid is a foliation groupoid if and only if it is Morita equivalent to an étale Lie groupoid.*

Remark 12. Since the property of being proper is invariant under Morita equivalences, the above theorem implies that an orbifold groupoid is a groupoid which is Morita equivalent to a proper étale Lie groupoid.

If $\mathcal{G}$ is an étale Lie groupoid, then each arrow of g induces a germ of a diffeomorphism as follows: let U be an open neighbourhood of g in $\mathcal{G}$ such that $s_{|U}$ is a diffeomorphism and let $\mathrm{Eff}(g)$ be the germ of $t \circ (s_{|U})^{-1} : s(U) \longrightarrow t(U)$ at $s(g)$. Restricting to the isotropy group $\mathcal{G}_x$ of a point $x \in M$, we obtain a map

$$\mathrm{Eff}_{|\mathcal{G}_x} : \mathcal{G}_x \longrightarrow Diff_x(M), \tag{1.1}$$

where $Diff_x(M)$ denotes the group of germs at x of diffeomorphisms of M.

Definition 1.50. An Lie groupoid $\mathcal{G} \rightrightarrows M$ is **effective** if it is étale and for all $x \in M$, the map (1.1) is injective.

Finally, the description of orbifolds in terms of groupoids is the following.

Definition 1.51. An **orbifold atlas** on a topological space X is given by an orbifold groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $f : M/\mathcal{G} \longrightarrow X$.

Two orbifold atlases $(\mathcal{G}, f)$ and $(\mathcal{H}, f')$ are **equivalent** if $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Morita equivalent, and the homeomorphism $\Phi : N/\mathcal{H} \longrightarrow M/\mathcal{G}$ induced by the Morita equivalence (see Lemma 1.28) satisfies $f \circ \Phi = f'$.

Definition 1.52. An **orbifold** is a topological space equipped with an equivalence class of orbifold atlases.

Note that by Theorem 1.49, any orbifold groupoid is Morita equivalent to a proper étale groupoid, so that any orbifold X can be represented by a proper étale groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $M/\mathcal{G} \longrightarrow X$.

Finally, the relation between classical orbifolds and orbifold groupoids is the following. Any classical orbifold atlas $\mathcal{U}$ for a classical orbifold gives rise to an effective proper Lie groupoid $\Gamma(\mathcal{U})$, such that equivalent atlases give rise to Morita equivalent effective proper groupoids [72, Prop. 5.29]. The next result singles out which orbifold groupoids arise from classical orbifold atlases.

Proposition 1.53 (part of Corollary 5.31 in [72])**.** *Any proper effective groupoid $\mathcal{G} \rightrightarrows M$ is weakly equivalent to the Lie groupoid $\Gamma(\mathcal{U})$ associated to a classical orbifold atlas $\mathcal{U}$ on $M/\mathcal{G}$.*

Remark 13. We have not defined morphisms between orbifolds, but let us mention that the notion of orbifold in terms of orbifold groupoids is more suited to do so than the classical definition. However, a large part of the literature actually defines not the category, but the 2-category of orbifolds - see [54, Ch. 8] for a review of the several definitions of orbifold, and of the category or 2-category of orbifolds in the literature.

1.5 Orbispaces, proper Lie groupoids and differentiable stacks

We discuss spaces which are locally modelled on quotients of Euclidean spaces by smooth actions of compact Lie groups. It is clear that a definition of such a space should generalize that of an orbifold, since any finite group is compact.

Orbispaces and proper groupoids

Using exactly the same definitions as for classical orbifold charts and classical orbifold atlases, but replacing finite groups by compact Lie groups in the definition of orbifold chart, one arrives at the notions of classical orbispace chart and classical orbispace atlas. It is possible to define a notion of (classical) orbispace in terms of these orbispace atlases, by saying that a (classical) orbispace is a second-countable Hausdorff space together with an equivalence class of orbispace atlases, satisfying some technical conditions - see [85] for the precise definition.

We follow a different approach, generalizing the definition of orbifolds via proper étale Lie groupoids (similarly to [31]).

Definition 1.54. An **orbispace atlas** on a topological space X is given by a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $f : M/\mathcal{G} \longrightarrow X$.

Two orbispace atlases $(\mathcal{G}, f)$ and $(\mathcal{H}, f')$ are **equivalent** if $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Morita equivalent, and the homeomorphism $\Phi : N/\mathcal{H} \longrightarrow M/\mathcal{G}$ induced by the Morita equivalence (see Lemma 1.28) satisfies $f \circ \Phi = f'$.

Definition 1.55. An **orbispace** is a topological space equipped with an equivalence class of orbispace atlases.

Given any proper Lie groupoid $\mathcal{G} \rightrightarrows M$, the orbispace associated to it by using the identity map on the orbit space $M/\mathcal{G}$ as atlas is denoted suggestively by $M//\mathcal{G}$.

Remark 14. As a consequence of the Linearization theorem for proper Lie groupoids (Theorem 1.39), any proper Lie groupoid defines a classical orbispace, defined in terms of orbispace charts on its orbit space. However, unlike the case of orbifolds, it is not known whether every classical orbispace can be realized as the orbispace associated to a proper Lie groupoid.

The following result is a converse to Lemma 1.28: it says that a morphism that preserves transverse geometry is a weak equivalence. This result can also be interpreted as a sort of inverse function theorem for orbispaces. If one interprets the normal representations as describing the “tangent space” of an orbispace, then it says that a map between groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ induces an isomorphism between the orbispaces they represent if and only if it induces a homeomorphism between the underlying topological spaces, and isomorphims at the level of “tangent spaces”.

Theorem 1.56 ([31], Theorem 4.3.1)**.** *Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Lie groupoids and $\phi : \mathcal{G} \to \mathcal{H}$ be a Lie groupoid morphism. Then ϕ is a weak equivalence if and only if it induces a homeomorphism between the orbit spaces $M/\mathcal{G}$ and $N/\mathcal{H}$ and isomorphisms between the normal representations $\mathcal{G}_x \curvearrowright \mathcal{N}_x \to H_{\phi_0(x)} \curvearrowright \mathcal{N}_{\phi_0(x)}$, for all $x \in M$.*

Remark 15 (Some topological properties of orbispaces)*.* We have already seen in Proposition 1.32 that the orbit space X of a proper groupoid is Hausdorff. Since the quotient map of any groupoid is an open map, we can also conclude that X is second-countable and locally compact. Having all the above properties, X is normal and paracompact as well.

Differentiable stacks

Dropping the condition of properness in the definition of orbispace atlas, we arrive at the notion of differentiable stack - while orbispaces generalize orbifolds, differentiable stacks can be seen as a generalization of Haefliger's leaf spaces:

Definition 1.57. A **differentiable stack atlas** on a topological space X is given by a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $f : M/\mathcal{G} \longrightarrow X$.

Two differentiable stack atlases $(\mathcal{G}, f)$ and $(\mathcal{H}, f')$ are **equivalent** if $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Morita equivalent, and the homeomorphism $\Phi : N/\mathcal{H} \longrightarrow M/\mathcal{G}$ induced by the Morita equivalence (see Lemma 1.28) satisfies $f \circ \Phi = f'$.

A **differentiable stack** is a topological space equipped with an equivalence class of orbispace atlases. The differentiable stack associated to a Lie groupoid $\mathcal{G} \rightrightarrows M$ is denoted by $M//\mathcal{G}$.

Remark 16. We warn the reader about the fact that we are avoiding all technicalities related with defining morphisms (and 2-morphisms) between differentiable stacks (and orbispaces), but we implicitly identify isomorphic differentiable stacks (and isomorphic orbispaces). Nonetheless, the definitions presented here are sufficient for the scope of this thesis. We refer to [5, 48, 68] for comprehensive introductions to the theory of differentiable stacks.

It is immediate from the previous definitions that orbispaces are examples of differentiable stacks, and orbifolds are examples of orbispaces.

1.6 Smooth functions on Orbispaces

We now discuss some basics on smooth functions on the orbit space of a proper Lie groupoid. Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid with orbit space X.

Definition 1.58. The **algebra of smooth functions on** X is defined as

$$C^\infty(X) := \{f : X \to \mathbb{R} \mid f \circ \pi \in C^\infty(M)\},$$

where $\pi : M \longrightarrow X$ denotes the canonical projection map.

The **sheaf of smooth functions on** X, denoted by $\mathcal{C}^\infty_X$, is defined by letting

$$\mathcal{C}^\infty_X(U) := C^\infty(\pi^{-1}(U)/\mathcal{G}_{|\pi^{-1}(U)}).$$

Note that the pullback map $\pi^* : C^\infty(X) \longrightarrow C^\infty(M)$ identifies the algebra of smooth functions on X with the algebra of $\mathcal{G}$-invariant smooth functions on M, denoted by $C^\infty(M)^{\mathcal{G}-\mathrm{inv}}$.

Constructing functions on the orbit space

The orbit space X of a proper Lie groupoid is Hausdorff, second-countable, and locally compact (see Remark 15). Using these properties, we are be able to guarantee the existence of several useful functions on X.

Proposition 1.59. *The algebra $C^\infty(X)$ is **normal**, i.e., for any disjoint closed subsets $A, B \subset X$ there is a function $f \in C^\infty(X)$ with values in $[0,1]$ such that $f_{|A} = 0$ and $f_{|B} = 1$.*

Proof. Let $A, B \subset X$ be closed and disjoint. The sets $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are closed and disjoint in M. Since M is a manifold, we can find a function $h \in C^\infty(M)$ separating these two sets. Now average h with respect to a Haar system on $\mathcal{G}$. We obtain in this way an invariant smooth function $\tilde{h}$ on M which still separates $\pi^{-1}(A)$ and $\pi^{-1}(B)$. Being invariant, it corresponds via the pullback by the projection π to a function $f \in C^\infty(X)$ that separates A and B. □

The fact that $C^\infty(X)$ is normal can be used to prove the existence of useful smooth functions on X: partitions of unity and proper functions. Before we see how, let us recall a helpful classical result, the Shrinking lemma. We say that a cover $\{U_i\}_{i\in I}$ of a space X is locally finite if for each $x \in X$ has a neighbourhood V_x such that there are finitely many indices $i \in I$ for which $V_x \cap U_i \neq \emptyset$.

Lemma 1.60 (Shrinking lemma)**.** *Let X be a paracompact Hausdorff space. Then X is normal and for any open cover $\{U_i\}_{i\in I}$ of X there is a locally finite open cover $\{V_i\}_{i\in I}$ with the property that $\{\overline{V_i} \subset U_i\}$ for all $i \in I$.*

Proof. See for example [36]. $\square$

Proposition 1.61 (Partitions of unity). *For any open cover $\mathcal{U}$ of X there is a smooth partition of unity subordinated to $\mathcal{U}$.*

Proof. One possible proof goes by the standard argument used for the classical version of this result, for continuous functions on a paracompact Hausdorff space [36]. It first involves using the Shrinking lemma twice to obtain locally finite open covers $\{V_n\}$ and $\{W_n\}$ of X such that $\overline{W}_n \subset V_n$ and $\overline{V}_n \subset U_n$ for each n. Secondly, since $C^\infty(X)$ is normal, we can choose functions $f_n : X \to [0,1]$ such that $f_{n|X-V_n} = 0$ and $f_{n|\overline{W}_n} = 1$. Since the covers used are locally finite, we can define functions $g_n = f_n / \sum f_n$, which form a partition of unity subordinated to $\mathcal{U}$.

There is also a proof using the classical version of the result. Take the open cover $\pi^{-1}(\mathcal{U})$ of M defined by the preimages by the projection map of the opens of $\mathcal{U}$, and consider a smooth partition of unity (f_n) subordinated to it, which exists because M is a manifold. Now average each function with respect to a Haar system to obtain a new family of functions $(\tilde{f}_n)$, which are invariant. Then $\tilde{g_n} = \tilde{f}_n / \sum \tilde{f}_n$ is a smooth invariant partition of unity subordinated to $\pi^{-1}(\mathcal{U})$, so it induces a smooth partition of unity subordinated to $\mathcal{U}$. $\square$

Proposition 1.62 (Existence of proper functions). *Let X be the orbit space of a proper groupoid. There exists a smooth proper function $f : X \to \mathbb{R}$.*

Proof. Let $\{U_n\}$ be a locally finite countable open cover of X such that each U_n has compact closure. Using the Shrinking lemma twice, find open covers $\{V_n\}$ and $\{W_n\}$ of X such that $\overline{W}_n \subset V_n$ and $\overline{V}_n \subset U_n$ for each n. Let $f_n \in C^\infty(X)$ be a function separating $X - V_n$ and $\overline{W}_n$. This means that $\mathrm{supp}(f_n) \subset U_n$ and $f_n = 1$ on a neighbourhood of $\overline{W}_n$. By local finiteness of the covers we can define the function $f = \sum_n n f_n$, which is proper. Indeed, if $K \in \mathbb{R}$ is compact, then it is bounded above by some integer m, and so $f^{-1}(K)$ is a closed subspace of the compact $\overline{W}_1, \cup \ldots \cup \overline{W}_m$, and so it is compact itself. $\square$

1.7 Riemannian metrics on groupoids

In this section we briefly discuss some metrics that can be defined on the base and space of arrows of a Lie groupoid. These are studied extensively in [32] and [86], which are the sources for the results in this section, presented here without proof.

Riemannian metrics on objects

Definition 1.63. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A Riemannian metric η on M is called **transversally invariant** if for each orbit $\mathcal{O}$ the metric induced on the normal bundle $N\mathcal{O}$ is invariant under the canonical action of $\mathcal{G}_{\mathcal{O}}$.

Proposition 1.64 (Proposition 3.14 of [86]). *If $\mathcal{G} \rightrightarrows M$ is a proper Lie groupoid, there is a transversally invariant Riemannian metric on M.*

We have already seen that for a proper groupoid the orbits are embedded submanifolds. The existence of a transversally invariant metrics for proper groupoids guarantees that the partition by the orbits has some extra structure.

Definition 1.65. A partition $\mathcal{F}$ of a Riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a **singular Riemannian foliation** (cf. [75]) if

1. $\mathcal{F}$ is a **singular foliation**: for each leaf L and each $v \in T_xL$ there is a smooth vector field X on M tangent to the leaves, such that $X(p) = v$.
2. Every geodesic perpendicular to one leaf is perpendicular to every leaf it meets.

Proposition 1.66 (Proposition 6.4 in [86]). *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and η a transversally invariant Riemannian metric on M. Then the connected components of the orbits of $\mathcal{G}$ form a singular Riemannian foliation with respect to this metric.*

Riemannian metrics on arrows

Recall that we denote by $\mathcal{G}^{(n)}$ the space of strings of n composable arrows of $\mathcal{G}$,

$$\mathcal{G}^{(n)} = \{(g_1, \dots, g_n) \in \mathcal{G}^n \mid s(g_i) = t(g_{i+1}),\ i = 1, \dots, n-1\}.$$

Definition 1.67. The **nerve** of a Lie groupoid $\mathcal{G}$ is the simplicial manifold for which the manifold of n-simplices is $\mathcal{G}^{(n)}$, with face maps defined by

$$\delta_i : \mathcal{G}_n \to \mathcal{G}_{n-1},\ \delta_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \leq i \leq n-1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

and degeneracy maps defined by $s_i(g_1, \dots, g_n) = (g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n)$.

Definition 1.68. Let (P, g_P) and (M, g_M) be Riemannian manifolds. A **Riemannian submersion** $\pi : P \to M$ is a submersion such that the fibres are equidistant or, equivalently: for any $x \in M$ and $p \in \pi^{-1}(x)$ the map $d_p\pi : T_p(\pi^{-1}(x))^{\perp} \to T_xM$ is an isometry.

Remark 17. Given a Riemannian submersion $(P, g_P) \to (M, g_M)$, the metric g_M is determined by the metric g_P.

Definition 1.69. A **n-metric** on a Lie groupoid [32] is a Riemannian metric $\eta^{(n)}$ on $\mathcal{G}^{(n)}$ which is invariant under the action of S_n on $\mathcal{G}^{(n)}$ and for which all the face maps $\mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ are Riemannian submersions.

In the particular case of $n = 1$, the definition of an n-metric is simply that of a Riemannian metric on $\mathcal{G}$ such that the source is a Riemannian submersion and the inversion is an isometry (and so the target is also a Riemannian submersion), as in [41]. The focus of [32], however, is on 2-metrics, as they can be used to prove a generalization of the Linearization theorem for proper Lie groupoids (Theorem 1.39).

Theorem 1.70 (Theorem 1 in [32]). *Any proper Lie groupoid admits a 2-metric.*

When $\mathcal{G} \rightrightarrows M$ is a Lie groupoid an $S \subset M$ is an embedded invariant submanifold, the restriction of $\mathcal{G}$ to S is a Lie groupoid $\mathcal{G}_S \rightrightarrows S$. The local model of $\mathcal{G}$ around S is $N\mathcal{G}_S \rightrightarrows NS$, the normal bundle of $\mathcal{G}_S$ inside $\mathcal{G}$, which is a groupoid over the normal bundle NS of S inside M.

Theorem 1.71 (Theorem 2 in [32]). *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid equipped with a 2-metric η, and let $S \subset N$ be an embedded invariant submanifold. Then the exponential map of η defines a linearization of $\mathcal{G}$ around S, i.e., an isomorphism between a neighbourhood of $\mathcal{G}_S$ in $\mathcal{G}$ and a neighbourhood of $\mathcal{G}_S$ inside the local model $N\mathcal{G}_S$.*

Chapter 2

Orbispaces as smooth spaces

In this chapter we discuss some ways in which singular spaces (i.e., spaces which are not necessarily manifolds) arising naturally in geometry (e.g. quotients) can be seen as having a "smooth" structure, so that one can attempt to study them with techniques that are more commonly applied to smooth manifolds.

There are numerous different approaches to what a "smooth" structure on a space can be, depending on the nature of the spaces one deals with. Whatever definition one chooses to use, one should try to strike a balance: it should coincide with the classical theory in the case of spaces that are already smooth, i.e., manifolds, be general enough to be applicable to the spaces we want to study, but still be simple and restrictive enough to let us draw interesting consequences.

One way of describing a smooth structure on a manifold is via atlases. Generalizations of this idea lead to the notions of classical orbifold or orbispace atlases, as discussed in sections 1.4 and 1.5. We will not explore this way of describing smooth structures further, but more information can be found in [85].

A different (but equivalent) way to define the smooth structure on a smooth manifold is by describing its structure sheaf - thus seeing it as a topological space with a sheaf of "smooth functions". Generalizing this approach, one can think of a smooth structure on a space X as a description of what should be considered to be the collection of smooth functions on X.

Of course, to make this idea precise (and useful), one needs to point some reasonable axioms that the prescribed "collection of smooth maps" should satisfy. One could for example ask it to be a subalgebra of the algebra of continuous functions, or to be a sheaf of algebras. Different choices of "collections of smooth functions" and of axioms governing them lead to different ways of dealing with singular spaces. In this chapter we give a short overview of three such approaches, that of C^∞-differentiable spaces, of C^∞-schemes, and of Sikorski spaces; in particular, we provide the background that is necessary in order to understand how orbispaces fit into each theory.

A common aspect to the three theories is that they deal with studying the differential geometry of a space X via the "collection of smooth functions on X", in a way analogous to algebraic geometry.

Instead of using a collection of functions on X, one could describe a "smooth" structure on X by using functions from other spaces *into* X. A theory that arises in this way is that of diffeological spaces. Diffeological spaces are not discussed in this text, but their relevance and usefulness for the study of orbispaces is detailed in [105].

2.1 Manifolds and locally ringed spaces

We start by recalling some background on the more algebro-geometric approach to studying smooth manifolds using the language of locally ringed spaces, as well as some classical results about smooth functions on manifolds that will be later generalized to other spaces. In this section we follow the exposition of [78]. Unless otherwise mentioned, all manifolds are considered to be finite dimensional, Hausdorff and second-countable.

Ringed spaces

Definition 2.1. A **ringed $\mathbb{R}$-space** (also called ringed space) is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of $\mathbb{R}$-algebras on X, called the **structure sheaf** of the ringed space. A **morphism of ringed spaces** is a pair

$$(\phi, \phi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consisting of a continuous map $\phi : X \to Y$ and a morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ of sheaves on Y (or equivalently, a morphism $\phi^\sharp : \phi^* \mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on X).

Remark 18. In general, the notion of ringed space allows for the structure sheaf to be a sheaf of unital rings, but since all the structure sheaves we use in this text are actually sheaves of $\mathbb{R}$-algebras, we restrict to this class.

Example 2.2. The following are two essential examples:

1. Any topological space X is a ringed space if we take $\mathcal{O}_X$ to be equal to $\mathcal{C}_X$, the sheaf of continuous functions.

2. Any smooth manifold M is a ringed space, if we take $\mathcal{O}_M$ to be equal to the sheaf of smooth functions $\mathcal{C}^\infty_M$.

In many examples (just like $(X, \mathcal{C}_X)$ and $(M, \mathcal{C}^\infty_M)$), the structure sheaf is a subsheaf of the sheaf of continuous functions on a topological space, i.e, $\mathcal{O}_X$ is such that for each open U of X, $\mathcal{O}_X(U)$ is a subalgebra of $\mathcal{C}(U)$.

Definition 2.3. A **reduced ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a subsheaf of $\mathbb{R}$-algebras of the sheaf $\mathcal{C}_X$ of continuous functions on X which contains all constant functions.

A **morphism of reduced ringed spaces** $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\phi : X \to Y$ such that

$$\phi^* f := f \circ \phi \in \mathcal{O}_X(\phi^{-1}(U)),$$

for all $f \in \mathcal{O}_Y(U)$. It follows that ϕ induces a morphism of sheaves

$$\phi^* : \mathcal{O}_Y \to \phi_* \mathcal{O}_X.$$

Both the examples of topological spaces and of smooth manifolds, seen as ringed spaces, are clearly reduced ringed spaces. An important class of ringed spaces is that of locally ringed spaces.

Definition 2.4. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that the stalk $\mathcal{O}_{X,x}$ at any point $x \in X$ is a local ring, i.e., it has a unique maximal ideal, denoted by $\mathfrak{m}_x$.

A **morphism of locally ringed spaces** is a morphism of ringed spaces

$$(\phi, \phi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that $\phi^\sharp(\mathfrak{m}_y) \subset \mathfrak{m}_{\phi(y)}$. This condition means that

$$(\phi^* f)(x) = 0 \Leftrightarrow f(\phi(x)) = 0.$$

A morphism (ϕ, ϕ^*) is an isomorphism if ϕ is a homeomorphism and ϕ^* is an isomorphism of sheaves.

Remark 19. Any reduced ringed space is automatically a locally ringed space, the unique maximal ideal of the stalk at a point being the ideal of germs that vanish at that point.

In this way, topological spaces and smooth manifolds can be treated as locally ringed spaces. In fact, we can use the language of locally ringed spaces to give an alternative (equivalent) definition of smooth manifolds. To do so, we start by noting that $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$ is a locally ringed space, where $\mathcal{C}^\infty_{\mathbb{R}^n}$ is the sheaf of smooth functions on opens of $\mathbb{R}^n$; it is easy to check that any morphism of locally ringed spaces $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n}) \to (\mathbb{R}^m, \mathcal{C}^\infty_{\mathbb{R}^m})$ is just given by a smooth map $\mathbb{R}^n \to \mathbb{R}^m$.

Definition 2.5 (Manifolds as locally ringed spaces)**.** A **smooth manifold of dimension** n is a locally ringed space $(M, \mathcal{O}_M)$ such that M is Hausdorff, second-countable, and has a cover by open subsets U_i with the property that each restriction $(U_i, \mathcal{O}_{M|U_i})$ is isomorphic to an open subset of $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$.

Proposition 2.6. *The category of smooth manifolds (as defined as in Definition 2.5) and of morphisms of locally ringed spaces between them is isomorphic to the category of smooth manifolds and smooth maps defined in terms of atlases.*

See for example [79, Ch. 7] for a proof of the previous proposition. An important property of smooth manifolds, which makes the isomorphism from the previous proposition easier to check, is that a smooth manifold $(M, \mathcal{O}_M)$ can be reconstructed from the ring $C^\infty(M)$ of global smooth functions on M. This is a particular case of a general process by which we associate to any $\mathbb{R}$-algebra A a locally ringed space $(\operatorname{Spec}_r A, \tilde{A})$, called the real spectrum of A, giving a functor from the category of $\mathbb{R}$-algebras to the category of locally ringed spaces; in the case of a smooth manifold $(M, \mathcal{O}_M)$, we obtain that $(\operatorname{Spec}_r C^\infty(M), \widetilde{C^\infty(M)})$ is isomorphic to $(M, \mathcal{O}_M)$.

The real spectrum of an algebra

We recall the construction of the real spectrum of an $\mathbb{R}$-algebra A. To start with, let us see how to construct the underlying set. An ideal $\mathfrak{m}$ of A is called a **real ideal** if it is a maximal ideal and $A/\mathfrak{m} \cong \mathbb{R}$. A **character** on A is a morphism of $\mathbb{R}$-algebras $\chi : A \to \mathbb{R}$; the kernel of any character is a real ideal, so there is a natural bijection between the set of characters on A and the set of real ideals of A.

Definition 2.7. Let A be an $\mathbb{R}$-algebra. The **real spectrum** of A is the set

$$\operatorname{Spec}_r A := \operatorname{Hom}(A, \mathbb{R}) = \{\text{real ideals of } A\}.$$

Given a point x in $\operatorname{Spec}_r A$, the corresponding character is denoted by χ_x and the corresponding ideal by $\mathfrak{m}_x$.

Any element $f \in A$ defines a real valued function $\hat{f}$ (but also denoted by f if there is no risk of confusion) on $\operatorname{Spec}_r A$, given by

$$\hat{f}(x) := \chi_x(f) = [f] \in A/\mathfrak{m}_x \cong \mathbb{R}.$$

In this way we have that

$$\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$$

and the corresponding character χ_x is the evaluation map at x.

The topology that we consider on the real spectrum $\operatorname{Spec}_r A$ is the **Gelfand topology**, which is the smallest one such that the functions $\hat{f} : \operatorname{Spec}_r A \to \mathbb{R}$ are continuous, for all $f \in A$.

Remark 20. For any subset I of an $\mathbb{R}$-algebra A, we define the **zero-set** of I to be

$$(I)_0 := \{x \in \operatorname{Spec}_r A \mid f(x) = 0, \ \forall f \in I\}.$$

By definition, these subsets of $\operatorname{Spec}_r A$ are the closed subsets of the **Zariski topology** on $\operatorname{Spec}_r A$. The Gelfand topology is always finer than the Zariski topology, although they agree in some cases, as we will see below.

Finally, the structure sheaf on $\operatorname{Spec}_r A$ is the sheaf associated to the presheaf that associates to an open subset $U \subset \operatorname{Spec}_r A$ the ring A_U defined as the localization (i.e., ring of fractions - see [4]) of A with respect to the multiplicative system of all elements $f \in A$ such that $\hat{f}$ does not vanish at any point of U. The stalk at any point coincides with the localization of A with respect to the multiplicative system $\{f \in A \mid \hat{f}(x) \neq 0\}$. The resulting locally ringed space $(\operatorname{Spec}_r A, \tilde{A})$ is called **the real spectrum of** A (it has the same name as the underlying set, but the meaning is usually clear from the context).

As mentioned before, a manifold can be recovered from its ring of functions. To start with, the following theorem shows we can recover the underlying topological space of a manifold.

Theorem 2.8. *Let M be a smooth manifold. Then the map*

$$\chi : M \to \operatorname{Spec}_r C^\infty(M),$$

given by $\chi(p)(f) = f(p)$ (i.e., $\chi(p)$ is the evaluation at p) is a homeomorphism.

For a proof see for example [78] or [79], but also the proof of Theorem 2.11 below which generalizes this result. From the proof it also follows that for a smooth manifold, the Gelfand and the Zariski topologies coincide.

That the structure sheaf associated to the ring $C^\infty(M)$ by localization coincides with the sheaf $\mathcal{C}^\infty_M$ of smooth functions on opens of M is a consequence of the following result (see e.g. [78] for a modern exposition of the proof).

Theorem 2.9 (Localization theorem)**.** *Let M be a smooth manifold and $U \subset M$ an open. For any differentiable function f on U there exist global differentiable functions g, h on M, such that h does not vanish on U and $f = g/h$ on U, i.e.,*

$$C^\infty(U) = C^\infty(M)_U.$$

Finally, smooth maps between manifolds can also be recovered from algebra maps between the rings of functions.

Theorem 2.10 (Theorem 2.3 in [78])**.** *For any two manifolds M and N there is a natural bijection*

$$C^\infty(M, N) \to \mathrm{Hom}_{\mathbb{R}-\mathrm{alg}}(C^\infty(N), C^\infty(M))$$

given by $\phi \mapsto \phi^*$.

An approach to equipping singular spaces with smooth structures is to choose a class $\mathcal{A}$ of $\mathbb{R}$-algebras, such that the singular spaces we want to consider can be modelled by the real spectrum of algebras in $\mathcal{A}$. Moreover, $\mathcal{A}$ should include the algebra $C^\infty(M)$, for any manifold M. The spaces that can be studied in this way are those locally ringed spaces which are locally isomorphic to the real spectrum of algebras in $\mathcal{A}$. There is an ample choice of which kind of algebras should be taken to belong to $\mathcal{A}$ (see for example [54] or Appendix 2 of [74] for a discussion on possible models).

Since our goal is to equip the orbit space X of a proper groupoid $\mathcal{G}$ with a smooth structure, we should ask that the spectrum of $C^\infty(X)$ (see Definition 1.58) is locally isomorphic to the spectrum of an algebra in $\mathcal{A}$. We could also go further and require that the algebra $C^\infty(X)$ itself is in $\mathcal{A}$. We discuss both possibilities in the next two subsections. But first let us show that, in any case, we can recover the underlying topological space X from the algebra $C^\infty(X)$. The result generalizes statement 2.8 and the proof is exactly the same as the one for manifolds, relying simply on the existence of proper functions on X (Proposition 1.62) and on the fact that $C^\infty(X)$ is normal (Proposition 1.59).

Theorem 2.11. *Let X be the orbit space of a proper Lie groupoid. Then the natural map $\phi : X \to Spec_r C^\infty(X)$ given by $x \mapsto ev_x$ is a homeomorphism.*

Proof. The map ϕ is clearly injective since $C^\infty(X)$ is point-separating. To check surjectivity, let $\chi \in \mathrm{Spec}_r\, C^\infty(X)$. According to Proposition 1.62, we can choose a proper function $f \in C^\infty(X)$, and then the level set $K = f^{-1}(\chi(f))$ is compact. Suppose that χ is not in the image of ϕ, i.e., it is not given by evaluation at a point. Then for each point $y \in X$ there is a function $f_y \in C^\infty(X)$ such that $f_y(y) \neq \chi(f_y)$. The sets

$$U_y = \{x \in X \mid f_y(x) \neq \chi(f_y)\}$$

cover K, which is compact, so we can take a finite subcover of it, $U_{y_1}, \ldots, U_{y_k}$. Consider now the function

$$g = (f - \chi(f))^2 + \sum_{i=1}^{k} (f_{y_i} - \chi(f_{y_i}))^2.$$

It is easy to see that $\chi(g) = 0$. But g is a nowhere vanishing smooth function on X, so it is invertible and we have that

$$1 = \chi(1) = \chi\left(g\frac{1}{g}\right) = \chi(g)\chi\left(\frac{1}{g}\right),$$

so that $\chi(g) \neq 0$. Thus we have a contradiction and ϕ must be surjective.

The map ϕ is continuous, since if $U = \hat{f}^{-1}(V)$, with $f \in C^\infty(X)$ and V open in $\mathbb{R}$ is some basic open for the Gelfand topology, then $\phi^{-1}(U) = f^{-1}(V)$ is open in X.

To finish checking that ϕ is a homeomorphism, let ϕ induce a topology on X which we also call the Gelfand topology. Given a set $Y \subset X$ which is closed in the original topology of X, we can consider the ideal I_Y of all functions of $C^\infty(X)$ vanishing on Y. Since $C^\infty(X)$ is normal, we have that $Y = \{x \in X \mid f(x) = 0 \ \ \forall\ f \in I_Y\}$, so Y is also closed in the Gelfand topology. □

2.2 Framework I: Differentiable spaces

In this section we discuss the category of differentiable spaces. These are spaces that are locally modelled on the spectrum of differentiable algebras, i.e., algebras of the form $C^\infty(\mathbb{R}^n)/I$, where I is an ideal of $C^\infty(\mathbb{R}^n)$ which is closed with respect to the weak Whitney topology (cf. e.g. [52]). Differentiable spaces appeared in the work of Spallek [97] and also as a particular case of the theory of C^∞-schemes [35, 54, 74], that is discussed in the next section. A study of differentiable spaces, analogous to the basics of scheme theory in algebraic geometry, is discussed in detail in the book [78]; this is the main reference for this chapter. See *loc. cit.* for the proofs of the results quoted below.

2.2.1 Some general theory of differentiable spaces

Fréchet topology, closed ideals and differentiable algebras

Let M be a smooth manifold and let $\{K_i\}_{i\in\mathbb{N}}$ be a countable family of compact subsets of M such that each K_i is contained in the domain of some chart $(U_i, x_1, \ldots, x_n)$ and such that M is covered by the interior of the K_i's. The family of seminorms

$$||f||_{K_i,r} = \max_{\substack{p\in K_i \\ |\alpha|<r}} 2^r \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(p)\right|$$

induces a locally convex topology on $C^\infty(M)$, independent of the choices made, making it into a Fréchet algebra, called the Fréchet topology on $C^\infty(M)$ (also called the weak Whitney topology [52]).

We deal with ideals of $C^\infty(M)$ that are closed in the Fréchet topology. An example of such a closed ideal is the ideal $\mathfrak{m}_p$ of all smooth functions vanishing at a point $p \in M$. The following classical result of Whitney characterizes the closed ideals of $C^\infty(M)$ in terms of conditions on the jets of smooth functions (cf. e.g. [61] for a proof).

Theorem 2.12 (Whitney's spectral theorem)**.** *Let M be a smooth manifold and $\mathfrak{a}$ an ideal of $C^\infty(M)$. Then*

$$f \in \bar{\mathfrak{a}} \ \Leftrightarrow\ j_x f \in j_x(\mathfrak{a}), \forall x \in M,$$

where $j_x(\mathfrak{a}) = \{j_x g \mid g \in \mathfrak{a}\}$.

Definition 2.13. An $\mathbb{R}$-algebra A is called a **differentiable algebra** if it is isomorphic to the quotient $C^\infty(\mathbb{R}^n)/\mathfrak{a}$, where $\mathfrak{a}$ is a closed ideal (with respect to the Fréchet topology). Morphisms of differentiable algebras are simply morphisms of the underlying $\mathbb{R}$-algebras.

Remark 21. Differentiable algebras also appear in [74], where they are treated as a particular class of C^∞-rings (called there closed C^∞-rings).

Example 2.14 (Open or closed subsets of Euclidean space)**.** If $U \subset \mathbb{R}^n$ is an open subset, then $C^\infty(U)$ is a differentiable algebra.

If $Y \subset \mathbb{R}^n$ is a closed subset, denote by $\mathfrak{p}_Y$ the ideal of all smooth functions on $\mathbb{R}^n$ which vanish on Y. Define

$$A_Y := \{f_{|Y} \mid f \in \mathcal{C}^\infty(\mathbb{R}^n)\}.$$

Then A_Y is a differentiable algebra because $A_Y \cong C^\infty(\mathbb{R}^n)/\mathfrak{p}_Y$ via the map $[f] \mapsto f_{|Y}$, and the ideal $\mathfrak{p}_Y$ is closed, since $\mathfrak{p}_Y = \bigcap_{p \in Y} \mathfrak{m}_p$.

Example 2.15 (Smooth manifolds)**.** As a particular case of the previous example, using Whitney's embedding theorem (see e.g. [52]) we conclude that the algebra of smooth functions on any manifold is a differentiable algebra.

Differentiable spaces

We now study the real spectrum of a differentiable algebra (see the discussion in Section 2.1). The point is that for a differentiable algebra A, the real spectrum behaves quite similarly to a manifold. For example, closed (resp. open) subsets of $\operatorname{Spec}_r A$ share many features with closed (resp. open) subsets of a smooth manifold; this is made more precise by the next two propositions.

Proposition 2.16 (Closed subsets - Lemma 3.1 in [78])**.** *If A is a differentiable algebra and $Y \subset \operatorname{Spec}_r A$ is a closed subset, then Y is a zero-set, i.e., there is an element $a \in A$ such that $Y = (a)_0$.*

Note that the Gelfand and the Zariski topologies on $\operatorname{Spec}_r A$ coincide for any differentiable algebra A, since any closed subset of $\operatorname{Spec}_r A$ is a zero-set. For open subsets, the following results extend Theorems 2.8 and 2.9 to the setting of differentiable algebras.

Proposition 2.17 (Open subsets - Proposition 3.2 in [78])**.** *If A is a differentiable algebra and $U \subset \operatorname{Spec}_r A$ is an open subset, then we have a homeomorphism*

$$U \cong \operatorname{Spec}_r A_U,$$

where A_U denotes the localization of A with respect to the multiplicative system of elements of A that vanish nowhere in U.

It is also important to note that for a differentiable algebra A and an open subset $U \subset \operatorname{Spec}_r A$, the localization A_U is again a differentiable algebra [78, Thm. 3.7].

Finally, the following result is essential when considering the spectrum of a differentiable algebra as a locally ringed space.

Theorem 2.18 (Localization theorem for differentiable algebras)**.** *Let A be a differentiable algebra and let* $(\mathrm{Spec}_r A, \tilde{A})$ *be its real spectrum. Then for any open subset* $U \subset \mathrm{Spec}_r A$ *we have that*

$$\tilde{A}(U) = A_U.$$

So we see that the essential properties that allow to reconstruct a manifold from its ring of smooth functions as the real spectrum still hold in the more general setting of differentiable algebras.

Definition 2.19. An **affine differentiable space** is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the real spectrum $(\mathrm{Spec}_r A, \tilde{A})$ of some differentiable algebra A. By the Localization theorem, A must be isomorphic to $\mathcal{O}_X(X)$.

Definition 2.20. A **differentiable space** is a locally ringed space $(X, \mathcal{O}_X)$ for which every point x of X has a neighbourhood U such that $(U, \mathcal{O}_{X|U})$ is an affine differentiable space. Such opens are called **affine opens**.

Morphisms of differentiable spaces between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are simply defined to be morphisms of locally ringed spaces between them (Definition 2.4). Sections of the sheaf $\mathcal{O}_X$ over an open subset $U \subset X$ are called **differentiable functions** on U.

Example 2.21. Any smooth manifold M is an example of an affine differentiable space. As discussed before, given a manifold M, its algebra of smooth functions $C^\infty(M)$ is a differentiable algebra; the manifold $(M, \mathcal{C}^\infty_M)$ is isomorphic, as a locally ringed space, to the real spectrum of $C^\infty(M)$.

Example 2.22. If $(X, \mathcal{O}_X)$ is an affine differentiable space and U is an open subset of X then $(U, \mathcal{O}_{X|U})$ is an affine differentiable space.

Example 2.23. Let $(X, \mathcal{O}_X)$ be a differentiable space and $Y \subset X$ a closed subset. If $\mathcal{I}_Y$ is the sheaf of differentiable functions vanishing on Y, then $(Y, \mathcal{O}_X/\mathcal{I}_Y)$ is a differentiable space.

Affine differentiable spaces can be explicitly described, at least as a topological space, as follows.

Lemma 2.24. *Let I be an ideal of an $\mathbb{R}$-algebra A. Then there is a natural homeomorphism*

$$Spec_r(A/I) \cong (I)_0 \subset \mathrm{Spec}_r A.$$

Proposition 2.25 (Structure of affine spaces - Proposition 2.13 in [78])**.** *Let A be an algebra of the form $A = C^\infty(\mathbb{R}^n)/\mathfrak{a}$, for any ideal $\mathfrak{a} \subset C^\infty(\mathbb{R}^n)$. Then there is a homeomorphism*

$$\mathrm{Spec}_r A = (\mathfrak{a})_0 \subset \mathbb{R}^n.$$

As mentioned in the previous section, morphisms $M \to N$ between smooth manifolds seen as differentiable spaces are simply smooth maps, and these correspond to algebra maps $C^\infty(N) \to C^\infty(M)$. A similar result is valid for the more general setting of affine differentiable spaces.

Theorem 2.26 (Morphisms into affine spaces - Theorem 3.18 in [78]). *If $(X, \mathcal{O}_X)$ is a differentiable space and $(Y, \mathcal{O}_Y)$ is an affine differentiable space, then*

$$\operatorname{Hom}(X, Y) \cong \operatorname{Hom}_{\mathbb{R}-\mathrm{alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)), \quad (\phi, \phi^\sharp) \mapsto \phi^\sharp.$$

As a particular case of this result, we obtain a characterization of morphisms from a differentiable space to an Euclidean space.

Corollary 2.27 (Morphisms into Euclidean space). *If $(X, \mathcal{O}_X)$ is a differentiable space, then we have an isomorphism*

$$\operatorname{Hom}(X, \mathbb{R}^n) \cong \bigoplus_{i=1}^{n} \mathcal{O}_X(X), \quad (\phi, \phi^\sharp) \mapsto (\phi^\sharp(x_1), \ldots, \phi^\sharp(x_n)).$$

Reduced differentiable spaces

We have seen in the general discussion about the real spectrum of an algebra A that any element $f \in A$ can be seen as a continuous function $\hat{f}$ on $\operatorname{Spec}_r A$. We now focus on the case in which the assignment $f \mapsto \hat{f}$ is injective.

Definition 2.28. A differentiable space $(X, \mathcal{O}_X)$ is said to be **reduced** if for any open subset U of X and any $f \in \mathcal{O}_X(U)$ we have

$$f = 0 \Leftrightarrow \hat{f}(x) = 0, \ \forall x \in X.$$

Being reduced is a local condition: If every point of x has a reduced open neighbourhood then X is reduced. Note that, by definition, if $(X, \mathcal{O}_X)$ is a reduced differentiable space, then the map $\mathcal{O}_X(U) \to C(U)$, $f \mapsto \hat{f}$ is injective for any open U; hence $(X, \mathcal{O}_X)$ is a reduced ringed space.

Let $(\phi, \phi^\sharp)$ be a morphism of differentiable spaces (Definition 2.20) between reduced differentiable spaces. Then it can be checked (cf. [78]) that $(\phi, \phi^\sharp)$ is a morphism of reduced ringed spaces (Definition 2.3), i.e., that $\phi^\sharp = \phi^*$.

Example 2.29. Smooth manifolds are reduced differentiable spaces. The example 2.23 of a differentiable space $(Y, \mathcal{O}_X/\mathcal{I}_Y)$, where Y is a closed subset of a differentiable space X, is also reduced.

We now look at an explicit description of affine reduced differentiable spaces.

Definition 2.30. Let Y be a topological subspace of $\mathbb{R}^n$. A **smooth function on** Y is a continuous function $f : Y \to \mathbb{R}$ for which every point $y \in Y$ has an open neighbourhood U_y in $\mathbb{R}^n$ such that f coincides on $Y \cap U_y$ with the restriction of a smooth function on U_y. The smooth functions on Y form the ring $C^\infty(Y)$. Denote by $\mathcal{C}_Y^\infty$ the sheaf of continuous functions on Y defined by $\mathcal{C}_Y^\infty(V) := C^\infty(V)$, for all $V \subset Y$.

Proposition 2.31 (Structure of reduced affine spaces - [78], Prop. 3.22). *Let $A = C^\infty(\mathbb{R}^n)/\mathfrak{a}$ be a differentiable algebra. If the affine differentiable space $Y = \operatorname{Spec}_r A$ is reduced, then $\mathcal{O}_Y = \mathcal{C}_Y$.*

Remark 22. We knew already from Proposition 2.25 that in the conditions of the previous proposition, $Y \cong (\mathfrak{a})_0$ as a topological space, because $A = C^\infty(\mathbb{R}^n)/\mathfrak{a}$ is affine. The new information we get from knowing that Y is reduced is the characterization of the structure sheaf. In conclusion, reduced differentiable spaces are reduced ringed spaces which are locally isomorphic to $(Z, \mathcal{C}_Z^\infty)$, for some closed subset Z of $\mathbb{R}^n$.

Functions on differentiable spaces

Let us mention some properties of differentiable spaces that extend classical results for smooth manifolds. These guarantee the existence of smooth functions satisfying some requirements. To start with, let us mention that for any differentiable algebra, the sheaf $\tilde{A}$ is a **soft sheaf**, i.e., sections over closed subsets of $\operatorname{Spec}_r A$ may be extended to global sections.

Proposition 2.32 (Lemma 4.2 in [78])**.** *Let $(X, \mathcal{O}_X)$ be a Hausdorff differentiable space. Let $K \subset X$ be any compact subset and let U be an open neighbourhood of K. Then, there exists a global differentiable function f on X, with $0 \leq f \leq 1$, such that f is constant equal to 1 on a neighbourhood of K, and the support of f is contained in U.*

The previous result can be used to construct useful smooth functions on any Hausdorff and second-countable differentiable space: partitions of unity and proper functions (see e.g. [78, Sec. 4.1] for the proofs):

Theorem 2.33 (Partitions of unity)**.** *Let $(X, \mathcal{O}_X)$ be a Hausdorff and second-countable differentiable space. Any open cover of X admits a subordinated partition of unity.*

Corollary 2.34 (Proper functions)**.** *Let $(X, \mathcal{O}_X)$ be a Hausdorff second-countable differentiable space. Then there exists a differentiable function $f \in \mathcal{O}_X(X)$ which is proper, as a continuous function on X.*

These play an important role, for example, in the proof of the Embedding theorem for differentiable spaces (Theorem 2.41 below; see [78] for the proof).

Finally, just as with smooth manifolds, closed subsets of Hausdorff second-countable differentiable spaces can be separated by a global smooth function:

Corollary 2.35 (Corollary 4.6 in [78])**.** *Let $(X, \mathcal{O}_X)$ be a Hausdorff and second-countable differentiable space. Then $\mathcal{O}_X(X)$ is normal, i.e., closed subsets of X can be separated by functions in $\mathcal{O}_X(X)$.*

Subspaces and embeddings

We now discuss the Embedding theorem for differentiable spaces, which characterizes which differentiable spaces can be seen as a subspace of some affine space $\mathbb{R}^n$.

Definition 2.36. Let $(X, \mathcal{O}_X)$ be a differentiable space and let $Y \subset X$ be a locally closed subspace. Let $\mathcal{I}$ be a sheaf of ideals of $\mathcal{O}_{X|Y}$ and set $\mathcal{O}_X/\mathcal{I} := (\mathcal{O}_{X|Y})/\mathcal{I}$.

We say that $(Y, \mathcal{O}_X/\mathcal{I})$ is a **differentiable subspace** of $(\mathcal{O}_X, X)$ if it is a differentiable space. It is said to be an **open** differentiable subspace if Y is open in X and $\mathcal{I} = 0$. It is said to be a **closed** differentiable subspace if Y is closed in X.

Definition 2.37. An **embedding** of differentiable spaces is a morphism of differentiable spaces $(\phi, \phi^\sharp) : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$ such that

1. $\phi : Y \longrightarrow X$ induces a homeomorphism of Y onto a locally closed subspace of X
2. $\phi^\sharp : \phi^*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ is surjective.

It is called a **closed embedding** if additionally $\phi(Y)$ is closed in X.

Proposition 2.38 (Closed subspaces - Proposition 5.6 in [78]). *Let I be a closed ideal of a differentiable algebra A. Then $\operatorname{Spec}_r(A/I)$ is a closed differentiable subspace of $\operatorname{Spec}_r A$. Conversely, any closed differentiable subspace of $\operatorname{Spec}_r A$ is defined by a unique closed ideal.*

We already know from Proposition 2.25 that any affine differentiable space is homeomorphic to a subspace of some $\mathbb{R}^n$; the next corollary says that the embedding respects the differentiable structure.

Corollary 2.39. *Any affine differentiable space is isomorphic to some closed differentiable subspace of an affine space $\mathbb{R}^n$.*

Finally, let us state the Embedding theorem for differentiable spaces. For that, we need to recall the notions of tangent space and embedding dimension of a differentiable space at a point.

Definition 2.40. Let p be a point of a differentiable space $(X, \mathcal{O}_X)$ and let $\mathfrak{m}_p$ be the unique maximal ideal of $\mathcal{O}_{X,p}$. The **tangent space of X at p** is defined as:

$$T_pX = Der(\mathcal{O}_{X,p}, \mathcal{O}_{X,p}/\mathfrak{m}_p).$$

The dimension of T_pX is called the **embedding dimension** of X at p.

Theorem 2.41 (Embedding theorem for differentiable spaces - cf. [78]). *A differentiable space is affine if and only if it is Hausdorff, second-countable, and has bounded embedding dimension.*

2.2.2 Orbispaces as differentiable spaces

We look at how orbit spaces of proper groupoids can be seen as differentiable spaces. We start by discussing the smooth structure on orbit spaces of representations of compact Lie groups.

Orbit spaces of representations of compact Lie groups

Let G be a compact Lie group and let V be a representation of G.

Definition 2.42. The **algebra of smooth functions on** V/G is defined as

$$C^\infty(V/G) := \{f : V/G \to \mathbb{R} \mid f \circ \pi \in C^\infty(V)\},$$

where $\pi : V \longrightarrow V/G$ denotes the canonical projection map.

The **sheaf of smooth functions on** V/G, denoted by $\mathcal{C}^\infty_{V/G}$, is defined by letting

$$\mathcal{C}^\infty_{V/G}(U) := C^\infty(\pi^{-1}(U)/G).$$

It is natural to identify the algebra of smooth functions on the orbit space, $C^\infty(V/G)$, with the algebra of G-invariant smooth functions on V, via the pullback map π^*.

We now explain how V/G can be seen as an affine differentiable space. The first step in this direction is given by the following classical result of Schwarz [93].

Theorem 2.43 (Schwarz). *Let G be a compact Lie group and V a representation of G. Let $p_1, \dots, p_k$ be generators of the algebra of invariant polynomials $\mathbb{R}[V]^G$. Then $p : V \to \mathbb{R}^k$ defined by $p = (p_1, \dots, p_k)$ induces an isomorphism*

$$p^* C^\infty(\mathbb{R}^k) \cong C^\infty(V)^G.$$

This allows us to make sense of V/G as a differentiable space. To start with, the map p from the theorem is constant along orbits; so it induces a map on V/G, denoted by $\tilde{p} : V/G \to \mathbb{R}^k$.

Lemma 2.44. *With the notation from Schwarz's theorem, the map $p : V \to \mathbb{R}^k$ is proper and it induces a closed embedding (of topological spaces)*

$$\tilde{p} : V/G \to \mathbb{R}^k.$$

Remark 23. In fact, since p is a polynomial map, the image of V/G is naturally a semialgebraic set. Its semialgebraic structure can be described explicitly [89].

We can rephrase Schwarz's theorem in the language of differentiable spaces as follows.

Theorem 2.45. *Let G be a compact Lie group and V a representation of G. Then*

1. *$(V/G, \mathcal{C}^\infty_{V/G})$ is a reduced affine differentiable space;*
2. *the map $\tilde{p} : V/G \to \mathbb{R}^k$ is a closed embedding of differentiable spaces.*

Orbit spaces of proper Lie groupoids

We prove that the orbit space X of a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ is a differentiable space. We also see that the result only depends on the Morita equivalence class of $\mathcal{G}$. As such, the differentiable space structure is really associated to the orbispace presented by $\mathcal{G}$.

We know from the Linearization theorem (Theorem 1.39) that any point in X has a neighbourhood homeomorphic to a space of the form V/G, where V is a representation of a compact Lie group G. The idea is to upgrade this homeomorphism to an isomorphism of locally ringed spaces and to use the differentiable space structure on V/G described in the previous section.

Theorem 2.46. *The orbit space X of a proper Lie groupoid $\mathcal{G} \rightrightarrows M$, together with the sheaf $\mathcal{C}^\infty_X$ on X (Definition 1.58), is a reduced differentiable space. Moreover, it is affine if and only if it has bounded embedding dimension.*

Proof. Let $\mathcal{O} \in X$ and let x be a point in the orbit $\mathcal{O}$. Consider an open subset $U \subset M$ containing $\mathcal{O}$, such that $\mathcal{G}_U \cong \mathcal{N}_{\mathcal{O}}(\mathcal{G})_V$, as in the Linearization theorem (Theorem 1.39). Under this isomorphism, let $\Sigma \subset U$ be the slice (see def. 1.37) at x corresponding to the open $V \cap \mathcal{N}_x$ of the normal space to $\mathcal{O}$ at x. It also holds that

$$C^\infty(U)^{\mathcal{G}} \cong C^\infty(V)^{\mathcal{N}_{\mathcal{O}}(\mathcal{G})}.$$

Moreover, since we are only considering invariant functions, we could work with invariant opens instead - if $\tilde{V}$ is the saturation of V, we have that

$$C^\infty(\tilde{V})^{\mathcal{N}_{\mathcal{O}}(\mathcal{G})} \cong C^\infty(V)^{\mathcal{N}_{\mathcal{O}}(\mathcal{G})}.$$

So from now on assume that $\tilde{V} = P_x \times_{\mathcal{G}_x} W$, where P_x is the s-fibre of $\mathcal{G}$ at x and $W \subset \mathcal{N}_x$ is an invariant open subset. Note that W intersects all the orbits in V (it is actually a slice at x). We can define an algebra isomorphism

$$\phi : C^\infty(\tilde{V})^{\mathcal{N}_\mathcal{O}(\mathcal{G})} \to C^\infty(W)^{\mathcal{G}_x}$$

by restriction to the slice W. More precisely, given a function $f \in C^\infty(\tilde{V})^{\mathcal{N}_\mathcal{O}(\mathcal{G})}$ define $\phi(f)(w) = f([1_x, w])$. The invariance of $\phi(f)$ follows from that of f and it is also easy to check that ϕ is an injective algebra homomorphism. An explicit inverse can be given by $\phi^{-1}(f)([p, w]) = f(w)$. Once again it is easy to check that it is well defined, and an inverse to ϕ. To see that it is invariant we use that a point of V belonging to the orbit of $[p, w]$ must be of the form $[q, w]$. To summarize, we have isomorphisms

$$C^\infty(\pi(U)) \cong C^\infty(U)^\mathcal{G} \cong C^\infty(V)^{\mathcal{N}_\mathcal{O}(\mathcal{G})} \cong C^\infty(W/\mathcal{G}_x).$$

Since $\mathcal{G}_U$ is Morita equivalent to $\mathcal{G}_x \ltimes W$, it holds, as explained in Lemma 1.28, that $\pi(U) \cong W/\mathcal{G}_x$; what we have proved above is that in fact, we also obtain an isomorphism of reduced ringed spaces

$$\big(\pi(U), \mathcal{C}^\infty_{\pi(U)}\big) \cong \big(W/\mathcal{G}_x, \mathcal{C}^\infty(W)^{\mathcal{G}_x}\big).$$

From the discussion of the previous section, we know that Schwarz's theorem implies that $\big(W/\mathcal{G}_x, \mathcal{C}^\infty(W)^{\mathcal{G}_x}\big)$ is a reduced affine differentiable space (Theorem 2.45). We have thus proved that the reduced ringed space $(X, \mathcal{C}^\infty_X)$ is locally isomorphic to a reduced affine differentiable space, so it is a reduced differentiable space.

Since $(X, \mathcal{C}^\infty_X)$ is a Hausdorff, second-countable, and paracompact differentiable space, the Embedding theorem for differentiable spaces (Theorem 2.41) tells us that it is affine if and only if it has bounded embedding dimension. □

Remark 24. Note that a direct consequence of the last statement is that the algebra of smooth functions on the orbit space X of a proper groupoid is a differentiable algebra if and only if X has bounded embedding dimension.

A case in which X has bounded embedding dimension is for example when X has only finitely many Morita types, a notion that is discussed in Section 3.4.

Corollary 2.47 (Morita invariance)**.** *If $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, their orbit spaces are isomorphic as differentiable spaces.*

Proof. We already knew that the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$ are homeomorphic (Lemma 1.28). From the proof of Theorem 2.46, we see that as a differentiable space, the orbit space of a groupoid is locally isomorphic, around a point $\mathcal{O}$, to the orbit space of the isotropy representation at a point of the orbit $\mathcal{O}$, which is invariant under Morita equivalences. □

2.3 Framework II: C^∞-Schemes

In this section we briefly discuss C^∞-schemes. These are spaces locally modelled on the spectrum of a C^∞-ring. They have appeared as models for synthetic differential geometry in [35, 74].

We have already seen examples of such spaces, as any differentiable algebra is also a C^∞-ring (and so any differentiable space is a C^∞-scheme).

It is worth mentioning that C^∞-rings are much more general than differentiable algebras - for example, from the theory of the previous section, we know that the algebra of smooth functions on the orbit space of a proper groupoid is a differentiable algebra if and only if the orbit space has bounded embedding dimension. But we will see that it is always a C^∞-ring. On the other hand, although C^∞-rings are very general, those that arise as algebras of smooth functions on X still belong to a somewhat restrictive class - that of locally fair C^∞-rings. Some references for this material are [54, 74].

2.3.1 Some general theory of C^∞-rings and C^∞-Schemes

Definition 2.48. A C^∞**-ring** is a set $\mathfrak{C}$ together with operations $\phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for each $n \geq 0$ (also denoted by $\phi_f^{\mathfrak{C}}$) and for each $f : \mathbb{R}^n \to \mathbb{R}$, satisfying the following conditions.

1. Consider natural numbers $m, n \geq 0$, and smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $g : \mathbb{R}^m \to \mathbb{R}$. Define a smooth function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

Then we have that the following diagram commutes:

$$\begin{tikzcd} \mathfrak{C}^n \arrow[r, "(\phi_{f_1}, \ldots, \phi_{f_m})"] \arrow[dr, "\phi_h"'] & \mathfrak{C}^m \arrow[d, "\phi_g"] \\ & \mathfrak{C} \end{tikzcd}$$

2. For the coordinate functions $x_i : \mathbb{R}^n \to \mathbb{R}$ we have $\phi_{x_i}(c_1, \ldots, c_n) = c_i$, for all $c_1, \ldots, c_n \in \mathfrak{C}$.

A morphism of C^∞-rings is a map $F : \mathfrak{C} \to \mathfrak{D}$ such that for any $c_1, \ldots, c_n \in \mathfrak{C}$ and any $f \in C^\infty(\mathbb{R}^n)$ it holds that $F(\phi_f^{\mathfrak{C}}(c_1, \ldots, c_n)) = \phi_f^{\mathfrak{D}}(F(c_1), \ldots, F(c_n))$.

Proposition 2.49 (Proposition 1.2 in [74])**.** *Let $\mathfrak{C}$ be a C^∞-ring and I an ideal of $\mathfrak{C}$ (with $\mathfrak{C}$ considered as an $\mathbb{R}$-algebra). Then there is a unique C^∞-ring structure on the quotient $\mathfrak{C}/I$, such that the quotient map $\pi : \mathfrak{C} \to \mathfrak{C}/I$ is a C^∞-ring morphism.*

As an immediate consequence of this proposition we obtain the following.

Corollary 2.50. *Any differentiable algebra is a C^∞-ring.*

It is not hard to see that C^∞-rings are much more general that differentiable algebras. For example, the algebra of continuous functions on any topological space is a C^∞-ring.

Definition 2.51. A C^∞-ring $\mathfrak{C}$ is called **finitely generated** if there are $c_1, \ldots, c_n$ in $\mathfrak{C}$ which generate $\mathfrak{C}$ under all C^∞ operations.

A C^∞-ring $\mathfrak{C}$ is called a C^∞**-local ring** if it has a unique maximal ideal $\mathfrak{m}$ and $\mathfrak{C}/\mathfrak{m} \cong \mathbb{R}$.

Example 2.52. The ring $C_p^\infty(\mathbb{R}^n)$ of germs at p of functions on $\mathbb{R}^n$ is a C^∞-local ring.

Definition 2.53. A C^∞**-Ringed space** $(X, \mathcal{O}_X)$ is a topological X space together with a sheaf $\mathcal{O}_X$ of C^∞-rings on it. A **local** C^∞**-Ringed space** $(X, \mathcal{O}_X)$ is a C^∞-ringed space for which the stalks $\mathcal{O}_{X,x}$ are local rings for all $x \in X$.

A **morphism of** C^∞**-Ringed spaces** is a pair

$$(\phi, \phi^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

consisting of a continuous map $\phi : X \to Y$ and a morphism $\phi^\sharp : \mathcal{O}_Y \to \phi_* \mathcal{O}_X$ of sheaves on Y (or equivalently, a morphism $\phi^\sharp : \phi^* \mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of C^∞-rings on X).

Definition 2.54. An **affine** C^∞**-scheme** is a local C^∞-ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\operatorname{Spec}_r \mathfrak{C}$ as a local C^∞-ringed space, for some C^∞-ring $\mathfrak{C}$.

Definition 2.55. A C^∞**-scheme** is a local C^∞-ringed space $(X, \mathcal{O}_X)$ for which X can be covered by open sets U_i such that each $(U_i, \mathcal{O}_{X|U_i})$ is an affine C^∞-scheme. **Morphisms of** C^∞**-schemes** are just morphism of C^∞-Ringed spaces.

Definition 2.56. A **locally fair** C^∞-scheme is a C^∞-scheme $(X, \mathcal{O}_X)$ for which X can be covered by open sets U_i such that each $(U_i, \mathcal{O}_{X|U_i})$ is isomorphic to $\operatorname{Spec}_r \mathfrak{C}_i$, where $\mathfrak{C}_i$ is a finitely generated C^∞-ring.

2.3.2 Orbispaces as C^∞-Schemes

Putting together results from the previous sections, we can describe the smooth structure on the orbit space X of a proper groupoid in the language of C^∞-schemes. Although we already knew that X is a C^∞-scheme (since it is a differentiable space) we show that it is always affine, as a C^∞-scheme.

Proposition 2.57. *Let X be the orbit space of a proper Lie groupoid $\mathcal{G} \rightrightarrows M$. Then the algebra $C^\infty(X)$ is a C^∞-ring and $(X, \mathcal{C}_X^\infty)$ is a locally fair affine C^∞-scheme.*

Proof. It is clear that $C^\infty(X)$ is a C^∞-ring : the operation ϕ_f associated with a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is simply given by composition with f.

We also already know from Theorem 2.11 that

$$ev : X \cong \operatorname{Spec}_r C^\infty(X)$$

is a homeomorphism. In fact, almost by definition, ev is also an isomorphism of reduced ringed spaces: Let $f \in \mathcal{O}_{\operatorname{Spec}_r C^\infty(X)}(U)$ and recall that $\mathcal{O}_{\operatorname{Spec}_r C^\infty(X)}(U)$ is the localization of $C^\infty(X)$ with respect to the multiplicative system of those functions of $C^\infty(X)$ that vanish nowhere on U,. We then know that $f = \frac{g}{h}$ with $g, h \in C^\infty(X)$ and h non-vanishing on U, so it is easy to see that $ev^* f \in C_X^\infty(ev^{-1}(U))$. This implies that $(X, \mathcal{C}_X^\infty)$ is an affine C^∞-scheme.

Finally, in the proof of Theorem 2.46, we have already seen that X is locally isomorphic to differentiable spaces (hence C^∞-schemes) of the form $(V/G, \mathcal{C}_{V/G})$, where V is a representation of a compact Lie group G. Schwarz's theorem (Theorem 2.43) says precisely that the algebra of G-invariant functions on V is a finitely generated C^∞-ring, where V is a representation of a compact Lie group G, and we know that $(V/G, \mathcal{C}_{V/G})$ is affine; hence $(V/G, \mathcal{C}_{V/G})$ is isomorphic to the real spectrum of a finitely generated C^∞-ring. In other words, $(X, \mathcal{C}_X)$ is a locally fair C^∞-scheme. $\square$

Remark 25 (Morita invariance). Note that, since differentiable spaces form a subcategory of the category of C^∞-schemes, Corollary 2.47 implies that the structure of C^∞-scheme on the orbit space of a proper Lie groupoid is invariant under Morita equivalences.

2.4 Framework III: Sikorski spaces

The third notion of smooth structure that we discuss is that of a Sikorski space [94] (also called differential space in the literature). This type of smooth structure will be revisited later on, when we discuss smooth stratifications of the orbit space (Section 3.5). The main references used for this section are [96, 104].

2.4.1 Some general theory of Sikorski spaces

Definition 2.58. A **Sikorski space** is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a non-empty set of continuous real-valued functions on X, satisfying:

1. X has the weakest topology such that all the elements of $\mathcal{F}$ are continuous.
2. (Locality) Let $f : X \to \mathbb{R}$ be a function such that for all $x \in X$ there is a neighbourhood U of x and a function $g \in \mathcal{F}$ such that $f_{|U} = g_{|U}$. Then $f \in \mathcal{F}$.
3. (Smooth compatibility) If $F \in C^\infty(\mathbb{R}^n)$ and $f_1, \ldots, f_n \in \mathcal{F}$, then the composition $F(f_1, \ldots, f_n)$ belongs to $\mathcal{F}$.

The elements of $\mathcal{F}$ are called **smooth functions** on X.

Remark 26. There are some immediate observations we can draw from the previous definition.

1. Since the composition of elements of $\mathcal{F}$ with translations and rescalings is again in $\mathcal{F}$, the topology of X is generated by the open subsets of the form $f^{-1}(0,1)$, for $f \in \mathcal{F}$.
2. The smooth compatibility condition ensures that $\mathcal{F}$ is a commutative $\mathbb{R}$-algebra and that it contains all constant functions.
3. The locality condition guarantees that $\mathcal{F}$ induces a sheaf of continuous functions $\tilde{\mathcal{F}}$ on X: for any open $U \subset X$, let $\tilde{\mathcal{F}}(U)$ be the set of all functions $f : U \to \mathbb{R}$ such that for all $x \in X$ there is a neighbourhood V of x in U and a function $g \in \mathcal{F}$ such that $f_{|V} = g_{|V}$. In this way $(X, \tilde{\mathcal{F}})$ is a reduced ringed space.

Definition 2.59. A **smooth map** between Sikorski spaces $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ is any continuous map $\phi : X \to Y$ with the property that $f \circ \phi \in \mathcal{F}_X$ for all $f \in \mathcal{F}_Y$.A **diffeomorphism** is a smooth homeomorphism with a smooth inverse.

Definition 2.60. Let $(X, \mathcal{F}_X)$ be a Sikorski space. A **Sikorski subspace** of $(X, \mathcal{F}_X)$ is a Sikorski space $(Y, \mathcal{F}_Y)$, where Y is a topological subspace of X, and $\mathcal{F}_Y$ is generated by restrictions of functions of $\mathcal{F}_X$ to Y, i.e., $f \in \mathcal{F}_Y$ if and only if for all $y \in Y$ there is a neighbourhood U of y in X and a function $g \in \mathcal{F}$ such that $f_{|U \cap Y} = g_{|U \cap Y}$.

Definition 2.61. Let $(X, \mathcal{F}_X)$ be a Sikorski space and let R be an equivalence relation on X. The **quotient Sikorski space** of X with respect to R is the Sikorski space $(X_R, \mathcal{F}_R)$, where

1. X_R is the set of equivalence classes for R;
2. $\mathcal{F}_R = \{f \in \mathcal{F} \mid \pi^* f \in \mathcal{F}\}$, where $\pi : X \to X_R$ is the canonical projection;
3. the topology on X_R is the smallest one making all functions in $\mathcal{F}_R$ continuous.

Note that, in general, the topology induced on X_R by $\mathcal{F}_R$ may be distinct from the quotient topology. The following result provides a sufficient condition for the two topologies to coincide.

Proposition 2.62 (Proposition 2.1.11 in [96])**.** *Let $(X, \mathcal{F}_X)$ be a Sikorski space, let R be an equivalence relation on X and let $(X_R, \mathcal{F}_R)$ be the quotient Sikorski space. Then the topology induced by $\mathcal{F}_R$ on X_R coincides with the quotient topology if for every subset $U \subset X_R$, open for the quotient topology, and every point $y \in U$, there exists a function $f \in \mathcal{F}_R$ such that $f(y) = 1$ and f vanishes outside of U.*

Mostow spaces

There is also a sheaf-theoretical generalization of the notion of a Sikorski space due to Mostow [76]:

Definition 2.63. A **Mostow space** is a reduced ringed space $(X, \mathcal{F}_X)$ such that for any open $U \subset X$, the algebra $\mathcal{F}_X(U)$ satisfies the smooth compatibility condition from Definition 2.58.

A **morphism of Mostow spaces** is simply a morphism of reduced ringed spaces between Mostow spaces.

By point 3 of Remark 26, we can associate to any Sikorski space $(X, \mathcal{F})$ a Mostow space $(X, \tilde{\mathcal{F}})$. Indeed, Mostow spaces are very similar to Sikorski spaces, but not exactly the same. We mention here some simple relations. See [50] for a detailed comparison and for the proofs of the following two results.

Proposition 2.64. *The assignment $(X, \mathcal{F}) \rightsquigarrow (X, \tilde{\mathcal{F}})$ defines a full embedding $\mathcal{I} : \mathbf{Sik} \hookrightarrow \mathbf{Mos}$ of the category of Sikorski spaces into the category of Mostow spaces; here we are defining $\mathcal{I}$ to be the identity on morphisms (considering the morphisms as maps between the underlying sets).*

Proposition 2.65. *There is a global section functor $\Gamma : \mathbf{Mos} \to \mathbf{Sik}$ defined as follows: for an object $(X, \mathcal{F}_X)$, let $\Gamma((X, \mathcal{F}_X)) = (\tilde{X}, \mathcal{F}_X(X))$, where*

1. *$\mathcal{F}_X(X)$ is the algebra of global sections of the sheaf $\mathcal{F}_X$;*
2. *$\tilde{X}$ is the set X endowed with the smallest topology making all the functions of $\mathcal{F}_X(X)$ continuous.*

We set Γ to be the identity on morphisms (by considering the morphisms as maps between the underlying sets).

Actually, we can say a bit more about the relation between Sikorski and Mostow spaces:

Proposition 2.66. *The functor Γ is left adjoint to $\mathcal{I}$.*

Proof. First note that we have that $\Gamma\mathcal{I} = \mathrm{id}_{\mathbf{Sik}}$. Now let $(X, \mathcal{F}_X)$ be a Mostow space and consider the Sikorski space $\Gamma((X, \mathcal{F}_X)) = (\tilde{X}, \mathcal{F}_X(X))$. By the definition of Γ, the topology of $\tilde{X}$ is coarser than the topology of X, so the identity map

$$\mathrm{id}_X : X \longrightarrow \tilde{X}$$

is continuous. It induces a morphism $(X, \mathcal{F}_X) \xrightarrow{\mathrm{id}_X} \mathcal{I}\Gamma(X, \mathcal{F}_X) = (\tilde{X}, \widetilde{\mathcal{F}_X(X)})$ in **Mos**: if $f \in \widetilde{\mathcal{F}_X(X)}(U) = \tilde{\mathcal{F}}_X(U)$, then the sheaf property for $\mathcal{F}_X$ implies that $f \in \mathcal{F}_X(U)$. It is a simple verification to see that the assignment

$$(X, \mathcal{F}_X) \rightsquigarrow ((X, \mathcal{F}_X) \xrightarrow{\mathrm{id}_X} \mathcal{I}\Gamma(X, \mathcal{F}_X))$$

defines a unit of the adjunction. $\square$

Subcartesian spaces

Definition 2.67. A **subcartesian space** is a Hausdorff Sikorski space X such that every point $x \in X$ has a neighbourhood diffeomorphic to a subset of some $\mathbb{R}^n$.

Remark 27. Subcartesian spaces are sometimes called locally affine Sikorski spaces in the literature, or are defined in other (equivalent) ways. The precise connection between the various definitions is made clear in [104].

Definition 2.68. Let $(S, \mathcal{F}_S)$ be a subcartesian space and let X be a derivation of $\mathcal{F}_S$. An **integral curve** of X through a point x of S is a curve $c : I \longrightarrow S$ such that I is an interval containing 0 and

$$\frac{d}{d\epsilon} f(c(\epsilon)) = X(f)(c(t))$$

for all $f \in \mathcal{F}_S$ and all $\epsilon \in I$. If the domain of c is maximal with this property, we say that c is a **maximal integral curve**. By convention, the map $c : \{0\} \to S,\ 0 \mapsto x$ is an integral curve of any derivation.

Theorem 2.69 (Theorem 3.2.1 in [96])**.** *Let $(S, \mathcal{F}_S)$ be a subcartesian space and let X be a derivation of $\mathcal{F}_S$. Then for any x in S, there is a unique maximal integral curve c of X through x such that $c(0) = x$.*

Although derivations of $\mathcal{F}_S$ always admit maximal integral curves, they may fail to induce local one-parameter groups of local diffeomorphisms. The ones that do are called vector fields:

Definition 2.70. Let $(S, \mathcal{F}_S)$ be a subcartesian space. A **vector field on** S is a derivation X of $\mathcal{F}_S$ such that for every $x \in S$, there exists a neighbourhood U of x in S and $\epsilon > 0$ such that for all $t \in (-\epsilon, \epsilon)$, the map $\exp(tX)$ is defined on U and its restriction to U is a diffeomorphism of U with an open subset of S.

