# Aspects of random geometric graphs: pursuit-evasion and treewidth

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# Aspects of random geometric graphs: pursuit-evasion and treewidth

# Aspecten van stochastische meetkundige grafen: achtervolging-ontwijking en boombreedte

(met een samenvatting in het Nederlands)

Proefschrift

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*There is no permanent place for ugly mathematics.*

G.H. Hardy

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*To my family*

# Chapter 1

# Introduction

This thesis belongs to the interface between probability and combinatorics. To be more precise, it mainly covers some topics in random geometric graphs. In particular, I studied the following two problems:

- Cops and Robbers games on percolated random geometric graphs;
- Treewidth of random geometric graphs and percolated grids.

This thesis is based on the following two research papers:

- Chasing Robbers on percolated random geometric graphs; joint work with T.Müller and P. Prałat [71].
- On the treewidth of random geometric graphs and percolated grids; joint work with T.Müller [70].

The first paper [71] studied one pursuit-evasion game on random geometric graphs: *Cops and Robbers* game; the second paper [70] investigated the treewidth of random geometric graphs.

In this chapter, we first review some background of our research, then present the main contributions.

## 1.1 Background

### 1.1.1 Random (geometric) graphs

There are two main random graph models so far: random graphs and random geometric graphs. We will briefly present some properties of these two models separately in this section.

#### 1.1.1.1 Random graphs

The theory of random graphs originated in a series of papers [39, 41] published in the period 1959-1968 by Paul Erdös and Alfred Rényi. Over the fifty years that have passed since then, the random graph theory has developed into an independent and fast-growing branch of discrete mathematics. The interest in this field has been rapidly growing in recent years driven by many applications both in theoretical and practical aspects. Recent studies [84] show that a lot of networks can be modeled by random graphs, leading to a milestone in the research in this field. Nowadays, random graph models and related techniques are used in telecommunication networks [109], computer networks [11], neural networks [66], even in sociology [85] and interbank payment networks [86]. As a consequence, random graph theory becomes an extremely active topic which is seated at the intersection of graph theory, probability theory, computer science and so on. In this section, we first review some well-known results.

There are two basic random graph models: the binomial model and the uniform model. It is originating in the simple models introduced by Erdös [40].

**Definition 1.1** (Binomial random graph)**.** Given a real number $p$, $0 \le p \le 1$, the *binomial random graph*, denoted by $G(n,p)$, is defined by taking as $\Omega$ the set of all graphs on vertex set $[n] = \{1, ..., n\}$ and setting

$$\mathbb{P}(G) = p^{e_G}(1-p)^{\binom{n}{2}-e_G}, \qquad G \in \Omega.$$

in which $e_G = |E(G)|$ is the number of edges of $G$.

A sketch of binomial random graph can be found in Figure 1.1.

**Definition 1.2** (Uniform random graph)**.** Given an integer $m$, $0 \le m \le \binom{n}{2}$, the *uniform random graph*, denoted by $G(n,m)$, is denoted by take as $\Omega$ the set of all graphs on the vertex set $[n] = \{1, ..., n\}$ with exactly $m$ edges and as $\mathbb{P}$ the uniform probability on $\Omega$,

$$\mathbb{P}(G) = \binom{\binom{n}{2}}{m}^{-1}, \qquad G \in \Omega.$$

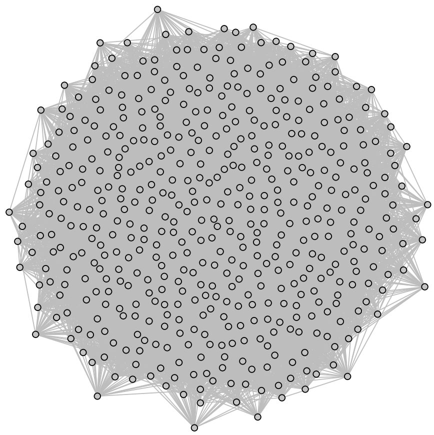

FIGURE 1.1: Binomial Random Graph $G(n,p) : n = 500, p = 0.1$

We should note that the two basic models in many cases asymptotically equivalent, provided that $\binom{n}{2}p$ is close to $m$.

The most influential results about random graphs which has ever appeared was from the seminal paper by Erdös and Rényi [38]. In this paper, they studied the evolution of $G(n,m)$ as $m$ goes from 0 to $\binom{n}{2}$, identifying main features of the random graph. The most striking results related to random graph model is the phase transition. That is, the short period of the random graph evolution when the size of the largest component of $G(n,m)$ rapidly grows from $\Theta(\log n)$ to $\Theta(n)$. We have the following more precise theorem:

**Theorem 1.3** (Theorem 5.4, [58]). *Let $np = c$, where $c > 0$ is a constant.*

*(i) If $c < 1$, then a.a.s. the largest component of $G(n,p)$ has at most $\frac{3}{(1-c)^2} \log n$ vertices.*

*(ii) Let $c > 1$, then $G(n,p)$ contains a giant component of $\Theta(n)$ vertices. Furthermore, a.a.s. the size of the second largest component of $G(n,p)$ is at most $\frac{16c}{(1-c)^2}\log n$.*

There are a great many of variants of these two classic random graph models $G(n,p)$ and $G(n,m)$ since they were first introduced. Here we just give a brief survey.

A very natural extension of $G(n,p)$ is to change the same $p$ into different $p_{ij}$ for all pairs $i,j \in [n]$, i.e., this graph become *inhomogeneous.* One of the special cases in this model is the *Kronecker random graph*, which was first proposed by Leskovec et al. [69], basing on Kronecker matrix multiplication and then a lot of typical graph properties, like the connectivity and giant component, were investigated rigorously by Mahdian and Xu [75], the later result on the emergence of giant component was improved by Horn and Radcliffe [57]. Moreover, a novel correspondence between the adjacencies in a general stochastic Kronecker graph and the action of a fixed Markov chain was developed by Radcliffe and J.Young [98], generalizing the arguments of Horn and Radcliffe on the emergence of the giant component. Very recently, the degree distribution and the subgraphs of stochastic Kronecker graphs were investigated by Kang, Karonski, Koch and Makai [60]. Another interesting generalization is the *Chung-Lu model*, also called expected degree model, introduced by Chung and Lu in [28], where the edge probability $p_{ij}$ depend on weights assigned to vertices. This extension also can be considered as a model for "Real World Networks", which is a very popular topic nowadays.

The most general model of inhomogeneous random graph so far was introduced by Bollobás, Janson and Riordan in their seminal paper [15]. In this paper, they present a wide range of results describing various properties of the inhomogeneous random graphs models. They give a necessary and sufficient condition for the existence of a giant component, show its uniqueness and determine the asymptotic number of edges in the giant component. Moreover, they study the robustness of the component, i.e., they show that its size does not change much if we add or delete a few edges. They also establish bounds on the size of small components, the asymptotic distribution of the number of vertices of given degree and study the distances between vertices. In the last section, they also explore the phase transition phenomenon where the giant component first emerges.

Other models include *the configuration model* and *the random intersection graph.* For more information, see the very recent book by Frieze and Karonski [46].

There has recently been an increased interest in the networks, partially due to the development of social networks and Internet. The well-known journal –internet mathematics, is possibly the product of this powerful development.

Two well-known properties of real networks are: *small world* and *power law.* Bollobás and Chung [14] studied the following graph model: added a random matching to a ring of $n$ vertices with nearest neighbor connection. They showed the resulting graph has diameter $\sim \log_2 n$. Ten years later, Watts and Strogatz [108] found that random long distance connection could drastically reduce the diameter of the network, which was called *small world model* later, see Figure 1.2. More or less at the same time, Barabási and Albert [7] found another interesting phenomena: the actor collaboration graph and the World Wide Web had degree distributions that were power laws $p_k \sim Ck^{-\gamma}$ as $k \to \infty$ ($\gamma$ is some positive constant). After their paper, a large number of examples with power law degree distributions were identified, which are also called *scale-free* random graphs. In [7], Barabási and Albert introduced the famous preferential attachment model in order to give a mechanistic explanation for the power law, see Figure 1.3. Cooper and Frieze [30] then proposed a more general model in which: old nodes sometimes generate new edges, and choices are sometimes made uniformly instead of by preferential attachment. See, for example, the book by Durrett [36] for more information on dynamics of random graphs.

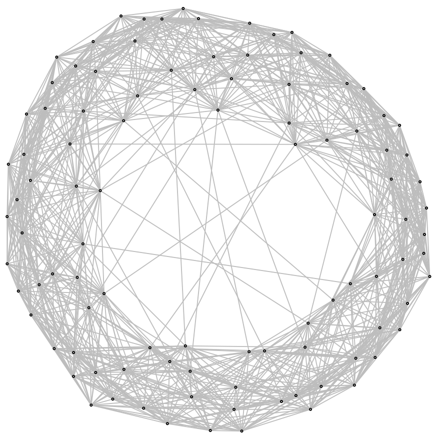

Figure 1.2: WS Model.

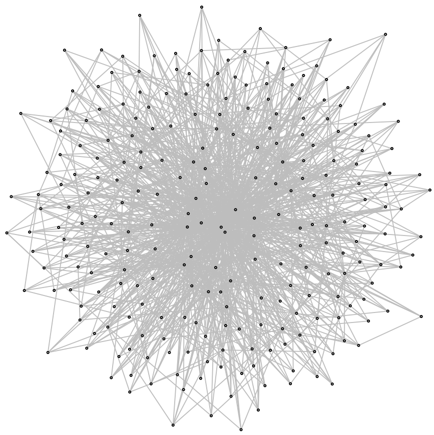

FIGURE 1.3: BA Model.

#### 1.1.1.2 Random geometric graphs

Gilbert introduced a different random graph model, which is called *random plane networks* (see [49]), later are called *random geometric graphs*. In this model, the vertices have some (random) geometric layout and the edges are determined by the position of these vertices.

The definition of *random geometric graph* goes as follows:

**Definition 1.4** (Random geometric graph, [92])**.** Let $||\cdot||$ be the Euclidean norm on $\mathbb{R}^d$, and $r \in \mathbb{R}^+$. Let $f$ be some specified probability density function on $\mathbb{R}^d$, and let $X_1, X_2, ...$ be independently and identically distributed $d-$dimensional variables with common density $f$. Let $\mathcal{X}_n = \{X_1, X_2, ..., X_n\}$. We denote by $RGG(\mathcal{X}_n; r)$ the undirected graph with vertex set $\mathcal{X}_n$ and with undirected edges connecting all those pairs $\{X_i, X_j\}$ with $||X_i - X_j|| \leq r$. We call this a random geometric graph.

See Figure 1.4 for a depiction: we draw $n = 500$ points uniformly at random from unit square $[0, 1]^2$ and fix $r = 0.1$.

*Remark* 1.5. The definition of random geometric graph above is very general. Actually, we will use some specific versions of this definition. In Chapter 2, the vertices of random

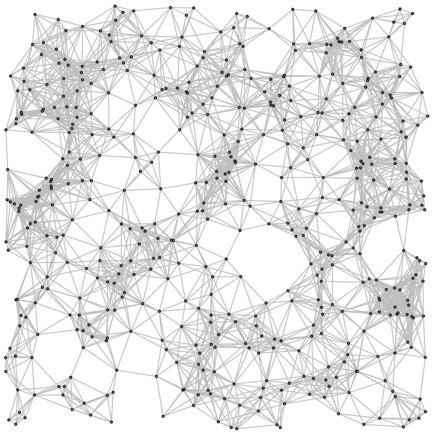

FIGURE 1.4: Random geometric graph in unit square $[0,1]^2$.

geometric graph are drawn uniformly at random from unit square $[0,1]^2$; In Chapter 3, the vertices are drawn uniformly at random from $[0,\sqrt{n}]^2$. To make it easy to follow, we will repeat the specific definitions again in the corresponding parts of this thesis.

These years have witnessed quite a lot of work on random geometric graphs, both on theoretical results and applications. One very important motivation for the study of random geometric graphs is multivariate statistics, especially some spatial and scan statistics.

Many classification techniques in statistical community and machine learning community are based on this spatial structure of the random geometric graphs. One of the oldest methods of clusters analysis is *single linkage* or *nearest-neighbor* cluster method. The single-linkage clusters at level $r$ are the connected components of $RGG(\mathcal{X};r)$: a graph with vertex set $\mathcal{X}$ and whose edges link just those pairs of distance at most $r$. As a consequence, the components of $RGG(\mathcal{X};r)$ form a partition of the data, and every component is one cluster, leading a clustering. The main aim in this method is to specify the parameter $r$, since different $r$'s lead to different geometric structures. For more information, see [106].

Besides statistical applications, many geometric aspects of this graph model have been extensively studied in the context of stochastic geometry and random fields, for example, see [101, 107].

In most of the settings, we consider random geometric graphs with vertices that are from some probability distributions or point processes, for example, the uniform distribution and Poisson point process. Except in the most trivial cases, exact formulae for properties of $RGG(\mathcal{X}_n; r)$ tend to be very unwieldy and cumbersome, which are too complicated to be of much use. Sometimes they even do not have some closed form at all, especially in more than one dimension. This is why most of the results in this area focus on *asymptotic* properties of $RGG(\mathcal{X}_n; r)$ for some sequence of parameter $r_n$, in most of cases, tending to zero when the size of the vertex set goes to infinity (for example, $r_n = (1/n)^{1/d}$, $r_n = (c \log n/n)^{1/d}$).

When $r = r_n$ is a function of the order of the random geometric graphs, there are two interesting limiting regimes for $(r_n)$. One of these is the *thermodynamic limit* in which $r_n \sim cn^{-1/d}$ ($c$ is a constant), so that the expected degree of a typical vertex tends to a constant. The second limiting regime of special interest is the *connectivity regime*, in which $r_n \sim c\left(\frac{\log n}{n}\right)^{1/d}$ with $c$ is a constant. In this case, the typical degree grows logarithmically in $n$. In fact, a lot of research are done on the random geometric model in these two limiting regimes, see [92].

As we have already presented in Theorem 1.3 for the random graphs: if the mean vertex degree is fixed at a value exceeding a critical value of 1, then the giant component emerges containing a non-vanishing proportion of vertices of graph. A similar phenomenon occurs for random geometric graphs at some $r_c$, which is still not known yet, for more arguments, see Chapter 9 and 10 of [92].

There are many variants and generalizations of the classic random geometric graph $RGG(\mathcal{X}_n; r)$. Here we just mention two of them.

One is called *soft random geometric graphs*, which was introduced and studied by Penrose [93]. The definition goes as follows: Given a measurable function $\phi : \mathbb{R}^d \to [0, 1]$ that is symmetric (i.e., satisfies $\phi(x) = \phi(-x)$ for all $x \in \mathbb{R}^d$), and given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, let $G_\phi(\mathcal{X})$ be the random graph with vertex set $\mathcal{X}$, obtained when each potential edge $\{x, y\}$(with $x, y \in \mathcal{X}$ and $x \neq y$) is present in the graph with probability $\phi(x - y)$, independently of all other pairs. One special case is when $\phi(x) = p\mathbf{1}_{[0,r]}(|x|)$. The graph obtained from this function is very interesting, which can be viewed as the intersection of the random geometric graph and the Erdös-Rényi random graph. Actually, in Chapter 2, we call this graph *percolated random geometric graph*, and will play a "Cops and Robber" game on this graph.

The other variant of the classic random geometric graph is called *random geometric irrigation graph* or *Bluetooth graph*, which was first given by Broutin, Devroye, Fraiman and Lugosi [24]. Let $G_n = (V, E)$ be a finite undirected graph on $|V| = n$ vertices and edge set $E$. A *random irrigation subgraph* $S_n = (V, \widehat{E})$ of $G_n$ is obtained as follows: Let $2 \leq c_n \leq n$ be a positive integer. For every vertex $v \in V$, we pick randomly and independently, without replacement, $c_n$ edges from $E$, each adjacent to $v$. These edges form the set of edges $\widehat{E} \subset E$ of the graph $S_n$ (if the degree of $v$ in $G_n$ is less than $c_n$, all edges adjacent to $v$ belong to $\widehat{E}$). It is obvious the structure of $S_n$ depends on that of the underlying graph $G_n$. The case $G_n = K_n$ was investigated by Fenner and Frieze [42]. To be more precise, for $c_n = c \geq 2$, they proved that $S_n$ is $c$-connected (for both vertex- and edge-connectedness) with high probability. The *random geometric irrigation graph* is the random irrigation subgraph of random geometric graphs $RGG(\mathcal{X}; r)$. In a series of papers [24–26], Broutin et al. studied the connectivity and optimal sparsification of this model.

### 1.1.2 Combinatorial game theory

Combinatorial game theory is the branch of combinatorics (and game theory) that studies games on complete information such as Tic-Tac-Toe and Hex. It is this challenge that motivates the development of *Combinatorial Game Theory.* For a comprehensive understanding of this amazing theory, we recommend the great book by J.Beck [8].

Two different types of combinatorial games are: *positional* games (like Tic-Tac-Toe game, Maker-Breaker game) and *pursuit-evasion* or *graph searching* games (like Cops-Robbers game). We will introduce Cops and Robbers game briefly in this section.

