A unitary structure for the graded quotient of conformal coblocks.

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A unitary structure for the graded quotient of conformal coblocks.

(met een samenvatting in het Nederlands)

Proefschrift

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Introduction

Fix a simple complex algebraic group G and a positive integer ℓ, called *level*. To a complex projective curve C we associated, in a natural way, a finite dimensional vector space; this construction is smooth, in the sense that a family of curves $\mathcal{C}/B$ gives rise to a vector bundle of finite rank over B. Moreover, the vector bundle comes with a natural flat *projective holomorphic connection*, that is locally given by a connection with scalar curvature. A comment on the adjective *projective*: what we will construct is globally only defined as projective holomorphic connection, but can locally be represented by a true holomorphic connection.

There are two ways to construct this vector bundle and the projective holomorphic connection. The first one, historically, uses a representation-theoretic approach, and has been baptized the vector bundle and (projective) connection as the *vector bundle of conformal blocks* and the *Wess-Zumino-Witten* (WZW) connection, respectively. The standard reference for this is [25], but this paper is notoriously difficult to read. Several papers, such as [19], [26], [2], (partly) redo [25], and the reader may find these more accessible.

The second approach is called geometric quantization, and started with [14]. In this construction the vector bundle and (projective) connection are called the *generalized* θ*-functions* and the *Hitchin connection*. It has been known for some time that conformal blocks and generalized θ-functions are essentially the same, for there is a natural isomorphism between them that is called the *Verlinde* isomorphism. More recently, Y. Laszlo also proved in [17] that this isomorphism identifies the connections.

In this thesis we introduce a natural filtration on the vector bundle of conformal coblocks (the dual of the vector bundle of conformal blocks), and discuss the graded quotient. It has a natural connection with scalar curvature and a unitary structure that is compatible with it. From a certain point of view, this connection is the graded equivalent of the WZW (projective) connection. It remains unclear to me what can be inferred form this regarding the unitary of the WZW connection itself.

In order to make the comparison between the WZW connection and its "graded counterpart", both are constructed using the same approach. As a whole, both of these constructions are rather technical and lengthy, but the key idea of the common approach is very simple and elegant: let $\mathcal{C}$ be a complex curve, p a point (or set of points) on $\mathcal{C}$, and $\mathcal{P}/\mathcal{C}$ a principal G-bundle with flat connection. From this geometric data we extract two kinds of data. The first is a complex presymplectic[1] Lie algebra $\mathcal{K}_\mathfrak{g}$ that (almost) encodes the entire geometry. The second is the singular cohomology of $\mathcal{C}$ with coefficients in the adjoint bundle of $\mathcal{P}/\mathcal{C}$. This data is then used to construct a projective representations of the algebra of vectorfields on a pointed formal disc near p - we will describe this construction in general.

Let $(H, (\cdot,\cdot))$ be a symplectic complex vector space. Then $\widehat{H} := H \oplus \mathbb{C}$ has a natural structure of a Lie algebra, by requiring that

$$[a \oplus r, a' \oplus r] = 0 \oplus (a, a'), \quad a, a' \in H,\ r, r' \in \mathbb{C}.$$

Using the symplectic form, we can identify an endomorphism α of H with an element $E(\alpha)$ of $H^{\otimes 2}$, and in particular an infinitesimal symplectic endomorphism with an element of $\mathrm{Sym}_2 H$. This allows us to define

$$\tau : \mathfrak{sp}(H) \to U\widehat{H}[\hbar^{-1}] : \alpha \mapsto \frac{1}{\hbar} E(\alpha),$$

where $H^{\otimes 2}$ is mapped to $U\widehat{H}$ by the obvious map[2]. Here $U\widehat{H}$ is the universal enveloping algebra of $\widehat{H}$, and $U\widehat{H}[\hbar^{-1}]$ the polynomial algebra in the formal variable $\hbar^{-1}$ over the ring $U\widehat{H}$. The key point is that τ is an algebra morphism, *up to a scalar*. Stated differently, the algebra generated by the image of τ is a central extension of $\mathfrak{sp}(H)$; by construction, this central extension acts on any $U\widehat{H}[\hbar^{-1}]$ module. This projective representation is used to define the connection on the vector bundle of conformal coblocks. The construction of the WZW connection is more involved, but nevertheless essentially follows the same theme.

[1] A *presymplectic* form is an antisymmetric bilinear form; if it is nondegenerate we call it *symplectic*.

[2] This is the composition of the inclusions $H^{\otimes 2} \hookrightarrow \widehat{H}^{\otimes 2} \hookrightarrow \bigotimes_\bullet \widehat{H}$ with the natural quotient map $\bigotimes_\bullet \widehat{H} \to U\widehat{H}$

When reading this thesis, reader may find himself lost at times due to the multitude of technical details. Of course, sometimes this is caused by fact that the author also lost himself in the details. However, it also seems to be part of the nature of the subject: though the end results are elementary geometric objects (vector bundle and connection), the way to get there consists of many small technical steps and uses a variety of notions from geometry and algebra. Below is an outline of this thesis that will hopefully allow the reader to keep track of "where he is" and to give him an idea of "where all this is going".

In the first chapter we briefly review a number of notions that we will use in the rest of the chapters: pointed curves, G-bundles on them, moduli spaces of these, loop groups and the Picard group of the moduli space of G-bundles.

In the second chapter we introduce generalized θ-functions, along with the Hitchin (projective) connection. The rest of the thesis is independent of this chapter, with the exception of a very short discussion of the Verlinde isomorphism at the very end.

The third chapter describes how to associate certain algebraic data to a pointed curve endowed with a G-bundle. This algebraic output will serve as input for the next two chapters.

The fourth chapter is devoted to the definition of a certain Fock module, that is defined in terms of the cohomological data of that comes with a unitary structure a compatible connection of scalar curvature. In the final chapter, a quotient of this Fock module will be identified with the vector bundle of graded conformal coblocks.

In chapter five we define the conformal blocks and the WZW (projective) connection.

The final chapter compares the output of chapter 2,4,5. In particular, it is shown here that (a quotient of) the Fock module from chapter 4 can be identified with the graded quotient of the conformal coblocks from chapter 5. It is argued that with respect to this identification, the connection on the Fock module can be thought of as a "graded counterpart" of the WZW connection.

A final note: unless stated otherwise, everything will be described in the language of locally ringed spaces[3]. These will be either in the category of $\mathbb{C}$-schemes, or in the complex analytic category (we will be a bit sloppy with the distinction). In either case, a "map" will be a morphism of locally ringed

[3] A reference for this is [13], chapter 2

spaces; if we mean a C^∞-map, we will explicitly say so. Furthermore, if $(X, \mathcal{O}_X)$ is such a locally ringed space, θ_X denotes the sheaf of local vector fields on X, Ω_X its dual and ω_X the highest nontrivial exterior power of Ω_X. If $f : X \to Y$ is a map, then $\theta_{Y/X}$ denotes the sheaf of local vertical vector fields on Y; $\Omega_{Y/X}$ and $\omega_{Y/X}$ have a similar meaning. Also, if $\mathcal{F}$ is an $\mathcal{O}_Y$ module, then $f^{-1}\mathcal{F}$ denotes the sheaf theoretic pullback and $f^*\mathcal{F}$ the coherent pullback.