Definition 2.71. Let $(S, \mathcal{F}_S)$ be a subcartesian space, let $\mathfrak{F}$ be a family of vector fields on S and let $x \in S$. The **orbit of the family $\mathfrak{F}$ through** x is the set of all points $y \in S$ of the form

$$y = \exp(t_n X_n) \circ \ldots \circ \exp(t_1 X_1)(x),$$

for some $t_1, \ldots, t_n \in \mathbb{R}$ and some $X_1, \ldots, X_n \in \mathfrak{F}$.

Theorem 2.72 (Theorem 3.4.5 in [95])**.** *Any orbit O of a family $\mathfrak{F}$ of vector fields on a subcartesian space S is a smooth manifold. Moreover, in the topology of O given by the manifold structure, the Sikorski structure induced on O by the inclusion on S coincides with the manifold structure.*

2.4.2 Orbispaces as Sikorski spaces

Proposition 2.73. *The orbit space X of a proper Lie groupoid $\mathcal{G} \rightrightarrows M$ is a subcartesian space.*

Proof. First of all, as a quotient, X can be endowed with the quotient Sikorski space structure $(X, C^\infty(X))$. The topology induced by $C^\infty(X)$ coincides with the quotient topology because X is a normal space, so it is in the conditions of Proposition 2.62.

We have already seen (for example in the proof of Theorem 2.46) that $(X, C^\infty(X))$ is locally isomorphic to the quotient of a representation of a compact group which, by Schwarz's theorem (Theorem 2.43), is isomorphic to a subspace of some $\mathbb{R}^n$. Hence $(X, C^\infty(X))$ is subcartesian. □

Corollary 2.74 (Morita invariance)**.** *If two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent, their orbit spaces X and Y are isomorphic as subcartesian spaces*

Proof. X and Y are homeomorphic (Lemma 1.28); the algebra of smooth functions can be seen as the algebra of global sections of the structure sheaf, which is invariant under Morita equivalences (Corollary 2.47). □

2.5 Comparison of the frameworks

We now survey some of the relations between the three kinds of "smooth spaces" discussed in this chapter. Relations between these theories and other models for "smooth" structures can be found in [76, 78, 104]. In the end of this section we present a diagram illustrating the relations discussed here.

Differentiable spaces and C^∞-schemes

We start by relating the categories of differentiable spaces and of C^∞-schemes. At the level of algebras we have the following.

Proposition 2.75. *The category of differentiable algebras is a full subcategory of the category of C^∞-rings.*

Proof. As was mentioned before (Corollary 2.50), any differentiable algebra is also a C^∞-ring. A morphism of differentiable algebras is always a morphism of C^∞-rings (cf. [78, Cor. 2.21]); hence differentiable algebras form, indeed, a subcategory of the category of C^∞-rings. It is a full subcategory because, as it is clear from the definitions, any morphism of C^∞-rings (Definition 2.48) between differentiable algebras is immediately a morphism of $\mathbb{R}$-algebras. □

Corollary 2.76. *The category of differentiable spaces is a full subcategory of the category of C^∞-schemes.*

Proof. As a consequence of Proposition 2.75, differentiable spaces form a full subcategory of the category of local C^∞-ringed spaces.

This implies that any affine differentiable space is automatically an affine C^∞-scheme; hence any differentiable space can be seen as a C^∞-scheme.

Since differentiable spaces form a full subcategory of the category of local C^∞-ringed spaces, the stated result follows from the fact that any morphism of differentiable spaces is a morphism of C^∞-schemes (Definition 2.55). This is the case because by definition, morphism of C^∞-schemes are just morphisms of local C^∞-ringed spaces. □

It is worth keeping in mind that C^∞-rings are much more general than differentiable algebras. As an extreme example, note that the algebra of continuous functions on any topological space is a C^∞-ring.

Even if we restrict our attention to rings of the form $C^\infty(\mathbb{R}^n)/I$, these are C^∞-rings for any ideal I (Proposition 2.49), while they are differentiable algebras only when I is closed with respect to the Fréchet topology. For a more concrete example, the ring $C^\infty_p(\mathbb{R}^n)$ of germs at p of functions on $\mathbb{R}^n$ is a C^∞-ring but not a differentiable algebra, except in some trivial cases.

C^∞-schemes and Sikorski spaces

Non-reduced C^∞-schemes are never Sikorski spaces. So in order to compare the theories, we should focus only on reduced C^∞-schemes. Let us start by pointing out a simple relation.

Proposition 2.77. *If $(X, \mathcal{O}_X)$ is a reduced affine C^∞-scheme, then $(X, \mathcal{O}_X(X))$ is a Sikorski space.*

Proof. The topology on X is already the one induced by $\mathcal{O}_X(X)$ (the Gelfand topology); smooth compatibility is a consequence of $\mathcal{O}_X(X)$ being a C^∞-ring; locality is a consequence of the fact that $\mathcal{O}_X$ is a sheaf of continuous functions on X. □

Since a C^∞-scheme is covered by affine C^∞-schemes, we have the following consequence of Proposition 2.77.

Corollary 2.78. *Any reduced C^∞-scheme $(X, \mathcal{O}_X)$ is a Mostow differential space.*

On the other hand, if $(X, \mathcal{F})$ is a Sikorski space, then, because of the condition of smooth compatibility, $\mathcal{F}$ is a C^∞-ring; so $(X, \tilde{\mathcal{F}})$ is a C^∞-ringed space, where $\tilde{\mathcal{F}}$ denotes the sheaf associated to $\mathcal{F}$ - see Remark 26. It is not clear (to the author) when $(X, \tilde{\mathcal{F}})$ is a C^∞-scheme, but we will see some sufficient conditions for that to happen.

Note that for a general Sikorski space $(X, \mathcal{F})$, there might not be enough functions in $\mathcal{F}$ to separate points of X. For example, let $\mathcal{F}$ to be the algebra of G-invariant functions on a G-manifold M, and consider the Sikorski space $(M, \mathcal{F})$, where we take the topology on M to be generated by invariant open subsets of the original topology on M. Then we cannot separate points in the same orbit by means of functions in $\mathcal{F}$. When this happens, the map that associates to each point $x \in X$ the character on $\mathcal{F}$ given by evaluation at x is not injective. To avoid this situation, let us focus on the case where $\mathcal{F}$ is point-separating on X.

Proposition 2.79. *If $(X, \mathcal{F})$ is a Hausdorff Sikorski space such that $\mathcal{F}$ is normal (it separates closed subsets) and there is a function f in $\mathcal{F}$ with compact fibres, then the map $ev : X \longrightarrow \mathrm{Spec}_r \mathcal{F}$ is a homeomorphism.*

Proof. Since X is Hausdorff and $\mathcal{F}$ is normal, we know that $\mathcal{F}$ is point-separating, hence ev is injective. The rest of the proof follows exactly the proof of Theorem 2.11. There is only one missing step, which is to check that if f is a function in $\mathcal{F}$ that vanishes nowhere, then $\frac{1}{f}$ is also in $\mathcal{F}$. To do so, take an open cover of X given by subsets of the form $U_\epsilon = \{x \in X \mid f(x) > \epsilon\}$, where $\epsilon > 0$, and consider for each ϵ a smooth function $F_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ such that $F_\epsilon(x) = 1/x$ when $|x| > \epsilon$. By smooth compatibility, $F_\epsilon \circ f$ belongs to $\mathcal{F}$; since $(1/f)_{|U_\epsilon} = (F_\epsilon \circ f)_{|U_\epsilon}$, the function $1/f$ belongs to $\mathcal{F}$ by locality. □

Note that subspaces of $\mathbb{R}^n$ do satisfy the conditions from the previous proposition. Since subcartesian spaces are by definition covered by open subspaces diffeomorphic to subspaces of $\mathbb{R}^n$, we arrive at the following result.

Corollary 2.80. *If $(X, \mathcal{F})$ is a subcartesian space, then $(X, \tilde{\mathcal{F}}')$ is a C^∞-scheme.*

Here $\tilde{\mathcal{F}}'$ is the sheaf associated to $\mathcal{F}$ by localization, as explained in Section 2.1. It is not clear (to the author) whether the locality condition is enough to ensure that the sheaves $\tilde{\mathcal{F}}$ (from Remark 26) and $\tilde{\mathcal{F}}'$ coincide *a priori*. If X has the property that any open subset of X is σ-compact, i.e., is the union of countably many compact subsets, then the proof of the Localization theorem for smooth manifolds (Theorem 2.9) can be adapted to show that for any open subset $U \subset X$, the ring $\mathcal{F}_X(U)$ coincides with $\tilde{\mathcal{F}}(U)$, so that $(X, \mathcal{F}_X)$ is a C^∞-scheme.

Differentiable spaces and Sikorski spaces

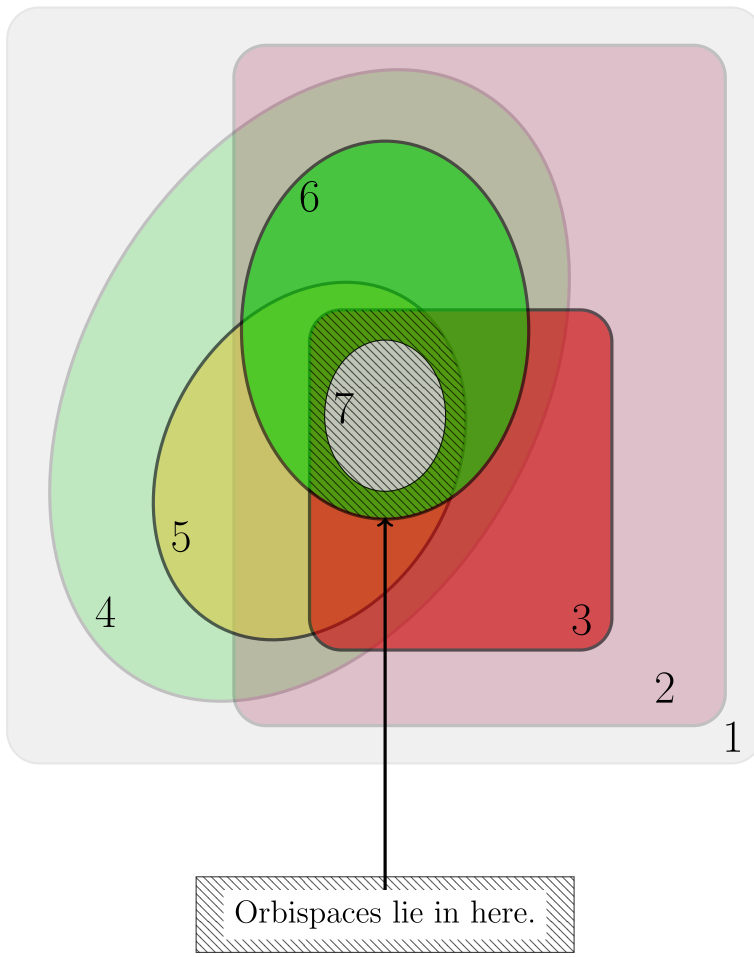
Since differentiable spaces are C^∞-schemes, we have the following consequence of Proposition 2.77.

Proposition 2.81. *Reduced affine differentiable spaces are subcartesian spaces. Reduced differentiable spaces are Mostow differential spaces.*

On the other hand, in general Sikorski spaces are far from being differentiable spaces. Even subcartesian spaces can be quite different from differentiable spaces. Roughly, differentiable spaces are locally modelled on *locally closed* subsets of Euclidean spaces (Corollary 2.39), while subcartesian spaces are locally modelled on *any* subsets of Euclidean spaces (Definition 2.67).

Comparison of some categories of “smooth” spaces

Note: In this diagram we consider only those schemes (and in particular differentiable spaces) which are reduced.



1. Mostow spaces
2. Sikorski spaces
3. Subcartesian spaces
4. C^∞-Schemes
5. Differentiable spaces
6. Affine C^∞-Schemes
7. Affine differentiable spaces

Chapter 3

Orbispaces as (differentiable) stratified spaces

In this chapter we study the canonical decomposition of the base and orbit space of a proper Lie groupoid. The decomposition is by smooth pieces that fit together in a prescribed way, as a stratification.

In the first three sections we recall some of the general theory of stratifications associated with proper Lie group actions. Most of this material is rather classical, but we try to clarify some points where the literature can sometimes be confusing. Some references for these sections are for example [37, 65, 84]. Throughout this chapter, we assume that X is a connected topological space and M is a connected manifold of dimension n. In the fourth and fifth sections we extend some of the theory of proper Lie group actions to proper Lie groupoids. We obtain in this way some of the results from [86] and a principal type theorem for proper Lie groupoids (Theorem 3.41).

3.1 Stratifications

Definition 3.1. Let X be a Hausdorff second-countable paracompact space. A **stratification** of X is a locally finite partition $\mathcal{S} = \{X_i \mid i \in I\}$ of X such that its members satisfy:

1. Each X_i, endowed with the subspace topology, is a locally closed, *connected* subspace of X, carrying a given structure of a smooth manifold;
2. (frontier condition) the closure of each X_i is the union of X_i with members of $\mathcal{S}$ of strictly lower dimension.

The members $X_i \in \mathcal{S}$ are called the **strata** of the stratification.

We will be studying in detail examples of stratifications coming from proper Lie group actions and proper Lie groupoids, but let us start by seeing some very simple examples.

1. Any manifold comes with the stratification by its connected components.
2. A manifold with boundary can be stratified by the connected components of its interior and the connected components of the boundary.

3. If M is a compact, connected, smooth manifold, then the cone on M,

$$CM = [0,1) \times M/\{0\} \times M$$

comes with a stratification with two strata: the vertex point and $(0,1) \times M$.

Remark 28. (Comments on the definition and comparison with the literature) When X is actually a smooth manifold, it is usual to require that the strata are submanifolds of X. Similarly, when X can be equipped with some sort of smooth structure, for example the ones described in Chapter 2, then it is natural to require some sort of compatibility between the smooth structure and the stratification. Section 3.5 is centred around this interplay.

Across the literature, it is possible to find quite a lot of variations on the definition of a stratification, typically so that the definition is most adapted to the problem under study. For example, some authors do not require X to be Hausdorff, paracompact, or second-countable. The two main conditions used here that are often not mentioned are connectedness of the strata (which is discussed in detail in Remark 29) and the requirement that strata included in the closure of another stratum have strictly lower dimension. Although the latter condition is often not required, without it we would be forced to consider pathological examples (e.g. the closed topologist's sine curve, and even more pathological ones - see [84, Ex. 1.1.12]) that do not occur anyway in our study of proper actions and proper Lie groupoids. On the other hand, some authors require further conditions on how the strata fit together, for example the conditions of topological local triviality, or the cone condition; these will be discussed in Section 3.5.

Remark 29 (On the condition of connectedness of the strata). The condition of connectedness of the strata is important not only as a technical condition but also conceptually. It is often present in the literature only implicitly, built into the definition of the partition that is to be studied. More precisely, one starts with a locally finite partition of X by locally closed submanifolds $\mathcal{P}$ and then one passes to the partition $\mathcal{P}^c$ by connected components of $\mathcal{P}$. The partition $\mathcal{P}^c$ is then checked to satisfy the frontier condition. One of the usual motivations for passing to connected components is that the elements of $\mathcal{P}$ might have components of different dimension. However, there is a much more fundamental reason to pass to connected components: in many important examples (such as the partition by orbit types - see Definition 3.9) the condition of frontier might not be satisfied unless we pass to connected components - see Example 3.11.

At a more conceptual level, the condition of connectedness also allows for a global implementation of Mather's approach to stratifications using germs of submanifolds (see e.g. [65, 84]), but without making reference to germs. We explain below the precise connection with Mather's approach. The main point to do so, present in the next lemma, is to understand when two partitions may give rise, after passing to connected components, to the same stratification.

Lemma 3.2. *Let $\mathcal{P}_i$, $i \in \{1,2\}$, be two partitions of X by smooth manifolds (whose connected components may have different dimensions) with the subspace topology; denote by $\mathcal{P}_i^c$ the new partition obtained by taking the connected components of the members of $\mathcal{P}_i$. Then $\mathcal{P}_1^c = \mathcal{P}_2^c$ if and only if, for each $x \in X$, there exists an open neighbourhood U of x in X such that*

$$P_1 \cap U = P_2 \cap U,$$

where $P_i \in \mathcal{P}_i$ are the members containing x.

Proof. For the direct implication, let $x \in X$ and let $A_i \in \mathcal{P}_i^c$ such that $x \in A_i$. Let $P_i \in \mathcal{P}_i$ be the members containing x. Then there are open subsets U_i of X such that U_i contains A_i but not the other connected components of P_i. The open neighbourhood $U = U_1 \cap U_2$ satisfies the second condition of the statement.

To prove the converse implication, it suffices to show that, for $A_i \in \mathcal{P}_i^c$ $(i \in \{1, 2\})$ with $A_1 \cap A_2 \neq \emptyset$, one must have $A_1 = A_2$. Let $P_i \in \mathcal{P}_i$ so that A_i is a connected component of P_i. We first show that $A_1 \cap A_2$ is open in A_1. Let $a \in A_1 \cap A_2$. By hypothesis, we find a neighbourhood U of a so that $U \cap P_1 = U \cap P_2$. Since A_1 is locally connected, we may assume that $U \cap A_1$ is connected. Then, since $U \cap A_1 \subset U \cap P_1 = U \cap P_2 \subset P_2$, we know that $U \cap A_1$ sits inside a connected component of P_2. Since $a \in U \cap A_1$ and A_2 is the connected component of P_2 containing a, we must have $U \cap A_1 \subset A_2$, hence $U \cap A_1 \subset A_1 \cap A_2$. This proves that $A_1 \cap A_2$ is open in A_1. Note that this implies that $\{A_1 \cap B \mid B \in \mathcal{P}_2^c\}$ is a partition of A_1 by open subspaces hence, by the connectedness of A_1, it must coincide with one of the members of this family - and that is necessarily the non-empty $A_1 \cap A_2$. Hence $A_1 \subset A_2$ and the reverse inclusion is proved similarly. $\square$

Definition 3.3. A **decomposition** of a Hausdorff second-countable topological space X is a partition $\mathcal{P}$ satisfying all conditions from Definition 3.1 except possibly connectedness of the strata.

Example 3.4. Some decompositions cannot be made into a stratification in our sense by passing to connected components. For example, consider the decomposition of the plane $\mathbb{R}^2$ into three pieces A, B, and C: A equals the origin $\{0\}$, B equals the union of all circles centred at the origin, of radius equal to $1/n$, with $n \in \mathbb{N}$, and $C = \mathbb{R}^2 \backslash A \cup B$. Passing to connected components, we would lose local finiteness of the partition.

Mather's approach using germs leads to the following alternative definition of stratification (cf. [65, 84]), that we designate by germ-stratification.

Definition 3.5. A **germ-stratification** of a topological space X is a rule which assigns to each $x \in X$ a germ $\mathcal{S}_x$ of a closed subset of X, such that, for each $x \in X$, there is a neighbourhood U of x and a decomposition $\mathcal{P}$ of U, with the property that for all $y \in U$, $\mathcal{S}_y$ is the germ of the piece of $\mathcal{P}$ containing y.

It is clear that, given any decomposition $\mathcal{S}$ on X, we can produce a germ-stratification by assigning to each $x \in X$ the germ of the piece of the decomposition containing x. A result of Mather [65, Lemma 2.2] states that any germ-stratification arises in this way. Lemma 3.2 guarantees that as long as we restrict to those decompositions that are stratifications and their corresponding germ-stratifications, this correspondence is indeed bijective. Accordingly, germ-stratifications are usually simply called stratifications in the literature.

Definition 3.6. Given a stratification $\mathcal{S}$ there is a natural partial order on the strata given by

$$S \leq T \;\Leftrightarrow\; S \subset \overline{T}.$$

The union of all maximal strata (with respect to this order) forms a subspace $M^{\mathcal{S}-\mathrm{reg}} \subset M$ called the **$\mathcal{S}$-regular part of** M.

The following lemma shows that maximality of a stratum is a local condition (cf. [23]).

Lemma 3.7. *A stratum $S \in \mathcal{S}$ is maximal if and only if it is open. The regular part $M^{\mathcal{S}-\mathrm{reg}}$ is open and dense in M.*

Proof. Assume that S is a maximal stratum which is not open. Let $x \in S$ lie outside the interior of S. Choose a neighbourhood V of x in M which intersects with only finitely many members of $\mathcal{S}$. Since x is not in the interior of S, by choosing a sequence of neighbourhoods $V \supset V_0 \supset V_1 \supset \ldots$ that shrink to x, we can find $x_n \in V_n \setminus S$ for each n. We obtain in this way a sequence $(x_n)_{n\geq 0}$ converging to x, with $x_n \in V \setminus S$. Each x_n belongs to one of the finitely many members of $\mathcal{S}$ which meets V, so after passing to a subsequence we may assume that $x_n \in T$ for all n, for some $T \in \mathcal{S}$. It follows that $x \in \overline{T}$, hence $S \cap \overline{T} \neq \emptyset$, and so $S \subset \overline{T}$. From the maximality of S we have that $S = T$, which contradicts the fact that x_n is not in S.

For the converse, assume that S is open and $S \subset \overline{T}$ for some $T \in \mathcal{S}$. If $S \neq T$, it follows that S is a stratum of dimension strictly less than that of T, which cannot be the case since S is open.

The regular part $M^{\mathcal{S}-\mathrm{reg}}$ is clearly open, being a union of open strata. Given an arbitrary $x \in M$, it belongs to at least one stratum; consider a strict chain $x \in S_1 < S_2 < \ldots < S_k$ which cannot be continued. Then S_k is maximal and $x \in \overline{S}_k$, hence x is in the closure of $M^{\mathcal{S}-\mathrm{reg}}$, proving that this space is dense. $\square$

Some natural questions about the regular part of M come to mind; how different is it from M? Is it connected? The following lemma tries to partially address these questions.

Lemma 3.8. *Let $\mathcal{S}$ be a stratification on a smooth manifold M, with no strata of codimension 1. Then the $\mathcal{S}$-regular part of M, denoted by M^{reg}, is connected.*

Proof. Let x and y be two points of M^{reg} and consider a smooth curve $\gamma : [0,1] \to M$ connecting x and y (recall that by M is connected by assumption). The image of γ is compact, so it can be covered by a finite number of open subsets of M, each of which intersects finitely many strata. Let U be the union of those open subsets. Then by the Transversality homotopy theorem (cf. [43, p. 70]), it is possible to find a map $\gamma' : [0,1] \to U$ which is homotopic to γ and transverse to all the finitely many strata of codimension greater than 1 in U, which means that it misses them. Since there are no strata of codimensions 1, the image of the map γ' must be completely contained in the union of the strata of codimension 0, which is precisely M^{reg}. $\square$

3.2 Proper group actions: the canonical stratification.

We recall an important example of a stratification, associated to a proper action of a Lie group G on a manifold M. This example serves as both motivation and background for the study of stratifications on proper groupoids. We take the standard approach of first defining a natural partition $\mathcal{P}$ on M associated to the action and then passing to the partition by connected components $\mathcal{P}$. For the whole of this section, let G be a Lie group acting properly on a smooth manifold M.

Orbit types and the canonical stratification

Definition 3.9. The **orbit type equivalence** is the equivalence relation on M given by

$$x \sim y \iff G_x \sim G_y \text{ (i.e. } G_x \text{ and } G_y \text{ are conjugate in } G).$$

The **partition by orbit types**, denoted by $\mathcal{P}_\sim(M)$, is the resulting partition (each member of $\mathcal{P}_\sim(M)$ is called an **orbit type**).

The reason for the terminology is that $x \sim y$ is equivalent to the fact that the orbits through x and y are diffeomorphic as G-manifolds. The members of this partition can be indexed by conjugacy classes (H) of subgroups H of G. To each such conjugacy class corresponds the orbit type

$$M_{(H)} = \{x \in M \mid G_x \sim H\} \in \mathcal{P}_\sim(M).$$

Remark 30. Points in the same orbit belong to the same orbit type, so the partition $\mathcal{P}_\sim(M)$ descends to a partition by orbit types $\mathcal{P}_\sim(M/G)$ of the orbit space.

By passing to the connected components of the members of $\mathcal{P}_\sim(M)$ and of $\mathcal{P}_\sim(M/G)$, we obtain stratifications of M and of M/G. Moreover, by passing to the quotient, the projection of the strata on M coincide with the strata on M/G. In other words, passing to the quotient commutes with taking connected components of the orbit types. A proof of these facts can be found for example in [37, 84].

Definition 3.10. The canonical stratification on M (respectively M/G) associated to the action of G is the partition of M (respectively of M/G) by connected components of the members of $\mathcal{P}_\sim(M)$ (respectively $\mathcal{P}_\sim(M/G)$) and is denoted by $\mathcal{S}_G(M)$ (respectively $\mathcal{S}(M/G)$).

Remark 31. It is important to note that, as mentioned in Remark 29, the passage to connected components of $\mathcal{P}_\sim(M)$ is really necessary to guarantee that we end up with a stratification. Besides the obvious problem that orbit types may be disconnected, there is the more serious issue that the frontier condition may not be satisfied (between orbit types). For example, there may be orbit types whose closure contains some, but not all of the fixed points of the action, as in the following example.

Example 3.11 (Orbit types do not satisfy frontier condition)**.** Consider the finite group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm 1\} \times \{\pm 1\}$, acting on $\mathbb{R}P^2$ by $(\epsilon, \eta) \cdot [x : y : z] = [x : \epsilon y : \eta z]$. For the subgroup $H = \mathbb{Z}_2 \times \{1\}$, we see that

$$\mathbb{R}P^2_{(\mathbb{Z}_2 \times \{1\})} = \{[x : 0 : z] \in \mathbb{R}P^2 \mid x \neq 0 \neq z\},$$

so its closure contains the fixed points $[1 : 0 : 0]$ and $[0 : 0 : 1]$, but not the fixed point $[0 : 1 : 0]$.

Comparison with other partitions

There are other partitions of M that induce the same stratification $\mathcal{S}(M/G)$, by passing to connected components. Let us start with the simplest one to describe.

Definition 3.12. The **partition by isotropy isomorphism classes** on M, denoted by $\mathcal{P}_\cong(M)$, is defined by the equivalence relation on M given by

$$x \cong y \iff G_x \cong G_y \text{ (Lie group isomorphism).}$$

The members of this partition can be indexed by isomorphism classes $[H]$ of subgroups H of G. To each such conjugacy class corresponds the orbit type

$$M_{[H]} = \{x \in M \mid G_x \cong H\} \in \mathcal{P}_{\cong}(M).$$

There is an obvious inclusion $M_{(H)} \subset M_{[H]}$ which is strict in general, as the following example shows.

Example 3.13. Let $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm 1\} \times \{\pm 1\}$ act on $\mathbb{R}^2$ by $(\epsilon, \eta) \cdot (x, y) = (\epsilon x, \eta y)$. The subgroups $H_1 = \mathbb{Z}_2 \times \{1\}$ $H_2 = \{1\} \times \mathbb{Z}_2$ are isomorphic but not conjugate, and arise as isotropy groups of the points $(0, 1)$ and $(1, 0)$ respectively. Hence

$$\mathbb{R}^2_{(\mathbb{Z}_2 \times \{1\})} \neq \mathbb{R}^2_{[\mathbb{Z}_2 \times \{1\}]}.$$

Proposition 3.14. *After passing to connected components, $\mathcal{P}_{\cong}(M)$ induces the same stratification on M as $\mathcal{P}_{\sim}(M)$.*

Proof. Using Lemma 3.2, it is enough to check that given any $x \in M$, there is an open neighbourhood U of x such that $M_{[G_x]} \cap U = M_{(G_x)} \cap U$. Using the normal form $[G \times_{G_x} \mathcal{N}_x]$ around x given by the Tube theorem (Theorem 1.34), and since points belonging to the same orbit are equivalent for both $\cong$ and $\sim$, it is enough to compare $\cong$ and $\sim$ for points on a neighbourhood of x in $\mathcal{N}_x$. Therefore, we have reduced the problem to checking that given a representation V of a compact subgroup $K \subset G$, it holds that $V_{[K]} = V_{(K)}$. The following lemma guarantees that this is always the case. □

Lemma 3.15. *If H is a closed subgroup of a compact Lie group K with the property that H is isomorphic (or just diffeomorphic) to K, then $H = K$.*

Proof. For dimensional reasons we see that H and K must have the same Lie algebra; from this it follows that their connected components containing the identity, H^0 and K^0, coincide. The fact that H is diffeomorphic to K implies that they also have the same (finite) number of connected components, from which the statement follows. □

There is yet another partition that induces the canonical stratification: the partition by **local types** (cf. [37, Def. 2.6.5]). Its members are indexed by equivalence classes of pairs (H, V), where H is a subgroup of G and V is a representation of H; two such pairs (H, V) and (H', V') are equivalent if H is conjugate to H' by some $g \in G$ and $V \cong V'$ by an isomorphism compatible with Ad_g.

Definition 3.16. The **partition by local types** on M, denoted by $\mathcal{P}_{\approx}(M)$, is defined by the equivalence relation on M given by

$$x \approx y \iff (G_x, \mathcal{N}_x) \approx (G_y, \mathcal{N}_y),$$

where $\mathcal{N}_x$ is the normal representation at x.

Remark 32. The reason for the terminology is that, as a consequence of the Tube theorem (Theorem 1.34), x and y belong to the same local type if and only if the orbits through x and y admit equivariantly-diffeomorphic neighbourhoods.

By the same arguments as in the proof of Proposition 3.14 above, we conclude the following.

Proposition 3.17. *After passing to connected components, $\mathcal{P}_{\approx}(M)$ induces the canonical stratification on M.*

Unlike the case for the two partitions discussed before, all the connected components of a local type have the same dimension (which can be seen using the normal form given by the Tube theorem). However, $\mathcal{P}_{\approx}(M)$ may still fail to satisfy the frontier condition: the same counterexample as for $\mathcal{P}_{\sim}(M)$ (Example 3.11) works here as well.

The difference between the three partitions discussed is that they group the strata of $\mathcal{S}_G(M)$ in different ways: one has

$$\mathcal{P}_{\cong}(M) \prec \mathcal{P}_{\sim}(M) \prec \mathcal{P}_{\approx}(M)$$

in the sense that each member of $\mathcal{P}_{\cong}(M)$ is a union of members of $\mathcal{P}_{\sim}(M)$, etc.

The infinitesimal canonical stratification

We recall another interesting stratification on M, which appears as the infinitesimal version of the canonical stratification from the previous section. The idea is that replacing the isotropy Lie groups G_x by their Lie algebras $\mathfrak{g}_x$, one obtains similar (but in general different) partitions of M.

Definition 3.18. The **infinitesimal orbit type equivalence** is the equivalence relation on M given by

$$x \underset{\text{inf}}{\sim} y \iff \mathfrak{g}_x \sim \mathfrak{g}_y \;\; \text{(i.e. } \mathfrak{g}_x \text{ and } \mathfrak{g}_y \text{ are conjugate in } \mathfrak{g}\text{).}$$

The **partition by infinitesimal orbit types** is the resulting partition, denoted by $\mathcal{P}_{\underset{\text{inf}}{\sim}}(M)$ (each member of $\mathcal{P}_{\sim}(M)$ is called an **infinitesimal orbit type**).

As before, it is easy to see that points in the same orbit belong to the same infinitesimal orbit type, so that we obtain a partition by infinitesimal orbit type on the orbit space.

Definition 3.19. The **infinitesimal canonical stratification** on M, denoted by $\mathcal{S}_G^{\text{inf}}(M)$, is the partition of M by connected components of the members of $\mathcal{P}_{\underset{\text{inf}}{\sim}}(M)$.

Remark 33. Note that, unlike the partition by orbit types, the partition by infinitesimal orbit types does not induce a stratification on the orbit space M/G. Indeed, the members of the partition on the orbit space may fail to be manifolds. For example, in the case of the action of $\mathbb{Z}_2$ on $\mathbb{R}$ by reflection at the origin, all points have the same infinitesimal orbit type, but the orbit space is not a manifold.

Similarly, we define infinitesimal versions of the partitions by isotropy isomorphism classes and by local types - define the equivalence relations $\underset{\text{inf}}{\approx}$ and $\underset{\text{inf}}{\cong}$ by replacing the isotropy Lie groups G_x by their Lie algebras $\mathfrak{g}_x$ in the definitions. The infinitesimal analogue of Lemma 3.15 is obvious, and therefore we obtain that the partitions $\mathcal{P}_{\underset{\text{inf}}{\sim}}(M)$ and $\mathcal{P}_{\underset{\text{inf}}{\cong}}(M)$ induce the canonical infinitesimal stratification $\mathcal{S}_G^{\text{inf}}(M)$.

3.3 Proper actions - principal and regular types

We have seen several partitions inducing the same stratifications $\mathcal{S}_G(M)$ and $\mathcal{S}_G^{\inf}(M)$. This allows us to use different partitions when proving results about the stratification, using whichever is more convenient for the proof.

It is interesting to distinguish which notions are intrinsic to the stratification, and which are particular to one of the partitions giving rise to it; similarly, we can wonder about whether a given result on the stratification can be strengthened to a result on one of the partitions giving rise to it.

We now recall some properties of the maximal strata of the canonical stratification and point out the relation with the partitions that give rise to it.

Definition 3.20. The **principal part** of M is defined to be the $\mathcal{S}_G(M)$-regular part of M and is denoted by $M^{\mathrm{princ}} := M^{\mathcal{S}_G(M)-\mathrm{reg}}$. The orbits inside M^{princ} are called **principal orbits**.

In order to check whether a point $x \in M$ belongs to a principal orbit, we use Lemma 3.7 and a tube $G \times_{G_x} V$ around x. We arrive at the condition that V^{G_x} is open in V, which leads to the following characterization.

Lemma 3.21. *For any point $x \in M$, the following are equivalent:*

1. *$x \in M^{\mathrm{princ}}$*
2. *the action of G_x on the normal space to the orbit is trivial.*

In this case all the orbits $G \cdot y$ through points y close to x are diffeomorphic $G \cdot x$.

By definition, M^{princ} is intrinsically associated to the canonical stratification. But to understand it better, we recall a related notion, defined in terms of the partition $\mathcal{P}_\sim(M)$ by orbit types. First of all, note that there is a partial order on the orbit types that is analogous to the ordering on the strata:

$$M_{(H)} \geq M_{(K)} \Longleftrightarrow K \text{ is } G-\text{conjugate to } \text{ a subgroup of } H.$$

The maximal orbit types (with respect to this order) are called principal orbit types. They are related to M^{princ} by the Principal orbit type theorem (cf. [37, Thm. 2.8.5], or Subsection 3.4.4 for a generalization of this theorem for proper Lie groupoids). The theorem states that $\mathcal{P}_\sim(M)$ admits one and only one maximal orbit type: there exists a unique conjugacy class, denoted (H_{princ}) such that any isotropy group G_x of the action contains a conjugate of H_{princ}. In terms of the stratification, this means that

$$M^{\mathrm{princ}} = M_{(H_{\mathrm{princ}})}.$$

Hence the maximal strata of $\mathcal{S}_G(M)$ are precisely the connected components of the principal orbit type. We also see that, although it was originally defined in terms of $\mathcal{P}_\sim(M)$, the notion of principal orbit type only depends on the stratification.

Moreover, the Principal orbit type theorem also states that, even when M^{princ} is not connected, the quotient $M_{(H_{\mathrm{princ}})}/G$ is always connected. Hence the stratification $\mathcal{S}(M/G)$ of the quotient M/G has one and only one maximal (principal) stratum.

Alternatively, we could proceed similarly but using $\mathcal{P}_{\cong}(M)$ instead of $\mathcal{P}_{\sim}(M)$; in that case, the partial order to consider is

$$M_{[H]} \geq M_{[K]} \Longleftrightarrow K \text{ is isomorphic to a subgroup of } H,$$

which is, indeed, a partial order by Lemma 3.15. The corresponding version (for $\mathcal{P}_{\cong}(M)$) of the Principal orbit type theorem results in a unique maximal element, which is precisely $M_{[H_{\text{princ}}]}$.

Proposition 3.22. *The maximal elements of $\mathcal{P}_{\cong}(M)$ and of $\mathcal{P}_{\sim}(M)$ coincide, i.e.,*

$$M_{[H_{\text{princ}}]} = M_{(H_{\text{princ}})}.$$

Proof. It is always true that $M_{(H)} \subset M_{[H]}$, so we are left with checking the converse. If x belongs to $M_{[H_{\text{princ}}]}$ we know that $G_x \cong H_{\text{princ}}$. However, we do also know that G_x is conjugate to a subgroup of H_{princ}. Using Lemma 3.15 again, we see that the subgroup must be the entire H_{princ}, and so G_x is conjugate to it; hence x is also in $M_{(H_{\text{princ}})}$. □

Finally, let us mention that, using the partition by local types, one would equally find that there is a single maximal local type, which again coincides with M^{princ} (cf. [37, Cor. 2.8.6]).

Let us now focus on the infinitesimal canonical stratification. The associated regular part is denoted:

$$M^{\text{reg}} := M^{\mathcal{S}_G^{\text{inf}}(M)-\text{reg}}.$$

In complete similarity with Lemma 3.21, using a the normal form given by the Tube theorem (Theorem 1.34), we find:

Lemma 3.23. *For any point $x \in M$, the following are equivalent:*

1. *$x \in M^{\text{reg}}$;*
2. *the infinitesimal action of $\mathfrak{g}_x$ on the normal space to the orbit is trivial;*
3. *the action of G_x^0 on the normal space to the orbit is trivial;*
4. *all the orbits through points close to x have the same dimension,*

In this case all the orbits $G \cdot y$ through points y close to x are coverings of $G \cdot x$;

The next proposition shows that, although the infinitesimal canonical stratification behaves worse than the canonical stratification (e.g. it does not induce a stratification on the quotient), it does have some advantages over $\mathcal{S}_G(M)$.

Proposition 3.24. *The infinitesimal canonical stratification satisfies:*

1. *$\mathcal{S}_G^{\text{inf}}(M)$ does not contain codimension 1 strata.*
2. *M^{reg} is connected.*
3. *$\mathcal{S}_G^{\text{inf}}(M)$ has one and only one maximal strata.*

Proof. Assume that S is a stratum of codimension one; S is a connected component of a subspace of type

$$M_{[\mathfrak{h}]} = \{x \in M \mid \mathfrak{g}_x \cong \mathfrak{h}\}$$

for some Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $x \in S$; we may assume that $G_x = H$. Let $U \subset G \times_H V$ be a tube around x, with x represented by $(e, 0)$. Then

$$U \cap M_{[\mathfrak{h}]} \cong \{y = [a, v] \in G \times_H V \mid G \times_{G_x} \mathfrak{h}_v = h\} = G \times_H V^{\mathfrak{h}}.$$

To achieve codimension 1, $V^{\mathfrak{h}}$ must be of codimension 1 in V. Let W be the complement of $V^{\mathfrak{h}}$ with respect to an H-invariant metric on V; then $W^{\mathfrak{h}} = 0$, hence W is a non-trivial one dimensional representation of the compact connected Lie group H^0, which is impossible. The fact that 1) implies 2) follows from Lemma 3.8; 3) is just a reformulation of 2). $\square$

3.4 Proper groupoids: the canonical (Morita type) stratifications

In this section we introduce the canonical stratification associated to a proper Lie groupoid. This generalizes the canonical stratification induced by a proper Lie group action. As in the case of proper actions, a proper Lie groupoid induces a stratification not only on its base, but also on the orbit space. Let us mention already that one of the essential properties of the stratification on the orbit space is that it is Morita invariant (see Remark 34), meaning that it is intrinsically associated to the orbispace presented by the groupoid.

3.4.1 Morita types and the canonical stratification

Let $\mathcal{G}$ be a proper Lie groupoid over M and denote its orbit space by X. In order to describe the canonical stratifications induced by $\mathcal{G}$, start by considering the following equivalence relation on M.

Definition 3.25. The **Morita type equivalence** is the equivalence relation on M given by

$$x \sim_{\mathcal{M}} y \iff \mathcal{N}(\mathcal{G}_{\mathcal{O}_x}) \text{ and } \mathcal{N}(\mathcal{G}_{\mathcal{O}_y}) \text{ are Morita equivalent},$$

where $\mathcal{N}(\mathcal{G}_{\mathcal{O}_x})$ and $\mathcal{N}(\mathcal{G}_{\mathcal{O}_y})$ denote the local models of $\mathcal{G}$ around the orbits through x and y. The **partition by Morita types**, denoted by $\mathcal{P}_{\mathcal{M}}(M)$, is defined to be the resulting partition. Each member of $\mathcal{P}_{\mathcal{M}}(M)$ is called an **Morita type**.

Definition 3.26. The canonical stratification on M, denoted by $\mathcal{S}_{\mathcal{G}}(M)$, is the partition on M obtained by passing to connected components of $\mathcal{P}_{\mathcal{M}}(M)$. In Section 3.4.3 we see that this partition is, indeed, a stratification.

We can give the following equivalent, more concrete, description of Morita types.

Lemma 3.27. *We can rewrite the Morita type equivalence as*

$$x \sim_{\mathcal{M}} y \iff \quad (\mathcal{G}_x, \mathcal{N}_x) \cong (\mathcal{G}_y, \mathcal{N}_y),$$

where $(\mathcal{G}_x, \mathcal{N}_x) \cong (\mathcal{G}_y, \mathcal{N}_y)$ means that there is an isomorphism $\phi : \mathcal{G}_x \to \mathcal{G}_y$ and a compatible isomorphism of representations between $\mathcal{N}_x$ and $\mathcal{N}_y$.

Proof. Recall that $\mathcal{N}(\mathcal{G}_{\mathcal{O}_x})$ and $\mathcal{N}(\mathcal{G}_{\mathcal{O}_y})$ are Morita equivalent to $\mathcal{G}_x \ltimes \mathcal{N}_x$ and $\mathcal{G}_y \ltimes \mathcal{N}_y$, respectively. These in turn are Morita equivalent to each other if and only if there is an isomorphism $\phi : \mathcal{G}_x \to \mathcal{G}_y$ and an isomorphism of representations between $\mathcal{N}_x$ and $\mathcal{N}_y$, compatible with ϕ. $\square$

In other words, the partition by Morita types is indexed by equivalence classes of pairs (H, V) where H is a Lie group, V is a representation of H, and two pairs are equivalent in the way described above: $H \cong H'$ and $V \cong V'$ in a compatible way. In this case, we write $(H, V) \cong (H', V')$. Set $[H, V]$ for the equivalence the equivalence class of the pair (H, V). If $[H, V] = \alpha$, then the element of $\mathcal{P}_{\mathcal{M}}(M)$ corresponding to α is

$$M_{(\alpha)} = \{x \in M \mid [\mathcal{G}_x, \mathcal{N}_x] = \alpha\} \in \mathcal{P}_{\mathcal{M}}(M).$$

We also denote by $M_{(x)}$ the Morita type of a point $x \in M$.

Remark 34. By definition, the Morita type of a point depends only on the Morita equivalence class of of the local model of the groupoid in a neighbourhood of its orbit. This implies that two points of the same orbit belong to the same Morita type and hence we also obtain a partition by Morita types on the orbit space, $\mathcal{P}_{\mathcal{M}}(X)$. The projection map $\pi : M \to X$ takes Morita types in M to Morita types in X; we use the notation $X_{(\alpha)} = \pi(M_{(\alpha)})$ for Morita types in the orbit space. Note that this partition on X is really associated to the orbispace X presented by $\mathcal{G}$ and not to $\mathcal{G}$ itself. Indeed, by its very definition, the Morita type of $\mathcal{O} \in X$ only depends on Morita invariant information.

Definition 3.28. The canonical stratification on the orbit space $M/\mathcal{G}$, denoted by $\mathcal{S}(M/\mathcal{G})$, is the partition on $M/\mathcal{G}$ obtained by passing to connected components of the partition by Morita types in the orbit space. In Section 3.4.3 we see that this partition is, indeed, a stratification.

Remark 35. By passing to the corresponding germ-stratification, $\mathcal{S}(M/\mathcal{G})$ corresponds to the canonical germ-stratification of [86].

3.4.2 Comparison with other equivalence relations

The notion of partition by isotropy isomorphism classes (see Definition 3.12) still makes sense for a general Lie groupoid, so we can compare the partitions $\mathcal{P}_{\cong}(M)$ and $\mathcal{P}_{\mathcal{M}}(M)$ (see Definition 3.25). In the case of an action groupoid of a proper Lie group action, it also makes sense to compare these partitions with the ones by orbit types and by local types. It is easy to see that

$$\mathcal{P}_{\cong}(M) \prec \mathcal{P}_{\mathcal{M}}(M) \prec \mathcal{P}_{\approx}(M).$$

Example 3.29. There is no such relation comparing $\mathcal{P}_{\mathcal{M}}(M)$ and $\mathcal{P}_{\sim}(M)$ in general. For example, for the action of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm 1\} \times \{\pm 1\}$ on $\mathbb{R}^2$ by $(\epsilon, \eta) \cdot (x, y) = (\epsilon x, \eta y)$, the Morita types of $(1, 0)$ and $(0, 1)$ are the same, while their orbit types are different.

On the other hand, for the action of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\pm 1\} \times \{\pm 1\}$ on $\mathbb{R}P^2$ by

$$(\epsilon, \eta) \cdot [x : y : z] = [x : \epsilon y : \eta z],$$

all fixed points have the same orbit type, but the Morita types of the fixed points $[1 : 0 : 0]$ and of $[0 : 1 : 0]$ are different. Indeed, the isotropy representation at $[1 : 0 : 0]$ is isomorphic with the representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}^2$ given by $(\epsilon, \eta) \cdot (x, y) = (\epsilon x, \eta y)$, while the isotropy representation at $[0 : 1 : 0]$ is isomorphic with the representation given by $(\epsilon, \eta) \cdot (x, y) = (\epsilon x, \epsilon\eta y)$.

Nonetheless, as soon as we pass to connected components, Morita types induce the same stratification as the other partitions.

Proposition 3.30. *Let $\mathcal{G}$ be an action groupoid associated with a proper Lie group action. The partition by Morita types on M induces, after passing to connected components, the canonical stratification associated with the action.*

Proof. The situation is completely analogous to the comparison between $\mathcal{P}_{\sim}(M)$ and $\mathcal{P}_{\cong}(M)$: apply Lemma 3.2 and use the normal form given by the Tube theorem (Theorem 1.34) to compare $\mathcal{P}_{\mathcal{M}}(M)$ with $\mathcal{P}_{\approx}(M)$. Looking at the normal form means we consider the associated bundle $G \times_K V$, where V is a representation V of a compact subgroup $K \subset G$. We have to look at the points $v \in V$ with the property that K_v is conjugate to K, (for $\approx$), or with the property that K_v is isomorphic to K (for $\sim_{\mathcal{M}}$), and additionally that the normal representation of K_v on $\mathcal{N}_v$ is isomorphic to the representation of K on V in a compatible way with the isomorphisms of K_v and K; once more, the first condition reduces to the condition $K_v = K$ because of Lemma 3.15. The conditions on the compatibility of the isomorphism of the representation become the same in both cases - compatibility with the identity map of K. $\square$

The following result relates the Morita types for a proper Lie groupoid and the Morita types for the action groupoids given by the local model.

Lemma 3.31 (Reduction to Morita types on a slice)**.** *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and let $x \in M$. Then there are invariant opens U around x in M and W around 0 in $\mathcal{N}_x$ (which we identify with a slice at x) such that the intersection of the Morita types for $\mathcal{G}$ with U are given by the saturation of the Morita types for the linear action of $\mathcal{G}_x$ on W.*

Proof. By the Linearization theorem for proper groupoids (Theorem 1.39), there are invariant open subsets U around $\mathcal{O}_x$ in M and W around 0 in $\mathcal{N}_x$ such that $\mathcal{G}_U$ is Morita equivalent to the action groupoid $\mathcal{G}_x \ltimes W$. We identify W with a slice S at x in M. By this Morita equivalence, the Morita types in U coincide with the saturation of the Morita types for the isotropy action of $\mathcal{G}_x$ on the slice S, and consequently correspond to the Morita types for the linear action of $\mathcal{G}_x$ on W. $\square$

Using the notation from the previous lemma, let F be the orthogonal complement to $W^{\mathcal{G}_x}$ with respect to an invariant inner product. The isomorphism

$$W^{\mathcal{G}_x} \times F \to W, \quad (w, f) \mapsto w + f$$

is equivariant. There is also an equivariant diffeomorphism

$$\phi : \mathbb{R}_+ \times \Sigma \to F \backslash \{0\},$$

where Σ denotes the unit sphere in F, with respect to the inner product used above; $\mathcal{G}_x$ acts on $\mathbb{R}_+ \times \Sigma$ by $g(r, p) = (r, gp)$.

Lemma 3.32. *Using the notation introduced above and identifying W with a slice at x, the intersection of the Morita type of x with U is the $\mathcal{G}$-saturation of $W^{\mathcal{G}_x}$. Each other Morita type is given by the $\mathcal{G}$-saturation of*

$$W^{\mathcal{G}_x} + \phi(\mathbb{R}_+ \times T) \subset W,$$

where T is a Morita type for the action of $\mathcal{G}_x$ on Σ.

Proof. The Morita type of the point x (which is identified with $0 \in W$) for the action of $\mathcal{G}_x$ on W is given by $W_{(x)} = W^{\mathcal{G}_x}$. To see this, note that every point of $W_{(x)}$ has an isotropy group isomorphic to $\mathcal{G}_x$, hence equal to $\mathcal{G}_x$ by Lemma 3.15, so that $W_{(x)} \subset W^{\mathcal{G}_x}$. The converse inclusion can be deduced from the fact that all points of $W^{\mathcal{G}_x}$ have the same orbit type, hence the same Morita type. We can assume that this is the case since for a small enough open containing 0, the members of $\mathcal{P}_\sim(W)$ and $\mathcal{P}_\mathcal{M}(W)$ containing 0 coincide. The first statement of the lemma follows by applying Lemma 3.31.

The second part of the statement is obtained as a consequence of Lemma 3.31 and of the fact that the isomorphism $W^{\mathcal{G}_x} \times F \to W$ and the diffeomorphism $\phi : \mathbb{R}_+ \times \Sigma \to F\backslash\{0\}$ are equivariant. □

Remark 36 (Morita types on a neighbourhood of a point). On the same note as the previous result, if we are only interested in looking at how Morita types look like in a small neighbourhood of a point x in the base, then we can use the simpler local model for $\mathcal{G}$ around x given by Proposition 1.40. This local model is the product of a pair groupoid $O \times O \rightrightarrows O$ with an action groupoid $\mathcal{G}_x \ltimes W \rightrightarrows W$, where W is an invariant ball centred at 0 in $\mathcal{N}_x$, on which the compact group $\mathcal{G}_x$ acts linearly. So we see that the intersection of each Morita type in M with the neighbourhood $O \times W$ of x is of the form $O \times T$ where T is a Morita type in W for the action groupoid $\mathcal{G}_x \ltimes W \rightrightarrows W$.

3.4.3 The canonical stratification

We focus on proving that the partitions by Morita types induce stratifications on M and X.

Proposition 3.33. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid with only one Morita type. Then the orbit space X is a smooth manifold and the canonical projection $\pi : M \to X$ is a submersion, whose fibres are the orbits.*

Proof. We already know from Proposition 2.46 that $(X, C^\infty(X))$ is a differentiable space, so in order to show that it is a smooth manifold it is enough to show that every point $\mathcal{O} \in X$ has a neighbourhood U such that $(U, C^\infty(U))$ is a smooth manifold. Consider $\mathcal{O} \in X$ and $x \in M$ such that $\pi(x) = \mathcal{O}$. We have seen in the proof of Proposition 2.46 that $\mathcal{O}$ has a neighbourhood U in X such that

$$(U, C^\infty(U)) \cong (W/\mathcal{G}_x, C^\infty(W)^{\mathcal{G}_x}),$$

where W is an open in $\mathcal{N}_x$ containing the origin, which is invariant for the isotropy representation of $\mathcal{G}_x$ on $\mathcal{N}_x$. To avoid confusion let us use the notation $H := \mathcal{G}_x$. By Morita equivalence, since $\mathcal{G}$ only has one Morita type, the same holds for $\mathcal{G}_x \ltimes W$. This implies that for every $v \in W$, the isotropy group H_v is isomorphic to H, hence $H_v = H$, by Lemma 3.15. This means that H acts trivially on W and so

$$(U, C^\infty(U)) \cong (W/\mathcal{G}_x, C^\infty(W)^{\mathcal{G}_x}) = (W, C^\infty(W))$$

is a smooth manifold, and so is X.

To check that π is a submersion, as this is a local property, we can use a neighbourhood $V \subset P_x \times_{\mathcal{G}_x} W$ of x in the local model. Since $\mathcal{G}_x$ acts trivially on W then $P_x \times_{\mathcal{G}_x} \mathcal{N}_x = \mathcal{O} \times W$. The restriction of the projection π to V is then given by $\pi(y, v) = v$, hence π is a submersion. □

Given a Morita type $M_{(\alpha)}$, we denote the restriction of $\mathcal{G}$ to $M_{(\alpha)}$ by $\mathcal{G}_{(\alpha)} := s^{-1}(M_{(\alpha)})$. It is clear that $\mathcal{G}_{(\alpha)}$ is a groupoid over $M_{(\alpha)}$. Recall that we denote its orbit space by $X_{(\alpha)}$. The next result ensures smoothness of all these objects.

Proposition 3.34. *The Morita type $M_{(\alpha)}$ is a smooth submanifold of M, the groupoid $\mathcal{G}_{(\alpha)} \rightrightarrows M_{(\alpha)}$ is a Lie groupoid, and the orbit space $X_{(\alpha)}$ is a smooth manifold.*

Proof. Let $x \in M_{(\alpha)}$ and let $g \in \mathcal{G}_{(\alpha)}$. By the Linearization theorem (Theorem 1.39), there are opens U around $\mathcal{O}_x$ in M and V around $\mathcal{O}_x$ in $\mathcal{N}\mathcal{O}_x$ such that $\mathcal{G}_U \cong \mathcal{N}(\mathcal{G}_{\mathcal{O}_x})_V$ (note that $\mathcal{G}_U$ is an open in $\mathcal{G}$ containing g).

The idea of the proof is the following: all the conditions we need to verify are local - that $M_{(\alpha)}$ and $\mathcal{G}_{(\alpha)}$ are submanifolds of M and $\mathcal{G}$, that structure maps restrict to smooth maps and that $s, t : \mathcal{G}_{(\alpha)} \to M_{(\alpha)}$ are submersions. This means that it is enough to check them on the opens V and $\mathcal{N}(\mathcal{G}_{\mathcal{O}_x})_V$, slightly abusing notation by thinking of them as opens around x and g. Checking that the multiplication restricts to a smooth map is done in an analogous way as for the other maps, but in a neighbourhood of a composable pair, in the local model.

The situation becomes simple in the linear local model: indeed, $U \cap M_{(\alpha)}$ corresponds to $V \cap (\mathcal{N}\mathcal{O}_x)_{(\alpha)}$; to check that this is a submanifold of $\mathcal{N}\mathcal{O}_x$ we might as well substitute V by its saturation $\tilde{V}$. Using the bundle description $\tilde{V} \cap N(\mathcal{G}_{\mathcal{O}_x}) \cong P_x \times_{\mathcal{G}_x} W$ and Lemma 3.32, we have that $\tilde{V} \cap (\mathcal{N}\mathcal{O}_x)_{(\alpha)}$ is equal to the saturation of $W^{\mathcal{G}_x}$ inside $P_x \times_{\mathcal{G}_x} W$ and so it is a submanifold, which is diffeomorphic to $\mathcal{O}_x \times W^{\mathcal{G}_x}$. Its dimension is constant along $M_{(\alpha)}$, so $M_{(\alpha)}$ is a submanifold of M.

Moreover, $\mathcal{G}_{(\alpha)} \cap \mathcal{G}_U$ corresponds to $(N(\mathcal{G}_{\mathcal{O}_x})_V)_{(\alpha)}$; for our purposes (local verifications) we can substitute V by $\tilde{V}$, since $(N(\mathcal{G}_{\mathcal{O}_x})_V)_{(\alpha)}$ is a neighbourhood of g inside of $(N(\mathcal{G}_{\mathcal{O}_x})_{\tilde{V}})_{(\alpha)}$. Using the description

$$N(\mathcal{G}_{\mathcal{O}_x}) \cong (P_x \times P_x) \times_{\mathcal{G}_x} \mathcal{N}_x,$$

we are interested in the restriction of $(P_x \times P_x) \times_{\mathcal{G}_x} W)$ to

$$\tilde{V} \cap (N\mathcal{O}_x)_{(\alpha)} \cong \mathcal{O}_x \times W^{\mathcal{G}_x},$$

which is exactly $Gauge(P_x) \times W^{\mathcal{G}_x} \rightrightarrows \mathcal{O}_x \times W^{\mathcal{G}_x}$ - a Lie subgroupoid of $N(\mathcal{G}_{\mathcal{O}_x})$. Therefore, we conclude that on a neighbourhood of g, $\mathcal{G}_{(\alpha)}$ is a smooth submanifold of $\mathcal{G}$, that the restriction of all the structure maps of $\mathcal{G}$ (except possibly the multiplication) to it are smooth, and that the restrictions of the source and target are submersions.

To conclude, one can proceed in a completely analogous way to check that the restriction of the multiplication map is smooth: one only needs to check it a neighbourhood of a composable pair $(g, h) \in \mathcal{G}_{(\alpha)} \cap \mathcal{G}^{(2)}$; it is enough to work in the local model, where such a neighbourhood can be constructed starting from $N(\mathcal{G}_{\mathcal{O}_x})_V \times N(\mathcal{G}_{\mathcal{O}_x})_V$, since $N(\mathcal{G}_{\mathcal{O}_x})_V$ is a neighbourhood of both g and h, and then the restriction of the multiplication coincides with the multiplication of $Gauge(P_x) \times W^{\mathcal{G}_x}$.

This proves that $\mathcal{G}_{(\alpha)} \rightrightarrows M_{(\alpha)}$ is a Lie groupoid and therefore Proposition 3.33 tells us that $X_{(\alpha)}$ is a smooth manifold. $\square$

Proposition 3.35. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then the partitions of M and of $X = M/\mathcal{G}$ by connected components of Morita types are stratifications.*

Proof. Using the local model for a groupoid around a point x in the base, we have seen in Remark 36 how to index Morita types in an open around x by the Morita types for the action groupoid associated to the action of $\mathcal{G}_x$ on the linear slice W. In such a case, we have seen in Proposition 3.30 that the partition by connected components of Morita types coincides with the canonical stratification $\mathcal{S}_{\mathcal{G}_x}(W)$, which is locally finite. We have also seen in Proposition 3.34 that the elements of the partitions by Morita types on M and X are smooth manifolds, with the subspace topology.

Let us check the condition of frontier. Let T_1 and T_2 be connected components of Morita types $M_{(\alpha)}$ and $M_{(\beta)}$ respectively and suppose that $\overline{T_1} \cap T_2 \neq \emptyset$.

First of all, let us start by checking that the condition of frontier for T_1 and T_2 holds on a neighbourhood U of any point $x \in \overline{T_1} \cap T_2$, i.e., that $T_2 \cap U \subset \overline{T_1} \cap U$. For this we use the local model of $\mathcal{G}$ around the point x, as in Proposition 1.40. As explained in Remark 36, the Morita types in such a neighbourhood of x are identified with the product of an open ball O in $\mathcal{O}_x$ with the Morita types for the action groupoid $\mathcal{G}_x \ltimes W$, where W is an invariant open ball in $\mathcal{N}_x$; similarly, passing to connected components, $T_i \cap U_0$ is given by $O \times (T_i \cap W)$, for $i = 1, 2$. So it is enough to check that $y \in \overline{T_1}$ for all points $y \in W \cap T_2$. We conclude that this holds from the fact that the partition by connected components of Morita types for an action groupoid on the slice S induces the canonical stratification on S associated to the action of $\mathcal{G}_x$.

Let y be any other point in T_2 and consider any continuous path γ from x to y in T_2. We can cover the image of γ by finitely many open subsets U_i of M, centred at points p_i along γ, such on each of them, the frontier condition for T_1 and T_2 holds, because of the same considerations made above for x. This implies that $y \in \overline{T_1}$ and so $T_2 \subset \overline{T_1}$.

The continuity of π ensures that the condition of frontier also holds in the orbit space, so both the partition of M and of X by the connected components of the Morita types define stratifications. □

As mentioned before, these stratifications are called the canonical stratifications of M and X and are denoted by $\mathcal{S}_{\mathcal{G}}(M)$ and $\mathcal{S}(X)$. Recall that the reason for omitting $\mathcal{G}$ in the notation for the canonical stratification on X is that this stratification is Morita invariant.

3.4.4 Principal and regular types

In this section we discuss principal and regular orbits and give a proof of the Principal type theorem for proper Lie groupoids (Theorem 3.41), generalizing the corresponding result for proper Lie group actions.

In analogy with the case of proper actions, we denote by M^{princ} the $\mathcal{S}_{\mathcal{G}}(M)$-regular part of M and orbits inside M^{princ} are called **principal orbits**. Similarly we denote by $X^{\text{princ}} \subset X$ the $\mathcal{S}(X)$-regular part of X. As before, we get a criteria for when a point $x \in M$ lies in M^{princ} - combining Lemma 3.7 and the local model of $\mathcal{G}$ around x, given as in Proposition 1.40 by $(O \times O) \times (\mathcal{G}_x \ltimes W) \rightrightarrows O \times W$, we arrive at the condition that $W^{\mathcal{G}_x}$ is open in W. This leads to:

Lemma 3.36. *For a point $x \in M$, the following are equivalent:*

1. *$x \in M^{\text{princ}}$*

2. *the action of $\mathcal{G}_x$ on the normal space to the orbit is trivial.*

However, one difference appears: while for a proper action, all orbits close enough to a principal orbit are diffeomorphic to it, for a proper groupoid this might not be the case. The simplest example is obtained by considering the Lie groupoid associated to a submersion between connected manifolds. It is proper and all orbits are principal, so the similar statement as for proper actions holds if and only if all fibres of the submersion are diffeomorphic. In general, this property holds for the principal orbits of a proper groupoid if and only if the restriction of the groupoid to M^{princ} is source-locally trivial, which is the case for action groupoids.

At this stage we know that M^{princ} is open, dense, and a union of Morita types (hence invariant). The Principal type theorem tells us a bit more about its geometry, but before we get into it and as preparation for its proof, let us look at an infinitesimal version of the concepts of canonical stratification and principal orbits.

The **infinitesimal canonical stratification** on M (and X) given by a proper Lie groupoid is constructed exactly like the stratification $\mathcal{S}_{\mathcal{G}}(M)$, substituting everywhere the isotropy group $\mathcal{G}_x$ and isotropy action on $\mathcal{N}_x$, at a point x, by the corresponding isotropy Lie algebra $\mathfrak{g}_x$ and induced Lie algebra action of $\mathfrak{g}_x$ on $\mathcal{N}_x$. The corresponding stratifications are denoted by $\mathcal{S}_{\mathcal{G}}^{\text{inf}}(M)$ and $\mathcal{S}^{\text{inf}}(X)$.

The constructions and results are very reminiscent of the ones for proper actions: we denote by M^{reg} and M^{reg} the $\mathcal{S}_{\mathcal{G}}^{\text{inf}}(M)$-regular part of M and the $\mathcal{S}^{\text{inf}}(X)$-regular part of X, respectively; orbits in M^{reg} are called **regular** and the following result characterizes them:

Lemma 3.37. *For any point $x \in M$, the following are equivalent:*

1. *$x \in M^{\text{reg}}$,*
2. *the infinitesimal action of $\mathfrak{g}_x$ on the normal space to the orbit is trivial,*
3. *the action of G_x^0 on the normal space to the orbit is trivial,*
4. *all the orbits through points close to x have the same dimension,*

The proof of this lemma is straightforward by using Lemma 3.7 and the local model of Proposition 1.40 for the Lie groupoid $\mathcal{G}$ around x.

The main difference with the case of proper actions is that it is no longer true that all the orbits $\mathcal{O}_y$ through points y close to a regular point x are coverings of $\mathcal{O}_x$. It is still true in the case of source-locally trivial groupoids.

From the definitions it is immediate that $M^{\text{princ}} \subset M^{\text{reg}}$ and so also that $X^{\text{princ}} \subset X^{\text{reg}}$. It is also clear that like for M^{princ}, it holds that M^{reg} is open, dense and consists of a union of Morita types. We look into connectedness properties for M^{princ} and M^{reg}. With Lemma 3.8 in mind, we start by looking at codimension 1 Morita types.

Lemma 3.38 (Structure of codimension 1 strata)**.** *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid and $x \in M$. Suppose that the codimension of $M_{(x)}$ in M is 1. Then*

1. *All orbits in $M_{(x)}$ are regular, i.e., $M_{(x)} \subset M^{\text{reg}}$;*
2. *F is 1-dimensional, where F is an orthogonal complement to $(\mathcal{N}_x)^{\mathcal{G}_x}$ in $\mathcal{N}_x$, for any $\mathcal{G}_x$-invariant inner product on $\mathcal{N}_x$, as in Lemma 3.32; $\mathcal{G}_x$ acts non-trivially on F as $O(1, \mathbb{R}) = \mathbb{Z}_2$.*

3. *The orbits near $\mathcal{O}_x$ which are not in $M_{(x)}$ are principal.*

4. *There is a neighbourhood U of x in M such that the intersection of each orbit which is not in $M_{(x)}$ with U is a two-fold covering of $\mathcal{O}_x \cap U$.*

Proof. We can use Lemma 3.31 to restrict our attention to a slice at x. We have that $\dim M_{(x)} = \dim \mathcal{O}_x + \dim \mathcal{N}_x^{\mathcal{G}_x}$. In the case that the codimension of $M_{(x)}$ is 1, F must be 1-dimensional. Now $\mathcal{G}_x$ acts non-trivially on F by orthogonal transformations, so it must act as multiplication by $\{+1, -1\}$. This proves part 2 of the lemma.

Since the orbits near $\mathcal{O}_x$ are equal to the saturation of their intersection of a slice, which can be identified with $\mathcal{N}_x$, then the dimension of the orbits is constant near $\mathcal{O}_x$, proving part 1 of the lemma.

For part 3, observe that the orbits near $\mathcal{O}_x$ which are not in $M_{(x)}$ must be the saturation of an element $v \in F\backslash\{0\}$. All such points have the same Morita type, so they all belong to the same open Morita type and therefore to M^{princ}. Part 4 is then an easy consequence of the local model for the groupoid around x (Proposition 1.40). $\square$

Corollary 3.39. *Let $\mathcal{G}$ be a Lie groupoid over M. Then*

1. *$\mathcal{S}_{\mathcal{G}}^{\text{inf}}(M)$ does not contain codimension 1 strata.*

2. *M^{reg} is not only dense and open in M, but also connected.*

3. *hence $\mathcal{S}_G^{\text{inf}}(M)$ has one and only one maximal stratum.*

Proof. Part 1 is a reformulation of part 1 of Lemma 3.38; part 2 follows from part 1 and Lemma 3.8; part 3 is just a reformulation of part 2. $\square$

Unlike the stratification $\mathcal{S}_{\mathcal{G}}^{\text{inf}}(M)$, the canonical stratification $\mathcal{S}_{\mathcal{G}}(M)$ can have codimension 1 strata. However, Lemma 3.38 also implies that on a small neighbourhood of a regular point x belonging to a codimension 1 stratum, all the orbits not contained in this stratum belong to M^{princ}; the stratum through x then disconnects M^{princ} into two half-spaces, which are permuted by the isotropy action. In other words, passing to the orbit space we conclude that a neighbourhood of $\mathcal{O}_x$ in X looks like a neighbourhood of the boundary point $\mathcal{O}_x$ on a manifold with boundary.

Example 3.40. Perhaps the simplest illustration of this situation is the following. Consider an action of the circle on the Möbius band (and its associated action groupoid), so that all orbits are regular and only the central orbit is not principal - its Morita type consists of only itself and has codimension 1. We are then in conditions of Lemma 3.38 and it is clear that the orbit space is a manifold with boundary: a closed half-line.

Theorem 3.41 (Principal Morita type theorem)**.** *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then X^{princ} is not only open and dense but also connected, hence $\mathcal{S}(X)$ has a single maximal stratum.*

Proof. Let x and y be points on two principal orbits and consider a path γ in X connecting them. since the image of γ is compact, by local finiteness of the stratification, it meets finitely many strata. As a consequence, we can use a version the Transversality homotopy theorem [43, p. 72] to obtain a curve γ', between x and y, which is transverse to all strata. This means that it misses all strata of codimension 2 or higher, and it meets finitely many strata of codimension 1 transversally.

If p is a point in the intersection of a codimension 1 stratum with the image of γ', we know by Lemma 3.38 that a neighbourhood U of $\mathcal{O}_p$ in X looks like a neighbourhood of a boundary point in a closed half-space, in which $\mathcal{O}_p$ is a boundary point (and all orbits not on the boundary are principal). Since γ' is transverse to the stratum of p, we can assume that the part of the image of $\pi \circ \gamma'$ that lies in U only touches the boundary at $\mathcal{O}_p$, so we can homotope $\pi \circ \gamma$ so that its image in U misses the boundary completely, hence it is contained in X^{princ}. Doing this to the curve $\pi \circ \gamma'$ in X finitely many times, we obtain a new curve connecting $\mathcal{O}_x$ to $\mathcal{O}_y$, completely contained in X^{princ}. $\square$

In the case of action groupoid associated to a proper Lie group action, Theorem 3.41 becomes the classical Principal orbit type theorem mentioned in Section 3.2. With some mild assumptions we obtain an easy consequence about M^{princ} as well:

Corollary 3.42. *If $\mathcal{G} \rightrightarrows M$ is a proper Lie groupoid with connected orbits (for example, if it has connected s-fibres), then M^{princ} is connected, hence $\mathcal{S}_{\mathcal{G}}(M)$ has a single maximal stratum.*

3.5 Stratified spaces with a smooth structure

When a space X admits both a stratification and a smooth structure, it is natural to ask how the strata fit together with respect to the smooth structure. For example, if X is a smooth manifold, it is natural to require that the strata are submanifolds.

In general, one way to proceed would be to think of X as a stratified space and then give it some smooth structure in a way compatible with the stratification, for example via an appropriate atlas. This approach is studied in detail in [84]. Alternatively, one could think of X as a "smooth" space (for example by giving it one of the structures described in Chapter 2) and then consider a stratification on X that respects the smooth structure. We follow this second approach.

3.5.1 Stratified differentiable spaces

Definition 3.43. A **stratified differentiable space** is a differentiable space $(X, \mathcal{O}_X)$ together with a stratification $\mathcal{S}$ on X such that the inclusion of each stratum is an embedding of differentiable spaces.

As remarked in [86], the notion of a reduced stratified differentiable space is the same as that of a stratified space with smooth structure of Pflaum, defined in terms of singular charts (cf. [84, Sec. 1.3]).

For a general stratification, the only condition we have required on how the strata should fit together is the frontier condition. In the presence of smooth structure, there are several other conditions that are often imposed, the most common of them being Whitney's conditions (A) and (B).

Definition 3.44. Let M be a smooth manifold and let $\mathcal{S}$ be a stratification on M. We say that a pair of strata (R, S) satisfies the **Whitney condition (A)** if the following condition holds:

(A) Let (x_n) be any sequence of points in R such that x_n converges to a point $x \in S$ and $T_{x_n}S$ converges to $\tau \subset T_xM$ in the Grassmannian of $(\dim R)$-dimensional subspaces of TM. Then $T_xS \subset \tau$.

Let $\phi : U \longrightarrow \mathbb{R}^n$ be a local chart around $x \in S$. We say that (S,T) satisfies the **Whitney condition (B)** at x with respect to the chart (U,ϕ) if satisfies the following condition:

(B) Let (x_n) be a sequence as in (A), with $x_n \in U \cap R$ and let (y_n) be a sequence of points of $U \cap S$, converging to x, such that the sequence of lines

$$\overline{\phi(x_n)\phi(y_n)}$$

converges in projective space to a line ℓ. Then $(d_x\phi)^{-1}(\ell) \subset \tau$.

The pair (S,T) is said to satisfy the Whitney condition (B) if the above condition holds for any point $x \in S$ and any chart around x.

Although we have used charts in the definition of Whitney condition (B), the condition is actually independent of the chart chosen - if (S,T) satisfy Whitney (B) with respect to a chart around x, they do so with respect to any chart as well [84, Lemma 1.4.4].

Remark 37. Note that Whitney's conditions are local, by definition, and they also make sense for a stratification on a subspace of $\mathbb{R}^n$. In this way we can make sense of when a reduced stratified differentiable space X satisfies Whitney's conditions: by Proposition 2.31 such a space is always covered by open subspaces $(U_i, \mathcal{O}_{X|U_i})$ isomorphic to $(Z, \mathcal{C}^\infty_Z)$, for some closed subset Z of $\mathbb{R}^n$. The subspace Z_i has a decomposition $\mathcal{P}_i$ induced by the decomposition on U_i (By restricting to an open, we might not have a stratification, but only a decomposition; that is not a problem since all that we want is to check that Whitney's conditions hold for the pieces of the decomposition on Z_i corresponding to the strata on X).