#### 1.1.2.1 Cops and Robbers game

The game of cops and robbers, is one *pursuit-evasion* game on graphs, which dates back to the early 1980s, introduced independently by Nowakowski and Winkler [87] and Quilliot [97]. To be more specifically, the game is played with a set of cops (they are controlled by a single player, but the cops can move to different vertices in some manner according to the command of this player each time) trying to capture the robber (controlled by the other player). The cops and robber are restricted to vertices, and they move each round to neighboring vertices: each move is seen by both players, i.e., this is a game with complete information. The cops win if they manage to occupy the same vertex as the robber, and the robber wins if she avoids this forever. The smallest number of cops needed to capture the robber is the *cop number*, which is our main interest in this area. It turns out such a seemly-simple game leads to a very complicated and deep theory. Bonato and Nowakowski mentioned in their monograph [20]: Despite the fact the game is nearly three decades old, the last five years however, have seen an explosive growth in research in this field. Some newer work settles some old problems, while novel approaches, both probabilistic, structural, and algorithmic, have emerged on this classic game on graphs.

There are a lot of beautiful and deep results in this topic, we just mention some of which related to our research in Chapter 2.

We will always assume that $G$, on which the game is played, is undirected, simple, connected, and finite.

The original version of cops and robbers game [87, 97] concerned a single cop chasing the robber. These papers characterized the structure of *cop-win* graphs for which a single cop can have a winning strategy. In fact, the *cop-win* graphs enjoy a very useful property, called *dismantlability*.

For $v \in V(G)$, the neighborhood of $v$ is $N(v) = \{u \in V : (u, v) \in E\}$ and the *closed neighborhood* of $v$ is $\bar{N}(v) = \{v\} \cup N(v)$. For $u \in V(G)$ and for some $v \in V(G) \setminus \{u\}$, if $\bar{N}(u) \subset \bar{N}(v)$, we say $u$ is a *pitfall*. A graph is *dismantlable* if we can reduce $G$ to a single vertex by successively removing pitfalls. The characterization of cop-win graphs goes as follows:

**Theorem 1.6** ([87, 97])**.** *$G$ is dismantlable if and only if $c(G) = 1$.*

For the multi cops case, one of the classic results was proved by Aigner and Fromme [1]: three cops is sufficient to catch the robber on planar graph.

**Theorem 1.7** ([1])**.** *If $G$ is planar and connected, then the cop number $c(G) \leq 3$.*

Various authors have studied the cop number of families of graphs, for examples, see [45, 79, 83]. The most important open problem in this area is Meyniel's conjecture (communicated by Frankl [44]). It states that $c(n) = O(\sqrt{n})$, where $c(n)$ is the maximum of $c(G)$ over all $n$-vertex connected graphs. If true, the estimate is best possible as one can construct a graph based on the finite projective plane with the cop number of order at least $\Omega(\sqrt{n})$, for example, see detailed arguments in section 3.3 of [20]. Up until recently, the best known upper bound of $O(n \log \log n / \log n)$ was given in [44]. This was improved to $c(n) = O(n/\log n)$ in [27]. Today we know that the cop number is at most $n 2^{-(1+o(1))\sqrt{\log_2 n}}$ (which is still $n^{1-o(1)}$) for any connected graph on $n$ vertices (a result obtained independently by Lu and Peng [73] and Scott and Sudakov [102], see also [47] for some extensions).

Recently, Beveridge, Dudek, Frieze and Müller [12] studied the cop number of geometric graphs in $\mathbb{R}^2$. Given points $x_1, ....x_n \in \mathbb{R}^2$ and $r \in \mathbb{R}^+$, the *geometric graph* $G = G(x_1, ..., x_n; r)$ has vertices $V(G) = \{1, ..., n\}$ and $ij \in E(G)$ if and only if $||x_i - x_j|| \leq r$. This model is widely used to model ad-hoc wireless networks [53]. Analogously with the result on planar graphs, they proved a constant upper bound on the cop number of 2-dimensional geometric graphs.

**Theorem 1.8** ([12])**.** *If $G$ is a connected geometric graph in $\mathbb{R}^2$, then $c(G) \leq 9$.*

The geometric graphs are frequently non-planar. But with some geometric techniques, Beveridge et al. [12] proved this theorem by an adaptation of the proof of Theorem 1.7. This adaptation requires three cops on a geometric graph to play the role of a single cop on a planar graph.

Random graphs, which was introduced by Erdös and Rényi in their a series of pioneering work on this subjects in 1960s (see [38, 39]), play an important role both as displaying asymptotic and typical properties of graphs, and as a beautiful theory in their own right. For a deep and comprehensive study, see the monographs by Bollobas [13], Janson, Luczak and Rucinski [58] and Frieze and Karonski [46].

Gilbert introduced a different random graph model, which is called *random geometric graphs* (see [49]). In this model, the vertices have some (random) geometric layout and the edges are determined by the position of these vertices, which are obviously different from the Erdös -Rényi model. Most of the theoretical results on random geometric graphs can be found in the monograph by Penrose [92].

As the development of the random structures, it is very natural to transfer the classic cop and robber games into random versions. The cop number of the classic Erdös and Rényi random graph models $G(n, p)$ were investigated by several authors, see [16, 19, 21, 74]. The most interesting result on the cop number of random graph when $np = n^{\alpha+o(1)}$, where $0 < \alpha < 1$, is the recent striking and surprising Zig-Zag Theorem of Luczak and Pralat [74].

Here we list some of the results mentioned above as follows: For constant $p \in (0, 1)$ or $p = p(n) = o(1)$, we define

$$\mathbb{L}n = \log_{\frac{1}{1-p}} n.$$

The cop number of random graph $G(n, p)$ was studied in [19] for constant $p$ stating that the cop number $c(G(n, p))$ concentrates on $\Theta(\log n)$.

**Theorem 1.9** (Constant $p$, [19])**.** *Let $0 < p < 1$ be fixed. For every real $\epsilon > 0$, a.a.s.*

$$(1 - \epsilon)\mathbb{L}n \leq c(G(n, p)) \leq (1 + \epsilon)\mathbb{L}n.$$

*In particular,*

$$c(G(n,p)) = \Theta(\log n).$$

The more difficult problem is to determine the cop number of $G(n,p)$, where $p = p(n)$ is a function of $n$. We now show some work on the cop number $c(G(n,p))$ of $G(n,p)$ for variable $p = p(n)$.

**Theorem 1.10** ([21]). *(1) Suppose that $p \geq p_0$ where $p_0$ is the smallest $p$ for which*

$$\frac{p^2}{40} \geq \frac{\log\left((\log^2 n)/p\right)}{\log n}$$

*holds. Then a.a.s.*

$$\mathbb{L}n - \mathbb{L}\left((p^{-1}\mathbb{L}n)(\log n)\right) \leq c(G(n,p)) \leq \mathbb{L}n - \mathbb{L}\left((\mathbb{L}n)(\log n)\right) + 2.$$

*(2) If $(2\log n)/\sqrt{n} \leq p = o(1)$ and $\omega(n)$ is any function tending to infinity, then a.a.s.*

$$\mathbb{L}n - \mathbb{L}\left((p^{-1}\mathbb{L}n)(\log n)\right) \leq c(G(n,p)) \leq \mathbb{L}n + \mathbb{L}\left(\omega(n)\right).$$

The results above seem a little cumbersome, especially for the choice of $p = p(n)$. An example of a suitable $p$ in the result above would be $p = (\log n)^{-1/4}$.

**Theorem 1.11** (Concentration of cop number, [21]). *If $p = n^{-o(1)}$ and $p < 1$, then a.a.s.*

$$c(G(n,p)) = (1 + o(1))\mathbb{L}n.$$

**Theorem 1.12** ([16]). *If $p(n) \geq 2.1\log n/n$, then a.a.s.*

$$\frac{1}{(np)^2} n^{\frac{1}{2}\frac{\log\log(np)-9}{\log\log(np)}} \leq c(G(n,p)) \leq 160000\sqrt{n}\log n.$$

Actually, a byproduct of the theorem above is proving Meyniel's conjecture for random graphs, up to a logarithmic factor of $n$ from the upper bound.

Now we present the famous Zig-Zag Theorem from Luczak and Pralat [74].

**Theorem 1.13** (Zig-Zag Theorem, [74]). *Let $0 < \alpha < 1$, and $d = d(n) = np = n^{\alpha+o(1)}$.*

*(i) If $\frac{1}{2j+1} < \alpha < \frac{1}{2j}$ for some $j \geq 1$, then a.a.s.*

$$c(G(n,p)) = \Theta(d^j).$$

*(ii) If $\frac{1}{2j} < \alpha < \frac{1}{2j-1}$ for some $j \geq 1$, then a.a.s.*

$$\Omega\left(\frac{n}{d^j}\right) = c(G(n,p)) = O\left(\frac{n\log n}{d^j}\right).$$

The cops and robbers game in the scenario of random geometric graphs (see formal introduction of random geometric graphs in the Section 1.1.1.2) were also studied by Beveridge et al. [12]. They first show the cop number for dense random geometric graphs is at most 2 by adapting a winning two cops strategy on the grid to obtain a winning strategy on random geometric graph: let $X_1, ..., X_n$ be $n$ points on $[0,1]^2$ drawn uniformly at random, two points $X_i, X_j$ are adjacent when the distance between them is within some $r > 0$, i.e., $||X_i - X_j|| \leq r$. Denote the probability space of random geometric graphs by $\mathcal{G}_2(n,r)$.

**Theorem 1.14** ([12]). *There is a constant $K > 0$ such that the following holds. If $G \in \mathcal{G}_2(n,r)$ on $[0,1]^2$ with $r \geq K(\log n/n)^{1/4}$, then a.a.s. $c(G) \leq 2$.*

In particular, they improved over the bound of Theorem 1.8 when the random geometric graph is sufficiently dense. The essence of this proof is to adapt known pursuit evasion strategies of static graph to the geometric setting. By the dismantablity criterion of Theorem 1.6, they proved that a sufficiently dense geometric graph also requires a single cop.

The following result for standard random geometric graphs was obtained in [12].

**Theorem 1.15** ([12]). *There exists an absolute constant $c > 0$ so that the following holds. If $G \in \mathcal{G}_2(n,r)$ on $[0,1]^2$ with $r^5 > c\frac{\log n}{n}$ then a.a.s. $c(G) = 1$.*

This result was proven independently by Alon and Pralat [3] using a graph pursuit algorithm in the spirit of [105] and [67]. Actually, in the same paper, they got the cop number of random geometric graph in $\mathbb{R}^d$ for all $d \geq 2$.

**Theorem 1.16** ([3]). *For fixed $d > 1$, there exists a constant $c_d$ such that if $G \in \mathcal{G}_d(n,r)$ with $r^{3d-1} > c_d\frac{\log n}{n}$, then a.a.s. $c(G) = 1$.*

### 1.1.3 Percolation theory

#### 1.1.3.1 Discrete percolation

Percolation theory was proposed by Broadbent and Hammersley [23], in order to model the flow of fluid in a porous medium with random blocked channels. From one point of view, percolation theory can be seen as the study of the component structures of some random subgraphs of graphs. The underlying graph can be any graph, although many of the work on percolation so far focus on the case that the underlying graph is a lattice or a lattice-like graph. To obtain the random subgraph we keep vertices or edges independently with some probability $p \in [0, 1]$ and delete them independently with probability $1-p$ ($p$ may be different for different edges or vertices). For example, the classic Erdös-Rényi random graph $G(n, p)$ can be considered as a random subgraph with the underlying graph $K_n$, i.e., a complete graph with $n$ vertices. Then Theorem 1.3 can be considered as a percolation argument on random graph $G(n, c/n)$. For more information on percolation theory, see the well-known books [17, 51, 64]. For percolation on general graphs that beyond lattices, Benjamini and Schramm gave a very comprehensive survey with many conjectures [10], more advanced information can be found in the book of Benjamini [9].

We first introduce some basic concepts of percolation theory and then show some fundamental results concerning them, the notations and definitions in this section are mainly borrowed from [17].

In the context of percolation theory, vertices and edges are called *sites* and *bonds*, and components are called *clusters*. When the random subgraph is obtained by keeping vertices, we call it *site percolation*; when we keep edges, *bond percolation*. In either case, the sites or bonds kept are called *open* and those delete are called *closed*; the *state* of a site or bond is open if it is kept, and closed otherwise. In site percolation, the *open subgraph* is the subgraph induced by the open sites; in bond percolation, the *open subgraph* is formed by the open edges and *all vertices*.

In this thesis, we only consider unoriented bond percolation, i.e., percolation on an unoriented graph $G$. We will assume $G = (V, E)$ is connected, infinite and locally finite (i.e., every vertex has finite degree) and simple (without loops and multi-edges).

For every $e \in E$, we open it with probability $p_e \in [0, 1]$, independently of all others, which gives a probability measure $\mathbb{P}$ on the set of subgraphs of $G$. Then we give the formal introduction for this product measure.

For a simple and undirected graph $G = (V, E)$, a bond configuration is a function $\omega : E \to \{0, 1\}, e \to \omega_e$; we write $\Omega = \{0, 1\}^E$ for the set of all bond configurations. A bond $e$ is *open* in the configuration $\omega$ if and only if $\omega_e = 1$, so the configurations correspond to open subgraphs. Let $\Sigma$ be the $\sigma$-field on $\Omega$ generated by the cylindrical sets

$$C(F, \sigma) = \{\omega \in \Omega : \omega_f = \sigma_f \ for \ f \in F\},$$

where $F$ is a finite subset of $E$ and $\sigma \in \{0,1\}^F$. Let $\mathbf{p} = (p_e)_{e\in E}$, with $0 \le p_e \le 1$ for every bond $e$. We denote by $\mathbb{P}_\mathbf{p}$ the probability measure on $(\Omega, \Sigma)$ induced by

$$\mathbb{P}_\mathbf{p}\left(C(F,\sigma)\right) = \prod_{f\in F,\ \sigma_f=1} p_f \prod_{f\in F,\ \sigma_f=0} (1-p_f).$$

In the measure $\mathbb{P}_\mathbf{p}$, the states of the bonds are independent, with the probability that $e$ is open equal to $p_e$; thus, for two disjoint sets $F_0$ and $F_1$ of bonds, we have

$$\mathbb{P}_\mathbf{p}(\textit{the bonds in } F_1 \textit{ are open and those in } F_0 \textit{ are closed}) = \prod_{f\in F_1} p_f \prod_{f\in F_0} (1-p_f).$$

We call $\mathbb{P}_\mathbf{p}$ an *independent bond percolation measure* on $G$.

An *open path* is a path (i.e., a self-avoiding walk) in the open subgraph. For sites $x$ and $y$, we write "$x \to y$" for the event that there is an open path from $x$ to $y$, and $\mathbb{P}(x \to y)$ for the probability of this event in the measure under consideration, We shall write "$x \to \infty$" for the event that there is an infinite open path starting from $x$.

An *open cluster* is a component of the open subgraph. Since we only consider graphs that are locally finite, an open cluster is infinite if and only if for every site $x$ in the cluster, the event $\{x \to \infty\}$ holds. Given a site $x$, we write $C_x$ for the open cluster containing $x$, if there is one; otherwise, $C_x$ is the empty set. Thus $C_x = \{y \in V : x \to y\}$ is the set of sites $y$ for which there is an open path from $x$ to $y$.

When $p_e = p \in [0,1]$ for all $e \in E$. We define $\theta_x(p)$ be the probability that $C_x$ is infinite, so $\theta_x(p) = \mathbb{P}_p(x \to \infty)$, it is easy to note that, in general, this value depends on the underlying graph $G$. More formally, we define

$$\theta_x(p) = \mathbb{P}_p(|C_x| = \infty),$$

where $|C_x| = |V(C_x)|$ is the number of sites in $C_x$. In fact, we only consider some special graphs: *vertex transitive* graphs.

**Definition 1.17** (Vertex transitive graphs)**.** (i) Let $G = (V, E)$ be a graph. A bijection $g : V \to V$ such that $\{g(u), g(v)\} \in E$ if and only if $\{u, v\} \in E$ is called a graph automorphism. The set of all automorphisms of $G$ is denoted by Aut(G).

(ii) A graph $G$ is called vertex transitive if for every pair $u, v \in V$, there exists a graph automorphism mapping $u$ to $v$.

In short, every vertex in a transitive graph plays a same role as others. Then we can write $\theta(p)$ for $\theta_x(p)$ for any site $x$. The quantity $\theta(p)$ is called *percolation probability.*

From the definition of $\theta(p)$, it is easy to see that $\theta(p)$ is an increasing function of $p$. Thus there is a value $p_c \in [0,1]$, which are called *critical probability*, such that if $p < p_c$, then $\theta(p) = 0$, and if $p > p_c$, then $\theta(p) > 0$.

The most amazing property of percolation is that the component structure of the open subgraph undergoes a dramatic change as $p$ increases past $p_c$. More precisely, for $p < p_c$, the probability of the event $E$ that there exists an infinite open cluster is 0; while for $p > p_c$, the probability is 1. To see this, note that the event $E$ is independent of the states of any finite set of bonds, then by Kolmogorov's 0-1 law, we can conclude that $\mathbb{P}_p(E)$ is either 0 or 1.