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Chapter 1

Moduli of curves and G-bundles

In this chapter, we introduce the geometric setting that will form the basis for the rest of this thesis. In particular, we define the notions of pointed curves, principal G-bundles over them and families thereof. We will discuss the moduli of these object, and later on introduce loop groups to describe the moduli space of G-bundles.

1.1 (Families of) N-pointed curves.

Fix an integer $N > 0$. Let $(C, p = (p_1, \ldots, p_N))$ be a pointed smooth projective connected curve over $\mathbb{C}$ of genus g. Here $(p_1, \ldots, p_N)$ is an ordered set of points in C. *We always assume the points* $p_1, \ldots, p_N$ *to be distinct and closed.* If (C', p') is another one, then they are said to be isomorphic iff there is an isomorphism $\phi : C \to C'$ over $\mathbb{C}$ such that $\phi(p_i) = p'_i$, $i = 1, \ldots, N$.

A *family* (over a smooth variety B) of these objects is a pair $(\mathcal{C}/B, p)$, where $\mathcal{C}/B : \mathcal{C} \to B$ is a proper surjective map and $p = (p_1, \ldots, p_N)$ is an N-tuple of disjoints sections, such that $(\mathcal{C}_b, p_b)$ is a complex pointed smooth projective connected curve of genus g for every closed $b \in B$ (here $\mathcal{C}_b$ denotes the fiber over $b \in B$). If $\mathcal{C}'/B' : \mathcal{C}' \to B'$, $p' = (p'_1, \ldots, p'_N) : B \to \mathcal{C}^n$ is another such family, then a morphism from $(\mathcal{C}/B, p)$ to $(\mathcal{C}'/B', p')$ is pair of maps

$\phi : \mathcal{C} \to \mathcal{C}'$, $\psi : B \to B'$ such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{C}' \\ {\scriptstyle \mathcal{C}/B}\downarrow \uparrow{\scriptstyle p} & & {\scriptstyle p'}\uparrow \downarrow{\scriptstyle \mathcal{C}'/B'} \\ B & \xrightarrow{\psi} & B' \end{array}$$

is a Cartesian diagram for the downward pointing arrow, and a commutative diagram for the upward pointing ones. A morphism of such families is an isomorphism if it has an inverse, which is the case iff ϕ is bijective.

In the following, by a (N*-)pointed curve, we shall mean an* N*-pointed smooth projective connected genus* g *curve over* $\mathbb{C}$*, and a family of (*N*-)pointed curves will be a family of* N*-pointed smooth projective connected genus* g *complex curves.*

Continuing with the previous notation, we suppose a family of pointed curves $(\mathcal{C}/B, p)$ is given. For any map $f : B' \to B$ we have a pullback square

$$\begin{array}{ccc} f^*\mathcal{C} := B' \times_B \mathcal{C} & \longrightarrow & \mathcal{C} \\ {\scriptstyle f^*p}\uparrow \downarrow{\scriptstyle f^*(\mathcal{C}/B)} & & {\scriptstyle \mathcal{C}/B}\downarrow \uparrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B, \end{array}$$

where f^*p is the universal morphism induced by the pair of maps $\mathrm{id} : B' \to B$, $p \circ f$. In the left column is a family of pointed curves, which we call the *pull back of* $(\mathcal{C}/B, p)$ *along* f and denote $f^*(\mathcal{C}/B, p)$. A family of pointed curves $(\mathcal{C}^u/B^u, p^u)$ is called *universal* iff every family of pointed curves is, up to unique isomorphism, the pullback of $(\mathcal{C}^u/B^u, p^u)$ along a unique map to B_u: every $(\mathcal{C}/B, p)$ as before is isomorphic to $f^*(\mathcal{C}^u/B^u, p^u)$ for some unique smooth $f : B \to B^u$.[1] Such a family does not exist, but there is a family that is "locally universal", as we will see shortly.

1.1.1 $\mathcal{T}_{g,N}$, $\mathcal{C}_{g,N}$ and $\mathcal{M}_{g,N}$

A N-pointed curve (of genus g) corresponds to a complex structure on a N-pointed smooth compact orientable genus g real surface. The diffeomorphism type of the latter is unique, so one can think of an arbitrary pointed curve (C, p) as a complex structure on a fixed smooth compact oriented surface Σ_g with an N tuple of distinct labeled points. Moreover, almost complex structures on Σ_g are always integrable, so *isomorphism types of* N*-pointed*

[1] This is of course just saying that $(\mathcal{C}^u/B^u, p^u)$ is a universal object in the category of families of pointed curves.

curves are in 1-1 correspondence with almost complex structures on Σ_g *up to diffeomorphisms preserving* p.

Definition 1.1.1. Let $\mathrm{Diff}(\Sigma_g, p)$ denote the group of diffeomorphisms of Σ_g that fix the elements of p, and let $\mathrm{Diff}^0(\Sigma_g, p)$ be the subgroup of diffeomorphisms that are isotopic to the identity. The Teichmüller space is defined as

$$\mathcal{T}_{g,N} = \{\text{almost complex structures on } \Sigma_g\} / \mathrm{Diff}^0(\Sigma_g, p).$$

and the mapping class group as $\Gamma_{g,N} = \mathrm{Diff}(\Sigma_g, p)/\mathrm{Diff}^0(\Sigma_g, p)$. Their quotient, as orbifold, is denoted

$$\mathcal{M}_{g,N} := \mathcal{T}_{g,N}/\Gamma_{g,N}.$$

These three objects have been, and still are, the object of extensive study, and much is already know about them. We list some of their properties:

For $3g - 3 + N > 0$, the Teichmüller space $\mathcal{T}_{g,N}$ has a natural structure of a contractible $3g - 3 + N$ dimensional complex manifold,

with respect to this structure, $\Gamma_{g,N}$ acts holomorphically and proper discontinuously so that $\mathcal{M}_{g,N}$ receives an orbifold structure,

there is a family of N-pointed curves $(\mathcal{C}_{g,N}/\mathcal{T}_{g,N}, p_{g,N})$, called the Teichmüller family, such that any family of N-pointed curves is locally a pullback thereof,

$\mathcal{M}_{g,N}$ has a natural structure of a quasi-projective irreducible variety, compatible with the complex structure induced from $\mathcal{T}_{g,N}$.