Definition 3.45. Let $(X, \mathcal{O}_X, \mathcal{S})$ be a reduced stratified differentiable space. We say that $\mathcal{S}$ is a **Whitney stratification** if X is covered by opens U_i, such that, in the notation of Remark 37 above, $(U_i, \mathcal{O}_{X|U_i})$ is isomorphic to $(Z, \mathcal{C}^\infty_{Z_i})$, and all the induced stratifications $\mathcal{S}_i$ satisfy Whitney's conditions (A) and (B).

The idea is that Whitney's conditions are local conditions about how the strata fit together that permit drawing important global information about the stratification. Let us mention one particularly important example of this idea: stratifications satisfying Whitney's conditions are locally trivial (see Proposition 3.48 below).

Definition 3.46. A **morphism of stratified spaces** is a continuous map $f : X \longrightarrow Y$ between stratified spaces with the property that for every stratum S of X there is a stratum R_S of Y such that $f(S) \subset Y$ and the restriction $f_{|S} : S \longrightarrow R_S$ is smooth.

Let X_1 and X_2 be two topological spaces, with stratifications $\mathcal{S}_i$ on X_i, $i = 1, 2$. Then the products of the form $S \times R$ with $S \in \mathcal{S}_1$ and $R \in \mathcal{S}_2$ form a stratification on $X_1 \times X_2$.

Definition 3.47. A stratification $\mathcal{S}$ on a space S is called **topologically locally trivial** if for every $x \in X$ there is an open neighbourhood U of x in X, a stratification $\mathcal{S}_F$ on a space F, a point $0 \in F$ and an isomorphism of stratified spaces

$$\phi : (S \cap U) \times F \longrightarrow U,$$

where S is the stratum of $\mathcal{S}$ containing x, such that the stratum of $\mathcal{S}_F$ containing 0 is simply $\{0\}$, and such that $\phi(y, 0) = y$ for any $y \in S \cap U$. In this case F is called the **typical fibre** over x. When F is a cone, $F = CL$, we say that L is the **link** of x.

If L is locally trivial with cones as typical fibres, and that holds again for the links in the points of L, and so on, we say that $(X, \mathcal{S})$ is a **cone space**.

Proposition 3.48. *[Thom-Mather [64, 99]] Any Whitney stratified reduced differentiable space is locally trivial, with cones as typical fibres.*

3.5.2 Orbispaces as stratified differentiable spaces

We now focus on the smooth properties of the canonical stratifications associated to a proper Lie groupoid.

Proposition 3.49. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then M and the orbit space $X = M/\mathcal{G}$, together with the canonical stratifications, are stratified differentiable spaces. Moreover, the canonical stratifications of M and X are Whitney stratifications.*

Proof. We have seen before that each connected component of a Morita type is a submanifold of M so, together with its canonical stratification, M is a stratified differentiable space. Moreover, using the local model of $\mathcal{G}$ around a point in M, as in Remark 36, we see that since the stratification of the action of $\mathcal{G}_x$ on $\mathcal{N}_x$ is a Whitney stratification, the same holds for the canonical stratification on M.

Now let us focus on the orbit space. We already know that the strata are locally closed subspaces of X. It is a local problem to Verify that they are embedded in X as differentiable spaces and satisfy Whitney's conditions. Let x in X and let U be a neighbourhood of x in X such that the intersection of the canonical stratification with U coincides with the canonical stratification associated with a compact Lie group representation (again, by the local description of Morita types as in Remark 36)

The statement now follows from a result of Bierstone [7], which states that the orbit space of a representation of a compact group has a Whitney stratification (which coincides with the canonical stratification). □

Corollary 3.50. *Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then the canonical stratifications on M and X are locally trivial with cones as typical fibres. Moreover, M is a cone space.*

Proof. The first statement is a direct consequence of Proposition 3.48. But actually, we already had an explicit description of this fact, by combining Lemma 3.32 and Remark 36. For the stratification on M we find that the typical fibre over x is the orthogonal complement F to $\mathcal{N}_x^{\mathcal{G}_x}$ with respect to an invariant inner product for the $\mathcal{G}_x$-action on $\mathcal{N}_x$; the link of x is the unit sphere Σ of F. Since the stratification on Σ is the canonical stratification for the $\mathcal{G}_x$-action, it has the same properties, so M, with the canonical stratification, is a cone space. □

With the notation of the proof of the previous corollary, on X we have that the typical fibre over the orbit $\mathcal{O}_x$ is $F/\mathcal{G}_x$ and the link of $\mathcal{O}_x$ is $\Sigma/\mathcal{G}_x$.

Remark 38. The proof that the canonical stratifications on M and X are Whitney stratifications, using the language of germ-stratifications, first appeared in [86]. The authors of *loc. cit.* also use the canonical stratification to prove that the orbit space of a proper groupoid can be triangulated, and a deRham theorem for the basic cohomology of a proper Lie groupoid.

Another example of a stratified differentiable space arising in the study of proper Lie groupoids is that of the inertia groupoid of a proper Lie groupoid [39, 40].

3.5.3 Orbispaces as stratified subcartesian spaces

We now briefly discuss stratifications on subcartesian spaces coming from orbits of families of vector fields (see Section 2.4) and their relevance for orbispaces.

Proposition 3.51 (Proposition 4.1.2 in [96]). *The partition of a subcartesian space S by orbits of the family of all vector fields on S satisfies the frontier condition.*

Corollary 3.52 (Corollary 4.1.3 in [96]). *The partition of a subcartesian space S by orbits of the family of all vector fields on S is a stratification if and only if the partition is locally finite and the orbits are locally closed.*

An important example of a case where orbits of all vector fields do indeed define a stratification is that of orbit spaces of proper actions.

Theorem 3.53 (Theorem 4.3.10 in [96]). *Let G be a lie group acting properly on a manifold M. The partition of the subcartesian space M/G by orbits of the family of all vector fields on M/G is the canonical stratification $\mathcal{S}(M/G)$.*

Note that this means that for the orbit space of a proper lie group action, the canonical stratification is completely determined by the smooth structure on M/G. Since the orbit space of a proper Lie groupoid is locally diffeomorphic (as a subcartesian space) to the orbit space of a representation of a compact Lie group, we obtain the following.

Corollary 3.54. *Let X be the orbit space of a proper Lie groupoid. Then the canonical stratification $\mathcal{S}(X)$ coincides with the partition by the orbits of the family of all vector fields on X, seen as a subcartesian space.*

In the case of a classical orbifold, i.e., one which is presented by an effective proper groupoid, the following result of Watts shows that we can actually recover all the information from the smooth structure:

Theorem 3.55 (Main theorem in [106]). *Given an effective orbifold X, an orbifold atlas for it can be constructed out of invariants of the ring of smooth functions $C^\infty(X)$.*

Chapter 4

Measures and volumes on differentiable stacks

Differentiable stacks have recently received increased attention. Intuitively, they allow us to treat very singular spaces by looking at non-singular ones endowed with extra structure. While they are inspired by the notion of stack from algebraic geometry, differentiable stacks are nowadays very often seen as Morita equivalence classes of Lie groupoids; the intuition is that the stack associated to a Lie groupoid $\mathcal{G}$ over a manifold M models the space of orbits $M/\mathcal{G}$ of $\mathcal{G}$ (very singular in general!). Accordingly, one uses the notation $M//\mathcal{G}$ to denote the stack represented by $\mathcal{G}$. From this point of view, the framework that differentiable stacks provide for studying singular spaces is a rather obvious extension of Haefliger's philosophy [46] on studying leaf spaces (and the transverse geometry of foliations); while Haefliger's understanding of leaf spaces was as Morita equivalence classes of étale groupoids, differentiable stacks allow for more general groupoids/singular spaces. In this chapter we will show that a rather straightforward extension of Haefliger's approach to transverse measures for foliations [45] allows one to talk about measures and geometric measures (densities) for differentiable stacks. For densities, when we compute the resulting volumes, we will recover the formulas that are taken as definition by Weinstein [110]; also, using our viewpoint, we prove the conjecture left open in *loc.cit.*

We would also like to stress that the original motivation for writing the paper that gave rise to this chapter comes from the study of Poisson manifolds of compact type [21, 22, 23]; there, the space of symplectic leaves comes with two interesting measures (the integral affine and the Duistermaat-Heckman measures); although the leaf space is an orbifold, the measures are of a "stacky" nature that goes beyond Haefliger's framework. From this point of view, this work puts those measures into the general framework of measures on stacks; see also the comments below and our entire Section 4.5.

The measures we work with in this chapter are Radon measures on locally compact Hausdorff spaces X, interpreted as positive linear functionals on the space $C_c(X)$ of compactly supported continuous functions, or $C_c^\infty(X)$ if X is a manifold (the basic definitions are collected in Section 4.1). Therefore, for a Lie groupoid $\mathcal{G} \rightrightarrows M$ (representing the stack $M//\mathcal{G}$), the main question is how to define the space $C_c^\infty(M//\mathcal{G})$; this is done in Haefliger's style, by thinking about "compactly supported smooth functions on the orbit space $M/\mathcal{G}$". To make sure that the resulting notion of measure (called transverse measure for $\mathcal{G}$) is intrinsic to the stack, we have to prove the following (for a more precise statement, please see Theorem 4.19):

Theorem 4.1. *The notion of transverse measure for Lie groupoids is invariant under Morita equivalences.*

On manifolds X, one usually works with "geometric measures", i.e., measures induced by densities $\rho \in \mathcal{D}(X)$, i.e., sections of the density bundle $\mathcal{D}_X$ (so that the resulting integration is based on the standard Lebesgue integration). We will explain (again by "general non-sense") that the same discussion applies to differentiable stacks. For a groupoid $\mathcal{G} \rightrightarrows M$, the object that is central to the discussion (and which replaces the density bundle $\mathcal{D}_X$ from the classical case) is the transverse density bundle

$$\mathcal{D}^{\mathrm{tr}}_{\mathcal{G}} := \mathcal{D}_{A^*} \otimes \mathcal{D}_{TM} \quad \text{(morally, it is } \mathcal{D}_{M//\mathcal{G}})$$

(the tensor product of the density bundle of the dual of the Lie algebroid A of $\mathcal{G}$ and the density bundle of TM). The remark that this is a representation of $\mathcal{G}$ which behaves functorially under Morita equivalence is important conceptually because such representations L of $\mathcal{G}$ represent vector bundles $L//\mathcal{G}$ on the stack $M//\mathcal{G}$. In turn, the sections of $L//\mathcal{G}$ are defined as $\mathcal{G}$-invariant sections of L. Therefore, densities (= geometric measures) on the stack $M//\mathcal{G}$ correspond to (positive) invariant sections of bundle $\mathcal{D}^{\mathrm{tr}}_{\mathcal{G}}$; we will follow Haefliger's viewpoint and call them transverse densities for $\mathcal{G}$. Such a transverse density σ can be decomposed as $\sigma = \rho^{\vee} \otimes \tau$, there ρ is a density on the Lie algebroid and τ one on M; such decompositions are useful for writing down explicit formulas for the resulting measures.

While the general theory may seem rather abstract, it becomes more concrete when applied to proper stacks. While general stacks can be viewed as generalization of Haefliger's leaf spaces, proper stacks generalize orbifolds. In the groupoid language, this means that we look at groupoids $\mathcal{G} \rightrightarrows M$ which are proper in the sense that the map $(s, t) : \mathcal{G} \to M$ is proper (s is the source map, and t is the target). Thinking of Lie groupoids as generalizations of Lie groups or Lie group actions, proper groupoids are generalizations of compact groups or proper Lie group actions, respectively. Section 4.4 is entirely devoted to the proper case. In this case the orbit space

$$B := M/\mathcal{G}$$

(with the quotient topology) is not so pathological any more: it is a locally compact Hausdorff space and one can even make a very satisfactory sense of "smooth functions on B" (as those functions that, when pulled-back to M, are smooth). In particular, measures on B make sense in the classical sense. We will prove:

Theorem 4.2. *For proper groupoids, transverse measure are in 1-1 correspondence with measures on the orbit space.*

For a more detailed statement, please see Theorem 4.30. This theorem is rather surprising, and also a bit disappointing, since it says that, from the point of view of measure theory, proper stacks can be treated as ordinary topological spaces. Still, the point of view of groupoids/stacks does bring something new and interesting into the story: the understanding of geometric measures (densities), which depends on the full stack structure and not just on the underlying space. This gain is not only conceptual - it also allows us to build, out of concrete sections of vector bundles, namely from the transverse densities $\sigma = \rho^{\vee} \otimes \tau$ mentioned above, measures on the orbit space for which the resulting integration can be computed by rather explicit Weyl-type formulas:

$$\int_M f(x)\, d\mu_\tau(x) = \int_B \left(\int_{\mathcal{O}_b} f(y)\, d\mu_{\rho_{\mathcal{O}_b}}(y) \right) d\mu_\sigma(b), \tag{4.1}$$

This is explained in our Proposition 4.31, which is one of the main results of this chapter (see also the comment that follows the proposition).

Another digression is provided by the case of symplectic groupoids, which is treated in Section 4.5. This case presents several particularities, due to the presence of symplectic/Poisson geometry. In particular, we will recast the measures from [22]:

Theorem 4.3. *Any regular proper symplectic groupoid $(\mathcal{G}, \Omega)$ over M carries a canonical transverse density, hence a canonical induced measure μ_{aff} (called the affine measure) on the leaf space $B = M/\mathcal{G}$.*

If $\mathcal{G}$ is s-proper (but not necessarily regular), then it carries a canonical transverse density, hence a canonical induced measure μ_{DH} (called the Duistermaat-Heckman measure) on B, obtained by pushing forwards the Liouville measure associated to Ω.

See Corollary 4.38 and Subsection 4.5.1. In [22] it is shown that the two measures are related by a Duistermaat-Heckman formula. As application of our framework, we will provide a new proof of the Weyl-type integration formula from [22] (as an application of the general formula (4.1) mentioned above) and we will prove the characterization of the affine measure μ_{aff} that is left as a conjecture in [110].

Returning to the general theory, a surprising (at first) feature of transverse measures (geometric or not!) is that:

Theorem 4.4. *Any transverse measure for a Lie groupoid satisfies the Stokes formula (with respect to algebroid differentiation); hence it gives rise to a closed algebroid current (and, conversely, if the s-fibres are connected).*

The precise statement is given in Proposition 4.42 and 4.44. The surprise come from the fact that such formulas are usually satisfied only by geometric measures. The explanation is the fact that the Stokes formula in this setting involves derivatives only in the longitudinal direction. As a consequence, one can reinterpret transverse measures as certain currents in the world of Lie algebroids (as stated in Proposition 4.44); when applied to foliations, this construction gives rise to the Ruelle-Sullivan current associated to a transverse measure.

Conceptually, the space $C_c^\infty(M//\mathcal{G})$ that we use to define the notion of transverse measure is best understood in terms of differentiable cohomology with compact supports (which, modulo re-indexing, will also be called "differentiable homology"); one has, directly from the definitions,

$$C_c^\infty(M//\mathcal{G}) = H_0(\mathcal{G}) = H_c^r(\mathcal{G}),$$

where r is the dimension of the s-fibres of $\mathcal{G}$. We will prove

Theorem 4.5. *$H_\bullet(\mathcal{G}) = H_c^{r-\bullet}(\mathcal{G})$ is Morita invariant and it is related to the compactly supported cohomology of the Lie algebroid by a Van Est map*

$$VE^\bullet : H_c^\bullet(A, \mathfrak{o}_A) \to H_c^\bullet(\mathcal{G}).$$

If the s-fibres of $\mathcal{G}$ are homologically k-connected, where $k \in \{0, 1, \ldots, r-1\}$ (and r is the rank of A), then $VE^0, \ldots, VE^k$ are isomorphisms.

This is Theorem 4.47 and Theorem 4.49 in the main body. This provides the tools to prove some of the results mentioned above (e.g. the Stokes formula and the reinterpretation in terms of currents).

The main body of the chapter provides the details for the story we have just described. However, along the way, we also have to revisit some of the standard material that is, we believe, not so clearly treated in the existing literature. One such standard material, that we organize in an appendix (since we believe to be of independent interest), concerns Haar systems for Lie groupoids $\mathcal{G}$, i.e., families

$$\mu = \{\mu_x\}$$

of measures on the s-fibres of $\mathcal{G}$, that are invariant under right translations. In the "geometric case", i.e., when the measures come from densities, this boils down to (certain) sections of the density bundle $\mathcal{D}_A$ of the algebroid A of $\mathcal{G}$; we call them Haar densities. When trying to use such Haar systems/density to perform "standard averaging", one is lead to the notion of properness of μ: which means that the source map is proper when restricted to the support of μ. We will show:

Proposition 4.6. *For any Lie groupoid $\mathcal{G}$ over a manifold M, the following are equivalent:*

1. *$\mathcal{G}$ is proper.*
2. *$\mathcal{G}$ admits a proper Haar system.*
3. *$\mathcal{G}$ admits a proper Haar density.*

In particular, if $\rho \in C^\infty(M, \mathcal{D}_A)$ is a full Haar density then there exists a function $c \in C^\infty(M)$ such that $c \cdot \rho$ is a proper Haar density which, moreover, may be arranged to be normalized. Such a c is called a ***cut-off function for*** *ρ.*

Cut-off functions are very often used to perform averaging in order to produce invariant objects (functions, metrics, etc); the standard reference for the existence of such functions is [101]; our previous proposition is the result of our attempt to understand the proof from *loc. cit.* and the actual meaning of such cut-off functions.

Another piece of standard material that we revisit concerns the notion of modular class of Lie algebroids and Lie groupoids; in particular, we point out that the representation Q_A [38] that is commonly used when talking about modular classes is a rather unfortunate choice, since it is not really an object associated to the stack (in general, it is not a representation of the groupoid!); as we will explain, one should use the transverse density bundle $\mathcal{D}_A^{\mathrm{tr}}$ instead of Q_A.

Finally, we would like to point our that our approach is very much compatible with the non-commutative approach to measures on orbit spaces [15, 16], as the "localization at units" of the non-commutative measures arising as traces on the convolution algebras (see Propositions 4.18 and 4.25). Of course, the non-commutative viewpoint calls for more: understanding higher traces (Hochschild/cyclic cohomology computations) and other localizations (not just at units) in more geometric terms. We hope that our study of transverse measures/densities is a useful step in that direction; it would be interesting to combine it with the related computations of Pflaum-Posthuma-Tang from [87].

Notation and conventions: We denote by $\Gamma(E) = C^{\infty}(M, E)$ the space of smooth sections of a vector bundle E over a manifold M, and by $C_c^{\infty}(M, E)$ the space of smooth sections with compact support of E.

Throughout this text we assume that given a Lie groupoid $\mathcal{G}$, all its source-fibres have the same dimension. The notation related to groupoids and algebroids can be found in the Appendix to this chapter.

4.1 Measures on manifolds

4.1.1 Radon measures

To fix the notation, we start by recalling some background on measures on manifolds. As working definition we will adopt the following:

Definition 4.7. Let X be a locally compact Hausdorff topological space. A **(Radon) measure** on X is a linear functional

$$\mu : C_c(X) \to \mathbb{R}$$

which is positive in the sense that for any $f \in C_c(X)$ with $f \geq 0$ one has $\mu(f) \geq 0$.

Via the Riesz duality theorem (cf. e.g. [91, Ch. 2]), this notion is equivalent to the more intuitive one in terms of set-measures, which we recall for completeness; the set-measure version will stay in the back of our mind, but will not be used other than for notational purposes: we deal with measures

$$\mu : \mathcal{B}(X) \to [0, \infty]$$

defined on the σ-algebra generated by the topology of X, with the property that μ is finite on compact sets and is inner regular, in the sense that for every Borel set B we have

$$\mu(B) = \sup \{\mu(K) \mid K \subset B,\ K - \text{compact}\}. \tag{4.2}$$

This condition implies that μ is uniquely determined by what it does on compact sets; it is also determined by what it does on open sets, since it follows that, for any compact K,

$$\mu(K) = \inf \{\mu(U) \mid K \subset U,\ U - \text{open}\}. \tag{4.3}$$

Such a measure gives rise to an integration map

$$I_{\mu} : C_c(X) \to \mathbb{R}, \quad I_{\mu}(f) = \int_X f(x)\ d\mu(x).$$

defined on the space of compactly supported continuous functions on X, i.e., a measure in the sense of the previous definition. For the converse, the way in which μ can be recovered from I_{μ} comes from the formula

$$\mu(U) = \sup\{I_{\mu}(f) \mid f \in C_c(X), 0 \leq f \leq 1, \mathrm{supp(f)} \subset U\}$$

valid for all open subsets $U \subset X$ (together with the previous remark that μ is uniquely determined by what it does on open subsets).

Remark 39 (a handy notation). For a Radon measure μ on X, although we will not use explicitly the corresponding set-measure, we will use the formula

$$\mu(f) = \int_X f(x)\, d\mu(x)$$

as a notation that indicates which is the function on X that μ acts on. For instance, if μ is a Radon measure on a group G, then writing $\int_G f(gh, h^2g^3)\, d\mu(h)$ indicates that g is fixed and one applies μ to the function $G \ni h \mapsto f(gh, h^2g^3)$.

Remark 40 (automatic continuity + a guiding idea for our approach). The definition that we adopted is often convenient to work with because it is purely algebraic. However, $C_c(X)$ carries its standard topology: the inductive limit topology that arises by writing $C_c(X)$ as the union over all compacts $K \subset X$ of the spaces $C_K(X)$ of continuous functions supported in K, where each such space is endowed with the sup-norm; convergence $f_n \to f$ in this topology means convergence in the sup-norm together with the condition that there exists a compact $K \subset M$ such that $\mathrm{supp}(f_n) \subset K$ for all n.

The main reason that continuity is not mentioned in the definition is because it follows from the positivity condition (see also below). However, it is very important. For instance, if one wants to allow more general measures (real-valued), then one gives up on the positivity condition but one requests continuity.

In contrast with the continuity property (which is important but not explicitly required), the space $C_c(X)$ on which μ is defined is not so important. A very good illustration of this remark is the fact that, if X is a manifold, we could use the space $C_c^\infty(X)$ (which is more natural in the smooth setting) to define the notion of measure. All that matters is that the (test) spaces $C_c(X)$ and $C_c^\infty(X)$ contain enough "test functions" to model the topology of X (hence to characterize completely the Radon measures). This remark is important for our approach to transverse measures, approach that is centred around the question: what is the correct space of "transverse test functions"?

The following brings together some known results that we were alluding to; we include the proof for completeness and also for later use.

Proposition 4.8. *For any smooth manifold M,*

1. *$C_c^\infty(M)$ is dense in $C_c(M)$.*
2. *any positive linear functional on $C_c(M)$ is automatically continuous; and the same holds for $C_c^\infty(M)$ with the subspace topology.*
3. *the construction $\mu \mapsto \mu|_{C_c^\infty(M)}$ induces a 1-1 correspondence:*

$$\{\text{(Radon) measures on } M\} \stackrel{\sim}{\longleftrightarrow} \{\text{positive linear functionals on } C_c^\infty(M)\}.$$

Proof. The density statement is well-known, but here is a less standard argument: use Stone-Weierstrass (cf. [30, Thm. 10.A.2]). When M is compact this is straightforward. In general, consider $\mathcal{A} = C_c^\infty(M)$ as a subspace of the space of continuous functions vanishing at infinity

$$C_0(M) := \{f \in C(M) \mid \forall\, \epsilon > 0\ ,\ \exists\, K \subset M \text{ compact, such that } |f_{|M\setminus K}| < \epsilon\};$$

the Stone-Weierstrass theorem for locally compact Hausdorff spaces (as it follows from the one for compact spaces applied to the one-point compactification of M and the unitization of $\mathcal{A}$) tells us that $\mathcal{A}$ is dense in $C_0(M)$ with respect to the sup-norm, provided:

- $\mathcal{A}$ is a subalgebra of $C_0(M)$.
- $\mathcal{A}$ does not vanish anywhere (i.e., for each $x \in M$, there exists $f \in \mathcal{A}$ with $f(x) \neq 0$).
- $\mathcal{A}$ separates the points of M: for each $x, y \in M$ distinct, one can find $f \in \mathcal{A}$ with $f(x) \neq f(y)$.

Since this is clearly satisfied by our $\mathcal{A}$, we deduce that for any $f \in C_0(M)$ one can find a sequence $(f_n)_{n\geq 1}$ in $C_c^\infty(M)$ that converges to f in the sup-norm. If f has compact support K then, choosing $\phi \in C_c^\infty(M)$ with $\phi|_K = 1$, we have that $\{\phi \cdot f_n\}_{n\geq 1}$ converges uniformly to $\phi \cdot f = f$ and the support of all these functions are inside the compact $\mathrm{supp}(\phi)$; hence the last convergence holds in $C_c(M)$.

For part 2 recall that the continuity of a function μ on $C_c(M)$ means that for every compact $K \subset X$, there exists a constant C_K such that

$$|\mu(f)| \leq C_K ||f||_{\mathrm{sup}} \quad \forall\ f \in C_K X).$$

To see that this is implied by the positivity of μ, for $K \subset X$ compact, we choose a function $\phi_K \in C_c(X)$ that is 1 on K, for any $f \in C_K(X)$ we have that

$$-||f||_{\mathrm{sup}}\phi_K \leq f \leq ||f||_{\mathrm{sup}}\phi_K$$

hence, by the positivity of μ,

$$|\mu(f)| \leq C_K ||f||_{\mathrm{sup}}, \quad \text{where } C_K = \mu(\phi_K).$$

Exactly the same argument (just that one chooses ϕ_K smooth) works for $C_c^\infty(M)$.

With parts 1 and 2 at hand, part 3 is basically the standard result that continuous functionals on a dense subspace of a locally convex vector space extend uniquely to continuous linear functional on the entire space; starting with μ on $C_c^\infty(M)$, the extension $\tilde{\mu}$ is (and must be given by) $\tilde{\mu}(f) = \lim_{n\to\infty} \mu(f_n)$ where $\{f_n\}_{n\geq 1}$ is a sequence in $C_c^\infty(M)$ converging to f (the continuity of μ implies that the sequence $\{\mu(f_n)\}_{n\geq 1}$ is Cauchy, hence the limit is well defined and is easily checked to be independent of the choice of the sequence). What does not follow right away from the standard argument is the positivity of $\tilde{\mu}$, but this is arranged by a simple trick: for $f \in C_c(M)$ one can find a sequence of smooth functions g_n such that $g_n^2 \to f$ (because $\sqrt{f} \in C_c(M)$), and then $\tilde{\mu}(f)$ is the limit of $\mu(g_n^2) \geq 0$. □

Sometimes one would like to integrate more than just compactly supported functions. One way to do so is by working with measures with compact supports. The notion of support of a measure μ, denoted by $\mathrm{supp}(\mu)$ can be defined since measures on X form a sheaf (due to the existence of partitions of unity). Explicitly, it is the smallest closed subspace of X with the property that

$$\mu(f) = 0 \quad \forall f \in C_c(X \setminus \mathrm{supp}(\mu)).$$

Note that, when X is a manifold, one could also use $C_c^\infty(X)$ to characterize the support. In the next lemma, continuity refers to the topology on $C(X)$ of uniform convergence on compacts.

Lemma 4.9. *For a measure μ on X, the following are equivalent:*

1. *μ is compactly supported.*
2. *μ has an extension (necessarily unique) to a linear continuous map*

$$\mu : C(X) \to \mathbb{R}$$

(and, for manifolds, the similar statement that uses $C^\infty(X)$).

Proof. Assume first that μ has compact support and choose $\phi \in C_c(X)$ that is 1 on the support. Define the extension of μ to $C(X)$ by

$$\tilde{\mu}(f) := \mu(\phi \cdot f).$$

This is an extension because, for $f \in C_c(X)$, one has that $f - \phi \cdot f$ has compact support inside $X \setminus \mathrm{supp}(\mu)$, hence $\mu(f - \phi \cdot f) = 0$ and then $\mu(f) = \tilde{\mu}(f)$. Also, $\tilde{\mu}$ is continuous because if $f_n \to f$ in $C(X)$ (uniform convergence on compacts) then clearly $\phi \cdot f_n \to \phi \cdot f$ in $C_c(X)$. The extension is unique because $C_c(X)$ is dense in $C(X)$: choosing an exhaustion $\{K_n\}_{n\geq 1}$ of X by compacts and $\phi_n \in C_c(X)$ that is 1 on K_n, then for any $f \in C(X)$, $\{\phi_n \cdot f\}_{n\geq 1}$ is a sequence of compactly supported continuous functions that converges to f uniformly on compacts.

For the converse, assume that we have such an extension, still denoted by μ. Using an exhaustion of X as above, the topology on $C(X)$ is the one induced by the increasing family of norms given by the sup-norms on the K_n's. Hence the continuity of μ translates into the existence of a compact K and a constant $C > 0$ such that

$$|\mu(f)| \leq C \cdot \sup\{|f(x)| \ : \ x \in K\}.$$

It is clear that, for $f \in C_c(X \setminus K)$ one must have $\mu(f) = 0$, hence $\mathrm{supp}(\mu) \subset K$ must be compact. □

4.1.2 Geometric measures: densities

In the case of manifolds there is a distinguished class of measures to consider: the ones for which, locally, the corresponding integration is the standard integration (of compactly supported smooth functions) on $\mathbb{R}^n$. This brings us to the notion of densities on manifolds, that we recall next (cf. [6] for a textbook account).

First of all, recall that a **density** on a finite dimensional vector space V is a map

$$\rho : \Lambda^{\mathrm{top}} V \to \mathbb{R}$$

with the property that

$$\rho(\lambda v) = |\lambda|\rho(v) \quad \forall\ \lambda \in \mathbb{R},\ v \in V.$$

We denote by $\mathcal{D}_V$ the resulting **space of densities** of V. This is a 1-dimensional vector space, therefore isomorphic to $\mathbb{R}$, but not canonically. It is closely related to the **top-exterior space** $\Lambda^{\mathrm{top}} V^*$ and the **orientation space** $\mathfrak{o}_V$ of V; this is best seen by realizing these spaces in a similar way:

$$\Lambda^{\mathrm{top}} V^* = \{\omega : \Lambda^{\mathrm{top}} V \to \mathbb{R} \mid \omega(\lambda v) = \lambda\omega(v) \quad \forall\ \lambda \in \mathbb{R},\ v \in V\},$$

$$\mathfrak{o}_V = \{\epsilon : \Lambda^{\text{top}} V \to \mathbb{R} \mid \epsilon(\lambda v) = \text{sign}(\lambda)\epsilon(v) \quad \forall\, \lambda \in \mathbb{R}^*,\ v \in V\},$$

which gives rise to a canonical isomorphism

$$\mathcal{D}_V \otimes \mathfrak{o}_V \cong \Lambda^{\text{top}} V^*.$$

(alternatively, to avoid working with maps on $\Lambda^{\text{top}} V$ which are neither linear nor smooth, one can represent the elements of these spaces as $\mathbb{R}$-valued functions on the space of frames of V, $\xi : \text{Fr}(V) \to \mathbb{R}$, satisfying the invariance conditions of type $\xi(e \cdot A) = \delta(A)\xi(e)$ for some suitable chosen group homomorphism $\delta : GL_r \to \mathbb{R}^*$; in this equation we use the canonical action of elements $A \in GL_r$ on frames $e \in \text{Fr}(V)$; using $\delta = \det$ we obtain $\Lambda^{\text{top}} V$, using $\delta = |\det|$ we obtain $\mathcal{D}_V$ and using $\delta = \text{sign} \circ \det$ we obtain $\mathfrak{o}_V$).

It is clear that for any $\omega \in \Lambda^{\text{top}} V^*$ its absolute value $|\omega|$ is a density, and all densities are of these type; hence, in some sense, $\mathcal{D}_V$ is the absolute value of $\Lambda^{\text{top}} V^*$. This can be made more precise by introducing the **absolute value of a 1-dimensional vector space** W as

$$|W| = \{T : W^* \to \mathbb{R} \mid T(\lambda\xi) = |\lambda| T(\xi) \quad \forall\, \lambda \in \mathbb{R},\ \xi \in W^*\},$$

so that we have the following, which is often the notation used in the literature for the space of densities:

$$\mathcal{D}_V = |\Lambda^{\text{top}} V^*|.$$

For 1-dimensional vector spaces W_1 and W_2, one has a canonical isomorphism $|W_1| \otimes |W_2| \cong |W_1 \otimes W_2|$ (in particular $|W^*| \cong |W|^*$). From the properties of $\Lambda^{\text{top}} V^*$ (or by similar arguments), one obtains canonical isomorphisms:

1. $\mathcal{D}_V^* \cong \mathcal{D}_{V^*}$ for any vector space V.
2. For any short exact sequence of vector spaces

   $$0 \to V \to U \to W \to 0,$$

   one has an induced isomorphism

   $$\mathcal{D}_U \cong \mathcal{D}_V \otimes \mathcal{D}_W.$$

   In particular, for any vector spaces V and W, $\mathcal{D}_{V \oplus W} \cong \mathcal{D}_V \otimes \mathcal{D}_W$.

One special feature of $\mathcal{D}_V$ is that we can talk about **positive** or **strictly positive** densities. Recall also that, for any $r \in \mathbb{R}$, one has the space of r**-densities** on V:

$$\mathcal{D}_V^r := \{\rho : \Lambda^{\text{top}} V \to \mathbb{R} \mid \rho(\lambda v) = |\lambda|^r \rho(v) \quad \forall\, \lambda \in \mathbb{R},\ v \in V\}.$$

These spaces interact with each other by canonical isomorphisms

$$\mathcal{D}_V^r \otimes \mathcal{D}_V^s \cong \mathcal{D}_V^{r+s}.$$

Since the previous discussion is canonical (free of choices), it can applied (fibrewise) to vector bundles over a manifold M so that, for any such vector bundle E, one can talk about the associated line bundles over M

$$\mathcal{D}_E,\ \Lambda^{\text{top}} E^*,\ \mathfrak{o}_E$$

and the previous isomorphisms continue to hold at this level. However, at this stage, only $\mathcal{D}_E$ is trivializable, and even that is in a non-canonical way.

Definition 4.10. A **density on a manifold** M is any section of the density bundle $\mathcal{D}_{TM}$. We denote by $\mathcal{D}(M)$ the space of densities on M.

The main point about densities is that they can be integrated in a canonical fashion: one has an integration map

$$\int_M : C_c^\infty(M, \mathcal{D}_{TM}) \to \mathbb{R} \tag{4.4}$$

satisfying the usual linearity conditions; locally, for densities ρ supported on a coordinate chart $(x_1, \ldots, x_n)$, writing $\rho = f|dx_1 \ldots dx_n|$, the integral of ρ is the usual integral of f. Of course, the integral is well-defined because of the change of variables formula

$$\int_{h(U)} f = \int_U f \circ h|Jac(h)|$$

and the fact that the coefficient f of ρ changes precisely to $f \circ h|Jac(h)|$. In what follows, the integration (4.4) of compactly supported densities will be called **the canonical integration on** M.

Remark 41 (why densities?). It is customary to avoid working with densities by making an extra choice that allows one to rewrite the density bundle. For instance, the choice of a Riemannian metric induces a volume density which trivializes the density bundle and then one can integrate functions. Or the choice of an orientation trivializes the orientation bundle, hence it identifies the density bundle with the volume bundle and one can integrate top-differential forms. In general, there are no problems in making such choices (one needs to require at most orientability). However, for our purposes, it is useful to proceed in a fully canonical way; indeed, when looking at transverse measures, there will be many line bundles (at least six) that may be trivialized under mild conditions; however, such trivializations would obscure large parts of the discussion (and would make some proofs extremely tricky).

Definition 4.11. For a positive density $\rho \in \mathcal{D}(M)$, we define the associated Radon measure by

$$\mu_\rho : C_c^\infty(M) \to \mathbb{R}, \quad \mu_\rho(f) = \int_M f \cdot \rho.$$

Remark 42. According to the notation introduced above, we will also use the notation

$$\int_M f(x) \; d\mu_\rho(x)$$

where $d\mu_\rho$ may be viewed as just a notational symbol. The advantage of this notation is explained in Remark 39.

4.1.3 Some basic examples/constructions

Example 4.12 (Haar measures and densities)**.** Let G be a Lie group. Recall that a **right Haar measure** on G is any non-zero measure on G that is invariant under right translations (and similarly for left Haar measures). This notion makes sense for general locally compact Hausdorff topological groups, and a highly non-trivial theorem says that such measures exist and are unique up to multiplication by scalars. However, for Lie group the situation is much easier; for instance, for the existence, one can search among the geometric measures, i.e., the ones induced by densities ρ on G. The invariance condition means that ρ is obtained from its value at 1, $\rho_1 \in \mathcal{D}_{\mathfrak{g}}$ (where $\mathfrak{g}$ is the Lie algebra of G), by right translation. Hence Haar densities on G are in 1-1 correspondence with non-zero-elements of the 1-dimensional vector space of densities on $\mathfrak{g}$.

Recall also that, while the right and left Haar measures are in general different, for compact groups, they coincide. Actually, in the compact case, it follows that there exists a unique right (and left) Haar measure on G for which G has volume 1. This is called **the Haar measure** of the compact group G. Of course, to obtain it, one just starts with any non-zero density ρ_1 on $\mathfrak{g}$ and one rescales it by the resulting volume of G. Note that, due to this normalization condition, when dealing with a non-connected compact Lie group G, the Haar density of G differs from the one of the identity component G^0 by a factor which is the number of connected components of G. We refer to [37, Sec. 3.13] for a textbook account on Haar measures and densities on Lie groups.

With the basic understanding that when passing from groups to groupoids the source-fibres of the groupoid are used for making sense of right translations, there is a rather obvious generalization of the notion of Haar measures/densities to the world of groupoids - called Haar systems/densities. Although this is well known, some of the basic facts are hard to find or have been overlooked. They are collected in the (self-contained) Appendix to this chapter.

Example 4.13 (pushforward measures/densities)**.** If $\pi : P \to B$ is a proper map (the preimages by π of compacts are compact), then measures I on P can be pushed-down to measures $\pi_!(I)$ on B: the properness condition implies that composition with π takes $C_c^\infty(B)$ to $C_c^\infty(P)$ and then one can define:

$$\pi_!(I)(f) = I(f \circ \pi),$$

or, in the integral notation (cf. Remark 39),

$$\int_B f(b)\ d\mu_{\pi_!(I)}(b) := \int_P f(\pi(p))\ d\mu_I(p).$$

If π is a submersion, then the construction $I \mapsto \pi_!(I)$ takes geometric measures on P to geometric measures on B. The key remark is that integration over the fibres makes sense (canonically!) for densities, to give a map

$$\pi_! = \int_{\text{fibres}} : \mathcal{D}(P) \to \mathcal{D}(B).$$

Here is the definition of $\pi_!(\rho)(b)$ for a density ρ on P and for $b \in B$: restrict ρ to the fibre $P_b = \pi^{-1}(b)$; one finds for each $p \in P_b$:

$$\rho(p) \in \mathcal{D}_{T_pP} \cong \mathcal{D}_{T_pP_b} \otimes \mathcal{D}_{T_bB},$$

where the last isomorphism is the one induced by the short exact sequence

$$0 \to T_pP_b \to T_pP \to T_bB \to 0.$$

Hence we can interpret

$$\rho|_{P_b} \in \mathcal{D}(P_b) \otimes \mathcal{D}_{T_bB}; \tag{4.5}$$

integrating over P_b (compact because π is proper) we find the desired element

$$\pi_!(\rho)(b) := \int_{P_b} \rho \in \mathcal{D}_{T_bB}.$$

The fact that this operation is compatible with the one on measures, i.e., that $\pi_!(I_\rho) = I_{\pi_!(\rho)}$, follows from the Fubini formula for densities:

$$\int_P \rho = \int_B \pi_!(\rho) \quad \left(= \int_B \int_{\text{fibres}} \rho\right)$$

(which follows in turn from the standard Fubini formula on Euclidean spaces and the fact that π is locally a projection).

Example 4.14. (invariant densities/measures) The previous discussion can be continued further in the case of a (right) principal G-bundle

$$\pi : P \to B$$

for a compact Lie group G (cf. [37, Sec. 3.13]). One of the main outcomes will be that the construction $I \mapsto \pi_!(I)$ induces a bijection

$$\{\text{invariant measures/densities on } P\} \xleftrightarrow{\sim} \{\text{measures/densities on } B\}.$$

Let us concentrate here on the case of densities. What is special in this case is that each fibre P_b ($b \in B$) carries a canonical "Haar density" ρ^b_{Haar}. One way to see this is by using the identifications of the fibres P_b with G: choosing $p \in P_b$, $m_p : G \to P_b$, $g \mapsto pg$ is a diffeomorphism through which we can transport ρ^G_{Haar} to a density on P_b; the bi-invariance of ρ^G_{Haar} implies that the result does not depend on the choice of $p \in P_b$. Using ρ^b_{Haar}, arbitrary densities on P can be decomposed, via (4.5) , as

$$\rho(p) = \rho^b_{\text{Haar}} \otimes \rho_B(p) \ \text{ for some } \ \rho_B(p) \in \mathcal{D}_{T_bB}.$$

A simple check shows that ρ is invariant if and only if $\rho_B(p)$ does not depend on p but only on $b = \pi(p)$. Of course, in this case $\rho_B = \pi_!(\rho)$ (because of the normalization of the Haar density of G). Therefore

$$\pi_! : \mathcal{D}(P)^G \cong \mathcal{D}(B).$$

is an isomorphism. For $\rho_B \in \mathcal{D}(B)$, we denote by

$$\rho_{\text{Haar}} \otimes \rho_B \in \mathcal{D}(P)^G$$

the corresponding invariant density on P.

4.2 Transverse measures

In this section we introduce the notion of transverse measures for groupoids as a rather straightforward generalization of Haefliger's interpretation of transverse measures for foliations (and of étale groupoids).

4.2.1 Making sense of "orbit spaces"; transverse objects

Intuitively, given a groupoid $\mathcal{G}$ over M, transverse structures on $\mathcal{G}$ are structures that are intrinsic to the geometry of $\mathcal{G}$ that lives in the direction transverse to the orbits of $\mathcal{G}$; more suggestively (but less precisely), one may think that they are structures associated to the orbit space

$$M/\mathcal{G} := M/(x \sim y \text{ iff } \exists\, g \in \mathcal{G} \text{ from } x \text{ to } y).$$

Of course, the actual quotient *topological space* $M/\mathcal{G}$ may be very pathological and uninteresting; for that reason, when one refers to the "orbit *space*" one often has in mind much more than just the topological space itself. Making sense of "singular spaces" (like $M/\mathcal{G}$) in a satisfactory but precise way is at the very origin of various approaches to geometry. We recall here a few.

Leaf spaces as étale groupoids: This is Haefliger's approach to the study of the transverse geometry of foliated spaces $(M, \mathcal{F})$, giving a satisfactory meaning to "the spaces of leaves $M/\mathcal{F}$" [44, 46]. The first point is that any foliated manifold $(M, \mathcal{F})$ has an associated holonomy groupoid $\mathrm{Hol}(M, \mathcal{F})$; it is a groupoid over M whose arrows are determined by germs of holonomy transformations; in particular, two points in M are connected by an arrow if and only if they belong to the same leaf. In this way the leaf space $M/\mathcal{F}$ is realized as the orbit space of a groupoid, and one may think that $\mathrm{Hol}(M, \mathcal{F})$ represents (as some kind of "desingularization") $M/\mathcal{F}$. The second point is that this representative can be simplified by restricting to a complete transversal $T \subset M$ of the foliation:

$$\mathrm{Hol}_T(M, \mathcal{F}) := \mathrm{Hol}(M, \mathcal{F})|_T$$

will have the same orbit space (at least intuitively) and has a rather special extra property: it is étale, in the sense that its source and target maps are local diffeomorphisms. These properties make it possible to study étale groupoids (and their orbit spaces) very much like as one studies usual manifolds (think of a manifold M as the étale groupoid over M with only identity arrows).

To make the entire story precise, one also has to give a precise meaning to "two groupoids give rise to (or model) the same orbit space". This is precisely what the notion of Morita equivalence of groupoids does; the basic example is the holonomy groupoid $\mathrm{Hol}(M, \mathcal{F})$ of a foliation being Morita equivalent to $\mathrm{Hol}_T(M, \mathcal{F})$; hence, staying within the world of étale groupoids, Haefliger's philosophy is: the leaf space $M/\mathcal{F}$ is represented by an étale groupoid (namely $\mathrm{Hol}_T(M, \mathcal{F})$ for some complete transversal T), which is well-defined up to Morita equivalence.

Differentiable stacks: Stacks originate in algebraic geometry, where they are used to model moduli spaces that are not well-defined otherwise (note the similarity with leaf spaces). The topological and smooth versions of the theory were studied in more detail only later (see for example [5, 48, 68]), and it was immediately noticed that the resulting notion of "differentiable stack" can be represented by Lie groupoids, two Lie groupoids representing the same stack if and only if they are Morita equivalent. Actually, a large part of existing literature views (by definition) differentiable stacks as Morita equivalence classes of Lie groupoids; the Morita class (stack) of a Lie groupoid $\mathcal{G}$ over M is then suggestively denoted by $M//\mathcal{G}$. In our opinion, this is a rather obvious extension of Haefliger's philosophy from étale to general Lie groupoids (but, sadly enough, Haefliger is often forgotten by the literature on differentiable stacks, despite the fact that relationships with the algebro-geometric point of view was made explicit already in [69]). We use the groupoid point of view on stacks; with these in mind, what we do here is again rather straightforward: just extend Haefliger's approach to transverse integration from étale groupoids to general Lie groupoids.

Non-commutative spaces: Another approach to "singular spaces" is provided by Connes' non-commutative geometry [16]; while standard spaces (e.g. compact Hausdorff spaces) are fully characterized by their (commutative) algebra of continuous scalar-valued functions, the idea is that non-commutative algebras (possibly with extra structure) should be interpreted as algebras of functions on a "non-commutative space". Therefore, when one deals with a singular space X, one does not look at its points or at its topological/smooth structure (usually ill-behaved) but one tries to model it via a non-commutative algebra. In general, this modelling step is not precisely defined and it very much depends on the specific X one looks at. However, in most examples, one uses groupoids as an intermediate step.

More precisely: one first realizes X as the orbit space of a Lie groupoid $\mathcal{G}$ (i.e., one makes sense of X as a differentiable stack) and then one appeals to a standard construction that associates to a Lie groupoid $\mathcal{G}$ over M a (usually non-commutative) algebra, namely the **convolution algebra** of the groupoid. Before we recall its definition let us mention that, intuitively, the convolution algebra of $\mathcal{G}$ should be thought of as a "non-commutative model" for the algebra of (compactly supported) functions on the singular orbit space $M/\mathcal{G}$ (or better: of the stack $M//\mathcal{G}$). For that reason, we will denote the convolution algebra by $\mathcal{NC}_c^\infty(M//\mathcal{G})$. As a vector space, it is simply $C_c^\infty(\mathcal{G})$ - the space of compactly supported smooth functions on $\mathcal{G}$. The convolution product is, in principle, given by a convolution formula

$$\text{“}(u_1 \star u_2)(g) = \int_{g_1 g_2 = g} u_1(g_1) u_2(g_2)\text{”}. \tag{4.6}$$

The simplest case when this makes sense is when $\mathcal{G}$ is étale, case in which

$$(u_1 \star u_2)(g) = \sum_{g_1 g_2 = g} u_1(g_1) u_2(g_2).$$

For general Lie groupoids, it is customary to choose a full Haar system (see Definition 4.63) or, even better, to take advantage of the smooth structure and start with a full Haar density $\rho \in C^\infty(M, \mathcal{D}_A)$ (Haar densities are recalled in the appendix - see Definition 4.64). Using the induced integration (of functions) along the s-fibres, one can now make precise sense of (4.6) as:

$$(u_1 \star_\rho u_2)(g) = \int_{s^{-1}(x)} u_1(gh^{-1}) u_2(h)\, d\mu_{\vec{\rho}}(h) \quad \text{where } x = s(g). \tag{4.7}$$

It is not difficult to see that the choice of the full Haar density does not affect the isomorphism class of the resulting convolution algebra. However, at the price of becoming a bit more abstract, but keeping (4.6) as intuition, one can proceed intrinsically as follows (cf. e.g. [16]): consider the bundle $\mathcal{D}_A^{1/2}$ of half densities, pull it back to $\mathcal{G}$ via s and t and define

$$\mathcal{NC}_c^\infty(M//\mathcal{G}) := C_c^\infty(\mathcal{G}, t^*\mathcal{D}_A^{1/2} \otimes s^*\mathcal{D}_A^{1/2}). \tag{4.8}$$

Given u_1 and u_2 in this space, we look at the expression (4.6). For

$$x \overset{g_1}{\leftarrow} z \overset{g_2}{\leftarrow} y$$

such that $g = g_1 g_2$, $u(g_1)u(g_2)$ makes sense as a tensor product; since

$$\mathcal{D}^{1/2}_{A,z} \otimes \mathcal{D}^{1/2}_{A,z} = \mathcal{D}_{A,z} \cong \mathcal{D}_{T_{g_2}\mathcal{G}},$$

(where we use the right translations for the last identification), we see that

$$u(g_1)v(g_2) \in \mathcal{D}^{1/2}_{A,x} \otimes \mathcal{D}_{T_{g_2}\mathcal{G}} \otimes \mathcal{D}^{1/2}_{A,y}; \tag{4.9}$$

hence, when $g = g_1 g_2$ is fixed, we deal with a density in the $g_2 \in s^{-1}(y)$ argument; integrated, this gives rise to

$$(u_1 \star u_2)(g) \in \mathcal{D}^{1/2}_{A,x} \otimes \mathcal{D}^{1/2}_{A,y},$$

hence to a new element $u_1 \star u_2$ in our space (4.8).

4.2.2 Transverse integration à la Haefliger

The notion of transverse measure and transverse integration is best understood in the case of foliated manifolds $(M, \mathcal{F})$; the idea is that such a measure should measure the size of transversals to the foliation in a way that is invariant under holonomy transformations (so that, morally, they measure not the transversals but the subspaces that they induce in the leaf space). This idea can be implemented either by working directly with set-measures, as done e.g. by Plante [88], or by working with the dual picture (via the Riesz theorem) in terms of linear functionals, as done by Haefliger [45].

Here we follow Haefliger's approach, but applied to a general Lie groupoid $\mathcal{G}$ over a manifold M. The main point is to define what deserves to be called "the space of compactly supported smooth functions on the orbit space $M/\mathcal{G}$". It is instructive to first think about the meaning of "the space of smooth functions on the orbit space $M/\mathcal{G}$"; indeed, there is an obvious candidate, namely

$$C^\infty(M//\mathcal{G}) := C^\infty(M)^{\mathcal{G}-\mathrm{inv}},$$

the space of smooth functions on M that are $\mathcal{G}$-invariant (i.e., constant on the orbits of $\mathcal{G}$). In this way, intuitively, a "smooth function on $M/\mathcal{G}$" is represented by its pullback to M. One may try to add compact supports to the previous discussion (to define $C^\infty_c(M//\mathcal{G})$), but one encounters a serious problem: smooth invariant functions may fail to have compact support (this happens already for actions of groups). Moreover, this approach would be too naive; the point is that the theory with compact supports should not be seen just as a subtheory of the one without support conditions, but as a dual theory (think e.g. of DeRham cohomology and its version with compact supports, which are related by Poincare duality).

Example 4.15 (A simple example that illustrates the general philosophy)**.** Probably the best example is that of an action groupoid $\mathcal{G} = \Gamma \ltimes M$ associated to an action of a discrete group Γ on a manifold M. As for smooth functions, there is an obvious approach to "measures on the orbit space": measures on M, $\mu : C^\infty_c(M) \to \mathbb{R}$, which are Γ-invariant in the sense that

$$\mu(\gamma^*(f)) = \mu(f) \quad \forall\ f \in C^\infty_c(M),\ \gamma \in \Gamma,$$

where γ^* denotes the induced action on smooth functions ($\gamma^*(u)(x) = u(\gamma \cdot x)$). Therefore, if we want to represent such measures as linear functionals, we have to consider the quotient of $C_c^\infty(M)$ by the linear span of elements of type $f - \gamma^*(f)$. This is a construction that is defined more generally for actions of Γ on a vector space V; the resulting quotient is denoted by V_Γ and is know as the space of co-invariants (dual to the space of invariants V^Γ - cf. e.g. [107]). Therefore we arrive at the following model for the space of compactly supported smooth functions on the orbit space:

$$C_c^\infty(M//\Gamma) := C_c^\infty(M)_\Gamma, \tag{4.10}$$

so that invariant measures correspond to (positive) linear functionals on $C_c^\infty(M//\Gamma)$.

When the action is free and proper, then $B = M/\Gamma$ is itself a manifold, and the integration over the fibres of the canonical projection π,

$$\pi_! : C_c^\infty(M) \to C_c^\infty(B), \quad \pi_!(u)(b) = \sum_{x \in \pi^{-1}(b)} u(x),$$

descends to an isomorphism of our model (4.10) with $C_c^\infty(B)$. In contrast, the invariant part of $C_c^\infty(M)$ is trivial if Γ is infinite.

For general actions, $C_c^\infty(M)^\Gamma$ has a nicer description if Γ is finite; actually, in that case, $V^\Gamma \cong V_\Gamma$ for any Γ-vector space V (where the isomorphism is induced by the canonical projection and the inverse by averaging). Also, although M/Γ may fail to be a manifold, it is a locally compact Hausdorff space, and there is a satisfactory definition for the spaces of smooth functions on M/Γ:

$$C^\infty(M/\Gamma) := \{f \in C(M/\Gamma) \mid f \circ \pi \text{ is smooth}\},$$

$$C_c^\infty(M/\Gamma) := \{f \in C^\infty(M/\Gamma) \mid \text{supp}(f) - \text{compact}\}.$$

We deduce that, if Γ is finite,

$$C_c^\infty(M//\Gamma) \cong C_c^\infty(M)^\Gamma \cong C_c^\infty(M/\Gamma),$$

hence the elements of $C_c^\infty(M//\Gamma)$ can be seen as compactly supported functions on M/Γ. This is a phenomena that is at the heart of our separate section on proper groupoids.

Example 4.16 (Étale groupoids; leaf spaces)**.** The Definition (4.10) (and the heuristics behind it) extend without any problem to general étale groupoids $\mathcal{G}$ over M. First of all one defines

$$C^\infty(M//\mathcal{G}) := C^\infty(M)^{\mathcal{G}-\text{inv}},$$

the space of smooth functions on M that are $\mathcal{G}$-invariant (i.e., constant on the orbits of $\mathcal{G}$). Equivalently, but easier to dualize, we have

$$C^\infty(M//\mathcal{G}) = \text{Ker}\,(s^* - t^* : C^\infty(M) \to C^\infty(\mathcal{G})),$$

where $s^*(u) = u \circ s$ and similarly for t^*. Dual to s^* we have:

$$s_! : C_c^\infty(\mathcal{G}) \to C_c^\infty(M), \quad s_!(u)(x) = \sum_{s(g)=x} u(g)$$

and similarly $t_!$. One then defines

$$C_c^\infty(M//\mathcal{G}) := \mathrm{Coker}(s_! - t_! : C_c^\infty(\mathcal{G}) \to C_c^\infty(M)). \tag{4.11}$$

Let us also mention that, like $C^\infty(M//\mathcal{G})$, $C_c^\infty(M//\mathcal{G})$ is invariant under Morita equivalences; this is implicit in Haefliger's work and will be discussed in full generality later in the chapter.

This construction can be applied to the étale holonomy groupoids associated to a foliation $(M, \mathcal{F})$: for any complete transversal T one considers $C_c^\infty(T//\mathcal{F}) := C_c^\infty(T//\mathrm{Hol}_T(M, \mathcal{F}))$. These are precisely the spaces denoted $\Omega_c^0(T/\mathcal{F})$ by Haefliger [45] and used to handle transverse integration; our definition is basically the same as his, just that we use the groupoids explicitly. Morita invariance shows that, up to canonical isomorphisms, $C_c^\infty(T//\mathcal{F})$ does not depend on the choice of T; it serves as a model for "$C_c^\infty(M/\mathcal{F})$". Of course, transverse measures are now understood as positive linear functionals on $C_c^\infty(T//\mathcal{F})$. One point that is not clarified by Haefliger and which may serve as motivation for extending the theory to more general groupoids is whether there is an intrinsic description of "$C_c^\infty(M/\mathcal{F})$" defined directly using objects that live on M (note that the similar question for "$C^\infty(M/\mathcal{F})$" has a simple and rather obvious answer: smooth functions on M that are constant on the leaves).

4.2.3 Transverse measures: the general case

With all the previous background at hand, it is clear how to proceed, in Haefliger's style, to define transverse integration for a general Lie groupoid $\mathcal{G}$ over M. One only needs to make sense of the integration over the s and t-fibres of $\mathcal{G}$. This issue is identical to the one that one encounters for making sense of the convolution product for general Lie groupoids (the last part of Subsection 4.2.1); so, it is not surprising that one can proceed as there: either fix a strictly positive density $\rho \in C^\infty(M, \mathcal{D}_A)$ to define a version of $C_c^\infty(M//\mathcal{G})$ that depends on ρ but whose isomorphism class does not depend on ρ, or to provide a more abstract but intrinsic version of $C_c^\infty(M//\mathcal{G})$. We prefer a main definition which is choice-free.

Hence, similar to the discussion on convolution products, we use the density bundle associated to A and its pullbacks to $\mathcal{G}$ via the source and target maps. One has a canonical integration over the s-fibres map

$$s_! : C_c^\infty(\mathcal{G}, t^*\mathcal{D}_A \otimes s^*\mathcal{D}_A) \to C_c^\infty(M, \mathcal{D}_A); \tag{4.12}$$

indeed, for u in the domain of this map and $x \in M$, the restriction of u to $s^{-1}(x)$ is a section of

$$(t^*\mathcal{D}_A)|_{s^{-1}(x)} \otimes \mathcal{D}_{A,x} \cong \mathcal{D}_{T(s^{-1}(x))} \otimes \mathcal{D}_{A,x}.$$

This allows us to integrate $u|_{s^{-1}(x)}$ and obtain an element in $\mathcal{D}_{A,x}$; varying x, this gives rise to $s_!(u)$. Of course, the same reasoning makes sense of $t_!$.

Definition 4.17. Define the **intrinsic model for compactly supported smooth functions** on the orbit space as

$$C_c^\infty(M//\mathcal{G}) := C_c^\infty(M, \mathcal{D}_A)/\mathrm{Im}(s_! - t_!).$$

A **transverse measure** for $\mathcal{G}$ is a positive linear functional on $C_c^\infty(M//\mathcal{G})$, i.e., a linear map

$$\mu : C_c^\infty(M, \mathcal{D}_A) \to \mathbb{R}$$

which is positive and which satisfies the invariance condition

$$\mu \circ s_! = \mu \circ t_!. \tag{4.13}$$

Remark 43 (some intuition). The intuition is that elements $\rho \in C_c^\infty(M, \mathcal{D}_A)$ represent "compactly supported smooth functions on $M/\mathcal{G}$" as follows: while sections ρ of $\mathcal{D}_A$ can be interpreted as invariant family of densities $\{\rho^x\}$ (as in the Appendix - see (4.47)), their integrals (when defined) will give an invariant function of x, hence a function on $M/\mathcal{G}$:

$$\mathrm{Av}(\rho) : M/\mathcal{G} \to \mathbb{R},\ \mathrm{Av}(\rho)(\mathcal{O}_x) = \int_{s^{-1}(x)} \rho^x.$$

Of course, there are problems to make this precise, and that is the reason for working on M. Nevertheless, we will see that such problems will disappear in one important case: for proper Lie groupoids. That is the subject of Section 4.4.

Remark 44 (the relationship with the non-commutative approach). It is instructive to compare our more classical approach with the non-commutative one (see Subsection 4.2.1). Then one has to look at the non-commutative model $\mathcal{NC}_c^\infty(M//\mathcal{G})$ (see (4.8)) and look at linear maps τ which are **traces**, i.e., satisfy

$$\tau(u \star v) = \tau(v \star u), \quad \forall\ u, v \in \mathcal{NC}_c^\infty(M//\mathcal{G}). \tag{4.14}$$

A general well-known phenomena is that classical approaches are recovered as "localization at units" of the non-commutative ones (see e.g. [10]). Hence, one expects that our notion of transverse measure induces, by restricting to units, such traces. And, indeed, this fits perfectly with our approach (and even with our use of $\mathcal{D}_A$). First of all, for $u \in \mathcal{NC}_c^\infty(M//\mathcal{G})$, its restriction to units gives elements

$$u(1_x) \in \mathcal{D}_{A,x}^{1/2} \otimes \mathcal{D}_{A,x}^{1/2} = \mathcal{D}_{A,x},$$

so that $u|_M \in C_c^\infty(M, \mathcal{D}_A)$. Hence any linear μ as before induces

$$\tilde{\mu} : \mathcal{NC}_c^\infty(M//\mathcal{G}) \to \mathbb{R},\ u \mapsto \mu(u|_M).$$

Proposition 4.18. *μ satisfies the invariance condition (4.13) if and only if $\tilde{\mu}$ satisfies the tracial condition (4.14).*

Proof. Applying the definition of $u \star v$ at points 1_x and comparing the resulting formula with the one defining $s_!$ we find that

$$(u \star v)|_M = s_!(\phi),$$

where $\phi(g) = u(g^{-1})v(g)$ defines an element in $C_c^\infty(\mathcal{G}, t^*\mathcal{D}_A \otimes s^*\mathcal{D}_A)$. Similarly, or as a consequence, we have $(v \star u)|_M = t_!(\phi)$. This explains the statement (strictly speaking, one has to check that any compactly supported ϕ can be written as $\phi(g) = u(g^{-1})v(g)$ for some compactly supported u and v. Trivializing the density bundles this becomes a question about functions, which is clear: just take $v = \phi$ and $u \in C_c^\infty(\mathcal{G})$ any function which is 1 on the inverse of the support of ϕ.) $\square$

Remark 45 (the more down to earth approach). If we choose a strictly positive density $\rho \in C^{\infty}(M, \mathcal{D}_A)$ (see the Appendix), then one can use ρ to trivialize all the density bundles. Then $s_!$ becomes identified with

$$s_!^{\rho} : C_c^{\infty}(\mathcal{G}) \to C_c^{\infty}(M), \quad s_!^{\rho}(u)(x) = \int_{s^{-1}(x)} u(g) \overrightarrow{\rho}\,(g)$$

(for $\overrightarrow{\rho}$, see (4.47) in the Appendix); similarly for $t_!$, hence $C_c^{\infty}(M//\mathcal{G})$ is identified with its more concrete (but ρ-dependent) model

$$C_c^{\infty}(M//\mathcal{G}, \rho) := \mathrm{Coker}(s_!^{\rho} - t_!^{\rho} : C_c^{\infty}(\mathcal{G}) \to C_c^{\infty}(M)).$$

It is instructive to prove directly that a different density ρ' induces an isomorphic space. That follows from the following commutative diagram

$$\begin{tikzcd} C_c^{\infty}(\mathcal{G}) \arrow[r, "s_!^{\rho} - t_!^{\rho}"] \arrow[d, "s^*(\frac{\rho}{\rho'})t^*(\frac{\rho}{\rho'})"'] & C_c^{\infty}(M) \arrow[d, "\frac{\rho}{\rho'}"] \\ C_c^{\infty}(\mathcal{G}) \arrow[r, "s_!^{\rho'} - t_!^{\rho'}"'] & C_c^{\infty}(M) \end{tikzcd}$$

where the vertical maps (which are given by multiplication by the indicated functions) are isomorphisms. Hence the desired isomorphism is the one induced by the multiplication by $\frac{\rho}{\rho'}$:

$$\frac{\rho}{\rho'} : C_c^{\infty}(M//\mathcal{G}, \rho) \to C_c^{\infty}(M//\mathcal{G}, \rho').$$

We see that, if we want to avoid using the density bundle in order to define transverse measures, then we have to work with pairs (ρ, τ) where ρ is as above and τ is a positive linear functional on $C_c^{\infty}(M//\mathcal{G}, \rho)$ and we have to identify two such pairs if $\tau' \circ \frac{\rho}{\rho'} = \tau$. A suggestive notation for the class of (ρ, τ) would be τ/ρ. Of course, the use of $\mathcal{D}_A$ allows us to make sense of this quotient as a transverse measure: the one defined as the composition

$$C_c^{\infty}(M, \mathcal{D}_A) \xrightarrow{\frac{1}{\rho}} C_c^{\infty}(M) \xrightarrow{\tau} \mathbb{R}.$$

Let us state right away one of the most basic properties of our definition: Morita invariance. This will follow from a more general result - Theorem 4.47 of Section 4.7.

Theorem 4.19. *Any Morita equivalence between two groupoids $\mathcal{G}$ (over M) and $\mathcal{H}$ (over N) gives rise to a (functorial) isomorphism between $C_c^{\infty}(M//\mathcal{G})$ and $C_c^{\infty}(N//\mathcal{H})$ and induces a 1-1 correspondence between the transverse measures on $\mathcal{G}$ and those of $\mathcal{H}$.*

Example 4.20 (the case of submersions)**.** When looking at "transverse notions", i.e., notions that morally live on the orbit space, the "test case" is provided by the groupoids whose orbit spaces B are already smooth or, more precisely, by the groupoids that are (Morita) equivalent to smooth manifolds B. Such groupoids are associated to smooth submersions $\pi : P \to B$. Explicitly, any such submersion π gives rise to a Lie groupoid $\mathcal{G}(\pi)$:

$$\mathcal{G}(\pi) = P \times_B P = \{(p, q) \in P \times P \mid \pi(p) = \pi(q)\},$$

with the source and target being the second, respectively the first, projection, and the multiplication $(p,q)\cdot(q,r)=(p,r)$. The Lie algebroid of $\mathcal{G}(\pi)$ is the sub-bundle of TP consisting of vectors that are tangent to the fibres of π,

$$\mathcal{F}(\pi) := \mathrm{Ker}\,(d\pi) \subset TP.$$

In this case the intuition mentioned in Remark 43 works without problems and tells us that we should look at the fibre integration map

$$\int_{\pi-\text{fibres}} : C_c^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)}) \to C_c^\infty(B), \tag{4.15}$$

Even more, it motivates the use of the density bundle starting with the question: how can one represent compactly supported smooth functions on B by functions on P, in a canonical way? Of course, the main problem is then to understand the kernel of this map. The definition of $C_c^\infty(M//\mathcal{G})$ suggests the answer: look at

$$s_! - t_! : C_c^\infty(P\times_B P, t^*\mathcal{D}_{\mathcal{F}(\pi)} \otimes s^*\mathcal{D}_{\mathcal{F}(\pi)}) \to C_c^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)}).$$

Lemma 4.21. *For the groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : P \to B$, the integration over the fibre (4.15) is surjective and its kernel is the image of $s_! - t_!$; therefore it induces an isomorphism*

$$C_c^\infty(P//\mathcal{G}(\pi)) \cong C_c^\infty(B).$$

Proof. Let $C_{\pi-c}^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)})$ be the space of sections of $\mathcal{D}_{\mathcal{F}(\pi)}$ which have fibrewise compact supports. We claim that the integration over the fibre, now viewed as a map

$$\int_\pi := \int_{\pi-\text{fibres}} : C_{\pi-c}^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)}) \to C^\infty(B),$$

is surjective. Since $\int_\pi \pi^*(f)u = f\int_\pi u$, it suffices to show that there exists:

$$c \in C_{\pi-c}^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)}) \ \text{ such that } \ \int_\pi c = 1.$$

To see this, choose an open cover $\{V_i\}_{i\in I}$ of B such that, for each i, there exists an open $U_i \subset P$ with the property that $\pi|_{U_i}$ is diffeomorphic to the projection $V_i \times \mathbb{R}^q \to V_i$. On each U_i choose

$$c_i \in C_{\pi|_{U_i}-c}^\infty(U_i, \mathcal{D}_{\mathcal{F}(\pi)})$$

such that $\int_{\pi|_{U_i}} c_i = 1$. Choose also a partition of unity $\{\eta_i\}_{i\in I}$ with η_i supported in V_i. Set now $c = \sum_i \eta_i c_i$.

The surjectivity from the statement now follows immediately using c: for $f \in C_c^\infty(B)$, $\pi^*(f)\cdot c$ has compact support and $\int_\pi \pi^*(f)\cdot c = f$. Let now $u \in C_c^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)})$ such that $\int_{\pi-\text{fibres}} u = 0$. Construct now

$$v \in C_c^\infty(P\times_B P, t^*\mathcal{D}_{\mathcal{F}(\pi)} \otimes s^*\mathcal{D}_{\mathcal{F}(\pi)}),\ v(p,q) = \phi(\pi(q))u(p)\otimes c(q)$$

where $\phi \in C_c^\infty(B)$ is chosen to be 1 on $\pi(\mathrm{supp}(u))$. The role of ϕ is to make v have compact support. Compute now:

$$s_!(v)(p) = \phi(\pi(b))c(p)\int_\pi u = 0,\ t_!(v)(q) = u(p)\phi(\pi(p))\int_\pi c = u(p);$$

therefore u is in the image of $s_! - t_!$. $\square$

4.3 *Geometric* transverse measures (transverse densities)

4.3.1 The transverse volume and density bundles

We are now interested on geometric transverse measures, i.e., the analogues of the (Radon) measures induced by densities on manifolds. Looking first at linear functionals on $C_c^\infty(M, \mathcal{D}_A)$ that may arise from sections of a vector bundle combined with the canonical integration on M (like for standard densities) one immediately is led to the transverse density bundle:

Definition 4.22. For a Lie algebroid A over M we define the **transverse density bundle** of A as

$$\mathcal{D}_A^{\mathrm{tr}} := \mathcal{D}_{A^*} \otimes \mathcal{D}_{TM}.$$

Given a section σ of $\mathcal{D}_A^{\mathrm{tr}}$, **a decomposition of** σ is any writing of σ of type $\sigma = \rho^\vee \otimes \tau$ with $\rho \in C^\infty(M, \mathcal{D}_A)$ strictly positive and τ a density on M (where $\rho^\vee$ is the dual section induced by ρ).

Similarly one can define the **transverse volume and orientation bundles** of A

$$\mathcal{V}_A^{\mathrm{tr}} := \mathcal{V}_{A^*} \otimes \mathcal{V}_{TM} = \Lambda^{\mathrm{top}} A \otimes \Lambda^{\mathrm{top}} T^*M, \quad \mathfrak{o}_A^{\mathrm{tr}} := \mathfrak{o}_{A^*} \otimes \mathfrak{o}_{TM}$$

and the usual relations between these bundles continue to hold in this setting; e.g.:

$$\mathcal{D}_A^{\mathrm{tr}} = |\Lambda^{\mathrm{top}} A \otimes \Lambda^{\mathrm{top}} T^*M| = |\mathcal{V}_A^{\mathrm{tr}}|.$$

One of the main properties of these bundles is that they are representations of A and, even better, of $\mathcal{G}$, whenever $\mathcal{G}$ is a Lie groupoid with algebroid A; hence they do deserve the name of "transverse" vector bundles. We now describe the canonical action of $\mathcal{G}$ on the transverse density bundle $\mathcal{D}_A^{\mathrm{tr}}$ (for the other two the description is identical). We have to associate to any arrow $g : x \to y$ of $\mathcal{G}$ a linear transformation

$$g_* : \mathcal{D}_{A,x}^{\mathrm{tr}} \to \mathcal{D}_{A,y}^{\mathrm{tr}}.$$

The starting point is the short exact sequence induced by the differentials of the source map and right-translations (4.45)

$$0 \to A_y \to T_g\mathcal{G} \xrightarrow{ds} T_xM \to 0,$$

which induces an isomorphism

$$\mathcal{D}(T_g\mathcal{G}) \cong \mathcal{D}(A_y) \otimes \mathcal{D}(T_xM). \tag{4.16}$$

Using the similar isomorphism at g^{-1} and the fact that the differential of the inversion map gives an isomorphism $T_g\mathcal{G} \cong T_{g^{-1}}\mathcal{G}$, we find an isomorphism

$$\mathcal{D}(A_y) \otimes \mathcal{D}(T_xM) \cong \mathcal{D}(A_x) \otimes \mathcal{D}(T_yM).$$

and therefore an isomorphism:

$$\mathcal{D}(A_x^*) \otimes \mathcal{D}(T_xM) \cong \mathcal{D}(A_y^*) \otimes \mathcal{D}(T_yM),$$

and this defines the action g_* we were looking for (it is straightforward to check that this defines indeed an action).

The infinitesimal action of A on these vector bundles, obtained by differentiation of the action of $\mathcal{G}$, only depends on the Lie algebroid; at least for $\mathcal{V}_A^{\mathrm{tr}}$, this action is described in [38].

Definition 4.23. A **transverse density** for the Lie groupoid $\mathcal{G}$ is any $\mathcal{G}$-invariant section of the transverse density bundle $\mathcal{D}_A^{\mathrm{tr}}$. When we want to stress the structure of $\mathcal{G}$-representation present on $\mathcal{D}_A^{\mathrm{tr}}$, we will denote it by $\mathcal{D}_{\mathcal{G}}^{\mathrm{tr}}$.