From the very early days of percolation theory in 1950s, one of the main challenges of this theory was the rigorous determination of the *critical probability* on 2-d lattice $p_c = p_c(\mathbb{L}^2)$, i.e., the underlying graph is the 2-d lattice $\mathbb{L}^2$, which was an open problem for many years. Hammersley suggested that this value might be 1/2 by some Monte Carlo simulation experiments; then further evidence for this was given by Domb [33], Elliot, Heap, Morgan and Rushbrooker [37], and Domb and Sykes [34].

The first major result on this topic was due to Harris [56], who proved that $p_c \geq 1/2$. Then the next important step towards the solution of this long-standing conjecture was done by Russo [100] and by Seymour and Welsh [104], the main technique they used there is called "Russo-Seymour-Welsh argument". It states that the crossing probability for a large enough rectangle goes to one for $p > p_c$. The final conclusion was given by Kesten [63]: he gave an ingenious proof that $p_c = 1/2$, basing on the Russo-Seymour-Welsh argument. To be more clearly, Harris [56] proved by a geometric construction that $\theta(1/2) = 0$, whence $p_c(\mathbb{L}^2) \geq 1/2$; Kesten [63] proved the complementary inequality. We combine these results into one theorem as follows:

**Theorem 1.18** ([56, 63]). *The critical probability of bond percolation $p_c(\mathbb{L}^2)$ on the square lattice equals $\frac{1}{2}$. Furthermore, $\theta(1/2) = 0$.*

The percolation theory mentioned above is *independent* percolation, i.e., every bond $e$ is kept with probability $p_e$, independent of all other bonds. It is very natural to also consider *dependent* percolation models, which will be used in Chapter 3. A bond percolation measure on a graph $G$ is *$k$-independent* if the states of sets $S$ and $T$ of bonds are independent, whenever $S$ and $T$ are at graph distance at least $k$. The case $k = 1$ is nontrivial, which means no bond in $S$ shares a site with a bond in $T$. We just mention one theorem related to the existence of some probability above which there exists an infinite cluster in the *dependent* percolation model on $\mathbb{L}^2$.

**Lemma 1.19** ([17]). *There exists a $p_0 < 1$ such that of $\bar{\mathbb{P}}$ is a 1-dependent bond percolation measure on $\mathbb{L}^2$ in which each bond is open with probability at least $p_0$, then* $\bar{\mathbb{P}}(|C_0| = \infty) > 0$.

#### 1.1.3.2 Continuum percolation

All the arguments above are related to discrete percolation models, i.e., the underlying graphs are discrete lattices $\mathbb{L}^d$ and general graph $G = (V, E)$. Actually, shortly after Broadbent and Hammersley [23] started the percolation theory and Erdös and Rényi's seminal papers on random graph theory [39, 41], Gilbert gave another random graph model with some layout and geometric structure [49], which was called random geometric graph (see Section 1.1.1.2). The *continuum percolation* model was defined originally in this same paper.

We introduce one of continuum percolation models: the *Boolean model.* Instead of giving formal mathematical details, we just try to say in words what this model is all about, as the definitions, for example the percolation probability, the critical probability can be fixed similarly as in the discrete setting.

Let begin the informal definition of *Boolean model.* Assume $X$ is a stationary point process (Intuitively, a point process $X$ defined on a subset $A$ of $\mathbb{R}^d$ is said to be stationary if the distribution of the number of points lying in $A$ depends on the size of $A$ but not its location) in $\mathbb{R}^d$. Each point of $X$ is the center of one closed ball in the Euclidean metric with a random radius in such a way that radii corresponding to different points are independent of each other and are identically distributed. The radii are also independent of $X$. In this way, $\mathbb{R}^d$ is partitioned into two regions: the *occupied* region, which is the region covered by at least one closed ball, and the *vacant* region, which is just the complement of the occupied region. The occupied region is denoted by $C$. Both the occupied and vacant regions consist of connected components, which are the main concern for almost all the results on continuum percolation theory. The connected components in the occupied region will be called *occupied components*; and connected components in the vacant region are called *vacant components.* For a subset $A \subset \mathbb{R}^d$, we denote by $W(A)$ the union of all occupied components which have non-empty intersection with $A$. The special case is when $A = \{0\}$, We write $W := W(\{0\})$. We then call $W$ the occupied component of the origin. Of course, similarly we can define the corresponding notation for the vacant region. This is the definition of Boolean model. Then the classic percolation theory is to answer some problems like: is there a positive probability that the occupied or vacant component of the origin is unbounded? A lot of question can be asked in some similar way as in the discrete setting. For more background on continuum percolation, we refer to the book [78].

### 1.1.4 Treewidth

Many problems that are NP-hard for arbitrary graphs can be linear or polynomial time solvable if the input is restricted to graphs with some constant upper bound on the treewidth, for example Hamiltonian circuit, vertex cover, independent set, etc. The celebrated Courcelle's Theorem [31] states that any problem that can be expressed in monadic second order logic, can be solved in linear time for the class of graphs with bounded treewidth. This motivates the study of this parameter and other tree-like parameters on graphs, see [65]. In the research community, these parameters of random graphs and random geometric graphs are investigated, e.g., the tree-depth of random graphs is studied in [94] and the treewidth of random geometric graphs is studied in [80].

The treewidth was introduced independently by Halin in [54] and by Robertson and Seymour in [99]. It is a parameter that measures the similarity between a graph and the class of trees in general and is commonly used as a parameter in the parameterized complexity analysis of graph algorithms.

There are several equivalent ways to define the treewidth; to be consistent with this thesis, we only mention two of them. The one defined from the size of the largest vertex set in a tree decomposition of the graph, which in fact is used in our research in Chapter 3; the other one is defined from a point view of describing a strategy for a pursuit-evasion game on the graph, which is related to the *Cop-Robber* game in Chapter 2. For more information on the theoretical aspects of treewidth, we recommend the book by Diestel [32].

The formal definition of treewidth via tree decomposition goes as follows:

For a graph $G = (V, E)$ on $n$ vertices, we call $(T, \mathcal{W})$ a *tree decomposition* of $G$, where $\mathcal{W}$ is a set of vertex subsets $W_1, ..., W_s \subset V$, called *bags*, and $T$ is a forest with vertices in $\mathcal{W}$, such that:

(i) $\cup_{i=1}^{s} W_i = V$;

(ii) For any $e = uv \in E$ there exists a set $W_i \in \mathcal{W}$ such that $u, v \in W_i$;

(iii) For any $v \in V$, the subgraph of $T$ induced by the set of $W_i$ with $v \in W_i$ is connected.

The *width* of a tree-decomposition is $w(T, \mathcal{W}) = \max_{1 \le i \le s} \mid W_i \mid -1$, and the *treewidth* of a graph $G$ can be defined as

$$\mathrm{tw}(G) = \min_{(T,\mathcal{W})} w(T, \mathcal{W}).$$

See Figure 1.5 for a depiction of tree decomposition.

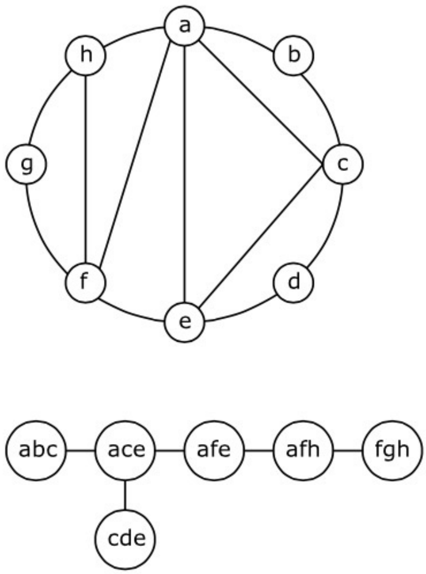


FIGURE 1.5: Tree decomposition.

As far as we know, the first result on treewidth of random graphs was given by Kloks in [65].

**Theorem 1.20** ([65]). *For any sequence $m(n)$ with $\frac{m(n)}{n} > 1.18$ for all $n$, there exists a constant $\delta_1 > 0$ such that*

$$\lim_{n \to \infty} \mathbb{P}[\mathrm{tw}(G(n, m)) > \delta_1 n] = 1.$$

Kloks also commented that it was not known whether his lower bound 1.18 can be further improved. It seems no further result appeared since then until 2006, Gao [48] established an improved lower bound on the threshold for a random graph to have a linear treewidth.

**Theorem 1.21** ([48]). *For any sequence $m(n)$ with $\frac{m(n)}{n} > 1.081$ for all $n$, there exists a constant $\delta_2 > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}[\mathrm{tw}(G(n,m)) > \delta_2 n] = 1.$$

The treewidth of random geometric graphs was first studied by Mitsche and Perarnau [80]. The random geometric graphs model $G(n,r)$ they studied is defined on a square $[0,\sqrt{n}]^2$: Let $V$ be a set of $n$ points $X_1,..,X_n$ i.i.d. uniformly at random from the square $\mathcal{S}_n = [0,\sqrt{n}]^2$ and join $X_i$ and $X_j$ by an edge if their Euclidean distance is at most $r$, in which $r = r(n)$ is a nonnegative real number. One of the most well-known phenomena in random geometric graphs is the "sudden emergence of a giant component". By this we mean that there exists critical value $r_c$ such that if $\limsup r < r_c$ then, a.a.s., every component of $G(n,r)$ has $O(\log n)$ vertices, whereas if $\liminf r > r_c$ then, a.a.s, there exists a "giant" component of with $\Omega(n)$ vertices. In particular, they proved the theorem as follows:

**Theorem 1.22** ([80]). *Let $r_c$ be the threshold radius for the appearance of the giant component in random geometric graph $G(n,r)$. A.a.s.,*

- *For $0 < r < r_c$, $\mathrm{tw}\left(G(n,r)\right) = \Theta\left(\frac{\log n}{\log\log n}\right)$;*
- *For $r > c$, $\mathrm{tw}\left(G(n,r)\right) = \Theta(r\sqrt{n})$, in which $c$ is a sufficiently large constant.*

In the last section of [80], Mitsche and Perarnau also conjectured the latter can be extended into the whole supercritical regime, that is, they conjectured that

$$\mathrm{tw}\,(G) = \Theta(r\sqrt{n}), \qquad for\ all\ r > r_c.$$

This conjecture is equivalent to the existence of a phase transition for the treewidth of random geometric graphs at critical value $r = r_c$.

We investigated this conjecture, and answered it in the affirmative in Chapter 3.

*Remark* 1.23. In fact, the treewidth $\mathrm{tw}(G)$ of a graph $G$ is also related to the cop number $c(G)$ of $G$ in some kind of Cops and Robbers game on graphs (under the name of *Helicopter Cops and Robbers*). To our best knowledge, the first Cops and Robbers results that connected with concept of treewidth was given by Seymour and Thomas [103]. In [103], they defined the rules for *Helicopter Cops and Robbers* on a graph as follows: the cops choose vertices, then the robber chooses a vertex; the positions of both players are known to each other. The moves are almost simultaneous: the cops announce which ones will move and are "transported by helicopter" to the new positions; that is, they are not on the graph for a period of time. And a cop can move to any vertex, not only its neighbors. During this time, the robber can move from her present position to any vertex that is reachable by a path that does not go through a vertex occupied by a cop still on the graph. The robber could also remain on the same vertex. The game is over if a cop occupies the same vertex as the robber. Seymour and Thomas [103] then gave a characterization of treewidth in terms of this game: a graph $G$ has treewidth at most $k$ if $k+1$ cops can capture the robber in *Helicopter Cops and Robbers.*

### 1.1.5 The probabilistic method

The probabilistic method, introduced by Erdös in [40] is a powerful tool for tracking many problems in discrete mathematics, especially in extremal combinatorics. Roughly speaking, the method works as follows: In order to prove that a structure with a certain desired property exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this space with positive probability. In other words, the probabilistic method is a non-constructive method for proving the existence of a prescribed kind of mathematical structure.

One of the probabilistic tools we will use for several times in this thesis is the second moment technique, i.e., using Chebyshev's Inequality.

**Theorem 1.24** (Chebyshev's Inequality)**.** *Let $X$ be a random variable, and $\mu = \mathbb{E}(X)$, $\sigma^2 = \mathbb{E}[(X-\mu)^2]$.For positive $\lambda$,*

$$\mathbb{P}[|X-\mu| \geq \lambda\sigma] \leq \frac{1}{\lambda^2}.$$

Then we generally apply the Chebyshev's Inequality in asymptotic terms.

**Lemma 1.25.** *If $Var[X] = o(\mathbb{E}[X]^2)$, then $X > 0$ a.a.s.*

**Lemma 1.26.** *If $Var[X] = o(\mathbb{E}[X]^2)$, then $X \sim \mathbb{E}[X]$ a.a.s.*

However, in many cases, the statement that a random variable is not too far from its mean is too weak. We want to show a much stronger statement: under some proper conditions, the tail probability should decrease exponentially. This powerful technique is the Chernoff bound.

**Theorem 1.27** (Chernoff bound, [58])**.** *If $X \in Bin(n,p)$ and $\lambda = np$, then, with $H(x) = (1+x)\log(1+x) - x$, $x \geq -1$, we have*

$$\mathbb{P}\left(X \geq \mathbb{E}[X] + t\right) \leq \exp\left(-\lambda H\left(\frac{t}{\lambda}\right)\right) \leq \exp\left(-\frac{t^2}{2(\lambda + t/3)}\right), \quad t \geq 0;$$

*and*

$$\mathbb{P}\left(X \leq \mathbb{E}[X] - t\right) \leq \exp\left(-\lambda H\left(\frac{-t}{\lambda}\right)\right) \leq \exp\left(-\frac{t^2}{2\lambda}\right), \quad t \leq 0.$$

For more information on the probabilistic method, we recommend the book by Alon and Spencer [5].

## 1.2 Main Results

In this section, we present the main theorems of this thesis.

### 1.2.1 Cops and Robbers games on percolated random geometric graphs

This chapter is based on a joint paper with T.Müller and P. Prałat [71].

In this chapter, we study the cop number of some random geometric graphs. To be more precise: there are two players, a set of $k$ *cops*, where $k \geq 1$ is a fixed integer, and the *robber*. The cops begin the game by occupying any set of $k$ vertices (in fact, for a connected $G$, their initial position does not matter). The robber then chooses a vertex, and the cops and robber move in alternate rounds. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current positions. The players know each others' current locations. The cops win and the game ends if at least one of the cops eventually occupies the same vertex as the robber; otherwise, that is, if the robber can avoid this indefinitely, she wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on $G$.

To be more precise, we study the vertex pursuit game of *Cops and Robbers*, in which cops try to capture a robber on the vertices of a *percolated* random geometric graph $\mathcal{G}(n,r,p)$, in which $n$ vertices are chosen uniformly at random and independently from $[0,1]^2$, and two vertices are adjacent with probability $p$ if the Euclidean distance between them is at most $r$. If the distance is bigger than $r$, then they are never adjacent. We present asymptotic results for the cop number of the game of Cops and Robber played on $\mathcal{G}(n,r,p)$ for a wide range of $p = p(n)$ and $r = r(n)$.

We say that an event holds *asymptotically almost surely* (*a.a.s.*) if its probability tends to one as $n$ goes to infinity.

In particular, we get the following theorem:

**Theorem 1.28.** *For every $\epsilon > 0$, and functions $p = p(n)$ and $r = r(n)$ so that $p^2r^2 \geq n^{-1+\epsilon}$ and $p \leq 1 - \epsilon$ we have that a.a.s. $c(\mathcal{G}(n, r, p)) = \Theta\left(\frac{\log n}{p}\right)$.*

We find this result quite surprising, since the asymptotics of the cop number for a large range of the parameters does not depend on $r$ but only on $p$.

### 1.2.2 Treewidth of random geometric graphs and percolated grids

This chapter is based on a joint paper with T.Müller [70].

The parameters of random structures are studied extensively in these years. It turns out that the treewidth of some random structures undergoes some phase transition at a critical point. One of the main contributions in this chapter is the proof of the conjecture given by Mitsche and Perarnau [80]:

**Theorem 1.29.** *If $r = r(n)$ is such that $\liminf r > r_c$, where $r_c$ is the critical value for the emergence of the giant component, then, a.a.s. as $n \to \infty$, $\mathrm{tw}(G(n, r)) = \Theta(r\sqrt{n})$.*

Our proof of Theorem 1.29 makes good use of a comparison to bond percolation on $\mathbb{Z}^2$. We will denote by $\Gamma(k, p)$ the restriction of this process to the $k \times k$ integer grid. I.e., $\Gamma(k, p)$ has vertex set $[k]^2$ and for every pair of points $u, v \in [k]^2$ with Euclidean distance equal to one, we add an edge with probability $p$, independently of the choices for all other pairs. (Here and in the rest of the paper we use the notation $[k] := \{1, \ldots, k\}$.) For the proof of Theorem 1.29 we only need to consider the treewidth of $\Gamma(k, p)$ when $p$ is very close to one, but with a little bit of extra work we are able to obtain the following result in addition to Theorem 1.29.