1.1.2 (Co)tangent space of a miniversal family

Suppose we are given a family $(\mathcal{C}/B, p)$ of pointed curves, then we denote by $\theta_{\mathcal{C}}$ the sheaf of local vectorfields on $\mathcal{C}$ and by $\theta_{\mathcal{C}/B}$ the subsheaf *vertical vectorfields* over B. The latter can be characterized by the property that it consists of vectorfields that kill $(\mathcal{C}/B)^{-1}\mathcal{O}_B$. The "points" p determine another subsheaf of $\theta_{\mathcal{C}}$: the vectorfields that are tangent to p, and we denote it by $\theta_{\mathcal{C}}(\log p)$. Consider the short exact sequence

$$0 \longrightarrow \theta_{\mathcal{C}/B}(-p) \longrightarrow \theta_{\mathcal{C}}(\log p) \longrightarrow (\mathcal{C}/B)^*\theta_B \longrightarrow 0\ ,$$

where $\theta_{\mathcal{C}/B}(-p)$ is the sheaf of vectorfields that are both vertical over B and tangent to p, i.e. the vertical vectorfields that vanish at p. By pushing down to B we obtain the exact sequence

$$0 \longrightarrow (\mathcal{C}/B)_*\theta_{\mathcal{C}/B}(-p) \longrightarrow (\mathcal{C}/B)_*\theta_{\mathcal{C}}(\log p) \longrightarrow (\mathcal{C}/B)_*(\mathcal{C}/B)^*\theta_B$$
$$\xrightarrow{\delta} R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B}(-p) \longrightarrow R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}}(\log p) \qquad ,$$

where the connecting map δ is called the Kodaira-Spencer map. Since projective curves only have constant global functions, $(\mathcal{C}/B)_*(\mathcal{C}/B)^*\theta_B = \theta_B$.

We call $(\mathcal{C}/B, p)$ *miniversal* if δ is an isomorphism. From Serre duality it follows that if $(\mathcal{C}/B, p)$ is miniversal, i.e. $\delta : \theta_B \simeq R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B}(-p)$, then we also have that $\Omega_B \simeq (\mathcal{C}/B)_* \operatorname{Sym}_2 \Omega_{\mathcal{C}/B}(p)$.

Lemma 1.1.2. *The Teichmüller family* $(\mathcal{C}_{g,N}/\mathcal{T}_{g,N}, p_{g,N})$ *is miniversal.*

For more details regarding this section, we refer the reader to [20]. *In the rest of this chapter, we fix a miniversal family of pointed curves* $(\mathcal{C}/B, p)$.

1.2 The moduli space of G-bundles over a curve

In the following, let G *be a simple complex algebraic group with Lie algebra* $\mathfrak{g}$. We remark that simple algebraic implies that $Z(G)$, the center of G, is finite. The approach taken in this section is based on [17]. Below we introduce the notions of principal G-bundle, families thereof, and subsequently review their moduli spaces; references for this material are [22, 1].

1.2.1 Principal bundles and stability

Let C be a curve, then $P/C : P \to C$ is called a *principal* G*-bundle* if P has a free proper right G-action that commutes with P/C and if the quotient of P by G is isomorphic to C. Two principal G-bundles $P/C, P'/C$ are called isomorphic if there is a G equivariant morphism $P \to P'$ that commutes with the projections. In particular, the right action of an element of $Z(G)$, the center of G, is an automorphism of P/C.

For any representation $\rho : G \to V$ of G, there is an associated vector bundle

$$\rho_{P/C} : P \times_G V \to C.$$

In particular for the adjoint representation, we have a vector bundle $\mathrm{Ad}_{P/C}$, simply called the *adjoint bundle*. The determinant of $\mathrm{Ad}_{P/C}$, i.e. its highest nontrivial exterior power, is trivial: the determinant of the adjoint representation of G is a character of G and hence trivial, since G is simple. Therefore $\deg \mathrm{Ad}_{P/C} = 0$ (the degree of a vector bundle is by definition the degree of its top nontrivial exterior power).

We define the *slope* of a vector bundle $\xi : F \to C$ over C by

$$\frac{\deg \xi}{\operatorname{rank} \xi}.$$

A vector bundle ξ is called *semi-stable* iff for any proper holomorphic subbundle ξ' we have

$$\frac{\deg \xi'}{\operatorname{rank} \xi'} \leq \frac{\deg \xi}{\operatorname{rank} \xi},$$

and it is called *stable* if the inequality is always strict. Moreover, we say that a semistable vector bundle ξ is *polystable* if it is a direct sum of subbundles of the same slope. Note that for $\mathrm{Ad}_{P/C}$ the stability condition reduces to $\deg \xi' < 0$ for any proper holomorphic subbundle ξ' of $\mathrm{Ad}_{P/C}$.

For principal G-bundles we also have the notion of stability, semistability and polystability, but they coincide with that of their adjoint bundles. A finer notion of stability for principal bundles is regular stability: we say that P/C is *regularly stable* if it is stable and its automorphism group is $Z(G)$. As was noted above, $Z(G)$ is always contained in the automorphism group, so regular stability means stable and minimal automorphism group.

1.2.2 $\mathcal{M}_G^0$

The notions above extend to families of G-bundles over curves. Of course, we must first explain what a family means in this case: a *family of* G*-bundles over* C *with base space* B, is a G-bundle $P \to C_B := C \times B$. Such a *family is called stable, semi-stable, regularly stable* if every $P_{C\times\{b\}}$ has that property, $b \in B$. A *morphism of families of* G*-bundles over* C from $P \to C_B$ to $P' \to C_{B'}$ is a map that commutes with the projections. Given a family $P/C_B : P \to C_B$ and a map $\phi : B' \to B$, we pull back P/C_BB along ϕ in the obvious manner and thus obtain a family $\phi^*P \to C_{B'}$. Finally, a family is called universal if every other family arises (up to unique isomorphism) as a pullback.

Let g denote the genus of the curve C. We list some facts about moduli spaces of G-bundles over C under the assumption that either $g \geq 3$ or $g = 2$ and G not of type $SL(2)$, $PSL(2)$.

The set of isomorphism classes of regularly stable G-bundles over C, denoted $M_G^{rs}(C)$, has a natural structure of a connected quasiprojective variety.

There is a universal G-bundle $P^u(C)/M_G^{rs}(C) \times C : P^u(C) \to M_G^{rs}(C) \times C$.

There is a natural compactification $M_G^{ss}(C)$ of $M_G^{rs}(C)$ with boundary of codimension ≥ 2, i.e. there is a natural (singular) projective variety $M_G^{ss}(C)$, together with an open embedding $M_G^{rs}(C) \hookrightarrow M_G^{ss}(C)$, such that the complement of the image is a closed subvariety of codimension ≥ 2.

For the reader who is familiar with moduli spaces, the compactification mentioned is obtained as the coarse moduli space of semi-stable G-bundles.

We now let the base curve C vary in a family: let $\mathcal{C}/B : \mathcal{C} \to B$ be a family of curves (we forget the points for the moment).

- There are respectively quasiprojective and projective morphisms

$$\mathcal{M}_G^{rs}(\mathcal{C}/B)/B : \mathcal{M}_G^{rs}(\mathcal{C}/B) \to B, \quad \mathcal{M}_G^{ss}(\mathcal{C}/B)/B : \mathcal{M}_G^{ss}(\mathcal{C}/B) \to B,$$

  both flat, and such that $\mathcal{M}_G^{rs}(\mathcal{C}/B)_b \simeq M_G^{rs}(\mathcal{C}_b)$, $\mathcal{M}_G^{ss}(\mathcal{C}/B)_b \simeq M_G^{ss}(\mathcal{C}_b)$, $b \in B$.

- There is a principal G-bundle $\mathcal{P}^u(\mathcal{C}/B) \to \mathcal{M}_G^{rs}(\mathcal{C}/B) \times_B \mathcal{C}$, such that the restriction to $\mathcal{M}_G^{rs}(C) \times C$ is isomorphic to $P_u(\mathcal{C}_b)$, $b \in B$.