4.3.2 Transverse densities as transverse measures

Returning to the relevance of the transverse density bundle to transverse measures, note that the canonical pairing between $\mathcal{D}_A$ and $\mathcal{D}_{A^*}$ and the integration of densities on M allows us to interpret any section σ of $\mathcal{D}_A^{\mathrm{tr}}$ as a linear map

$$\mu_\sigma : C_c^\infty(M, \mathcal{D}_A) \to \mathbb{R}.$$

Explicitly, if $\sigma = \rho^\vee \otimes \tau$ is a decomposition of σ,

$$\mu_\sigma(f \cdot \rho) = \int_M f \cdot \tau = \int_M f(x) d\mu_\tau(x) \quad \forall\ f \cdot \rho \in C_c^\infty(M, \mathcal{D}_A).$$

Of course, the question is: when does μ_σ define a transverse measure (i.e., descends to $C_c^\infty(M//\mathcal{G})$)? We have:

Proposition 4.24. *For a section $\sigma \in C^\infty(M, \mathcal{D}_{\mathcal{G}}^{\mathrm{tr}})$, the following are equivalent:*

1. $\mu_\sigma \circ s_! = \mu_\sigma \circ t_!$ *(i.e., σ gives rise to a transverse measure, if it is positive).*
2. *σ is an invariant section of $\mathcal{D}_{\mathcal{G}}^{\mathrm{tr}}$ (i.e., it is a transverse density).*

Therefore, (positive) transverse densities appear as transverse measures of geometric type. The proof of the proposition will be given together with the proof of Proposition 4.25.

4.3.3 The compatibility with other view-points

Here we give two more equivalent characterizations of the invariance of a section $\sigma \in C^\infty(M, \mathcal{D}_A^{\mathrm{tr}})$ which show the compatibility of this condition with the points of view of measure groupoids [47] and of non-commutative geometry [16]. However, unlike the previous proposition, the next one depends on a decomposition $\sigma = \rho^\vee \otimes \tau$ (see Definition 4.22) that we now fix. In this situation, we have:

1. an induced measure $\mu_{\mathcal{G}}$ (to be defined) on the manifold $\mathcal{G}$ of arrows and, from the point of view of measure groupoids, the main condition to require is that $\mu_{\mathcal{G}}$ is invariant under the inversion map. Of course, since we are in the geometric setting (densities rather than general measures), $\mu_{\mathcal{G}} = \mu_{\rho_{\mathcal{G}}}$ will be associated to a density $\rho_{\mathcal{G}}$ on $\mathcal{G}$. Finally, $\rho_{\mathcal{G}}$ is defined as the density whose value at an arrow $g : x \to y$ of $\mathcal{G}$ equals to $\rho_y \otimes \tau_x$ modulo the isomorphism (4.16).

2. the concrete realization $(C_c^\infty(\mathcal{G}), \star_\rho)$ (using ρ) of the convolution algebra and, using τ, the associated integration over the units

$$\tilde{\mu}_\tau : C_c^\infty(\mathcal{G}) \to \mathbb{R},\ u \mapsto \mu_\tau(u|_M) = \int_M u(1_x)\tau(x).$$

   The interesting condition is, again, the trace condition.

Proposition 4.25. *Given $\sigma = \rho^\vee \otimes \tau$ as above, the following are equivalent:*

2. *σ is invariant (i.e., it is a transverse density).*

3. *the measure $\mu_{\rho_\mathcal{G}}$ on $\mathcal{G}$ is invariant under the inversion map.*

4. *$\tilde{\mu}_\tau$ is a trace on the algebra $(C_c^\infty(\mathcal{G}), \star_\rho)$.*

Proof. (of Propositions 4.24 and 4.25) We show that 1-4 from the last two propositions are all equivalent. In order to make clear what does it mean for the section σ to be invariant, let us write in a more explicit fashion the definition of the representation of $\mathcal{G}$ on $\mathcal{D}^{\mathrm{tr}}_{\mathcal{G}}$, given in Section 4.3.1. If $g : x \to y$ is an arrow of $\mathcal{G}$, we use the canonical identifications

$$\mathcal{D}(A_x^*) \otimes \mathcal{D}(A_y^*) \otimes \mathcal{D}(A_y) \otimes \mathcal{D}(T_xM) \stackrel{c_{23}}{\to} \mathcal{D}^{\mathrm{tr}}_{A,x}$$
$$\mathcal{D}(A_x^*) \otimes \mathcal{D}(A_y^*) \otimes \mathcal{D}(A_x) \otimes \mathcal{D}(T_yM) \stackrel{c_{13}}{\to} \mathcal{D}^{\mathrm{tr}}_{A,y},$$

where c_{ij} denotes the map given by contraction of the i-th and j-th factors of the tensor product in the left-hand side.

Since ρ is strictly positive, under these identifications $\sigma_x \in \mathcal{D}^{\mathrm{tr}}_{A,x}$ corresponds to $\rho_x^\vee \otimes \rho_y^\vee \otimes \rho_y \otimes \tau_x$ and σ_y corresponds to $\rho_x^\vee \otimes \rho_y^\vee \otimes \rho_x \otimes \tau_y$. The action $g_* : \mathcal{D}^{\mathrm{tr}}_{A,x} \to \mathcal{D}^{\mathrm{tr}}_{A,y}$, when written in terms of these identifications, is given by the tensor product of the identity map of $\mathcal{D}(A_x^*) \otimes \mathcal{D}(A_y^*)$ with the map di_g induced by the differential of the inversion of $\mathcal{G}$ at g. Therefore, σ is an invariant section if and only if for all $g \in \mathcal{G}$ with $g : x \to y$, it holds that

$$di_g(\rho_y \otimes \tau_x) = \rho_x \otimes \tau_y.$$

This is the same as the condition that $i^*(\rho_\mathcal{G}) = \rho_\mathcal{G}$ which, of course, is equivalent to the fact that the associated measure $\mu_{\rho_\mathcal{G}}$ is invariant under the inversion of $\mathcal{G}$. This proves the equivalence of 2 and 3.

Of course, the equivalence between 1 and 4 is just a rewriting of Proposition 4.18 after we identify the intrinsic convolution algebra with $(C_c^\infty(\mathcal{G}), \star_\rho)$. The proof will be complete if we establish the equivalence of 3 and 4. For $u \in C_c^\infty(M)$ write $u^*(g) = u(g^{-1})$, so that condition 3 becomes

$$\mu_{\rho_\mathcal{G}}(f) = \mu_{\rho_\mathcal{G}}(f^*) \tag{4.17}$$

for all $f \in C_c^\infty(\mathcal{G})$. Using the definition of $\rho_\mathcal{G}$ we obtain the formula for the associated integration

$$\mu_{\rho_\mathcal{G}}(f) = \int_\mathcal{G} f(g)\rho_\mathcal{G}(g) = \int_\mathcal{G} f(g)\rho_{t(g)} \otimes \tau_{s(g)} = \int_M \left(\int_{s^{-1}(x)} f(g)\, d\mu_{\overrightarrow{\rho}}(g) \right) \tau_x.$$

As in the proof of Proposition 4.18, it suffices to require (4.17) on elements of type $f = u^*v$ (hence $f(g) = u(g^{-1})v(g)$). But, on such elements, the previous formula combined with the definition (4.7) of $\star_\rho$ gives $\int_M (u \star_\rho v)(1_x)\tau_x = \tilde{\mu}_\tau(u \star_\rho v)$; then, on $f^* = v^*u$ we obtain $\tilde{\mu}_\tau(v \star_\rho u)$. Hence 3 is equivalent to $\tilde{\mu}_\tau$ being a trace. □

Remark 46. The non-commutative view-point tells us that one should look not only at traces, but also at higher versions of them; this is related to the Hochschild and cyclic cohomology of the convolution algebra [16] (see also [10]) from which traces emerge in degree zero. On the other hand also the invariant sections of $\mathcal{D}_A^{\mathrm{tr}}$ show up as degree zero elements in a cohomology: the differentiable cohomology of $\mathcal{G}$ with coefficients in $\mathcal{D}_A^{\mathrm{tr}}$. It is natural to expect that, via restriction at units, the differentiable cohomology in higher degrees will give rise to higher (non-commutative) traces; such higher traces will be "geometric" in the same way that measures induced by densities are. Our comments are very much in line with the work of Pflaum-Posthuma-Tang [87].

4.3.4 The regular case

We now have a closer look at the regular case, i.e., when all the orbits of $\mathcal{G}$ have the same dimension. In this case the connected components of the orbits form a regular foliation on the base manifold M; we denote it by $\mathcal{F}$ (hence, as an involutive sub-bundle of TM, it is the image of the anchor of the algebroid A of $\mathcal{G}$). This case comes with some special features:

1. The isotropy and the normal representations: The first special feature is that there is a canonical decomposition

$$\mathcal{D}_A^{\mathrm{tr}} \cong \mathcal{D}_{\mathfrak{g}^*} \otimes \mathcal{D}_\nu,$$

where $\mathfrak{g}$ and ν are the isotropy and the normal representations of $\mathcal{G}$, respectively (cf. e.g. [38]); as vector bundles,

$$\mathfrak{g} = \mathrm{Ker}\,(\rho), \quad \nu = TM/\mathcal{F}.$$

It is well-known that these are representations of $\mathcal{G}$. For the action of an arrow $g : x \to y$ on $\mathfrak{g}$ we use the adjoint action $\mathrm{Ad}_g(a) = gag^{-1}$ going from the isotropy group of $\mathcal{G}$ at x to the one at y; by differentiation, we obtain the action going from $\mathfrak{g}_x$ to $\mathfrak{g}_y$. For the action on ν, start with $v \in \nu_x$ and choose a curve $g(t) : x(t) \to y(t)$ with $g(0) = g$ and such that $\dot{x}(0) \in T_xM$ represents v; then $g \cdot v \in \nu_y$ is the vector represented by $\dot{y}(0) \in T_yM$.

Because of regularity we have two exact sequences of vector bundles over M

$$0 \to \mathfrak{g} \to A \xrightarrow{\rho} \mathcal{F} \to 0, \quad 0 \to \mathcal{F} \to TM \to \nu \to 0,$$

hence canonical isomorphism

$$\mathcal{D}_A \cong \mathcal{D}_\mathfrak{g} \otimes \mathcal{D}_\mathcal{F}, \ \mathcal{D}_{TM} \cong \mathcal{D}_\mathcal{F} \otimes \mathcal{D}_\nu.$$

Combining these, and working out the actions, we have:

Lemma 4.26. *These isomorphisms induce an isomorphism of representations of $\mathcal{G}$*

$$\mathcal{D}_A^{\mathrm{tr}} \cong \mathcal{D}_{\mathfrak{g}^*} \otimes \mathcal{D}_\nu. \tag{4.18}$$

For $\sigma \in \Gamma(\mathcal{D}_A)$, $\kappa \in \Gamma(\mathcal{D}_\mathfrak{g})$ nowhere vanishing and $\beta \in \Gamma(\mathcal{D}_\nu)$ we write

$$\sigma \equiv \kappa^\vee \otimes \beta \tag{4.19}$$

if the two elements correspond to each other by the previous isomorphism.

2. The foliation groupoid: Since the naive orbit space of $\mathcal{G}$ is the naive leaf space of the foliation $\mathcal{F}$, it can be described also as the orbit space of groupoids that are smaller than $\mathcal{G}$ - such as the holonomy or the monodromy groupoid of the foliation (or anything else that integrates $\mathcal{F}$). Actually, there is a such a smaller groupoid that is naturally associated to $\mathcal{G}$: the quotient of $\mathcal{G}$ obtained by dividing out the action of the connected components $\mathcal{G}_x^0$ of the isotropy groups $\mathcal{G}_x$ (for the smoothness, see Proposition 2.5 in [71]). We denote it by

$$\mathcal{E} = \mathcal{E}(\mathcal{G}) \tag{4.20}$$

and call it the foliation groupoid associated to $\mathcal{G}$. Note that, for this groupoid, its Lie algebroid is $\mathcal{F}$, hence its transverse density bundle is

$$\mathcal{D}_{\mathcal{E}}^{\mathrm{tr}} = \mathcal{D}_{\mathcal{F}^*} \otimes \mathcal{D}_{TM} \cong \mathcal{D}_{\nu}.$$

Since $\mathcal{G}$ and $\mathcal{E}$ have the same (naive) orbit space, it is natural to compare the notions of transverse measures/densities for $\mathcal{G}$ and $\mathcal{E}$. Note that, by the previous isomorphisms, the transverse density bundles of $\mathcal{G}$ and $\mathcal{E}$ are related by

$$\mathcal{D}_{\mathcal{G}}^{\mathrm{tr}} \cong \mathcal{D}_{\mathfrak{g}^*} \otimes \mathcal{D}_{\mathcal{E}}^{\mathrm{tr}}. \tag{4.21}$$

Hence, to talk about (strictly positive) transverse densities for both groupoids, we have to assume the existence of $\kappa \in \Gamma(\mathcal{D}_{\mathfrak{g}})$ strictly positive. On the other hand, since $\mathcal{D}_A = \mathcal{D}_{\mathfrak{g}} \otimes \mathcal{D}_{\mathcal{F}}$, the pairing with $\kappa^\vee \in \Gamma(\mathcal{D}_{\mathfrak{g}^*})$ induces a map

$$\kappa^\vee : C_c^\infty(M, \mathcal{D}_A) \to C_c^\infty(M, \mathcal{D}_{\mathcal{F}}).$$

Putting everything together, the next result follows by a straightforward computation:

Proposition 4.27. *If $\kappa \in \Gamma(\mathcal{D}_{\mathfrak{g}})$ is strictly positive and invariant, then the pairing with $\kappa^\vee$ descends to an isomorphism*

$$\kappa^\vee : C_c^\infty(M//\mathcal{G}) \xrightarrow{\sim} C_c^\infty(M//\mathcal{E}).$$

Also, via the relation $\sigma \equiv \kappa^\vee \otimes \beta$ (see (4.19), κ induces a 1-1 correspondence between

- *invariant sections β of $\mathcal{D}_\nu$ (i.e., transverse densities for $\mathcal{E}$)*
- *invariant sections σ of $\mathcal{D}_A^{\mathrm{tr}}$ (i.e., transverse densities for $\mathcal{G}$).*

Moreover, the transverse measure μ_σ for $\mathcal{G}$ agrees with the transverse measure μ_β for $\mathcal{E}$, via the isomorphism $\kappa^\vee$; i.e., one has a commutative diagram:

$$\begin{tikzcd}
C_c^\infty(M//\mathcal{G}) \arrow[d, "\kappa^\vee"'] \arrow[dr, "\mu_\sigma"] & \\
 & \mathbb{R} \\
C_c^\infty(M//\mathcal{E}) \arrow[ur, "\mu_\beta"'] &
\end{tikzcd}.$$

4.4 Intermezzo: the case of proper Lie groupoids

In this section we will show that "the general non-sense" from the previous sections takes a much more concrete (but not so obvious) form in the case of proper groupoids. Throughout this section we fix a proper groupoid $\mathcal{G}$ over M, and we consider the orbit space

$$B := M/\mathcal{G}, \quad \text{with quotient map } \ \pi : M \to B.$$

Properness implies that B is Hausdorff and locally compact; however, B has much more structure. The upshot of this section is that the abstract transverse measures discussed in the previous two sections descend to (standard) measures on B. We start with the existence result for transverse densities.

Proposition 4.28. *Any proper Lie groupoid admits strictly positive transverse densities.*

Proof. This is an immediate application of averaging, completely similar to our illustrative Lemma 4.67: using any proper Haar system μ, if one starts with a strictly positive section $\rho \in C^\infty(M, \mathcal{D}_A^{\mathrm{tr}})$, then

$$\mathrm{Av}_\mu(\rho)_x := \int_{s^{-1}(x)} g^*\rho_{t(g)}\ d\mu^x(g)$$

remains strictly positive and it is equivariant: for $a : x \to y$,

$$a^*\mathrm{Av}_\mu(\rho)_y = \int_{s^{-1}(y)} a^*g^*\rho_{t(g)}\ d\mu^y(g)$$

and then, using $a^*g^* = (ga)^*$, $t(g) = t(ga)$ and invariance of μ, the previous expression is

$$\int_{s^{-1}(y)} (ga)^*\rho_{t(ga)}\ d\mu^y(g) = \int_{s^{-1}(x)} g^*\rho_{t(g)}\ d\mu^x(g) = \mathrm{Av}_\mu(\rho)_x.$$

Hence the existence of invariant transverse densities and measures will follow from the existence of proper Haar systems (Proposition 4.66). $\square$

4.4.1 Classical measures on the orbit space B

We have already mentioned that $B = M/\mathcal{G}$ is a quite nicely behaved space. Although it is not a manifold in general, in many respects it behaves like one. For instance, one has a well-behaved algebra of smooth functions:

$$C^\infty(B) := \{f : B \to \mathbb{R} \mid f \circ \pi \in C^\infty(M)\},$$

(and one can show that one can recover the topological space B out of the algebra $C^\infty(B)$ as its spectrum - Theorem 2.11). The algebra of compactly supported smooth functions is defined as usual, by just adding the compact support condition:

$$C_c^\infty(B) := \{f \in C^\infty(B) \mid \mathrm{supp}(f) - \mathrm{compact}\}.$$

Of course, one can define similarly $C_c^k(B)$ for any integer $k \geq 0$; for $k = 0$, since B is endowed with the quotient topology, one recovers the usual space of compactly supported continuous functions on the space B. As in the case of smooth manifolds (Proposition 4.8), one has:

Proposition 4.29. *If $\mathcal{G}$ is a proper Lie groupoid over M, $B = M/\mathcal{G}$, then*

1. *$C_c^\infty(B)$ is dense in $C_c(B)$.*
2. *any positive linear functional on $C_c(B)$ is automatically continuous; and the same holds for $C_c^\infty(B)$ with the subspace topology.*
3. *the construction $I \mapsto I|_{C_c^\infty(B)}$ induces a 1-1 correspondence:*

$$\{\textit{measures on } B\} \stackrel{\sim}{\longleftrightarrow} \{\textit{positive linear functionals on } C_c^\infty(B)\}.$$

Proof. The proof of Proposition 4.8 was written so that it can be applied, word by word, also to this context. The only ingredient used there and which is still missing for us is the fact that $C^\infty(M//\mathcal{G})$ separates closed subsets, which is precisely our main illustration of the averaging procedure (Lemma 4.67 in the Appendix). □

4.4.2 Transverse measures ≡ classical measures on B

Next, we relate transverse measures to classical measures on B. The main point is that the intuition described in Remark 43 can now be made precise; consider the intrinsic averaging:

$$\mathrm{Av}_{\mathcal{G}} = \mathrm{Av} : C_c^\infty(M, \mathcal{D}_A) \to C_c^\infty(B),\ \mathrm{Av}(\rho)(\pi(x)) = \int_{s^{-1}(x)} \overrightarrow{\rho}|_{s^{-1}(x)}. \tag{4.22}$$

Theorem 4.30. *The averaging map (4.22) induces an isomorphism*

$$\mathrm{Av}_{\mathcal{G}} = \mathrm{Av} : C_c^\infty(M//\mathcal{G}) \stackrel{\sim}{\to} C_c^\infty(B). \tag{4.23}$$

In particular, this induces a 1-1 correspondence

$$\{\textit{transverse measures on } \mathcal{G}\} \stackrel{\sim}{\longleftrightarrow} \{\textit{measures on } B = M/\mathcal{G}\}.$$

For the inverse of Av, *choose any proper normalized Haar density ρ (Def. 4.65) and then, for $f \in C_c^\infty(B)$, the corresponding element in $C_c^\infty(M//\mathcal{G})$ is the one represented by $\pi^*(f)\rho \in C_c^\infty(M, \mathcal{D}_A)$.*

Proof. To check that the map Av is well-defined, recall that if two elements of $C_c^\infty(M, \mathcal{D}_A)$ represent the same class in $C_c^\infty(M//\mathcal{G})$, they differ by a density in the image of $(s_! - t_!)(u)$. If, as in Remark 45, we fix any strictly positive density $\rho_0 \in C^\infty(M, \mathcal{D}_A)$ to trivialize all the density bundles, we then check that

$$\mathrm{Av}((s_!^\rho - t_!^\rho)(u))(\pi(x)) = \int_{s^{-1}(x)} \left(\int_{s^{-1}(t(g))} u(g)\overrightarrow{\rho_0} - \int_{s^{-1}(t(g))} u(g^{-1})\overrightarrow{\rho_0} \right) \overrightarrow{\rho_0},$$

which is equal to zero, being the difference of two double integrals that have the same value. This is because both correspond to taking the integral of u over all arrows g whose source (and target) belong to the orbit $\pi(x)$.

Next, choosing any proper normalized Haar density ρ, the inverse to Av, given as above by mapping $f \in C_c^\infty(B)$ into the class of $\pi^*(f)\rho$ in $C_c^\infty(M//\mathcal{G})$ is well defined. Indeed, for such an f, there is a compact subset K of M such that $\pi(K)$ is the support of f. This means that the support of π^*f is the saturation of K, and the intersection of it with the support of ρ is compact, because ρ is a proper Haar density, so $\pi^*(f)\rho$ has compact support as well.

Finally, to see that this is indeed the inverse to Av, we first check that

$$\begin{aligned}\mathrm{Av}([\pi^*(f)\rho])(\pi(x)) &= \int_{s^{-1}(x)} \pi^*(f)(t(g))\overrightarrow{\rho} \\ &= f(\pi(x)) \int_{s^{-1}(x)} \overrightarrow{\rho} = f(\pi(x)).\end{aligned}$$

Next, suppose that the proper normalized Haar density ρ is of the form $\rho = c \cdot \rho'$, where c is a cut-off function and ρ' is full. Then, for any other density $h \cdot \rho' \in C_c^\infty(M, \mathcal{D}_A)$, we have that

$$\begin{aligned}(h \cdot \rho' - \pi^*\mathrm{Av}(h \cdot \rho')\rho)(x) &= h(x)\rho'(x) - \left(\int_{s^{-1}(x)} h(t(g))\overrightarrow{\rho'}\right) c(x)\rho'(x) \\ &= \left(\int_{s^{-1}(x)} h(x)c(t(g))\overrightarrow{\rho'} - \int_{s^{-1}(x)} c(x)h(t(g))\overrightarrow{\rho'}\right)\rho'(x),\end{aligned}$$

and this expression represents zero in $C_c^\infty(M//\mathcal{G})$. Indeed, using ρ' to trivialize the density bundles, it becomes precisely $(s_!^{\rho'} - t_!^{\rho'})(s^*h \cdot t^*c)$. $\square$

4.4.3 The measure on B induced by a transverse density

Putting the previous constructions together we see that a positive transverse density $\sigma \in C^\infty(M, \mathcal{D}_A^{\mathrm{tr}})$ induces a transverse measure for $\mathcal{G}$, hence a measure on B, which we will still denote by

$$\mu_\sigma : C_c^\infty(B) \to \mathbb{R}.$$

Of course, this measure can be made explicit. To do so, recall here in an explicit form the notion of cut-off function for a full Haar density ρ, as it follows from Proposition 4.66: it is a smooth positive function c on M satisfying:

1. the restriction of s to $t^{-1}(\mathrm{supp}(c))$ is proper (as a map to M).
2. $\int_{s^{-1}(x)} c(t(g))\, d\mu_\rho^x(g) = 1$ for all $x \in M$.

Proposition 4.31. *If σ is a positive transverse density, then the measure μ_σ induced on B is given by*

$$\int_B h(b)\, d\mu_\sigma(b) = \int_M c(x)h(\pi(x))\, d\mu_\tau(x) \tag{4.24}$$

where, to write the right hand side, we choose a (any) decomposition $\sigma = \rho^\vee \otimes \tau$ (Definition 4.22) and we choose c to be a (any) cut-off function for ρ. Moreover, μ_σ is uniquely characterized by the formula

$$\int_M f(x)\, d\mu_\tau(x) = \int_B \left(\int_{\mathcal{O}_b} f(y)\, d\mu_{\rho_{\mathcal{O}_b}}(y)\right) d\mu_\sigma(b),$$

for all $f \in C_c^\infty(M)$, where $\rho_{\mathcal{O}}$ denotes the density induced by ρ on the orbit (cf. Subsection 4.9.4).

Note that it is rather remarkable that the right hand side of the formula (4.24) for μ_σ does not depend on the choice of the decomposition or of the cut-off function.

Proof. The formula defining μ_σ on $C_c^\infty(B)$ is obtained simply by transferring the definition of μ_σ on $C_c^\infty(M//\mathcal{G})$ through the isomorphism $C_c^\infty(B) \xrightarrow{\sim} C_c^\infty(M//\mathcal{G})$ described in Theorem 4.30, which maps $f \in C_c^\infty(B)$ to the class of $\pi^*(f)(c \cdot \rho)$ in $C_c^\infty(M//\mathcal{G})$.

In order to prove the second statement of the theorem, use the description of μ_σ from the first part, and the discussion on $\rho_{\mathcal{O}}$ from Subsection 4.9.4, to check that

$$\begin{aligned}
\int_B \left(\int_{\mathcal{O}_b} f(y)\, d\mu_{\rho_{\mathcal{O}_b}}(y) \right) d\mu_\sigma(b) &= \int_M c(x) \left(\int_{\mathcal{O}_x} f(y)\, d\mu_{\rho_{\mathcal{O}_b}}(y) \right) d\mu_\tau(x) \\
&= \int_M c(x) \left(\int_{s^{-1}(x)} f(t(g))\, d\mu_{\overrightarrow{\rho}}(g) \right) d\mu_\tau(x) \\
&= \int_M \left(\int_{s^{-1}(x)} c(s(g)) f(t(g))\, d\mu_{\overrightarrow{\rho}}(g) \right) d\mu_\tau(x).
\end{aligned}$$

The last integral is the same as the integral of the function $(s^*c) \cdot (t^*f)$ over $\mathcal{G}$ with respect to the density $\rho_{\mathcal{G}}$ induced by ρ and τ. Since $\sigma = \rho^\vee \otimes \tau$ is invariant, we know from Proposition 4.25 that $\rho_{\mathcal{G}}$ is invariant under the inversion map, so that

$$\begin{aligned}
\int_B \left(\int_{\mathcal{O}_b} f(y)\, d\mu_{\rho_{\mathcal{O}_b}}(y) \right) d\mu_\sigma(b) &= \int_M \left(\int_{s^{-1}(x)} c(t(g)) f(s(g))\, d\mu_{\overrightarrow{\rho}}(g) \right) d\mu_\tau(x) \\
&= \int_M f(x) \left(\int_{s^{-1}(x)} (c(t(g))\, d\mu_{\overrightarrow{\rho}}(g) \right) d\mu_\tau(x) \\
&= \int_M f(x)\, d\mu_\tau(x)
\end{aligned}$$

□

4.4.4 Some interesting consequences

Here are some immediate but interesting consequences of Theorem 4.30 and the integral formula from Proposition 4.31.

Corollary 4.32. *With the same notation as above, if $\mathcal{G}$ is compact and σ is a transverse density then, for any $h \in C^\infty(B)$,*

$$\int_M h(\pi(x))\, d\mu_\tau(x) = \int_B h(b) \cdot \mathrm{Vol}(\mathcal{O}_b, \mu_{\mathcal{O}_b})\, d\mu_\sigma(b).$$

In particular, the volume function $\mathrm{Vol}_\rho : b \mapsto \mathrm{Vol}(\mathcal{O}_b, \mu_{\mathcal{O}_b})$ is a continuous function on B, smooth in the sense that its pullback to M is smooth, and its integral with respect to μ_σ is $\mathrm{Vol}(M, \mu_\tau)$.

Note that, in terms of set-measures, this reads as

$$\mu_\sigma = \frac{1}{\mathrm{Vol}_\rho} \pi_!(\mu_\tau).$$

Corollary 4.33. *With the same notation as above, if $\mathcal{G}$ is compact and σ is a transverse density, then the volume of $B = M/\mathcal{G}$ with respect to μ_σ is given by*

$$\mathrm{Vol}(B, \mu_\sigma) = \int_M \frac{1}{\mathrm{Vol}(\mathcal{O}_x, \mu_{\mathcal{O}_x})}\, d\mu_\tau(x).$$

We see that this expression for the volume of B with respect to μ_σ corresponds to the definition given by Weinstein (cf. [110, Def. 2.3, Thm. 3.2]) for the volume of the stack $M//\mathcal{G}$ with data (ρ, τ).

4.4.5 The proper regular case

We now assume that $\mathcal{G}$ is both regular and proper and we continue the discussion from Subsection 4.3.4 which compared the transverse measures for $\mathcal{G}$ with the transverse measures for the foliation groupoid $\mathcal{E} = \mathcal{E}(\mathcal{G})$ associated to $\mathcal{G}$. We have seen that the two correspond to each other modulo isomorphisms

$$\kappa^\vee : C_c^\infty(M//\mathcal{G}) \to C_c^\infty(M//\mathcal{E}) \tag{4.25}$$

induced by strictly positive invariant sections $\kappa \in \Gamma(\mathcal{D}_\mathfrak{g})$ (Proposition 4.27). In this (regular, proper) case there are two new features that are present:

- Both $C_c^\infty(M//\mathcal{G})$ and $C_c^\infty(M//\mathcal{E})$ are isomorphic to $C_c^\infty(B)$, by the averaging of Theorem 4.30 applied to $\mathcal{G}$ and $\mathcal{E}$, respectively.
- There is a preferred choice of a strictly positive invariant section κ: since the isotropy groups $\mathcal{G}_x$ are compact, their Lie algebras $\mathfrak{g}_x$ come with induced Haar densities; we consider the ones associated to the identity components of $\mathcal{G}_x$ - they define a strictly positive invariant section
$$\kappa_{\mathrm{Haar}} \in \Gamma(\mathcal{D}_\mathfrak{g}).$$

Note that choosing the Haar densities associated to the identity components (rather than of the entire $\mathcal{G}_x$'s) is essential for the smoothness of κ_{Haar}; indeed, while the bundle consisting of the full isotropy groups may fail to be smooth, passing to identity components does produce a smooth bundle of Lie groups (cf. [71, Prop. 2.5]). Using the normalization and the invariance conditions for the Haar densities, the following is immediate:

Proposition 4.34. *κ_{Haar} is the only strictly positive invariant section of $\mathcal{D}_\mathfrak{g}$ with the property that the isomorphism (4.25) becomes the identity after identifying $C_c^\infty(M//\mathcal{G})$ and $C_c^\infty(M//\mathcal{F})$ with $C_c^\infty(B)$.*

Putting this and Proposition 4.27 together we find in particular:

Corollary 4.35. *The relation $\sigma \equiv \kappa_{\mathrm{Haar}}^\vee \otimes \beta$ (see (4.19)) induces a bijection between*

- *invariant sections β of $\mathcal{D}_\nu$ (i.e., transverse densities for $\mathcal{E}$)*
- *invariant sections σ of $\mathcal{D}_A^{\mathrm{tr}}$ (i.e., transverse densities for $\mathcal{G}$).*

Moreover, the measures μ_σ and μ_β induced on B by σ and β coincide.

$$\begin{tikzcd}
& C_c^\infty(M//\mathcal{G}) \arrow[dl, "Av_{\mathcal{G}}"', "\sim"] \arrow[dr, "\mu_\sigma"] \arrow[dd, "\kappa^\vee_{\mathrm{Haar}}"', "\sim"] & \\
C_c^\infty(B) & & \mathbb{R} \\
& C_c^\infty(M//\mathcal{E}) \arrow[ul, "Av_{\mathcal{E}}", "\sim"'] \arrow[ur, "\mu_\beta"'] &
\end{tikzcd}.$$

4.5 Intermezzo: the case of symplectic groupoids

4.5.1 The general case

In this section we look at another particular class of groupoids: those that come from Poisson Geometry, i.e., symplectic groupoids [17, 55, 108, 111]. As mentioned in the introduction, this case and its relevance to compactness in Poisson Geometry [21, 22, 23] was one of our original motivations for the paper that gave rise to this chapter. Recall (cf. e.g. [17]) that a symplectic groupoid $(\mathcal{G}, \Omega)$ is a Lie groupoid $\mathcal{G}$ endowed with a symplectic form $\Omega \in \Omega^2(\mathcal{G})$ which is multiplicative. In this case the Lie algebroid of A is, as a vector bundle, canonically isomorphic to T^*M by

$$A \cong T^*M, \quad \alpha_x \mapsto (X_x \mapsto \Omega_x(\alpha_x, X_x)),$$

where we identify $x \in M$ with $1_x \in \mathcal{G}$. Therefore we obtain, as vector bundles,

$$\mathcal{D}_A^{\mathrm{tr}} = \mathcal{D}_{TM} \otimes \mathcal{D}_{TM} \tag{4.26}$$

and similarly when replacing $\mathcal{D}$ by $\mathcal{V}$ or $\mathfrak{o}$.

In the remaining part of this section we fix a symplectic groupoid $(\mathcal{G}, \Omega)$ which we assume to have connected s-fibres; we will also consider the Poisson structure on the base of $\mathcal{G}$ uniquely determined by the fact that the source map is a Poisson map; for $f \in C^\infty(M)$ we will denote by X_f the corresponding Hamiltonian vector field on M. We say that a density τ on M is invariant under the Hamiltonian flows if

$$L_{X_f}(\tau) = 0 \quad \forall\, f \in C^\infty(M).$$

Proposition 4.36. *The correspondence*

$$\tau \in \Gamma(\mathcal{D}_{TM}) \mapsto \sigma := \tau \otimes \tau \in \Gamma(\mathcal{D}_A^{\mathrm{tr}})$$

induces a bijection between strictly positive:

i. transverse densities σ for $\mathcal{G}$.

ii. densities τ on M which are invariant under the Hamiltonian flows.

Furthermore, $\mathcal{D}_{TM}$ can be made into a representation of $\mathcal{G}$ in a canonical way, uniquely determined by the fact that (4.26) becomes an isomorphism of representations of $\mathcal{G}$ and the action preserves the (fibrewise) positivity of densities. With respect to this action, a density τ on M is invariant as a section of $\mathcal{D}_{TM}$ if and only if τ is invariant under the Hamiltonian flows.

Proof. In general, the representation of $\mathcal{G}$ on $\mathcal{D}^{\mathrm{tr}}_A$ induces, by differentiation, a representation of A, i.e., a flat A-connection ∇ on $\mathcal{D}^{\mathrm{tr}}_A$. Using the formula for ∇ from [38] (but using $\mathcal{D}^{\mathrm{tr}}_A$ instead of $\mathcal{V}^{\mathrm{tr}}_A$), in the case where $A = T^*M$ is the Lie algebroid of the symplectic groupoid $\mathcal{G}$, we have that

$$\nabla_{df}(\tau_1 \otimes \tau_2) = (\mathcal{L}_{X_f}\tau_1) \otimes \tau_2 + \tau_1 \otimes (\mathcal{L}_{X_f}\tau_2).$$

Therefore, if τ is a strictly positive density on M, invariant under the Hamiltonian flows, then $\nabla_{df}(\tau \otimes \tau) = 0$, for any $f \in C^\infty(M)$. This implies that $\tau \otimes \tau$ is invariant with respect to the representation of $A = T^*M$: locally any 1-form α can be written as $\alpha = udf$, where $f, u \in C^\infty(M)$; invariance follows from the condition that $\nabla_{udf}(\tau \otimes \tau) = u\nabla_{df}(\tau \otimes \tau) = 0$. Since $\mathcal{G}$ is source-connected, $\tau \otimes \tau$ is invariant with respect to the representation of $\mathcal{G}$ as well. Moreover, the mapping $\tau \mapsto \tau \otimes \tau$ is injective among strictly positive densities.

Conversely, if σ is a strictly positive transverse density for $\mathcal{G}$, we can choose a decomposition of the form $\sigma = \tau \otimes u\tau$, where $u \in C^\infty(M)$ is strictly positive. Then $\tau' = \sqrt{u}\tau$ is a strictly positive smooth density on M and $\sigma = \tau' \otimes \tau'$. Since σ is a transverse density for $\mathcal{G}$, it holds that $\nabla_{df}\sigma = 0$ for any $f \in C^\infty(M)$. In other words,

$$0 = \nabla_{df}\sigma = (\mathcal{L}_{X_f}\tau') \otimes \tau' + \tau' \otimes (\mathcal{L}_{X_f}\tau')$$

Writing $\mathcal{L}_{X_f}\tau' = v\tau'$, where $v \in C^\infty(M)$, we conclude that $2v\tau' \otimes \tau' = 0$; since τ' is strictly positive, it must hold that $v = 0$; therefore τ' is invariant under the Hamiltonian flows.

In order to make $\mathcal{D}_{TM}$ into a representation of $\mathcal{G}$ in a canonical way, such that (4.26) becomes an isomorphism of representations of $\mathcal{G}$ and the action preserves the (fibrewise) positivity of densities, pick any strictly positive density τ on M. Then it must hold that for any arrow $g : x \to y$ in $\mathcal{G}$,

$$(g \cdot \tau_x) \otimes (g \cdot \tau_x) = g \cdot (\tau_x \otimes \tau_x).$$

Since the action of $\mathcal{G}$ on $\mathcal{D}^{\mathrm{tr}}_A$ preserves the positivity of sections, there exists a positive smooth function $c^\tau \in C^\infty(\mathcal{G})$, such that for any arrow $g \in \mathcal{G}$, it holds that

$$g \cdot (\tau_x \otimes \tau_x) = c^\tau(g)(\tau_y \otimes \tau_y),$$

where g has source x and target y; we are then forced to set

$$g \cdot \tau_x = \sqrt{c^\tau(g)}\tau_y,$$

in order to ensure that the action preserves the positivity of sections. Since $\mathcal{D}_{TM}$ is a line bundle, the representation of $\mathcal{G}$ is completely determined by the action on τ.

The resulting representation does not depend on the choice of τ: if τ' is another strictly positive density on M, then $\tau' = f\tau$, where f is a positive smooth function on M. Then

$$g \cdot (\tau'_x \otimes \tau'_x) = f(x)^2 g \cdot (\tau_x \otimes \tau_x) = f(x)^2 c^\tau(g)(\tau_y \otimes \tau_y) = \left(\frac{f(x)}{f(y)}\right)^2 c^\tau(g)(\tau'_y \otimes \tau'_y),$$

and on the other hand

$$g \cdot (\tau'_x \otimes \tau'_x) = c^{\tau'}(g)(\tau'_y \otimes \tau'_y),$$

so $c^{\tau'}(g) = \left(\frac{f(x)}{f(y)}\right)^2 c^{\tau}(g)$. Therefore

$$\sqrt{c^{\tau'}(g)}\tau'_y = \frac{f(x)}{f(y)}\sqrt{c^{\tau}(g)}\tau'_y = f(x)\sqrt{c^{\tau}(g)}\tau_y = g \cdot \tau'_x.$$

Finally, since (4.26) is now an isomorphism of representations and the action preserves positivity of densities, a density τ on M is invariant with respect to the representation of $\mathcal{G}$ on $\mathcal{D}_{TM}$ if and only if $\tau \otimes \tau$ is an invariant section of $\mathcal{D}_A^{\mathrm{tr}}$.

Since the source-fibres of $\mathcal{G}$ are connected, so are the orbits of $\mathcal{G}$; therefore, they coincide with the symplectic leaves of M. This implies that in order for τ to be invariant under the Hamiltonian flows, then along each orbit of $\mathcal{G}$ it has to either be strictly positive, strictly negative or constant equal to zero. When τ is strictly positive, we already know that it is invariant under the Hamiltonian flows if and only if $\tau \otimes \tau$ is transverse; the same holds under the assumption that τ is strictly negative.

If τ vanishes at a point $x \in M$, then in order to be invariant under the Hamiltonian flows it must vanish along the symplectic leaf containing x which is the same as the orbit of $\mathcal{G}$ containing x; hence, this is the same condition as the one we obtain by asking that $\tau \otimes \tau$ be a transverse density. □

Remark 47 (on the existence). By the general theory (Proposition 4.28), in the proper case strictly positive transverse densities do exist. However, it is interesting to search for natural ones, associated to the symplectic/Poisson geometry that is present in this context. Searching for a canonical σ is the same thing as searching for a canonical τ or, setting $\rho = \tau^{\vee}$, searching for a Haar density

$$\rho \in \Gamma(\mathcal{D}_A)$$

(see Definition 4.64 in the Appendix). With the intuition that such Haar densities are "smooth" families of densities along the orbits (see Corollary 4.68 in the Appendix), there is an obvious candidate: the one that uses the Liouville forms of the symplectic leaves! However, the resulting family is not smooth, unless we are in the regular case (and we choose the normalizations carefully). Nevertheless, in the general case one can still make sense of the resulting measure (just that it might not come from a transverse density); see [23].

Despite the previous remark, when asking a slightly stronger condition on $\mathcal{G}$, namely that of s-properness, one does obtain a canonical transverse density. First one integrates along the s-fibres the Liouville density of the groupoid:

$$\rho_{DH}^M := \int_{s-\mathrm{fibres}} \frac{|\Omega^{\mathrm{top}}|}{\mathrm{top}!}$$

(note that, since we do not assume any orientability, although the Liouville density comes from a volume form, integration over the s-fibres gives only a density). As explained in [21, 22], ρ_{DH}^M is an invariant density on M; hence it induces (by the previous proposition) a canonical (strictly positive) transverse density for $\mathcal{G}$,

$$\sigma_{DH} = \rho_{DH}^M \otimes \rho_{DH}^M \in \Gamma(\mathcal{D}_A^{\mathrm{tr}}),$$

and an associated transverse measure μ_{DH} ("DH" stands for Duistermaat-Heckman).

Remark 48. Even when the procedure from the previous remark works, the resulting transverse density is different from σ_{DH}; the two are related by a Duistermaat-Heckman formula (see [22] and also our next subsection).

4.5.2 The regular case

In the regular case there is yet another description of transverse densities: using the normal bundle

$$\nu = TM/\mathcal{F}$$

of the symplectic foliation $\mathcal{F}$ associated to Poisson structure π on M; recall that, as an involutive sub-bundle of TM, $\mathcal{F}$ is the image of $\pi^\sharp : T^*M \to TM$ (the Poisson bivector interpreted as a linear map). As for any foliation, ν comes with an action of the holonomy groupoid of $\mathcal{F}$; hence one can talk about sections of $\mathcal{D}_\nu$ which are invariant under holonomy.

Of course, we are in the setting of Subsection 4.3.4: the symplectic groupoid $\mathcal{G}$ is regular, $\mathcal{F}$ is the associated foliation and the resulting action of $\mathcal{G}$ on ν (see the subsection) factors through the holonomy action. What is special in this case is that the representation $\mathfrak{g} = \mathrm{Ker}\,(\pi^\sharp)$ is just the dual of ν. Hence the isomorphism (4.18) becomes

$$\mathcal{D}_A^{\mathrm{tr}} \cong \mathcal{D}_\nu \otimes \mathcal{D}_\nu, \tag{4.27}$$

an isomorphism of representations of $\mathcal{G}$. More explicitly: the Liouville forms induced by the leafwise symplectic forms define a section

$$\frac{|\omega^{\mathrm{top}}|}{\mathrm{top}!} \in \Gamma(\mathcal{D}_\mathcal{F})$$

which trivializes $\mathcal{D}_\mathcal{F}$ hence, using again the identification $\mathcal{D}_{TM} = \mathcal{D}_\mathcal{F} \otimes \mathcal{D}_\nu$, it induces

$$\mathcal{D}_{TM} \cong \mathcal{D}_\nu, \quad \frac{|\omega^{\mathrm{top}}|}{\mathrm{top}!} \otimes \beta \longleftrightarrow \beta.$$

We now apply Proposition 4.27; given the fact that $\mathfrak{g} = \nu^*$, the (positive) sections κ of $\mathcal{D}_\mathfrak{g}$ appearing in the proposition will be written as duals $\beta^\vee$ of sections β of $\mathcal{D}_\nu$. To avoid confusions, the resulting map $\kappa^\vee$ (now given by pairing with β) will be denoted by $\langle \beta, \cdot \rangle$. Combining the proposition also with Proposition 4.36, we find:

Proposition 4.37. *Consider the relations*

$$\Gamma(\mathcal{D}_{TM}) \ni \tau = \frac{|\omega^{\mathrm{top}}|}{\mathrm{top}!} \otimes \beta \in \Gamma(\mathcal{D}_\mathcal{F} \otimes \mathcal{D}_\nu)$$

$$\Gamma(\mathcal{D}_A^{\mathrm{tr}}) \ni \sigma = \tau \otimes \tau \in \Gamma(\mathcal{D}_{TM} \otimes \mathcal{D}_{TM})$$

and $\sigma \equiv \beta \otimes \beta$ modulo the isomorphism (4.27). These induce bijections between strictly positive:

1. *transverse densities σ for $\mathcal{G}$*

2. *densities τ on M invariant under the Hamiltonian flows.*

3. sections β of $\mathcal{D}_\nu$ invariant under holonomy (= transverse densities for $\mathcal{E}$).

Moreover, in this case the pairing with β descends to an isomorphism

$$\langle \beta, \cdot \rangle : C_c^\infty(M//\mathcal{G}) \xrightarrow{\sim} C_c^\infty(M//\mathcal{E})$$

which relates the transverse measure μ_σ for $\mathcal{G}$ with the transverse measure μ_β for $\mathcal{E}$ through the commutative diagram

$$\begin{tikzcd}
C_c^\infty(M//\mathcal{G}) \arrow[d, "\langle\beta,\cdot\rangle"', "\sim"] \arrow[dr, "\mu_\sigma"] & \\
 & \mathbb{R} \\
C_c^\infty(M//\mathcal{E}) \arrow[ur, "\mu_\beta"'] &
\end{tikzcd}.$$

4.5.3 The proper regular case

Under the condition that the symplectic groupoid $\mathcal{G}$ is both regular and proper we can further specialize the discussion from Subsection 4.4.5 and use Corollary 4.35. In this case we have at our disposal the Haar densities associated to the identity components of the isotropy groups $\mathcal{G}_x$, which give rise to a strictly positive section

$$\beta_{\mathrm{Haar}} \in \Gamma(\mathcal{D}_\nu).$$

This is just the dual of κ_{Haar} from Subsection 4.4.5 (as above, in the Poisson case we pass from $\mathfrak{g} = \nu^*$ to ν).

Corollary 4.38. *Any proper regular symplectic groupoid $(\mathcal{G}, \Omega)$ carries a canonical transverse density: the one corresponding to β_{Haar}, i.e., the σ that corresponds to $\beta_{\mathrm{Haar}} \otimes \beta_{\mathrm{Haar}}$ modulo the isomorphism (4.27) or, equivalently, corresponding to the density on M given by*

$$\frac{|\omega^{\mathrm{top}}|}{\mathrm{top}!} \otimes \beta_{\mathrm{Haar}}.$$

Moreover, the measure μ_σ induced by σ on $B = M//\mathcal{G}$ coincides with the measure μ_β induced by β on $B = M//\mathcal{E}$.

$$\begin{tikzcd}
 & C_c^\infty(M//\mathcal{G}) \arrow[dl, "Av_{\mathcal{G}}"', "\sim"] \arrow[dd, "\langle\beta_{\mathrm{Haar}},\cdot\rangle"', "\sim"] \arrow[dr, "\mu_\sigma=\mu_{\mathrm{aff}}"] & \\
C_c^\infty(B) & & \mathbb{R} \\
 & C_c^\infty(M//\mathcal{E}) \arrow[ul, "Av_{\mathcal{E}}", "\sim"'] \arrow[ur, "\mu_{\beta_{\mathrm{Haar}}}"'] &
\end{tikzcd}.$$

Remark 49. Let us also point out that, in the case of symplectic groupoids, the Haar density β_{Haar} has a nice Poisson geometric interpretation. For instance, if the symplectic groupoid has

1-connected s-fibres, then the variation of the leafwise symplectic areas give rise (because of properness) to a lattice in ν (a transverse integral affine structure), see e.g. [110]; then β_{Haar} is just the corresponding density. If also the leaves are simply connected - condition that ensures that $B = M/\mathcal{G}$ is smooth, then this induces an integral affine structure on B and β_{Haar} is the canonical density associated to it. A similar interpretation holds for general symplectic groupoids $\mathcal{G}$ - see [22]. For this reason, the resulting measure on B is called the affine measure induced by $\mathcal{G}$, denoted by μ_{aff}. This is related to the Duistermaat-Heckman measure from Subsection 4.5.1 by a Duistermaat-Heckman formula - see [22].

Of course, one can use Proposition 4.31 to obtain a Weyl-type integration formula for μ_{aff}. One obtains the following result from [22] (but with a different proof):

Corollary 4.39. *For the affine measure μ_{aff} on B, denoting by μ_M the measure on M induced by the corresponding density on M (τ above), one has*

$$\int_M f(x)\ d\mu_M(x) = \int_B \iota(b) \left(\int_{\mathcal{O}_b} f(y)\ d\mu_{\mathcal{O}_b}(y) \right)\ d\mu_{\text{aff}}(b),$$

for all $f \in C_c^\infty(M)$, where $\mu_{\mathcal{O}}$ denotes the Liouville measure of the symplectic leaf $\mathcal{O}$, $\iota(b) = \iota(x)$ is the number of connected components of $\mathcal{G}_x$ with $x \in \mathcal{O}_b$ (any).

Proof. We just have to be careful with computing the resulting densities on the orbits: they arise when looking at the principal $\mathcal{G}_x$-bundles $t : s^{-1}(x) \to \mathcal{O}_x$ and decomposing the resulting density on $s^{-1}(x)$ using the Haar density associated to $\mathcal{G}_x$ (see (4.51) in the Appendix). This differs from the density β_{Haar} that we used before precisely by the factor $\iota(x)$ (see Example 4.12). And the final conclusion is that the resulting measure on $\mathcal{O}_x$ is $\iota(x)$ times the Liouville measure $\mu_{\mathcal{O}_x}$. □

Corollary 4.40. *If $\mathcal{G}$ is compact then the affine volume of $B = M/\mathcal{G}$ is given by*

$$\text{Vol}(B, \mu_\sigma) = \int_M \frac{1}{\iota(x)\text{Vol}(\mathcal{O}_x, \mu_{\mathcal{O}_x})}\ d\mu_M(x).$$

Finally, let us point out the following immediate consequence of Proposition 4.34, which proves the Conjecture 5.2 from [110] (without the simplifying assumptions from *loc.cit*).

Corollary 4.41. *Let σ be a transverse density of the regular proper symplectic groupoid $(\mathcal{G}, \Omega)$ (denoted by λ in [110]). Write $\sigma = \tau \otimes \tau$ with τ an invariant density on M and write $\tau = \frac{|\omega^{\text{top}}|}{\text{top}!} \otimes \beta$ with $\beta \in \Gamma(\mathcal{D}_\nu)$ invariant. Then $\mu_\sigma = \mu_\beta$ (an equality of measures on B) if and only if $\beta = \beta_{\text{Haar}}$.*

4.6 Stokes formula and (Ruelle-Sullivan) currents

We now return to the general theory. In this section we point out that, as in the case of our motivating example of foliations, transverse measures for groupoids give rise to closed r-currents on the base manifold M, where r is the dimension of the s-fibres (or the rank of the Lie algebroid); one advantage of Haefliger's approach is that it makes such constructions rather obvious.

4.6.1 Stokes formula and Poincare duality for usual densities

The main property of the canonical integration of densities (4.4), which distinguishes it from other linear functionals on $C_c^{\infty}(M, \mathcal{D}_M)$, is the Stokes formula. To state it for general densities, one first re-interprets the sections of $\mathcal{D}_M$ as top-forms with values in the orientation bundle,

$$C_c^{\infty}(M, \mathcal{D}_{TM}) = \Omega_c^{\mathrm{top}}(M, \mathfrak{o}_M), \tag{4.28}$$

The orientation bundle comes with a flat connection. By a flat vector bundle we mean a vector bundle E together with a fixed flat connection

$$\nabla : C^{\infty}(M, TM) \times C^{\infty}(M, E) \to C^{\infty}(M, E),\ (X, e) \mapsto \nabla_X(e).$$

They provide the geometric framework for local coefficients; the main point for us is that, for such E, one has the spaces of E-valued forms

$$\Omega^{\bullet}(M, E) = C^{\infty}(M, \Lambda^{\bullet}T^*M \otimes E), \tag{4.29}$$

the flat connection ∇ gives rise to a DeRham operator d_{∇} on $\Omega^{\bullet}(M, E)$ given explicitly by the standard Koszul-type formula:

$$\begin{aligned} \mathrm{d}_{\nabla}\omega(X_1, \ldots, X_{k+1}) = &\sum_i (-1)^{i+1}\nabla_{X_i}(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})) + \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}), \end{aligned} \tag{4.30}$$

and the flatness of ∇ is equivalent to $d_{\nabla}^2 = 0$. Therefore one can talk about DeRham cohomology with coefficients in E, denoted by $H^{\bullet}(M, E)$. Back to densities, one uses the DeRham differential with coefficients in $\mathfrak{o}_M$

$$d : \Omega_c^{\mathrm{top}-1}(M, \mathfrak{o}_M) \to \Omega_c^{\mathrm{top}}(M, \mathfrak{o}_M),$$

and the Stokes formula for the canonical integration reads, via (4.28):

$$\int_M d\omega = 0 \quad \text{for all } \omega \in \Omega_c^{\mathrm{top}-1}(M, \mathfrak{o}_M).$$

Equivalently, $\int_M$ descends to a linear map

$$\int_M : H_c^{\mathrm{top}}(M, \mathfrak{o}_M) \to \mathbb{R}.$$

The fact that the domain is always 1-dimensional shows that the Stokes formula characterizes the integration of densities uniquely, up to multiplication by scalars.

4.6.2 Stokes for transverse measures

For a Lie algebroid A, the basic constructions that allows us to talk about the Stokes formula have an obvious A-version, mainly by replacing the tangent bundle TM by A; one obtains the notion of A-flat vector bundle (E, ∇) (implement the mentioned replacement in (4.29)), also known as representations of A, A-differential forms with values in E, $\Omega^\bullet(A, E) = C^\infty(M, \Lambda^\bullet A^* \otimes E)$, DeRham differential given by the same Koszul formula as above, the cohomology of A with coefficients in E, $H^\bullet(A, E)$; for details, see e.g. [18, 38]. Of course, considering only compactly supported sections one obtains the cohomology with compact supports (with coefficients in an arbitrary representation E), denoted

$$H_c^\bullet(A, E).$$

Via the anchor map $\sharp : A \to TM$, any flat vector bundle over M can be seen as a representation of A (this applies in particular to vector bundles of type $\mathfrak{o}_E$ where E is any vector bundle over M). One has the analogue of (4.28) describing the domain of definition for transverse measures:

$$C_c^\infty(M, \mathcal{D}_A) = \Omega_c^{\mathrm{top}}(A, \mathfrak{o}_A)$$

where top stands for the rank of A. Therefore we will use the flat vector bundle $\mathfrak{o}_A$ and the DeRham operator:

$$d_A : \Omega_c^{\mathrm{top}-1}(A, \mathfrak{o}_A) \to \Omega_c^{\mathrm{top}}(A, \mathfrak{o}_A).$$

Proposition 4.42. *Any transverse measure μ for $\mathcal{G}$ a Lie groupoid satisfies the Stokes formula*

$$\mu(d_A\omega) = 0 \quad \forall\, \omega \in \Omega_c^{\mathrm{top}-1}(A, \mathfrak{o}_A).$$

Equivalently, but at the level of cohomology: the quotient map

$$\Omega_c^{\mathrm{top}}(A, \mathfrak{o}_A) = C_c^\infty(M, \mathcal{D}_A) \to C_c^\infty(M//\mathcal{G})$$

descends to a surjection

$$VE : H_c^{\mathrm{top}}(A, \mathfrak{o}_A) \to C_c^\infty(M//\mathcal{G}). \tag{4.31}$$

The proof will be given in Subsection 4.7.3. Similarly to the classical case, this implies that a transverse measure μ descends to a linear functional on $H_c^{\mathrm{top}}(A, \mathfrak{o}_A)$ hence, composing with the pairing induced by wedge products, one obtains an $\mathbb{R}$-valued pairing,

$$\langle \cdot, \cdot \rangle_\mu : H^k(A) \times H_c^{\mathrm{top}-k}(A, \mathfrak{o}_A) \xrightarrow{\wedge} H_c^{\mathrm{top}}(A, \mathfrak{o}_A) \xrightarrow{\mu} \mathbb{R}.$$

called **the Poincare pairing induced by** μ. In many interesting examples this pairing is non-degenerate; however, that cannot happen in general since VE may fail to be an isomorphism (but see Theorem 4.44 below).

4.6.3 Reformulation in terms of (Ruelle-Sullivan) currents

The discussion above on the canonical integration of densities can be slightly reformulated in terms of currents [29]. Recall that a p-**current** on a manifold M is a continuous linear map

$$\xi : \Omega_c^p(M) \to \mathbb{R},$$

where the continuity is in the distributional sense, i.e., it uses the inductive limit topology arising from writing

$$\Omega^p_c(M) = \cup_{K-\text{compact}} \Omega^p_K(M),$$

where the space $\Omega^p_K(M)$ of p-forms supported in K is endowed with the topology of uniform convergence for all derivatives. The resulting spaces $\Omega_p(M)$ of p-currents fit into a chain complex

$$\Omega_0(M) \stackrel{d^*}{\to} \Omega_1(M) \stackrel{d^*}{\to} \dots$$

where $d^*(\xi) = \xi \circ d$. By construction, the resulting homology $H_\bullet(M)$ is in duality with the compactly supported DeRham cohomology; moreover, this pairing induces a canonical isomorphism

$$H_\bullet(M) \cong H^\bullet_c(M)^*. \tag{4.32}$$

Similarly one talks about currents on M with values in a flat vector bundle E and the homology $H_\bullet(M, E)$. Back to densities we see that the canonical integration becomes a top-current on M with coefficients in $\mathfrak{o}_M$, $\int_M \in \Omega_{\text{top}}(M, \mathfrak{o}_M)$, and the Stokes formula says that this is a closed current. Hence it gives rise to a completely canonical homology class

$$\left[\int_M\right] \in H_n(M, \mathfrak{o}_M), \tag{4.33}$$

where M is assumed to be connected, of dimension n. When M is compact and oriented, the orientation trivializes $\mathfrak{o}_M$ and $H_\bullet(M)$ is canonically isomorphic to singular homology (associate to a singular p-chain $\sigma : \Delta^p \to \mathbb{R}$ the p-current $C_\sigma(\omega) = \int_{\Delta^p} \sigma^*\omega$); with these identifications, $[\int_M]$ becomes the standard fundamental class $[M] \in H_n(M)$ of the compact oriented manifold M.

Remark 50. Note also that, via the isomorphism (4.32), the Poincare duality can now be stated in terms of currents as an isomorphism

$$H^\bullet(M) \cong H_{n-\bullet}(M, \mathfrak{o}_M). \tag{4.34}$$

In turn, this follows easily by a sheaf theoretic argument. Indeed, the complex computing $H_\bullet(M, \mathfrak{o}_M)$ can be arranged into a chain complex augmented by $\mathbb{R}$,

$$\mathbb{R} \to \Omega_n(M, \mathfrak{o}_M) \to \Omega_{n-1}(M, \mathfrak{o}_M) \to \dots\ ,$$

where the first map takes a scalar λ to $\lambda \cdot \int_M$. This complex is local, i.e., can be seen as a complex of sheaves over M; as such, it is actually a resolution of $\mathbb{R}$ by fine sheaves (basically the Poincare lemma), therefore it computes the cohomology of M, giving rise to the Poincare duality isomorphism (4.34).

The basic constructions from the previous discussion have an obvious generalization to algebroids, allowing us to talk about the space $\Omega_p(A, E)$ of A-currents of degree p with coefficients in a representation E of A and the resulting homology $H_\bullet(A, E)$. With this, a transverse measure μ can be interpreted as a top A-current, $\mu \in \Omega_{\text{top}}(A, \mathfrak{o}_A)$, and the Stokes formula can be reformulated as:

Corollary 4.43. *(reformulation of Proposition 4.42) Any transverse measure, interpreted as a top A-current, is closed.*

In particular, when A is oriented, composing with the anchor $\sharp$ induces a map

$$\sharp_* : \Omega_r(A) \to \Omega_r(M) \quad (r = \text{the rank of } A),$$

hence any transverse measure μ of $\mathcal{G}$ induces an r-current on M, called **the Ruelle-Sullivan current** on M induced by μ. Of course, the standard notion [92] is obtained in the case of foliations. Finally, we have the following converse of the previous corollary.

Proposition 4.44. *If $\mathcal{G}$ is a Lie groupoid with connected s-fibres then (4.31) is an isomorphism. As a consequence, transverse measures on $\mathcal{G}$ are the same thing as closed* top *A-currents with coefficients in $\mathfrak{o}_A$,*

$$\mu \in \Omega_{\text{top}}(A, \mathfrak{o}_A),$$

which are positive in the sense that, if $\rho \in \Omega_c^{\text{top}}(A, \mathfrak{o}_A) = C_c^\infty(M, \mathcal{D}_A)$ is positive as a density, then $\mu(\rho) \geq 0$.

The proof of this result and of Proposition 4.42 are given in the next section, once we gain more insight into the actual phenomena.

4.7 Cohomological insight and the Van Est isomorphism

Another way to understand the intrinsic model $C_c^\infty(M//\mathcal{G})$ (and then its relationship with algebroid cohomology, including the previous two propositions) is via a compactly supported version of differentiable cohomology. The relationship with the compactly supported cohomology of the algebroid (i.e., Propositions 4.42 and 4.44, and their proofs) will go via a compactly supported version of the Van Est isomorphism.

4.7.1 Differentiable cohomology with compact supports

A version of differentiable cohomology with compact supports was very briefly sketched in [18] (Remark 4 there). While *loc.cit* made use of a Haar system, in our context, there is a clear intrinsic model. For each $k \geq 0$ integer, we denote by $\mathcal{G}_k$ the manifold consisting of strings

$$x_0 \overset{g_1}{\leftarrow} x_1 \dots x_{k-1} \overset{g_k}{\leftarrow} x_k \tag{4.35}$$

of k composable arrows of $\mathcal{G}$. Recall that these spaces are related by the face maps

$$\delta_i : \mathcal{G}_k \to \mathcal{G}_{k-1},\ \delta_i(g_1, \dots, g_k) = \begin{cases} (g_2, \dots, g_k) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k) & \text{if } 1 \leq i \leq k-1 \\ (g_1, \dots, g_{k-1}) & \text{if } i = k \end{cases}$$

On each of the spaces $\mathcal{G}_k$ one considers the line bundle

$$\mathcal{D}^{k+1} = p_0^*\mathcal{D}_A \otimes \dots \otimes p_k^*\mathcal{D}_A$$

where $p_i : \mathcal{G}_k \to M$ ($0 \leq i \leq k$) takes a k-string (4.35) to x_i. The spaces $C_c^\infty(\mathcal{G}_k, \mathcal{D}^{k+1})$ come with the differential

$$\delta = \sum_{i=0} (-1)^i \delta_{i!} : C_c^\infty(\mathcal{G}_k, \mathcal{D}^{k+1}) \to C_c^\infty(\mathcal{G}_{k-1}, \mathcal{D}^k).$$

Since we end up with a chain complex we will talk about the **differentiable homology** of $\mathcal{G}$ and to use the notation $H_{\bullet}^{\mathrm{diff}}(\mathcal{G})$. However, it is sometimes useful to think of it as a cohomology with compact supports; then we use the notation

$$H_c^{\bullet}(\mathcal{G}) := H_{r-\bullet}(\mathcal{G}),$$

where r is the rank of the Lie algebroid of $\mathcal{G}$. For instance, in this way, the Van Est map will become a map between cohomologies with compact support, preserving the degree. With these, $C_c^{\infty}(M//\mathcal{G})$ is simply the homology in degree zero:

$$C_c^{\infty}(M//\mathcal{G}) = H_0^{\mathrm{diff}}(\mathcal{G}) = H_c^{r}(\mathcal{G}).$$

Let us also remark that, as in the case of differentiable cohomology, there is also a version $H_c^{\bullet}(\mathcal{G}, E) := H_{r-\bullet}(\mathcal{G}, E)$ with coefficients in a representation E of $\mathcal{G}$. The only subtlety is that (still as in the case of differentiable cohomology) one has to use the action of E in order to define the differential. For instance, in the lowest degree, while the integration over the s-fibres with coefficients,

$$s_!^E : C_c^{\infty}(\mathcal{G}, s^*E \otimes t^*\mathcal{D}_A \otimes s^*\mathcal{D}_A) \to C_c^{\infty}(M, E \otimes \mathcal{D}_A),$$

is defined exactly as (4.12)), for $t_!^E$ one first composes with the isomorphism $s^*E \cong t^*E$ induced by the action of $\mathcal{G}$ on E (so that $s_!^E$ and $t_!^E$ are defined on the same space). Similarly in higher degrees.

Example 4.45 (the case of submersions, continued)**.** Let us continue the discussion for the groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : P \to B$ (see Example 4.20). For this groupoid, the representations of $\mathcal{G}(\pi)$ are simply pullbacks of vector bundles E over B endowed with the tautological action (the arrow from x to y, which exists only when $\pi(x) = \pi(y)$, sends $(\pi^*E)_x = E_{\pi(x)}$ to $(\pi^*E)_y = E_{\pi(y)}$ by the identity map). Note that this can also be seen as a consequence of the fact that $\mathcal{G}(\pi)$ is Morita equivalent to B viewed as a groupoid with only unit arrows.

While Lemma 4.21 (and its obvious version with coefficients) can be seen as a computation of degree zero homology, we have the following (itself a particular case of Morita invariance, but used in the proof of the main theorem):

Lemma 4.46. *For the groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : P \to B$,*

$$H_c^k(\mathcal{G}(\pi), E) = H_{r-k}(\mathcal{G}(\pi), E) \cong \begin{cases} C_c^{\infty}(B, E) & \text{if } k = r \\ 0 & \text{otherwise} \end{cases}$$

where r is the rank of the Lie algebroid of $\mathcal{G}$.

This lemma is it the compactly supported version of Lemma 1 from [18], and it follows by the same type of arguments as there.

Theorem 4.47. *The homology $H_{\bullet}(\mathcal{G}, \cdot)$ is Morita invariant.*

Proof. The proof goes exactly as the proof of the Morita invariance of differentiable cohomology from [18]: given a Morita equivalence between $\mathcal{G}$ (over M) and $\mathcal{H}$ (over N), i.e., a principal bibundle P (with $\alpha : P \to M$, $\beta : P \to N$) one forms a double complex $C_{\bullet,\bullet}(P)$ together with quasi-isomorphisms

$$C_\bullet(\mathcal{G}) \stackrel{\int_\alpha}{\leftarrow} \mathrm{tot}(C_{\bullet,\bullet}(P)) \stackrel{\int_\beta}{\rightarrow} C_\bullet(\mathcal{H});$$

the fact that the two maps are quasi-isomorphism is ensured by the fact that each column $C_{p,\bullet}(P)$ comes with an augmentation $\int_\alpha : C_{p,\bullet}(P) \to C_p(\mathcal{G})$ compatible with the differentials, and similarly for the rows. Explicitly, $C_{p,q}(P)$ is built on the space $P_{p,q}$ - the subspace of $\mathcal{G}^p \times P \times \mathcal{H}^q$ consisting of elements $(g_1, \ldots, g_p, x, h_1, \ldots, h_q)$ with the property that the product $g_1 \ldots g_p x h_1 \ldots h_q$ is defined. On each such space one has a line bundle $\mathcal{D}_{p,q}$ defined as follows: one pulls-back the line bundle $\mathcal{D}_{\mathcal{G}}^{p+1}$ that appears in the definition of $C_\bullet(\mathcal{G})$ via the projection $P_{p,q} \to \mathcal{G}_p$, similarly for $\mathcal{D}_{\mathcal{H}}^{q+1}$; then, on P, one has the pullback algebroid $C = \alpha^! A \cong \beta^! B$ (see e.g. [18, Ex. 5]) and one pulls-back $\mathcal{D} := \mathcal{D}_C$ to $P_{p,q}$; with these,

$$\mathcal{D}_{p,q} = \mathcal{D}_{\mathcal{G}}^{p+1} \otimes \mathcal{D} \otimes \mathcal{D}_{\mathcal{H}}^{q+1}.$$

Then $C_{p,q}(P)$ is the space of compactly supported sections over this bundle. Fixing p, the differential of the p-column is defined so that it becomes the differentiable complex with compact supports of the action groupoid associated to the right action of $\mathcal{H}$ on $P_{p,0}$; since this action is principal, the previous lemma implies that the cohomology of the p-column is zero everywhere, except in degree zero where it is $C_p(\mathcal{G})$, with the isomorphism induced by $\int_\alpha$. Similarly for the rows, but using $\mathcal{H}$ instead of $\mathcal{G}$. Hence the conclusion follows. $\square$

Remark 51. Of course, the same holds in the presence of coefficients. Also, with arguments similar to the ones from [18], these isomorphisms are compatible with respect to the tensor product of bibundles (hence they are functorial). Even more, if P is only principal as an $\mathcal{H}$-bundle, i.e. a generalized morphism from $\mathcal{G}$ to $\mathcal{H}$, then a careful look at the previous argument (especially at the coefficients) gives rise to the "integration over the P-fibres map" (which is functorial in P).

Remark 52 (Co-invariants). The definition of $C_c^\infty(M//\mathcal{G})$ can also be thought of as making sense of "the space of $\mathcal{G}$-coinvariants associated to $C_c^\infty(M)$" (see Example 4.15). This viewpoint becomes important when looking for compactly supported versions of statements with no conditions on the support. For that purpose, we use a more suggestive notation and we extend the notion to a slightly larger setting. First of all, for a representation E of $\mathcal{G}$ define the space of coinvariants $C_c^\infty(M, E)_{\mathcal{G}-\mathrm{coinv}}$ (dual to the more obvious space $C^\infty(M, E)^{\mathcal{G}-\mathrm{inv}}$ of invariants) as

$$C_c^\infty(M, E)_{\mathcal{G}-\mathrm{coinv}} := H_0(\mathcal{G}, E).$$

Morally, while representations E of $\mathcal{G}$ can be thought of as vector bundles $E/\mathcal{G}$ over $M/\mathcal{G}$, $C^\infty(M, E)^{\mathcal{G}-\mathrm{inv}}$ plays the role of "$C^\infty(M/\mathcal{G}, E/\mathcal{G})$" and $C_c^\infty(E)_{\mathcal{G}-\mathrm{coinv}}$ plays the role of "$C_c^\infty(M/\mathcal{G}, E/\mathcal{G})$". These heuristics become precise in the case of the groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : P \to B$ when, by Lemma 4.21, the fibre integration induces an isomorphism

$$C_c^\infty(\pi^* E)_{\mathcal{G}(\pi)-\mathrm{coinv}} \cong C_c^\infty(B, E). \tag{4.36}$$

We will use the same notation in a slightly more general context: if $\mathcal{G}$ acts on a manifold P (say from the right) and E is a $\mathcal{G}$-equivariant bundle over P (i.e., $\mathcal{G}$ acts also on E, linearly in the fibres, compatible with the action on P). To put ourselves in the previous setting, one considers the associated action groupoid $P \rtimes \mathcal{G}$ and interprets E as a representation of it. The resulting space of $P \rtimes \mathcal{G}$-coinvariants will still be denoted by $C_c^\infty(P, E)_{\mathcal{G}-\mathrm{coinv}}$.

Corollary 4.48. If $\pi : P \to B$ is a principal $\mathcal{G}$-bundle over a manifold B, then $\mathcal{G}$-equivariant vector bundles on P are necessarily of type π^*E with E a vector bundle over B (so that the action on π^*E is the tautological action) and

$$C_c^\infty(P, \pi^*E)_{\mathcal{G}-\mathrm{coinv}} \cong C_c^\infty(B, E).$$

Moreover, in degrees $k > 0$, $H_k(P \rtimes \mathcal{G}, \pi^*E) = 0$.

Proof. In this situation one has a diffeomorphism $P \times_M \mathcal{G} \cong P \times_B P$, $(p, g) \mapsto (p, pg)$, and this identifies the action groupoid with the groupoid $\mathcal{G}(\pi)$. The statement becomes that of Lemma 4.46. □

Next we mention another important property of the differentiable cohomology with compact supports: the Van Est theorem.

Theorem 4.49. *For any Lie groupoid $\mathcal{G}$, with Lie algebroid denoted by A, and for any representation E of $\mathcal{G}$, there is a canonical map*

$$VE^\bullet : H_c^\bullet(A, E \otimes \mathfrak{o}_A) \to H_c^\bullet(\mathcal{G}, E).$$

Moreover, if $k \in \{0, 1, \ldots, r-1\}$ where r is the rank of A and if the fibres of s are homologically k-connected, then $VE^0, \ldots, VE^k$ are isomorphisms. The same is true for $k \geq r$ except for the case when some s-fibre is compact and orientable.