**Theorem 1.30.** *If $p \in (0, 1)$ is fixed then, a.a.s. as $k \to \infty$:*

$$\mathrm{tw}\left(\Gamma(k, p)\right) = \begin{cases} \Theta(k) & \text{if } p > 1/2\ ; \\ \Theta(\sqrt{\log k}) & \text{if } p < 1/2\ . \end{cases}$$

Combined with Theorem 1.22, we conclude that the treewidth of random geometric graphs also has a phase transition phenomenon. In other words, we have

**Corollary 1.31.** *A.a.s,*

$$\mathrm{tw}(G(n, r)) = \begin{cases} \Theta\left(\frac{\log n}{\log\log n}\right) & \text{if } 0 < \liminf r \leq \limsup r < r_c, \\ \Theta\left(r\sqrt{n}\right) & \text{if } \liminf r > r_c. \end{cases}$$

As mentioned in the last section of [80], their methods can not be extended to the whole supercritical regime because it requires the knowledge of the exact threshold value $r_c$ of the appearance of the giant component in $G(n, r)$, which is still not known yet. Compared to methods in [80] which mostly used the combinatorial arguments, our proofs are mainly based on the famous Russo-Seymour-Welsh argument (e.g., see [100, 104]), the *1-independent* percolation (see [17]), and the domination by product measures (see [72]). In other words, our methods rely more on probability techniques.

## 1.3 Notations

In this thesis, we use the following standard notation for the asymptotic behavior of the relative order of magnitude of two sequences of numbers $a_n$ and $b_n$, depending on the parameter $n \to \infty$. For simplicity we assume $b_n > 0$ for all sufficiently large $n$.

- $a_n = O(b_n)$ as $n \to \infty$ if there exist constants $C$ and $n_0$ such that $|a_n| \leq C|b_n|$ for $n \geq n_0$, i.e., if the sequence $a_n/b_n$ is bounded, except possibly for some small values of $n$ for which the ratio may be false.
- $a_n = \Omega(b_n)$ as $n \to \infty$ if there exist constants $c$ and $n_0$ such that $a_n \geq cb_n$ for $n \geq n_0$. If $a_n \geq 0$, this is equivalent to $b_n = O(a_n)$.
- $a_n = \Theta(b_n)$ as $n \to \infty$ if there exist constants $C, c > 0$ and $n_0$ such that $cb_n \leq a_n \leq Cb_n$ for $n \geq n_0$, i.e., $a_n = O(b_n)$ and $a_n = \Omega(b_n)$.

We say that an event holds *asymptotically almost surely* (*a.a.s.*) if its probability tends to one as $n$ goes to infinity.

# Chapter 2

# Chasing Robbers on random geometric graphs

This chapter is based on a joint paper with T.Müller and P. Prałat [71].

In this chapter, we study the vertex pursuit game of *Cops and Robbers*, in which cops try to capture a robber on the vertices of a graph. The minimum number of cops required to win on a given graph $G$ is called the cop number of $G$. We focus on $\mathcal{G}(n, r, p)$, a percolated random geometric graph in which $n$ vertices are chosen uniformly at random and independently from $[0, 1]^2$, and two vertices are adjacent with probability $p$ if the Euclidean distance between them is at most $r$. We present asymptotic results for the game of Cops and Robber played on $\mathcal{G}(n, r, p)$ for a wide range of $p = p(n)$ and $r = r(n)$.

## 2.1 Introduction and main results

The game of *Cops and Robbers*, introduced independently by Nowakowski and Winkler [87] and Quilliot [97] more than thirty years ago, is played on a fixed graph $G$. We will always assume that $G$ is undirected, simple, and finite. There are two players, a set of $k$ *cops*, where $k \geq 1$ is a fixed integer, and the *robber*. The cops begin the game by occupying any set of $k$ vertices (in fact, for a connected $G$, their initial position does not matter). The robber then chooses a vertex, and the cops and robber move in alternate rounds. The players use edges to move from vertex to vertex. More than one cop is allowed to occupy a vertex, and the players may remain on their current positions. The players know each others current locations. The cops win and the game ends if at least one of the cops eventually occupies the same vertex as the robber; otherwise, that is, if the robber can avoid this indefinitely, she wins. As placing a cop on each vertex guarantees that the cops win, we may define the *cop number*, written $c(G)$, which is the minimum number of cops needed to win on $G$. The cop number was introduced by Aigner and Fromme [1] who proved (among other things) that if $G$ is planar, then $c(G) \leq 3$. The most important open problem in this area is Meyniel's conjecture (communicated by Frankl [44]). It states that $c(n) = O(\sqrt{n})$, where $c(n)$ is the maximum of $c(G)$ over all $n$-vertex connected graphs. If true, the estimate is best possible as one can construct a graph based on the finite projective plane with the cop number of order at least $\Omega(\sqrt{n})$. Up until recently, the best known upper bound of $O(n \log\log n/\log n)$ was given in [44]. This was improved to $c(n) = O(n/\log n)$ in [27]. Today we know that the cop number is at most $n2^{-(1+o(1))\sqrt{\log_2 n}}$ (which is still $n^{1-o(1)}$) for any connected graph on $n$ vertices (a result obtained independently by Lu and Peng [73] and Scott and Sudakov [102], see also [47] for some extensions). If one looks for counterexamples for Meyniel's conjecture it is natural to study first the cop number of random graphs. Recent years have witnessed significant interest in the study of random graphs from that perspective [16] confirming that, in fact, Meyniel's conjecture holds asymptotically almost surely for binomial random graphs [96] as well as for random $d$-regular graphs [95]. For more results on vertex pursuit games such as *Cops and Robbers*, the reader is directed to the monograph [20].

In this chapter, we consider a *percolated random geometric graph* $\mathcal{G}(n, r, p)$ which is defined as a random graph with vertex set $V = \{X_1, X_2, \ldots, X_n\}$ in which the $X_i$-s are chosen uniformly at random and independently from $[0,1]^2$, and a pair of vertices within Euclidean distance at most $r$ appears as an edge with probability $p$, and they will never be adjacent if the distance between them are larger than $r$, independently for each such a pair. In particular, for $p = 1$ we get a *(classic) random geometric graph* $\mathcal{G}(n, r)$—see, for example, the monograph [92]. Random geometric graphs $\mathcal{T}_d(n, r, p)$ and $\mathcal{T}_d(n, r)$ on $d$-dimensional unit torus are defined similarly. Percolated random geometric graphs were recently studied by Penrose [93] under the name *soft random geometric graphs*. In [93] the connectivity of percolated random geometric graphs was considered, and in particular it was shown that the probability of being connected is governed by the probability of having no isolated vertices, much like in the case of the Erdős-Rényi model or the (unpercolated) classical random geometric graph model.

As typical in random graph theory, in this paper we shall focus on asymptotic properties of $\mathcal{G}(n, r, p)$ as $n \to \infty$, where $p = p(n)$ and $r = r(n)$ may and usually do depend on $n$. We say that an event in a probability space holds *asymptotically almost surely* (*a.a.s.*) if its probability tends to one as $n$ goes to infinity.

The following result for classic random geometric graphs was obtained independently in [12] and in [3].

**Theorem 2.1** ([3, 12])**.** *There exists an absolute constant $c > 0$ so that if $r^5 > c\frac{\log n}{n}$ then a.a.s. $c(\mathcal{G}(n,r)) = 1$.*

In [12], the known necessary and sufficient condition for a graph to be cop-win (see [87] for more details) is used; that is, it is shown that the random geometric graph is what is called dismantlable a.a.s. The proof in [3] is quite different, provides a tight $O(1/r^2)$ bound for the number of rounds required to catch the robber, and can be generalized to higher dimensions. In the proof an explicit strategy for the cop is introduced and it is shown that it is a winning one a.a.s. Essentially the same proof also gives a generalization of the result to higher dimensions. In [12] it was also shown that every (not necessarily random) connected geometric graph has cop number at most nine, that a.a.s. $c(\mathcal{G}(n,r)) \leq 2$ if $r^4 > c \log n/n$ for some absolute constant $c$, and that there are sequences $r$ for which $\mathcal{G}(n,r)$ is a.a.s. connected while its cop-number is strictly larger than one.

In this chapter, we consider the cop number of percolated random geometric graphs. In particular, we will prove the following result.

**Theorem 2.2.** *For every $\varepsilon > 0$, and functions $p = p(n)$ and $r = r(n)$ so that $p^2r^2 \geq n^{-1+\varepsilon}$ and $p \leq 1-\varepsilon$ we have that a.a.s. $c(\mathcal{G}(n,r,p)) = \Theta\left(\frac{\log n}{p}\right)$.*

We find this result quite surprising, since the asymptotics of the cop number for a large range of the parameters does not depend on $r$ but only on $p$.

We conjecture that, under the conditions of our theorem, a.a.s. the cop number is $(1+o(1))\log_{1/(1-p)} n$.

## 2.2 Proof of Theorem 2.2

For $0 < p \leq 1-\varepsilon$ for some $\varepsilon > 0$, it is convenient to define

$$\mathbb{L} = \mathbb{L}(n) := \log_{1/(1-p)} n,$$

and to state our intermediate results in terms of $\mathbb{L}$. Note that $\mathbb{L} = \Theta\left(\frac{\log n}{p}\right)$.

We will use the following version of *Chernoff bound.* (For more details, see Theorem 1.27.) Suppose that $X \in \mathrm{Bin}(n,p)$ is a binomial random variable with expectation

$\mu = np$. If $0 < \delta < 1$, then

$$\mathbb{P}[X < (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right),$$

and if $\delta > 0$,

$$\mathbb{P}[X > (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2+\delta}\right).$$

The lower and upper bounds are proved separately in the following two subsections.

### 2.2.1 Lower bound of Theorem 2.2

For the proof of the lower bound, we employ the following adjacency property that was used for dense binomial random graphs [22]. For a fixed $k > 0$ an integer, we say that a graph $G$ is $(1,k)$-*existentially closed* (or $(1,k)$-*e.c.*) if for each $k$-set $S$ of vertices of $G$ and vertex $u \notin S$, there is a vertex $z \notin S \cup \{u\}$ not joined to a vertex in $S$ and joined to $u$. If $G$ is $(1,k)$-*e.c.*, then $c(G) > k$. (The robber may use the property to construct a winning strategy against $k$ cops; she escapes to a vertex not joined to any vertex occupied by a cop.) Hence, to prove the lower bound in Theorem 2.2 it suffices to prove the following.

**Lemma 2.3.** *Writing $k := \lfloor \varepsilon \mathbb{L}/2 \rfloor$ -where $\epsilon > 0$ is as provided by conditions of Theorem 2.2- we have that, a.a.s., $\mathcal{G}(n,r,p)$ is $(1,k)$-e.c. In particular, a.a.s. $c(\mathcal{G}(n,r,p)) > k$.*

*Proof.* Let $s(u)$ be the number of vertices within Euclidean distance $r$ from $u$. It follows easily from Chernoff bound that there exists a function $t = t(n) = \Omega(r^2 n)$ such that a.a.s. for every vertex $u \in V(G)$, $s(u) \geq t$. Since our goal is to show a result that holds a.a.s. we may assume that this property holds deterministically. More precisely, we think of revealing the graph in two stages. In the first stage, we reveal only the locations of the points, in the second we reveal the relevant coin flips. In the remainder of the proof all mention of probability, expectation, etc., will be with respect to the situation where we have passed the first stage and it turned out that $s(u) \geq t$ for all $u \in V$. In other words, the only randomness we consider is in the coin flips deciding which pairs of points at Euclidean distance at most $r$ will become the edges of our graph.

Fix $S$, a $k$-subset of vertices and a vertex $u$ not in $S$. For a vertex $x \in V(G) \setminus (S \cup \{u\})$ that is within distance $r$ of $u$, the probability that $x$ is joined to $u$ and to no vertex of $S$ is at least $p(1-p)^k$ (note that this is a lower bound only, since $y \in S$ is adjacent to $x$ with probability $p$, provided that the distance between them is at most $r$; otherwise, there are not adjacent). Since edges are chosen independently, the probability that no such vertex can be found for this particular $S$ and $u$ is at most

$$(1-p(1-p)^k)^{t-k-1} = (1-p(1-p)^k)^{\Omega(r^2 n)}.$$

(Note that $r^2 n = \Omega(((\log n)/p)^2) = \Omega((k\log n)/p) \gg k$.)

Let $X$ be the random variable counting the number of $S$ and $u$ for which no such $x$ can be found. (Remember that this is after we have revealed the locations of the points.) We then have that

$$\begin{aligned}
\mathbb{E}(X) &\leq n\binom{n}{k}\left(1-p(1-p)^k\right)^{\Omega(r^2 n)} \\
&\leq n^{k+1}\exp[-\Omega(p(1-p)^k nr^2)] \\
&= \exp\left[(k+1)\log n - \Omega(n^{-\varepsilon/2}\cdot pnr^2)\right] \\
&\leq \exp\left[O(\log^2 n/p) - \Omega(n^{-\varepsilon/2}\cdot pnr^2)\right] \\
&= o(1),
\end{aligned}$$

where in the third line we have used the definition of $k$ and the last inequality follows from $p^2r^2 \geq n^{-1+\varepsilon}$ (which implies that $\log^2 n/p \ll n^{-\varepsilon/2}\cdot pnr^2$). This concludes the proof of the lemma. □

### 2.2.2 Upper bound of Theorem 2.2

In this section we show that, a.a.s., $21000\mathbb{L}$ cops suffice to catch the robber. Before presenting a winning strategy of the cops, we give some preparatory lemmas.

#### 2.2.2.1 Preliminaries

Let $V, E$ be the vertex set and edge set of $\mathcal{G}(n,r,p)$ respectively, and for $v \in V$, let $N(v) = \{w \in V : (w,v) \in E\}$. Recall that we say that a set of vertices $A \subseteq V$ *dominates* another set of vertices $B \subseteq V$ if every vertex of $B$ is adjacent to some vertex of $A$. Throughout this chapter we will denote by $B(x,s) = \{y \in \mathbb{R}^2 : ||y-x|| \leq s\}$ the ball of radius $s$ around $x$.

**Lemma 2.4.** *A.a.s., for every $v, w \in V$ with $\|v-w\| \leq 0.99\cdot r$, there is a subset $A \subseteq N(v)$ with $|A| \leq 1000\mathbb{L}$ that dominates $\{w\}\cup N(w)$.*

*Proof.* We will consider the number of "bad" (ordered) pairs $(v,w) \in V^2$ such that $\|v-w\| \leq 0.99\cdot r$, yet no set $A$ as required by the lemma exists. We will compute the probability that $(X_1, X_2)$ form such a bad pair. To do this, we reveal the graph in three stages. In the first stage we reveal $V$ (the positions of the points). In the second stage, we reveal all edges that have $X_1$ as an endpoint (i.e. all coin flips that determine these edges). In the third stage, we reveal all other edges (coin flips).

Let us condition of the event that $\|X_1 - X_2\| \leq 0.99 \cdot r$. (Note this does not affect the locations of the other points nor the status of any of the coin flips). We now define, for $i, j \in \{-1, +1\}$:

$$\begin{aligned} B_{i,j} &:= B(X_2 + i(r/10^{10})e_1 + j(r/10^{10})e_2, r/10^{10}), \\ U_{i,j} &:= N(X_1) \cap B_{i,j}. \end{aligned}$$

(Here, of course, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. See Figure 2.1 for a depiction.)

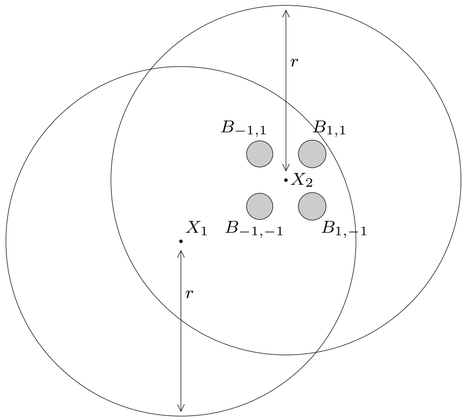


FIGURE 2.1: The definition of the $U_{i,j}$. (Not to scale.)

The $B_{i,j}$ have been chosen so that, no matter where in the unit square $X_2$ falls, for every $z \in B(X_2, r) \cap [0, 1]^2$ there is at least one pair $(i, j) \in \{-1, 1\}^2$ such that $B_{i,j} \subseteq B(z, r) \cap [0, 1]^2$.