We call $\mathcal{M}_G^{rs}(\mathcal{C}/B)/B$ the *family of moduli spaces of regularly stable* G*-bundles for the family* $\mathcal{C}/B$. In the next subsection we will discuss the (co)tangent sheaves of $M_G^{rs}(C)$ and $\mathcal{M}_G^{rs}(\mathcal{C}/B)/B$.

1.2.3 (Co)tangent spaces to the moduli space of G-bundles

Let C be a curve of genus $g > 1$. Abbreviate $P^u(C)$ to P and $M_G^{rs}(C)$ to M for the moment, and consider the diagram

$$\begin{array}{ccc} P & & \\ \downarrow & & \\ Y := M \times C & \xrightarrow{\pi_C} & C \\ \downarrow{\scriptstyle \pi_M} & & \\ M & & \end{array} .$$

We write $(\pi_{P/Y*}\theta_P)^G$ for the G-invariant part of $\pi_{P/Y*}\theta_P$. Then

$$0 \longrightarrow \mathrm{Ad}_{P/Y} \longrightarrow (\pi_{P/Y*}\theta_P)^G \longrightarrow \theta_Y \longrightarrow 0$$

is exact. Pushing down by π_M yields the exact sequence

$$0 \longrightarrow \pi_{M*}\mathrm{Ad}_{P/Y} \longrightarrow \pi_{M*}(\pi_{P/Y*}\theta_P)^G \longrightarrow \pi_{M*}\theta_Y \xrightarrow{\delta_G}$$

$$R^1\pi_{M*}\mathrm{Ad}_{P/Y} \longrightarrow R^1\pi_{M*}(\pi_{P/Y*}\theta_P)^G.$$

Lemma 1.2.1. *We have that*

$$\pi_{M*}\mathrm{Ad}_{P/Y} = 0, \quad \pi_{M*}\theta_Y = \theta_M,$$

and that the connecting map δ_G induces isomorphisms

$$\theta_M \simeq R^1\pi_{M*}\mathrm{Ad}_{P/Y}, \quad \Omega_M \simeq \pi_{M*}(\mathrm{Ad}^*_{P/Y} \otimes \pi_C^*\omega_C).$$

The first claim follows from a fiberwise argument: over every point $m \in M$, we have that $H^0(Y_m, \mathrm{Ad}_{P/Y})$ is the space of infinitesimal automorphism of P/Y_m, which must be equal to $Z(\mathfrak{g})$ since P/Y_m is regularly stable. But G is simple, so $Z(\mathfrak{g}) = 0$. For the second statement, note that $\theta_Y = \pi_M^*\theta_M \oplus \pi_C^*\theta_C$, and that $\pi_{M*}\pi_C^*\theta_C = H^0(C, \theta_C) \otimes \mathcal{O}_M = 0$, because genus > 1 curves do not have infinitesimal automorphisms. It follows that $\pi_{M*}\theta_Y = \pi_{M*}\pi_M^*\theta_M = \theta_M \otimes H^0(C, \mathcal{O}_C) = \theta_M$. The third statement does not admit such a short explanation; we refer the reader to the literature, e.g. [6],[7]. To derive the final statement, one just applies the Serre duality

$$R^1\pi_{M*}\mathrm{Ad}_{P/Y} = \pi_{M*}(\mathrm{Ad}^*_{P/Y} \otimes \pi^*_C\omega_{Y/M})^*,$$

and the identification $\omega_{Y/M} = \pi^*_C\omega_C$.

We now let our curve C vary in a family of genus g curves (we call this *the relative case*): for a fixed family of curves $\mathcal{C}/B : \mathcal{C} \to B$ we make the abbreviations $\mathcal{M} = \mathcal{M}^{rs}_G(\mathcal{C}/B)$, $P = P^u(\mathcal{C}/B)$ and have a commuting diagram

$$\begin{array}{ccc} P & & \\ \downarrow & & \\ \mathcal{Y} := \mathcal{M} \times_B \mathcal{C} & \xrightarrow{\pi_{\mathcal{C}}} & \mathcal{C} \\ \downarrow{\scriptstyle \pi_{\mathcal{M}}} & & \downarrow \\ \mathcal{M} & \longrightarrow & B \end{array} .$$

Furthermore, the short exact sequence

$$0 \longrightarrow \mathrm{Ad}_{P/\mathcal{Y}} \longrightarrow (\pi_{P/\mathcal{Y}*}\theta_{P/B})^G \longrightarrow \theta_{\mathcal{Y}/B} \longrightarrow 0$$

gives rise to a long exact sequence

$$0 \longrightarrow \pi_{\mathcal{M}*}\mathrm{Ad}_{P/\mathcal{Y}} \longrightarrow \pi_{\mathcal{M}*}(P/\mathcal{Y})_*\theta_{P/B})^G \longrightarrow \pi_{\mathcal{M}*}\theta_{\mathcal{Y}/B} \xrightarrow{\delta_G}$$

$$R^1\pi_{\mathcal{M}*}\mathrm{Ad}_{P/\mathcal{Y}} \longrightarrow R^1\pi_{\mathcal{M}*}(\mathcal{P}/\mathcal{Y})_*\theta_{P/B})^G.$$

Lemma 1.2.2. *We have that*

$$\pi_{1*}\mathrm{Ad}_{P/\mathcal{Y}} = 0, \quad \pi_{1*}\theta_{\mathcal{Y}/B} = \theta_{\mathcal{M}/B},$$

and that the connecting map δ_G induces isomorphisms

$$\theta_{\mathcal{M}} \simeq R^1\pi_{\mathcal{M}*}\mathrm{Ad}_{P/\mathcal{Y}}, \quad \Omega_{\mathcal{M}/B} \simeq \pi_{\mathcal{M}*}(\mathrm{Ad}^*_{P/\mathcal{Y}} \otimes \pi^*_{\mathcal{C}}\omega_{\mathcal{C}/B}).$$

These claims can be proved by applying lemma 1.2.1 over every point of B, but we will not make this precise.

1.2.4 Relation with moduli space of flat connections

The relation between moduli of holomorphic vector bundles and moduli of flat connections goes back to Narasimhan and Seshadri's paper [21]. Their results have been generalized in various ways, in particular to the setting of principal G-bundles; see e.g. [23], [1], [16].

Let $P/C : P \to C$ be a principal G-bundle over a curve and let G^c be a compact form of G. A C^∞*-connection on* P/C is a C^∞-map $\theta_C \to (P/C)_*^G\theta_P$ that is a section of the derivative of P/C. If G^c is a compact form of G, then we say that a principal G^c-bundle P^c/C is obtained by *reduction of the structure group of* P/C *to* G^c if P is C^∞-isomorphic to $P^c \times_{G^c} G$. A C^∞-connection on such a reduction induces one on P/C, and we say that a C^∞-connection on P/C is *Hermitian* if it is induced by a C^∞-connection on P^c/C, for some P^c/C as just described. Furthermore, a Hermitian connection on P/C is called *Einstein* if its curvature is central. Note that for a semi-simple Lie group the center of its Lie algebra is trivial, so that the Einstein condition amounts to flatness.