Example 4.50 (the case of submersions, continued)**.** Let us continue our discussion on the basic example associated to a submersion $\pi : P \to B$. In this case the algebroid is $A = \mathrm{Ker}\,(d\pi) = \mathcal{F}(\pi)$ - the foliation induced by π. While the resulting differentiable cohomology vanishes except in degree r (the rank of A), we are looking at a vanishing result for the foliated cohomology with compact supports. For trivial coefficients, the complex under discussion is

$$(\Omega_c^\bullet(\mathcal{F}(\pi), \mathfrak{o}_{\mathcal{F}(\pi)}), d_\pi)$$

where $d_\pi = d_{\mathcal{F}(\pi)}$ is now just DeRham differentiation along the fibres of π. One may think that the role of the orientation bundle is to make the top degree

$$\Omega_c^{\mathrm{top}}(\mathcal{F}(\pi), \mathfrak{o}_{\mathcal{F}(\pi)}) = C_c^\infty(P, \mathcal{D}_{\mathcal{F}(\pi)}),$$

the domain of the canonical fibre integration $\int_\pi$ with values in $C^\infty(B)$. For the version with coefficients, necessarily of type π^*E for some vector bundle E over B (cf. Example 4.45), $(\Omega_c^\bullet(\mathcal{F}(\pi), \mathfrak{o}_{\mathcal{F}(\pi)} \otimes \pi^*E), d_\pi)$ uses the flat $\mathcal{F}(\pi)$-connection which is the infinitesimal counterpart of the tautological action of $\mathcal{G}(\pi)$ on π^*E; this is uniquely characterized by the condition that sections of type π^*s are flat. The following particular case of the Van Est isomorphism will actually be used in the proof of the theorem. Note that it also clarifies why the degree r is special: for an r-dimensional manifold F, $H_c^0(F, \mathfrak{o}_F)$ is non-zero if and only if F is compact and orientable.

Lemma 4.51. *For the foliation $\mathcal{F}(\pi)$ associated to a submersion $\pi : P \to B$, if the fibres of π are homologically k-connected, where $k \in \{0, 1, \ldots, r-1\}$, then the following sequence is exact:*

$$\Omega_c^{r-k-1}(\mathcal{F}(\pi), \mathfrak{o}_{\mathcal{F}(\pi)} \otimes \pi^* E) \xrightarrow{d_\pi} \ldots \xrightarrow{d_\pi} \Omega_c^r(\mathcal{F}(\pi), \mathfrak{o}_{\mathcal{F}(\pi)} \otimes \pi^* E) \xrightarrow{\int_\pi} C_c^\infty(B, E) \to 0.$$

(hence the corresponding compactly supported cohomology is zero in degrees $r-k$, ..., $r-1$ and is $C_c^\infty(B, E)$ in degree r). The same is true for $k = r$ except for the case when some fibre of π is compact and orientable.

Proof. This is just the version of compact supports of Theorem 2 from [18] applied to the zero Lie algebroid; the first few lines of the proof in *loc.cit.* adapt immediately to compact supports. Alternatively, one can first show that the Poincare pairing (with respect to any positive measure on B) is non-degenerate and obtain our lemma as a consequence of the result from [18]. □

4.7.2 The compactly supported A-DeRham complex via forms along s-fibres

The definition of algebroid cohomology often comes with the remark that the defining complex can be interpreted as the complex of right-invariant forms along the s-fibres of $\mathcal{G}$. This makes the complex $\Omega^\bullet(A)$ into a subcomplex of the DeRham complex associated to the foliation $\mathcal{F}(s) := \mathrm{Ker}\,(ds)$ induced by s (fibrewise differential forms):

$$t^* : (\Omega^\bullet(A), d_A) \hookrightarrow (\Omega^\bullet(\mathcal{F}(s)), d_s),$$

(Hence $d_s = d_{\mathcal{F}(s)}$ is the DeRham differential of $\mathcal{F}(s)$ viewed as a Lie algebroid or, equivalently, DeRham differential along the fibres of s). Recalling that the right translations allow us to extend a section α of A to a vector field $\overrightarrow{\alpha}$ on $\mathcal{G}$ (tangent to the s-fibres) and to identify t^*A with $\mathcal{F}(s)$ (so that $t^*\alpha$ corresponds to $\overrightarrow{\alpha}$), the inclusion above identifies $\omega \in \Omega^\bullet(A)$ with the foliated form $\overrightarrow{\omega}$ defined by $\overrightarrow{\omega}(\overrightarrow{\alpha_1}, \ldots) = \omega(\alpha_1, \ldots)$; the foliated forms of type $\overrightarrow{\omega}$ are precisely the foliated forms that are right-invariant, i.e., $\omega \mapsto \overrightarrow{\omega}$ gives an isomorphism

$$t^* : (\Omega^\bullet(A), d_A) \cong (\Omega^\bullet(\mathcal{F}(s)), d_s)^{\mathcal{G}-\mathrm{inv}}.$$

For the later discussions, it is worth keeping in mind the structure that allows us to talk about $\mathcal{G}$-invariance: one has a space $P = \mathcal{G}$ together with a vector bundle $E = \mathcal{F}(s)$ over P, and $\mathcal{G}$ acts on both from the right: on P by right translations, while the action on $\mathcal{F}(s) = t^*A$ is the tautological one ($(t^*A)_{ag} = (t^*A)_a$).

We would like to understand the compactly supported counterpart of the previous discussion; the main point is that, in the resulting dual picture, subcomplexes (like $\Omega^\bullet(A)$ sitting as a subcomplex of $(\Omega^\bullet(\mathcal{F}(s)), d_s)$) will turn into quotient complexes, the inclusions will turn to into integrations over fibres and invariants into coinvariants. Let us also allow as coefficients any representation E of A. The relevant complex at the level of $\mathcal{G}$ is then

$$(\Omega_c^\bullet(\mathcal{F}(s), t^*E \otimes s^*\mathcal{D}_A), d_s),$$

where $\mathcal{F}(s)$ is the foliation induced by the source map s. Regarding the coefficients, we have mentioned above that pullbacks by s are canonically representations of $\mathcal{F}(s)$; for t^*E, the $\mathcal{F}(s)$-action is the pullback of the action of A on E: $\nabla_{\overrightarrow{\alpha}}(t^*e) = t^*\nabla_\alpha(e)$ for $\alpha \in C^\infty(M, A)$ and

$e \in C^\infty(M, E)$). Recalling the identifications $\mathcal{F}(s) = t^*A$, $s^*A = \mathcal{F}(t)$, we see that

$$\Omega_c^\bullet(\mathcal{F}(s), t^*E \otimes s^*\mathcal{D}_A) = C_c^\infty(\mathcal{G}, t^*(\Lambda^\bullet A^* \otimes E) \otimes \mathcal{D}_{\mathcal{F}(t)})$$

hence the integration along the t-fibres makes sense as a map

$$\int_t : \Omega_c^\bullet(\mathcal{F}(s), t^*E \otimes s^*\mathcal{D}_A) \to C_c^\infty(M, \Lambda^\bullet A^* \otimes E) = \Omega_c^\bullet(A, E). \tag{4.37}$$

Note also that we are in the position of talking about $\mathcal{G}$-coinvariants of $\Omega_c^\bullet(\mathcal{F}(s), t^*E)$, in the sense of Remark 52 where, as above, $\mathcal{G}$ acts on $\mathcal{G}$ from the right and $\mathcal{F}(s)$ is viewed as an equivariant $\mathcal{G}$-bundle over $\mathcal{G}$.

Lemma 4.52. *The map $\int_t$ is a surjective morphism of cochain complexes which descends to an isomorphism*

$$\Omega_c^\bullet(\mathcal{F}(s), t^*E \otimes s^*\mathcal{D}_A)_{\mathcal{G}-\mathrm{coinv}} \xrightarrow{\sim} \Omega_c^\bullet(A, E).$$

Proof. The fact that $\int_t$ descends to a degreewise isomorphism follows from Corollary 4.48 since the action of $\mathcal{G}$ on $\mathcal{G}$ is principal, with quotient map $t : \mathcal{G} \to M$. The main issue is the compatibility with the differentials. For that we use the Koszul formula for the differential d_A (and the analogous formula for d_s):

$$\begin{aligned} \mathrm{d}_A\omega(\alpha_1, \ldots, \alpha_{k+1}) = &\sum_i (-1)^{i+1} \nabla_{\alpha_i}(\omega(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_{k+1})) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_{k+1}), \end{aligned}$$

where ∇ stands for the action of A on E. Given $\omega \in \Omega_c^\bullet(\mathcal{F}(s), t^*E \otimes s^*\mathcal{D}_A)$, denote $\overline{\omega} = \int_t \omega \in \Omega_c^\bullet(A, E)$. Explicitly, for $\alpha_1, \ldots, \alpha_k \in C^\infty(M, A)$, $\omega(\overrightarrow{\alpha_1}, \ldots, \overrightarrow{\alpha_k}) \in C_c^\infty(\mathcal{G}, t^*E \otimes \mathcal{D}_{\mathcal{F}(t)})$ and

$$\overline{\omega}(\alpha_1, \ldots, \alpha_k) = \int_t \omega(\overrightarrow{\alpha_1}, \ldots, \overrightarrow{\alpha_k}) \in C^\infty(M, E).$$

We have to show that $\overline{d_s\omega} = d_A\overline{\omega}$. Writing this out using the previous formulas, and using that $[\overrightarrow{\alpha}, \overrightarrow{\beta}] = \overrightarrow{[\alpha, \beta]}$, we see that the identities we have to check will follow provided we prove the commutativity of the following diagram:

$$\begin{tikzcd} C_c^\infty(\mathcal{G}, t^*E \otimes \mathcal{D}_{\mathcal{F}(t)}) \arrow[r, "\nabla_{\overrightarrow{\alpha}}"] \arrow[d, "\int_t"'] & C_c^\infty(\mathcal{G}, t^*E \otimes \mathcal{D}_{\mathcal{F}(t)}) \arrow[d, "\int_t"] \\ C_c^\infty(M, E) \arrow[r, "\nabla_\alpha"'] & C_c^\infty(M, E) \end{tikzcd}$$

Writing the elements in the right upper corner as $t^*e \otimes \rho$ with $e \in C^\infty(M, E)$, $\rho \in C_c^\infty(M, \mathcal{D}_{\mathcal{F}(t)})$ (which are mapped by $\int_t$ to $e \cdot \int_t \rho$) we see that one may assume that E is the trivial representation. In this case the bottom ∇_α becomes the Lie derivative along $\sharp\alpha$, the image of α by the anchor map of A. For the upper horizontal arrow, $\nabla_{\overrightarrow{\alpha}}$, recall that it is the canonical $\mathcal{F}(s)$-connection on $s^*A = \mathcal{F}(t)$, hence it is uniquely determined by the Leibniz identity and the fact that sections of type $s^*\beta \cong \overleftarrow{\beta}$ are flat (where $\overleftarrow{\beta}$ is the left-invariant vector field induced by

β). On the other hand, since $[\overrightarrow{\alpha}, \overleftarrow{\beta}] = 0$ holds in general, we see that the usual Lie derivative $L_{\overrightarrow{\alpha}}$ (defined as variations along the flow $\phi^{\epsilon}_{\overrightarrow{\alpha}}$) has exactly the same properties; hence $\nabla_{\overrightarrow{\alpha}} = L_{\overrightarrow{\alpha}}$. Therefore, denoting $\overrightarrow{\alpha} = \tilde{V}$, $\sharp\alpha = V$, the new diagram becomes

$$\begin{tikzcd} C^{\infty}_c(M, \mathcal{D}_{\mathcal{F}(t)}) \arrow[r, "L_{\tilde{V}}"] \arrow[d, "\int_t"'] & C^{\infty}_c(M, \mathcal{D}_{\mathcal{F}(t)}) \arrow[d, "\int_t"] \\ C^{\infty}_c(M) \arrow[r, "L_V"'] & C^{\infty}_c(M) \end{tikzcd}$$

As such, the diagram is commutative for any submersion $t : \mathcal{G} \to M$ between two manifolds and any vector field $\tilde{V}$ on $\mathcal{G}$ that is t-projectable to V. which, in turn, can be interpreted as the infinitesimal counterpart of the diagram involving the flows:

$$\begin{tikzcd} C^{\infty}_c(M, \mathcal{D}_{\mathcal{F}(t)}) \arrow[r, "\phi^{\epsilon}_{\tilde{V}}"] \arrow[d, "\int_t"'] & C^{\infty}_c(M, \mathcal{D}_{\mathcal{F}(t)}) \arrow[d, "\int_t"] \\ C^{\infty}_c(M) \arrow[r, "\phi^{\epsilon}_V"'] & C^{\infty}_c(M) \end{tikzcd},$$

The commutativity of the last diagram (hence also of the others) follows immediately from the fact that the flow of $\tilde{V}$ covers the flow of V and the invariance of the integration of densities. □

4.7.3 The missing proofs

Here we give the proofs of the Propositions 4.44 and 4.42 and of Theorem 4.49.

Proof. (of Proposition 4.42) We have to show that

$$\mathrm{Im}(d_A : \Omega^{r-1}_c(A, \mathfrak{o}_A) \to \Omega^r_c(A, \mathfrak{o}_A)) \subset \mathrm{Im}(s_! - t_!).$$

Let $u = d_A(v)$ in the left hand side; to show that it is on the right hand side, we will use the following diagram

$$\begin{tikzcd} C^{\infty}_c(M, \mathcal{D}) & C^{\infty}_c(\mathcal{G}, \mathcal{D}^2) = \Omega^r_c(\mathcal{F}(s), \mathfrak{o}_{\mathcal{F}(s)} \otimes s^*\mathcal{D}) \arrow[l, "s_!"'] \arrow[d, "t_!"'] & \Omega^{r-1}_c(\mathcal{F}(s), \mathfrak{o}_{\mathcal{F}(s)} \otimes s^*\mathcal{D}) \arrow[l, "d_s"'] \arrow[d, "t_!"'] \\ & \Omega^r_c(A, \mathfrak{o}_A) & \Omega^{r-1}_c(A, \mathfrak{o}_A) \arrow[l, "d_A"'] \end{tikzcd}$$

In this diagram, the first horizontal sequence is of the type considered in Lemma 4.51 (hence the composition of the two maps is zero, and the sequence is even exact if the s-fibres are connected). The vertical maps are the ones of type (4.37) (for $E = \mathfrak{o}_A$), they are surjective and the square is commutative by Lemma 4.52. We can now look at v as an element sitting in the bottom right corner and write it as $\int_t \xi$ for some ξ in the upper right corner. Consider then $w = d_s(\xi)$. It follows that $u = t_!(w)$ and $s_!(w) = 0$. Hence u is in the image of $s_! - t_!$. □

Proof. (of Proposition 4.44) Here we have to prove the reverse inclusion. So, let $u \in \mathrm{Im}(s_! - t_!)$. It is clear that the argument from the previous proof can be reversed (because the s-fibres are connected), to conclude that $u \in \mathrm{Im}(d_A)$, provided we can show that we can write $u = t_! w$ for some w that is killed by $s_!$. For that we work on another diagram:

$$\begin{tikzcd}
C_c^\infty(\mathcal{G}, \mathcal{D}^2) \arrow[d, "\delta"'] & C_c^\infty(\mathcal{G}_2, \mathcal{D}^3) \arrow[l, "\delta_{0\,!}"'] \arrow[d, "\delta'"'] \\
C_c^\infty(M, \mathcal{D}) & C_c^\infty(\mathcal{G}, \mathcal{D}^2) = C_c^\infty(\mathcal{G}, t^*\mathcal{D}_A \otimes s^*\mathcal{D}_A) \arrow[l, "s_!"'] \arrow[d, "t_!"'] \\
 & C_c^\infty(M, \mathcal{D}_A)
\end{tikzcd}$$

where $\delta = s_! - t_!$, $\delta' = \delta_{1\,!} - \delta_{2\,!}$ and $\delta_i : \mathcal{G}_2 \to \mathcal{G}$ (defined in general in Subsection 4.7.1) are given by

$$\delta_0(g, h) = h,\ \ \delta_1(g, h) = gh,\ \ \delta_2(g, h) = g.$$

The functoriality of the fibre integration (Fubini) and the obvious identities $s \circ \delta_0 = s \circ \delta_1$ and $t \circ \delta_0 = s \circ \delta_2$ imply that the diagram is commutative. Moreover, the left vertical sequence is exact by Lemma 4.46 applied to $E = \mathcal{D}_A$ since it is part of the sequence computing the compactly supported differentiable cohomology with coefficients for the action groupoid associated to the right action of $\mathcal{G}$ on $\mathcal{G}$. Look now at u sitting in the lowest left corner; the hypothesis is that $u = \delta(v)$ for some v. Write $v = \delta_!^0(\xi)$ for some $\xi \in C_c^\infty(\mathcal{G}_2, \mathcal{D}^3)$ and consider $w' = \delta'(\xi)$, Then $u = s_!(w')$ and also $t_!(w') = 0$ since $t_! \circ \delta' = 0$. Of course, using the inversion ι of the groupoid, $w = \iota^*(w')$ will have the desired properties. $\square$

Proof. (of the Van Est isomorphism - Theorem 4.49) For notational simplicity we assume that E is the trivial representation. As in [18], we use a double complex argument. The (augmented) double complex will extend the diagrams from the previous two proofs:

$$\begin{tikzcd}
\cdots \arrow[d, "\delta"'] & \cdots \arrow[d, "\delta"'] & \cdots \arrow[d, "\delta"'] & \cdots \\
C_c^\infty(\mathcal{G}_2, \mathcal{D}^3) \arrow[d, "\delta"'] & C_c^\infty(\mathcal{G}_3, \mathcal{D}^4) \arrow[l, "\delta_{0\,!}"'] \arrow[d, "\delta"'] & C_{1,2} \arrow[l, "d_{\delta_0}"'] \arrow[d, "\delta"'] & \cdots \arrow[l, "d_{\delta_0}"'] \\
C_c^\infty(\mathcal{G}, \mathcal{D}^2) \arrow[d, "\delta"'] & C_c^\infty(\mathcal{G}_2, \mathcal{D}^3) \arrow[l, "\delta_{0\,!}"'] \arrow[d, "\delta"'] & C_{1,1} \arrow[l, "d_{\delta_0}"'] \arrow[d, "\delta"'] & \cdots \arrow[l, "d_{\delta_0}"'] \\
C_c^\infty(M, \mathcal{D}) & C_c^\infty(\mathcal{G}, \mathcal{D}^2) = \Omega_c^r(\mathcal{F}(s), \mathfrak{o}_{\mathcal{F}(s)} \otimes s^*\mathcal{D}) \arrow[l, "\delta_{0\,!} = s_!"'] \arrow[d, "t_!"'] & \Omega_c^{r-1}(\mathcal{F}(s), \mathfrak{o}_{\mathcal{F}(s)} \otimes s^*\mathcal{D}) \arrow[l, "d_s"'] \arrow[d, "t_!"'] & \cdots \arrow[l, "d_s"'] \\
 & \Omega_c^r(A, \mathfrak{o}_A) & \Omega_c^{r-1}(A, \mathfrak{o}_A) \arrow[l, "d_A"'] & \cdots \arrow[l, "d_A"']
\end{tikzcd}$$

The first vertical row and the bottom horizontal one compute the cohomologies from the statement, and they are the augmentations of the rows/columns of the actual double complex $C_{\bullet,\bullet}$. Explicitly, the double complex is defined as:

$$C_{i,k} = C_c^\infty(\mathcal{G}_{k+1}, p_0^*(\Lambda^{r-i} A^* \otimes \mathfrak{o}_A) \otimes p_1^*\mathcal{D}_A \otimes \ldots \otimes p_{k+1}^*\mathcal{D}_A).$$

For the differentials, let us look at the columns and the rows separately. Start with the i^{th} column, for a fixed i. We will describe it as the complex computing the compactly supported cohomology of a groupoid $\tilde{\mathcal{G}}$ with coefficients in a representation that depends on i. For that, we consider the action of $\mathcal{G}$ on itself from the right, which makes $\mathcal{G}$ into a (right) principal $\mathcal{G}$-bundle over M with projection $t : \mathcal{G} \to M$,

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "t"'] \arrow[drr, "s"] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\ M & & & M \end{tikzcd}$$

Define $\tilde{\mathcal{G}}$ to be the groupoid associated to this action. Hence $\tilde{\mathcal{G}} = \mathcal{G}_2$ viewed as a groupoid over $\mathcal{G}$ with

$$(g, h) \xrightarrow{s=\delta_1} gh, \ (g, h) \xrightarrow{t=\delta_2} g$$

and with the multiplication

$$(g, h) \cdot (gh, k) = (g, hk).$$

The algebroid $\tilde{A}$ of $\tilde{\mathcal{G}}$ is then (because of the principal bundle setting)

$$\tilde{A} = \mathcal{F}(t) \cong s^*A.$$

The representation of $\tilde{\mathcal{G}}$ that is relevant here is a tautological one, i.e., coming by the pullback along the principal bundle projection, i.e., by t; more precisely, it is:

$$E_i = t^*(\Lambda^{r-i}A^* \otimes \mathfrak{o}_A).$$

It is now straightforward to check that, as vector spaces,

$$C_{i,k} = C^k_c(\tilde{\mathcal{G}}, E_i);$$

this defines the augmented i^{th} column; by Corollary 4.48, this augmented column will be exact.

We now fix k and we describe the k^{th} row. For that, we first remark that $\delta_0 : \mathcal{G}_{k+1} \to \mathcal{G}_k$ is a left principal $\mathcal{G}$-bundle over M, with the action defined along p_0:

$$\begin{tikzcd} \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & \mathcal{G}_{k+1} \arrow[dl, "p_0"'] \arrow[dr, "\delta_0"] & \\ M & & \mathcal{G}_k \end{tikzcd}$$

This gives an identification of $p_0^*(A)$ with the foliation $\mathcal{F}(\delta_0)$, so that

$$C_{i,k} = \Omega^{r-i}_c(\mathcal{F}(\delta_0), \mathfrak{o}_{\mathcal{F}(\delta_0)} \otimes p_1^*\mathcal{D}_A \otimes \ldots \otimes p_{k+1}^*\mathcal{D}_A) = \Omega^{r-i}_c(\mathcal{F}(\delta_0), \mathfrak{o}_{\mathcal{F}(\delta_0)} \otimes (\delta_0)^*\mathcal{D}_{k+1}).$$

We see that we deal with spaces of the type that appear in Lemma 4.51 - and this interpretation describes the horizontal differentials d_{δ_0} so that (by the lemma) the augmented rows have vanishing cohomology in the relevant degrees.

Of course, one still has to check the compatibility of the (augmented) horizontal and the vertical differentials; in low degree, this is contained in the previous two proofs; in arbitrary degrees it is tedious but straightforward (and rather standard- see e.g. [18]); the trickiest part is for the squares in the bottom (involving $t_!$), but that was dealt with in Lemma 4.52. $\square$

4.8 The modular class(es) revisited

The definition of the modular class of a Lie algebroid always comes with the slogan, inspired by various examples, that it is "the obstruction to the existence of a transverse measure". Here we would like to point out that the transverse density bundle $\mathcal{D}_A^{\mathrm{tr}}$ and our discussions makes this slogan precise. In particular, we point out that the canonical representation Q_A [38] (see also below) that is commonly used in the context of modular classes should actually be replaced by $\mathcal{D}_A^{\mathrm{tr}}$.

Throughout this section $\mathcal{G}$ is a Lie groupoid over M and A is its Lie algebroid. We will be using the transverse density bundle $\mathcal{D}_A^{\mathrm{tr}}$, volume bundle $\mathcal{V}_A^{\mathrm{tr}}$ and orientation bundle $\mathfrak{o}_A^{\mathrm{tr}}$, viewed as representations of $\mathcal{G}$ as explained in Section 4.3. Let us mention, right away, the relation between these bundles. As vector bundles over M, we already know (see Subsection 4.1.2) that there are canonical vector bundle isomorphism between

- $\mathcal{D}_A^{\mathrm{tr}}$ and $\mathcal{V}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$.
- $\mathcal{V}_A^{\mathrm{tr}}$ and $\mathcal{D}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$.
- $\mathfrak{o}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$ and the trivial line bundle.
- $\mathfrak{o}_A^{\mathrm{tr}}$ and $(\mathfrak{o}_A^{\mathrm{tr}})^*$.

Lemma 4.53. *All these canonical isomorphisms are isomorphisms of representations of $\mathcal{G}$ (where the trivial line bundle is endowed with the trivial action, i.e., it is the trivial representation of $\mathcal{G}$).*

Proof. Given the way that the action of $\mathcal{G}$ was defined (Section 4.3), the direct check can be rather lengthy and painful. Here is a more conceptual way to do it. The main remark is that these actions can be defined in general, whenever we have a functor F which associates to a vector space V a 1-dimensional vector space $F(V)$ and to a (linear) isomorphism $f : V \to W$ an isomorphism $F(f) : F(V) \to F(W)$ such that:

1. F commutes with taking duals, i.e., denoting by D the functor that takes duals, the functors $F \circ D$ and $D \circ F$ are isomorphic through a natural transformation $\eta : F \circ D \to D \circ F$ (hence η is an isomorphism on each V).
2. for any short exact sequence $0 \to U \to V \to W \to 0$ there is an induced isomorphism between $F(V)$ and $F(U) \otimes F(W)$, which is natural in the obvious sense.

Let us call such F's "good functors". The construction from Section 4.3 shows that for any good functor F,

$$F_A^{\mathrm{tr}} := F(A^*) \otimes F(TM)$$

is a representation of $\mathcal{G}$. Given two good functors F and F', an isomorphism $\eta : F \to F'$ will be called good if it is compatible with the natural transformations from 1. and 2. above. It is clear then that, for any such η, there is an induced map η^{tr} that is an isomorphism between F_A^{tr} and ${F'}_A^{\mathrm{tr}}$, as representations of $\mathcal{G}$. It should also be clear that, for any two good functors F and F', so is their tensor product. We see that we are left with proving that certain isomorphisms involving the functors $\mathcal{D}$, $\mathcal{V}$ and $\mathfrak{o}$ (e.g. $\mathcal{D} \cong \mathcal{V} \otimes \mathfrak{o}$) are good in the previous sense; and that is straightforward. □

4.8.1 The modular class of $\mathcal{G}$

Let us concentrate on the question of whether $\mathcal{G}$ admits a strictly positive transverse density (these are the "measures" in the slogan mentioned at the start of the section, or "geometric measures" in our terminology). First of all, choose any strictly positive section σ of $\mathcal{D}_A^{\mathrm{tr}}$. Then any other such section is of type $e^f\sigma$ for some $f \in C^\infty(M)$; moreover $e^f\sigma$ is invariant if and only if

$$e^{f(y)}\sigma(y) = e^{f(x)}g(\sigma(x))$$

for all $g : x \to y$ an arrow of $\mathcal{G}$. Considering

$$c_\sigma(g) := ln\left(\frac{\sigma(y)}{g(\sigma(x))}\right),$$

we obtain a smooth function $c_\sigma \in C^\infty(\mathcal{G})$ and one can check right away that it is a 1-cocycle, i.e.,

$$c_\sigma(gh) = c_\sigma(g) + c_\sigma(h)$$

for all g and h composable. The condition on f that we were considering reads:

$$f(x) - f(y) = c_\sigma(g)$$

for all $g : x \to y$, i.e., $c_\sigma = \delta(f)$ in the differentiable cohomology complex $(C^\bullet_{\mathrm{diff}}(\mathcal{G}), \delta)$. Furthermore, an easy check shows that the class $[c_\sigma] \in H^1_{\mathrm{diff}}(\mathcal{G})$ does not depend on the choice of σ. Therefore it gives rise to a canonical class

$$\mathrm{mod}(\mathcal{G}) \in H^1_{\mathrm{diff}}(\mathcal{G}),$$

called **the modular class of the Lie groupoid** $\mathcal{G}$, such that:

Lemma 4.54. *$\mathcal{G}$ admits a strictly positive transverse density iff* $\mathrm{mod}(\mathcal{G}) = 0$.

With this, Proposition 4.28 (and its proof) is just about the vanishing of differentiable cohomology of proper groupoids.

The construction of $\mathrm{mod}(\mathcal{G})$ can be seen as a very particular case of the construction from [18] of characteristic classes of representations of $\mathcal{G}$, classes that live in the odd differentiable cohomology of $\mathcal{G}$. A similar construction of the modular class of $\mathcal{G}$ appears in the context of modular classes of representations up to homotopy [67]. We are interested here only on the classes in degree one associated to 1-dimensional representations L, denoted here

$$\theta_{\mathcal{G}}(L) \in H^1_{\mathrm{diff}}(\mathcal{G}).$$

Here is the direct description, similar to definition of $\mathrm{mod}(\mathcal{G})$. Assume first that L is trivializable as a vector bundle and we choose a nowhere vanishing section σ. Then, for $g : x \to y$, we can write

$$g(\sigma(x)) = \tilde{c}_\sigma(g)\sigma(x) \qquad (\tilde{c}_\sigma(g) \in \mathbb{R}^*)$$

and this defines a function

$$\tilde{c}_\sigma : \mathcal{G} \to \mathbb{R}^* \tag{4.38}$$

that is a groupoid homomorphism. The cocycle of interest is

$$c_\sigma = \ln(|\tilde{c}_\sigma|) : \mathcal{G} \to \mathbb{R}; \tag{4.39}$$

its cohomology class does not depend on the choice of σ and defines $\theta_{\mathcal{G}}(L)$. It is clear that for two such representations L_1 and L_2 (trivializable as vector bundles) one has

$$\theta_{\mathcal{G}}(L_1 \otimes L_2) = \theta_{\mathcal{G}}(L_1) + \theta_{\mathcal{G}}(L_2). \tag{4.40}$$

This indicates how to proceed for a general L: consider the representation $L \otimes L$ which is (noncanonically) trivializable and define:

$$\theta_{\mathcal{G}}(L) := \frac{1}{2}\theta_{\mathcal{G}}(L \otimes L). \tag{4.41}$$

Note that the multiplicativity formula for $\theta_{\mathcal{G}}$ remains valid for all L_1 and L_2. Note that, by construction,

Lemma 4.55. *One has* $\mathrm{mod}(\mathcal{G}) = \theta_{\mathcal{G}}(\mathcal{D}_A^{\mathrm{tr}})$.

Remark 53 (a warning). One should be aware that it is not true (even if L is trivializable as a vector bundle) that $\theta_{\mathcal{G}}(L)$ is the obstruction to L being isomorphic to the trivial representation. The fact that Lemma 4.54 holds is due to the fact that the transverse density bundle is more than trivializable: one can also talk about positivity of sections of $\mathcal{D}_A^{\mathrm{tr}}$. In other words, $\mathcal{D}_A^{\mathrm{tr}}$ is an oriented representation of $\mathcal{G}$ (with the action preserving the orientation) which can be trivialized as an oriented bundle.

The tendency in existing literature, at least for the infinitesimal version of the modular class (see below), is to use simpler representations instead of $\mathcal{D}_A^{\mathrm{tr}}$. Here we would like to clarify the role of the transverse volume bundle $\mathcal{V}_A^{\mathrm{tr}}$: can one use it to define $\mathrm{mod}(\mathcal{G})$? In short, the answer is: yes, but one should not do it because it would give rise to the wrong expectations (because of the previous warning!). We summarize this into the following:

Proposition 4.56. *For any groupoid $\mathcal{G}$,* $\mathrm{mod}(\mathcal{G}) = \theta_{\mathcal{G}}(\mathcal{V}_A^{\mathrm{tr}})$. *However, it is not true that that* $\mathrm{mod}(\mathcal{G}) = 0$ *happens if and only if $\mathcal{G}$ admits a transverse volume form (i.e., a nowhere vanishing $\mathcal{G}$-invariant section of $\mathcal{V}_A^{\mathrm{tr}}$).*

For the last part note that counterexamples are already provided by manifolds M, viewed as groupoids with unit arrows only. Indeed, in this case the associated transverse (density, volume) bundles are the usual bundles of M; hence the modular class is zero even if M is not orientable. For the first part of the proposition, using the multiplicativity (4.40) of $\theta_{\mathcal{G}}$ and the canonical isomorphisms discussed at the beginning of the section, we have to show that

$$\theta_{\mathcal{G}}(\mathfrak{o}_A^{\mathrm{tr}}) = 0. \tag{4.42}$$

In turn, this follows by applying again the multiplicativity of $\theta_{\mathcal{G}}$ and the canonical isomorphism between $\mathfrak{o}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$ and the trivial representation.

4.8.2 The modular class of A (and why not to use Q_A)

The construction of the modular class of a Lie algebroid A, introduced by Evens, Lu and Weinstein [38], is based on the geometry of a certain 1-dimensional representation of the Lie algebroid A. First of all there is an infinitesimal version of the construction of the characteristic class $\theta_{\mathcal{G}}(L)$ of a groupoid representation, version that associates to any 1-dimensional representation L of A a class $\theta_A(L) \in H^1(A)$. In a bit more detail: first one uses the analogue of (4.41) to reduce the construction to the case when L is trivializable as a vector bundle; then, for such L, one chooses a nowhere vanishing section σ and one writes the infinitesimal action ∇ of A on L as

$$\nabla_\alpha(\sigma) = c_\sigma(\alpha) \cdot \sigma,$$

therefore defining c_σ as an element

$$c_\sigma(L) \in \Omega^1(A).$$

Similar to the previous discussion, the flatness of ∇ implies that $c_\sigma(L)$ is a closed A-form and its cohomology class does not depend on the choice of σ; therefore it defines a class, called the characteristic class of L, and denoted

$$\theta_A(L) \in H^1(A).$$

Note that the situation is simpler than at the level of $\mathcal{G}$: for L trivializable as a vector bundle, $\theta_A(L) = 0$ if and only if L is isomorphic to the trivial representation of A (compare with the warning from Remark 53!).

The previous construction can be applied to various (canonical) 1-dimensional representations of A, all producing the same result. For instance, since the representations of $\mathcal{G}$ are automatically representations of A (by differentiation), one could use the transverse density and volume bundles of A. However, the common choice in the existing literature (starting with [38]) is the line bundle

$$Q_A = \Lambda^{\mathrm{top}} A \otimes |\Lambda^{\mathrm{top}} T^* M|.$$

The way in which this is a representation of A is explained in [38]; equivalently, one writes

$$Q_A = \mathcal{D}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A \tag{4.43}$$

in which both terms are representations of A: the transverse density bundle was already discussed, while $\mathfrak{o}_A$ is a representation of A since it is a flat vector bundle over M. With this, one defines:

Definition 4.57. The **modular class of a Lie algebroid** A, denoted by $\mathrm{mod}(A)$, is the characteristic class of Q_A.

However, we would like to point out that the choice of Q_A is rather unfortunate. One of the reasons is the fact that, in general, Q_A is not a representation of the groupoid $\mathcal{G}$. Indeed, with (4.43) in mind, we are looking at whether $\mathfrak{o}_A$ can be made into a representation of $\mathcal{G}$; but $\mathfrak{o}_A$ are the typical examples of algebroid representations that do not come from groupoid ones. That is clear already in the case of the pair groupoid of a manifold M, whose representations are automatically trivial as vector bundles, but for which $\mathfrak{o}_A = \mathfrak{o}_{TM}$ is not trivializable if M is not orientable.

Another reason that the previous definition is not the best one is the fact that it may give rise to the wrong expectations regarding the interpretation of $\mathrm{mod}(A)$ as an obstruction class (and, indeed, there are several claims in the existing literature that are not quite right). Our message is that one should use $\mathcal{D}_A^{\mathrm{tr}}$ or $\mathcal{V}_A^{\mathrm{tr}}$ (to resolve the first issue mentioned above), with a preference for $\mathcal{D}_A^{\mathrm{tr}}$ (because it also clarifies the second issue). The fact that, to define the modular class of the Lie algebroid, one can use any of these representations is ensured by the following:

Lemma 4.58. *The representations Q_A, $\mathcal{D}_A^{\mathrm{tr}}$ and $\mathcal{V}_A^{\mathrm{tr}}$ of A have the same characteristic class (namely $\mathrm{mod}(A)$).*

The fact that the representation $\mathcal{D}_A^{\mathrm{tr}}$ of $\mathcal{G}$ (!) can be used to compute the modular class immediately implies:

Corollary 4.59. *For any Lie groupoid $\mathcal{G}$, the Van Est map in degree 1,*

$$VE : H^1_{\mathrm{diff}}(\mathcal{G}) \to H^1(A)$$

sends the modular class of $\mathcal{G}$ to the modular class of A. In particular, if A is integrable by a unimodular Lie groupoid (e.g. by a proper Lie groupoid), then its modular class vanishes.

Finally, the other advantage of $\mathcal{D}_A^{\mathrm{tr}}$, namely that it is trivializable as a vector bundle, allows for a characterization of the modular class as an obstruction class:

Corollary 4.60. *For an integrable Lie algebroid A, if the s-fibres of $\mathcal{G}$ are 1-connected,* $\mathrm{mod}(A)$ *is the obstruction to the existence of a strictly positive transverse density of $\mathcal{G}$.*

Proof. (of the Lemma 4.58 and the corollaries) Given the isomorphisms $\mathcal{D}_A^{\mathrm{tr}} \cong \mathcal{V}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$ and (4.43) and the multiplicativity of θ_A, to prove the lemma it suffices to show that $\theta_A(\mathfrak{o}_A^{\mathrm{tr}}) = 0$ and similarly for $\mathfrak{o}_A$; using again multiplicativity, it suffices to show that $\theta_A(\mathfrak{o}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}) = 0$ - which is true because $\mathfrak{o}_A^{\mathrm{tr}} \otimes \mathfrak{o}_A^{\mathrm{tr}}$ is isomorphic to the trivial representation (Lemma 4.53). And similarly for $\mathfrak{o}_A$ just that, this time, $\mathfrak{o}_A \otimes \mathfrak{o}_A$ is isomorphic to the trivial line bundle already as flat vector bundles over M. Corollary 4.59 is just a particular case of [18]. For Corollary 4.60 we use the fact that the assumptions on the s-fibres implies that VE from the previous corollary is an isomorphism and the obstruction property of $\mathrm{mod}(\mathcal{G})$ (Lemma 4.54). □

4.8.3 Transverse orientability and the first Stiefel-Whitney class of $\mathcal{G}$

It is interesting to look back at the construction of the characteristic class $\theta_{\mathcal{G}}(L)$ of a 1-dimensional representation L of $\mathcal{G}$. The reason for the warning mentioned in Remark 53 comes from the fact that, when passing from $\tilde{c}_\sigma$ to c_σ in (4.39), one loses information related to orientability.

We return to the discussion around (4.39); in particular, we assume that L is a 1-dimensional representation of $\mathcal{G}$ that is trivializable as a vector bundle, and σ is a nowhere vanishing section of L. Then one can either:

- consider the entire

$$\tilde{c}_\sigma : \mathcal{G} \to \mathbb{R}^*$$

- consider only the part of $\tilde{c}_\sigma$ that is not contained in c_σ, i.e.,

$$\text{sign} \circ \tilde{c}_\sigma : \mathcal{G} \to \mathbb{Z}_2,$$

where we identify $\mathbb{Z}_2$ with $\{-1, 1\} \subset \mathbb{R}^*$.

Both of them are (differentiable) cocyles on $\mathcal{G}$ with coefficients in a (abelian Lie) group. Such cocycles give rise to classes in the cohomology groups $H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*)$ and $H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2)$, which are abelian groups but no longer vector spaces. As before, the resulting cohomology classes are independent of σ; we denote them by

$$\tilde{\theta}_{\mathcal{G}}(L) \in H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*), \quad w(L) \in H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2),$$

and we will call them the **extended characteristic class of** L, and the **Stiefel-Whitney class of** L, respectively. All these classes can be put together using the decomposition of the group $\mathbb{R}^*$ as

$$\mathbb{R}^* \cong \mathbb{R} \times \mathbb{Z}_2, \quad \lambda \mapsto (\ln(|\lambda|), \text{sign}(\lambda))$$

and the induced isomorphism

$$H^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) \cong H^1_{\text{diff}}(\mathcal{G}) \times H^1_{\text{diff}}(\mathcal{G}, \mathbb{Z}_2).$$

With this, the extended class of L is

$$\tilde{\theta}_{\mathcal{G}}(L) = (\theta_{\mathcal{G}}(L), w(L)).$$

Note that, by construction, $\tilde{\theta}_{\mathcal{G}}(L)$ is trivial (equal to the identity of the group) if and only if L is isomorphic to the trivial representation, and $w(L)$ is trivial if and only if L is $\mathcal{G}$-orientable, i.e., if L admits an orientation with the property that the action of $\mathcal{G}$ is orientation-preserving. A warning however: the classes $\tilde{\theta}_{\mathcal{G}}(L)$ and $w(L)$ have been defined so far only when L is trivializable as a vector bundle; moreover, these constructions cannot be extended to general L's while preserving their main properties. Hence, when it comes to the canonical representations of $\mathcal{G}$, one can apply them only to $\mathcal{D}_A^{\text{tr}}$, for which one obtains

$$w(\mathcal{D}_A^{\text{tr}}) = 0, \quad \tilde{\theta}_{\mathcal{G}}(\mathcal{D}_A^{\text{tr}}) = (\text{mod}(\mathcal{G}), 0).$$

For the previous discussion we assumed that L was trivializable as a vector bundle. To handle general L's one can use covers $\mathcal{U}$ of M by open subsets over which L is trivializable. Such an open cover induces a groupoid $\mathcal{G}_{\mathcal{U}}$ over the disjoint union of the open subsets in $\mathcal{U}$, obtained by pulling-back $\mathcal{G}$ along the canonical map from the disjoint union into M. The pullback $L_{\mathcal{U}}$ of L is a representation of $\mathcal{G}_{\mathcal{U}}$ and, by the choice of $\mathcal{U}$, one has well-defined classes

$$\tilde{\theta}(L_{\mathcal{U}}) = (\theta(L_{\mathcal{U}}), w(L_{\mathcal{U}})) \in H^1_{\text{diff}}(\mathcal{G}_{\mathcal{U}}, \mathbb{R}^*) \cong H^1_{\text{diff}}(\mathcal{G}_{\mathcal{U}}) \times H^1_{\text{diff}}(\mathcal{G}_{\mathcal{U}}, \mathbb{Z}_2).$$

Note that, since $\mathcal{G}_{\mathcal{U}}$ is Morita equivalent to $\mathcal{G}$, when passing from L to $L_{\mathcal{U}}$ (as representations) one does not loose any information. To obtain a class that is independent of the covers one proceeds as usual and one passes to the filtered colimit (with respect to the refinement of covers) and defines

$$\check{\mathrm{H}}^1_{\text{diff}}(\mathcal{G}, \mathbb{R}^*) = \lim_{\to \mathcal{U}} H^1_{\text{diff}}(\mathcal{G}_{\mathcal{U}}, \mathbb{R}^*),$$

and similarly $\check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{Z}_2)$. Note that the Morita invariance of differentiable cohomology with coefficients in $\mathbb{R}$ implies that the restriction to open subsets induces an isomorphism

$$H^1_{\mathrm{diff}}(\mathcal{G}) \cong H^1_{\mathrm{diff}}(\mathcal{G}_{\mathcal{U}})$$

(that sends $\theta_{\mathcal{G}}(L)$ to $\theta_{\mathcal{G}_{\mathcal{U}}}(L_{\mathcal{U}})$); we conclude that there are induced cohomology classes

$$\check{\theta}_{\mathcal{G}}(L) \in \check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{R}^*), \quad w(L) \in \check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{Z}_2)$$

and canonical isomorphism of groups

$$\check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{R}^*) \cong H^1_{\mathrm{diff}}(\mathcal{G}) \times \check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{Z}_2)$$

such that:

- $\check{\theta}_{\mathcal{G}}(L) = (\theta_{\mathcal{G}}(L), w(L))$.
- $\check{\theta}_{\mathcal{G}}(L)$ is trivial if and only if L is isomorphic to the trivial representation.

Actually, $\check{\theta}_{\mathcal{G}}$ gives an isomorphism between the group $\mathrm{Rep}^1(\mathcal{G})$ of isomorphism classes of 1-dimensional representations (with the tensor product) with the Čech-type cohomology with coefficients in $\mathbb{R}^*$:

$$\check{\theta}_{\mathcal{G}} : \mathrm{Rep}^1(\mathcal{G}) \xrightarrow{\sim} \check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{R}^*). \tag{4.44}$$

Denoting $\mathbb{R}^* = GL_1(\mathbb{R})$ by H, this is a particular case of the interpretation of $\mathcal{G}$-equivariant principal H-bundles in terms of transition functions (see for example [53]), valid for any Lie group H, interpretation that is itself at the heart of Haefliger's work on the transverse geometry of foliations [44]. While w is trivial on $\mathcal{D}^{\mathrm{tr}}_A$, it gives rise to interesting information when applied to the transverse volume bundle or, equivalently, to the orientation one (the two classes coincide because of the multiplicativity of w).

Definition 4.61. The **transverse first Stiefel-Whitney class** of $\mathcal{G}$ is defined as

$$w_1^{\mathrm{tr}}(\mathcal{G}) := w(\mathcal{V}^{\mathrm{tr}}_A) = w(\mathfrak{o}^{\mathrm{tr}}_A) \in \check{\mathrm{H}}^1_{\mathrm{diff}}(\mathcal{G}, \mathbb{Z}_2).$$

As a consequence of the previous discussion we state here the following:

Corollary 4.62. *One has:*

1. *$\mathcal{G}$ is transversally orientable iff $w_1^{\mathrm{tr}}(\mathcal{G}) = 1$.*
2. *$\mathcal{G}$ admits transverse volume forms iff $\mathrm{mod}(\mathcal{G}) = 0$ and $w_1^{\mathrm{tr}}(\mathcal{G}) = 1$.*

4.9 Appendix: Haar systems and cut-off functions, revisited

Haar systems are complementary to the transverse measures discussed here. However, they are used several times; moreover, the notion of Haar systems is of independent interest. In some sense this is just the extension from groups to groupoids of the notion of Haar measure. This is a well-known theory [90]. We recall here some of the basic definitions; this will allow us to fix some notation, but also gives us the opportunity of clarifying a few points that are perhaps not so well-known.

4.9.1 Haar systems

Throughout this appendix $\mathcal{G}$ is a Lie groupoid over a manifold M. To talk about right-invariance, one first remarks that the right multiplication by an arrow $g : x \to y$ of $\mathcal{G}$ is no longer defined on the entire $\mathcal{G}$ (as for groups) but only between the s-fibres:

$$R_g : s^{-1}(y) \to s^{-1}(x). \tag{4.45}$$

Hence, in principle, right-invariance is a property for families of objects that live on the s-fibres of $\mathcal{G}$.

Definition 4.63. A **smooth Haar system** on a Lie groupoid $\mathcal{G}$ is a family

$$\mu = \{\mu^x\}_{x \in M}$$

of non-zero measures μ^x on $s^{-1}(x)$ which is right-invariant and smooth, i.e.:

1. via any right-translation (4.45) by an element $g : x \to y$, μ^x is pulled-back to μ^y; or, in the integral notation (Remark 39),
$$\int_{s^{-1}(y)} f(hg)\ d\mu^y(h) = \int_{s^{-1}(x)} f(h)\ d\mu^x(h).$$
2. for any $f \in C_c^\infty(\mathcal{G})$, the function obtained by integration over the s-fibres,
$$M \ni x \mapsto \mu^x(f|_{s^{-1}(x)}) = \int_{s^{-1}(x)} f(h)\ d\mu^x(h) \in \mathbb{R}$$
is smooth.

The Haar system is called **full** if the support each μ^x is the entire $\mu^{-1}(x)$.

In general, the support of Haar system μ is defined as

$$\mathrm{supp}(\mu) = \cup_{x \in M} \mathrm{supp}(\mu_x) \subset \mathcal{G}.$$

Due to the invariance of μ, $\mathrm{supp}(\mu)$ is right $\mathcal{G}$-invariant, i.e., $ag \in \mathrm{supp}(\mu)$ for all $a \in \mathrm{supp}(\mu)$ and $g \in \mathcal{G}$ composable. Therefore $\mathrm{supp}(\mu)$ is made of t-fibres, hence it is determined by its t-projection, which is denoted

$$\mathrm{supp}_M(\mu) := t(\mathrm{supp}(\mu)).$$

More precisely, we have:

$$\mathrm{supp}(\mu) = t^{-1}(\mathrm{supp}_M(\mu)). \tag{4.46}$$

Remark 54. In the existing literature one often restricts to what we call here full Haar systems. However, some important constructions (e.g. already for the averaging process for proper groupoids) are based on systems that are not full. For us, the fullness is replaced by the condition that each μ_x is non-trivial. Note also that this condition has consequences on the supports, expressed in one of the following equivalent ways

- $s|_{\mathrm{supp}(\mu)} : \mathrm{supp}(\mu) \to M$ is surjective.
- the $\mathcal{G}$-saturation of $\mathrm{supp}_M(\mu)$ is the entire M.

4.9.2 Geometric Haar systems (Haar densities)

We now pass to Haar systems that are of geometric type, i.e., for which each μ^x comes from a density on $s^{-1}(x)$; this will bring us to sections

$$\rho \in C^\infty(M, \mathcal{D}_A)$$

of the density bundle associated to the Lie algebroid A of $\mathcal{G}$. To see this, recall the construction of A: it is the vector bundle over M whose fibre above a point $x \in M$ is the tangent space at the unit 1_x of the s-fibre above x (where $s : \mathcal{G} \to M$ stands for the source map):

$$A_x = T_{1_x} s^{-1}(x).$$

(Globally, A is the restriction to M, via the unit map $M \hookrightarrow \mathcal{G}$, of the bundle $T^s\mathcal{G} = \mathrm{Ker}\,(ds)$ of vector tangent to the s-fibres, also denoted by $\mathcal{F}(s)$). With this, the right translation (4.45) associated to an arrow $g : x \to y$ induces, after differentiation at the unit at y, an isomorphism:

$$R_g : A_y \to T_g s^{-1}(x) = T^s_g\mathcal{G}.$$

In this way any section $\alpha \in \Gamma(A)$ gives rise to a vector field $\overrightarrow{\alpha}$ on $\mathcal{G}$ and this identifies $\Gamma(A)$ with the space of vector fields on $\mathcal{G}$ that are tangent to the s-fibres and invariant under right translations (and the algebroid bracket of A, a Lie bracket $[\cdot, \cdot]$ on $\Gamma(A)$, is the one induced by the usual Lie bracket of vector fields on $\mathcal{G}$: $\overrightarrow{[\alpha, \beta]} = [\overrightarrow{\alpha}, \overrightarrow{\beta}]$).

Stated in the spirit of the previous definition, this reinterprets elements $\alpha \in \Gamma(A)$ as families of vector fields on the s-fibres of $\mathcal{G}$, that are right-invariant and smooth. It is clear that the same reasoning applies to sections $\rho \in \Gamma(\mathcal{D}_A) = C^\infty(M, \mathcal{D}_A)$, so that such sections can be interpreted as families of densities on the s-fibres of $\mathcal{G}$, that are right-invariant and smooth. Explicitly, right translating ρ we obtain a (right-invariant) density on the vector bundle $\mathcal{F}(s)$,

$$\overrightarrow{\rho} \in C^\infty(\mathcal{G}, \mathcal{D}_{\mathcal{F}(s)}),$$

hence a family of densities

$$\rho^x := \overrightarrow{\rho}|_{s^{-1}(x)} \in \mathcal{D}(s^{-1}(x)) \quad (x \in M). \tag{4.47}$$

Note that, for the usual notion of support of sections of vector bundles, we have the analogue of (4.46):

$$\mathrm{supp}(\overrightarrow{\rho}) = t^{-1}(\mathrm{supp}(\rho)).$$

Of course, the two are related.

Definition 4.64. A **Haar density** for $\mathcal{G}$ is any positive density $\rho \in C^\infty(M, \mathcal{D}_A)$ of the Lie algebroid A of $\mathcal{G}$ with the property that the $\mathcal{G}$-saturation of its support is the entire M. It is called a full Haar density if it is strictly positive.

The previous discussion shows that these correspond to (full) Haar systems of geometric type, i.e., for which each of the measures is induced by a density. For ρ such a Haar density, we will denote by

$$\mu_\rho = \{\mu_{\rho^x}\}_{x \in M}$$

the corresponding Haar system (given by (4.47)).

Remark 55. Sections $\rho \in C^\infty(M, \mathcal{D}_A)$ can be multiplied by smooth functions $f \in C^\infty(M)$. At the level of measures we find

$$\mu_{f\rho}(g) = f(t(g))\mu_\rho(g).$$

Hence we are led to the operation of multiplying a Haar system μ by a positive function $f \in C^\infty(M)$ which, on $\mathcal{G}$, corresponds to the standard multiplication by $t^*(f)$. Note that

$$\mathrm{supp}_M(f \cdot \mu) \subset \mathrm{supp}(f)$$

hence, even if μ is full, for $f \cdot \mu$ to be a Haar system one still has to require that the $\mathcal{G}$-saturation of the support of f is M.

4.9.3 Proper Haar systems; proper groupoids

One of the standard uses of the Haar measure of compact Lie groups is to produce G-invariant objects out of arbitrary ones, by averaging. On the other hand it is well-known that the correct generalization of compactness when going from Lie groups to groupoids is properness. Recall here that a Lie groupoid $\mathcal{G}$ over M is called proper if the map

$$(s, t) : \mathcal{G} \to M \times M$$

is proper, i.e., for every $K, L \subset M$ compacts, the subspace $\mathcal{G}(K, L) \subset \mathcal{G}$ of arrows that start in K and end in L is compact. The point is that averaging arguments do work well for proper groupoids. However, there is a small subtlety due to the fact that we would have to integrate over the s-fibres which may fail to be compact even if $\mathcal{G}$ is proper. For that reason, we need Haar systems $\mu = \{\mu^x\}$ in which each μ^x is compactly supported (see Lemma 4.9).

Definition 4.65. A Haar system on a Lie groupoid $\mathcal{G}$ is said to be **proper** if

$$s|_{\mathrm{supp}(\mu)} : \mathrm{supp}(\mu) \to M$$

is proper. A Haar density ρ is said to be proper if μ_ρ is proper, i.e., if the restriction of s to $t^{-1}(\mathrm{supp}(\rho))$ is proper.

In this case one can talk about the volume of the s-fibres,

$$\mathrm{Vol}(s^{-1}(x), \mu_x) = \int_{s^{-1}(x)} d\mu^x$$

and this defines a smooth function on M which is strictly positive; hence, rescaling μ by it (in the sense of the previous remark), we obtain a new proper Haar system satisfying the extra condition

$$\mathrm{Vol}(s^{-1}(x), \mu_x) = 1.$$

Such proper Haar systems are called **normalized**. Similarly for Haar densities.

Note that full Haar systems cannot be proper even for proper non-compact groupoids. Also, while full Haar systems exist on all Lie groupoids, proper ones do not. Actually, we have:

Proposition 4.66. *For a Lie groupoid $\mathcal{G}$, the following are equivalent:*

1. $\mathcal{G}$ is proper.

2. $\mathcal{G}$ admits a proper Haar system.

3. $\mathcal{G}$ admits a proper Haar density.

In particular, if $\rho \in C^\infty(M, \mathcal{D}_A)$ is a full Haar density then there exists a function $c \in C^\infty(M)$ such that $c \cdot \rho$ is a proper Haar density which, moreover, may be arranged to be normalized. Such c is called a ***cut-off function for*** *ρ (cf. [101]).*

Proof. The fact that $\mathcal{G}$ is proper if a proper Haar system μ exists will follow from the properties of $\mathrm{supp}(\mu)$ (right $\mathcal{G}$-invariance plus the fact that the restriction of s to $\mathrm{supp}(\mu)$ is proper and surjective) or, equivalently, those of $T := \mathrm{supp}_M(\mu)$:

- the $\mathcal{G}$-saturation of T is the entire M
- $\mathcal{G}(K, T)$ (arrows that start in K and end in T) is compact if K is.

Indeed, the first property implies that for any $K, L \subset M$ one has

$$\mathcal{G}(K, L) \subset \mathcal{G}(K, T)^{-1} \cdot \mathcal{G}(L, T)$$

(for g in the left hand side, choose any a from $t(g)$ to an element in T and write $g = a^{-1} \cdot (ag)$); therefore, if if K and L are compacts, then so will be $\mathcal{G}(K, L)$ (as closed inside a compact).

Recall that a full Haar density ρ always exists and if we multiply it by a positive function c then the M-support of the resulting Haar system coincides with $\mathrm{supp}(c\rho) = \mathrm{supp}(c)$. Therefore, to close the proof, it suffices to show that if $\mathcal{G}$ is proper then one finds a smooth function c such that its support $T := \mathrm{supp}(c)$ has the properties mentioned above. We first construct T. Recall that a slice through $x \in M$ is any $T \subset M$ that intersects transversally all the orbits that it meets, intersects the orbit through x only at x and its dimension equals the codimension of that orbit. We construct T as a union of such slices (enough, but not too many). For that we need the basic properties of slices for proper groupoids (cf. e.g. [28]): through each point of M one can find a slice and, for any slice Σ, its saturation $\mathcal{G}\Sigma$ is open in M (in particular, $\pi(\Sigma)$ is open in $M/\mathcal{G}$). Since $M/\mathcal{G}$ is paracompact, we find a family $\{\Sigma_i'\}_{i \in I}$ of locally compact slices such that $V_i' := \pi(\Sigma_i')$ defines a locally finite cover of $M/\mathcal{G}$. As usual, we refine this cover to a new cover $\{V_i\}_{i \in I}$ with $\overline{V}_i \subset V_i'$ and write $V_i = \pi(\Sigma_i)$ with $\Sigma_i = \pi^{-1}(V_i) \cap \Sigma_i'$. Note that

$$\pi(\overline{\Sigma}_i) \subset \overline{\pi(\Sigma_i)} = \overline{V}_i \subset V_i' = \pi(\Sigma_i')$$

hence $\overline{\Sigma}_i$ is contained in the saturation of Σ_i' (but may fail to be contained in Σ_i'!).

We claim that $T := \cup_i \overline{\Sigma}_i$ will have the desired properties; the saturation property is clear (even before taking closures) since the V_i's cover $M/\mathcal{G}$. Next, the compactness of $\mathcal{G}(K, T)$ when K is compact: if $B(K, \overline{\Sigma}_i)$ is non-empty, then $\pi(K)$ must intersect $\pi(\overline{\Sigma}_i)$ hence also V_i'; but, since $\pi(K)$ is compact, it can intersects only a finite number of V_i'; therefore we find indices $i_1, \ldots, i_k$ such that

$$B(K, T) = B(K, \overline{\Sigma}_{i_1}) \cup \ldots \cup B(K, \overline{\Sigma}_{i_1}),$$

hence $B(K, T)$ is compact. Note that the proof shows a bit more: instead of Σ_i we can use any C_i relatively compact, with

$$\Sigma_i \subset C_i \subset \overline{C}_i \subset \mathcal{G}\Sigma_i',$$

and then $T = \cup_i \overline{C}_i$ still has the desired properties. This is important for constructing c since not every closed subspace can be realized as the support of a smooth function. Since each $\overline{\Sigma}_i$ is compact and sits inside the open $\mathcal{G}\Sigma_i'$, we find $c_i : M \to [0,1]$ smooth, supported in $\mathcal{G}\Sigma_i'$ with $c_i > 0$ on $\overline{\Sigma}_i$. Hence, in the previous construction we can set $C_i = \{c_i > 0\}$ to define T. By construction, $\pi(\mathrm{supp}(c_i))$ sits inside V_i', hence $\{c_i\}_{i\in I}$ is locally finite and $c = \sum_i c_i$ makes sense as a smooth function. Moreover, $\{c \neq 0\} = \cup_i \{c_i \neq 0\}$ is a locally finite union, hence

$$\mathrm{supp}(c) = \cup_i \mathrm{supp}(c_i) = T.$$

□

Let us also illustrate the averaging technique with one simple example.

Lemma 4.67. *If $\mathcal{G}$ is a proper Lie groupoid over M and $A, B \subset M$ are two closed disjoint subsets that are saturated, then there exists a smooth function $f : M \to [0,1]$ that $\mathcal{G}$-invariant (i.e., constant on the orbits of $\mathcal{G}$) such that $f|_A = 0$, $f|_B = 1$.*

Without the invariance condition, this is a basic property of smooth functions on manifolds. The idea is that, choosing any f as above but possibly non-invariant, one just replaces it by its average with respect to a proper normalized Haar system μ:

$$\mathrm{Av}_\mu(f)(x) = \int_{s^{-1}(x)} f(t(g))\ d\mu^x(g).$$

While it is clear that $\mathrm{Av}(f)$ vanishes on A, the normalization implies that it is 1 on B. Also the invariance is immediate: if x and y are in the same orbit, i.e., if there exists an arrow $a : x \to y$ then, using the invariance of μ and $t(ga) = t(g)$, we find

$$\mathrm{Av}_\mu(f)(x) = \int_{s^{-1}(x)} f(t(g))\ d\mu^x(g) = \int_{s^{-1}(y)} f(t(ga))\ d\mu^y(g) = \mathrm{Av}_\mu(f)(y).$$

By similar techniques one proves the existence of invariant metrics on $\mathcal{G}$-vector bundles, or of other geometric structures.

4.9.4 Induced measures/densities on the orbits

The slogan that Haar systems and densities are related to measure theory along the orbits can be made very precise in the case of proper groupoids: in that case they induce (in a canonical fashion) measures/densities on the orbits (and conversely!). To explain this, let $\mathcal{G}$ be a proper groupoid over M and let $\mu = \{\mu^x\}$ be a Haar system for $\mathcal{G}$. Above each $x \in M$ one has the isotropy Lie group $\mathcal{G}_x$ consisting of arrows that start and end at x and the s-fibre above $x \in M$ is a principal $\mathcal{G}_x$-bundle over the orbit $\mathcal{O}_x$ of $\mathcal{G}$ through x, with projection map

$$t : s^{-1}(x) \to \mathcal{O}_x. \tag{4.48}$$

The invariance of the family $\{\mu^x\}$ implies that each μ^x is an invariant measure on this bundle hence, since $\mathcal{G}_x$ is compact, μ^x corresponds to a measure on $\mathcal{O}_x$ (cf. Example 4.14), namely $\mu_{\mathcal{O}_x} = t_!(\mu^x)$ or, in the integral formulation,

$$\int_{\mathcal{O}_x} f(y)\ d\mu_{\mathcal{O}_x}(y) = \int_{s^{-1}(x)} f(t(g))\ d\mu^x(g).$$

Using again the invariance of μ, we see that each such $\mu_{\mathcal{O}_x}$ depends only on the orbit itself and not on the point x in the orbit. Therefore any Haar system μ determines (and is determined by) a family of measures on the orbits of $\mathcal{G}$,

$$\{\mu_{\mathcal{O}}\}_{\mathcal{O}-\text{orbit of } \mathcal{G}}. \tag{4.49}$$

Of course, this is compatible with "geometricity", so that any Haar density $\rho \in C^{\infty}(M, \mathcal{D}_A)$ determines (and is determined by) a family of densities on the orbits of $\mathcal{G}$,

$$\{\rho_{\mathcal{O}} \in \mathcal{D}(\mathcal{O})\}_{\mathcal{O}-\text{orbit of } \mathcal{G}}. \tag{4.50}$$

in such a way that $(\mu_\rho)_{\mathcal{O}} = \mu_{\rho_{\mathcal{O}}}$. Explicitly, given ρ, the density $\rho_{\mathcal{O}}$ on the orbit through x is $t_!(\rho^x)$, where $\rho^x = \overrightarrow{\rho}|_{s^{-1}(x)}$ is the induced density on the s-fibre- see (4.47). Also, the discussion from Example 4.14 tells us how to recover ρ from this family (and similarly for μ): consider $\mathfrak{g}_x =$ the isotropy Lie algebra at x (the Lie algebra of $\mathcal{G}_x$), with corresponding Haar density denoted by μ_{Haar} and then

$$\rho(x) \in \mathcal{D}_{A_x} \cong \mathcal{D}_{\mathfrak{g}_x} \otimes \mathcal{D}(T_x\mathcal{O}) \ \text{ equals to } \mu_{\text{Haar}} \otimes \rho_{\mathcal{O}}(x). \tag{4.51}$$

Remark 56. Therefore, for proper groupoids, a Haar system μ (Haar density ρ) is the same thing as a family of measures (densities) on the orbits, satisfying a certain smoothness condition that ensures that the reconstructed μ is smooth. Working this out we find the condition that for any $f \in C^{\infty}_c(\mathcal{G})$, the function

$$M \ni x \mapsto \int_{\mathcal{O}_x} \int_{\mathcal{G}_{x,y}} f(g)\ d\mu^{\text{Haar}}_{x,y}(g) d\mu_{\rho_{\mathcal{O}_x}}(y) \in \mathbb{R}$$

is smooth. Here $\mathcal{G}_{x,y}$ is the space of arrows that start at x and end at y; these show up as the fibres of (4.48), hence they come with a canonical Haar density (cf. Example 4.14) obtained by transporting the Haar density on $\mathcal{G}_x$ and $d\mu^{\text{Haar}}_{x,y}$ stands for the resulting integration.

Corollary 4.68. *If $\mathcal{G}$ is proper, there is a 1-1 correspondence between:*

1. *Haar densities ρ of $\mathcal{G}$.*

2. *families (4.50) of non-trivial densities on the orbits of $\mathcal{G}$, smooth in the previous sense.*

Similarly for Haar systems.

The situation is even nicer in the regular case, i.e., when the orbits have the same dimension. For the version with densities, we make use of the foliation on M by the (connected components of the) orbits of $\mathcal{G}$, identified with the vector bundle $\mathcal{F} \subset TM$ of vectors tangent to the orbits (which can also be described as the image of the anchor map $\sharp : A \to TM$).

Corollary 4.69. *If $\mathcal{G}$ is proper and regular, there is a 1-1 correspondence between*

1. *Haar densities ρ of $\mathcal{G}$,*

2. *positive densities of the bundle $\mathcal{F}$, $\rho_{\mathcal{F}} \in C^{\infty}(M, \mathcal{D}_{\mathcal{F}})$, with the property that the support of $\rho_{\mathcal{F}}$ meets each orbit of $\mathcal{G}$.*

Moreover, ρ is full if and only if $\rho_{\mathcal{F}}$ is strictly positive, and ρ is proper if and only if the intersection of the support of $\rho_{\mathcal{F}}$ with each leaf is compact.

For Haar systems we obtain:

Corollary 4.70. *If $\mathcal{G}$ is proper and regular, there is a 1-1 correspondence between:*

1. *Haar systems μ of $\mathcal{G}$.*
2. *families of non-trivial measures $\mu_{\mathcal{O}}$ on the orbits of $\mathcal{G}$ which are smooth in the sense that, for any $f \in C^\infty_c(M)$, the function*

$$x \mapsto \int_{\mathcal{O}_x} f(y)\ d\mu_{\mathcal{O}_x}(y)$$

   is smooth.

The last two corollaries imply that, for proper regular Lie groupoids with connected s-fibre, having a Haar system/density for $\mathcal{G}$ is equivalent to having one for $\mathcal{F}$, where the notion of Haar measure/density for a foliation $\mathcal{F}$ is the one described by points 2. of the previous corollaries.

Chapter 5

Deformations of Lie groupoids

A central problem in geometry is that of understanding the behaviour of geometric structures under deformations; each class of geometric structures comes with its deformation theory, including a cohomology theory that controls such deformations. The aim of the paper that gave rise to this chapter is to investigate the cohomology theory controlling deformations of a large class of geometric structures and to use it to prove several rigidity results. The geometric structures that we have in mind are those that can be modelled by Lie groupoids and it includes Lie groups (and bundles of such), Lie group actions on manifolds, foliations, the symplectic groupoids of Poisson geometry, etc. In other words, we study the deformation theory of Lie groupoids $\mathcal{G}$ and the resulting deformation cohomology $H^*_{\text{def}}(\mathcal{G})$. The cohomology is built in such a way that deformations of a groupoid $\mathcal{G}$ give rise to 2-cocycles inducing elements in $H^2_{\text{def}}(\mathcal{G})$; 1-cochains that transgress these 2-cocycles (when they exist) are then used to produce flows that allow us to prove rigidity results. Intuitively, this allows us to think of $H^2_{\text{def}}(\mathcal{G})$ as the first order approximation (i.e., tangent space) of the moduli space of deformations of $\mathcal{G}$. We became aware of the existence of such a cohomology while searching for a geometric proof of Zung's linerization theorem for proper Lie groupoids [28]; the same cohomology also arises naturally when looking at the VB-interpretation of the adjoint representation [42].

Before we give more details on deformations and rigidity results, let us first describe some of the main properties/results regarding the deformation cohomology. The first one we would like to mention is Morita invariance:

Theorem 5.1 (Morita invariance)**.** *If two Lie groupoids are Morita equivalent, then their deformation cohomologies are isomorphic.*

Intuitively, this means that the deformation cohomology only depends on "the transverse geometry of the groupoid". This is very much related to Haefliger's philosophy/approach to the transverse geometry of foliations via the associated étale groupoids; the role of the groupoid(s) was to model (desingularize) leaf spaces of foliations; the notion of Morita equivalence of groupoids comes in for a simple reason: there is no canonical étale groupoid modelling a given leaf space, but several of them (e.g. any complete transversal to the foliation gives rise to one) - and the fact that two groupoids correspond to the same leaf space can be recognized by the fact that they are Morita equivalent. In other words, Haefliger's point of view is that the transverse geometry of foliations is the part of the geometry of étale groupoids which is Morita invariant. A slight generalization of this philosophy is the interpretation of Lie groupoids as "atlases" for differentiable stacks; again, two groupoids correspond to the same stack if and only if they are Morita equivalent. Hence the previous theorem allows us to talk about "the deformation cohomology of differentiable stacks".

The deformation cohomology $H^*_{\mathrm{def}}(\mathcal{G})$ is also related to differentiable cohomologies of $\mathcal{G}$. As we will recall in Subsection 5.1.3, $H^*_{\mathrm{diff}}(\mathcal{G}, E)$ (also simply denoted by $H^*(\mathcal{G}, E)$) makes sense as soon as we have a representation E of $\mathcal{G}$. It is not true that, in general, $H^*_{\mathrm{def}}(\mathcal{G})$ is isomorphic to $H^*(\mathcal{G}, E)$ for some representation E of $\mathcal{G}$ (and this is very much related to the fact that "the adjoint representation of $\mathcal{G}$" does not make sense as a representation in the usual sense - see also below). However, $H^*_{\mathrm{def}}(\mathcal{G})$ can often be related to differentiable cohomology. This is best illustrated in the regular case, i.e., when all the orbits of $\mathcal{G}$ have the same dimension. In this case, $\mathcal{G}$ comes with two natural representations in the classical sense: the normal representation ν (on the normal bundle of the orbits) and the isotropy representation $\mathfrak{i}$ (made of the Lie algebras $\mathfrak{i}_x$ of the Lie groups $\mathcal{G}_x$ consisting of arrows that start and end at $x \in M$). These are recalled in more detail in Subsections 5.3.1 and 5.3.4. We will show that $H^*_{\mathrm{def}}(\mathcal{G})$ is related to the standard differentiable cohomology with coefficients in these representations by a long exact sequence:

Proposition 5.2 (The regular case)**.** *The deformation cohomology of a regular Lie groupoid $\mathcal{G}$ fits into a long exact sequence*

$$\cdots \longrightarrow H^k(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\pi} H^{k-1}(\mathcal{G}, \nu) \xrightarrow{K} H^{k+1}(\mathcal{G}, \mathfrak{i}) \longrightarrow \cdots,$$

where ν and $\mathfrak{i}$ are the normal and the isotropy representation of $\mathcal{G}$, respectively.

The spaces of invariants

$$\Gamma(\mathfrak{i})^{\mathrm{inv}} := H^0(\mathcal{G}, \mathfrak{i}) \text{ and } \Gamma(\nu)^{\mathrm{inv}} := H^0(\mathcal{G}, \nu)$$

are of independent interest; with some extra care (explained in Subsections 5.3.1 and 5.3.4), they make sense even in the non-regular case. In our analysis in low degrees we will see that, in general, $H^0_{\mathrm{def}}(\mathcal{G})$ is always isomorphic to $\Gamma(\mathfrak{i})^{\mathrm{inv}}$ and one has a low degrees exact sequence (Proposition 5.28):

$$0 \longrightarrow H^1(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^1_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\pi} H^0(\mathcal{G}, \nu) \xrightarrow{K} H^2(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^2_{\mathrm{def}}(\mathcal{G}).$$

Another important property of $H^*_{\mathrm{def}}(\mathcal{G})$, which will be essential also for proving rigidity results, is its behaviour for proper groupoids. Recall that a Lie groupoid $\mathcal{G}$ over a manifold M is said to be proper if $\mathcal{G}$ is Hausdorff and the map $(s, t) : \mathcal{G} \longrightarrow M \times M$, which associates to an arrow its source and target, is proper; this generalizes the compactness of Lie groups and the properness of Lie group actions. While proper groupoids are the first candidates for rigidity phenomena, an essential step in proving such rigidity theorems is to understand the behaviour of $H^*_{\mathrm{def}}(\mathcal{G})$ in low degrees. Of particular importance will be the vanishing of $H^2_{\mathrm{def}}(\mathcal{G})$ which, due to its interpretation as the tangent space to the moduli space of deformations, could be called "infinitesimal rigidity". However, the understanding of $H^1_{\mathrm{def}}(\mathcal{G})$ (related to families of automorphisms) and even of $H^0_{\mathrm{def}}(\mathcal{G})$, will be important as well. We will show:

Theorem 5.3 (Vanishing)**.** *Let $\mathcal{G}$ be a proper Lie groupoid. Then*

$$H^k_{\mathrm{def}}(\mathcal{G}) = 0 \textit{ for all } k \geq 2$$

while $H^1_{\mathrm{def}}(\mathcal{G}) \cong \Gamma(\nu)^{\mathrm{inv}}$, $H^0_{\mathrm{def}}(\mathcal{G}) \cong \Gamma(\mathfrak{i})^{\mathrm{inv}}$.