Observe that, conditioning on the event that the position of $X_2$ is such that $B_{i,j} \subseteq [0, 1]^2$, we have

$$|U_{i,j}| \stackrel{d}{=} \text{Bin}(n - 2, p\pi(r/10^{10})^2).$$

In particular, $\mathbb{E}|U_{i,j}| = \Omega(pnr^2) = \Omega(n^{\varepsilon}/p) \gg \mathbb{L}$. Using Chernoff bound, it follows that

$$\mathbb{P}(|U_{i,j}| < \mathbb{E}|U_{i,j}|/2) \leq \exp[-\Omega(n^{\varepsilon})].$$

Note that, to find the $U_{i,j}$ we have to reveal the first two stages, but we do not need to reveal the coin flips corresponding to potential edges not involving $X_1$. Assuming that in the first two stages we managed to find $U_{i,j}$'s of size at least half of the expected size, we can now fix, for each $i,j \in \{-1,1\}$ with $B_{i,j} \subseteq [0,1]^2$, an arbitrary subset $A_{i,j} \subseteq U_{i,j}$ with $|A_{i,j}| = 250\mathbb{L}$. We let $A$ be the union of these $A_{i,j}$'s. Since each $z \in B(X_2, r) \cap [0,1]^2$ satisfies $A_{i,j} \subseteq B(z,r)$ for at least one pair $(i,j) \in \{-1,1\}^2$, the probability that there is a vertex $X_j \in N(X_2) \cup \{X_2\}$ not connected by an edge to any vertex of $A$ is at most $n(1-p)^{250\mathbb{L}} = n^{-249}$. It follows that

$$\mathbb{P}((X_1, X_2) \text{ is a bad pair}) \leq 4e^{-\Omega(n^{\varepsilon})} + n^{-249} \leq 2n^{-249},$$

the last inequality holding for sufficiently large $n$. This shows that the expected number of bad pairs is at most $\binom{n}{2} 2n^{-249} = o(1)$. The lemma follows by Markov's inequality. □

**Lemma 2.5.** *A.a.s., for every $v \in V$ and every $z \in B(v,r) \cap [0,1]^2$ there is a vertex $w \in N(v) \cap B(z, r/1000)$.*

*Proof.* We dissect $[0,1]^2$ into squares of side $s := 1/\lceil \frac{10^{10}}{r} \rceil$ (note $s \leq r/10^{10}$ and $s = \Theta(r)$). Observe that if $v, z \in [0,1]^2$ with $\|v - z\| \leq r$ then there is at least one square of our dissection contained in $B(v,r) \cap B(z, r/1000)$. It thus suffices to count the number $Z$ of "bad pairs" consisting of a vertex $v$ and a square $S$ of the dissection contained in $B(v,r)$ such that $N(v) \cap S = \emptyset$, and to show this number is zero a.a.s. Note that the number of squares is $O(1/r^2) = O(n)$. Hence we have

$$\begin{aligned} \mathbb{E}Z &= O(n^2) \cdot (1 - ps^2)^{n-1} = O(n^2) \cdot \exp[-\Omega(pnr^2)] \\ &= \exp[O(\log n) - \Omega(n^{\varepsilon})] = o(1), \end{aligned}$$

and the proof of the lemma is finished by Markov's inequality. □

**Lemma 2.6.** *A.a.s., for every $v, w \in V$ with $\|v - w\| \leq 1.99 \cdot r$ there is a vertex $u$ such that $uv, uw \in E$ and $\|u - (v+w)/2\| \leq r/1000$.*

*Proof.* We use the same dissection into small squares of side $s := 1/\lceil \frac{10^{10}}{r} \rceil$ as in the proof of the previous lemma. Note that if $v, w \in [0,1]^2$ then $B((v+w)/2, r/1000)$ contains at least one square of the dissection. It thus suffices to count the number of "bad triples" $Z$ consisting of two vertices $v \neq w$ at distance at most $1.99r$ and one square $S$ of the dissection that is contained in $B((v+w)/2, r/1000)$, such that $N(v) \cap N(w) \cap S = \emptyset$. We have

$$\begin{aligned} \mathbb{E}Z &\leq O(n^3) \cdot (1 - p^2 s^2)^{n-2} = O(n^3) \cdot \exp[-\Omega(p^2 nr^2)] \\ &= \exp[O(\log n) - \Omega(n^{\varepsilon})] = o(1), \end{aligned}$$

proving the lemma. □

The (easy) proof of the next, standard and elementary, observation is left to the reader.

**Lemma 2.7.** *Suppose that $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$ are such that $\|x_1 - x_2\|, \|y_1 - y_2\| \leq r$ and the line segments $[x_1, x_2], [y_1, y_2]$ cross. Then $\|x_i - y_j\| \leq r/\sqrt{2}$ for at least one pair $(i, j) \in \{1, 2\}^2$.*

We say that a cop $C$ *controls* a path $P$ in a graph $G$ if whenever the robber steps onto $P$, then she either steps onto $C$ or is caught by $C$ on his responding move. The readers should keep in mind that "control" is in relation to the current location of the robber, with the cop allowed to move around in response to the robber.

Let $\mathrm{diam}(G)$ denote the diameter of the graph. The terminology "shortest path" will always refer to the graph distance (as opposed to say the sum of the edge-lengths). Aigner and Fromme in [1] proved the following useful result.

**Lemma 2.8** (Lemma 4, [1])**.** *Let $G$ be any graph, $u, v \in V(G)$, $u \neq v$ and $P = \{u = v_0, v_1, \ldots v_s = v\}$ a shortest path between $u$ and $v$. A single cop $C$ can control $P$ after a finite number of moves.*

In fact, it takes cop $C$ at most $\mathrm{diam}(G)$ moves to reach $P$, and then at most $s$ moves to take control of $P$. We have the modified version of the above lemma as follows:

**Lemma 2.9.** *Let $G$ be any graph, $u, v \in V(G)$, $u \neq v$ and $P = \{u = v_0, v_1, \ldots v_s = v\}$ a shortest path between $u$ and $v$. A single cop $C$ can control $P$ after at most* $\mathrm{diam}(G) + s$ *moves.*

*Proof.* See proof of Lemma 4 in [1]. □

#### 2.2.2.2 The cop's strategy

In the sequel, since we aim for a statement that holds a.a.s., we assume that we are given a realization of $\mathcal{G}(n, p, r)$ that is connected (which is true a.a.s. for our choice of parameters as, for instance, follows from the work of Penrose [93]) and for which the conclusions of Lemmas 2.4, 2.5 and 2.6 hold. We will show that under these conditions, a team of $21000\mathbb{L}$ cops is able to catch the robber. This will clearly prove the upper bound of Theorem 2.2.

Our strategy is an adaptation of the strategy of Aigner and Fromme showing $c(G) \leq 3$ for connected planar graphs. We will have three teams $T_1, T_2, T_3$ of cops, each consisting of $7000\mathbb{L}$ cops that are each charged with guarding a particular shortest path.

In more detail, for $v_0 \neq v_m \in V$, a team $T_i$ that patrols a shortest path $P = v_0 v_1 \dots v_m$ is divided into 7 subteams $T_{i,-3}, T_{i,-2}, T_{i,-1}, T_{i,0}, T_{i,1}, T_{i,2}, T_{i,3}$ of $1000\mathbb{L}$ cops each. These subteams will move in unison (i.e. the cops in a particular subteam will always be on the same vertex of $P$). The team $T_{i,0}$ moves exactly according to the strategy given in the proof of Lemma 2.9. That is, after an initial period, the $T_{i,0}$-cops are able to move along $P$ in such a way that, whenever the robber steps onto a vertex $v_k \in P$ then either the entire team $T_{i,0}$ is already on $v_k$ or they are on $v_{k-1}$ or $v_{k+1}$. Team $T_{i,j}$ will be $j$ places along $T_{i,0}$ (i.e. if $T_{i,0}$ is on $v_k$ then $T_{i,j}$ is on $v_{k+j}$). If this is not possible because $T_{i,0}$ is too close to the respective endpoint of $P$ then $T_{i,j}$ just stays on that endpoint (i.e. if $T_{i,0}$ is on $v_k$ and $k + j > m$ then $T_{i,j}$ is on $v_m$ and if $k + j < 0$ then $T_{i,j}$ stays on $v_0$). We now claim that the robber can not cross (in the sense that the edge she uses crosses an edge of $P$ when both are viewed as line segments) the path $P$ without getting caught by the cops of team $T_i$.

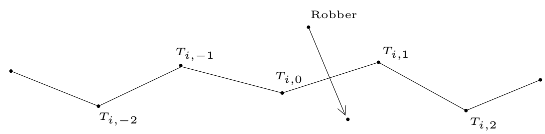


FIGURE 2.2: The robber tries to cross a path guarded by team $T_i$.

To see this, we first observe that if the robber moves along an edge that crosses some edge of $P$, then either her position before the move or her position right after the move is within distance at most $r/\sqrt{2}$ of some vertex of $P$ by Lemma 2.7. Next, we remark that whenever the robber steps onto a vertex $u$ within distance $0.99 \cdot r$ of some vertex $v_k \in P$, then the cops can catch her in at most two further moves. This is because from $u$, the robber could move to $v_k$ in at most two moves (Lemma 2.6). As the cops of subteam $T_{i,0}$ follow the strategy prescribed in the proof of Lemma 2.9, they are guaranteed to be on one of $v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}$ when the robber arrives on $u$. But then there must be some team $T_{i,j}$ that inhabits the vertex $v_k$ at the very moment when the robber arrived on $u$. This team now acts as follows: at the time the robbers arrives on $u$, the subteam occupies the set $A$ provided by Lemma 2.4 (this one time the subteam do not all stay on the same vertex; instead they "spread" following the strategy implied by the lemma) and in the next move the cops are able to catch the robber, since they now dominate the closed neighbourhood of the vertex she inhabits. Thus, each of our three teams can indeed prevent the robber from crossing a chosen path (after an initiation phase). What is more, the robber can never get to within distance $0.99r$ of any vertex of such a path.

We can now mimic the strategy that Aigner and Fromme [1] developed for catching the robber on connected planar graphs using three cops. The idea is to confine the robber

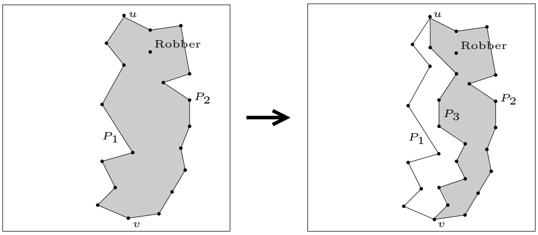


FIGURE 2.3: The adapted Aigner-Fromme strategy.

in smaller and smaller subgraphs of our graph, until finally the cops apprehend her. We start by taking two vertices $u, v$. We let $P_1$ be the shortest $uv$-path, and we let $P_2$ be the shortest $uv$-path in the graph with all internal vertices of $P_1$, and all edges that cross $P_1$ removed. (Using Lemmas 2.5 and 2.6 it is easily seen that at least one such path exists: every vertex has enough neighbors to construct such paths.) Note that $P_1 \cup P_2$ constitutes a Jordan curve and hence $\mathbb{R}^2 \setminus (P_1 \cup P_2)$ consists of two connected regions, the interior and the exterior. Once the game starts, we send $T_1$ to patrol $P_1$ and $T_2$ to patrol $P_2$. After an initial phase, the robber will either be trapped in the interior region or the exterior region of $\mathbb{R}^2 \setminus (P_1 \cup P_2)$. Let us denote the region she is trapped on by $R$. If it happens that every vertex inside $R$ is within distance $0.99r$ of some vertex of $P_1 \cup P_2$ then we are done by the previous argument. Let us thus assume this is not the case. We then proceed as follows: we remove all vertices which are neither on $P_1 \cup P_2$ nor inside $R$, and we remove all edges that cross $P_1$ or $P_2$. (Conceivably there can be vertices that lie inside $R$ but with an edge between them that passes through $P_1 \cup P_2$.) We let $P_3$ be a $uv$-path in the remaining graph that is shortest among all $uv$-paths that are distinct from $P_1, P_2$. (To see that at least one such path exists, we first find a vertex $u_1 \in R$ that has distance at least $0.99r$ to every vertex of $P_1 \cup P_2$. Then we use Lemma 2.5 and 2.6 to construct vertex-disjoint paths between $u_1$ and two distinct vertices of $P_1 \cup P_2$.) See Figure 2.3 for a depiction.

Note that $P_3$ does not cross $P_1$ or $P_2$ (but it may share some edges with them). In particular, $R \setminus P_3$ consists of two or more connected parts, each of which is either bounded by (parts of) $P_1$ and $P_3$ or by (parts of) $P_2$ and $P_3$. We now send $T_3$ to patrol $P_3$. After an initial phase, the robber will be caught in one of the connected parts $R'$ of $R \setminus P_3$. Without loss of generality $R'$ is bounded by $P_2, P_3$. Discarding unneeded parts of $P_2, P_3$ (namely those that do not bound $R'$) and relabelling we can also assume that $P_2, P_3$ only meet in their endpoints $u, v$. If every vertex inside $R'$ is within distance $0.99r$ of a vertex of $P_2, P_3$ we are again done. Otherwise, the team $T_1$ abandons guarding path $P_1$, we remove all vertices not on $P_2, P_3$ or inside $R'$ and all edges that cross $P_2$ or $P_3$, we find a $uv$-path $P_4$ in the remaining graph, shortest among all $uv$-paths different from $P_2, P_3$, and we let $T_1$ patrol $P_4$. Now $P_4$ will dissect $R'$ into two or more connected paths, and we repeat the procedure to either catch the robber or restrict her to an even smaller region.

It is clear that in each iteration of this process, at least one edge is removed from the subgraph under consideration. Hence the process must stop eventually. In other words, the robber will get caught eventually. This concludes the proof of (the upper bound of) Theorem 2.2.

### 2.2.3 Some remarks about the proofs

As mentioned earlier, we conjecture that the $\Theta(\log n/p)$ in Theorem 2.2 can in fact be improved to $(1+o(1))\log_{1/(1-p)} n$.

We suspect that the $p^2$ term in the conditions for Theorem 2.2 is just an artefact of the proof and that the result should in fact hold when $pr^2 \geq n^{-1+\varepsilon}, p \leq 1-\varepsilon$.

Let us also remark that bounding $p$ away from one is essential for our result, as can be seen for instance from Theorem 2.1 or the result in [12] that connected geometric graphs have bounded cop number. An interesting avenue of further investigation would thus be to see what goes on when $p \to 1$ as $n \to \infty$. Clearly some sort of phase change must occur, depending on the speed at which $p$ approaches one.

We remark that the proof of the lower bound in Theorem 2.2 readily generalizes to arbitrary dimensions (replacing $r^2$ by $r^d$ everywhere), but that the reasoning using in the upper bound proof is essentially two-dimensional. We would be very interested to learn of a proof technique that does work for all dimensions.

# Chapter 3

# Treewidth of random geometric graphs and percolated grids

This chapter is based on a joint work with T.Müller [70].

In this chapter, we consider the treewidth of two related random graph models. The first one is the random geometric graph, where we drop $n$ points onto the square $[0, \sqrt{n}]^2$ and connect pairs of points by an edge if their distance is at most $r = r(n)$. The second one is the percolated grid, in which we take a $k \times k$-grid and keep each edge with probability $p$, independently of all other edges.

We show that, with probability tending to one as $k \to \infty$, the treewidth of the percolated grid is $\Theta(k)$ if $p > 1/2$ and $\Theta(\sqrt{\log k})$ if $p < 1/2$. By reusing part of the proof of this last result, we also prove a conjecture of Mitsche and Perarnau [80] stating that, with probability going to one as $n \to \infty$, the treewidth the random geometric graph is $\Theta(r\sqrt{n})$ when $\liminf r > r_c$, where $r_c$ is the threshold radius for the appearance of the giant component.

## 3.1 Introduction and main results

The *random geometric graph* $G(n, r)$ is the random graph obtained by taking $n$ points $X_1, \dots, X_n$ i.i.d. uniformly at random from the square $[0, \sqrt{n}]^2$, and joining $X_i$ and $X_j$ by an edge if their Euclidean distance is at most $r$. Here $r = r(n)$ may and often does depend on $n$. To avoid having to deal with annoying trivial special cases we assume $r \leq \sqrt{2n}$ throughout the paper. The study of random geometric graphs essentially goes back to Gilbert [49] who defined a very similar model in 1961. For this reason random geometric graphs are often also called the *Gilbert model* of random graphs.

Random geometric graphs have been the subject of a considerable research effort in the last two decades. As a result, detailed information is now known on various aspects such as ($k$-)connectivity [90, 91], the largest component [92], the chromatic number and clique number [77, 81], the (non-)existence of Hamilton cycles [6, 82], monotone graph properties in general [50] and the simple random walk on the graph [29]. One of the most well-known phenomena in random geometric graphs is the "sudden emergence of a giant component". By this we mean that there exists critical value $r_c$ such that if $\limsup r < r_c$ then, a.a.s., every component of $G(n, r)$ has $O(\log n)$ vertices, whereas if $\liminf r > r_c$ then, a.a.s, there exists a "giant" component of with $\Omega(n)$ vertices. Here and in the rest of the paper, we say that a sequence of events $(E_n)_n$ holds *asymptotically almost surely* (abbreviation: a.a.s.) if $\lim_{n\to\infty} \mathbb{P}(E_n) = 1$. The exact value of $r_c$ is not known at this time, but simulations suggest that the exact value is approximately 1.2 (see [92]). For more details, and proofs, on the giant component phenomenon and background on random geometric graphs in general we refer the reader to the monograph [92].