Theorem 1.2.3 ([1], Theorem 3.7)**.** *A principal* G*-bundle* P/C *is polystable iff it admits a Hermite-Einstein connection. This connection is unique if it exists.*

Another result of [1] says that if P/C is stable, then it is also polystable. From this we derive the following consequence

Corollary 1.2.4. *If* G *is a simple Lie group, then any regularly stable* G*-bundle over a curve admits a unique flat Hermitian connection.*

1.3 Loop groups

The following considerations are crucial for the rest of this thesis: it allows for the geometric data of pointed curves and principal G-bundles over them, to be translated into algebraic data. We will investigate this in greater detail in Chapter 3.

Let (C, p) be a N-pointed curve. Then $C^0 := C \backslash p$ is affine and we write $A(C^0) := H^0(C^0, \mathcal{O}_{C^0})$ for its ring of regular functions. Thus C is covered by the affine subschemes C^0 and the completion of C at p: $\widehat{C}_p = \text{Spec } \widehat{\mathcal{O}}_{C,p}$. Here $\widehat{\mathcal{O}}_{C,p}$ denotes the completion of the semilocal ring $\mathcal{O}_{C,p}$ with respect to the intersection of the ideals defining the $p_i \in p$. The intersection $\widehat{C}_p^0 = C^0 \cap \widehat{C}_p$ is also affine and equal to Spec $\widehat{\mathcal{K}}_{C,p}$, where $\widehat{K}_{C,p}$ is the ring of fractions of the semilocal ring $\widehat{\mathcal{O}}_{C,p}$. One can think of $\widehat{C}_p$ as an infinitesimal disk in C centered at p, and of $\widehat{C}_p^0$ as this disk with p deleted.

If $t_1, \ldots, t_N$ are local parameters at $p_1, \ldots, p_N$, respectively, then

$$\widehat{\mathcal{O}}_{C,p} = \bigoplus_{i=1}^{N} \mathbb{C}[[t_i]], \quad \widehat{\mathcal{K}}_{C,p} = \bigoplus_{i=1}^{N} \mathbb{C}((t_i)),$$

and $A(C^0)$ is some subalgebra of $\bigoplus i = 1^N \mathbb{C}((t_i))$.

Now suppose we are given a principal G-bundle $P/C : P \to C$, then P/C is trivial over C^0 and $\widehat{C}_p$ because they are affine. This can be seen as follows: take a faithful representation ρ of G and consider the bundle $\rho_{P/C}$. Since G is simple, it has trivial determinant. Furthermore, since C is a nonsingular curve, $A(C^0)$ is a Dedekind domain. Hence, over C^0, $\rho_{P/C}$ corresponds to a projective module over $A(C^0)$ with trivial determinant, so by ([11]), it is free. It follows that P is obtained by gluing the trivial bundles on the cover $C^0, \widehat{C}_p$ over the intersection $\widehat{C}_p^0$.

Definition 1.3.1.

$$\begin{aligned} LG(C,p) &:= \mathrm{Mor}_{\mathrm{Spec}\ \mathbb{C}}(\widehat{C}_p^0, G), \\ L^{\geq}G(C,p) &:= \mathrm{Mor}_{\mathrm{Spec}\ \mathbb{C}}(\widehat{C}_p, G), \\ LG(C^0) &:= \mathrm{Mor}_{\mathrm{Spec}\ \mathbb{C}}(C^0, G), \end{aligned}$$

where $\mathrm{Mor}_{\mathrm{Spec}\ \mathbb{C}}(\cdot,\cdot)$ denotes the group of morphisms of $\mathbb{C}$-schemes.

We call $LG(C,p)$ the loop group of G. Since $\widehat{\mathcal{O}}_{(C,p)} = \bigoplus i = 1^N \mathbb{C}[[t_i]]$ if t_i is a formal coordinate at p_i, $LG(C,p)$ and $L^{\geq}G(C,p)$ do not depend on (C,p) up to isomorphism. The isomorphisms however, are noncanonical, so we insist on including (C,p) in the notation.

Returning to our previous discussion, the transition function on $\widehat{C}_p^0$ between trivializations of P/C over $C^0, \widehat{C}_p$ is given by an element of $LG(C,p)$. Conversely, for any element of $LG(C,p)$ we can construct a principal G-bundle. Since changing a trivialization over C^0 or $\widehat{C}_p$ does not change the isomorphism type of the bundle, the element in $LG(C,p)$ corresponding to the isomorphism class of P is determined up to automorphisms over C^0 and $\widehat{C}_p$, i.e. up to elements of $LG(C^0)$ and $L^{\geq}G(C,p)$ respectively. When viewing this change of trivialization as an action of $LG(C,p)$, we have that $L^{\geq}G(C,p)$ and

$\mathrm{LG}(C^0)$ act on from different sides on $\mathrm{LG}(C,p)$ by multiplication - we convene that $\mathrm{LG}(C^0)$ acts on the left and $L^{\geq}G(C,p)$ on the right (this choice just depends on the ordering of the cover C^0, $\widehat{C}_p$). Hence the isomorphism types of principal G-bundles on C are in bijective correspondence with

$$\mathrm{LG}(C^0)\backslash \mathrm{LG}(C,p)/L^{\geq}G(C,p).$$

Even more is true:

Theorem 1.3.2. *(Y. Laszlo, C. Sorger [18])*
The moduli stack of principal G-bundles over C, denoted $\underline{\mathcal{M}}_G(C)$, is isomorphic to the quotient stack

$$\mathrm{LG}(C^0)\backslash \mathrm{LG}(C,p)/L^{\geq}G(C,p).$$

It is known that $\mathrm{LG}(C,p)$, $L^{\geq}G(C,p)$ and $\mathrm{LG}(C^0)$ are simply connected; this is a consequence of the fact that G, being a simple finite dimensional complex group, is 2-connected. Hence if we let $\mathrm{LG}^{rs}(C,p)$ be the open part of $\mathrm{LG}(C,p)$ corresponding to regularly stable bundles, then $\mathrm{LG}^{rs}(C,p)$ is connected: its quotient by the connected groups $L^{\geq}G(C,p)$ and $\mathrm{LG}(C^0)$, i.e. $\mathcal{M}_G^{rs}(C)$, is connected.

We extend the arguments above to families: let $(\mathcal{C}/B, p)$ be a family of N-pointed curves over B. Define the B-scheme

$$\mathcal{C}^0 := \mathcal{C}\backslash p,$$

and let $\mathcal{C}^0/B$ be its structure morphism.