Besides the relevance to deformations, there is yet another guiding principle that is worth having in mind when thinking about $H^*_{\mathrm{def}}(\mathcal{G})$: it makes sense of the (differentiable) cohomology with coefficients in "the adjoint representation" (think e.g. about the case of Lie groups [80]). This guiding principle is very useful when computing the deformation cohomology in terms of ordinary differentiable cohomology. The last quotes indicate the subtleties that arise when looking for "the adjoint representation of a Lie groupoid". Indeed, as remarked already in the early years of Lie groupoids, one of the subtleties which make Lie groupoids harder to handle than Lie groups is the fact that the notion of adjoint representation does not make sense when restricting to the classical notion of representation. With the more recent concept of "representation up to homotopy" [2], we now understand that:

- the adjoint representation Ad of a Lie groupoid $\mathcal{G}$ makes sense intrinsically as an isomorphism class of representations up to homotopy of $\mathcal{G}$.
- to represent Ad by an actual representation up to homotopy Ad_σ one needs to choose an Ehresmann connection σ on $\mathcal{G}$ (this will be recalled in Section 5.8).
- by the very definition of representations up to homotopy, they serve as coefficients for differentiable cohomology. However, to have an explicit model for the cohomology with coefficients in the adjoint representation, one still has to choose a connection σ and consider the associated $H^*(\mathcal{G}, \mathrm{Ad}_\sigma)$.

One of the main features of $H^*_{\mathrm{def}}(\mathcal{G})$ is that it provides an intrinsic model for these cohomologies, independent of any auxiliary choices:

Theorem 5.4 (The adjoint representation)**.** *If $\mathcal{G}$ is a Lie groupoid, then for any connection σ on $\mathcal{G}$ one has a canonical isomorphism of $H^*_{\mathrm{def}}(\mathcal{G})$ with $H^*(\mathcal{G}, \mathrm{Ad}_\sigma)$.*

We would like to point out that we have decided to write the paper in such a way that it does not assume prior knowledge of the adjoint representation as a representation up to homotopy. On the contrary, using the paradigm that "the adjoint representation serves as coefficients of the cohomology theory that controls deformations" (i.e., of $H^*_{\mathrm{def}}(\mathcal{G})$), we will slowly guide the reader towards the final outcome. In particular, the reader will encounter along the way several comments on "the adjoint representation", slowly revealing its full structure.

There is yet another way to look at $H^*_{\mathrm{def}}(\mathcal{G})$, related to the fact that the theory of Lie groupoids comes with an infinitesimal counterpart - that of Lie algebroids. From this point of view, our cohomology is the global analogue of the deformation cohomology $H^*_{\mathrm{def}}(A)$ of the Lie algebroid A of $\mathcal{G}$, which was studied in [27]. In the same way that the differentiable cohomology of a Lie group(/oid) is related to the cohomology of the corresponding Lie algebra(/oid) by a Van Est map, we will prove that:

Theorem 5.5 (Van Est isomorphisms)**.** *Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A. There exists a canonical chain map (the Van Est map)*

$$\mathcal{V} : C^*_{\mathrm{def}}(\mathcal{G}) \longrightarrow C^*_{\mathrm{def}}(A).$$

Moreover, if $\mathcal{G}$ has k-connected s-fibres then the map induced in cohomology

$$\mathcal{V} : H^p_{\mathrm{def}}(\mathcal{G}) \longrightarrow H^p_{\mathrm{def}}(A)$$

is an isomorphism for all $p \leq k$.

It is worth insisting a bit more on the similarities and differences with the corresponding infinitesimal theory $H^*_{\mathrm{def}}(A)$ from [27]. As there, one of the main subtleties of $H^*_{\mathrm{def}}(\mathcal{G})$ comes from the fact that the adjoint representation is only defined as a representation up to homotopy. However, the setting of Lie groupoids, because of its non-linear nature, comes with even more subtleties - to the extent that we revisit even the very definition of Lie groupoids - giving rise to the Appendix of this chapter. On the other hand, and in contrast with the infinitesimal theory, for $H^*_{\mathrm{def}}(\mathcal{G})$ one can prove the vanishing result mentioned above, with direct consequences to rigidity.

As we have already mentioned, one of the main motivations for studying the deformation cohomology $H^*_{\mathrm{def}}(\mathcal{G})$ comes from its relevance to deformations and rigidity results. We are interested in general deformations:

Definition 5.6. Let $\mathcal{G} \rightrightarrows M$ be a groupoid over M, with structure maps denoted by s, t, m, i, u (the source, target, multiplication, inversion, unit map, respectively). A **(smooth) deformation of $\mathcal{G}$** is a family

$$\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon \mid \epsilon \in I\} \text{ of groupoids } \mathcal{G}_\epsilon \rightrightarrows M_\epsilon$$

smoothly parameterized by ϵ in an open interval I containing 0, such that $\mathcal{G}_0 = \mathcal{G}$ as groupoids. We will denote the structure maps of $\mathcal{G}_\epsilon$ by $s_\epsilon, t_\epsilon, m_\epsilon, i_\epsilon, u_\epsilon$.

The deformation is called **strict** if $\mathcal{G}_\epsilon = \mathcal{G}$ as manifolds; it is called s**-constant** if, furthermore, s_ϵ does not depend on ϵ. The **constant deformation** is the one with $\mathcal{G}_\epsilon = \mathcal{G}$ as groupoids.

Two deformations $\{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ and $\{\mathcal{G}'_\epsilon \mid \epsilon \in I'\}$ are called **equivalent** if there exists a family of groupoid isomorphisms $\phi^\epsilon : \mathcal{G}_\epsilon \longrightarrow \mathcal{G}'_\epsilon$, smoothly parameterized by ϵ in an open interval containing 0, such that $\phi_0 = \mathrm{Id}$. We say that $\tilde{\mathcal{G}}$ is **trivial** if it is equivalent to the constant deformation.

In general, a family $\{M_\epsilon \mid \epsilon \in I\}$ of manifolds smoothly parameterized by ϵ can be understood as a manifold $\tilde{M}$ together with a submersion $\tilde{\pi} : \tilde{M} \longrightarrow I$, so that M_ϵ is just the fibre of π above ϵ; similarly for families of groupoids - see Definition 5.9.

As we shall see, any s-constant deformation as above induces a deformation cocycle

$$\xi_0 \in C^2_{\mathrm{def}}(\mathcal{G})$$

whose cohomology class depends only on the equivalence class of the deformation. It is interesting to keep in mind that the *single* cocycle ξ_0 encodes the variation of *all* the structure maps $t_\epsilon, m_\epsilon, i_\epsilon, u_\epsilon$; to be able to do that, we first have to revisit the very definition of groupoids and note that everything is encoded in the source map and the operation $\overline{m}(g, h) = gh^{-1}$; the resulting precise axioms are worked out in the Appendix.

A similar construction applies to general deformations (i.e., not necessarily s-constant, and not even strict); the price to pay for the greater generality is that we will no longer have a 2-cocycle that is canonical (i.e., independent of auxiliary choices), but only a canonical cohomology class in $H^2_{\mathrm{def}}(\mathcal{G})$. Using these cocycles/classes and the vanishing theorem stated above, we will deduce several rigidity theorems. We mention here:

Theorem 5.7. *Any strict deformation of a compact groupoid is trivial.*

This theorem can be seen as mutual generalization of the results of Palais on rigidity and deformations of actions of Lie groups [82, 83], and those of Nijenhuis and Richardson [80] on deformations of Lie groups. In fact, our Proposition 5.18 shows that in the case of an *action groupoid*, our deformation cohomology sits in a long exact sequence which relates the deformation cohomology of a Lie group (as in [80]), and the deformation cohomology of an action of a fixed group (as in [83]).

As we have already mentioned, the deformation cohomology $H^*_{\text{def}}(\mathcal{G})$ arises naturally when studying a related phenomena, namely the linearization of Lie groupoids. In essence, this is due to a very simple remark: for any real function $f = f(x)$ which vanishes at 0, its linearization around 0 can be written as a limit of rescales of f:

$$f'(0)x = \lim_{\epsilon \to 0} f_\epsilon \quad \text{where } f_\epsilon(x) = \frac{1}{\epsilon} f(\epsilon x).$$

A groupoid version of this is that the linearization of a Lie groupoid around a fixed point comes with a canonical (strict) deformation whose members for $\epsilon \neq 0$ are isomorphic to the original groupoid (around the fixed point). Using this and a local version of the last theorem, we immediately deduce a generalization of "Zung's linearization theorem", which was proved recently using different methods by del Hoyo and Fernandes in [32]

Theorem 5.8 (Linearization theorem)**.** *If $\mathcal{G}$ is an s-proper groupoid and $N \subset M$ is invariant, then $\mathcal{G}$ is linearizable around N.*

Zung's theorem corresponds to the special case when N is a fixed point of $\mathcal{G}$ (see also [28] for more on the relation to Zung's theorem).

The relationship between $H^*_{\text{def}}(\mathcal{G})$ and deformations also give rise to variation maps for families of Lie groupoids, very much in the spirit of the Kodaira-Spencer map associated to a family of complex manifolds [56] and other similar variation maps. We will be looking at families of groupoids in the following sense:

Definition 5.9. A **family of Lie groupoids parameterized by a smooth manifold** B,

$$\mathcal{G} \rightrightarrows M \xrightarrow{\pi} B,$$

consists of a Lie groupoid $\mathcal{G}$ over a manifold M and a surjective submersion π from M to B such that $\pi \circ s = \pi \circ t$. For $b \in B$ we will denote by $\mathcal{G}_b$ the resulting groupoid over the fibre $M_b = \pi^{-1}(b)$ of π above b. We say that it is a **proper family** if $\mathcal{G}$ is proper.

Two families $\mathcal{G} \rightrightarrows M \xrightarrow{\pi} B$ and $\mathcal{G}' \rightrightarrows M' \xrightarrow{\pi'} B$ are said to be **isomorphic** if there exists an isomorphism of groupoids $F : \mathcal{G} \longrightarrow \mathcal{G}'$ with base map $f : M \longrightarrow M'$ compatible with π and π' (i.e., $\pi' \circ f = \pi$).

Looking at the variation of the groupoids $\mathcal{G}_b$ in directions of curves γ in B (i.e., applying the previous ideas to the deformations $\{\mathcal{G}_{\gamma(\epsilon)}\}$ of $\mathcal{G}_{\gamma(0)}$), we obtain the variation maps of the family,

$$\mathrm{Var}_b : T_bB \longrightarrow H^2_{\text{def}}(\mathcal{G}_b).$$

Again, it is possible to prove several rigidity results for families; we mention here the simplest one:

Theorem 5.10 (Local triviality of compact families)**.** *Any compact family of Lie groupoids is locally trivial, i.e., with the previous notation, for any $b \in B$, there exists a neighbourhood U of b in B such that the resulting family parameterized by U is isomorphic to the trivial family $\mathcal{G}_b \times U$.*

5.1 The deformation complex

In this section we introduce the deformation complex of a Lie groupoid.

5.1.1 Some notation/terminology

We start by fixing some notation/terminology. For a Lie groupoid $\mathcal{G} \rightrightarrows M$, we denote by s, t, m, u, and i its source, target, multiplication, unit, and inversion maps respectively. When there is no danger of ambiguity, we write $m(g, h) = gh$, $i(g) = g^{-1}$ and we identify $x \in M$ with the corresponding unit $u(x) \in \mathcal{G}$. We also write $g : x \longrightarrow y$ to indicate that $g \in \mathcal{G}$, $x = s(g)$, $y = t(g)$. The s and t-fibres above $x \in M$ are denoted

$$\mathcal{G}(x, -) = s^{-1}(x),\ \mathcal{G}(-, x) = t^{-1}(x).$$

For $g : x \longrightarrow y$ in $\mathcal{G}$, we consider the corresponding right translation

$$R_g : \mathcal{G}(y, -) \longrightarrow \mathcal{G}(x, -)$$

and similarly the left translation L_g which maps t-fibres to t-fibres. Their differentials will be denoted by r_g and l_g, respectively.

Recall that the Lie algebroid A of $\mathcal{G}$ is, as a vector bundle, the restriction of $T^s\mathcal{G} = \mathrm{Ker}\,(ds)$ to M (pulled-back via the unit map $u : M \hookrightarrow \mathcal{G}$), so that

$$A_x = T_x\mathcal{G}(x, -) \ \text{ for all } x \in M.$$

The anchor of A is the vector bundle map $\sharp : A \longrightarrow TM$ given by the differential of t. Using right translations, any $\alpha \in \Gamma(A)$ induces a right-invariant vector field $\overrightarrow{\alpha}$ on $\mathcal{G}$ (necessarily tangent to the s-fibres, so that right-invariance makes sense):

$$\overrightarrow{\alpha}(g) = r_g \alpha_{t(g)}$$

This construction identifies $\Gamma(A)$ with the space $\mathfrak{X}^s_{\mathrm{inv}}(\mathcal{G})$ of right-invariant vector fields on $\mathcal{G}$; in turn, this induces the Lie algebra bracket $[\cdot, \cdot]$ on $\Gamma(A)$. Similarly, any $\alpha \in \Gamma(A)$ induces a left-invariant vector field $\overleftarrow{\alpha}$ on $\mathcal{G}$, given by

$$\overleftarrow{\alpha}(g) = l_g \circ di(\alpha_{s(g)}). \tag{5.1}$$

5.1.2 The deformation complex

We are now ready to introduce the deformation complex of a Lie groupoid. We will need the division map $\bar{m}$ of $\mathcal{G}$,

$$\bar{m}(p, q) = pq^{-1},\ \text{for all } p, q \in \mathcal{G},\ \text{such that } s(p) = s(q);$$

Its advantage over the multiplication map m, especially when it comes to deformations, is explained in Subsection 5.4.2. We also consider the space of strings of k-composable arrows

$$\mathcal{G}^{(k)} = \{(g_1, \dots, g_k) \mid s(g_i) = t(g_{i+1}) \text{ for all } 1 \leq i \leq k-1\}.$$

Definition 5.11. The **deformation complex** $(C^\bullet_{\text{def}}(\mathcal{G}), \delta)$ of the Lie groupoid $\mathcal{G}$, whose cohomology is denoted by $H^*_{\text{def}}(\mathcal{G})$, is defined as follows. For $k \geq 1$, the k-cochains $c \in C^k_{\text{def}}(\mathcal{G})$ are the smooth maps

$$c : \mathcal{G}^{(k)} \longrightarrow T\mathcal{G}, \quad (g_1, \dots, g_k) \mapsto c(g_1, \dots, g_k) \in T_{g_1}\mathcal{G},$$

which are s-projectable in the sense that

$$ds \circ c(g_1, g_2, \dots, g_k) =: s_c(g_2, \dots, g_k)$$

does not depend on g_1; the resulting s_c is called the s-projection of c. The differential of $c \in C^k_{\text{def}}(\mathcal{G})$ is defined by

$$\begin{aligned}(\delta c)(g_1, \dots, g_{k+1}) = &- d\bar{m}(c(g_1 g_2, \dots, g_{k+1}), c(g_2, \dots, g_{k+1})) \\ &+ \sum_{i=2}^{k} (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) + (-1)^{k+1} c(g_1, \dots, g_k).\end{aligned}$$

For $k = 0$, $C^0_{\text{def}}(\mathcal{G}) := \Gamma(A)$ and the differential of $\alpha \in \Gamma(A)$ is defined by

$$\delta(\alpha) = \overrightarrow{\alpha} + \overleftarrow{\alpha} \in C^1_{\text{def}}(\mathcal{G}).$$

Note that, for $k = 0$, one may think of a section of A as a map $c : \mathcal{G}^{(0)} = M \longrightarrow T\mathcal{G}$, with $c(1_x) = c_x \in T_{1_x}\mathcal{G}$, such that $ds \circ c_x = 0_x$.

Lemma 5.12. *$(C^\bullet_{def}(\mathcal{G}), \delta)$ is, indeed, a cochain complex.*

Proof. First of all, δ is well-defined, i.e., $\delta(c) \in C^{k+1}_{\text{def}}(\mathcal{G})$ for $c \in C^k_{\text{def}}(\mathcal{G})$; indeed, applying ds to the formula for $\delta(c)(g_1, \dots, g_{k+1})$ (and using $s(\bar{m}(a, b)) = t(b)$) one finds that $\delta(c)$ is s-projectable with s-projection

$$\begin{aligned}s_{\delta c}(g_2, \dots, g_{k+1}) = &- dt(c(g_2, \dots, g_{k+1})) + \\ &+ \sum_{i=2}^{k} (-1)^i s_c(g_2, \dots, g_i g_{i+1}, \dots, g_{k+1}) + (-1)^{k+1} s_c(g_2, \dots, g_k).\end{aligned} \tag{5.2}$$

For $\alpha \in \Gamma(A)$, it holds that $\delta(\alpha)$ is s-projectable to $\sharp(\alpha)$, the image of α by the anchor map, since $ds \circ r_g = 0$ and $ds \circ l_g \circ di = dt$.

To check that δ squares to zero, note that after cancelling the pairs of terms with opposite signs, the expression $\delta(\delta c)(g_1, \ldots, g_{k+2})$ becomes

$$\begin{aligned}
&\delta(\delta c)(g_1, \ldots, g_{k+2}) = \\
&= d\bar{m}[d\bar{m}(c(g_1g_2g_3, \ldots, g_{k+2}), c(g_3, \ldots, g_{k+2})), d\bar{m}(c(g_2g_3, \ldots, g_{k+2}), c(g_3, \ldots, g_{k+2}))] \\
&+ d\bar{m}\left(\sum_{i=3}^{k+1}(-1)^i c(g_1g_2, \ldots, g_ig_{i+1}, \ldots, g_{k+2}), \sum_{i=3}^{k+1}(-1)^i c(g_2, \ldots, g_ig_{i+1}, \ldots, g_{k+2})\right) \\
&- d\bar{m}(c(g_1g_2g_3, \ldots, g_{k+2}), c(g_2g_3, \ldots, g_{k+2})) \\
&+ \sum_{i=3}^{k+1}(-1)^{i+1} d\bar{m}(c(g_1g_2, \ldots, g_ig_{i+1}, \ldots, g_{k+2}), c(g_2, \ldots, g_ig_{i+1}, \ldots, g_{k+2})).
\end{aligned}$$

At this point, using the associativity axiom of the division map, i.e.,

$$\bar{m}(\bar{m}(g, k), \bar{m}(h, k)) = \bar{m}(g, h)$$

in the first line of the expression, and linearity of $d\bar{m}$ in the second line, we see that the first and second lines become the symmetric of the third and fourth, respectively. □

Example 5.13. We will see several (classes of) examples throughout this chapter. Let us mention here the simplest one: when $\mathcal{G}$ is a Lie group G (hence $M = \{*\}$ is a point). Then, using the trivialization $TG \cong G \times \mathfrak{g}$ induced by right translations ($\mathfrak{g}$ being the Lie algebra of G) we obtain an identification of $C^*_{\text{def}}(G)$ with the complex $C^*(G, \text{Ad})$ computing the differentiable cohomology of G with coefficients in the adjoint representation, hence

$$H^*_{\text{def}}(G) \cong H^*(G, \text{Ad}).$$

This is to be expected since the right hand side is the cohomology which controls deformations of Lie groups [80].

5.1.3 Differentiable cohomology

For a better perspective, and for the later use, let us recall here the ordinary differentiable cohomology of Lie groupoids. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid; for a representation $E \longrightarrow M$ of $\mathcal{G}$, the action $E_x \longrightarrow E_y$ induced by an arrow $g : x \longrightarrow y$ will be denoted by $v \mapsto g \cdot v$.

Definition 5.14. The **(differentiable) cohomology of the Lie groupoid $\mathcal{G}$ with coefficients in the representation** E, denoted by $H^*(\mathcal{G}, E)$, is the cohomology of the complex $(C^*(\mathcal{G}, E), \delta)$, where k-cochains are the smooth maps

$$u : \mathcal{G}^{(k)} \longrightarrow E, \quad (g_1, \ldots, g_k) \mapsto u(g_1, \ldots, g_k) \in E_{t(g_1)}$$

and the differential is given by

$$\begin{aligned}
(\delta u)(g_1, \ldots, g_{k+1}) &= g_1 \cdot u(g_2, \ldots, g_{k+1}) \\
&+ \sum_{i=1}^{k}(-1)^i u(g_1, \ldots, g_ig_{i+1}, \ldots, g_{k+1}) \\
&+ (-1)^{k+1} u(g_1, \ldots, g_k).
\end{aligned} \tag{5.3}$$

Note that, in degree zero, $H^0(\mathcal{G}, E) = \Gamma(E)^{\text{inv}}$ is the space of sections of E that are invariant with respect to the action of $\mathcal{G}$. When E is the trivial line bundle with the trivial action, the resulting complex is denoted by $C^*(\mathcal{G})$. It comes together with a graded product - the cup product, given by

$$(u \cdot v)(g_1, \ldots, g_{k+k'}) = u(g_1, \ldots, g_k)v(g_{k+1}, \ldots, g_{k+k'}) \tag{5.4}$$

for $u \in C^k(\mathcal{G})$, $v \in C^{k'}(\mathcal{G})$. The same formula makes $C(\mathcal{G}, E)$ into a right graded $C(\mathcal{G})$-module; the fact that E is a representation is encoded in the differential of $C(\mathcal{G}, E)$, which makes it into a (right) $C(\mathcal{G})$- DG-module. Note that, using precisely the same formulas for the cup-product and the same arguments, one has:

Lemma 5.15. *$(C^*_{\text{def}}(\mathcal{G}), \delta)$ is a (right) $(C^*(\mathcal{G}), \delta)$- DG-module.*

Remark 57. For the later use note also that the spaces $C^k(\mathcal{G}, E)$ make sense for any vector bundle $E \longrightarrow M$:

$$C^k(\mathcal{G}, E) = \Gamma(\mathcal{G}^{(k)}, t^*E)$$

there t on $\mathcal{G}^{(k)}$ picks the target of the first arrow. Also, whenever we have a quasi-action of $\mathcal{G}$ on E, i.e., a smooth operation that associates to any arrow $g : x \longrightarrow y$ a linear map $\lambda_g : E_x \longrightarrow E_y$ depending smoothly on g, one has an induced operator δ_λ on $C^*(\mathcal{G}, E)$ defined by exactly the same formulas as δ, but using the quasi-action; δ_λ is still a graded derivation on the $C(\mathcal{G})$-module $C(\mathcal{G}, E)$ (actually, any graded derivation is of type δ_λ for some quasi-action). The associativity of the quasi-action is equivalent to $\delta_\lambda^2 = 0$.

Remark 58. It will be often useful to consider a smaller complex computing deformation cohomology. The **normalized deformation complex** of a Lie groupoid $\mathcal{G}$ is the subcomplex $\widehat{C}^*_{\text{def}}(\mathcal{G})$ of the deformation complex $C^*_{\text{def}}(\mathcal{G})$, which in degree $k \geq 2$ is composed of those cochains $c \in C^k_{\text{def}}(\mathcal{G})$ satisfying

$$c(1_x, g_2, \ldots, g_k) = s_c(g_2, \ldots, g_k) \qquad \text{and} \qquad c(g_1, \ldots, 1_x, \ldots, g_k) = 0.$$

In degree 1, the only condition is that $c(1_x) = s_c(x)$, and in degree 0 there is no condition, i.e., $\widehat{C}^0_{\text{def}}(\mathcal{G}) = \Gamma(A)$. It is a simple computation to check that $\widehat{C}^*_{\text{def}}(\mathcal{G})$ is, indeed, a subcomplex - one only has to remember the expression for $s_{\delta c}$ (equation (5.2)).

The proof that $\widehat{C}^*_{\text{def}}(\mathcal{G})$ is quasi-isomorphic to $C^*_{\text{def}}(\mathcal{G})$ can be seen as a particular case of the proof of Theorem 5.58 (see Remark 84 and Proposition 5.65), so we postpone the discussion until then.

Remark 59 (Related to the adjoint representation). As we have already mentioned in the introduction, another guiding principle that is worth having in mind when thinking about $H^*_{\text{def}}(\mathcal{G})$ is that it plays the role of "differentiable cohomology with coefficients in the adjoint representation". The reason for the quotes is that there is no adjoint representation in the classical sense. Actually, one of the main problems with the very notion of representation is that there are only very few representations that come for free and make sense intrinsically for all Lie groupoids. The situation is slightly better in the regular case, when one has at hand the normal and the isotropy representations, denoted by ν and $\mathfrak{i}$ and recalled in the next two sections. However, even in this case, and although "the adjoint representation" is closely related to ν and $\mathfrak{i}$ (it contains them!), its structure is still subtle and requires the "up to homotopy" version of representations. This remark is the first one of a series of remarks that will guide the reader towards the full structure of the adjoint representation.

5.2 First examples

We now look at some basic examples. The order we choose is based on the simplicity of the structure of the adjoint representations involved.

5.2.1 Gauge groupoids

Recall that any principal G-bundle $\pi : P \longrightarrow M$ (G is a Lie group) has an associated gauge groupoid, which is the quotient of the pair groupoid $P \times P \longrightarrow P$ (with source and target the two projections) modulo the diagonal action of G:

$$\mathcal{G} = P \times_G P \rightrightarrows M.$$

Recall also that the adjoint bundle of P is defined as the vector bundle

$$P[\mathfrak{g}] = (P \times_G \mathfrak{g}) \cong \mathrm{Ker}(d\pi : TP/G \longrightarrow TM).$$

This bundle will also be be discussed later on, in the general context, when it will show up as the kernel of the anchor map. In this case the Lie algebroid of $\mathcal{G}$ is TP/G, its space of sections is $\mathfrak{X}(P)^G$, the Lie algebroid bracket of TP/G comes from the Lie bracket of vector fields on P and the anchor is induced by the differential of π. Important for us is the fact that $\mathrm{Ker}(\sharp) = P[\mathfrak{g}]$ is a representation of $\mathcal{G}$: indeed, any class $[p, q] \in \mathcal{G}$ viewed as an arrow from $x = \pi(p)$ to $y = \pi(q)$ induces the action

$$P[\mathfrak{g}]_x \longrightarrow P[\mathfrak{g}]_y, [p, u] \mapsto [q, u].$$

Proposition 5.16. *If G is a Lie group and $\mathcal{G}$ is the gauge groupoid associated to a principal G-bundle P, then there are canonical isomorphisms*

$$H^*_{\mathrm{def}}(\mathcal{G}) \cong H^*(\mathcal{G}, P[\mathfrak{g}]) \cong H^*(G, \mathfrak{g})$$

(where the last group is the differentiable cohomology of the Lie group G with coefficients in its adjoint representation).

There are various ways to look at this result. The first isomorphism will follow from our later general results (e.g. on the regular case); the isomorphism between the first and last groups follows directly from the Morita invariance of deformation cohomology (Theorem 5.58); the last isomorphism can also be seen as an immediate consequence of Morita invariance of differentiable cohomology [18].

Remark 60 (Related to the adjoint representation). In the spirit of Remark 59, we see that the candidate for "the adjoint representation" of the gauge groupoid is given by the adjoint bundle $P[\mathfrak{g}]$.

5.2.2 Foliation groupoids

Next we look at the Lie groupoids that arise from foliation theory (such as holonomy or homotopy groupoids of foliations), i.e., which integrate foliations; here we identify a foliation with its tangent bundle and we interpret it as a Lie algebroid with the inclusion as anchor. It then makes sense to talk about the integrability of a foliation by a groupoid. We see that a foliation groupoid is a Lie groupoid $\mathcal{G} \rightrightarrows M$ with the property that the anchor of the associated Lie algebroid is injective or, in a global formulation, with the property that the isotropy groups of $\mathcal{G}$ are discrete [26]. They come with a regular foliation $\mathcal{F}$ on M (the image of the anchor); the resulting normal bundle

$$\nu := TM/\mathcal{F}$$

is then a representation of $\mathcal{G}$, where the action is given by linear holonomy (see [26] but also our general discussion from Subsection 5.3.4 below) which, in turn, is a global manifestation of the (foliated) Bott connection on ν (cf. e.g. [49]). The resulting cohomology $H^*(\mathcal{G}, \nu)$ is the groupoid counterpart of the foliated cohomology $H^*(\mathcal{F}, \nu)$ which, in turn, was shown by Heitsch [49] to control deformations of the foliation (the two are related by a Van Est map - see [18]). Therefore the expectation that $H^*(\mathcal{G}, \nu)$ is related to deformations of foliation groupoids, i.e., to $H^*_{\mathrm{def}}(\mathcal{G})$. Moreover, while deformations of Lie groupoids give rise to 2-cocycles in deformation cohomology (think e.g. of Lie groups and see also below), deformations of foliations give rise to degree 1 classes in the cohomology with coefficients in ν; therefore one also expects a degree shift. And, indeed:

Proposition 5.17. *For any foliation groupoid $\mathcal{G} \rightrightarrows M$ one has canonical isomorphisms:*

$$H^*_{\mathrm{def}}(\mathcal{G}) \cong H^{*-1}(\mathcal{G}, \nu)$$

where the isomorphism sends a cocycle $c \in C^k_{\mathrm{def}}(\mathcal{G})$ into $[s_c]$ - the class modulo $\mathcal{F}$ of the s-projection of c (see Definition 5.11).

Proof. A simple computation shows that $c \mapsto [s_c]$ is a chain map. Since it is also surjective, it suffices to show that its kernel, call it C^*, is acyclic. So, assume that $c \in C^k$, i.e., $c \in C^k_{\mathrm{def}}(\mathcal{G})$ has the property that s_c takes values in $\mathcal{F}$. We show that $c = \delta(c')$ for some c' with the property that $s_{c'}$ takes values in $\mathcal{F}$ (we will actually achieve $s_{c'} = 0$). Namely we set

$$c'(g_1, \ldots, g_{k-1}) := -r_{g_1}(s_c(g_1, g_2, \ldots, g_{k-1})$$

where we identify $\mathcal{F}$ with the Lie algebroid of $\mathcal{G}$ to make sense of right translations. It is clear that $s_{c'} = 0$. We are left with showing that $c = \delta(c')$. Using the fact that the map $(ds, dt) : T\mathcal{G} \longrightarrow TM \times TM$ is injective, it suffices to show that $ds \circ c = ds \circ \delta(c')$ and similarly for dt. For the first one use again that ds kills c' and that $dt(r_g(\alpha)) = \sharp(\alpha) \cong \alpha$ for $\alpha \in \mathcal{F}$ and we see that, after applying (ds) to the formula for $\delta(c')(g_1, \ldots, g_k)$ we are left with

$$-dt(c'(g_2, \ldots, g_k)) = s_c(g_2, \ldots, g_k) = ds(c(g_1, \ldots, g_k)).$$

Hence we are left with showing that $dt \circ c = dt \circ \delta(c')$. Applying dt to the formula for $\delta(c')(g_1, \ldots, g_k)$ we find

$$-dt(c'(g_1 g_2, \ldots, g_k)) + \sum_{i=2}^{k-1} (-1)^i dt(c'(g_1, \ldots, g_i g_{i+1}, \ldots, g_k)) \\ + (-1)^k dt(c'(g_1, \ldots, g_k))$$

which, by the previous arguments, is

$$c_s(g_1g_2, g_3, \ldots, g_k)) + \sum_{i=2}^{k-1}(-1)^{i+1}c_s(g_1, g_2, \ldots, g_ig_{i+1}, \ldots, g_k) \\ + (-1)^{k+1}c_s(g_1, g_2, \ldots, g_{k-1})).$$

Comparing with the formula (5.2) for the s-projection of $\delta(c)$ (which vanishes because $\delta(c)$ does), we find precisely $dt(c(g_1, \ldots, g_k))$. □

Remark 61 (Related to the adjoint representation). In the spirit of Remarks 59 and 60 we see that the candidate for the adjoint representation of $\mathcal{G}$ in this case is $\nu[1]$- viewed as a graded representation of $\mathcal{G}$ concentrated in degree 1 (to make up for the shift in the proposition).

5.2.3 Action groupoids

One of the first classes of examples in which the deformation cohomology can be understood in terms of (differentiable) cohomology with coefficients in representations is that of action groupoids. So, let us assume that G is a Lie group acting on a manifold M. Recall that the action groupoid $\mathcal{G} = G \ltimes M \rightrightarrows M$ is the product $G \times M$, with $s(g, x) = x$, $t(g, x) = gx$ and $(g, x)(h, y) = (gh, y)$. The corresponding Lie algebroid is

$$A = \mathfrak{g}_M = \mathfrak{g} \times M,$$

the trivial vector bundle over M with fibre the Lie algebra $\mathfrak{g}$ of G, the anchor given by the infinitesimal action of $\mathfrak{g}$ on M

$$\sharp : \mathfrak{g}_M \longrightarrow TM,\ \sharp(v, x) = \frac{d}{d\epsilon}|_{\epsilon=0} \exp(tv)x,$$

and the bracket is uniquely determined by the Leibniz identity and the condition that, on constant sections c_v with $v \in \mathfrak{g}$, it restricts to the bracket of $\mathfrak{g}$:

$$[c_u, c_v] = c_{[u,v]}.$$

As is the convention with every Lie algebroid in this text, we identify the Lie algebra of G with the space of right-invariant vector fields on G. Note that a representation E of $\mathcal{G}$ is the same thing as an equivariant bundle over M; then $\Gamma(E)$ is naturally a G-module and $H^*(\mathcal{G}, E)$ can be interpreted as the resulting differentiable cohomology $H^*(G, \Gamma(E))$. The action groupoid $\mathcal{G}$ has two natural representation:

- $\mathfrak{g}_M$ itself, with the action induced by the adjoint action of G on $\mathfrak{g}$. Note that

  $$H^*(\mathcal{G}, \mathfrak{g}_M) = H^*(G, C^\infty(M, \mathfrak{g}))$$

  is a bundle-like version of $H^*(G; \mathfrak{g}_M)$, which controls deformations of the Lie group G, as explained by Richardson-Nijenhuis [80].

- TM, using the induced action of G on TM. Note that

$$H^*(\mathcal{G}, TM) = H^*(G, \mathfrak{X}(M))$$

arises in the work of Palais [83] as the cohomology controlling deformations of the Lie group action (keeping G fixed).

Therefore it is not surprising that, in this case, these cohomologies are closely related to the deformation cohomology of the groupoid:

Proposition 5.18. *The deformation cohomology of an action Lie groupoid $\mathcal{G} = G \ltimes M$ fits into a long exact sequence*

$$\cdots \longrightarrow H^{k-1}(\mathcal{G}, TM) \longrightarrow H^k_{\text{def}}(\mathcal{G}) \longrightarrow H^k(\mathcal{G}, \mathfrak{g}_M) \xrightarrow{\sharp_*} H^k(\mathcal{G}, TM) \longrightarrow \cdots$$

where $\sharp_*$ is induced by the infinitesimal action $\sharp : \mathfrak{g}_M \longrightarrow TM$.

Proof. It suffices to remark that there is a short exact sequence

$$0 \longrightarrow C^{k-1}(\mathcal{G}, TM) \xrightarrow{j} C^k_{\text{def}}(\mathcal{G}) \xrightarrow{\pi} C^k(\mathcal{G}, \mathfrak{g}_M) \longrightarrow 0$$

compatible with the differentials and to identify the boundary map of the long exact sequence with the map $\sharp_*$. The deformation cochains of $\mathcal{G}$ take a composable string $(\gamma_1, \ldots, \gamma_k)$ of $\mathcal{G}$, with $\gamma_1 = (g, x)$, to

$$T_{(g,x)}(G \times M) = T_g G \times T_x M \cong \mathfrak{g} \times T_x M = \mathfrak{g} \times T_{t(\gamma_2)} M$$

where the isomorphism is the one induced by the right translations of G. Moreover, the TM-component of the cochain, since it is the projection by the source map, depends only on $(\gamma_2, \ldots, \gamma_k)$; in other words, we obtain a decomposition

$$C^k_{\text{def}}(\mathcal{G}) \cong C^k(\mathcal{G}, \mathfrak{g}_M) \oplus C^{k-1}(\mathcal{G}, TM).$$

This is not compatible with the differentials, but the natural inclusion on the first component (called j) and projection on the second component (called π) are (a simple computation) - therefore the desired short exact sequence. The identification of $\sharp_*$ with the boundary map is straightforward. Or, a bit more conceptually: computing the differential of $C^k_{\text{def}}(\mathcal{G})$ with respect to the previous decomposition, one finds the formula $(c_1, c_2) \mapsto (\partial(c_1), -\sharp_*(c_1) - \partial(c_2))$, i.e., $C^*_{\text{def}}(\mathcal{G})$ is isomorphic to the mapping cone of $\sharp_*$. □

Remark 62 (Related to the adjoint representation). Note that the previous proof shows that $H^*_{\text{def}}(\mathcal{G})$ is isomorphic to the differentiable cohomology of $\mathcal{G}$ with coefficients in the cochain complex of representations:

$$\mathfrak{g}_M \xrightarrow{\sharp} TM, \tag{5.5}$$

($\mathfrak{g}_M$ in degree 0 and TM in degree 1). Hence, a remark on the adjoint representation similar to Remarks 60 and 61 shows that we have to look at even more general structures: cochain complexes of representations. Then (5.5) becomes the natural candidate for the adjoint representation in this case.

5.2.4 Bundles of Lie groups

We now take a closer look at the case of bundles of Lie groups parameterized by M; these correspond to Lie groupoids $\mathcal{G}$ over M for which the source map coincides with the target map; we denote them by

$$\pi : \mathcal{G} \longrightarrow M.$$

Its Lie algebroid is just the bundle of Lie algebras $\mathfrak{g}$ consisting of the Lie algebras $\mathfrak{g}_x$ of the Lie groups $\mathcal{G}_x$ (and the anchor is zero). A representation of $\mathcal{G}$ is then just a collection of representations of each of the groups $\mathcal{G}_x$ which are smoothly parameterized by $x \in M$ and fit into a vector bundle over M.There are two such representations of $\mathcal{G}$ that will be relevant for us: $\mathfrak{g}$ itself, endowed with the (fibrewise) adjoint actions and TM with the trivial action. The resulting cohomologies are related by a certain "curvature map"

$$K : H^*(\mathcal{G}, TM) \longrightarrow H^{*+2}(\mathcal{G}, \mathfrak{g})$$

which arises by evaluating a cohomology class with values in the Hom-bundle:

$$\mathrm{Var} \in H^2(\mathcal{G}, \mathrm{Hom}(TM, \mathfrak{g})).$$

Var arises as the obstruction to the existence of a (Ehresmann) connection on $\mathcal{G}$ which is compatible with the multiplication- in the sense that the induced parallel transport respects the group structure on the fibres. We call such connections multiplicative. To construct the class Var one starts with an arbitrary connection on $\pi : \mathcal{G} \longrightarrow M$, interpreted as a splitting $\sigma_g : T_{\pi(g)}M \longrightarrow T_g\mathcal{G}$ of $d\pi$ (smooth in $g \in \mathcal{G}$). The fact that σ is multiplicative is equivalent to the condition that, for any $x \in M$, $g, h \in \mathcal{G}_x$, $X_x \in T_xM$, one has

$$\sigma_{gh}(X_x) = (dm)(\sigma_g(X_x), \sigma_h(X_x)).$$

For our general σ (multiplicative or not), the difference between the two terms lives in the vertical space of π at gh, hence it is obtained by right translations (in $\mathcal{G}_x$) of an element in $\mathfrak{g}_x$; that element defines

$$\mathrm{Var}_\sigma(g, h)(X_x) \in \mathfrak{g}_x.$$

It is not difficult to check now that Var_σ is a differentiable cocycle, whose cohomology class does not depend on σ and whose vanishing is equivalent to the existence of a multiplicative connection. This defines Var.

Note also that Var induces at each $x \in M$ a linear map

$$\mathrm{Var}_x : T_xM \longrightarrow H^2(\mathcal{G}_x, \mathfrak{g}_x) \ (\ldots = H^2_{\mathrm{def}}(\mathcal{G}_x))$$

which encodes the variations of the group structure along directions in M (to be explained in Subsection 5.4.5 for general Lie groupoids).

Proposition 5.19. *For any bundle of Lie groups $\pi : \mathcal{G} \longrightarrow M$, interpreted as a groupoid with source and target equal to π, the deformation cohomology fits into a long exact sequence*

$$\cdots \longrightarrow H^k(\mathcal{G}, \mathfrak{g}) \xrightarrow{r} H^k_{\mathrm{def}}(\mathcal{G}) \longrightarrow H^{k-1}(\mathcal{G}, TM) \xrightarrow{K} H^{k+1}(\mathcal{G}, \mathfrak{g}) \longrightarrow \cdots .$$

Proof. Again, one has a short exact sequence

$$C^k(\mathcal{G}, \mathfrak{g}) \xrightarrow{r} C^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\pi} C^{k-1}(\mathcal{G}, TM)$$

where π takes the base component of a cochain, so that the kernel of π consists of $c \in C^k_{\mathrm{def}}(\mathcal{G})$ which take values in the vertical spaces of π, i.e., modulo right translations, come from $C^k(\mathcal{G}, \mathfrak{g})$. It is straightforward to check that π and r are chain maps (and it will be discussed in the more general context also later on). Hence we are left with identifying the boundary map in the induced long exact sequence,

$$\partial : H^{k-1}(\mathcal{G}, TM) \longrightarrow H^{k+1}(\mathcal{G}, \mathfrak{g}).$$

Consider a cocycle $u \in C^{k-1}(\mathcal{G}, TM)$. By the definition of ∂, one has to write $u = \pi(c)$ for some c, $\delta(c)$ comes, via r, from a cocycle $v \in C^{k+1}(\mathcal{G}, \mathfrak{g})$ and then $\partial([u]) = [v]$. Fixing a connection σ on $\mathcal{G}$ there is a canonical choice for c:

$$c(g_1, \ldots, g_k) = \sigma_{g_1}(u(g_2, \ldots, g_k)).$$

Compute now $\delta(c)(g_1, \ldots, g_{k+1})$. The first component in the resulting sum is

$$-d\bar{m}(\sigma_{g_1 g_2}(u(g_3, \ldots, g_{k+1}), \sigma_{g_2}(u(g_3, \ldots, g_{k+1})).$$

To handle this, note that, in general, for any pair of composable arrows (g, h) and $X \in T_{s(h)}M$,

$$\begin{aligned} d\bar{m}(\sigma_{gh}(X), \sigma_h(X)) &= d\bar{m}(dm(\sigma_g(X), \sigma_h(X)) + r_{gh}(\mathrm{Var}_\sigma(g,h)(X), \sigma_h(X)) \\ &= d\bar{m}(dm(\sigma_g(X), \sigma_h(X)), \sigma_h(X)) \\ &+ d\bar{m}(r_{gh}(\mathrm{Var}_\sigma(g,h)(X), 0_h) \\ &= \sigma_g(X) + r_g \mathrm{Var}_\sigma(g,h)(X) \end{aligned}$$

where for the first equality we have used the definition of Var_σ and for the last one the differentiated identities $\bar{m}(m(a,b), b) = b$ and $\bar{m}(agh, h) = ag = R_g(a)$. Hence the first in the sum from $\delta(c)$ is

$$-\sigma_{g_1}(u(g_3, \ldots, g_{k+1})) - r_{g_1}\mathrm{Var}_\sigma(g_1, g_2)(u(g_3, \ldots, g_{k+1})).$$

The other terms are

$$\sigma_{g_1}(u(g_2 g_3, \ldots, g_{k+1}) + \ldots + (-1)^{k+1} u(g_1, \ldots, g_k)).$$

Hence, adding up and using that $\delta(u)$ is zero, we find that

$$v(g_1, \ldots, g_{k+1}) = r_{g_1^{-1}}(\delta(c)(g_1, \ldots, g_{k+1}) = -\mathrm{Var}_\sigma(g_1, g_2)(u(g_3, \ldots, g_{k+1})).$$

$\square$

Remark 63 (Related to the adjoint representation). It is not so clear any more how to continue the series of remarks 60, 61 and 62 on the adjoint representation, so that it applies also to bundles of Lie groups. The trouble comes from the presence of variation (curvature). Indeed, while it is clear that the relevant graded representation is $\mathfrak{g}[0] \oplus TM[1]$ (with the zero differential), it is not so clear how to interpret K. Already at this point, for this very simple class of examples, we need the notion of representation up to homotopy (still to be recalled!).

5.2.5 Relation to Poisson geometry

Recall that a Poisson manifold is a manifold M equipped with a bivector $\pi \in \Lambda^2(T^*M)$ such that the Poisson bracket defined by $\{f, g\} = \pi(df, dg)$ satisfies the Jacobi identity on $C^\infty(M)$. It gives rise to a Lie algebroid structure on T^*M with anchor $\pi^\sharp$ given by $\beta(\pi^\sharp(\alpha)) = \pi(\alpha, \beta)$ for all 1-forms α and β and Lie bracket given by

$$[\alpha, \beta]_\pi = L_{\pi^\sharp(\alpha)}(\beta) - L_{\pi^\sharp(\beta)}(\alpha) - d(\pi(\alpha, \beta)).$$

The global counterpart of a Poisson manifold (whenever it exists) is a symplectic groupoid, i.e., a Lie groupoid Σ equipped with a symplectic form $\omega \in \Omega^2(\Sigma)$ which is multiplicative:

$$m^*\omega = pr_1^*\omega + pr_2^*\omega.$$

On the other hand, given any vector bundle A, there is a 1-1 correspondence between Lie algebroid structures on A and Poisson structures on the dual vector bundle A^* which are linear along the fibres. A Lie groupoid $\mathcal{G}$ with Lie algebroid A gives rise to a symplectic groupoid integrating A^* - namely $T^*\mathcal{G}$ (see [17] for the groupoid structure on $T^*\mathcal{G}$). Moreover, deformations of $\mathcal{G}$ give rise to deformations of the Poisson structure of A^*, which are controlled by the Poisson cohomology $H^*_\pi(A^*)$, which is closely related to the differentiable cohomology $H^*(T^*\mathcal{G})$ via a Van Est map (see [27]). Hence one expects a relation between $C^*_{\mathrm{def}}(\mathcal{G})$ and the differentiable cohomology complex $C^*(T^*\mathcal{G})$ (an inclusion!). We obtain in this way, the complex described by Gracia-Saz and Mehta in [42].

Indeed, $C^*_{\mathrm{def}}(\mathcal{G})$ can be identified with the subcomplex $C^*_{\mathrm{proj}}(T^*\mathcal{G})$ of $C^*(T^*\mathcal{G})$ consisting of **left-projectable linear cochains**. This subcomplex arises in [42] by looking at the VB-groupoid interpretation of the adjoint representation, where $C^*_{\mathrm{proj}}(T^*\mathcal{G})$ is introduced as the VB-groupoid complex of the groupoid $T\mathcal{G}$. Let us now look in more detail at the relation between these complexes.

First of all, there is a natural subcomplex $C^*_{\mathrm{lin}}(T^*\mathcal{G})$ of $C^*(T^*\mathcal{G})$, where k-cochains are those which are linear in $(T^*\mathcal{G})^{(k)}$. Inside $C^*_{\mathrm{lin}}(T^*\mathcal{G})$ there is the space of left-projectable linear cochains $C^*_{\mathrm{proj}}(T^*\mathcal{G})$, where a linear k-cochain u is called left-projectable if it satisfies 2 conditions: for any $(\xi_1, \ldots, \xi_k) \in (T^*\mathcal{G})^{(k)}$ such that $\xi_i \in T^*_{g_i}\mathcal{G}$,

1. the value of $u(\xi_1, \ldots, \xi_k)$ depends only on ξ_1 and on the base points $g_2, \ldots, g_k$, i.e.,

$$u(\xi_1, \ldots, \xi_k) = u(\xi_1, 0_{g_2}, \ldots, 0_{g_k});$$

2. the cochain u is left-invariant in the first argument, in the sense that

$$u(0_h \cdot \xi_1, \ldots, \xi_k) = u(\xi_1, \ldots, \xi_k),$$

whenever 0_h and ξ_1 are composable.

It is not hard to check from the definitions that the differential of $C^*(T^*\mathcal{G})$ restricts to $C^*_{\mathrm{lin}}(T^*\mathcal{G})$ and to $C^*_{\mathrm{proj}}(T^*\mathcal{G})$ making them into subcomplexes. Putting together Proposition 5.5 and Theorem 5.6 of [42] (in the case where the VB-groupoid of *loc. cit.* is $T\mathcal{G}$), one obtains:

Proposition 5.20. *There is an isomorphism of right $(C^*(\mathcal{G}), \delta)$-DG-modules*

$$\phi : C^*_{\mathrm{def}}(\mathcal{G}) \longrightarrow C_{\mathrm{proj}}(T^*\mathcal{G})$$

given by

$$\phi(c)(\xi_1, \ldots, \xi_k) = \xi_1(c(g_1, \ldots, g_k))$$

for any $\xi_1, \ldots, \xi_k \in T^*\mathcal{G}^{(k)}$ such that $\xi_i \in T^*_{g_i}\mathcal{G}$.

5.3 Low degrees

Here we look at the deformation cohomology in low degrees (mainly 0 and 1). This will already bring into the discussion the two natural representations of $\mathcal{G}$, namely the isotropy and the normal representation (both part of the adjoint representation!), and will also reveal the presence of curvature. Throughout this section we fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ with Lie algebroid denoted by A.

5.3.1 The isotropy representation

The discussion in degree zero will require the isotropy representation of $\mathcal{G}$. First of all, consider the isotropy bundle

$$\mathfrak{i} = \mathfrak{i}_A := \mathrm{Ker}(\sharp : A \longrightarrow TM).$$

The bracket of A induces a Lie algebra bracket on each fibre $\mathfrak{i}_x$ and, using the groupoid $\mathcal{G}$, $\mathfrak{i}_x$ is just the Lie algebra of the isotropy group at x (arrows starting and ending at x). Moreover, conjugation by $g : x \longrightarrow y$ in $\mathcal{G}$ induces, after differentiation at units, the "action" of $\mathcal{G}$ on $\mathfrak{i}$

$$\mathrm{Ad}_g : \mathfrak{i}_x \longrightarrow \mathfrak{i}_y.$$

When $\mathcal{G}$ is regular (in the sense that its orbits have the same dimension or, equivalently, that $\sharp$ has constant rank), $\mathfrak{i}$ is a (smooth) representation of $\mathcal{G}$. In general, it is only a set-theoretic representation of $\mathcal{G}$. However, one can still make sense of the space of its smooth (invariant) sections: define $\Gamma(\mathfrak{i})$ by requiring the smoothness as sections of A:

$$\Gamma(\mathfrak{i}) = \mathrm{Ker}(\sharp : \Gamma(A) \longrightarrow \Gamma(TM)),$$

and then

$$H^0(\mathcal{G}, \mathfrak{i}) = \Gamma(\mathfrak{i})^{\mathrm{inv}} := \{\alpha \in \Gamma(\mathfrak{i}) \mid \mathrm{Ad}_g(\alpha(x)) = \alpha(y) \;\; \forall\, g : x \longrightarrow y \text{ in } \mathcal{G}\}.$$

Proposition 5.21. *For any Lie groupoid $\mathcal{G}$, $H^0_{\mathrm{def}}(\mathcal{G}) \cong H^0(\mathcal{G}, \mathfrak{i}) = \Gamma(\mathfrak{i})^{\mathrm{inv}}$.*

Proof. We have to show that $\alpha \in \Gamma(A)$ satisfies $\overrightarrow{\alpha} + \overleftarrow{\alpha} = 0$ if and only if α is an invariant section of $\mathrm{Ker}(\sharp)$. After applying dt to the equation we see that $\sharp(\alpha)$ must be 0. This implies that $(di)(\alpha_x) = -\alpha_x$ for all $x \in M$ and then the condition $r_g(\alpha_y) + l_g(di)(\alpha_x) = 0$ for $g : x \longrightarrow y$ becomes the invariance of α. □

Remark 64. The action of $\mathcal{G}$ on $\mathfrak{i}$ has as infinitesimal counterpart the action of A given by the Lie bracket. The elements of $\Gamma(\mathfrak{i})$ that are invariant with respect to the infinitesimal action (i.e., $\alpha \in \Gamma(\mathfrak{i})$ with $[u, \alpha] = 0$ for all $u \in \Gamma(A)$) are precisely the ones in the centre $Z(\Gamma(A))$ of the Lie algebra $\Gamma(A)$. Hence, by the standard arguments, we have $\Gamma(\mathfrak{i})^{\mathrm{inv}} \subseteq Z(\Gamma(A))$ and equality holds when $\mathcal{G}$ has connected s-fibres.

5.3.2 $H^*(\mathcal{G}, \mathfrak{i})$ and its contribution to $H^*_{\mathrm{def}}(\mathcal{G})$

Even when $\mathfrak{i}$ is not of constant rank (i.e., $\mathcal{G}$ is not regular), one can still make sense of the differentiable cohomology $H^*(\mathcal{G}, \mathfrak{i})$. The defining complex $C^*(\mathcal{G}, \mathfrak{i})$ is defined as in Subsection 5.1.3, where the smoothness of the cochains is obtained by interpreting them as A-valued. The fact that the differential of $C^*(\mathcal{G}, \mathfrak{i})$ preserves smoothness follows e.g. from the relationship with the deformation complex: one has an inclusion (compatible with the differentials!):

$$r : C^*(\mathcal{G}, \mathfrak{i}) \longrightarrow C^*_{\mathrm{def}}(\mathcal{G}) \tag{5.6}$$

which associates to a differentiable cochain u with values in $\mathfrak{i}$ the deformation cochain c_u given by

$$c_u(g_1, \ldots, g_k) := r_{g_1}(c(g_1, \ldots, g_k)).$$

The induced map in cohomology,

$$r : H^*(\mathcal{G}, \mathfrak{i}) \longrightarrow H^*_{\mathrm{def}}(\mathcal{G}) \tag{5.7}$$

will be shown to be injective in degree 1, but may fail to be so in higher degrees.

5.3.3 Degree 1 and multiplicative vector fields

We now start looking at degree 1 by re-interpreting $H^1_{\mathrm{def}}(\mathcal{G})$ in terms of multiplicative vector fields. Recall that, for any groupoid $\mathcal{G} \rightrightarrows M$, $T\mathcal{G} \rightrightarrows TM$ is canonically a groupoid with structure maps the differentials of the structure maps of $\mathcal{G}$; with this, a vector field $X \in \mathcal{G}$ is called **multiplicative** if, as a map $X : \mathcal{G} \longrightarrow T\mathcal{G}$, it is a morphism of groupoids. In other words, X must be projectable both along s as well as along t to some vector field $V \in \mathfrak{X}(M)$, $du(V_x) = X_{u(x)}$ and

$$X_{gh} = dm(X_g, X_h)$$

for any pair (g, h) of composable arrows. A particular class of multiplicative vector fields are those of type $\overrightarrow{\alpha} + \overleftarrow{\alpha}$ with $\alpha \in \Gamma(A)$ - whose flows give rise to inner automorphisms of $\mathcal{G}$; therefore they are called **inner multiplicative vector fields**.

Proposition 5.22. *For any Lie groupoid $\mathcal{G}$ one has*

$$H^1_{\mathrm{def}}(\mathcal{G}) = \frac{\text{multiplicative vector fields on } \mathcal{G}}{\text{inner multiplicative vector fields on } \mathcal{G}}.$$

Proof. One has to check that a vector field $X \in \mathfrak{X}(\mathcal{G})$ is a cocycle in $C^1_{\mathrm{def}}(\mathcal{G})$ if and only if, as a map $\mathcal{G} \longrightarrow T\mathcal{G}$, it is a groupoid morphism; but this is an immediate consequence of the characterization of groupoid morphisms from the Appendix (Corollary 5.68). □

Note also that the multiplicativity of X is equivalent with the fact that the flow of X is compatible with the groupoid structure - and this is how multiplicativity (and H^1_{def}) will come in the proof of rigidity results. So, for later use, here is a more precise statement. We will use the following general notation: for a vector field $X \in \mathfrak{X}(M)$ we denote by ϕ_X its flow, $\phi^{\epsilon}_X(x) = \phi_X(x, \epsilon)$ and by

$$\mathcal{D}(X) = \{(x, \epsilon) \in M \times \mathbb{R} \mid \phi_X(x, \epsilon) \text{ is defined}\} \subset M \times \mathbb{R}$$

its domain. For each $\epsilon > 0$, we consider the ϵ-slice $\mathcal{D}_{\epsilon}(V) \subset M$ ($x \in M$ with the property that the integral curve of V through x is defined at time ϵ). The following will be needed later on.

Lemma 5.23. *If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid and $X \in \mathfrak{X}(\mathcal{G})$ is multiplicative, then the flow of X preserves the groupoid structure, wherever defined. More precisely, denoting by $V \in \mathfrak{X}(M)$ the base field of X, then, for any $\epsilon \geq 0$, $\mathcal{D}_\epsilon(X) \subset \mathcal{G}$ is an open subgroupoid of $\mathcal{G}$ with base $\mathcal{D}_\epsilon(V)$, and*

$$\phi_X^\epsilon : \mathcal{D}_\epsilon(X) \longrightarrow \mathcal{G}$$

is a morphism of groupoids covering the flow of V.

If $\mathcal{G}$ is proper then, moreover,

$$\mathcal{D}_\epsilon(X) = \mathcal{G}|_{\mathcal{D}_\epsilon(V)},$$

i.e., the flow $\phi_X^\epsilon(g)$ is defined precisely when $\phi_X^\epsilon(s(g))$ and $\phi_X^\epsilon(t(g))$ are (and this holds under the weaker hypothesis on X that it is s and t-projectable to some $V \in \mathfrak{X}(M)$).

Proof. The first part is well-known (see [60]). For the second part, the inclusion " $\subset$" is clear; we prove the reverse one. Let $g \in \mathcal{G}$, consider the maximal integral curve γ_s of V through $s(g)$, similarly γ_t, and let (a, b) be the intersection of the domains of γ_s and γ_t. Denoting by I the domain of the maximal curve γ of X through g, we know that $I \subset (a, b)$ and we have to show that equality holds. We show that for all $(u, v) \subset I$ with $a < u \leq v < b$, one must have $[u, v] \subset I$: indeed, for such u and v,

$$\gamma((u, v)) \subset \{a \in \mathcal{G} \mid s(a) \in \gamma_s([u, v]), t(a) \in \gamma_t([u, v])\}$$

where the last subspace of $\mathcal{G}$ is compact because $\mathcal{G}$ is proper. Since $\gamma((u, v))$ is relatively compact, it follows that $[u, v]$ is contained in I. $\square$

5.3.4 The normal representation ν and $H^0(\mathcal{G}, \nu)$

Next we have a closer look at $H^1_{\mathrm{def}}(\mathcal{G})$ in a way similar to that from Proposition 5.21. The discussion will bring into the picture another important "representation" of $\mathcal{G}$ - the normal one (another piece of the adjoint representation!). Consider the normal bundle

$$\nu := \mathrm{Coker}(\sharp) = TM/\sharp(A).$$

As in the case of $\mathfrak{i}$, this is a smooth vector bundle only in the regular case, but, in general, we can still talk about its fibres and, as we shall see, make sense of "its space of smooth (invariant) sections". First of all, still as for $\mathfrak{i}$, any arrow $g : x \longrightarrow y$ of $\mathcal{G}$ induces an action

$$\mathrm{Ad}_g : \nu_x \longrightarrow \nu_y.$$

Explicitly, for $v \in \nu_x$, one chooses a curve $g(\epsilon) : x(\epsilon) \longrightarrow y(\epsilon)$ in $\mathcal{G}$ with $g(0) = g$ and such that $\dot{x}(0) \in T_xM$ represents v, and then $\mathrm{Ad}_g(v)$ is the class of $\dot{y}(0) \in T_yM$. One can check that this construction does not depend on the choices involved. The following is immediate:

Lemma 5.24. *For any $g : x \longrightarrow y$, the action $\mathrm{Ad}_g : \nu_x \longrightarrow \nu_y$ is uniquely determined by the condition that, for any vector $X_g \in T_g\mathcal{G}$, it sends the class of $ds(X_g)$ modulo $\sharp(A)$ to that of $dt(X_g)$. In particular, for $V \in \mathfrak{X}(M)$, the condition that*

$$M \ni x \mapsto [V_x] \in \nu_x$$

is invariant is equivalent to the fact that for a (any) s-lift $X \in \mathfrak{X}(\mathcal{G})$ of V and any $g : x \longrightarrow y$, there exists $\eta(g) \in A_{t(g)}$ such that

$$V_{t(g)} = dt(X_g) + \sharp(\eta(g)). \tag{5.8}$$

In the general (i.e., the possibly non-regular) case, since ν is a quotient, making sense of (the space of) smooth sections of ν is more subtle than for $\mathfrak{i}$; and the same for defining $H^*(\mathcal{G}, \nu)$ - and here we will only take care of degree zero, i.e., making sense of "invariant sections". First of all, one defines

$$\Gamma(\nu) := \mathfrak{X}(M)/\mathrm{Im}(\sharp).$$

Note that any $[V] \in \Gamma(\nu)$ induces a set theoretic section $M \ni x \mapsto [V_x] \in \nu_x$, but the two objects are now different. Similarly, the invariance of $[V] \in \Gamma(\nu)$ is a stronger (or better: a smooth version) of the pointwise invariance of the induced set theoretical section. More precisely, with the last part of the previous lemma in mind, it is natural to define the invariance of $[V] \in \Gamma(\nu)$ by requiring that for a/any s-lift $X \in \mathfrak{X}(\mathcal{G})$ of V, the invariance equation (5.8) holds for some smooth section η over $\mathcal{G}$ of t^*A. Replacing X by X' given by $X'_g = X_g + r_g(\eta(g))$, we arrive at the following:

Definition 5.25. We say that $[V] \in \Gamma(\nu)$ is invariant if there exists a vector field $X \in \mathfrak{X}(\mathcal{G})$ which is both s as well as t-projectable to V - in which case we say that X is an (s,t)-lift of V. The resulting space of invariant elements is denoted

$$H^0(\mathcal{G}, \nu) = \Gamma(\nu)^{\mathrm{inv}} \subset \Gamma(\nu).$$

From the previous discussion, it is clear that, in the regular case, one recovers the usual space of sections of the smooth bundle ν and its invariant sections. In general, we obtain:

Lemma 5.26. *One has a natural linear map*

$$\pi : H^1_{\mathrm{def}}(\mathcal{G}) \longrightarrow \Gamma(\nu)^{\mathrm{inv}}$$

which associates to a multiplicative vector field X on $\mathcal{G}$ the class modulo Im($\sharp$) of the vector field on M associated with X.

Remark 65 (Related to the adjoint representation). It is instructive to keep in mind the (intuitive for now) interpretations of this discussion in terms of the adjoint representation. As indicated by the previous examples, both $\mathfrak{i}$ as well as ν contribute to the adjoint representation. The fact that $H^1_{\mathrm{def}}(\mathcal{G})$ is related to $H^0(\mathcal{G}, \nu)$ indicates that the contribution of ν involves a degree shift by one; so, a first guess would be that the adjoint representation is represented by $\mathfrak{i}[0] \oplus \nu[1]$ (the first one in degree 0, the second one in degree 1). There are two remarks to have in mind here:

- in order to stay within smooth vector bundles also in the non-regular case, one should think of $\mathfrak{i}[0] \oplus \nu[1]$ as the cohomology of the length two complex

$$0 \longrightarrow A \xrightarrow{\sharp} TM \longrightarrow 0$$

  with A in degree zero and TM in degree 1. The idea is to think of such complexes of vector bundles as being smooth representatives of their (possibly non-smooth) cohomology bundles, and then work with such complexes "up to homotopy" (quasi-isomorphisms).

- however, even in the regular case when no smoothness issues arise and one could (try to) use the graded representation $\mathfrak{i}[0] \oplus \nu[1]$, the resulting cohomology (in degree 1) would surject onto $H^0(\mathcal{G}, \nu)$ - which is certainly not the case for the deformation cohomology and π. This indicates a more subtle structure of the adjoint representation, related to the cokernel of π.

5.3.5 The first manifestation of curvature

Next, we look closer at the kernel and cokernel of π. As we shall see, this will bring the cohomology $H^*(\mathcal{G}, \mathfrak{i})$ back to our attention. Looking for the cokernel reveals the presence of "curvature". The precise meaning of "curvature" will become clear later on (see also the next remark); here we note its manifestation on the cohomology of lower degrees.

Lemma 5.27. *For $[V] \in \Gamma(\nu)^{\mathrm{inv}}$, choosing an (s,t)-lift $X \in \mathfrak{X}(\mathcal{G})$, $\delta(X) \in C^2_{\mathrm{def}}(\mathcal{G})$ takes values in the subcomplex (see (5.6))*

$$C^2(\mathcal{G}, \mathfrak{i}) \overset{r}{\hookrightarrow} C^2_{\mathrm{def}}(\mathcal{G}),$$

and it defines a cocycle in $H^2(\mathcal{G}, \mathfrak{i})$ whose cohomology class does not depend on the choice of X; hence one has an induced linear map (the cohomological curvature in degree 0)

$$K : \Gamma(\nu)^{\mathrm{inv}} \longrightarrow H^2(\mathcal{G}, \mathfrak{i}).$$

Proof. Note that the inclusion (5.6) identifies $C^*(\mathcal{G}, \mathfrak{i})$ with the subcomplex of $C^*_{\mathrm{def}}(\mathcal{G})$ consisting of deformation cochains that take values in $\mathrm{Ker}(ds) \cap \mathrm{Ker}(dt)$; therefore we will work only inside the deformation complex. Computing $ds(\delta(X)(g,h))$ we find

$$ds(d\bar{m}(X_{gh}, X_h)) - ds(X_g) = dt(X_h) - ds(X_g) = V_{t(h)} - V_{s(g)} = 0$$

and similarly for $dt(\delta(X)(g,h))$; hence $\delta(X)$ lives in the subcomplex. Moreover, if X' is another (s,t)-lift, then $c := X' - X$ is in the subcomplex hence $\delta(X') = \delta(X) + \delta(c)$ represent the same class in $H^2(\mathcal{G}, \mathfrak{i})$. Finally, this class only depends on the class $[V]$. Indeed, if $[V] = [V']$, then we find $\alpha \in \Gamma(A)$ such that $V' = V + \sharp(\alpha)$; then, if X is an (s,t)-lift of V, we can use $X' = X + \overrightarrow{\alpha} + \overleftarrow{\alpha} = X + \delta(\alpha)$ as the (s,t)-lift of V' but then $\delta(X') = \delta(X)$. $\square$

Remark 66 (Related to the adjoint representation). Continuing the previous remark on the adjoint representation, the presence of K is a manifestation of the more subtle structure of the adjoint representation: it shows that there is a certain interaction between ν and $\mathfrak{i}$, that allows one to move from ν to $\mathfrak{i}$, via a differentiable 2-cocycle (hence a 2-cocycle with values in $\mathrm{Hom}(\nu, \mathfrak{i})$ in the regular case, and some kind of cocycle with values in $\mathrm{Hom}(TM, A)$ in general).

Putting all the maps together, we have:

Proposition 5.28. *One has an exact sequence*

$$0 \longrightarrow H^1(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^1_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\pi} \Gamma(\nu)^{\mathrm{inv}} \xrightarrow{K} H^2(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^2_{\mathrm{def}}(\mathcal{G}).$$

Proof. For the injectivity of the first map r, assume that $c \in C^1_{\mathrm{def}}(\mathcal{G})$ comes from $C^1(\mathcal{G}, \mathfrak{i})$ (i.e., $c(g)$ is killed by both ds and dt, for all g) and it is also exact. Hence $c = \delta(\alpha)$ for some $\alpha \in \Gamma(A)$;

since dt sends $\delta(\alpha) = \overrightarrow{\alpha} + \overleftarrow{\alpha}$ to $\sharp(\alpha)$, we see that $\alpha \in \Gamma(\mathfrak{i})$ hence c is a coboundary in the subcomplex $C^*(\mathcal{G}, \mathfrak{i})$.

For the exactness at $H^1_{\mathrm{def}}(\mathcal{G})$, compute the kernel of π: it consists of classes $[X]$, where X is a multiplicative vector field on $\mathcal{G}$ with the property that its base field V is zero in $\Gamma(\nu)$, i.e., it is of type $V = \sharp(\alpha)$ for some $\alpha \in \Gamma(A)$; replacing X by $X - \delta(\alpha)$, we deal with classes $[X]$ with the property that the base field is zero, i.e., coming from the inclusion (5.6).

Next, we take care of the kernel of K: it consists of classes $[V] \in \Gamma(\nu)^{\mathrm{inv}}$ with the property that, choosing $X \in \mathfrak{X}(\mathcal{G})$ an (s,t)-lift of V, one has $\delta(X) = \delta(c) \in C^2_{\mathrm{def}}(\mathcal{G})$, for some c that is killed by ds and dt; replacing X by $X - c$, we see that we deal with classes $[V]$ which admit an (s,t)-lift Y that is closed in the deformation complex, i.e., which is multiplicative as a vector field. Hence $\mathrm{Ker}(K) = \mathrm{Im}(\pi)$.

Finally, the kernel of the last map r: by the definition of K, it is clear that $r \circ K = 0$; conversely, if $[c] \in H^2(\mathcal{G}, \mathfrak{i})$ is in the kernel of r, we have $r(c) = \delta(X) \in C^2_{\mathrm{def}}(\mathcal{G})$ for some $X \in C^1_{\mathrm{def}}(\mathcal{G})$; however, the fact that $r(c)$ (hence $\delta(X)$) takes values in the kernels of ds and dt, implies that X will be (s,t)-projectable to some $V \in \mathfrak{X}(M)$, hence, by the construction of K, $[c] = K([V])$. □

5.4 Degree 2 and deformations

In this section we indicate the relevance of deformation cohomology to deformations of Lie groupoids by explaining how such deformations give rise to 2-cocycles.

5.4.1 The case of (s,t)-constant deformations

Let $\{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ be a strict deformation of $\mathcal{G}$ (see the introduction); we would like to study the variation of the groupoid structure. This variation is, at least intuitively, measured by the variation of the structure maps, such as of the expressions of type $m_\epsilon(g,h)$ around $\epsilon = 0$. As mentioned in the appendix, to make sense of this, one encounters the problem that if (g,h) are composable with respect to the original groupoid structure, they may fail to be composable for $\mathcal{G}_\epsilon$. Although the appendix indicates the way to proceed (using $\bar{m}$), let us first assume first that s_ϵ and t_ϵ do not depend on ϵ and proceed in a more classical way.

In this case

$$-\frac{d}{d\epsilon}|_{\epsilon=0} m_\epsilon(g,h) \in T_{gh}\mathcal{G}$$

is well-defined for any $(g,h) \in \mathcal{G}^{(2)}$ and is killed by ds and dt (the choice of the sign will soon become clear). Being killed by ds means that it lives in the image of the right translation $R_{gh} : A_{t(g)} \longrightarrow T_{gh}\mathcal{G}$, while being killed also by dt means that it comes from the isotropy $\mathfrak{i}_{t(g)}$. Hence we end up with a differential cochain

$$u_0 \in C^2(\mathcal{G}, \mathfrak{i})$$

with the defining property

$$\frac{d}{d\epsilon}|_{\epsilon=0} m_\epsilon(g,h) = -r_{gh}(u_0(g,h)) \in T_{gh}\mathcal{G}.$$

Moreover, differentiating the associativity equation

$$m_\epsilon(m_\epsilon(g,h),k) = m_\epsilon(g, m_\epsilon(h,k))$$

with respect to ϵ at 0, we find that u_0 is a cocycle. As it will follow from the more general discussion, or checked directly (but see also [109] where strict deformations were first discussed):

Lemma 5.29. *The resulting cohomology class*

$$[u_0] \in H^2(\mathcal{G}, \mathfrak{i})$$

only depends on the equivalence class of the deformation.

Before we pass to the more general case of s-constant deformations, we consider the image ξ_0 of u_0 by the inclusion

$$r : C^2(\mathcal{G}, \mathfrak{i}) \hookrightarrow C^2_{\mathrm{def}}(\mathcal{G}),$$

(see (5.6)) or, explicitly,

$$\xi_0(g,h) = r_g(u_0(g,h)) = -r_{h^{-1}} \frac{d}{d\epsilon}|_{\epsilon=0} m_\epsilon(g,h).$$

For a more convenient formula, differentiate the identity $m_\epsilon(\bar{m}_\epsilon(m_0(g,h),h),h) = m_0(g,h)$ at $\epsilon = 0$:

$$\frac{d}{d\epsilon}|_{\epsilon=0} m_\epsilon(\bar{m}_0(gh,h),h) + (dm_0)_{g,h}\left(\frac{d}{d\epsilon}|_{\epsilon=0}\bar{m}_\epsilon(gh,h), 0_h\right) = 0;$$

using also the fact that in any groupoid $(dm)_{g,h}(X_g, 0_h) = r_h(X_g)$ and applying $r_{h^{-1}}$ we find

$$\xi_0(g,h) = \frac{d}{d\epsilon}|_{\epsilon=0}\bar{m}_\epsilon(gh,h)).$$

5.4.2 The case of s-constant deformations I: the direct approach

The advantage of the last expression is that it makes sense for all s-constant deformations. Of course, this is also very much in the spirit of the appendix, which teaches us that we should look at the variation of $\bar{m}_\epsilon$ (and only at it).