In the present paper, we consider the *treewidth* of random geometric graphs. The treewidth of a graph was introduced by Halin in [54] and independently but later by Robertson and Seymour in [99]. It is a graph parameter that in a sense measures how similar a given graph is to a tree. (We postpone the precise – and technical – definition of treewidth until the next section of the paper, in order to streamline the introduction.) Treewidth plays an important role in modern algorithmic graph theory. Many NP-hard algorithmic decision problems have for instance been shown to be polynomially solvable when restricted to the class of instances with a bounded treewidth. In fact, a striking result of Courcelle [31] states that any algorithmic decision problem that can be expressed in monadic second order logic, can be solved in linear time for the class of graphs with bounded treewidth. An example of a decision problem that is NP-hard in general and can be expressed in monadic second order is $k$-colourability (for any fixed $k$). As random geometric graphs have been used extensively as models for modeling communication networks, this motivated Mitsche and Perarnau [80] to consider the treewidth of random geometric graphs. They proved that if $r \in (0, r_c)$ is fixed then, a.a.s., $\text{tw}(G(n, r)) = \Theta(\log n/ \log\log n)$, while if $r > C$ where $C$ is a large constant then, a.a.s., $\text{tw}(G(n, r)) = \Theta(r\sqrt{n})$. Mitsche and Perarnau [80] also conjectured that the second result should extend all the way to the critical value. Here we will prove their conjecture:

**Theorem 3.1.** *If $r = r(n)$ is such that $\liminf r > r_c$, where $r_c$ is the critical value for the emergence of the giant component, then, a.a.s. as $n \to \infty$, $\text{tw}(G(n, r)) = \Theta(r\sqrt{n})$.*

Our proof of Theorem 3.1 makes use of a comparison to bond percolation on $\mathbb{Z}^2$. Recall that this refers to the infinite random graph obtained by retaining each edge of the familiar integer lattice with probability $p$ and discarding it with probability $1-p$, independently of the choices for all other edges. We will denote by $\Gamma(k,p)$ the restriction of this process to the $k \times k$ integer grid. I.e., $\Gamma(k,p)$ has vertex set $[k]^2$ and for every pair of points $u, v \in [k]^2$ with Euclidean distance equal to one, we add an edge with probability $p$, independently of the choices for all other pairs. (Here and in the rest of the paper we use the notation $[k] := \{1, \ldots, k\}$.) For the proof of Theorem 3.1 we only need to consider the treewidth of $\Gamma(k,p)$ when $p$ is very close to one, but with a little bit of extra work we are able to obtain the following result in addition to Theorem 3.1.

**Theorem 3.2.** *If $p \in (0,1)$ is fixed then, a.a.s. as $k \to \infty$:*

$$\operatorname{tw}(\Gamma(k,p)) = \begin{cases} \Theta(k) & \text{if } p > 1/2 \ ; \\ \Theta(\sqrt{\log k}) & \text{if } p < 1/2 \ . \end{cases}$$

Note that $k$ is the *square root* of the number of vertices of $\Gamma(k,p)$.

*Remark* 3.3. The treewidth for Erdös-Renyi random graph model $\mathcal{G}(n,p)$ and random geometric graph $\mathcal{RGG}(n,r)$ are different: it is known that in the Erdös-Renyi random graph model, as soon as the giant component appears, the graph has linear treewidth (see [68]). In contrast to this, our result shows that the treewidth of random geometric graph is same order as $\sqrt{n}$. This different behavior of the two models can be explained by their different expansion properties and the connection between balanced separators and treewidth. Classical random graphs have very good expansion properties, and thus it is difficult to find small separators of large sets of vertices. However, the geometric properties of the random geometric graphs imply the lack of large expanders. In other words, we can construct a tree decomposition with small bags for random geometric graphs but not for classic random graphs.

*Remark* 3.4. As already mentioned in [80], other parameters that are sandwiched between treewidth and treedepth clearly then also have the same asymptotic behavior for the random geometric graphs. For example, the path-width is upper bounded by treedepth and lower bounded by treewidth, as the treewidth and treedepth have the same asymptotic order for the random geometric graphs, we can conclude that the path width of random geometric graph has the same asymptotic order as the treewidth and treedepth.

*Remark* 3.5. It turns out that the Cops and Robber games on graphs is also related to the treewidth of graphs. An elegant upper bound of the cops number $c(G)$ was given in [59]:

$$c(G) \leq \lfloor \operatorname{tw}(G)/2 \rfloor + 1.$$

This bound is tight if the graph has small treewidth (up to treewidth 5). Moreover, Joret, Kaminski and Theis studied the cop number in the classes of graphs defined by forbidding one or more graphs as either subgraphs or induced subgraphs in [59]. For recently similar results on cops number in terms of tree decompositions, see [18].

*Remark* 3.6. In [2], Alon and Mehrabian consider a variant of the Cops and Robber game, in which the robber has unbounded speed, that is, can take any path from her vertex in her turn, but is not allowed to pass through a vertex occupied by a cop. They show the Cop number $c(G)$ of planar graph $G$ with order $n$ in this version satisfies

$$c(G) = \Theta(\mathrm{tw}(G)) = O(\sqrt{n}).$$

## 3.2 Notation and preliminaries

In this section, we give some definitions and results which we will need in the sequel. To make this chapter more readable, we repeat the definition of treewidth and some simple property.

We start with the precise definition of treewidth. For a graph $G = (V, E)$ on $n$ vertices, we call $(T, \mathcal{W})$ a *tree decomposition* of $G$, where $\mathcal{W}$ is a set of vertex subsets $W_1, ..., W_s \subset V$, called *bags*, and $T$ is a forest with vertices in $\mathcal{W}$, such that:

(i) $\cup_{i=1}^{s} W_i = V$;

(ii) For any $e = uv \in E$ there exists a set $W_i \in \mathcal{W}$ such that $u, v \in W_i$;

(iii) For any $v \in V$, the subgraph of $T$ induced by the set of $W_i$ with $v \in W_i$ is connected.

The *width* of a tree-decomposition is $w(T, \mathcal{W}) = \max_{1 \le i \le s} | W_i | -1$, and the *treewidth* of a graph $G$ can be defined as

$$\mathrm{tw}(G) := \min_{(T,\mathcal{W})} w(T, \mathcal{W}),$$

where the minimum is of course taken over all tree decompositions $(T, \mathcal{W})$ of $G$. From the definition of treewidth, one can see that the treewidth of a tree is one (one can put every edge in a bag), while the treewidth of a $k$-clique is $k - 1$ (all the vertices should be in the same bag). Let us also observe that if $H$ is a subgraph of $G$, then $\mathrm{tw}(H) \le \mathrm{tw}(G)$ (if $(W_i)_{1 \le i \le s}$ is a tree decomposition of $G$, then $(W_i \cap V(H))_{1 \le i \le s}$ is a tree decomposition of $H$), and if $G$ is a graph with connected components $G_1, ..., G_m$, then $\mathrm{tw}(G) = \max_{1 \le i \le m} \mathrm{tw}(G_i)$.

Given an edge $xy$ of graph $G$, the graph $G/xy$ is obtained from $G$ by *contracting* the edge $xy$. That is, to obtain $G/xy$, we identify the vertices $x$ and $y$ and remove all resulting loops and duplicate edges. A graph $H$ is a *minor* of $G$ if it is a subgraph of graph obtained from $G$ by a sequence of edge-contractions. Again, one can see from the definitions that if $H$ is a *minor* of $G$, $\mathrm{tw}(H) \leq \mathrm{tw}(G)$ (in fact, contractions of edges lead to small bags).

Alon, Seymour and Thomas [4] proved the following powerful result, bounding the treewidth of graphs without a given minor.

**Theorem 3.7** ([4])**.** *If $G$ does not have $H$ as a minor, then* $\mathrm{tw}(G) \leq |V(H)|^{\frac{3}{2}} \cdot \sqrt{|V(G)|}$.

In this paper we will make use of the following immediate corollary:

**Corollary 3.8.** *There exists a constant $C > 0$ such that every planar graph $G$ satisfies* $\mathrm{tw}(G) \leq C\sqrt{|V(G)|}$.

Throughout the paper we will denote by $\Gamma(k)$ $(:= \Gamma(k, 1))$ the $k \times k$ grid. The next observation appears as Exercise 16 in Chapter 12 of [32].

**Lemma 3.9.** $\mathrm{tw}(\Gamma(k)) = k$.

For one of our lower bounds on the treewidth, we will need the following lemma which links the treewidth of a graph and the existence of a partition of its vertex set with special properties. A vertex partition $V = \{A, S, B\}$ is a *balanced $k$-partition* if $|S| = k + 1$, there are no edges in $G$ between a vertex in $A$ and a vertex in $B$, and $\frac{1}{3}(n - k - 1) \leq |A|, |B| \leq \frac{2}{3}(n - k - 1)$. In this case, $S$ is called a *balanced separator.* The following result connecting balanced partitions and treewidth is due to Kloks [65], which provides a necessary condition for a graph to have a treewidth of certain size.

**Lemma 3.10** ([65])**.** *Let $G$ be a graph and suppose that* $\mathrm{tw}(G) \leq k \leq |V(G)| - 1$. *Then $G$ has a balanced $k$-partition.*

We say that $A \subseteq \{0, 1\}^n$ is an up-set if whenever we take a point of $A$ and we change one of its coordinates into a one, then the resulting point is still in $A$. We will use the following lemma later on.

**Lemma 3.11** (Harris' lemma, [55])**.** *Let $A, B \subseteq \{0, 1\}^n$ be up-sets and let $X = (X_1, \dots, X_n)$ be a vector of independent Bernoulli random variables. Then $\mathbb{P}(X \in A \cap B) \geq \mathbb{P}(X \in A)\mathbb{P}(X \in B)$.*

By a slight abuse of notation, throughout this paper we will denote the graph with vertex set $\mathbb{Z}^2$ and an edge $vw \in E(\mathbb{Z}^2)$ if and only if $\|v - w\| = 1$ also by $\mathbb{Z}^2$. Recall that bond percolation on $\mathbb{Z}^2$ refers to the random process where we keep each edge of $\mathbb{Z}^2$ with probability $p$ and discard it with probability $1-p$, independently of all other edges. The edges that are kept are referred to as *open* and the discarded edges as *closed*. If $R := \{a, \dots, b\} \times \{c, \dots, d\}$ is an axis-parallel rectangle, then we say that $R$ has a *horizontal crossing* if there is an open path that stays inside $R$ and connects the left side $\{a\} \times \{c, \dots, d\}$ to the right side $\{b\} \times \{c, \dots, d\}$. A vertical crossing is defined similarly. We denote by $H(R)$ the event that there is a horizontal crossing of $R$, and $V(R)$ the event that there is a vertical crossing of $R$. In the sequel, we will use the following well-known result on bond percolation on $\mathbb{Z}^2$ with $p > \frac{1}{2}$. A proof can for instance be found in [17] (Lemma 8 on page 64).

**Lemma 3.12.** *If $p > 1/2$ then $\lim_{k\to\infty} \mathbb{P}(H([3k] \times [k])) = 1$.*

In words, when $p > 1/2$ then the probability of crossing a $3k \times k$ rectangle in the long direction can be made arbitrarily close to one by making $k$ large.

Formally speaking, we can describe bond percolation on $\mathbb{Z}^2$ as a random vector $X$ taking values in $\{0,1\}^{E(\mathbb{Z}^2)}$. Here $X_e = 1$ if $e$ is open, and $X_e = 0$ otherwise. In the standard setup, the coordinates $X_e$ are i.i.d. Bernoulli random variables. Of course one can also consider more general bond percolation models in which the coordinates are not independent. We say that such a bond percolation model $Y$ is 1-*independent* if, for every pair of sets $S, T \subseteq E(\mathbb{Z}^2)$ with the property that no edge in $S$ shares an endpoint with any edge in $T$, the random vectors $(Y_e)_{e\in S}$ and $(Y_e)_{e\in T}$ are independent. Recall that a coupling of two random objects $X, Y$ is a joint probability space on which both are defined (and have the correct marginal distributions). The following result is a reformulation of a special case of a result by Liggett, Schonmann and Stacey [72]:

**Theorem 3.13** ([72])**.** *There exists a function $\pi : [0,1] \to [0,1]$ such that, $\lim_{p\uparrow 1} \pi(p) = 1$, and the following holds. Suppose that $Y$ follows a 1-independent bond percolation model on $\mathbb{Z}^2$ and set $p := \inf_{e\in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$. Then there exists a coupling of $Y$ with a standard (i.e. independent) bond percolation $X$ with $\mathbb{P}(X_e = 1) = \pi(p)$, such that, almost surely, $X_e \le Y_e$ for all $e \in E(\mathbb{Z}^2)$.*

In words, this last theorem says that every 1-independent bond percolation model contains the edges of an independent bond percolation model, and the edge probability $\pi(p)$ of this independent bond percolation approaches one as $p := \inf_{e\in E(\mathbb{Z}^2)} \mathbb{P}(Y_e = 1)$ approaches one.

When working with random geometric graphs, it is often useful to consider the *Poissonized* random geometric graph. We define $G_{\mathrm{Po}}(n,r)$ analogously to $G(n,r)$ except that now we we take the points of a Poisson process of intensity one on $[0, \sqrt{n}]^2$ and then build our graph on that as before. Equivalently, we can say that we throw $N_n \overset{d}{=} \mathrm{Po}(n)$ i.i.d. uniform points onto $[0, \sqrt{n}]^2$ and then build the graph on those as before. Working with the Poissonized version is often useful in proofs because of the convenient independence properties of the Poisson process. Recall that if $N_n \overset{d}{=} \mathrm{Po}(n)$ then $\mathbb{P}(N_n > (1+\varepsilon)n) = o(1)$, as can for instance be seen by Chebyshev's inequality. Using a straightforward coupling and rescaling, this gives the following observation:

**Corollary 3.14.** *There is a coupling such that for every $r = r(n)$, a.a.s.,$G_{Po}\left((1-\varepsilon)n, r\sqrt{1-\varepsilon}\right)$ is a subgraph of $G(n,r)$.*

It of course also makes sense to simply consider the random geometric graph built on a Poisson process $\mathcal{P}$ of intensity one on all of the plane $\mathbb{R}^2$. This is the well-known *continuum percolation* model defined originally by Gilbert [49]. We remark that Gilbert and several other sources in the literature fix $r = 1$ and allow the intensity of the Poisson process to vary, but it is easily seen that this is equivalent to the setting where we vary $r$ and the intensity of the Poisson process is fixed to be one. Note that $G_{\mathrm{Po}}(n,r)$ is just the restriction of continuum percolation to the square $[0, \sqrt{n}]^2$. Let us also remark that the critical $r_c$ for the "emergence of a giant component" in $G(n,r)$ is the same as the critical value for the existence of an infinite component in continuum percolation (see [92], Chapters 9 and 10). Similarly to the case of bond percolation on $\mathbb{Z}^2$, we can define crossing events for continuum percolation. Our definition follows Meester and Roy [78]. For $R \subseteq \mathbb{R}^2$ an axis-parallel rectangle, we say that $H(R)$ holds (i.e. there is a horizontal crossing of $R$) if there is a path between a point that is within $r/2$ of the left side of $R$, and a point within $r/2$ of the right side of $R$, and all other points of the path are either inside $R$ or within distance $r/2$ of $R$. Note that while some of the vertices of the path may lie outside of $R$, it is possible to draw a continuous curve between the right and left side, that is completely covered by the balls of radius $r/2$ around the vertices of the path. We have the following analogue of Lemma 3.12.

**Lemma 3.15** ([78], Corollary 4.1)**.** *If $r > r_c$ then $\lim_{a\to\infty} \mathbb{P}(H([0,3a] \times [0,a]) = 1$.*

We say that an event $A$ defined with respect to the Poisson process $\mathcal{P}$ is *increasing* if whenever $A$ is true for some set of points $X = \{x_1, x_2, \dots\} \subseteq \mathbb{R}^2$ (i.e. some realization of $\mathcal{P}$), then $A$ also holds for any set $X'$ that contains $X$. We have the following analogue of Lemma 3.11 above.

**Lemma 3.16** ([78], Theorem 2.2)**.** *If $A, B$ are increasing events (wrt. $\mathcal{P}$) then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$.*

## 3.3 The treewidth of the percolated grid $\Gamma(k, p)$

### 3.3.1 When $p$ is large

Instead of proving the $p > 1/2$ part of Theorem 3.2 directly, we first prove the following weaker version:

**Proposition 3.17.** *There exist constants $c > 0$ and $p < 1$ such that* $\operatorname{tw}(\Gamma(k,p)) \geq ck$ *a.a.s.*

If $A \subseteq \mathbb{Z}^2$ is finite and connected (as a subgraph of $\mathbb{Z}^2$) then there is a well defined "surrounding cycle" $\operatorname{surr}(A)$ in the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ (that separates $A$ from infinity, and every other cycle in $(\mathbb{Z}^2)^*$ that separates $A$ from infinity contains $\operatorname{surr}(A)$ in its interior). For $A \subseteq [k]^2$ connected, we define $\operatorname{outer}(A)$ to be the set of edges of $\Gamma(k)$ that cross $\operatorname{surr}(A)$. (See Figure 3.1 for a depiction.)

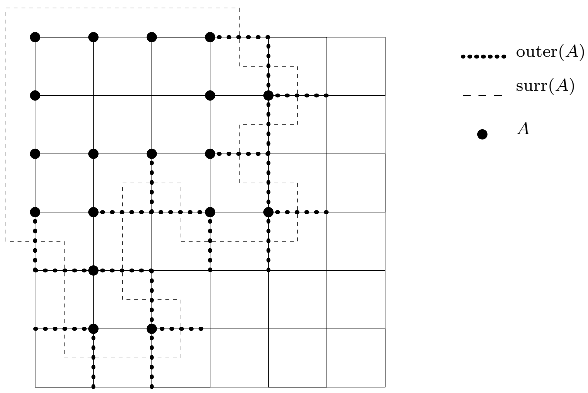


FIGURE 3.1: Depiction of $\operatorname{surr}(A)$ and $\operatorname{outer}(A)$ for a set $A \subseteq [7]^2$.