Definition 1.3.3. We define the following $\mathcal{O}_B$ algebras:

$$\mathcal{O} := \widehat{\mathcal{O}}_{\mathcal{C},p}, \quad \mathcal{A} := (\mathcal{C}^0/B)_*\mathcal{O}_{\mathcal{C}}, \quad \mathcal{K} := \widehat{\mathcal{K}}_{\mathcal{C},p}$$

and B-schemes

$$\widehat{\mathcal{C}}_p/B : \widehat{\mathcal{C}}_p := \mathrm{Spec}\ \mathcal{O} \to B, \quad \widehat{\mathcal{C}}^0_p/B : \widehat{\mathcal{C}}^0_p := \mathrm{Spec}\ \mathcal{K} \to B.$$

Locally over B, $\mathcal{C}^0$ is always affine, so we can cover B by small open sets U such that $\mathcal{C}^0_U = \mathrm{Spec}\ \mathcal{A}(U)$. Since this is functorial in U, $\mathcal{C}^0 = \mathbf{Spec}_{\mathcal{O}_B}\mathcal{A}$. We note that the assignment $(\mathcal{O},\mathcal{K},\mathcal{A}) \mapsto (\widehat{\mathcal{C}}_p/B, \widehat{\mathcal{C}}^0_p/B, \mathcal{C}^0/B)$ is functorial.

Now let $U \subseteq B$ be as before (so $\mathcal{C}^0_U$ is affine), then define

$$\begin{aligned}
\mathcal{L}G(\mathcal{C}/B, p)(U) &:= \mathrm{Mor}(\widehat{\mathcal{C}}^0_p|_U, G),\\
\mathcal{L}^{\geq}G(\mathcal{C}/B, p)(U) &:= \mathrm{Mor}(\widehat{\mathcal{C}}_p|_U, G)\\
\mathcal{L}G(\mathcal{C}^0/B)(U) &:= \mathrm{Mor}(\mathcal{C}^0_U, G).
\end{aligned}$$

The assignments $U \mapsto \mathcal{L}G(\mathcal{C}/B, p)(U), \mathcal{L}^{\geq}G(\mathcal{C}/B, p)(U), \mathcal{L}G(\mathcal{C}^0/B)(U)$ define a presheaves over B, with values in (complex ind-) groups over B.

Definition 1.3.4. Let $\mathcal{L}G(\mathcal{C}/B, p)$, $\mathcal{L}^{\geq}G(\mathcal{C}/B, p)$, $\mathcal{L}G(\mathcal{C}^0/B)$ be the sheaves associated to the presheaves determined by

$$U \mapsto \mathcal{L}G(\mathcal{C}/B, p)(U), \mathcal{L}^{\geq}G(\mathcal{C}/B, p)(U), \mathcal{L}G(\mathcal{C}^0/B)(U),$$

respectively. We call these (sheaves of) *loop groups*.

If $(\mathcal{C}_b, p_b)$ is the pointed curve over a closed point $b \in B$, then

$$\mathcal{L}G_b = LG(C, p), \quad \mathcal{L}^{\geq}G_b = L^{\geq}G(C, p), \quad \mathcal{L}G(\mathcal{C}^0/B)_b = LG(C^0)(C, p).$$

The quotient stack

$$\underline{\mathcal{M}}_G(\mathcal{C}/B, p) := \mathcal{L}G(\mathcal{C}^0/B)\backslash\mathcal{L}G(\mathcal{C}/B, p)/\mathcal{L}^{\geq}G(\mathcal{C}/B, p)$$

is the moduli space of G-bundles for the family $\mathcal{C} \to B$, in the sense that for any $b \in B$, the restriction to b is the moduli stack of G-bundles for $\mathcal{C}_b$ as introduced before: $\underline{\mathcal{M}}_G(\mathcal{C}_b)$. If we let $\mathcal{L}G^{rs}(\mathcal{C}/B)$ be the subsheaf of $\mathcal{L}G(\mathcal{C}/B)$ corresponding to regularly stable G-bundles, then by definition, $\mathcal{L}^{\geq}G, \mathcal{L}G(\mathcal{C}^0/B) \subseteq \mathcal{L}G^{rs}$. Clearly the right action of $\mathcal{L}^{\geq}G$ on $\mathcal{L}G$ is free and preserves $\mathcal{L}G^{rs}$, and the same goes for the left action of $\mathcal{L}G(\mathcal{C}^0/B)$. Since left and right action commute, the right action of $\mathcal{L}G(\mathcal{C}^0/B)$ descends to the quotient $\mathcal{L}G^{rs}/\mathcal{L}^{\geq}G$. It is, however, not free: the stabilizer at a point of $\mathcal{L}G^{rs}$ corresponds to the automorphism of the bundle represented by the point. For the regularly stable part, this is precisely $Z(G)$, so the stabilizer of the $\mathcal{L}G(\mathcal{C}^0/B)$ action on $\mathcal{L}G^{rs}/\mathcal{L}^{\geq}G$ is $Z(G)$ (as constant sheaf over B). It follows that $\mathcal{L}G(\mathcal{C}^0/B)^{red} := \mathcal{L}G(\mathcal{C}^0/B)/Z(G)$ acts freely on $\mathcal{L}G^{rs}/\mathcal{L}^{\geq}G$ and that we have a quotient scheme

$$\mathcal{M}^{rs}_G(\mathcal{C}/B) := \mathcal{L}G(\mathcal{C}^0)(\mathcal{C}/B)^{red}\backslash\mathcal{L}G^{rs}(\mathcal{C}/B)/\mathcal{L}^{\geq}G(\mathcal{C}/B).$$

This is the moduli space of regularly stable G-bundles for the family $\mathcal{C}/B$ as mentioned before, so $\mathcal{M}_G^{rs}(\mathcal{C}/B) \to B$ is a quasiprojective morphism. For any affine $U \subseteq B$, $\mathcal{M}_G^{rs}(\mathcal{C}/B)_U$ has a natural compactification with complement of codimension ≥ 2, as was stated before. These compactification are compatible in the sense that the glue to a projective morphism $\mathcal{M}_G^{ss}(\mathcal{C}/B) \to B$, together with open embedding of B schemes $\mathcal{M}_G^{rs}(\mathcal{C}/B) \to \mathcal{M}_G^{ss}(\mathcal{C}/B)$ whose complement is of codimension ≥ 2.

1.3.1 The generator of the Picard group and $\widehat{LG}$.

References for this subsection are [16] and [4]. For a curve C, the Picard group of $M_G^{ss}(C)$ is infinitely cyclic. As was mentioned earlier, the codimension of $M_G^{rs}(C)$ in $M_G^{ss}(C)$ is ≥ 2. However, $M_G^{ss}(C)$ does not have a smooth boundary, so Hartogs' theorem does not apply here. Nevertheless, line bundles on $M_G^{rs}(C)$ *do extend uniquely* to line bundles on $M_G^{ss}(C)$, and the same holds for isomorphisms between them. Thus

$$\mathrm{Pic}(M_G^{rs}(C)) = \mathrm{Pic}(M_G^{ss}(C)) = \mathbb{Z},$$

the first identification being canonical since it is induced by pullback; the second one is canonical because there is only one ample generator. For the relative situation $\mathcal{C}/B$ we have the following: pullback gives inclusions $\mathrm{Pic}(B) \subset \mathrm{Pic}(\mathcal{M}_G^{rs}(\mathcal{C}/B) = \mathrm{Pic}(\mathcal{M}_G^{ss}(\mathcal{C}/B))$ and

$$\mathrm{Pic}_{\mathcal{C}/B}(\mathcal{M}_G^{rs}(\mathcal{C}/B)) := \mathrm{Pic}(\mathcal{M}_G^{ss}(\mathcal{C}/B))/\,\mathrm{Pic}(B) \simeq \mathbb{Z}.$$