Definition 5.30. Given an s-constant deformation $\{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ of $\mathcal{G}$, the associated deformation cocycle is defined as

$$\xi_0 \in C^2_{\mathrm{def}}(\mathcal{G}) \text{ given by } : \quad \xi_0(g,h) = \frac{d}{d\epsilon}|_{\epsilon=0}\bar{m}_\epsilon(gh,h) \in T_g\mathcal{G}.$$

Lemma 5.31. *ξ_0 is indeed a cocycle and its cohomology class $[\xi_0] \in H^2_{\mathrm{def}}(\mathcal{G})$ depends only on the equivalence class of the deformation.*

Proof. The first part follows again from the associativity, but written in terms of the division:

$$\bar{m}_\epsilon(\bar{m}_\epsilon(u,w), \bar{m}_\epsilon(v,w)) = \bar{m}_\epsilon(u,v)$$

Indeed, by differentiating at $\epsilon = 0$ we we obtain

$$d\bar{m}_0\left(\frac{d}{d\epsilon}_{|\epsilon=0}\bar{m}_\epsilon(u,w), \frac{d}{d\epsilon}_{|\epsilon=0}\bar{m}_\epsilon(v,w)\right) + \\ + \left(\frac{d}{d\epsilon}_{|\epsilon=0}\bar{m}_\epsilon\right)(uw^{-1}, vw^{-1}) - \left(\frac{d}{d\epsilon}_{|\epsilon=0}\bar{m}_\epsilon\right)(u,v) = 0,$$

and by choosing $u = g_1g_2g_3$, $v = g_2g_3$ and $w = g_3$, this means precisely that

$$\delta\xi_0(g_1,g_2,g_3) = d\bar{m}_0(\xi_0(g_1g_2,g_3),\xi_0(g_2,g_3)) - \xi_0(g_1,g_2g_3) + \xi_0(g_1,g_2) = 0.$$

The second part follows similarly. If $\phi^\epsilon : \mathcal{G}_\epsilon \longrightarrow \mathcal{G}'_\epsilon$ is an equivalence of deformations let

$$X = \frac{d}{d\epsilon}_{|\epsilon=0}\phi^\epsilon \in C^1_{\mathrm{def}}(\mathcal{G}).$$

Then, by differentiating the expression

$$\bar{m}'_\epsilon(\phi^\epsilon(gh), \phi^\epsilon(h)) = \phi^\epsilon\bar{m}_\epsilon(gh,h)$$

we obtain that

$$\xi'_0(g,h) - \xi_0(g,h) = \delta X(g,h).$$

□

Example 5.32. Let $X \in \mathfrak{X}(\mathcal{G})$ be a vector field on a groupoid $\mathcal{G} \rightrightarrows M$ and assume it is s-projectable to some $V \in \mathfrak{X}(M)$ (hence $X \in C^1_{\mathrm{def}}(\mathcal{G})$). To avoid irrelevant technicalities, we assume X to be complete and let ϕ_X^ϵ be its flow. Then one obtains a new family of groupoid structures on $\mathcal{G}$ by pulling-back the original structure along ϕ_X^ϵ; hence

$$s_\epsilon = \phi_V^{-\epsilon}\circ s\circ\phi_X^\epsilon, \quad m_\epsilon = \phi^{-\epsilon}\circ m\circ(\phi_X^\epsilon,\phi_X^\epsilon), \quad \text{etc.}$$

This deformation is s-constant since X is s-projectable, and it is trivial by the very construction. Hence we know that the associated cocycle is exact; and, computing it, one finds precisely $\delta(X) \in C^2_{\mathrm{def}}(\mathcal{G})$.

Remark 67. One can remove the condition that s_ϵ is constant and treat general strict deformations; the price to pay for this generality is that we will no longer have a canonical 2-cocycle, but only a canonical 2-cohomology class. Let us indicate how to proceed in a direct fashion (an alternative route will be described in detail, for general deformations, a bit later). On follows the obvious idea: given an arbitrary strict deformation $\mathcal{G}_\epsilon$ of $\mathcal{G}$, replace it by an equivalent one, $\mathcal{G}'_\epsilon$, which is s-constant. That amounts to finding a family of diffeomorphisms $\phi^\epsilon : \mathcal{G} \longrightarrow \mathcal{G}$ so that, when we pullback the groupoid structure of $\mathcal{G}_\epsilon$ via ϕ^ϵ, the new structure $\mathcal{G}'_\epsilon$ has constant s. Assume for simplicity that ϕ^ϵ is the identity on units, so the condition is that $s_\epsilon \circ \phi^\epsilon$ does not depend on ϵ, and we look for ϕ^ϵ of type ϕ_X^ϵ - the "flow" of a time-dependent vector field $\tilde{X} = \{X^\epsilon\}$ on $\mathcal{G}$ (see the next paragraph for the notation). Differentiating with respect to ϵ, the desired equation becomes

$$(ds_\epsilon)(X^\epsilon(g)) + \frac{d}{d\epsilon}s_\epsilon(g) = 0. \tag{5.9}$$

Since each (ds_ϵ) is surjective, it is clear that such a family X^ϵ exists. Going backwards the only problem is the possible lack of completeness; however, the local flow is all that is needed to make sense of the 2-cocycle associated to the resulting m'_ϵ. The resulting cohomology class will only depend on the deformation we started with.

Next we indicate how the deformation cocycles can be used to establish rigidity results. But first some generalities on flows. By a time dependent vector field on a manifold M (or 1-parameter family of vector fields on M) we mean a family $\tilde{X} = \{X^\epsilon \mid \epsilon \in I\}$ consisting of vector fields $X^\epsilon \in \mathfrak{X}(M)$ depending smoothly on ϵ in an open interval $I \subset \mathbb{R}$ containing 0. Such a time-dependent vector field $\tilde{X}$ has a flow $\phi^{t,s}_{\tilde{X}}$ with a double dependence in the parameters; it consists of (local) diffeomorphisms

$$\phi^{t,s}_{\tilde{X}} : M \longrightarrow M,$$

the solutions to the system

$$\frac{d}{dt}\phi^{t,s}_{\tilde{X}}(x) = X(t, \phi^{t,s}_{\tilde{X}}(x)), \quad \phi^{s,s}_{\tilde{X}}(x) = x.$$

The flow relations are $\phi^{s,s}_{\tilde{X}} = \mathrm{Id}$, $\phi^{t,u}_{\tilde{X}}\phi^{u,s}_{\tilde{X}} = \phi^{t,s}_{\tilde{X}}$ - valid modulo the usual issues on the domains of definitions. When interested only on what happens for parameters close to 0, it suffices to consider the family of (local) diffeomorphisms of M given by

$$\phi^{\epsilon}_{\tilde{X}} := \phi^{\epsilon,0}_{\tilde{X}}$$

and then, for small parameters,

$$\phi^{s+\epsilon,s}_{\tilde{X}} = \phi^{s+\epsilon}_{\tilde{X}} \circ (\phi^{s}_{\tilde{X}})^{-1}.$$

When X is autonomous $\phi^{\epsilon}_{\tilde{X}}$ is the usual flow, and $\phi^{s+\epsilon,s}_{\tilde{X}} = \phi^{\epsilon}_{\tilde{X}}$ only depends on ϵ.

The flows of type $\phi^{t,s}_{\tilde{X}}$ are suited to relate the members $\mathcal{G}_s$ and $\mathcal{G}_t$ of s-constant deformations:

Proposition 5.33. *Let $\{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ be an s-constant deformation of $\mathcal{G}$, and consider the induced deformation cocycles $\xi_\epsilon \in C^2_{\mathrm{def}}(\mathcal{G}_\epsilon)$. Assume that for all ϵ small enough, one finds X^ϵ such that*

$$\delta(X^\epsilon) = -\xi_\epsilon \quad \text{in } C^2_{\mathrm{def}}(\mathcal{G}_\epsilon). \tag{5.10}$$

and assume that the resulting time dependent vector field $\tilde{X} = \{X^\epsilon\}$ is smooth. Then, for t and s close to 0, $\phi^{t,s}_{\tilde{X}}$ is a locally defined morphism from $\mathcal{G}_s$ to $\mathcal{G}_t$, covering the similar flow of $V^\epsilon = ds(X^\epsilon)$.

Moreover, if $\mathcal{G}$ is proper, then $\phi^{t,s}_{\tilde{X}}(g)$ is defined whenever $\phi^{t,s}_{\tilde{V}}(s(g))$ and $\phi^{t,s}_{\tilde{V}}(t(g))$ are.

This lemma can be proved directly but our reinterpretations from the next subsection (Proposition 5.35) will show that it is a consequence of Lemma 5.23 on flows of multiplicative vector fields.

5.4.3 The case of s-constant deformations II: reinterpretation using the groupoid $\tilde{\mathcal{G}}$

Here we provide another way of looking at the deformation cocycles, less intuitive but easier to work with. This is based on the reinterpretation of strict deformations of $\mathcal{G}$ as (germs of) families parameterized by an interval I: one can identify a strict deformation $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ with the groupoid

$$\tilde{\mathcal{G}} := \mathcal{G} \times I \quad \text{over } \tilde{M} := M \times I,$$

with structure maps

$$\tilde{s}(g, \epsilon) = (s_\epsilon(g), \epsilon),\ \tilde{m}((g, \epsilon), (h, \epsilon)) = (m_\epsilon(g, h), \epsilon),\ \text{etc.}$$

This allows us to reinterpret the deformation cocycle associated to an s-constant deformation as follows.

Proposition 5.34. *Let $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon\}$ be an s-constant deformation of the Lie groupoid $\mathcal{G}$. Then, interpreting $\frac{\partial}{\partial \epsilon}$ as an element of $C^1_{\mathrm{def}}(\tilde{\mathcal{G}})$ and considering*

$$\xi = \delta\left(\frac{\partial}{\partial \epsilon}\right) \in C^2_{\mathrm{def}}(\tilde{\mathcal{G}}),$$

one has

$$\xi_0 = \xi|_{\mathcal{G}_0} \in C^2_{\mathrm{def}}(\mathcal{G}_0).$$

Note that, implicit in this statement, is also the fact that the restriction of ξ to $\mathcal{G}_0 \subset \tilde{\mathcal{G}}$ takes values in $T\mathcal{G}_0 \subset T\tilde{\mathcal{G}}$ (warning: the operation of restricting elements of $C^*_{\mathrm{def}}(\tilde{\mathcal{G}})$ to $\mathcal{G}_0$, even when it produces elements in $C^*_{\mathrm{def}}(\mathcal{G}_0)$, is not compatible with the differentials).

Proof. To check the previous identity we use that, as a vector field on $\tilde{\mathcal{G}}$,

$$\frac{\partial}{\partial \epsilon}(g, 0) = \frac{d}{d\epsilon}|_{\epsilon=0}(g, \epsilon) \in T_{(g,0)}\tilde{\mathcal{G}} \tag{5.11}$$

hence on elements $g \equiv (g, 0)$, $h \equiv (h, 0)$,

$$\begin{aligned}
\xi((g, 0), (h, 0)) &= (d\bar{\tilde{m}})\left(\frac{\partial}{\partial \epsilon}(gh, 0), \frac{\partial}{\partial \epsilon}(h, 0)\right) - \frac{\partial}{\partial \epsilon}(g, 0) \\
&= \frac{d}{d\epsilon}|_{\epsilon=0}\bar{\tilde{m}}((gh, \epsilon), (h, \epsilon)) - \frac{\partial}{\partial \epsilon}(g, 0) \\
&= \frac{d}{d\epsilon}|_{\epsilon=0}(\bar{m}_\epsilon(gh, h), \epsilon) - \frac{\partial}{\partial \epsilon}(g, 0) \\
&= \frac{d}{d\epsilon}|_{\epsilon=0}\bar{m}_\epsilon(gh, h) = \xi_0(g, h)
\end{aligned}$$

□

Remark 68. The interpretation given in the proposition makes substantial use of the assumption that the deformation is s-constant, not only in the proof, but already right from the start when we interpreted $\frac{\partial}{\partial \epsilon}$ as an element of $C^1_{\mathrm{def}}(\tilde{\mathcal{G}})$. Indeed, when the source is not constant, this vector field on $\tilde{\mathcal{G}}$ is not even $\tilde{s}$-projectable. This is very much related to the choice of the family of vector fields X^ϵ in the previous subsection; actually, the equation (5.9) precisely means that the resulting vector field on $\tilde{\mathcal{G}}$:

$$\tilde{X}(g, \epsilon) = X^\epsilon(g) + \frac{\partial}{\partial \epsilon}(g, 0)$$

is s-projectable (... to $\frac{\partial}{\partial \epsilon} \in \mathfrak{X}(\tilde{M})$).

Next, we reinterpret Proposition 5.33 in terms of the groupoid $\tilde{\mathcal{G}}$. But first we need another general remark on time-dependent vector fields $\tilde{X} = \{X^\epsilon \mid \epsilon \in I\}$ on a manifold M; such an $\tilde{X}$ can be identified with the vector field on $M \times I$:

$$\tilde{X}(x, \epsilon) = X^\epsilon(x) + \frac{\partial}{\partial \epsilon};$$

then the flow $\phi^{t,s}_{\tilde{X}}$ mentioned above is related to the standard flow of $\tilde{X}$ by

$$\phi^\epsilon_{\tilde{X}}(x, s) = (\phi^{s+\epsilon,s}_{\tilde{X}}(x), s + \epsilon)$$

so that $\phi^{s+\epsilon,s}_{\tilde{X}}$ is seen as moving $M \times \{s\}$ to $M \times \{s + \epsilon\}$. We will apply this to the time dependent vector fields that arise in Lemma 5.33.

Proposition 5.35. *Consider an s-constant deformation as in Proposition 5.33 and the associated groupoid $\tilde{\mathcal{G}}$. Then a smooth family X^ϵ of vector fields on $\mathcal{G}$ satisfies the cocycle equations (5.10) if and only the induced vector field on $\tilde{\mathcal{G}}$*

$$\tilde{X}(g, \epsilon) = X^\epsilon(g) + \frac{\partial}{\partial \epsilon} \in \mathfrak{X}(\mathcal{G})$$

is multiplicative.

Proof. Start from the multiplicativity equation for $\tilde{X}$:

$$\tilde{X}(g) = (d\bar{\bar{m}})_{(g,\epsilon),(h,\epsilon)}(\tilde{X}(m_\epsilon(g, h), \epsilon), \tilde{X}(h, \epsilon)),$$

where $\bar{\bar{m}}$ is the division map associated to $\tilde{\mathcal{G}}$. Writing $\tilde{X}$ in terms of X^ϵ, we see that the expression on the right-hand side is the sum of two expressions:

$$(d\bar{\bar{m}})_{(g,\epsilon),(h,\epsilon)}(X^\epsilon(m_\epsilon(g, h)), X^\epsilon(h)) \text{ and } (d\bar{\bar{m}})_{(g,\epsilon),(h,\epsilon)}\left(\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon}\right).$$

Since X^ϵ is tangent to the fibre $\mathcal{G}_\epsilon$ where $\bar{\bar{m}}$ restricts to $\bar{m}_\epsilon$, the first expression is just

$$(d\bar{m}_\epsilon)_{(g,h)}(X^\epsilon(m_\epsilon(g, h)), X^\epsilon(h)).$$

For the second expression we use again (5.11) (and the obvious analogue at $\epsilon \neq 0$) and we find

$$\begin{aligned}(d\bar{\bar{m}})_{(g,\epsilon),(h,\epsilon)}\left(\frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon}\right) &= \frac{d}{ds}|_{s=0}\bar{\bar{m}}(m_\epsilon(g, h), \epsilon + s), (h, \epsilon + s)) \\ &= \frac{d}{ds}|_{s=0}\bar{m}_{\epsilon+s}(m_\epsilon(g, h), h), \epsilon + s) \\ &= \xi_\epsilon(g, h) + \frac{\partial}{\partial \epsilon}.\end{aligned}$$

Putting everything together, the multiplicativity equation for $\tilde{X}$ becomes:

$$X^\epsilon(g) = (d\bar{m}_\epsilon)_{(g,h)}(X^\epsilon(m_\epsilon(g, h)), X^\epsilon(h)) - \xi_\epsilon(g, h)$$

i.e., the equations (5.10). □

5.4.4 General deformations

The previous subsection indicates how to proceed in the case of general deformations. Let us first reformulate Definition 5.6 in terms of families of groupoids (Definition 5.9).

Definition 5.36. A **deformation of a Lie groupoid** $\mathcal{G}$ is a family of Lie groupoids

$$\tilde{\mathcal{G}} \rightrightarrows \tilde{M} \xrightarrow{\pi} I$$

parameterized by an open interval $I \subset \mathbb{R}$ containing 0 such that $\mathcal{G}_0 = \mathcal{G}$ (as Lie groupoids). Two such deformations $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ are said to be **equivalent** if, after eventually restricting them to smaller open intervals around the origin, they are isomorphic by an isomorphism that is the identity on $\mathcal{G}_0$.

A deformation is said to be **proper** if $\tilde{\mathcal{G}}$ is a proper groupoid.

We see that strict deformations of $\mathcal{G}$ correspond to deformations $\tilde{\mathcal{G}}$ with the property that, as manifolds, $\tilde{\mathcal{G}} = \mathcal{G} \times I$, $\tilde{M} = M \times I$ and π is the projection in the second factor. To construct the deformation class of a deformation, we use the reinterpretation given in Proposition 5.34. First we need an analogue of $\frac{\partial}{\partial \epsilon}$, which makes sense as an element of $C^1_{\mathrm{def}}(\tilde{\mathcal{G}})$.

Definition 5.37. Let $\tilde{\mathcal{G}}$ be a deformation of $\mathcal{G}$. A **transverse vector field** for $\tilde{\mathcal{G}}$ is any vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ which is s-projectable to a vector field $V \in \mathfrak{X}(M)$ which, in turn, is π-projectable to $\frac{d}{d\epsilon}$.

With this, we can generalize the construction from Proposition 5.34 as follows.

Proposition 5.38. *Let $\tilde{\mathcal{G}}$ be a deformation of $\mathcal{G}$. Then:*

(i) there exist transverse vector fields for $\tilde{\mathcal{G}}$.

(ii) for any $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ transverse, $\delta(\tilde{X}) \in C^2_{def}(\tilde{\mathcal{G}})$, when restricted to $\mathcal{G}_0$, induces a cocycle

$$\xi_0 \in C^2_{\mathrm{def}}(\mathcal{G}_0).$$

(iii) the cohomology class of ξ_0 does not depend on the choice of $\tilde{X}$.

Proof. To produce transverse vector fields one chooses any $\tilde{V} \in \mathfrak{X}(\tilde{M})$ that is π-projectable to $\frac{\partial}{\partial \epsilon}$ and any $\tilde{X} \in \mathfrak{X}(\tilde{\mathcal{G}})$ that is $\tilde{s}$-projectable to $\tilde{V}$. For (ii), given $\tilde{X}$, we have to show that for $(g,h) \in \mathcal{G}_0$, $\delta(\tilde{X})(g,h) \in T_g\tilde{\mathcal{G}}$ is tangent to $\mathcal{G}_0$, i.e., that it is killed by the differential of $\pi \circ \tilde{s}$; but

$$d\pi(d\tilde{s}(\delta(\tilde{X})(g,h))) = d\pi(d\tilde{t}(\tilde{X}(h)) - d\tilde{s}(\tilde{X}(g)))$$

and using that $d\pi \circ d\tilde{t} = d\pi \circ d\tilde{s}$ and that $\tilde{X}$ is $\tilde{s}$-projectable, the desired vanishing follows. Finally, assume that $\tilde{X}'$ is another transverse vector field. Then $Y := \tilde{X}' - \tilde{X}$ is killed by $d\pi \circ d\tilde{s}$, hence the values $Y(g)$ are already tangent to the fibre groupoids; in particular, on $\mathcal{G} = \mathcal{G}_0$, one has $Y_0 \in C^1_{\mathrm{def}}(\mathcal{G}_0)$ and the cocycles associated to $\tilde{X}'$ and $\tilde{X}$ are related by $\xi_0^{'} - \xi_0 = \delta(Y_0)$. $\square$

Definition 5.39. The resulting cohomology class $[\xi_0] \in H^2_{\mathrm{def}}(\mathcal{G})$ is called the **deformation class** associated to the deformation $\tilde{\mathcal{G}}$ of $\mathcal{G}$.

Remark 69. We see that, in terms of the groupoids $\tilde{\mathcal{G}}$, the natural approach to rigidity from the previous subsections takes a more algebraic but simpler form: one searches for vector fields $\tilde{X}$ on $\tilde{\mathcal{G}}$ which are both transverse as well as multiplicative, then one uses their flows $\phi^{\epsilon}_{\tilde{X}}$ to obtain isomorphisms between $\mathcal{G}_0$ and $\mathcal{G}_{\epsilon}$. Of course, there is the usual issue regarding the domain of definition of the flows; and an even more serious issue is the existence of multiplicative transverse vector fields $\tilde{X}$. By the long exact sequence of Proposition 5.28, the last issue is equivalent to the existence of $[V] \in \Gamma(\nu)^{\mathrm{inv}}$ which is in the kernel of the map K there; we see that the vanishing of cohomology in degree 2 greatly reduces this issue.

5.4.5 Families of groupoids and the variation map

Consider a family of Lie groupoids parameterized by a manifold B,

$$\mathcal{G} \rightrightarrows M \xrightarrow{\pi} B$$

(see the introduction). Using the induced groupoids $\mathcal{G}_b$ with $b \in B$, any curve $\gamma : I \longrightarrow B$, induces a deformation $\gamma^*\mathcal{G}$ of $\mathcal{G}_{\gamma(0)}$.

Proposition 5.40. *Let $\mathcal{G} \rightrightarrows M \longrightarrow B$ be a family of Lie groupoids, $b \in B$. Then, for any curve $\gamma : I \longrightarrow B$ with $\gamma(0) = b$, the deformation class of $\gamma^*\mathcal{G}$ at time* 0 *only depends on $\dot{\gamma}(0)$, and this defines a linear map*

$$Var_b : T_bB \longrightarrow H^2_{\mathrm{def}}(\mathcal{G}_b)$$

Proof. Let $v = \dot{\gamma}(0)$ and choose an arbitrary extension $\tilde{v} \in \mathfrak{X}(B)$ of v, and an s-projectable $\tilde{X} \in \mathfrak{X}(\mathcal{G})$ such that $d(\pi \circ s)(\tilde{X}) = \tilde{v}$. Define $[\tilde{\xi}_b] \in H^2_{\mathrm{def}}(\mathcal{G}_b)$ as the cohomology class of

$$\tilde{\xi}_b = (\delta\tilde{X})\big|_{\mathcal{G}_b^{(2)}}. \tag{5.12}$$

We will show that:

- The cohomology class $[\tilde{\xi}_b] \in H^2_{\mathrm{def}}(\mathcal{G}_b)$ does not depend on the choices of $\tilde{v}$ and $\tilde{X}$;
- Under the canonical identification of $(\gamma^*\mathcal{G})_0$ with $\mathcal{G}_b$, the deformation class $[\xi_0]$ coincides with $[\tilde{\xi}_b]$.

For the first statement, the proof that the class does not depend on the lift $\tilde{X}$ is identical to the proof of Proposition 5.38 (iii). To show that $[\tilde{\xi}_b]$ does not depend on the extension $\tilde{v}$, let $\tilde{v}'$ be another extension of v and choose a lift $\tilde{X}'$ of $\tilde{v}'$ such that $\tilde{X}(g) = \tilde{X}'(g)$ for all $g \in \mathcal{G}_b$. Then, for this choice of lift it follows from the linearity of δ that

$$\delta\tilde{X}(g,h) = \delta\tilde{X}'(g,h) \text{ for all } g,h \in \mathcal{G}_b^{(2)}.$$

Next we prove the second statement. Since the statement is local (in ϵ), we can assume with loss of generality the $\gamma : I \longrightarrow B$ is an embedding. We choose $\tilde{v}$ to coincide with $\dot{\gamma}$ on all points of the curve, and we take a lift $\tilde{X}$ of $\tilde{v}$ as above. Then the vector field

$$X_{(\epsilon,g)} = \frac{\partial}{\partial\epsilon} + \tilde{X}_g \in T_{(\epsilon,g)}\gamma^*\mathcal{G}$$

is transverse, and thus ξ_0 is the restriction to $(\gamma^*\mathcal{G})_0^{(2)}$ of δX.

However, since the multiplication and inversion on $\gamma^*\mathcal{G}$ are given by

$$m_\gamma(\epsilon, g), (\epsilon, h) = (\epsilon, gh), \quad (\epsilon, g)^{-1} = (\epsilon, g^{-1})$$

we obtain that

$$d\bar{m}_\gamma(X_{(0,gh)}, X_{(0,h)}) = \frac{\partial}{\partial \epsilon} + d\bar{m}(\tilde{X}_{gh}, \tilde{X}_h),$$

from where it follows that (for these choices of lift, and transverse vector field) $\tilde{\xi}_b = \xi_0$. □

Remark 70. Equation 5.12 in the proof above gives an alternative description of the variation map.

5.5 The proper case

Theorem 5.41. *Let $\mathcal{G}$ be a proper Lie groupoid. Then*

$$H^0_{\mathrm{def}}(\mathcal{G}) \cong \Gamma(\mathfrak{i})^{\mathrm{inv}}, \quad H^1_{\mathrm{def}}(\mathcal{G}) \cong \Gamma(\nu)^{\mathrm{inv}}, \quad \textit{and} \quad H^k_{\mathrm{def}}(\mathcal{G}) = 0 \textit{ for all } k \geq 2.$$

Proof. We proceed similarly to the proof of the vanishing of differentiable cohomology from [18]. As there we appeal to the fact that for any proper Lie groupoid $\mathcal{G}$ over M one can use a Haar system and a cut-off function to construct a family of linear maps

$$\int_{t^{-1}(x)} : C^\infty(t^{-1}(x)) \longrightarrow \mathbb{R}, \quad f \mapsto \int_{t^{-1}(x)} f \overset{\text{notation}}{=} \int_x f(g)\, dg$$

which depends smoothly on $x \in M$, (i.e., applying them to a smooth function on $\mathcal{G}$ produces a smooth function on M), is normalized (it sends the constant function 1 on $\mathcal{G}$ to 1 on M) and left-invariant (the integral does not change under composition with L_h, for any $h \in \mathcal{G}$).

For $k \geq 2$, let $c \in C^k_{\mathrm{def}}(\mathcal{G})$ be a cocycle and define a map $X : \mathcal{G}^{k-1} \longrightarrow T\mathcal{G}$ by

$$X(g_1, \ldots, g_{k-1}) = (-1)^k \int_{s(g_{k-1})} c(g_1, \ldots, g_{k-1}, h)\, dh. \tag{5.13}$$

The map X is an element of $C^{k-1}_{\mathrm{def}}(\mathcal{G})$ and we will now show that $\delta X = c$:

$$
\begin{aligned}
\delta X(g_1, \ldots, g_k) &= -d\bar{m}(X(g_1 g_2, \ldots, g_k), X(g_2, \ldots, g_k)) \\
&+ \sum_{i=2}^{k-1} (-1)^i X(g_1, \ldots, g_i g_{i+1}, \ldots, g_k) \\
&+ (-1)^k X(g_1, \ldots, g_{k-1}) \\
&= (-1)^k \int_{s(g_k)} -d\bar{m}(c(g_1 g_2, \ldots, g_k, h), c(g_2, \ldots, g_k, h)) \\
&+ \sum_{i=2}^{k-1} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_k, h)\ dh \\
&+ \int_{s(g_{k-1})} c(g_1, \ldots, g_{k-1}, h)\ dh \\
&\overset{\delta c = 0}{=} \int_{s(g_k)} -c(g_1, \ldots, g_{k-1}, g_k h) + c(g_1, \ldots, g_k)\ dh \\
&+ \int_{s(g_{k-1})} c(g_1, \ldots, g_{k-1}, h)\ dh \\
&\overset{\text{left-inv.}}{=} c(g_1, \ldots, g_k).
\end{aligned}
$$

Note that exactly the same formulas applied to cocycles in $C^*(\mathcal{G}, \mathfrak{i})$ imply that $H^k(\mathcal{G}, \mathfrak{i}) = 0$ for all $k \geq 0$ - and this is basically the proof of the vanishing of differentiable cohomology with coefficients in $\mathfrak{i}$ from [18], with the extra remark that the integrals still define smooth cochains: indeed, if $u \in C^k(\mathcal{G}, \mathfrak{i})$, it is smooth as a map

$$(g_1, \ldots, g_k) \mapsto u(g_1, \ldots, g_k) \in A_{t(g_1)}$$

hence so is

$$(g_1, \ldots, g_{k-1}) \mapsto \int_{s(g_{k-1})} u(g_1, \ldots, g_{k-1}, h)\ dh \in A_{t(g_1)}$$

($t(g_1)$ stays constant in the integration process). The vanishing of $H^*(\mathcal{G}, \mathfrak{i})$ in degrees 1 and 2 combined with the exact sequence from Proposition 5.28, implies the desired isomorphism $H^1_{\text{def}}(\mathcal{G}) \cong \Gamma(\nu)^{\text{inv}}$. The isomorphism $H^0_{\text{def}}(\mathcal{G}) \cong \Gamma(\mathfrak{i})^{\text{inv}}$ holds by Proposition 5.21. □

5.5.1 Some consequences (relevant for rigidity results)

Here are some consequences of the previous result (and its proof).

Corollary 5.42. *If $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ is a proper family of groupoids, $k \geq 2$ and and $u^\epsilon \in C^k_{\text{def}}(\mathcal{G}_\epsilon)$ is a smooth family of deformation cocyles, then the equations*

$$\delta(X^\epsilon) = u^\epsilon$$

admit solutions $X^\epsilon \in C^1_{\text{def}}(\mathcal{G})$ that are smooth with respect to ϵ.

Proof. This follows from the vanishing part of the theorem, applied to the full groupoid $\tilde{\mathcal{G}}$. □

Remark 71. It is important to realize that assuming only that each $\mathcal{G}_\epsilon$ is proper (which ensures that the equation for each ϵ has solution) does not suffice. One can actually find families of compact groupoids for which the previous corollary fails (because, as a family, it is not proper).

In order to handle general deformations, we know from the previous section (see e.g. Remark 69) that it is important to ensure the existence of multiplicative transverse vector fields.

Lemma 5.43. *Any proper family $\mathcal{G} \rightrightarrows M \xrightarrow{\pi} I$ of Lie groupoids parameterized by an interval I, admits a multiplicative vector field $X \in \mathfrak{X}(\mathcal{G})$ that is transverse (in the sense of Definition 5.37).*

Proof. By the isomorphism $H^1_{\text{def}}(\mathcal{G}) \cong \Gamma(\nu)^{\text{inv}}$ it suffices to show that one can find $V \in \mathfrak{X}(M)$ which is π-projectable to $\frac{\partial}{\partial \epsilon}$ and with the property that the induced $[V] \in \Gamma(\nu)$ is invariant. Start with any transverse vector field $\tilde{X} \in \mathfrak{X}(\mathcal{G})$; it will be s-projectable to some $W \in \mathfrak{X}(M)$. We define the vector field V on M by

$$V_x := \int_x dt(X_a)\ da.$$

We have

$$d\pi(V_x) = \int_x d\pi(dt(X_a))\ da = \int_x d\pi(ds(X_a))\ da = \int_x \frac{\partial}{\partial \epsilon}\ da = \frac{\partial}{\partial \epsilon}.$$

To check that $[V]$ is invariant, it suffices to find a vector field Y on $\mathcal{G}$ which is both s and t-projectable to V. We claim that

$$X'_g := \int_{t(g)} d\bar{m}(X(ga), X(a))\ da \in T_g\mathcal{G} \tag{5.14}$$

does the job. Indeed,

$$ds(X'_g) = \int_{t(g)} ds(d\bar{m}(X(ga), X(a)))\ da = \int_{t(g)} dt(X(a))\ da = W_{s(g)}$$

and a similar computation combined with the invariance of the integral shows that also $dt(X'_g) = W_{t(g)}$. □

Moreover, in order to handle deformations semi-locally (relative versions), one needs a relative version of the previous lemma.

Lemma 5.44. *With the same notation as in the previous lemma, if $N \subset M$ is a closed invariant submanifold so that $\pi|_N : N \longrightarrow I$ is still a submersion and $X_N \in \mathfrak{X}(\mathcal{G}|_N)$ is a given multiplicative transverse vector field, then X can be chosen so that it extends X_N.*

Proof. The proof is just a careful analysis of the previous proofs; here are the details. Start with X_N and its base vector field V_N. Choose an extension V of V_N that is π-projectable to $\frac{\partial}{\partial \epsilon}$ and choose $X \in \mathfrak{X}(\mathcal{G})$ which is s-projectable to V. We modify X in steps. First we make sure that X also extends X_N: since $X_N(g) - X(g)$ is killed by ds, it is of type $r_g(\eta(g))$ with $\eta(g) \in A_{t(g)}$ defined for $g \in \mathcal{G}|_N$; choosing a smooth extension $\tilde{\eta} \in \Gamma(\mathcal{G}, t^*A)$ of η one then replaces X by $g \mapsto X(g) + r_g(\tilde{\eta}(g))$.

Hence we may assume that X extends X_N and is s-projectable. One can also arrange it so that it is also t-projectable: the replacement is the one given by formula (5.14) - indeed, when applied to $g \in \mathcal{G}|_N$, the integration variable a stays in $\mathcal{G}|_N$, hence one deals with

$$\int_{t(g)} d\bar{m}(X_N(ga), X_N(a))\ da = \int_{t(g)} X_N(g)\ da = X_N(g)$$

where the multiplicativity of X_N was used.

Hence we may assume that X extends X_N and is s and t-projectable to some V. Then the expressions

$$d\bar{m}(X(gh), X(h)) - X(g) \in T_g\mathcal{G}$$

are killed by ds and dt hence they arise by right translations with respect to g of some elements

$$\zeta(g, h) \in \mathfrak{i}_{t(g)}.$$

As before one can find $\eta \in C^1(\mathcal{G}, \mathfrak{i})$ so that $\zeta = \delta(\eta)$. However, the integral formula for η (of type (5.13)) shows that η vanishes on elements of $\mathcal{G}|_N$ (because the multiplicativity of X_N implies that ζ does). We can now change X to

$$X'(g) = X(g) + r_g(\eta(g))$$

which has the same properties as X' and is also multiplicative; the multiplicativity is implicit in the last part of the proof of Theorem 5.41 but also follows easily by a direct computation. $\square$

Remark 72. Note that the last two lemmas apply (basically with no changes in the proof) to general proper families $\mathcal{G} \rightrightarrows M \xrightarrow{\pi} B$ parameterized by a manifold B; the conclusion is that any vector field W on B admits a multiplicative lift X to $\mathcal{G}$ (lift in the sense that X is projectable, via $\pi \circ s = \pi \circ t$ to W).

Moreover, using the explicit argument from the last proof (applied to $N = \emptyset$ and with I replaced by B) shows that the choice of X can be made smooth in W. Indeed, all the steps involved to produce X are given by explicit formulas that are clearly smooth, except maybe for the starting step which starts with the choice of a vector field V on M that is π-projectable to W and $X \in \mathfrak{X}(\mathcal{G})$ that is s-projectable to V. But that can be done smoothly as well, by fixing smooth splittings of $d\pi$ and ds (i.e., Ehresmann connections on π and s).

As analogue of Lemma 5.44 one can start with any sub-family $\mathcal{G}|_N \rightrightarrows N \xrightarrow{\pi} B_0$ with $B_0 \subset B$, $N \subset M$ submanifolds and N invariant and then, if $W \in \mathfrak{X}(B)$ is tangent to B_0 and one is given a multiplicative lift X_N of $W|_{B_0}$ to $\mathcal{G}|_N$, then one can find a multiplicative lift $X \in \mathfrak{X}(\mathcal{G})$ that extends X_N.

5.6 Applications to rigidity

As explained in the previous section, the main idea to obtain rigidity results is to use the vanishing of the deformation cohomology and the flows of the resulting vector fields. Here are some immediate illustrations of this idea. We start with the cases of (s,t)-constant and s-constant deformations.

Theorem 5.45. *The following hold true:*

- *Any (s,t)-constant deformation of a proper Lie groupoid is trivial.*
- *Any s-constant deformation of a compact Lie groupoid is trivial.*

Proof. It is clear that any deformation $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon\}$ as in the statement is proper. Then we follow the general plan:

- we appeal to Corollary 5.42 to obtain $-\tilde{X} = \{-X^\epsilon\}$ smoothly depending on X transgressing the deformation cocycles ξ_ϵ.
- we use the first part of Lemma 5.33 and the resulting flows $\phi^{t,s}_{\tilde{X}}$ as candidates for isomorphisms between $\mathcal{G}_s$ and $\mathcal{G}_t$.
- to make sure that $\phi^{u,v}_{\tilde{X}}$ is defined on the entire $\mathcal{G}_v$ for u and v small enough, we use the second part of Lemma 5.33; we are left with proving that $\phi^{u,v}_{\tilde{V}}$ is defined on the entire M for u and v small enough.

The very last part is clear in the second case since M is compact. For (s,t)-constant deformations, we have seen that the resulting deformation cocycles ξ_ϵ live in the subcomplex $C^2(\mathcal{G}_\epsilon, \mathfrak{i})$ and we can solve the equations (5.10) inside this subcomplex; i.e., we can arrange that $V^\epsilon = 0$ hence, again, there are no problems with the flow. □

Remark 73. It is not true that s-constant deformations of proper Lie groupoids are trivial. An example of this can be obtained by carefully analysing a construction that Palais gave in [81]. In *loc. cit* it is shown that for any non-trivial compact Lie group G, there exists an uncountable family of inequivalent G-actions on an Euclidean space $\mathbb{R}^n$, such that any of them have isomorphic linearized actions at a fixed point. More specifically:

- It is shown in [66] that there exists an open 3-manifold W which is not diffeomorphic to Euclidean 3-space, and such that $W \times \mathbb{R}^n$ is diffeomorphic to $n+3$-Euclidean space (this is really the important point!).
- If one considers the diagonal action on $W \times \mathbb{R}^n$ formed from a faithful representation of a compact group G on $\mathbb{R}^n$ and the trivial action of G on W, the corresponding (non-linear) G-action on $\mathbb{R}^{n+3}$ has a fixed point set diffeomorphic to W.
- If one takes a fixed point of the action (which without loss of generality can be assumed to be the origin) the homotety deformation to the linear action is a non-trivial deformation: the fixed point set of the linear action is a vector space of dimension 3 (i.e., $\mathbb{R}^3$) and hence cannot be diffeomorphic to the fixed point set of the non-linear action (which is diffeomorphic to W).
- In fact, in [66] it is shown that there is an uncountable family of non-diffeomorphic manifolds W_α with the property above. We thus obtain uncountably many inequivalent s-constant deformations of the proper action groupoid associated to the linearized action.

The authors would like to thank Rui Loja Fernandes for his interpretation and formulation of the previous example, which made the exposition much clearer.

The last part of the Theorem 5.45 also has relative version; moreover, passing to semi-local statements, one can deal with general proper groupoids. The first one in this direction is the following:

Theorem 5.46. *Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid, let $N \subset M$ be a closed invariant submanifold.*

Then for any s-constant deformation $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ of $\mathcal{G}$ which is constant on N ($\mathcal{G}_\epsilon|_N = \mathcal{G}|_N$ for all ϵ) and any compact interval $I_0 \subset I$ one can find a smooth family of groupoid isomorphisms

$$F_\epsilon : \mathcal{G}|_{U_0} \longrightarrow \mathcal{G}_\epsilon|_{U_\epsilon}, \quad (\epsilon \in I_0).$$

which restrict to the identity on $\mathcal{G}|_N$. Moreover, if M is compact then one may choose $U_\epsilon = M$.

Proof. We need to look a bit closer at the previous arguments. First of all, because of the hypothesis, the deformation cocycles ξ_ϵ vanish on all pairs (g, h) coming from $\mathcal{G}|_N$. Looking at the formulas (5.13) defining the transgressing cochains we see that also X^ϵ vanishes at points of $\mathcal{G}|_N$. Let us now move to the product $M \times I$ and interpret $\tilde{V}$ as a vector field there. Since $\tilde{V}$ agrees with $\frac{\partial}{\partial \epsilon}$ on $N \times I$, we see that $\phi^\epsilon_{\tilde{V}}(x, v) = \phi^{v+\epsilon,v}_{\tilde{V}}(x), v + \epsilon)$ is defined for all $x \in N$ as long as $v + \epsilon, v \in I$. In particular, $\phi^{\epsilon,0}_{\tilde{V}}(x)$ is defined for all $x \in N$, $\epsilon \in I$. With I_0 as in the statement, since any open in $M \times I$ containing $N \times I_0$ also contains $U_0 \times I_0$ for some open neighbourhood U_0 of N in M, we find such an U_0 such that $\phi^{\epsilon,0}_{\tilde{V}}(x)$ is defined for all $x \in U_0$ and $\epsilon \in I_0$. Set

$$U_\epsilon = \phi^{\epsilon,0}_{\tilde{V}}(U_0).$$

Of course, $\phi^{0,\epsilon}_{\tilde{V}}$ is defined on U_ϵ. All together, using again Lemma 5.23 we find that

$$\phi^{\epsilon,0}_{\tilde{V}} : \mathcal{G}|_{U_0} \longrightarrow \mathcal{G}_\epsilon|_{U_\epsilon}$$

is a groupoid isomorphism, with inverse $\phi^{0,\epsilon}_{\tilde{X}}$ defined on the entire $\mathcal{G}_\epsilon|_{U_\epsilon}$. Since X^ϵ is zero on $\mathcal{G}|_N$, this isomorphism is the identity on $\mathcal{G}|_N$.

We are left with proving the last part. But this follows from a general property of flows of vector fields $\tilde{V}$ on $M \times I$ when M is compact: $\phi^{\epsilon,0}_{\tilde{V}}(x)$ is defined for all $x \in M$ and $\epsilon \in I$. $\square$

The following shows that proper deformations are trivial locally around compact invariant submanifolds.

Theorem 5.47. *Let $\mathcal{G} \rightrightarrows M$ be a proper groupoid and let $N \subset M$ be a compact invariant submanifold. Then for any proper deformation $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon\}$ of $\mathcal{G}$, there exists a smooth family $\tilde{U} = \{U_\epsilon\}$ of opens $U_\epsilon \subset M_\epsilon$ such that N is contained in U_0 and such that the deformation $\tilde{\mathcal{G}}|_{\tilde{U}} = \{\mathcal{G}_\epsilon|_{U_\epsilon}\}$ is trivial.*

In particular, any proper deformation of a compact groupoid is trivial.

Proof. We work on the large groupoid $\tilde{\mathcal{G}} \rightrightarrows \tilde{M}$ to which we apply Lemma 5.43 to obtain a multiplicative transverse vector field $\tilde{X}$ with base field denoted by $\tilde{V}$. Since N is compact we can choose a smaller $I_0 \subset I$ and an open neighbourhood U_0 of N in M_0 such that $\phi^\epsilon_{\tilde{V}}(y)$ is defined for all $y \in U_0$, $\epsilon \in I_0$. We set

$$U_\epsilon = \phi^\epsilon_{\tilde{V}}(U_0) \quad (\epsilon \in I_0).$$

The second part of Lemma 5.23 insures that $\phi^{\epsilon}_{\tilde{X}}$ is defined on the entire $\mathcal{G}|_U = \mathcal{G}_0|_{U_0}$, while the multiplicativity of $\tilde{X}$ implies (cf. the first part of Lemma 5.23) that

$$\phi^{\epsilon}_{\tilde{V}} : \mathcal{G}|_U \longrightarrow \mathcal{G}_\epsilon|_{U_\epsilon}$$

is a groupoid isomorphism (note that the inverse $\phi^{-\epsilon}_{\tilde{X}}$ is defined on the entire $\mathcal{G}_\epsilon|_{U_\epsilon}$ again because of Lemma 5.23 and the fact that $\phi^{-\epsilon}_{\tilde{X}}$ is defined on U_ϵ). □

Theorem 5.46 can also be extended to general proper deformations.

Theorem 5.48. *Assume that $\tilde{\mathcal{G}} = \{\mathcal{G}_\epsilon \mid \epsilon \in I\}$ is a proper family of Lie groupoids and $\tilde{N} = \{N_\epsilon\}$ is a smooth family of closed submanifolds $N_\epsilon \subset M_\epsilon$ invariant with respect to $\mathcal{G}_\epsilon$ and such that $\mathcal{G}|_{\tilde{N}} = \{\mathcal{G}|_{N_\epsilon}\}$ is trivial, with given trivializing diffeomorphisms*

$$\psi_\epsilon : \mathcal{G}_0|_{N_0} \longrightarrow \mathcal{G}_\epsilon|_{N_\epsilon}.$$

Then for any I_0 which is the interior of a compact interval contained in I, there exists a smooth family $\tilde{U} = \{U_\epsilon \mid \epsilon \in I_0\}$ of open neighbourhoods of N_ϵ in M_ϵ and a smooth family of groupoid isomorphisms

$$F_\epsilon : \mathcal{G}_0|_{U_0} \longrightarrow \mathcal{G}_\epsilon|_{U_\epsilon}, \quad (\epsilon \in I_0).$$

extending ψ_ϵ.

Proof. We proceed as in the proof of Theorem 5.46 but working directly on the large groupoid $\tilde{\mathcal{G}}$; in particular, we interpret the smooth family $\{\psi_\epsilon\}$ as a map

$$\psi : \mathcal{G}|_N \times I \longrightarrow \tilde{\mathcal{G}}, \quad (g, \epsilon) \mapsto \psi(g, \epsilon) := \psi_\epsilon(g)$$

and similarly for the base map, denoted by f. For $g \in \tilde{\mathcal{G}}|_{\tilde{N}}$, consider $\epsilon = \pi \circ s(g)$, i.e., with the property that $g \in \mathcal{G}_\epsilon|_{N_\epsilon}$. Move g back to $\epsilon = 0$, i.e., consider $g_0 = \psi_\epsilon^{-1}(g) \in \mathcal{G}|_N$ and consider

$$\tilde{X}_0(g) := \frac{d}{dv}|_{v=\epsilon}\psi(g_0, v).$$

Since the values of the curve $v \mapsto \psi(g_0, v)$ belong to $\mathcal{G}_v|_{N_v} \subset \tilde{\mathcal{G}}|_{\tilde{N}}$, $\tilde{X}_0$ is a vector field on $\tilde{\mathcal{G}}|_{\tilde{N}}$ which is s and t-projectable to the similar vector field $\tilde{V}_0$ on $\tilde{N}$ (constructed using f instead of F). Also, it is clear that $\tilde{V}_0$ is π-projectable to $\frac{\partial}{\partial \epsilon}$. Since each ψ_ϵ is a groupoid homomorphism, we deduce that $\tilde{X}_0$ is a multiplicative transverse vector field on $\tilde{\mathcal{G}}|_{\tilde{N}}$. Use now Lemma 5.44 to extend $\tilde{X}_0$ to a similar vector field $\tilde{X}$ on $\tilde{\mathcal{G}}$. Note also that, for $g_0 \in \mathcal{G}|_N$, the curve $\gamma(\epsilon) = \psi(g_0, \epsilon)$ has

$$\dot{\gamma}(\epsilon) = \frac{d}{dv}|_{v=\epsilon}\psi(g_0, v) = \tilde{X}_0(\psi_\epsilon(g_0)) = \tilde{X}(\gamma(\epsilon))$$

hence the resulting flows $\phi^{\epsilon}_{\tilde{X}}$ satisfy $\phi^{\epsilon}_{\tilde{X}}(g_0) = \psi(g_0, \epsilon)$ for all ϵ and for $g_0 \in \mathcal{G}|_N$. Then one proceeds like in the previous two proofs, using the flow of $\tilde{X}$. □

Finally, let us present an application to the linearization problem. However, we would like to emphasize that our aim here is not so much to prove linearization results but to show that, for proper groupoids, the linearization follows from a stronger property: rigidity.

In general, for any Lie groupoid $\mathcal{G} \rightrightarrows M$ and any $N \subset M$ invariant submanifold, one can make sense of the linear normal form of $\mathcal{G}$ around N, denoted by $\mathcal{N}_N(\mathcal{G})$. For instance, when $N = \{x\}$ is a fixed point of $\mathcal{G}$ (i.e., all the arrows that start at x also end at x), the action of $\mathcal{G}$ on ν restricts to a linear action of the isotropy group $\mathcal{G}_x$ at x on the tangent space T_xM and the linear normal form of $\mathcal{G}$ around x, $\mathcal{N}_N(\mathcal{G})$, is the resulting action groupoid. For an arbitrary invariant $N \subset M$, similar to the action of $\mathcal{G}$ on ν, one has a canonical action of $\mathcal{G}|_N$ on the normal bundle $\mathcal{N}(N)$ of N in M and the linearization of $\mathcal{G}$ around N, denoted by $\mathcal{N}_N(\mathcal{G})$, is defined as the resulting action groupoid. A more conceptual description is obtained by considering the normal bundle $\mathcal{N}(\mathcal{G}|_N)$ of $\mathcal{G}|_N$ in $\mathcal{G}$; as for $T\mathcal{G} \rightrightarrows TM$ (and as a quotient of it), the differentials of the structure maps of $\mathcal{G}$ make $\mathcal{N}(\mathcal{G}|_N)$ into a groupoid over $\mathcal{N}(N)$, canonically isomorphic to $\mathcal{N}_N(\mathcal{G})$. Note that N, identified with the zero section of $\mathcal{N}_N(M)$, is invariant with respect to $\mathcal{N}_N(\mathcal{G})$. The linearization problem asks whether $\mathcal{G}$ and $\mathcal{N}_N(\mathcal{G})$ are isomorphic when restricted to neighbourhoods of N. When this happens one says that $\mathcal{G}$ is linearisable around N. The relationship with deformations is provided by the following type of topological remarks:

Lemma 5.49. *If $\mathcal{G}$ is an s-proper Lie groupoid and $N \subset M$ is an invariant submanifold, then there exists an open neighbourhood W of N in M and a smooth proper family $\{\mathcal{G}_\epsilon\}$ of groupoids over W such that:*

- $\mathcal{G}_1 = \mathcal{G}|_W$.
- $\mathcal{G}_0$ *is (isomorphic to) the linear model* $\mathcal{N}_N(\mathcal{G})$.
- $\mathcal{G}_\epsilon|_N = \mathcal{G}|_N$ *as groupoids for all* ϵ.

(even more: one can ensure that $\mathcal{G}_\epsilon$ is isomorphic to $\mathcal{G}|_W$ for all $\epsilon \neq 0$).

Proof. (sketch) One proceeds exactly as in the case of fixed points which was explained in full detail in [28]. First of all, the topological arguments from Proposition 2.2 of *loc.cit* go through using the following remarks: for a smooth map $\pi : P \longrightarrow M$ and $N \subset M$ a submanifold,

- If π is proper then any open $\mathcal{U} \subset P$ which contains $\pi^{-1}(N)$ also contains $\pi^{-1}(V)$ for some neighbourhood V of N.
- If π is a submersion, then one can find tubular neighbourhoods of N in M and of $\pi^{-1}(N)$ in P which are compatible with π in the sense that the restriction of π to the tubular neighbourhood corresponds to the map induced by $d\pi$ on the normal bundles.

One then finds an embedding of type

$$i : \mathcal{G}|_{\mathcal{N}(N)} \hookrightarrow \mathcal{N}_N(\mathcal{G}) = (\mathcal{G}|_N) \times_N \mathcal{N}(N)$$

which is the identity on $\mathcal{G}|_N$, where $\mathcal{N}(N)$ is identified with a tubular neighbourhood of $N \subset M$. Then work on the left side, making use of the multiplication by scalars in the fibres of the normal bundles to set

$$m_\epsilon(e, f) = \frac{1}{\epsilon} m(\epsilon e, \epsilon f)$$

(and similarly for the other structure maps), defined on

$$\mathcal{G}_\epsilon := \{e \in \mathcal{N}_N(\mathcal{G}) \mid \epsilon e \in \mathcal{G}|_{\mathcal{N}(N)}\}.$$

Like in [28], it is not difficult to see that the limit at $\epsilon = 0$ gives rise to the linearized groupoid structure on $\mathcal{G}_0 = \mathcal{N}_N(\mathcal{G})$ and that the resulting groupoid $\tilde{\mathcal{G}}$ is proper. □

We see that we are almost in the position of applying Theorem 5.46; however, for that one would first have to improve the previous topological argument to insure that the inclusion i above is an isomorphism, so that the resulting deformation is strict (the fact that the source is constant is clear). Instead, we can just apply Theorem 5.48 and we deduce the following linearization theorem (proved first in [32] - see Theorem 1.71):

Theorem 5.50. *If $\mathcal{G}$ is an s-proper groupoid and $N \subset M$ is closed and invariant, then $\mathcal{G}$ is linearizable around N.*

Finally, our techniques can be applied also to the study of families of Lie groupoids. As before, there are several variations (semi-local or relative versions). Here we present the most restrictive but simplest statement.

Theorem 5.51. *Any compact family $\mathcal{G} \rightrightarrows M \xrightarrow{\pi} B$ of Lie groupoids is locally trivial.*

Proof. Consider a point b_0 in B, and a coordinate chart (U, ψ) around b_0 in B, with $\psi = (x_1, \dots, x_n)$, such that ψ maps U to an open disc D around 0 in $\mathbb{R}^n$, and $\psi(b_0) = 0$. For any $b \in U$, we consider the vector field

$$\sum_{i=1}^{n} x_i(b) \frac{d}{dx_i}$$

on D, and the corresponding (via ψ) vector field W^b on U. The flow of W^b at time one maps b_0 to b and moreover, from the construction it is clear that the choice of W^b is smooth in b. As explained in Remark 72, the vector fields W^b on U admit multiplicative lifts X^b to $\mathcal{G}_{|U}$ and moreover the choice of X^b can be made smooth in W^b. By taking the flow of X^b at time 1 (which is defined for all $g \in \mathcal{G}_{b_0}$ because of compactness), we find a family of isomorphisms

$$\phi^1_{X^b} : \mathcal{G}_{b_0} \longrightarrow \mathcal{G}_b,$$

parameterized smoothly by $b \in U$, giving the desired local trivialization. □

Remark 74. Using the Reeb stability theorem (see e.g. [72]), it is enough that $\mathcal{G}_{b_0}$ be compact to ensure that there is a neighbourhood U of b_0 in B such that $\mathcal{G}_b$ is diffeomorphic to $\mathcal{G}_{b_0}$ for all $b \in U$. It then follows that the same proof above ensures that $\mathcal{G}$ is trivial in a (possibly smaller) neighbourhood of b_0.

Remark 75. Del Hoyo and Fernandes have recently obtained (independently) the theorem above [33]. Their proof follows from a version of the Ehresmann stability theorem for Lie groupoids.

5.7 The regular case

When $\mathcal{G}$ is a regular Lie groupoid, i.e., having all leaves of the same dimension, it has natural representations on the bundle of isotropy Lie algebras of $\mathcal{G}$, denoted by $\mathfrak{i}$ and on the normal bundle to the orbits, denoted by ν. An arrow $g \in \mathcal{G}$ acts on $\alpha \in \mathfrak{g}_{s(g)}$ by conjugation, $g \cdot \alpha = r_{g^{-1}} \circ l_g \alpha$ and it acts on $[v] \in \nu_{s(g)}$ by the isotropy representation: if $g(\epsilon)$ is a curve on $\mathcal{G}$ with $g(0) = g$ such that $[v] = \left[\frac{d}{d\epsilon}_{|\epsilon=0} s(g(\epsilon))\right]$, then $g \cdot [v] = \left[\frac{d}{d\epsilon}_{|\epsilon=0} t(g(\epsilon))\right]$.

Proposition 5.52. *The deformation cohomology of a regular Lie groupoid $\mathcal{G}$ fits into a long exact sequence*

$$\cdots \longrightarrow H^k(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} H^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\pi} H^{k-1}(\mathcal{G}, \nu) \xrightarrow{K} H^{k+1}(\mathcal{G}, \mathfrak{i}) \longrightarrow \cdots$$

where r is induced by the canonical inclusion (5.7)*, π associates to a deformation cocycle c the class modulo* $\mathrm{Im}(\sharp)$ *of the s-projection s_c of c, and K will be discussed below.*

Proof. We will construct two cochain complexes $\mathcal{C}$ and $\mathcal{A}$ which fit into two short exact sequences

$$0 \longrightarrow C^*(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} C^*_{\mathrm{def}}(\mathcal{G}) \xrightarrow{R} \mathcal{C}^* \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{C}^* \longrightarrow \mathcal{A}^* \xrightarrow{S} C^*(\mathcal{G}, \nu) \longrightarrow 0 \tag{5.15}$$

and with the property that $\mathcal{A}^*$ is acyclic. For $\mathcal{A}^*$, define

$$\mathcal{A}^k = C^k(\mathcal{G}, TM) \oplus C^{k-1}(\mathcal{G}, TM).$$

(see also Remark 57), with the differential given by

$$\delta(\phi, \psi) = (-\delta'(\phi), -\phi + \delta'(\psi)),$$

where

$$\delta' : C^k(\mathcal{G}, TM) \longrightarrow C^{k+1}(\mathcal{G}, TM),$$

$$(\delta'\phi)(g_1, \ldots, g_{k+1}) = \sum_{i=1}^{k} (-1)^{i+1} \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{k+1}) + (-1)^{k+1} \phi(g_1, \ldots, g_k).$$

One can check directly that $\mathcal{A}^*$ is acyclic. More conceptually: $\mathcal{A}^*$ is the cylinder of the complex $(C^*(\mathcal{G}, TM), \delta')$, and δ' is clearly acyclic (with homotopy $\phi(g_1, \ldots, g_k) \mapsto \phi(1, g_1, \ldots, g_k)$).

The map $S : \mathcal{A}^* \longrightarrow C^*(\mathcal{G}, \nu)$ is defined by

$$S(\phi, \sigma)(g_1, \ldots, g_k) = [\phi(g_1, \ldots, g_k)] - g_1 \cdot [\sigma(g_2, \ldots, g_k)]$$

where $[V] \in \nu$ denotes the class of the tangent vector $V \in TM$ and "$g_1 \cdot$" refers to the canonical action of $\mathcal{G}$ on ν. It is straightforward to check that S is compatible with the differentials. Next, $\mathcal{C}^*$ is defined as the kernel of S and $\sharp$ associates to a cochain $c \in C^k_{\mathrm{def}}(\mathcal{G})$ the pair

$$R(c) = (\phi_c, \psi_c) \in C^k(\mathcal{G}, TM) \oplus C^{k-1}(\mathcal{G}, TM)$$

characterized by:

$$\phi_c(g_1, \ldots, g_k) = dt(c(g_1, \ldots, g_k)), \quad \psi_c(g_2, \ldots, g_k) = ds(c(g_1, \ldots, g_k)).$$

Lemma 5.24 implies that R takes values in $\mathcal{C}^*$. Again, a straightforward computation shows that R is compatible with the differentials. It is also clear that $\mathrm{Ker}(R) = \mathrm{Im}(S)$. Finally, we show that R is surjective; let $(\phi, \psi) \in \mathcal{C}^k$. One can first find $c \in \mathcal{C}^k_{\mathrm{def}}(\mathcal{G})$ so that $\psi = \psi_c$ (e.g. use a splitting of $ds : T\mathcal{G} \longrightarrow TM$ to lift the expressions $\psi(g_2, \ldots, g_k)$ to $T_{g_1}\mathcal{G}$). Using again Lemma 5.24 we see that

$$g_1 \cdot [\psi(g_2, \ldots, g_k)] = [dt(c(g_1, \ldots, g_k)]$$

hence the condition that $(\phi, \psi) \in \mathcal{C}^k$ implies that $\phi - dt \circ c$ takes values in the image of the anchor map $\sharp$, hence (also using regularity) we can write $\phi = dt \circ c + \sharp \circ \xi$ from some $\xi \in C^k(\mathcal{G}, A)$. Then $c' \in C^k_{\mathrm{def}}(\mathcal{G})$ defined by

$$c'(g_1, \ldots, g_k) = c(g_1, \ldots, g_k) + r_{g_1}(\xi(g_2, \ldots, g_k))$$

is sent by R to (ϕ, ψ).

Back to the original sequences, consider the long exact sequences in cohomology that they induce. The one induced by (5.15) gives rise to isomorphisms

$$\partial : H^{k-1}(\mathcal{G}, \nu) \longrightarrow H^k(\mathcal{C}),$$

which combined with the one induced by (5.15) gives rise to a long exact sequence as in the statement. We still have to show that, modulo the isomorphism ∂, R is identified with π, i.e., in cohomology, $\partial \circ \pi = R$. □

Although the previous proof may seem rather ad-hoc at first sight, it becomes very natural if one follows the intuition given by the adjoint representation interpretation of deformation cohomology (see Remark 78 below). Furthermore, it reveals several new aspects (see the remarks below), including a possible route to the discovery of the full structure of the adjoint representation (next section).

Remark 76. (*A deformation complex for morphisms*) First of all, the complex $\mathcal{A}^*$ is similar to the deformation complex and this indicates the presence of a more general construction: one has a deformation complex $C^*_{\mathrm{def}}(F)$ associated to any morphism of Lie groupoids

$$F : \mathcal{G} \longrightarrow \mathcal{H}$$

(covering some base map $f : M \longrightarrow N$). With the notation from Remark 57,

$$C^k_{\mathrm{def}}(F) \subset C^k(\mathcal{G}, F^*T\mathcal{H})$$

so that k-cochains c are smooth maps

$$\mathcal{G}^{(k)} \ni (g_1, \ldots, g_k) \mapsto c(g_1, \ldots, g_k) \in T_{F(g_1)}\mathcal{H}.$$

The condition that c belongs to $C^k_{\mathrm{def}}(F)$ is that $ds(c(g_1, \ldots, g_k))$ does not depend on g_1. Moreover, the differential δ of $C^*_{\mathrm{def}}(F)$ is given by exactly the same formula as for the deformation complex, but using the division $\bar{m}$ of $\mathcal{H}$. Denote the resulting cohomology by $H^*_{\mathrm{def}}(F)$. Of course,

$$H^*_{\mathrm{def}}(\mathcal{G}) = H^*_{\mathrm{def}}(\mathrm{Id}_{\mathcal{G}}).$$

More generally, the relation between the deformation complexes of F, $\mathcal{G}$ and $\mathcal{H}$ is given by the maps

$$C^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{F_*} C^k_{\mathrm{def}}(F) \xleftarrow{F^*} C^k_{\mathrm{def}}(\mathcal{H}),$$

defined by $F_*(c)(g_1,\ldots g_k) = dF \circ c(g_1,\ldots,g_k)$ and $F^*(c')(g_1,\ldots,g_k) = c'(F(g_1),\ldots,F(g_k))$, for any $c \in C^k_{\mathrm{def}}(\mathcal{G})$ and $c' \in C^k_{\mathrm{def}}(\mathcal{H})$.

Similar to the deformation cohomology of $\mathcal{G}$, $H^*_{\mathrm{def}}(F)$ can be seen as "the differentiable cohomology of $\mathcal{G}$ with coefficients in $F^*\mathrm{Ad}_{\mathcal{H}}$ - the pullback by F of the adjoint representation of $\mathcal{H}$".

With this, the complex $\mathcal{A}^*$ from the previous proof is $C^*_{\mathrm{def}}(F)$ where $F = (s,t) : \mathcal{G} \longrightarrow \Pi := M \times M$ is the canonical map into the pair groupoid of M; hence it is related to $F^*\mathrm{Ad}_\Pi$; the acyclicity of $\mathcal{A}$ is related to the fact that Ad_Π, i.e., the complex $TM \xrightarrow{\mathrm{Id}} TM$, is acyclic.

Remark 77. (*The curvature map*) One can also go on and compute the "curvature map K" in the sequence. In degree 0 one finds the curvature discussed in Subsection 5.3.5. A careful analysis in higher degrees shows that K is the cup-product with a canonical cohomology class, still denoted by K,

$$K \in H^2(\mathcal{G}, \mathrm{Hom}(\nu, \mathfrak{i})).$$

Here we use the induced Hom-representation: for $g : x \longrightarrow y$, its action of $\xi : \nu_x \longrightarrow \mathfrak{i}_x$ is $g \cdot \xi : \nu_y \longrightarrow \mathfrak{i}_y$ given by

$$(g \cdot \xi)(v) = g \cdot \xi(g^{-1} \cdot v).$$

Also, the cup-product operation that we refer to is

$$C^k(\mathcal{G}, \mathrm{Hom}(\nu, \mathfrak{i})) \times C^{k'}(\mathcal{G}, \nu) \longrightarrow C^{k+k'}(\mathcal{G}, \mathfrak{i}),\ (\xi, v) \mapsto \xi \cdot v$$

defined by the same formula as (5.4) where the pointwise product of u and v is replaced by the evaluation of u on v. We will explain in the next section how the point of view of representations up to homotopy can be used to describe the class K (see Remark 81). However, it is worth having in mind that one can proceed directly and analyse K as it arises from the previous proposition; the analysis is not completely straightforward (e.g. one has to realise the relevance of connections on groupoids) but, ultimately, it reveals the full structure on $\mathrm{Ad} = A \oplus TM$ (and the notion of representation up to homotopy). Although we do not give here the full details of such a direct approach, we hope that our comments motivate and clarify the next section.

Remark 78. (*Yoneda extensions*) The heart of the previous proof is the exact sequence

$$0 \longrightarrow C^*(\mathcal{G}, \mathfrak{i}) \xrightarrow{r} C^*_{\mathrm{def}}(\mathcal{G}) \xrightarrow{R} \mathcal{A}^* \xrightarrow{S} C^*(\mathcal{G}, \nu) \longrightarrow 0.$$

With Remark 76 in mind, this sequence represents a sequence involving the adjoint and the related representations: $\mathfrak{i}$, $\mathrm{Ad}_{\mathcal{G}}$, Ad_Π and ν; at the level of chain complexes, one simply deals with:

$$\begin{array}{cccccccc}
 & \mathfrak{i} & \hookrightarrow & A & \xrightarrow{\sharp} & TM & \longrightarrow & \nu \\
 & \oplus & & \oplus & & \oplus & & \oplus \\
 & \underbrace{0} & \longrightarrow & \underbrace{TM} & \xrightarrow{Id} & \underbrace{TM} & \longrightarrow & \underbrace{0} \\
\mathcal{E}\colon & \mathfrak{i} & \longrightarrow & \mathrm{Ad}_{\mathcal{G}} & \longrightarrow & F^*\mathrm{Ad}_\Pi & \longrightarrow & \nu
\end{array}$$

This gives an interpretation of the curvature map from the previous proposition in terms of Yoneda extensions (in the sense of homological algebra), as the cup-product with the element in the Ext-group represented by $\mathcal{E}$; moreover, since we basically deal with vector bundles (for which $\mathrm{Hom}(E,F) = E^* \otimes F$) the relevant group $\mathrm{Ext}^2(\nu, \mathfrak{i})$ is simply $H^2(\mathcal{G}, \mathrm{Hom}(\nu, \mathfrak{i}))$. Again, all these can be made precise within the framework of representations up to homotopy, providing another way of looking at the cohomology class K.

5.8 Relation with the adjoint representation

In this section we describe the relationship between $H^*_{\mathrm{def}}(\mathcal{G})$ and the adjoint representation of [2]. Actually, we will explain that, once a connection σ is fixed, $C^*_{\mathrm{def}}(\mathcal{G})$ gives rise right away to a representation up to homotopy Ad_σ and then we will identify it with the adjoint representation from [2].

We fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ and we start by briefly recalling the notion of representation up to homotopy. As in Remark 57, for a vector bundle E over M we consider the space $C^k(\mathcal{G}, E)$ of E-valued differentiable cochains. If E is graded, then we consider $C(\mathcal{G}, E)^*$ with the total grading

$$C(\mathcal{G}, E)^n = \bigoplus_{k+l=n} C^k(G, E^l).$$

As in Subsection 5.1.3, the cup-product makes $C(\mathcal{G}, E)$ into a right graded $C(\mathcal{G})$-module. By definition, a **representation up to homotopy of** $\mathcal{G}$ is a graded vector bundle E together with a differential D on $C(\mathcal{G}, E)$ (the structure operator) which makes it into a differential graded $(C(\mathcal{G}), \delta)$-module (with respect to the total grading and the cup-products). This is precisely what the deformation complex of $\mathcal{G}$ gives us once we fix a splitting of the short exact sequence:

$$t^*A \xrightarrow{r} T\mathcal{G} \xrightarrow{ds} s^*TM. \tag{5.16}$$

Such a splitting can be seen as a right inverse $\sigma : s^*TM \longrightarrow T\mathcal{G}$ of (ds), i.e., an Ehresmann connection on the bundle $s : \mathcal{G} \longrightarrow M$; we say that σ is an Ehresmann connection on $\mathcal{G}$ if, moreover, at the units it coincides with the canonical splitting (du).

Lemma 5.53. *Consider the graded vector bundle*

$$\mathrm{Ad} = A \oplus TM$$

with A in degree 0 and TM in degree 1. Then any Ehresmann connection σ induces isomorphisms

$$I_\sigma : C^k_{\mathrm{def}}(\mathcal{G}) \cong C(\mathcal{G}, \mathrm{Ad})^k = C^k(\mathcal{G}, A) \oplus C^{k-1}(\mathcal{G}, TM), \quad c \longleftrightarrow (u, v)$$

characterized by

$$c(g_1, \ldots, g_k) = r_{g_1}(u(g_1, \ldots, g_k)) - \sigma_{g_1}(v(g_2, \ldots, g_k)). \tag{5.17}$$

Moreover, this is an is an isomorphism of right $C(\mathcal{G})$-modules (see Lemma 5.15 for the module structure on $C_{\mathrm{def}}(\mathcal{G})$). In particular, for any Ehresmann connection σ on $\mathcal{G}$, there is a unique operator D_σ on $C(\mathcal{G}, \mathrm{Ad})$ which makes Ad into a representation up to homotopy of $\mathcal{G}$ and such that I_σ is an isomorphism between $(C_{\mathrm{def}}(\mathcal{G}), \delta)$ and $(C(\mathcal{G}, \mathrm{Ad}), D_\sigma)$.

The resulting representation up to homotopy will be denoted by Ad_σ. To make it more explicit (and identify it with the one of [2]), let us first be more explicit about the structure of representations up to homotopy of length 2, i.e., of type

$$E = E^0 \oplus E^1$$

In this case, the structure operator

$$D : C^k(\mathcal{G}, E^0) \oplus C^{k-1}(\mathcal{G}, E^1) \longrightarrow C^{k+1}(\mathcal{G}, E^0) \oplus C^k(\mathcal{G}, E^1)$$

is necessarily of type

$$D(u, v) = (\delta_\lambda(u) + K \cdot v, -\delta_\lambda(v) + \partial(u))$$

where:

- $\partial : E^0 \longrightarrow E^1$ is a vector bundle morphism and we use the same letter for

$$\partial : C^k(\mathcal{G}, E^0) \longrightarrow C^k(\mathcal{G}, E^1),\ \partial(c) = \partial \circ c.$$

- λ is a quasi-action of $\mathcal{G}$ on $E = E^0 \oplus E^1$ acting componentwise and

$$\delta_\lambda : C^*(\mathcal{G}, E) \longrightarrow C^{*+1}(\mathcal{G}, E)$$

  is the induced operator (cf. Remark 57).