We will make use of the following straightforward observation. We include a proof for completeness.

**Lemma 3.18.** *Suppose that $A \subseteq [k]^2$ is connected (as a subgraph of $\Gamma(k)$) and does not contain a horizontal crossing of $[k]^2$. Then* $|\operatorname{outer}(A)| \geq \max(\sqrt{|A|}, |\operatorname{surr}(A)|/4)$.

*Proof.* First suppose that $A$ contains a vertical crossing. Since $A$ does not contain a horizontal crossing, $\operatorname{surr}(A)$ must contain a (dual) path that separates the left edge of $[k]^2$ from its right edge. This implies that $\operatorname{outer}(A)$ contains at least $k$ edges, i.e., $|\operatorname{outer}(A)| \geq k$. Hence $|\operatorname{outer}(A)| \geq k \geq \sqrt{|A|}$. Note that, since $A$ can intersect at most one of the vertical sides of $[k]^2$, we have that the number of edges of $\operatorname{surr}(A)$ that do not intersect edges of $\Gamma(k)$ is at most $|\operatorname{surr}(A)| - |\operatorname{outer}(A)| \leq 3k$, i.e., $|\operatorname{surr}(A)| \leq |\operatorname{outer}(A)| + 3k \leq 4|\operatorname{outer}(A)|$. This shows $|\operatorname{outer}(A)| \geq |\operatorname{surr}(A)|/4$.

Let us then assume that $A$ contains neither a horizontal nor a vertical crossing. Let $a := |\pi_x(A)|, b := |\pi_y(A)|$, where $\pi_x$, resp. $\pi_y$, denote the projection onto the $x$-axis, resp. the $y$-axis. Clearly we have that $|A| \leq a \cdot b$ and $|\operatorname{outer}(A)| \geq a + b$. Thus,

$$\sqrt{|A|} \leq \max(a, b) \leq a + b \leq |\operatorname{outer}(A)|.$$

Also note that $|\operatorname{surr}(A)| - |\operatorname{outer}(A)| \leq a+b$. So certainly $|\operatorname{outer}(A)| \geq |\operatorname{surr}(A)|/4$. $\square$

We say a set $A \subseteq [k]^2$ is *dirty* (wrt. $\Gamma(k, p)$) if

(i) $A$ is connected (as a subgraph of $\Gamma(k)$),

(ii) $A$ intersects at most three sides of $[k]^2$,

(iii) at most half of the edges of $\operatorname{outer}(A)$ are open in $\Gamma(k, p)$.

We say that a vertex $v \in [k]^2$ is *dirty* if it is contained in some dirty set.

**Lemma 3.19.** *There exists a $p_0 < 1$ such that whenever $p \geq p_0$ then, a.a.s., there are at most $k^2/10^{10}$ dirty vertices.*

*Proof.* Let $Y$ denote the number of vertices contained in some dirty set $A$ with $|\operatorname{surr}(A)| \geq k^{.01}$ and let $Z$ denote the number of vertices contained in some dirty set $A$ with $|\operatorname{surr}(A)| < k^{.01}$. It is easy to see that the number of cycles in $(\mathbb{Z}^2)^*$ that have length $\ell$ and that surround a given vertex $v \in \mathbb{Z}^2$ is at most $\ell 4^\ell$. This allows us to bound the expectation of $Y$ as follows:

$$\begin{aligned}
\mathbb{E}(Y) &\leq k^2 \sum_{\ell \geq k^{0.01}} 4^\ell \binom{\ell}{\ell/8} (1-p)^{\ell/8} \\
&\leq k^2 \sum_{\ell \geq k^{0.01}} 8^\ell (1-p)^{\ell/8} \\
&= \frac{k^2 \left(8(1-p)^{1/8}\right)^{k^{0.01}}}{1 - 8(1-p)^{1/8}} \\
&= o(1),
\end{aligned}$$

where we have used that $|\operatorname{outer}(A)| \geq |\operatorname{surr}(A)|/8$ in the first line, that $\binom{\ell}{\ell/8} \leq 2^\ell$ in the second line, and where the last equality holds provided $p_0$ was chosen sufficiently close to one (and $p \geq p_0$). In particular, for $p_0$ sufficiently close to one and $p > p_0$, we have that $Y = 0$ a.a.s.

Next, we consider $Z$. For $v \in [k]^2$, we denote by $E_v$ the event that $v$ is contained in a dirty $A$ with $|\operatorname{surr}(A)| < k^{.01}$. We have

$$\begin{aligned}
\mathbb{P}(E_v) &\leq \sum_{\ell \leq k^{0.01}} \ell 4^\ell \binom{\ell}{\ell/8} (1-p)^{\ell/8} \\
&\leq \sum_{\ell \leq k^{0.01}} \ell \left(8(1-p)^{1/8}\right)^\ell \\
&\leq \tfrac{8(1-p)^{1/8}}{\left(1-8(1-p)^{1/8}\right)^2}. \\
&\leq 10^{-11},
\end{aligned}$$

where the last inequality holds provided $p_0$ is chosen sufficiently close to one (and $p \geq p_0$).

On the other hand, we clearly have

$$\mathbb{P}(E_v) \geq (1-p)^4.$$

Hence, we have that $k^2(1-p)^4 \leq \mathbb{E}Z = \sum_v \mathbb{P}(E_v) \leq k^2/10^{11}$. In particular, $\mathbb{E}Z = \Theta(k^2)$.

Next we consider the second moment of $Z$. Observe that if $|u-v|_\infty \geq 3 \cdot k^{.01}$, then $E_u$ and $E_v$ are independent. (Here $|(x,y)|_\infty = \max(|x|,|y|)$ denotes the familiar $L_\infty$-norm.) This allows us to write

$$\begin{aligned}
\mathbb{E}Z^2 &= \sum_{u,v} \mathbb{P}(E_u \cap E_v) \\
&= \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty < 3k^{.01}} \mathbb{P}(E_u|E_v) + \sum_v \mathbb{P}(E_v) \sum_{|u-v|_\infty \geq 3k^{.01}} \mathbb{P}(E_u|E_v) \\
&\leq \sum_v \mathbb{P}(E_v) 36 k^{.02} + \sum_{u,v} \mathbb{P}(E_v)\mathbb{P}(E_u) \\
&= \mathbb{E}Z \cdot o(k^2) + (\mathbb{E}Z)^2 \\
&= (1+o(1))\,(\mathbb{E}Z)^2 .
\end{aligned}$$

This shows that $\operatorname{Var}(Z) = o\left((\mathbb{E}Z)^2\right)$. An application of Chebyshev's inequality shows:

$$\begin{aligned}
\mathbb{P}(Z > k^2/10^{10}) &\leq \mathbb{P}(|Z - \mathbb{E}Z| \geq \tfrac{9}{10}\mathbb{E}Z) \\
&\leq (\tfrac{10}{9})^2 \cdot \tfrac{\operatorname{Var}(Z)}{(\mathbb{E}Z)^2} \\
&= o(1).
\end{aligned}$$

In conclusion, we have seen that, when $p_0$ is sufficiently close to one and $p_0 < p \leq 1$, then $Y = 0$ a.a.s. and $Z \leq k^2/10^{10}$ a.a.s., which obviously implies the lemma. □

**Proof of Proposition 3.17:** Let us pick $1 > p > p_0$ with $p_0$ as provided by Lemma 3.19. Then, a.a.s., $\Gamma(k, p)$ has no more than $k^2/10^{10}$ dirty vertices. In the remainder of the proof we therefore assume we are given a subgraph $G \subseteq \Gamma(k)$ for which there are at most $k^2/10^{10}$ dirty vertices, but which is otherwise arbitrary. We will show that any such $G$ satisfies $\mathrm{tw}(G) \geq k/1000$.

Aiming for a contradiction, we assume that there exists some balanced partition $\{A, S, B\}$ of $V(G) = [k]^2$ with $|S| < k/1000$.

We first observe that we can assume without loss of generality that $A$ does not contain a horizontal crossing. For, if it does then $B$ cannot contain a vertical crossing (otherwise $A, B$ would not be disjoint). Hence, by applying symmetry (switching the roles of $A, B$ and rotating by 90 degrees) we can indeed assume $A$ does not contain a horizontal crossing. Observe that

$$|A| \geq k^2 - |B| - |S| \geq k^2 - 2k^2/3 - k/1000 \geq k^2/10.$$

Let $A_1, \ldots, A_m$ denote the connected components of $A$ (connected when considered as subgraphs of $G$). Let us set

$$\mathcal{I} := \{i : A_i \text{ is not dirty}\}, \quad A' := \bigcup_{i \in \mathcal{I}} A_i.$$

Since the total number of dirty vertices is less that $k^2/10^{10}$, we have

$$|A'| \geq |A| - k^2/10^{10} \geq k^2/100.$$

Note that every edge (in $G$) between a vertex of $A'$ and a vertex of $[k^2] \setminus A'$ must in fact connect a vertex of $A'$ to a vertex of $S$. Hence, it follows that

$$\begin{aligned} 4|S| &\geq \textstyle\sum_{i \in \mathcal{I}} |\operatorname{outer}(A_i)|/2 \\ &\geq \textstyle\sum_{i \in \mathcal{I}} \sqrt{|A_i|}/2 \\ &\geq \sqrt{|A'|}/2 \\ &\geq k/20. \end{aligned}$$

(Here we have used Lemma 3.18 for the second line and the concavity of the square root function for the third line.) So it follows that $4k/1000 \geq 4|S| \geq k/20$, a contradiction.

This shows that there is no balanced partition with $|S| < k/1000$, which implies that $\mathrm{tw}(G) \geq k/1000$ by Kloks' lemma (Lemma 3.10). ∎

### 3.3.2 When $p > 1/2$

We are now ready to prove the first part of Theorem 3.2 with the help of Proposition 3.17.

**Proof of Theorem 3.1, the case $p > 1/2$:** Our proof is an application of a standard technique for comparing supercritical percolation to percolation with $p$ close to one, by means of Lemma 3.12 and Theorem 3.13. See for instance [17], pages 74–75.

Let $p_0$ be as provided by Lemma 3.19, and let $\pi$ be as provided by Theorem 3.13. We now pick $p_1$ such that $\pi(p_1) > p_0$. By Lemma 3.12, we can find an $a \in \mathbb{N}$ such that $\mathbb{P}(H([3a] \times [a])) > \sqrt[3]{p_1}$.

For $R$ a $3a \times a$ rectangle, we define the event $E(R) := H(R) \cap V(R_L) \cap V(R_R)$ where $R_L$ denotes the leftmost $a \times a$ subrectangle, and $R_R$ denotes the rightmost $a \times a$ rectangle (see Figure 13 on page 74 of [17] for a depiction). If $R$ is a $a \times 3a$ rectangle the we define $E(R) := V(R) \cap H(R_B) \cap H(R_T)$ with $R_B$, resp. $R_T$, the bottom, resp. top, $a \times a$ subrectangle of $A$. Note that, by choice of $a$ and Harris' lemma, we have that $\mathbb{P}(E(R)) > p_1$ for every $3a \times a$ or $a \times 3a$ rectangle $R$.

We now define a (dependent) bond percolation model $Y$ on $\mathbb{Z}^2$ as follows. We declare the horizontal edge between $(i, j)$ and $(i+1, j)$ open in $Y$ if $E(\{2ai+1, \ldots, 2ai+3a\} \times \{2aj+1, \ldots, 2aj+a\})$ holds; and similarly the edge between $(i, j)$ and $(i, j+1)$ is open in $Y$ if $E(\{2ai+1, \ldots, 2ai+a\} \times \{2aj+1, \ldots, 2aj+3a\})$ holds. It is not difficult to see that $Y$ is in fact 1-independent. Hence, by Theorem 3.13, $Y \geq X$, where $X$ is standard (independent) percolation on $\mathbb{Z}^2$ with edge-probability $> p_0$.

We can view $\Gamma(k, p)$ as the restriction of the (independent, edge-probability $p$) percolation process to the $k \times k$ grid $[k]^2$. Similarly, we let $\Gamma_X$, resp. $\Gamma_Y$, denote the subgraph that $X$, resp. $Y$, defines on $[\ell]^2$, where $\ell := \lfloor k/2a \rfloor$. Note that we have chosen $\ell$ so that each of the rectangles corresponding to the edges of $\Gamma_Y$ are contained in $[k]^2$. Observe that by construction (and Proposition 3.17) we have that, a.a.s.:

$$\mathrm{tw}(\Gamma_Y) \geq \mathrm{tw}(\Gamma_X) = \Omega(\ell) = \Omega(k).$$

We remark that $\Gamma_Y$ is in fact a minor of $\Gamma(k,p)$ (under the natural coupling associated with the construction of $Y$). To see this, we can proceed as follows: If $E(R)$ holds with $R$ a $3a \times a$ rectangle that corresponds to some edge of $\Gamma_Y$, then we do a sequence of contractions that will identify all vertices of $R_L$ that participate in (horizontal or vertical) crossings of $R_L$ into a single vertex $x$, we produce a vertex $y$ via contractions in $R_R$ similarly, and then we contract the remaining edges of a long, horizontal crossing of $R$ into a single edge that connects $x$ and $y$. If we do this for each rectangle corresponding to an edge of $\Gamma_Y$, then discard any unneeded vertices (making sure to keep exactly one vertex in each $a \times a$ square that corresponds to a vertex $(i,j) \in [\ell]^2$ that was not incident to any edge of $Y$), then we obtain a graph isomorphic to $\Gamma_Y$.

Since $\Gamma_Y$ is a minor of $\Gamma(k,p)$, we have $\mathrm{tw}(\Gamma(k,p)) \geq \mathrm{tw}(\Gamma_Y) = \Omega(k)$, a.a.s., as required. ■

### 3.3.3 When $p < 1/2$

In this section, we prove the upper and lower bound of the treewidth $\Gamma(k,p)$ for $p < 1/2$. We need the following result from percolation theory, that is originally due to Kesten [61, 62].

**Theorem 3.20** ([61, 62])**.** *Consider bond percolation on $\mathbb{Z}^2$ and let $C_0$ denote the number of vertices in the cluster (component) of the origin. For each $p < 1/2$ there exists $\lambda(p) > 0$ such that*

$$\mathbb{P}(|C_0| \geq n) \leq e^{-n\lambda(p)},$$

*for all $n \geq 0$.*

This has the following easy consequence:

**Corollary 3.21.** *If $0 < p < 1/2$ then, a.a.s., all components of $\Gamma(k,p)$ have $O(\log k)$ vertices.*

*Proof.* Let us fix $0 < p < 1/2$ and let $\lambda(p)$ be as provided by Theorem 3.20. Let $K := 100/\lambda(p)$. Observe that, for every $v \in [k]^2$ and $\ell \in \mathbb{N}$, the probability that is in a component of order $\geq \ell$ in $\Gamma(k,p)$ is no more that the probability that $|C_0|$ exceeds $\ell$. Thus, we can conclude:

$$\begin{aligned}\mathbb{P}(\Gamma(k,p) \text{ has a component of size } \geq K \log k) &\leq k^2 \exp[-100 \log k] \\ &= \exp[-98 \log k] \\ &= o(1).\end{aligned}$$

□

Since the treewidth of a graph equals the maximum of the treewidth of its components, and all components of $\Gamma(k,p)$ are planar, the required upper bound for $\mathrm{tw}(\Gamma(k,p)$ in the case when $p < 1/2$ follows immediately using Corollary 3.8:

**Corollary 3.22.** *If* $0 < p < 1/2$ *then, a.a.s.,* $\mathrm{tw}(\Gamma(k,p)) = O(\sqrt{\log k})$.

The following lemma now completes the proof of Theorem 3.2.

**Lemma 3.23.** *Fix* $0 < p < 1/2$ *then, a.a.s.,* $\mathrm{tw}(\Gamma(k,p)) = \Omega(\sqrt{\log k})$.

*Proof.* We fix a $\varepsilon = \varepsilon(p)$ (small, to be determined later), and we set $\ell := \lceil \sqrt{\varepsilon \log k} \rceil$. We now fix $N := \lfloor k/(\ell+1) \rfloor^2 = \Omega(k^2/\log k)$ (vertex-) disjoint $\ell \times \ell$-subgrids $G_1, \ldots, G_N$ in $[k]^2$. We will say that the subgrid $G_i$ is *intact* if all of its edges are present in $\Gamma(k,p)$. By independence of the events that the $G_i$-s are intact, we have:

$$\begin{aligned}\mathbb{P}(\text{at least one } G_i \text{ is intact}) &= 1 - (1 - p^{2\ell(\ell-1)})^N \\ &\geq 1 - \exp[-Np^{2\ell(\ell-1)}] \\ &\geq 1 - \exp[-Np^{\ell^2}] \\ &= 1 - \exp[-Np^{\varepsilon \log k}].\end{aligned}$$

Next, note that

$$Np^{\varepsilon \log k} = \Omega\left(\frac{k^2}{\log k} \cdot \exp[\varepsilon \log p \log k]\right) = \Omega(\exp[2 \log k - \log \log k + \varepsilon \log p \log k]).$$

Hence, provided we chose $\varepsilon < -2/\log p$, we have that $Np^{\varepsilon \log k} \to \infty$ and hence also

$$\mathbb{P}(\text{at least one } G_i \text{ is intact}) = 1 - o(1).$$

Hence, by Lemma 3.9, and since $\mathrm{tw}(H) \leq \mathrm{tw}(G)$ if $H \subseteq G$, we have that $\mathrm{tw}(\Gamma(k,p)) \geq \ell = \Omega(\sqrt{\log k})$ a.a.s. □

Corollary 3.22 and Lemma 3.23 together give the $p < 1/2$ part of Theorem 3.2.