1.3.2 $\widehat{\mathcal{L}\mathfrak{g}}$

Fix a family of pointed curves $(\mathcal{C}/B, p)$. For brevity, we use the following abbreviations:

$$\begin{aligned} \mathcal{M}_G^{rs} &= \mathcal{M}_G^{rs}(\mathcal{C}/B), & \mathcal{L}G &= \mathcal{L}G(\mathcal{C}/B, p), \\ \mathcal{L}G^{rs} &= \mathcal{L}G^{rs}(\mathcal{C}/B, p), & \mathcal{L}^{\geq}G &= \mathcal{L}^{\geq}G(\mathcal{C}/B, p). \end{aligned}$$

We can construct explicit representatives for $\mathrm{Pic}_{\mathcal{C}/B}(\mathcal{M}_G^{rs})$ using a certain central extension of $\mathcal{L}G$. There is in fact, a 1-1 correspondence between the characters of $\mathcal{L}^{\geq}G$, central extensions of $\mathcal{L}G$ induced from $\mathcal{L}^{\geq}G$ and classes of line bundles on $\mathcal{M}_G^{rs}$, but we will not completely elaborate this. We start with the infinitesimal description:

Definition 1.3.5.

$$\begin{aligned}\mathcal{L}\mathfrak{g}(\mathcal{C}/B,p) &:= \mathrm{Lie}(\mathcal{L}G(\mathcal{C}/B,p)),\\ \mathcal{L}^{\geq}\mathfrak{g}(\mathcal{C}/B,p) &:= \mathrm{Lie}(\mathcal{L}^{\geq}G(\mathcal{C}/B,p)) \subset \mathcal{L}\mathfrak{g},\\ \mathcal{L}\mathfrak{g}(\mathcal{C}^0/B) &:= \mathrm{Lie}(\mathcal{L}G(\mathcal{C}/B,p)) \subset \mathcal{L}\mathfrak{g},\end{aligned}$$

where the Lie algebra functor should be interpreted in the obvious sheafified manner. These $\mathcal{O}_B$-modules are called *loop algebras*. We use the abbreviations analogous to those for the loop groups: $\mathcal{L}\mathfrak{g} = \mathcal{L}\mathfrak{g}(\mathcal{C}/B,p)$ and $\mathcal{L}^{\geq}\mathfrak{g} = \mathcal{L}^{\geq}\mathfrak{g}(\mathcal{C}/B,p)$. The correspondence between morphisms of affine schemes and ring homomorphisms gives us

$$\begin{aligned}\mathcal{L}\mathfrak{g} &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{K}, \mathcal{O}_B \otimes_{\mathbb{C}_B} \mathfrak{g}_B) = \mathcal{K} \otimes_{\mathbb{C}} \mathfrak{g},\\ \mathcal{L}^{\geq}\mathfrak{g} &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{O}, \mathcal{O}_B \otimes_{\mathbb{C}_B} \mathfrak{g}_B) = \mathcal{O} \otimes_{\mathbb{C}} \mathfrak{g},\\ \mathcal{L}\mathfrak{g}(\mathcal{C}^0/B) &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{A}, \mathcal{O}_B \otimes_{\mathbb{C}_B} \mathfrak{g}_B) = \mathcal{A} \otimes_{\mathbb{C}} \mathfrak{g},\end{aligned}$$

where $\mathfrak{g}_B$ is the constant sheaf on B determined by $\mathfrak{g}$ (recall that $\mathfrak{g} = \mathrm{Lie}(G)$), and similarly $\mathbb{C}_B$ the one determined by $\mathbb{C}$.

For an open subset $U \subseteq B$, $\mathcal{L}\mathfrak{g}(U)$ is generated by elements of the form $X \otimes f$, where $X \in \mathfrak{g}$ and $f \in \mathcal{K}(U)$. If $Y \otimes g$ is another section of $\mathcal{L}\mathfrak{g}$ over U of the same form, then

$$[X \otimes f, Y \otimes g] = [X,Y] \otimes fg.$$

Remark 1.3.6. For our simple Lie group G, there is a unique G-invariant bilinear form tr on $\mathfrak{g} = \mathrm{Lie}(G)$ such that the longest root, with respect to a choice of Borel subalgebra and positive roots, has length 2. Since tr is a nonzero multiple of the Killing form, it is nondegenerate.

Definition 1.3.7. Denote by $\widehat{\mathcal{L}\mathfrak{g}}$ the $\mathcal{O}_B$-module $\mathcal{L}\mathfrak{g} \oplus \mathcal{O}_B$ with Lie algebra structure given by ($X \otimes f$ and $Y \otimes g$ as above)

$$[X \otimes f \oplus b, Y \otimes g \oplus b'] = [X,Y] \otimes fg \oplus 0 + 0 \oplus \mathrm{tr}(X,Y)\,\mathrm{Res}_p(g\,df).$$

We write c for the global section $0 \oplus 1$, so $\widehat{\mathcal{L}\mathfrak{g}}$ is a central extension of $\mathcal{L}\mathfrak{g}$ by $\mathcal{O}_B$ with central generator c.

By the residue theorem for closed curves,

$$\widehat{\mathcal{L}^{\geq}\mathfrak{g}} := \mathcal{L}^{\geq}\mathfrak{g} \oplus \mathcal{O}_B, \quad \widehat{\mathcal{L}\mathfrak{g}}(\mathcal{C}^0/B) := \mathcal{L}\mathfrak{g}(\mathcal{C}^0/B) \oplus 0$$

are $\mathcal{O}_B$-subalgebras of $\widehat{\mathcal{L}\mathfrak{g}}$. We will identify $\widehat{\mathcal{L}\mathfrak{g}}(\mathcal{C}^0/B)$ with $\mathcal{L}\mathfrak{g}(\mathcal{C}^0/B)$ and write the latter for its image in $\widehat{\mathcal{L}\mathfrak{g}}$.

1.3.3 The fundamental representation

For every integer $\ell > 0$, which we will call *level*, there is a representation of $\widehat{\mathcal{L}\mathfrak{g}}$, called the (sheaf of) fundamental representations of level ℓ. Consider $\mathcal{O}_B$ as an $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ representation, by letting $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ acts via the character $\ell\chi$, where

$$\chi : \widehat{\mathcal{L}^{\geq}\mathfrak{g}} = \mathcal{L}^{\geq}\mathfrak{g} \oplus \mathcal{O}_B :\to \mathcal{O}_B : (x, f) \mapsto f.$$