- the "curvature term" K is a smooth section which associates to a pair (g, h) of composable arrows a linear map $K(g, h) : E^1_{s(h)} \longrightarrow E^0_{t(g)}$ and we use the cup-product operation

$$C^{k-1}(\mathcal{G}, E^1) \longrightarrow C^{k+1}(\mathcal{G}, E^0),\ c \mapsto K \cdot c$$

$$(K \cdot c)(g_1, \ldots, g_{k+1}) = K(g_1, g_2)(c(g_3, \ldots, g_{k+1})).$$

Note that the condition that $D^2 = 0$ breaks into

$$\partial \circ \lambda_g = \lambda_g \circ \partial \quad (\text{on } E^0), \tag{5.18}$$

$$\lambda_g\lambda_h - \lambda_{gh} + K(g, h) \circ \partial = 0 \quad (\text{on } E^0), \tag{5.19}$$

$$\lambda_g\lambda_h - \lambda_{gh} + \partial \circ K(g, h) = 0 \quad (\text{on } E^1), \tag{5.20}$$

$$\lambda_g K(h, k) - K(gh, k) + K(g, hk) - K(g, h)\lambda_k = 0 \quad (\text{on } E^1). \tag{5.21}$$

Hence, Lemma 5.53 implies that Ad comes with operators ∂, λ, K associated to σ. It is not difficult to check that ∂ is the anchor of A. We claim that λ and K coincide with the quasi-action and the basic curvature of σ, introduced in [2]:

- the quasi-actions associate to every $g : x \longrightarrow y$ the maps

$$\lambda_g : T_xM \longrightarrow T_yM,\ \lambda_g(X) = (dt)_g(\sigma_g(X)),$$

$$\lambda_g : A_x \longrightarrow A_y,\ \lambda_g(\alpha) = -\omega_g(\overleftarrow{\alpha}(g)),$$

  where $\overleftarrow{\alpha}$ is given by (5.1)) and where $\omega : T\mathcal{G} \longrightarrow t^*A$ is the induced left splitting of the sequence 5.16, $\omega_g(X) = r_{g^{-1}}(X - \sigma_g ds(X))$;

- the basic curvature, $K = K_\sigma^{\mathrm{bas}}$, associates to a pair of composable arrows

$$y \xleftarrow{g} \xleftarrow{h} x$$

and a vector $v \in T_x M$, the element

$$K(g,h)v \in A_y.$$

characterized by

$$r_{gh}(K(g,h)v) = \sigma_{gh}(v) - (dm)_{g,h}(\sigma_g(\lambda_h(v)), \sigma_h(v)) \in T_{gh}\mathcal{G}. \tag{5.22}$$

In other words, we have:

Proposition 5.54. *The structure operator of the representation up to homotopy $\mathrm{Ad}_\sigma = A \oplus TM$ arising from Lemma 5.53 is the one associated to the quasi-actions and the basic curvature of σ and the bundle map $\partial = \sharp : A \longrightarrow TM$, i.e.,*

$$D_\sigma(u,v) = (\delta_\lambda(u) + K \cdot v, -\delta_\lambda(v) + \sharp(u))$$

Proof. Before we do the actual computation, let us first point out three general formulas. First we note that, with respect to the decomposition induced by σ, the expressions of type $\overleftarrow{\beta}(g)$ correspond to the pair $(-\lambda_g(\beta), \sharp(\beta)) \in A_{t(g)} \oplus T_{s(g)}M$; indeed, the A component is obtained by applying ω_g, hence it is $-\lambda_g(\beta)$, while the TM-component is obtained by applying ds, hence it is $ds(l_g(di(\beta))) = dt(\beta) = \sharp(\beta)$. In conclusion,

$$\overleftarrow{\beta}(g) = -r_g(\lambda_g(\beta)) + \sigma_g(\sharp(\beta)).$$

Next, we claim that, for all $\alpha, \beta \in A$ and (g,h) composable arrows, one has

$$(d\bar{m})_{gh,h}(r_{gh}(\alpha), r_h(\beta)) = \overrightarrow{\alpha}(g) + \overleftarrow{\beta}(g) = r_g(\alpha) - r_g(\lambda_g(\beta)) + \sigma_g(\sharp(\beta)) \tag{5.23}$$

Due to the previous discussion, we only have to show the first equality. Since both $(r_{gh}(\alpha), 0_h)$ and $(0_g, r_h(\beta))$ are tangent to the domain $\mathcal{G}^{[2]}$ of $\bar{m}$, to prove this formula it suffices to consider separately the cases when $\beta = 0$ and then when $\alpha = 0$. When $\beta = 0$, the left hand side is the differential of the map $s^{-1}(x) \ni a \mapsto \bar{m}(r_{gh}(a), h) = ag = r_g(a)$ $(x = t(g))$, at the unit at x, applied to α; hence it gives $r_g(\alpha(x)) = \overrightarrow{\alpha}(g)$. Similarly, when $\alpha = 0$, we deal with the differential of the map $s^{-1}(y) \ni b \mapsto \bar{m}(gh, r_h(b)) = gb^{-1} = l_g(i(b))$ $(y = t(h) = s(g))$, at the unit at y applied to β, i.e., (see (5.1)), $\overleftarrow{\beta}(g)$.

Finally we also need the fact that, for (g,h) composable and $V \in T_{s(h)}M$,

$$(d\bar{m})_{gh,h}(\sigma_{gh}(V), \sigma_h(V)) = \sigma_g(\lambda_h(V)) + r_g(K(g,h)(V)) \tag{5.24}$$

Indeed, using the formula for $\sigma_{gh}(V)$ that results from (5.22), we find

$$(d\bar{m})_{gh,h}((dm)_{g,h}(\sigma_g(\lambda_h(V)), \sigma_h(V)), \sigma_h(V))) + (d\bar{m})_{gh,h}(r_{gh}(K(g,h)(V), 0_h).$$

The first term is just $\sigma_g(\lambda_h(V))$ because $\bar{m}(m(g,h),h) = g$, while the second term is $r_g(K(g,h)(V))$.

Consider now a deformation cochain c written as (5.17); we will compute $\delta(c)$ in terms of u and v.

The first term in the formula for $\delta(c)(g_1, \dots, g_{k+1})$ is

$$-(d\bar{m})(r_{g_1 g_2}(u(g_1 g_2, g_3, \dots, g_{k+1})) - \sigma_{g_1 g_2}(v(g_3, \dots, g_{k+1})), r_{g_2}(u(g_2, \dots, g_{k+1})) \\ - \sigma_{g_2}(v(g_3, \dots, g_{k+1})).$$

Using (5.23) we find that $-(d\bar{m})(r_{g_1 g_2}(u(g_1 g_2, g_3, \dots, g_{k+1})), r_{g_2}(u(g_2, \dots, g_{k+1})))$ is

$$-r_{g_1}(u(g_1 g_2, g_3, \dots, g_{k+1})) + r_{g_1}(\lambda_{g_1}(u(g_2, \dots, g_{k+1}))) - \sigma_{g_1}(\sharp(u(g_2, \dots, g_{k+1})))$$

and using (5.24) we find that $(d\bar{m})(\sigma_{g_1 g_2}(v(g_3, \dots, g_{k+1})), \sigma_{g_2}(v(g_3, \dots, g_{k+1})))$ is

$$\sigma_{g_1}(\lambda_{g_2}(v(g_3, \dots, g_{k+1}))) + r_{g_1}(K(g_1, g_2)(v(g_3, \dots, g_{k+1}))).$$

Hence the components of the first term in the formula for $\delta(c)$ are

$$-u(g_1 g_2, g_3, \dots, g_{k+1}) + \lambda_{g_1}(u(g_2, \dots, g_{k+1})) + K(g_1, g_2)(v(g_3, \dots, g_{k+1})) \in A_{t(g_1)},$$

$$-\sharp(u(g_2, \dots, g_{k+1})) + \lambda_{g_2}(v(g_3, \dots, g_{k+1})) \in T_{s(g_1)}M.$$

The next terms in the formula for $\delta(c)(g_1, \dots, g_{k+1})$ are, for each $i \in \{2, \dots, k\}$:

$$(-1)^i(r_{g_1}(u(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1})) - \sigma_{g_1}(v(g_2, \dots, g_i g_{i+1}, \dots, g_{k+1})))$$

hence give the components

$$(-1)^i u(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) \in A_{t(g_1)},$$

$$(-1)^{i+1} v(g_2, \dots, g_i g_{i+1}, \dots, g_{k+1}) \in T_{s(g_1)}M.$$

And similarly for the last term. All together, we see that the components corresponding to $\delta(c)$ are

$$\delta_\lambda(u)(g_1, \dots, g_{k+1}) + K(g_1, g_2)(v(g_3, \dots, g_{k+1})) \in A,$$

$$-\sharp(u(g_2, \dots, g_{k+1})) + \delta_\lambda(v)(g_2, \dots, g_{k+1}) \in TM.$$

Therefore, with respect to the splitting given by (5.17) (which introduces a minus sign on the TM component) we find

$$(\delta_\lambda(u) + K \cdot v, \sharp(u) - \delta_\lambda(v)).$$

□

Remark 79. When comparing with [2], note that one has to take care with adjusting signs: the formula in *loc. cit.* for the differential δu of a differentiable k-cochain (formula (5.3)) differs from ours by a factor of $(-1)^k$, and the formula for the cup product of a k-cochain u and a k'-cochain v (formula (5.4)) differs by a factor of $(-1)^{kk'}$. By sending $f \in C^l(\mathcal{G})$ and $(u, v) \in C(\mathcal{G}, \mathrm{Ad})^k$ to $(-1)^{\lfloor \frac{l}{2} \rfloor} f$ and $((-1)^{\lfloor \frac{k}{2} \rfloor} u, (-1)^{\lfloor \frac{k-1}{2} \rfloor} v)$ respectively, one obtains an isomorphism of chain complexes between the two versions, respecting the DG-module structures. Here $\lfloor x \rfloor$ denotes the largest integer that is smaller or equal to x.

Remark 80. It is straightforward to check that the isomorphism I_σ restricts to an isomorphism of right $C(\mathcal{G})$-modules between the normalized subcomplexes $(\widehat{C}_{\mathrm{def}}(\mathcal{G}), \delta)$ and $(\widehat{C}(\mathcal{G}, \mathrm{Ad}), D_\sigma)$. A cochain $c \in C^k(\mathcal{G}, \mathrm{Ad})$ is said to be normalized if $c(g_1, \ldots, g_k) = 0$ whenever any of the arrows g_i is a unit, and $\widehat{C}^k(\mathcal{G}, \mathrm{Ad})$ denotes the subspace of normalized k-cochains.

Remark 81. We now return to case of a regular Lie groupoid $\mathcal{G}$ and to the sequence from Proposition 5.52, in order to give another description of the curvature map K. Choose splittings $\mu : \nu \longrightarrow TM$ of the canonical projection π and $\tau : \mathrm{Im}(\sharp) \longrightarrow A$ of the anchor map, and a *compatible* Ehresmann connection σ - i.e., with the property that μ and τ are equivariant with respect to the canonical actions of $\mathcal{G}$ on ν and $\mathfrak{i}$, and the quasi-actions λ induced by σ on A and TM. Such a σ can always be obtained, by starting with any Ehresmann connection σ' on $\mathcal{G}$, and modifying it via a cochain $\eta \in C^1(\mathcal{G}, \mathrm{Hom}(TM, A))$, as explained in [42, Lemma 6.10].

Then, composing K_σ with μ and τ, we have that any pair (g, h) of composable arrows induces a map $\tau K_\sigma(g, h)\mu : \nu_{s(h)} \longrightarrow \mathfrak{i}_{t(g)}$; moving from $\nu_{s(h)}$ to $\nu_{t(g)}$ using the action by gh, we end up with a map $K_\sigma^{\mu,\tau}(g, h) : \nu_{t(g)} \longrightarrow \mathfrak{i}_{t(g)}$, therefore with a differentiable cochain

$$K_\sigma^{\mu,\tau} \in C^2(\mathcal{G}, \mathrm{Hom}(\nu, \mathfrak{i})).$$

It can be proved that $K_\sigma^{\mu,\tau}$ is a cocycle, the resulting cohomology class

$$[K_\sigma^{\mu,\tau}] \in H^2(\mathcal{G}, \mathrm{Hom}(\nu, \mathfrak{i}))$$

does not depend on the choice of σ, μ and τ (as long as σ is compatible with μ and τ), and the map K from Proposition 5.52 is given by the cup-product with this class.

5.9 Relation with the infinitesimal theory (the Van Est map)

In this subsection we show that the deformation cohomology $H^*_{\mathrm{def}}(\mathcal{G})$ is the global (groupoid) analogue of the similar cohomology $H^*_{\mathrm{def}}(A)$ from the deformation theory of Lie algebroids [27].

Recall first the definition of the deformation complex $(C^*_{\mathrm{def}}(A), \delta)$ (and the deformation cohomology $H^*_{\mathrm{def}}(A)$) of a Lie algebroid A over a manifold M. The k-cochains are antisymmetric multilinear maps $D : \Gamma(A)^k \longrightarrow \Gamma(A)$ which are **multiderivations**, i.e., such that there is a map $\sigma_D : \Gamma(A)^{k-1} \longrightarrow \mathfrak{X}(M)$, called the **symbol** of D, which is multilinear and satisfies

$$D(\alpha_1, \alpha_2, \ldots, f\alpha_k) = fD(\alpha_1, \alpha_2, \ldots, \alpha_k) + \mathcal{L}_{\sigma_D(\alpha_1, \ldots, \alpha_{k-1})}(f)\alpha_k$$

for any $f \in C^\infty(M)$ and $\alpha_i \in \Gamma(A)$. The differential $\delta : C^k_{\mathrm{def}}(A) \longrightarrow C^{k+1}_{\mathrm{def}}(A)$ is given by

$$\begin{aligned}\delta(D)(\alpha_1, \ldots, \alpha_{k+1}) = &\sum_i (-1)^{i+1}[\alpha_i, D(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{k+1})] \\ &+ \sum_{i<j} (-1)^{i+j} D([\alpha_i, \alpha_j], \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{k+1}).\end{aligned}$$

Next, we relate the deformation cohomology $H^*_{\mathrm{def}}(\mathcal{G})$ of a Lie groupoid $\mathcal{G}$ with the deformation cohomology $H^*_{\mathrm{def}}(A)$ of the Lie algebroid A of $\mathcal{G}$. This can be seen as a generalization of the more ordinary Van Est map relating differentiable cohomology to Lie algebroid cohomology. Given a section $\alpha \in \Gamma(A)$ we can define a map $R_\alpha : \widehat{C}^{k+1}_{\mathrm{def}}(\mathcal{G}) \longrightarrow \widehat{C}^{k}_{\mathrm{def}}(\mathcal{G})$ by $R_\alpha(c) = [c, \overrightarrow{\alpha}]_{|M}$ when $k = 0$ and by

$$R_\alpha(c)(g_1, \dots, g_k) = (-1)^k \frac{d}{d\epsilon}_{|\epsilon=0} c(g_1, \dots, g_k, \phi^\alpha_\epsilon(s(g_k))^{-1})$$

when $k \geq 1$, where ϕ^α_ϵ denotes the flow of the right-invariant vector field on $\mathcal{G}$ associated to α. The Van Est map is the map

$$\mathcal{V} : \widehat{C}^{*}_{\mathrm{def}}(\mathcal{G}) \longrightarrow C^{*}_{\mathrm{def}}(A)$$

defined by

$$(\mathcal{V}c)(\alpha_1, \dots, \alpha_k) = \sum_{\tau \in S_k} (-1)^{|\tau|} R_{\alpha_{\tau(k)}} \circ \dots \circ R_{\alpha_{\tau(1)}}(c).$$

Note that the Van Est map is only defined on the subcomplex of normalized deformation cochains.

In analogy to the usual Van Est theorem relating Lie groupoid cohomology to Lie algebroid cohomology, we obtain the following result:

Theorem 5.55. *For any Lie groupoid $\mathcal{G}$ the Van Est map $\mathcal{V}$ is a chain map; hence it induces a map in cohomology*

$$\mathcal{V} : H^p_{\mathrm{def}}(\mathcal{G}) \longrightarrow H^p_{\mathrm{def}}(A).$$

Moreover, if $\mathcal{G}$ has k-connected s-fibres then this map is an isomorphism in all degrees $p \leq k$.

In Section 5.8 we have seen the construction of the adjoint representation of a Lie groupoid, and the isomorphism of $C^*_{\mathrm{def}}(\mathcal{G})$ with $C^*(\mathcal{G}, \mathrm{Ad}_\sigma)$; the infinitesimal analogue of this construction gives rise to the adjoint representation ad_∇ of a Lie algebroid A [1]; it is proved in *loc. cit.* that $C^*_{\mathrm{def}}(A)$ is isomorphic to $C^*(A, \mathrm{ad}_\nabla)$.

We prove Theorem 5.55 by showing that under the isomorphisms of $C^*_{\mathrm{def}}(\mathcal{G})$ and $C^*_{\mathrm{def}}(A)$ with $C^*(\mathcal{G}, \mathrm{Ad}_\sigma)$ and $C^*(A, \mathrm{ad}_\nabla)$, it translates to the Van Est theorem of Arias Abad and Schätz [3, Thm. 4.7] for the Van Est map relating the complexes $C^*(\mathcal{G}, \mathrm{Ad}_\sigma)$ and $C^*(A, \mathrm{ad}_\nabla)$.

We use the concepts and notation from [1, 3, 27] needed for the proof. Similarly to the setting for Lie groupoids, $\Omega(A, E) = \Gamma(\Lambda^*(A^*) \otimes E)$ is a right graded $\Omega(A)$-module; a representation up to homotopy of A is a graded vector bundle E together with a differential D on $\Omega(A, E)$ which makes it into a differential graded $(\Omega(A), d_A)$-module.

The adjoint representation of a Lie algebroid A, seen as a representation up to homotopy, is given by the graded vector bundle $\mathrm{ad} = A \oplus TM$, where A has degree 0 and TM has degree 1, so that $\Omega(A, \mathrm{ad})^k = \Omega^k(A, A) \oplus \Omega^{k-1}(A, TM)$, and a differential D_∇ which is defined using a connection ∇ on A. When A is the Lie algebroid of a Lie groupoid $\mathcal{G}$, one can take ∇ to be induced by an Ehresmann connection σ on $\mathcal{G}$ by letting

$$\nabla_X \alpha = [\sigma(X), \overrightarrow{\alpha}]\big|_M, \tag{5.25}$$

for any vector field X on M and $\alpha \in \Gamma(A)$. The resulting chain complex is denoted by $C^*(A, \mathrm{ad}_\nabla)$.

We recall that the isomorphism $C_{\mathrm{def}}(A) \cong C^*(A, \mathrm{ad}_\nabla)$ induced by a connection ∇ on A is given by $\Psi_\nabla : C_{\mathrm{def}}(A) \longrightarrow C^*(A, \mathrm{ad}_\nabla)$, $D \mapsto (L_D, -\sigma_D)$, where L_D is defined by

$$L_D(\alpha_1, \ldots, \alpha_k) = D(\alpha_1, \ldots, \alpha_k) + (-1)^{k-1} \sum_i (-1)^i \nabla_{\sigma_D(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_k)} \alpha_i.$$

We now recall the definition of the Van Est map from [3], for cochains with values in representations up to homotopy of length 2. Let $E = E^0 \oplus E^1$ be a graded vector bundle over M, and α a section of A. Define the map $R^l_\alpha : \widehat{C}^{k+1}(\mathcal{G}, E^l) \longrightarrow \widehat{C}^k(\mathcal{G}, E^l)$ by

$$R^l_\alpha(c)(g_1, \ldots, g_k) = \frac{d}{d\epsilon}_{|\epsilon=0} c(g_1, \ldots, g_k, \phi^\alpha_\epsilon(s(g_k))^{-1}).$$

The Van Est map of [3] is the map $\mathcal{V}^{E^l} : \widehat{C}^k(\mathcal{G}, E^l) \longrightarrow \Omega^k(A, E^l)$ which is given by

$$(\mathcal{V}^{E^l} c)(\alpha_1, \ldots, \alpha_k) = (-1)^{kl} \sum_{\tau \in S_k} (-1)^{|\tau|} R^l_{\alpha_{\tau(k)}} \circ \ldots \circ R^l_{\alpha_{\tau(1)}}(c),$$

when $k \geq 1$ and by the identity map when $k = 0$. The Van Est map for the adjoint representation is obtained by taking $E^0 = A$ and $E^1 = TM$.

We will show that for an Ehresmann connection σ on $\mathcal{G}$, and the connection ∇ on A induced by σ (see formula 5.25), the Van Est map $\mathcal{V}$ of deformation cohomology corresponds to the Van Est map of [3], under the isomorphisms $I_\sigma : C_{\mathrm{def}}(\mathcal{G}) \longrightarrow C(\mathcal{G}, \mathrm{Ad})$ and $\Psi_\nabla : C_{\mathrm{def}}(A) \longrightarrow C(A, \mathrm{ad})$ induced by σ. Recall that $I_\sigma(c) = (\omega(c), -s_c)$, where $s_c = ds \circ c$ and $(\omega(c))(g_1 \ldots, g_k) = r^{-1}_{g_1}(c - \sigma \circ s_c)(g_1, \ldots, g_k)$. In order to accommodate for the sign convention of [3], we will actually use a slightly different map (see Remark 79)

$$\tilde{I}_\sigma(c) = ((-1)^{\lfloor \frac{k}{2} \rfloor} \omega(c), (-1)^{\lfloor \frac{k-1}{2} \rfloor + 1} s_c).$$

Lemma 5.56 (R_α in degree 1)**.** *For $c \in \widehat{C}^1_{\mathrm{def}}(\mathcal{G})$, it holds that*

$$R_\alpha(c) := [c, \overrightarrow{\alpha}]_{|M} = R^0_\alpha(\omega(c)) + \nabla_{s_c} \alpha. \tag{5.26}$$

Proof. Since $c(g) = r_g \omega(c)(g) + \sigma_g s_c(s(g))$, it is enough to prove the equation in the two following cases:

Case 1: $\omega(c) = 0$, or equivalently, $c = \sigma_g s_c$; in this case equation (5.26) holds since it is exactly the defining expression for ∇.

Case 2: $s_c = 0$, or equivalently, $\omega(c)(g) = r_{g^{-1}} c(g)$; in this case we want to prove that

$$[c, \overrightarrow{\alpha}]_{|M}(x) = R^0_\alpha(\omega(c))(x) := \frac{d}{d\epsilon}_{|\epsilon=0} r_{\phi^\alpha_\epsilon(x)} c(\phi^\alpha_\epsilon(x)^{-1}).$$

Given any 1-cochain, normalized or not, consider the section $\gamma_c = c_{|M} - s_c \in \Gamma(A)$. Define a projection $\pi : C^1_{\mathrm{def}}(\mathcal{G}) \longrightarrow \widehat{C}^1_{\mathrm{def}}(\mathcal{G})$ by $\pi(c) = c - \overrightarrow{\gamma_c}$. For a deformation cochain c satisfying $s_c = 0$, we will prove that $[c, \overrightarrow{\alpha}]_{|M}(x) = R^0_\alpha(\omega(c))(x)$, which in the case of a normalized cochain c gives the desired result. To do so, first note that $[\pi(c), \overrightarrow{\alpha}]_{|M}(x) - R^0_\alpha(\omega(c))(x)$ is $C^\infty(\mathcal{G})$-linear in c. Indeed,

$$
\begin{aligned}
[fc, \overrightarrow{\alpha}]_{|M}(x) - [\overrightarrow{\gamma_{fc}}, \overrightarrow{\alpha}]_{|M}(x) - R^0_\alpha(\omega(fc))(x) = \\
&= f[c, \overrightarrow{\alpha}]_{|M}(x) - f[\overrightarrow{\gamma_c}, \overrightarrow{\alpha}]_{|M}(x) - fR^0_\alpha(\omega(c))(x) \\
&- (\mathcal{L}_{\overrightarrow{\alpha}} f)\, c(x) + \left(\mathcal{L}_{\sharp(\alpha)} f\right) c(x) - (\mathcal{L}_{\overleftarrow{\alpha}} f)\, c(x),
\end{aligned}
$$

and the sum of the last three terms is zero since $\overrightarrow{\alpha}_x + \overleftarrow{\alpha}_x = \sharp(\alpha_x)$. To see how the term $(\mathcal{L}_{\overleftarrow{\alpha}} f)\, c(x)$ shows up by expanding $R^0_\alpha(\omega(fc))(x)$, it is enough to notice that $\phi^\alpha_\epsilon(x)^{-1} = \phi^{\overleftarrow{\alpha}}_\epsilon(x)$ and apply the chain rule.

Any cochain c with $s_c = 0$ is a linear combination of right-invariant ones, with coefficients in $C^\infty(\mathcal{G})$, and since $[\pi(c), \overrightarrow{\alpha}]_{|M}(x) - R^0_\alpha(\omega(c))(x)$ is $C^\infty(\mathcal{G})$-linear, it is enough to check that it is zero for a right-invariant c. This clearly holds because in this case both $[\pi(c), \overrightarrow{\alpha}]_{|M}(x)$ and $R^0_\alpha(\omega(c))(x)$ will be zero. $\square$

Note that Lemma 5.56 says precisely that the map $\mathcal{V}$ on $\widehat{C}^1_{\mathrm{def}}(\mathcal{G})$ corresponds to $(\mathcal{V}^A, \mathcal{V}^{TM})$ on $\widehat{C}(\mathcal{G}, \mathrm{Ad})^1$, through the isomorphisms $\tilde{I}_\sigma$ and Ψ_∇.

Lemma 5.57 (R_α in higher degrees)**.** *For all $k \geq 1$, the maps $R_\alpha : \widehat{C}^{k+1}_{def}(\mathcal{G}) \longrightarrow \widehat{C}^k_{def}(\mathcal{G})$ satisfy*

$$R_\alpha = \tilde{I}_\sigma^{-1} \circ (R^0_\alpha, -R^1_\alpha) \circ \tilde{I}_\sigma.$$

Proof. If $c \in \widehat{C}^{k+1}_{\mathrm{def}}(\mathcal{G})$, then

$$
\begin{aligned}
\left(\tilde{I}_\sigma^{-1} \circ (R^0_\alpha,\ -R^1_\alpha) \circ \tilde{I}_\sigma(c)\right)(g_1, \ldots, g_k) = \\
&= (-1)^{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k-1}{2} \rfloor} r_{g_1} R^0_\alpha(\omega(c))(g_1, \ldots, g_k) \\
&+ (-1)^{\lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{k-2}{2} \rfloor + 1} \sigma_{g_1} R^1_\alpha(s_c)(g_2, \ldots, g_k) \\
&= (-1)^k (r_{g_1} R^0_\alpha(\omega(c))(g_1, \ldots, g_k) + \sigma_{g_1} R^1_\alpha(ds \circ c)(g_1, \ldots, g_k)) \\
&= (-1)^k (r_{g_1}\omega_{g_1} + \sigma_{g_1} ds) \left(\frac{d}{d\epsilon}_{|\epsilon=0} c(g_1, \ldots, g_k, \phi^\alpha_\epsilon(s(g_k))^{-1})\right),
\end{aligned}
$$

where the last equality follows from checking that, since both ω_{g_1} and $(ds)_{g_1}$ are linear and do not depend on ϵ, they commute with the operation $\frac{d}{d\epsilon}_{|\epsilon=0}$ in the definitions of R^0_α, R^1_α and R_α. $\square$

Proof. (Theorem 10.1) We can compare the Van Est maps $\mathcal{V}$ and $(\mathcal{V}^A, \mathcal{V}^{TM})$ in arbitrary degree as follows. Consider a normalized deformation cochain c of degree k, and a permutation $\tau \in S_k$. Lemma 5.57 implies that

$$
\begin{aligned}
&R_{\alpha_{\tau(k)}} \circ \ldots \circ R_{\alpha_{\tau(1)}}(c) = \\
&= R_{\alpha_{\tau(k)}} \circ \tilde{I}_\sigma^{-1} \circ (R^0_{\alpha_{\tau(k-1)}} \circ \ldots \circ R^0_{\alpha_{\tau(1)}}, (-1)^{k-1} R^1_{\alpha_{\tau(k-1)}} \circ \ldots \circ R^1_{\alpha_{\tau(1)}}) \circ \tilde{I}_\sigma(c),
\end{aligned}
$$

and since we are applying $R_{\alpha_{\tau(k)}}$ to a deformation 1-cochain, using Lemma 5.56 we see that

$$
\begin{aligned}
R_{\alpha_{\tau(k)}} \circ \ldots \circ R_{\alpha_{\tau(1)}}(c) = (-1)^{\lfloor \frac{k}{2} \rfloor} R^0_{\alpha_{\tau(k)}} \circ R^0_{\alpha_{\tau(k-1)}} \circ \ldots \circ R^0_{\alpha_{\tau(1)}}(\omega(c)) \\
+ (-1)^{\lfloor \frac{k-1}{2} \rfloor + 1} \nabla_{(-1)^{k-1} R^1_{\alpha_{\tau(k-1)}} \circ \ldots \circ R^1_{\alpha_{\tau(1)}}(s_c)} \alpha_{\tau(k)}.
\end{aligned}
\tag{5.27}
$$

The Van Est map $\mathcal{V}$ is obtained by summing the previous expressions over all permutations in S_k. We can split this into a double sum, summing over $i = 1, \dots, k$ and over the permutations $\tau \in S_k$ such that $\tau(k) = i$. Using this resummation and equation (5.27), we have that

$$\begin{aligned}
(\mathcal{V}c)(\alpha_1, \dots, \alpha_k) &= \sum_{i=1}^{k} \sum_{\substack{\tau \in S_k \\ \tau(k)=i}} (-1)^{|\tau|} R_{\alpha_{\tau(k)}} \circ \dots \circ R_{\alpha_{\tau(1)}}(c) \\
&= (-1)^{\lfloor \frac{k}{2} \rfloor} \sum_{i=1}^{k} (-1)^{|\tau|} R^0_{\alpha_{\tau(k)}} \circ R^0_{\alpha_{\tau(k-1)}} \circ \dots \circ R^0_{\alpha_{\tau(1)}}(\omega(c)) \qquad (5.28)\\
&+ (-1)^{\lfloor \frac{k-1}{2} \rfloor + 1} \sum_{i=1}^{k} \sum_{\substack{\tau \in S_k \\ \tau(k)=i}} (-1)^{|\tau|} \nabla_{(-1)^{k-1} R^1_{\alpha_{\tau(k-1)}} \circ \dots \circ R^1_{\alpha_{\tau(1)}}(s_c)} \alpha_i
\end{aligned}$$

If $\tau \in S_k$ is a permutation with $\tau(k) = i$, by composing the cycle $r_i = (k \;\; k-1 \;\; \dots \;\; i+1 \;\; i)$ with it, we obtain a permutation $\tau' = r_i \circ \tau$ that fixes k, so it can be seen as element of S_{k-1} (and any element of S_{k-1} is of this form), for which we have $(-1)^{|\tau'|} = (-1)^{|\tau| + |r_i|} = (-1)^{|\tau| + (k-(i-1))}$. From this, and the definitions of $\mathcal{V}^A, \mathcal{V}^{TM}$ we see that (5.28) is equal to

$$(-1)^{\lfloor \frac{k}{2} \rfloor} \mathcal{V}^A \omega(c)(\alpha_1, \dots, \alpha_k) + (-1)^{\lfloor \frac{k-1}{2} \rfloor + 1} \sum_{i} (-1)^{|r_i|} \nabla_{\mathcal{V}^{TM} s_c(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_k)}(\alpha_i),$$

which is precisely the expression for $\Psi_\nabla^{-1} \circ (\mathcal{V}^A, \mathcal{V}^{TM}) \circ \tilde{I}_\sigma(c)(\alpha_1, \dots, \alpha_k)$, meaning that $\mathcal{V}$ does indeed correspond to $(\mathcal{V}^A, \mathcal{V}^{TM})$. To finish the proof, apply the Van Est theorem [3, Thm. 4.7] for $(\mathcal{V}^A, \mathcal{V}^{TM})$. □

Remark 82. A version of the theorem above, valid for more general representations up to homotopy, was proved by Arias Abad and Schätz in [3]. The main point of our theorem is that it relates *directly and canonically* the deformation complexes of groupoids and algebroids. Using the result of [3], one would only obtain the desired Van Est map after choosing connections and identifying the deformation complexes with the respective adjoint complex.

An alternative proof using the VB-interpretation of the adjoint cohomology (see [42] or Subsection 5.2.5) has been recently communicated to us by Cabrera and Drummond, and will appear in [12].

5.10 Morita invariance

In this section we prove that Morita equivalent Lie groupoids have isomorphic deformation cohomologies. This is useful not only for computations but also for conceptual reasons: it shows that the deformation cohomology of a Lie groupoid $\mathcal{G}$ is an invariant of the differentiable stack presented by $\mathcal{G}$.

Remark 83. In this direction, it is interesting to note that the notion of invariant vector fields up to isomorphism of [58] is related to $H^1_{\mathrm{def}}(\mathcal{G})$, where $\mathcal{G}$ is an action groupoid.

Theorem 5.58 (Morita invariance)**.** *If two Lie groupoids are Morita equivalent, then their deformation cohomologies are isomorphic.*

Proof. We recall that by Remark 6, it is enough to prove invariance under isomorphisms and under pullback by a surjective submersion $f : P \longrightarrow M$.

For both cases we use that given a morphism of Lie groupoids $F : \mathcal{G} \longrightarrow \mathcal{H}$, we have maps

$$C^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{F_*} C^k_{\mathrm{def}}(F) \xleftarrow{F^*} C^k_{\mathrm{def}}(\mathcal{H})$$

relating the deformation cohomologies of $\mathcal{G}$, F and $\mathcal{H}$ (Remark 76). When F is an isomorphism, F_* and F^* will be isomorphisms of chain complexes, so that takes care of invariance under isomorphims. We will now focus on the case where $f : P \longrightarrow M$ is a surjective submersion, and F is the induced map $F : f^*\mathcal{G} \longrightarrow \mathcal{G}$, $\tilde{g} = (p, g, q) \mapsto g$. We will prove that $C^*_{\mathrm{def}}(f^*\mathcal{G})$ is quasi-isomorphic to $C^*_{\mathrm{def}}(\mathcal{G})$ by showing that both are quasi-isomorphic to $C^*_{\mathrm{def}}(F)$.

Claim 5.59. F_* is a quasi-isomorphism between $C^*_{\mathrm{def}}(f^*\mathcal{G})$ and $C^*_{\mathrm{def}}(F)$.

Proof of Claim 1: Recall that $F_*c = dF \circ c$. We first note that F_* is surjective. In fact, if we chose a connection on TP, i.e., a right splitting Γ on the sequence

$$0 \longrightarrow \mathrm{Ker}\,(f) \longrightarrow TP \longrightarrow f^*TM \longrightarrow 0,$$

then given $c \in C^*_{\mathrm{def}}(F)$, if we set $\tilde{c} = (\Gamma(dt(c)), c, \Gamma(ds(c)))$, we obtain that $F_*\tilde{c} = c$.

We thus obtain a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, F_* \longrightarrow C^*_{\mathrm{def}}(f^*\mathcal{G}) \longrightarrow C^*_{\mathrm{def}}(F) \longrightarrow 0,$$

so to prove the claim it is enough to show that $\mathrm{Ker}\, F_*$ is acyclic.

Note that k-cochains in $\mathrm{Ker}\, F_*$ may be identified with pairs of maps

$$u : (f^*\mathcal{G})^k \longrightarrow \mathrm{Ker}\, df, \quad v : (f^*\mathcal{G})^{k-1} \longrightarrow \mathrm{Ker}\, df,$$

where $u(\tilde{g}_1, \ldots, \tilde{g}_k) \in \mathrm{Ker}\,(df)_{p_1}$, $v(\tilde{g}_2, \ldots, \tilde{g}_k) \in \mathrm{Ker}\,(df)_{q_1}$, and $\tilde{g}_i$ denotes the arrow (p_i, g_i, q_i).

A simple computation shows that under this identification, the differential of the deformation complex satisfies

$$\delta(u, 0) = (w, u),$$

for some $w : (f^*\mathcal{G})^{k+1} \longrightarrow \mathrm{Ker}\, df$ as above. Then, if (u, v) is a cocycle, it must actually be exact, with $(u, v) = \delta(v, 0)$. Indeed, $(u, v) - \delta(v, 0) = (w, 0)$, and $(w', w) = \delta(w, 0) = \delta(u, v) - \delta^2(v, 0) = 0$, so $w = 0$. □

Next, we reduce our problem of showing that $H^*_{\mathrm{def}}(f^*\mathcal{G}) \simeq H^*_{\mathrm{def}}(\mathcal{G})$ to the case where $f : P \longrightarrow M$ admits a global section $\sigma : M \longrightarrow P$. For this we will need the following Mayer-Vietoris argument, where we use the Čech groupoid $\check{\mathcal{G}}(\mathcal{U})$ associated to an open cover $\mathcal{U} = \{U_i\}$ of M (Example 1.27).

Claim 5.60 (Mayer-Vietoris Argument)**.** If $\mathcal{U}$ is an open cover of M, then $H^*_{\mathrm{def}}(\mathcal{G}) \simeq H^*_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U}))$.

Since the proof of this claim is formally identical to the proofs of claims 5.63 and 5.64 below (with the role of the section σ to be played by a partition of unity), we will postpone it until Proposition 5.66.

Claim 5.61. We may assume without loss of generality that the map $f : P \longrightarrow M$ admits a global section $\sigma : M \longrightarrow P$.

Proof of Claim 5.61: Take a cover $\mathcal{U} = \{U_i\}$ of M by open sets which admit local sections $\sigma_i : U_i \longrightarrow f^{-1}U_i$, and denote by $\mathcal{V}$ the open cover of P by the open sets $V_i = f^{-1}U_i$. Note that f induces a surjective submersion with a global section

$$\check{f} : \amalg V_i \longrightarrow \amalg U_i.$$

Moreover, the Čech groupoid $(\check{f^*\mathcal{G}})(\mathcal{V})$ of $f^*\mathcal{G}$ with respect to the cover $\mathcal{V}$ is isomorphic to the pullback groupoid $\check{f}^*(\check{\mathcal{G}}(\mathcal{U}))$ of the Čech groupoid $\check{\mathcal{G}}(\mathcal{U})$ by the surjective submersion $\check{f}$. Thus, by Claim 5.60, and invariance under isomorphism, we have

$$H^*_{\mathrm{def}}(\check{f}^*(\check{\mathcal{G}}(\mathcal{U}))) \simeq H^*_{\mathrm{def}}(f^*\mathcal{G}) \text{ and } H^*_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U})) \simeq H^*_{\mathrm{def}}(\mathcal{G}).$$

Thus, $H^*_{\mathrm{def}}(\mathcal{G}) \simeq H^*_{\mathrm{def}}(f^*\mathcal{G})$ if and only if $H^*_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U})) \simeq H^*_{\mathrm{def}}(\check{f}^*(\check{\mathcal{G}}(\mathcal{U})))$. □

From now until the end of the proof of the theorem we will assume that $f : P \longrightarrow M$ admits a global section $\sigma : M \longrightarrow P$. We then obtain a left inverse to F^*:

$$\Psi_\sigma : C^*_{\mathrm{def}}(F) \longrightarrow C^*_{\mathrm{def}}(\mathcal{G}),$$

$$\Psi_\sigma(c)(g_1, \dots, g_k) = c(\sigma(t(g_1)), g_1, \sigma(s(g_1)), \dots, \sigma(t(g_k)), g_k, \sigma(s(g_k))).$$

It follows that the map induced by F^* in cohomology is injective, and it is our task to show that it is surjective. For this we consider the following decreasing sequence of subcomplexes of $C^*_{\mathrm{def}}(F)$.

We will call a cochain $c \in C^k_{\mathrm{def}}(F)$ **strongly normalized in the** $j-th$ **position**, if it satisfies the following two conditions:

$$c(\tilde{g}_1, \dots, \tilde{g}_j, \dots, \tilde{g}_k) = c(\tilde{g}_1, \dots, \tilde{g}'_j, \dots, \tilde{g}_k) \text{ whenever } F(\tilde{g}_j) = F(\tilde{g}'_j) \tag{5.29}$$

and if $F(\tilde{g}_j)$ is a unit in $\mathcal{G}$, then

$$\begin{gathered} (\tilde{g}_1, \dots, \tilde{g}_j, \dots, \tilde{g}_k) = 0 \text{ if } 1 < j \leq k \\ c(\tilde{g}_1, \dots, \tilde{g}_k) \text{ is a unit if } j = 1. \end{gathered} \tag{5.30}$$

We denote by $\mathcal{N}^* \subset C^*_{\mathrm{def}}(F)$ the complex of **strongly normalized cochains**, i.e., those that are strongly normalized in all positions.

We obtain a decreasing sequence of subcomplexes

$$C^*_{\mathrm{def}}(F) = \mathcal{N}^*_0 \supseteq \cdots \supseteq \mathcal{N}^*_\ell \supseteq \cdots \supseteq \mathcal{N}^*,$$

where we set

$$\mathcal{N}^k_\ell = \{c \in C^k_{\mathrm{def}}(F) \mid c \text{ is strongly normalized in position } j \text{ for all } k - \ell < j\}.$$

Claim 5.62. The complex $\mathcal{N}^*$ is isomorphic to the complex $\widehat{C}^*_{\mathrm{def}}(\mathcal{G})$ of normalized cochains of $\mathcal{G}$.

Proof of Claim 5.62: The map $F^* : C^k_{\mathrm{def}}(\mathcal{G}) \longrightarrow C^k_{\mathrm{def}}(F)$ is given by

$$(F^*c)((p_1, g_1, q_1), \ldots, (p_k, g_k, q_k)) = c(g_1, \ldots, g_k)$$

, so restricting it to normalized cochains on $\mathcal{G}$, we obtain a map $F^* : \widehat{C}^k_{\mathrm{def}}(\mathcal{G}) \longrightarrow \mathcal{N}^k$. This restriction is injective because of surjectivity of f, and is surjective because of condition (5.29) which is valid for all $1 \leq \ell \leq k$. (Alternatively, the restriction of Ψ_σ to $\mathcal{N}^*$ is the inverse of F^*). $\square$

In order to show that F^* is surjective in cohomology, we will prove that every cocycle in $\mathcal{N}^k_\ell$ is cohomologous to a cocycle in $\mathcal{N}^k_{\ell+1}$. We break this into two steps (the next two claims).

Claim 5.63. If a cocycle $c \in C^k_{\mathrm{def}}(F)$ satisfies condition (5.30) for all $\ell \leq j \leq k$, then it also satisfies condition (5.29) for all $\ell \leq j \leq k$.

Proof of Claim 5.63: For $j \geq \ell$, non-dependence on q_j follows from spelling out the cocycle equation

$$\delta c((p_1, g_1, q_1), \ldots, (p_j, g_j, q_j), (q_j, 1, q_j'), (p_{j+1}, g_{j+1}, q_{j+1}), \ldots, (p_k, g_k, q_k)) = 0.$$

If $\ell > 1$, since c satisfies condition (5.30) in positions $\ell, \ldots, k$, only two terms survive, and the result is

$$c(\tilde{g}_1, \ldots, (q_j, g_{j+1}, q_{j+1}), \ldots, \tilde{g}_k) = c((\tilde{g}_1, \ldots, (q_j', g_{j+1}, q_{j+1}), \ldots, \tilde{g}_k),$$

where $\tilde{g}_i$ denotes the arrow (p_i, g_i, q_i).

If $\ell = 1$ and $j = 1$, the only non-zero terms in the cocycle condition are the first two, and the result is that

$$\begin{aligned}-d\bar{m}(c((p_1, g_1, q_1'), (q_1', g_2, q_2), \ldots, \tilde{g}_k), c((q_1, 1, q_1'), (q_1', g_2, q_2), \ldots \tilde{g}_k))&\\ + c((p_1, g_1, q_1), (q_1, g_2, q_2), \ldots, \tilde{g}_k) &= 0.\end{aligned}$$

Since c satisfies condition (5.30) in position 1, it follows that $c(q_1, 1, q_1', \ldots)$ is a unit, and thus

$$c((p_1, g_1, q_1'), (q_1', g_2, q_2), \ldots, \tilde{g}_k) = c((p_1, g_1, q_1), (q_1, g_2, q_2), \ldots, \tilde{g}_k).$$

Non-dependence on p_1 follows simillarly from the cocycle equation

$$\delta c((p_1, 1, p_1'), (p_1', g_1, q_1), \ldots, (p_k, g_k, q_k)) = 0.$$

Spelling it out, we obtain

$$-d\bar{m}(c((p_1, g_1, q_1), \ldots, \tilde{g}_k), c((p_1', g_1, q_1), \ldots, \tilde{g}_k)) = \sum_i U_i,$$

where condition (5.30) implies that each U_i is a unit of $T\mathcal{G}$. The result then follows by multiplying (in $T\mathcal{G}$) both sides of the equation by $c((p_1', g_1, q_1), \ldots, \tilde{g}_k)$ on the right. $\square$

The next claim concludes the proof of the theorem.

Claim 5.64. Every cocycle $c \in \mathcal{N}^k_\ell$ is cohomologous to a cocycle in $\mathcal{N}^k_{\ell+1}$.

Proof of Claim 5.64: The notation becomes quite heavy in the following computations, so we make some simplifications, which we now explain, and which will not be used outside of this proof. For arrows $\tilde{g}_i = (p_i, g_i, q_i)$, we use the notation $(\tilde{g}_1, \ldots, \tilde{g}_k) = (p_1, g_1, q_1, g_2, \ldots, g_{k-1}, q_{k-1}, g_k, q_k)$ which is not ambiguous since $p_{i+1} = q_i$. We also abbreviate the expression by inserting $\tilde{g}_i = (p_i, g_i, q_i)$ in parts of expressions where the arrow is clear from its context. Moreover, we will implicitly assume when applying a cochain to a string of arrows, that they are composable, and omit the points of M where we apply the unit map and the section σ if they are uniquely determined by this requirement (composability). So, for example, we would simplify the expression $c((p, g, \sigma(s(g)), (\sigma(s(g)), 1_{s(g)}, q), (q, h, r))$ to $c(p, g, \sigma, 1, q, h, r)$.

We consider first the case $\ell \geq 2$. Let $c \in \mathcal{N}^k_{k-\ell} \subset C^k_{\mathrm{def}}(F)$ be a cocycle. Consider $\varphi^\ell_\sigma(c) \in C^{k-1}_{\mathrm{def}}(F)$ given by

$$\begin{aligned}\varphi^\ell_\sigma(c)(p_1, g_1, q_1, \ldots, g_{k-1}, q_{k-1}) &= \\ &= (-1)^{\ell+1} c(p_1, g_1, q_1, \ldots, g_{\ell-1}, q_{\ell-1}, 1, \sigma, g_\ell, q_\ell, \ldots, g_{k-1}, q_{k-1}).\end{aligned}$$

We claim that $c + \delta\varphi^\ell_\sigma(c)$ belongs to $\mathcal{N}^k_{k-\ell+1}$. In fact, let us compute $(c + \delta\varphi^\ell_\sigma(c))(\tilde{g}_1, \ldots \tilde{g}_k)$ when $F(\tilde{g}_i) = 1$. If $i > \ell$, then since $c \in \mathcal{N}^k_{k-\ell}$, most terms cancel and the only non-zero terms are the ones resulting from the strings containing $\tilde{g}_{i-1}\tilde{g}_i$, and $\tilde{g}_i\tilde{g}_{i+1}$, both with opposite signs, i.e., up to a sign we obtain

$$\begin{aligned}&c(p_1, g_1, \ldots, q_{\ell-1}, 1, \sigma, g_\ell, \ldots, q_{i-2}, g_{i-1}, q_i, \ldots g_k, q_k) - \\ &\qquad c(p_1, g_1, \ldots, q_{\ell-1}, 1, \sigma, g_\ell, \ldots, q_{i-2}, g_{i-1}, q_{i-1}, \ldots g_k, q_k) = 0\end{aligned}$$

which vanishes because c satisfies condition (5.29) in position $i > \ell$. This show that $c + \delta\varphi^\ell_\sigma(c) \in \mathcal{N}^k_{k-\ell}$.

Now let $i = \ell$. We will show that if $F(\tilde{g}_\ell) = 0$, then $(c+\delta\varphi^\ell_\sigma(c))(\tilde{g}_1, \ldots, \tilde{g}_k) = 0$, i.e., $c+\delta\varphi^\ell_\sigma(c)$ satisfies condition (5.30) in position ℓ. It then follows from Claim 5.63 that $c+\delta\varphi^\ell_\sigma(c) \in \mathcal{N}^k_{k-\ell+1}$.

After a straightforward compution we obtain that

$$\begin{aligned}&(c + \delta\varphi^\ell_\sigma(c))(\tilde{g}_1, \ldots, g_{\ell-1}, q_{\ell-1}, 1, q_\ell, \ldots, \tilde{g}_k) = \\ &\qquad = (-1)^{\ell+1}\delta c(\tilde{g}_1, \ldots, g_{\ell-1}, q_{\ell-1}, 1, q_\ell, 1, \sigma, g_{\ell+1} \ldots, \tilde{g}_k) = 0\end{aligned}$$

which vanishes because c is a cocycle.

We are left with showing the cases $\ell = 1$. Let $c \in \mathcal{N}^k_{k-1}$ be a cocycle which is strongly normalized in positions $2, \ldots, k$. In this case we use the canonical splitting $T_{1_x}\mathcal{G} \cong A_x \oplus T_x M$, and denote by X^A the A-component of a vector $X \in T_{1_x}\mathcal{G}$, to define $\varphi^1_\sigma(c) \in C^{k-1}_{\mathrm{def}}(F)$ by

$$\varphi^1_\sigma(c)(p_1, g_1, q_1, \ldots, g_{k-1}, q_{k-1}) = r_{g_1} c(p_1, 1, \sigma, g_1, q_1, \ldots, g_{k-1}, q_{k-1})^A.$$

Similarly, if c has degree 1, define $\varphi^1_\sigma(c)$ by $\varphi^1_\sigma(c)(p) = c(p, 1, \sigma)^A$.

We wish to show that $(c + \delta\varphi^1_\sigma(c))(p_1, 1, q_1, g_2, \ldots, g_k)$ is a unit of $T\mathcal{G}$. Most terms of this expression are zero and we are left with

$$\begin{aligned}&(c + \delta\varphi^1_\sigma(c))(p_1, 1, q_1, g_2, \ldots, g_k) = \\ &\qquad = c(p_1, 1, q_1, g_2, \ldots, g_k) \\ &\qquad - d\bar{m}(r_{g_2} c(p_1, 1, \sigma, g_2, \ldots, g_k)^A, (r_{g_2} c(q_1, 1, \sigma, g_2, \ldots, g_k)^A)\end{aligned}$$

If we decompose

$$c(p_1, 1, q_1, g_2, \dots, g_k) = ds \circ c(p_1, 1, q_1, g_2, \dots, g_k) + c(p_1, 1, q_1, g_2, \dots, g_k)^A$$

and use the cocycle identity

$$\begin{aligned} -d\bar{m}(r_{g_2} c(p_1, 1, \sigma, g_2, \dots, g_k)^A, (r_{g_2} c(q_1, 1, \sigma, g_2, \dots, g_k)^A) \\ + c(p_1, 1, q_1, g_2, \dots, g_k)^A = (\delta c(p_1, 1, q_1, 1, \sigma g_2, \dots, g_k))^A = 0, \end{aligned}$$

we obtain that

$$(c + \delta\varphi^1_\sigma(c))(p_1, 1, q_1, g_2, \dots, g_k) = ds \circ c(p_1, 1, q_1, g_2, \dots, g_k) \text{ is a unit.}$$

The case $k = 1$ works exactly in the same way: $(c+\delta\varphi^1_\sigma(c))(p, 1, q)$ is the unit $ds \circ c(p, 1, q)$, plus the A-part of $\delta c(p, 1, q, 1, \sigma)$. □

It follows from the previous two claims that $F^* : H^k_{\mathrm{def}}(\mathcal{G}) \longrightarrow H^k_{\mathrm{def}}(F)$ is surjective, and thus an isomorphism. This concludes the proof of the theorem. □

Remark 84. In the course of the proofs of claims 5.63 and 5.64, we actually showed that every cocycle in $C^*_{\mathrm{def}}(F)$ is cohomologous to one in $\mathcal{N}^*$, which we identify with $\widehat{C}^*_{\mathrm{def}}(\mathcal{G})$ (and not simply $C^*_{\mathrm{def}}(\mathcal{G})$). Thus, applying the proof above to the identity map yields the following proposition.

Proposition 5.65. *The inclusion $\iota : \widehat{C}^*_{\mathrm{def}}(\mathcal{G}) \hookrightarrow C^*_{\mathrm{def}}(\mathcal{G})$ is a quasi-isomorphism.*

We now prove the Mayer-Vietoris argument used in the proof above (Claim 5.60)

Proposition 5.66 (Mayer-Vietoris Argument)**.** *If $\mathcal{U}$ is an open cover of M, then $H^*_{\mathrm{def}}(\mathcal{G}) \simeq H^*_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U}))$.*

Proof. The proof is formally identical to the proofs of Theorem 5.58 above (Claims 5.63 and 5.64) with the role of the section σ replaced by a partition of unity $\{\rho_j\}$ subordinate to the open cover $\mathcal{U} = \{U_j\}$. Here are the main ingredients:

- $\check{\mathcal{G}}(\mathcal{U})$ is the pullback of $\mathcal{G}$ by a submersion, so there is an obvious chain map

$$\pi^* : C^k_{\mathrm{def}}(\mathcal{G}) \longrightarrow C^k_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U}))$$

$$\pi^*(c)((i_1, g_1, j_1), \dots, (j_{k-1}, g_k, j_k)) = c(g_1, \dots, g_k).$$

- Using the partition of unity we obtain a left inverse $\Psi : C^k_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U})) \longrightarrow C^k_{\mathrm{def}}(\mathcal{G})$,

$$\Psi(c)(g_1, \dots, g_k) = \sum \rho_{i_1}\rho_{j_1} \cdots \rho_{j_k} c((i_1, g_1, j_1), \dots, (j_{k-1}, g_k, j_k)),$$

  where the sum is taken over all indexes i_1, j_r such that $t(g_1) \in U_{i_1}$ and $s(g_r) \in U_{j_r}$, with $r = 1, \dots, k$.

- It follows that π^* induces an injection in cohomology and all that is left to prove is that it is also surjective.

- The normalized complex $\widehat{C}^*_{\mathrm{def}}(\mathcal{G})$ can be identified via Ψ with a subcomplex $\mathcal{N}^*$ of strongly normalized cochains of $C^k_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U}))$. (Defined by conditions analogous to (5.29) and (5.30)). In order to check that the map induced by π^* in cohomology is surjective, it is enough to show that every cocycle $c \in C^k_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U}))$ is cohomologous to a cocycle in $\mathcal{N}^k$.

- We consider the descending sequence

$$C^*_{\mathrm{def}}(\check{\mathcal{G}}(\mathcal{U})) = \mathcal{N}^*_0 \supseteq \cdots \supseteq \mathcal{N}^*_\ell \supseteq \cdots \supseteq \mathcal{N}^*,$$

  defined as in the proof of Theorem 5.58.

- The statement and proof of Claim 5.63 in Theorem 5.58 follow identically with p_r replaced by i_r, and q_r replaced by j_r.

- The statement and proof of Claim 5.64 in Theorem 5.58 follow identically with φ^ℓ_σ replaced by

$$\begin{aligned}\varphi^\ell(c)(i_1, g_1, j_1, \ldots, g_k, j_k) &= \\ &= (-1)^{\ell+1} \sum_j \rho_j c(i_1, g_1, j_1, \ldots, g_{\ell-1}, j_{\ell-1}, 1, j, g_\ell, j_\ell, \ldots, g_{k-1}, j_{k-1})\end{aligned}$$

  for $\ell > 1$, where the sum is taken over all indices j such that $t(g_\ell) \in U_j$; similarly, $\varphi^1_\sigma(c)$ is replaced by

$$\varphi^1(c)(i_1, g_1, j_1, \ldots, g_k, j_k) = \sum_j \rho_j r_{g_1} c(i_1, 1, j, g_1, j_1, \ldots, g_k, j_k)^A.$$

□

Remark 85. We have recently learned that Theorem 5.58 can be obtained also by using an appropriate notion of Morita equivalence for VB-groupoids (and using the VB-groupoid interpretation of the deformation cohomology - see Subsection 5.2.5). This is being worked out in an ongoing project of del Hoyo and Ortiz [34].

5.11 Appendix: Another way of looking at groupoids

Since the main motivation (and applications) for the deformation cohomology comes from the study of deformations of Lie groupoids, in order to gain some insight into its definition it is worth contemplating a bit on the meaning of deformations (and of Lie groupoids). Given a groupoid $\mathcal{G}$, we want to allow deformations smoothly parameterized by some real parameter ϵ of all the structure maps: s_ϵ, t_ϵ, m_ϵ etc. The cohomology theory that controls deformations should incorporate the variation of the structure maps ($\frac{d}{d\epsilon}s_\epsilon$, etc); the cocycle conditions should be first order consequences of (i.e., obtained by applying $\frac{d}{d\epsilon}$ to) the various equations that the structure maps satisfy. There are two relevant points to be addressed right from the start:

- In the set of Lie groupoid axioms there is a certain redundancy (e.g. the target map is determined by the source and the inversion: $t = s \circ i$). In order to study deformations, it is natural to look for a minimal set of axioms.

- Variations of type "$\frac{d}{d\epsilon} m_\epsilon(g, h)$" are problematic because they make sense only under very restrictive conditions (e.g. when s_ϵ and t_ϵ do not depend on ϵ, so that the condition that g and h are composable does not depend on on ϵ).

Both points are answered by a very simple remark: it is better to use the division map $\bar{m}$ of $\mathcal{G}$ instead of the multiplication; moreover, all the structure maps of $\mathcal{G}$ can be recovered from only $\bar{m}$ and s - themselves satisfying some simple axioms.

Proposition 5.67. *Given the manifolds $\mathcal{G}$ and M, defining structure maps (m, s, t, u, i) making $\mathcal{G}$ into a Lie groupoid over M is equivalent to giving pairs $(s, \bar{m})$ consisting of surjective submersions $s : \mathcal{G} \longrightarrow M$ and $\bar{m} : \mathcal{G} \times_s \mathcal{G} \longrightarrow \mathcal{G}$ (defined on the space of pairs (g, h) with $s(g) = s(h)$) satisfying:*

(i) For all $g, h \in \mathcal{G}$ with $s(g) = s(h)$ one has

$$s(\bar{m}(g, h)) = s(\bar{m}(h, h)).$$

(i.e., the expressions of type $s(\bar{m}(g, h))$ only depend on h).

(ii) For all $g, h, k \in \mathcal{G}$ with $s(g) = s(h) = s(k)$ one has

$$\bar{m}(\bar{m}(g, k), \bar{m}(h, k)) = \bar{m}(g, h)$$

(note: the first expression makes sense because of (i)).

(iii) The restriction of s to $\bar{m}(\Delta) = \{\bar{m}(g, g) \mid g \in \mathcal{G}\}$ is injective.

Proof. We have to see how we can recover all the structure maps and their axioms given a pair $(s, \bar{m})$ as in 2. Note first that any $x \in M$ can be written as $s(\bar{m}(g, g))$ for some $g \in \mathcal{G}$. This follows by using the surjectivity of s and $\bar{m}$ and the fact that $s(\bar{m}(g, h)) = s(\bar{m}(h, h))$ (by (i)). We deduce that the map from (iii) is a bijection, and we denote by $u : M \longrightarrow \bar{m}(\Delta) \subset \mathcal{G}$ its inverse. By construction,

$$u(s(\bar{m}(g, h)) = \bar{m}(h, h)$$

for all $(g, h) \in \mathcal{G} \times_s \mathcal{G}$. The other structure maps are defined by

$$\begin{aligned} t(g) &= s(\bar{m}(g, g)) \\ i(g) &= \bar{m}(u \circ s(g), g) \\ m(g, h) &= \bar{m}(g, i(h)). \end{aligned}$$

Next, we check the groupoid axioms.

First, note that t is a surjective submersion since both s and $\bar{m}$ are. Note also that, by definition, $s(u(s(g))) = s(g)$. On the other hand, $s(i(h)) = s(\bar{m}(u(s(h)), h))$, so by the first axiom, $s(i(h)) = s(\bar{m}(h, h))$ which is $t(h)$ by definition, thus implying that m is well defined and $s \circ i = t$.

1. $(t \circ i(g) = s(g))$:

   Using the second axiom we can compute

$$\begin{aligned}(t \circ i)(g) &= s(\bar{m}(i(g), i(g))) \\ &= s(\bar{m}(\bar{m}(u(s(g)), g), \bar{m}(u(s(g)), g)) \\ &= s(\bar{m}(u(s(g)), u(s(g)))) \\ &= s(\bar{m}(\bar{m}(k,k), \bar{m}(k,k))) \\ &= s(\bar{m}(k,k)),\end{aligned}$$

   where $u(s(g)) = \bar{m}(k,k)$ and so $s(\bar{m}(k,k)) = s(g)$ as desired.

2. $(s(m(g,h)) = s(h)$, and $t(m(g,h)) = t(g))$:

   For $(g,h) \in \mathcal{G}^{(2)}$, we have

$$\begin{aligned}s(m(g,h)) &= s(\bar{m}(g, i(h))) \\ &= s(\bar{m}(i(h), i(h)) \\ &= t(i(h)) \\ &= s(h).\end{aligned}$$

   Here, the second equality is consequence of the first axiom, and the third is the definition of t.

   Also, we have that

$$\begin{aligned}t(m(g,h)) &= t(\bar{m}(g, i(h))) \\ &= s(\bar{m}(\bar{m}(g, i(h)), \bar{m}(g, i(h))) \\ &= s(\bar{m}(g,g)) \\ &= t(g).\end{aligned}$$

   This time, the third equality is due to the second axiom.

3. $(s \circ u = \mathrm{id}_M$, and $t \circ u = \mathrm{id}_M)$:

   $s \circ u = \mathrm{id}_M$ is just by definition. For the other, let $u(x) = \bar{m}(k,k)$. Then

$$\begin{aligned}(t \circ u)(x) &= t(\bar{m}(k,k))) \\ &= s(\bar{m}(\bar{m}(k,k), \bar{m}(k,k))) \\ &= s(\bar{m}(k,k)) \\ &= s(u(x)) \\ &= \mathrm{id}_M(x).\end{aligned}$$

4. $(m(i(g), g) = u(s(g), m(g, i(g)) = u(t(g)$, and $i^2 = \mathrm{id}_{\mathcal{G}})$:

Again, let $u(s(g)) = \bar{m}(k,k)$

$$\begin{aligned} m(i(g),g) &= \bar{m}(i(g),i(g))) \\ &= \bar{m}(\bar{m}(u(s(g)),g),\bar{m}(u(s(g)),g)) \\ &= \bar{m}(u(s(g)),u(s(g))) \\ &= \bar{m}(\bar{m}(k,k),\bar{m}(k,k))) \\ &= \bar{m}(k,k) = u(s(g)). \end{aligned}$$

Next, suppose that $g = \bar{m}(h,k)$. Then

$$i(g) = \bar{m}(u(s(\bar{m}(h,k))),\bar{m}(h,k)) = \bar{m}(\bar{m}(k,k))),\bar{m}(h,k)) = \bar{m}(k,h).$$

From this, it follows that $i^2 = \mathrm{id}_{\mathcal{G}}$, (and consequently that i is a bijection).

Moreover,

$$\begin{aligned} m(g,i(g)) &= s(\bar{m}(i(i(g)),i(g))) \\ &= u(s(i(g)) \\ &= u(t(g)). \end{aligned}$$

5. $(m(g,u(s(g))) = m(u(t(g)),g) = g)$:

   First we are to prove that $m(g,u(s(g))) = m(u(t(g)),g)$.

$$\begin{aligned} m(u(t(g)),g) &= \bar{m}(u(t(g)),i(g))) \\ &= \bar{m}(u(s(\bar{m}(g,g))),\bar{m}(u(s(g)),g)) \\ &= \bar{m}(\bar{m}(g,g),\bar{m}(u(s(g)),g)) \\ &= \bar{m}(g,u(s(g))), \end{aligned}$$

   but $i(u(s(g))) = \bar{m}(u(s(g)),u(s(g)))$ and we saw the latter was equal to $u(s(g))$ thus allowing us to remove the bar in the last equation.

   Finally, let $g = \bar{m}(h,k)$, where

$$t(g) = s(\bar{m}(\bar{m}(h,k),\bar{m}(h,k)) = s(\bar{m}(h,h)).$$

   Then

$$\begin{aligned} m(u(t(g)),g) &= \bar{m}(u(t(g)),i(g))) \\ &= \bar{m}(u(s(\bar{m}(h,h))),\bar{m}(k,h)) \\ &= \bar{m}(\bar{m}(h,h),\bar{m}(k,h)) \\ &= \bar{m}(h,k) = g. \end{aligned}$$

6. (m is associative):

Let $(g,h),(h,k) \in \mathcal{G}^{(2)}$. We compute

$$\begin{aligned}
m(g,m(h,k)) &= \bar{m}(g,i(\bar{m}(h,i(k)))) \\
&= \bar{m}(g,\bar{m}(i(k),h)) \\
&= \bar{m}(\bar{m}(g,i(h)),\bar{m}(\bar{m}(i(k),h)),i(h)) \\
&= \bar{m}(m(g,h),\bar{m}(\bar{m}(i(k),h)),m(u(t(h),i(h))) \\
&= \bar{m}(m(g,h),\bar{m}(\bar{m}(i(k),h)),\bar{m}(u(t(h),h)) \\
&= \bar{m}(m(g,h),\bar{m}(i(k),u(t(h)) \\
&= \bar{m}(m(g,h),\bar{m}(i(k),i(u(t(h))) \\
&= \bar{m}(m(g,h),m(i(k),u(t(h))) \\
&= \bar{m}(m(g,h),i(k))) \\
&= m(m(g,h),k)).
\end{aligned}$$

We turn now to the problem of whether the maps are smooth. We have already pointed out that t is a surjective submersion, in particular it is smooth. Notice that u fits in the following diagram,

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\Delta"] \arrow[dr, "t"'] & \mathcal{G} \times_s \mathcal{G} \arrow[r, "\bar{m}"] & \mathcal{G} \\
& M \arrow[ur, "u"'] &
\end{tikzcd}$$

and is therefore smooth. Also, since $s \circ u = \mathrm{id}_M$, $d(s \circ u)_x = \mathrm{Id}_x$. Let $X \in T_xM$, if $du_x(X) = 0$,

$$\begin{aligned}
ds_{u(x)}(du_x(X)) &= ds_{u(x)}(0) \\
d(s \circ u)_x(X) &= 0 \\
\mathrm{Id}_x(X) &= 0 \\
X &= 0.
\end{aligned}$$

Summing up, u is an injective smooth immersion onto $u(M) = \bar{m}(\Delta)$, and again since s is a left inverse and s is continuous u is a homeomorphism onto its image, i.e., an embedding. From the smoothness of u and $\bar{m}$, the smoothness of i and m follow.
Finally, i is its own inverse, and it is a diffeomorphism. □

Corollary 5.68. *Given two Lie groupoids $\mathcal{G}$ over M and $\mathcal{H}$ over N and smooth maps $F : \mathcal{G} \longrightarrow \mathcal{H}$, $f : M \longrightarrow N$, then (F, f) is a groupoid morphism if and only if*

$$s(F(g)) = f(s(g)),\ \ \bar{m}(F(g), F(h)) = F(\bar{m}(g, h)$$

for all $g, h \in \mathcal{G}$ with $s(g) = s(h)$.

Continuing our previous motivating comments coming from deformations, here is one more explanatory remark. Assume that we have a deformation $s_\epsilon, t_\epsilon, m_\epsilon, u_\epsilon, i_\epsilon$ with $s = s_\epsilon$ not depending on ϵ (s-constant deformation). Proposition 5.67 reveals that, in order to study the variation associated to the deformation, it is enough to concentrate on the variation of $\bar{m}_\epsilon$ (since s_ϵ is constant in ϵ). This will induce a cocycle $\xi_0 \in C^2_{\mathrm{def}}(\mathcal{G})$ (discussed in detail in Section 5.4). The fact that ξ_0 is a cocycle incorporates, as we wanted, precisely the first order consequences of the axioms for $(s_\epsilon, \bar{m}_\epsilon)$ mentioned in Proposition 5.67:

1. The equation from (i) will imply that $ds(\xi_0(g,h)) = ds(\xi(1,h))$, i.e., precisely the s-projectability defining the complex $C^*_{\mathrm{def}}(\mathcal{G})$.

2. The associativity equation from (ii) will imply $\delta(\xi_0) = 0$.

Remark 86 (more natural variables). The previous proposition and remark indicate that there is a more natural way of (re-)writing the deformation complex, so that it only makes use of s and $\bar{m}$ in its definition; we will denote it by $\bar{C}^*_{\mathrm{def}}(\mathcal{G})$ (a more suggestive notation would be $C^*_{\mathrm{def}}(\mathcal{G}, s, \bar{m})$). It arises by the standard change of variables

$$\mathcal{G}^{(k)} \overset{\sim}{\longleftrightarrow} \mathcal{G}^{[k]}, \quad (g_1, \ldots, g_k) \longleftrightarrow (a_1, \ldots, a_k)$$

which relates strings of composable arrows to strings of arrows with the same source:

$$a_i = g_i g_{i+1} \ldots g_k, \quad g_i = \begin{cases} a_i a_{i+1}^{-1} & \text{for } i \leq k-1 \\ a_k & \text{for } i = k \end{cases} \tag{5.31}$$

Explicitly, the k-cochains $u \in \bar{C}^k_{\mathrm{def}}(\mathcal{G})$ are the smooth maps

$$u : \mathcal{G}^{[k]} \longrightarrow T\mathcal{G}, (a_1, \ldots, a_k) \mapsto u(a_1, \ldots, a_k) \in T_{\bar{m}(a_1,a_2)}\mathcal{G} = T_{a_1 a_2^{-1}}\mathcal{G}$$

defined on the strings of arrows with the same source, with the property that

$$s(u(a_1, \ldots, a_k)) \in T_{s(\bar{m}(a_1,a_2))}\mathcal{G}$$

does not depend on a_1 (note: "the first axiom" for s and $\bar{m}$, i.e., (i) of Proposition 5.67, ensures that $s(\bar{m}(a_1, a_2))$ does not depend on a_1).

The differential of $u \in \bar{C}^k_{\mathrm{def}}(\mathcal{G})$ is

$$\begin{aligned}(\bar{\delta}u)(a_1, \ldots, a_{k+1}) = &-d\bar{m}(c(a_1, a_3, \ldots, a_{k+1}), c(a_2, a_3, \ldots, a_{k+1}) \\ &+ \sum_{i=3}^{k+1} (-1)^{i+1} c(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) + (-1)^{k+1} u(a_1 a_{k+1}^{-1}, \ldots, a_k a_{k+1}^{-1}).\end{aligned}$$

(for $k = 0$, one keeps the same definition as for $C^*_{\mathrm{def}}(\mathcal{G})$). The second axiom for s and $\bar{m}$, i.e., (ii) of Proposition 5.67, ensures that δ is well-defined and squares to zero. Of course, the change of variables (5.31) induces an isomorphism between $(C^\bullet_{\mathrm{def}}(\mathcal{G}), \delta)$ and $(\bar{C}^*_{\mathrm{def}}(\mathcal{G}), \bar{\delta})$.

Finally, let us mention that the reason we choose to use $C^*_{\mathrm{def}}(\mathcal{G})$ instead of $\bar{C}^*_{\mathrm{def}}(\mathcal{G})$ is that, searching in the existing literature involving cohomology of groupoids (and even of groups), we see that one always uses the $\mathcal{G}^{(k)}$'s (and the corresponding formulas) for the domains of the cochains. It is clear however that $\bar{C}^*_{\mathrm{def}}(\mathcal{G})$ and the entire view-point that arises from Proposition 5.67 is much more conceptual and we expect it to be useful in various related problems (e.g. for finding a non-linear analogue of the Gerstenhaber bracket that makes the deformation complex of a Lie algebroid into a DG Lie algebra structure and allows one to interpret Lie algebroid structures as Maurer-Cartan elements [27]; or in the study of higher groupoids, etc).

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Samenvatting

De kern van dit proefschrift bestaat uit twee studies van de differentieerbare meetkunde van Lie groepoïdes en de singuliere (i.e. niet gladde) ruimtes die zij modelleren: Eén gaat over maten op baanruimten en de ander over de deformatiecohomologie van Lie groepoïden. Ze zijn aangevuld met wat tentoonstellend materiaal (met name betreffende verschillende benaderingen van het begrip "baanruimte" en hun indeling in lagen).

We introduceren de begrippen "dwarsmaat" en "dwarsdichtheid" voor Lie groepoïdes. Dit veralgemeniseert de aanpak van Haefliger voor étale Lie groepoïdes. Vervolgens bewijzen we enkele fundamentele resultaten: We tonen Morita-invariantie aan voor algemene Lie groepoïdes. Dit stelt ons in staat om dwarsmaten als objecten op stacks te beschouwen. Verder bewijzen we een Stokes-achtige formule. Dit kunnen we beschrijven aan de hand van (Ruelle-Sullivan) algebroïde currents. Tot slot bouwen we een Van-Est-isomorfisme.

In het propere geval kunnen we de theorie terugbrengen tot klassieke Radonmaten op de onderliggende ruimte. We geven expliciete Weyl-achtige formules die Weinstein's begrip van volumes op differentieerbare stacks verduidelijken. We beschouwen ook de begrippen "modulaire klasse" en "Haarsysteem".

Vervolgens bestuderen we deformaties van Lie groepoïden. We introduceren de deformatiecohomologie van een Lie groepoïde. Dit geeft een intrinsiek model voor de cohomologie van een Lie groepoïde met waarden in de adjoint-representatie. We bewijzen enkele fundamentele eigenschappen van de deformatiecohomologie, waaronder Morita-invariantie, een Van-Est-isomorfisme, en een verdwijningsresultaat in het propere geval.

Tot slot gebruiken we de deformatiecohomologie om rigiditeits- en normale vormresultaten te bewijzen.

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Curriculum Vitae

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