## 3.4 Proof of Theorem 3.1

Since Mitsche and Perarnau [80] have already shown the result is true when $r = r(n)$ is larger than some fixed constant $C$, we only need to consider the case when $r_c < \liminf r \leq \limsup r \leq C$. Note that in this case $\Theta(r\sqrt{n})$ simplifies to $\Theta(\sqrt{n})$. Moreover, by monotonicity, we see that for any such sequence $r$, a.a.s., $\mathrm{tw}(G(n,r)) \leq \mathrm{tw}(G(n,C)) = O(\sqrt{n})$ by Mitsche and Perarnau's result. Hence, we only need to prove an a.a.s. lower bound for the treewidth of order $\Omega(\sqrt{n})$. Using Corollary 3.14 and monotonicity, Theorem 3.1 follows if we establish:

**Lemma 3.24.** *For every fixed $r > r_c$, we have $\mathrm{tw}(G_{Po}(n,r)) = \Omega(\sqrt{n})$ a.a.s.*

*Proof.* The proof is almost exactly the same as the proof of the $p > 1/2$ case of Theorem 3.2 above. Again, we let $p_0$ be as provided by Lemma 3.19, we let $\pi$ be as provided by Theorem 3.13, and we pick $p_1$ such that $\pi(p_1) > p_0$. Using Lemma 3.15, we find an $a$ such that $\mathbb{P}(H([0,3a] \times [0,a])) > \sqrt[3]{p_1}$. For $R$ a $3a \times a$ or $a \times 3a$ rectangle we define $E(R)$ as in the proof of the $p > 1/2$ case of Theorem 3.2 above. We again define a 1-independent bond percolation model $Y$ on $\mathbb{Z}^2$, by declaring the horizontal edge between $(i,j)$ and $(i+1,j)$ open in $Y$ if $E([2ai, 2ai+3a] \times [2aj, 2aj+a])$ holds; and the edge between $(i,j)$ and $(i,j+1)$ is open in $Y$ if $E([2ai, 2ai+a] \times [2aj, 2aj+3a])$ holds. (Note that 1-independence holds provided we chose $a$ sufficiently large.) Again Theorem 3.13 gives that $Y \geq X$, where $X$ is standard (independent) percolation on $\mathbb{Z}^2$ with edge-probability $> p_0$.

We set $k := \lfloor \sqrt{n}/2a \rfloor$, and we let $\Gamma_X$, resp. $\Gamma_Y$, be the restriction of $X$, resp. $Y$, to $[k]^2$. Arguing analogously to the way we did in the proof of the $p > 1/2$ case of Theorem 3.2, we see that $\Gamma_Y$ is in fact a minor of $G_{\mathrm{Po}}(n,r)$ (under the natural coupling we get from the construction of $Y$). Hence, using Proposition 3.17, we get that a.a.s.:

$$\mathrm{tw}(G_{\mathrm{Po}}(n,r)) \geq \mathrm{tw}(\Gamma_Y) \geq \mathrm{tw}(\Gamma_X) = \Omega(k).$$

Since $k = \Theta(\sqrt{n})$ this concludes the proof. □

*Remark* 3.25. Our method can easily be transferred into the torus case and cylinder case, since these two cases do not change the local structures!

## 3.5 Discussion and further work

Together with the work of Mitsche and Perarnau [80], our Theorem 3.1 provides an almost complete picture of the behaviour of the treewidth of random geometric graphs, up to the order of the leading constants:

**Corollary 3.26.** *A.a.s,*

$$\mathrm{tw}(G(n,r)) = \begin{cases} \Theta\left(\frac{\log n}{\log\log n}\right) & \text{if } 0 < \liminf r \leq \limsup r < r_c, \\ \Theta\left(r\sqrt{n}\right) & \text{if } \liminf r > r_c. \end{cases}$$

Interestingly, by a result of McDiarmid [76], the clique number of the random geometric graphs is a.a.s. equal to $(1+o(1))\log n/\log\log n$ when $r$ is constant. This gives rise to the following natural questions.

**Question 3.27.** *Suppose* $0 < \liminf r \leq \limsup r < r_c$.
*Is* $\mathrm{tw}(G(n,r)) = (1+o(1))\log n/\log\log n$ *a.a.s.?*
*Is* $\mathrm{tw}(G(n,r)) = \omega(G(n,r))$ *a.a.s.?*

Of course we would also be very interested to learn the precise leading constants for the supercritical case.

Another tantalizing question is what happens precisely at the critical point. Based on widely believed conjectures on the "critical exponents" for two-dimensional percolation (see [51], Chapters 9 and 10), we offer the following conjectures:

**Conjecture 3.28.** *A.a.s.,* $\mathrm{tw}(G(n,r_c)) = n^{\frac{91}{192}+o(1)}$.

**Conjecture 3.29.** *A.a.s.,* $\mathrm{tw}(\Gamma(k,1/2)) = k^{\frac{91}{96}+o(1)}$.

Moreover, the treewidth of classic Erdös-Renyi random graphs were studied by Kloks [65] and Gao [48], and the treedepth of the classic Erdös-Renyi random graphs have been studied by Perarnau and Serra in [94]. Some other width parameters are also introduced and investigated, for example, the rank-width of Erdös-Renyi random graphs was studied by Lee, Lee and Oum in [68].

In this chapter, we studied the treewidth of two random structures: percolated grids and random geometric graphs. We find that this parameter undergoes a phase transition at the critical value. Then it is natural to ask same question for other random structures: does the treewidth of these structures also have a phase transition phenomenon?

The essence of our proof is "Russo-Seymour-Welsh (RSW)" arguments and $k$-independent percolation (In fact, finite range dependent percolation model). So our techniques can be workable for some general random structures that enjoy the "Crossing Property". Unfortunately, RSW technique maybe can not be implemented in general random structures.

The two-dimensional Ising model has very particular importance in the area of statistical mechanics. This model of ferromagnetism is the first natural model where the existence of a phase transition, a common property to many statistical mechanics models, has been investigated by Peierls [89]. In a series of seminal papers, Onsager [88] computed several quantities associated with this model. Since then, the Ising model has been studied extensively.

The random-cluster model has emerged in recent years as a key tool in the mathematical study of ferromagnetism. It was invented by Kees Fortuin and Piet Kasteleyn [43] around 1970 as a unification of percolation, Ising models, and Potts models, and also as an extrapolation of electrical networks. In fact, it is a *dependent* percolation model for which the probability of a configuration is weighted by the number of clusters (connected components) that it contains. This percolation representation turned out to be extremely powerful in studying many statistical mechanics models including the Ising model.

We consider the *random-cluster* measure on arbitrary finite graphs. We mainly use the notations and definitions from [52]. Let $G = (V, E)$ be a finite and simple graph. Let $\Omega = \{0, 1\}^E$ be the state space, member of which are 0/1-vectors $\omega = (\omega(e) : e \in E)$. We call the edge $e$ is *open* in $\omega$ if $\omega(e) = 1$ and is *closed* if $\omega(e) = 0$. For $\omega \in \Omega$, let $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ denote the set of open edges. It is easy to see that there is a one-one correspondence between vectors $\omega \in \Omega$ and subsets $F \subset E$, given by $F = \eta(\omega)$. Let $k(\omega)$ is the the number of connected components of the graph $(V, \eta(\omega))$ and the isolated vertices included. We associate with $\Omega$ the $\sigma$-field $\mathcal{F}$ of all its subsets.

Then we introduce the formal definition of *random-cluster model*. A *random-cluster measure* on $G$ has two parameters satisfying $p \in [0, 1]$ and $q \in (0, \infty)$, and is defined as the measure $\phi_{p,q}$ on the measurable pair $(\Omega, \mathcal{F})$ given by

$$\phi_{p,q}(\omega) = \frac{1}{Z_{RC}} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \Omega,$$

where the partition function, $Z_{RC}$ is given by

$$Z_{RC} = \sum_{\omega \in \Omega} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}.$$

and $k(\omega)$ is the the number of connected components of the graph and the isolated vertices included. For more detailed information, we recommend the monograph by Grimmett [52].

Recently, Duminil-Copin, Hongler and Nolin [35] studied the crossing probability for the FK Ising model (random cluster model when $q = 2$) and proved a RSW-type bound on the crossing probability for this model at *criticality*, independent of the boundary conditions. In particular, they prove lower and upper bounds for crossing probabilities in rectangles of bounded aspect ratio. Moreover, these bounds are uniform in the size of rectangles and in the boundary conditions, which are analogues for the FK Ising model to the celebrated Russo-Seymour-Welsh (RSW) bounds for percolation [100, 104].

Let $R$ be rectangles of the form $[0, n] \times [0, m]$ for $n, m > 0$ and translations of them. Set $\mathcal{C}_v(R)$ be the event that there exists a *vertical crossing* in $R$, i.e., there is a open path from the bottom side $[0, n] \times \{0\}$ to the top side $[0, n] \times \{m\}$. In particular, Duminil-Copin, Hongler and Nolin [35] proved the following theorem:

**Theorem 3.30** (RSW-Type Crossing Bounds, [35]). *Let* $0 < \beta_1 < \beta_2$. *There exist two constants* $0 < c_- < c_+ < 1$ *(depending only on* $\beta_1$ *and* $\beta_2$*) such that for any rectangle* $R$ *with side lengths* $n$ *and* $m \in [\beta_1 n, \beta_2 n]$. *One has*

$$c_- \leq \mathbb{P}_{p_c}^{\varsigma}(\mathcal{C}_v(R)) \leq c_+,$$

*for any boundary conditions* $\varsigma$, *where* $\mathbb{P}_{p_c}^{\varsigma}$ *denote the random cluster measure on* $R$ *with parameters* $(p, q) = (p_c, 2)$ *and boundary conditions* $\varsigma$.

However, in the random cluster model, the statue of every "bond" depends on all other bonds, not just finite neighboring "bonds". So the *k-independent* percolation technique can not be applied here. The natural problem is: can we modify our method here to get the treewidth of the random cluster model?

# Samenvatting

Een *graaf* is een verzameling punten samen met lijnen die een (mogelijk lege) deelverzameling er van verbinden. Een *stochastische meetkundige graaf* is een graaf waarvan de posities van de punten volgens een gegeven kansverdeling in een bepaald gebied liggen (bijvoorbeeld het eenheidsvierkant $[0,1]^2$) en de lijnen bepaald worden door de afstanden tussen deze punten, wat wil zeggen dat twee punten verbonden zijn door een lijn wanneer hun afstand kleiner is dan een gegeven parameter $r$. Dit is het hoofdonderwerp van studie in dit onderzoek.

Een *pad* in een graaf is een eindige of oneindige rij lijnen die een rij punten, welke alle verschillend zijn van elkaar, verbindt. Een *deelgraaf* $H$ van een graaf $G$ is een graaf waarvan de punten een deelverzameling van de puntenverzameling van $G$, en de lijnen een deelverzameling van de lijnenverzameling van $G$ vormen. Een *component* $K$ is een deelgraaf waarvan iedere twee punten met elkaar verbonden zijn door een pad dat volledig bevat is in $K$, en zodanig dat $K$ niet bevat is een grotere deelgraaf met deze eigenschap.

Eén van de meest bekende verschijnselen in de theorie van stochastische meetkundige grafen is de zogenaamde 'plotselinge verschijning van de reuzecomponent'. Hiermee bedoelen we dat er een kritieke waarde $r_c$ bestaat voor $r$, zodanig dat als $r < r_c$, iedere component van de stochastische meetkundige graaf hoogstens $C \cdot \log n$ punten heeft voor een bepaalde positieve constante $C$, terwijl wanneer $r > r_c$ er een *reuzecomponent* bestaat met minstens $c \cdot n$ punten voor een positieve constante $c$. Hier is $n$ het aantal punten van de graaf. Deze bewering geldt met kans 1 wanneer $n$ naar oneindig gaat.

In deze scriptie hebben we twee aspecten van stochastische meetkundige grafen bestudeerd: achtervolging-ontwijking en boombreedte.

**Een achtervolging-ontwijkingspel: agenten-en-boeven op stochastische meetkundige grafen**

De combinatorische speltheorie is de tak van de combinatoriek (en de speltheorie) die spellen met complete informatie bestudeert, zoals Boter-kaas-en-eieren en Hex. In het eerste deel van deze scriptie hebben we een bepaald achtervolging-ontwijkingspel bestudeerd: het *agenten-en-boevenspel*. Het dateert uit de jaren 70 en is in afgelopen jaren uitvoerig bestudeerd.

In dit spel probeert een verzameling agenten, bestuurd door een enkele speler, een boef te vangen die wordt bestuurd door een andere speler. De posities van de agenten en de boef zijn beperkt tot de punten van een graaf en iedere ronde bewegen ze naar een naastgelegen punt. Hoewel de agenten door een enkele speler worden bestuurd, mogen ze iedere ronde naar verschillende punten bewegen, volgens de strategie van deze speler die beslist hoe ze samenwerken. Iedere zet is zichtbaar voor beide spelers, wat betekent dat dit een spel met complete informatie is. De agenten winnen wanneer het minstens één van hen lukt om hetzelfde punt als de boef te bezetten, en de boef wint wanneer het hem lukt dit tot in de eeuwigheid te vermijden. Het kleinste aantal benodigde agenten voor het vangen van de boef heet het *agentengetal* van de graaf, en dit is waar onze interesse hoofdzakelijk naar uit gaat.

Wij onderzochten dit spel op stochastische meetkundige grafen en het is ons gelukt om het agentengetal te vinden. Preciezer gezegd: we hebben een versie van het spel bestudeerd waar agenten proberen een boef te vangen op de punten van $\mathcal{G}(n, r, p)$, een gepercoleerde stochastische meetkundige graaf waar $n$ punten onafhankelijk en uniform verdeeld zijn gekozen uit $[0, 1]^2$, en twee punten buren zijn met kans $p$ wanneer hun Euclidische afstand hoogstens $r$ is. We verkregen het agentengetal van $\mathcal{G}(n, r, p)$ voor een breed bereik van $p = p(n)$ en $r = r(n)$.

**De boombreedte van stochastische meetkundige grafen**

Een *boom* is een graaf waarvan iedere twee punten verbonden zijn door precies één pad. Bomen zijn een klasse van eenvoudigere grafen met enkele zeer onderscheidende en fundamentele eigenschappen. Het is daarom interessant om te vragen in hoeverre deze eigenschappen kunnen worden overgebracht naar meer algemene grafen die zelf niet bomen zijn, maar er wel in zekere zin op lijken.

De *boombreedte* van een graaf is een getal dat gebruikt kan worden om te meten in welke mate de graaf op een boom lijkt. Grof gezegd lijkt een graaf meer op een boom, des te kleiner zijn boombreedte is. Verder kunnen in de informatica veel problemen die 'moeilijk' zijn voor algemene grafen opgelost worden in lineaire of polynomiale tijd als we alleen grafen beschouwen met een constante bovengrens op hun boombreedte.

In het tweede deel van deze scriptie hebben we de boombreedte van twee gerelateerde modellen van stochastische grafen beschouwd. Het eerste model is de stochastische meetkundige graaf, waar we $n$ punten laten vallen op het vierkant $[0, \sqrt{n}]^2$ en elk paar van twee punten verbinden met een lijn wanneer hun afstand hoogstens $r = r(n)$ is.

Het tweede model is het *gepercoleerde rooster*, waar we een $k \times k$-rooster nemen (dat wil zeggen, de puntenverzameling is $\{0, \ldots, k\} \times \{0, \ldots, k\}$ en er zijn lijnen tussen paren van punten met afstand 1) en iedere lijn met kans $p$ laten staan, onafhankelijk van alle andere lijnen. We lieten zien dat, met kans naderend naar 1 wanneer $k \to \infty$, de boombreedte van het gepercoleerde rooster van orde $k$ is als $p > 1/2$, en van orde $\sqrt{\log k}$ als $p < 1/2$. Door een deel van het bewijs van dit laatste resultaat te hergebruiken, bewezen we ook een vermoeden van Mitsche en Perarnan welke luidt dat, met kans naderend naar 1 wanneer $n \to \infty$, de boombreedte van de stochastische meetkundige graaf van orde $r\sqrt{n}$ is als $\liminf r > r_c$, waar $r_c$ de drempelwaarde is van de straal voor het verschijnen van de reuzecomponent.

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# Curriculum Vitae

Anshui Li was born on 22nd of March in 1986 in Zibo, Shandong Province, People's Republic of China. He got his bachelor degree and Master degree in mathematics in 2008 and 2011 separately. From Sep, 2011, he started a Ph.D study in probability in Utrecht, first with prof. dr. Alexander Gnedin. Then from Dec, 2012, he began to do research on random graphs under the supervision of dr. Tobias Müller.

During his research, he attended several workshops and seminars in the Netherlands and also abroad, which includes Kyoto, Durham, Eindhoven, Lunteren, Oberwalfach, Paris, Poznan and Pittsburgh.

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