So c acts as multiplication with ℓ, and $\mathcal{L}^{\geq}\mathfrak{g}$ as 0. Induce this up to $\widehat{\mathcal{L}\mathfrak{g}}$, and let

$$\mathbb{F}(\widehat{\mathcal{L}\mathfrak{g}}, \widehat{\mathcal{L}^{\geq}\mathfrak{g}})_\ell := U\widehat{\mathcal{L}\mathfrak{g}} \otimes_{U\widehat{\mathcal{L}^{\geq}\mathfrak{g}}} \mathcal{O}_B$$

be the resulting Verma module[2], which has a global section $v_\ell := 1 \otimes 1$ that is an $U(\widehat{\mathcal{L}\mathfrak{g}})$ generator. There is a unique maximal proper submodule $\mathcal{Z}'_\ell$, and we write

$$\mathcal{H}_\ell(\widehat{\mathcal{L}\mathfrak{g}}, \widehat{\mathcal{L}^{\geq}\mathfrak{g}})_\ell := \mathbb{F}(\widehat{\mathcal{L}\mathfrak{g}}, \widehat{\mathcal{L}^{\geq}\mathfrak{g}})_\ell / \mathcal{Z}'_\ell$$

for the irreducible quotient sheaf; it is called *the fundamental representation of level* ℓ. A more explicit way to define this quotient over an affine subset of B is the following: let $t_1, \ldots, t_N$ be coordinates of $\mathcal{C}$ at p over some affine $U \subseteq B$, and let X_θ be a highest root vector of $\mathfrak{g}$ (relative some choice of positive roots). Then

$$\left((X_\theta \otimes t_i^{-1})^{\ell+1} \oplus 0\right) \circ v_\ell, \; i = 1, \ldots, N$$

is an N-tuple of local sections of $\mathcal{V}_\ell$, and $\mathcal{Z}'_\ell$ is the $\mathcal{V}_\ell$-submodule that is locally generated, over $U\widehat{\mathcal{L}\mathfrak{g}}$, by these sections.

1.3.4 $\widehat{LG}$

The representation of $\widehat{\mathcal{L}\mathfrak{g}}$ on $\mathcal{H}_\ell$ gives a morphism of $\mathcal{O}_B$-algebras $\widehat{\mathcal{L}\mathfrak{g}} \to \mathrm{End}_{\mathcal{O}_B}(\mathcal{H}_\ell)$, which induces an algebra morphism $\mathcal{L}\mathfrak{g} \to \mathrm{End}_{\mathcal{O}_B}(\mathcal{H}_\ell)/\mathcal{O}_B$. The latter representation integrates locally to a morphism $LG(U) \to PGL(\mathcal{H}_\ell(U))$, U being a suitably small open subset of B, and does so in a functorial manner. We write $\mathcal{P}GL(\mathcal{H}_\ell)$ for the sheaf associated to the presheaf $U \mapsto PGL(\mathcal{H}_\ell(U))$,

[2]Perhaps it would be better to write $U_{\mathcal{O}_B}\widehat{\mathcal{L}\mathfrak{g}}$ to indicate that we actually mean the universal enveloping algebra of $\widehat{\mathcal{L}\mathfrak{g}}$ regarded as $\mathcal{O}_B$-module, but we trust that the notation is clear to reader.

$U \subseteq B$ small enough. So we have a morphism $\mathcal{L}G \to \mathcal{P}GL(\mathcal{H}_\ell)$ of sheaves of (ind-)schemes over B. More details can be found [3] (where the origin of the idea is attributed to Faltings).

Definition 1.3.8. Define $\widehat{LG}$ to be the pull back of LG along the usual central extension $\mathcal{G}L(\mathcal{H}_1)$ of $\mathcal{P}GL(\mathcal{H}_1)$ by $\mathcal{O}_B^\times$, the invertible part of $\mathcal{O}_B$, so that the following diagram is Cartesian:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_B^\times & \longrightarrow & \widehat{\mathcal{L}G} & \longrightarrow & \mathcal{L}G & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{O}_B^\times & \longrightarrow & \mathcal{G}L(\widetilde{\mathcal{H}}_1) & \longrightarrow & \mathcal{P}GL(\mathcal{H}_1) & \longrightarrow & 1.
\end{array}$$

We let $\widehat{\mathcal{L}^{\geq}G}$, $\widehat{\mathcal{L}G}(\mathcal{C}^0/B)$ be the preimages of $\mathcal{L}^{\geq}G$, $\mathcal{L}G(\mathcal{C}^0/B)$ under the projection $\widehat{\mathcal{L}G} \to \mathcal{L}G$, respectively.

The corresponding diagram of Lie algebras is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_B & \longrightarrow & \widehat{\mathcal{L}\mathfrak{g}} & \longrightarrow & \mathcal{L}\mathfrak{g} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_B & \longrightarrow & \mathrm{End}_{\mathcal{O}_B}(\mathcal{H}_1) & \longrightarrow & \mathrm{End}_{\mathcal{O}_B}(\mathcal{H}_1)/\mathcal{O}_B & \longrightarrow & 0
\end{array}$$

where in the upper row, the map $\widehat{\mathcal{L}\mathfrak{g}} \to \mathcal{L}\mathfrak{g}$ comes from reduction modulo $\mathcal{O}_B c$.

By construction, $\widehat{\mathcal{L}^{\geq}G}$ contains the central part by which $\widehat{\mathcal{L}G}$ extends $\mathcal{L}G$, so it follows that $\widehat{\mathcal{L}G}/\widehat{\mathcal{L}^{\geq}G} = \mathcal{L}G/\mathcal{L}^{\geq}G$. Hence, if we let $\widehat{\mathcal{L}G}^{rs}$ be the part of $\widehat{\mathcal{L}G}$ projecting to $\mathcal{L}G^{rs}$, then

$$\mathcal{L}G(\mathcal{C}^0/B)^{red}\backslash\widehat{\mathcal{L}G}^{rs}/\widehat{\mathcal{L}^{\geq}G} = \mathcal{L}G(\mathcal{C}^0/B)^{red}\backslash\mathcal{L}G^{rs}/\mathcal{L}^{\geq}G = \mathcal{M}_G^{rs}.$$

1.3.5 $\mathcal{L}$

We are now ready to define a generator of $\mathrm{Pic}_{\mathcal{C}/B}(\mathcal{M}_G^{rs}(\mathcal{C}/B))$. Note that $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ leaves $\mathcal{O}_B v_1$ invariant, and therefore $\widehat{\mathcal{L}^{\geq}G}$ does so as well. This action defines a character

$$e^\chi : \widehat{\mathcal{L}^{\geq}G} \to \mathcal{O}_B^\times,$$

which by definition splits

$$1 \longrightarrow \mathcal{O}_B^\times \underset{\chi}{\overset{}{\rightleftarrows}} \widehat{\mathcal{L}^{\geq} G} \longrightarrow \mathcal{L}^{\geq} G \longrightarrow 1\,.$$

As the notation suggests, e^χ is the exponentiation of χ. Use χ^{-1} to lift the right action of $\widehat{\mathcal{L}^{\geq} G}$ on $\widehat{\mathcal{L}G}^{\mathrm{rs}}$ to an action on $\widehat{\mathcal{L}G}^{\mathrm{rs}} \times \mathbb{C}$, i.e.

$$(g, z) \cdot h = (gh, \chi(h)^{-1} z)$$

for local sections (g, z) of $\widehat{\mathcal{L}G}^{\mathrm{rs}} \times_B \mathcal{O}_B$ and h of $\widehat{\mathcal{L}^{\geq} G}$. This action commutes with the projection $\widehat{\mathcal{L}G}^{\mathrm{rs}} \times \mathbb{C} \to \widehat{\mathcal{L}G}^{\mathrm{rs}}$ and is free, so that

$$(\widehat{\mathcal{L}G}^{\mathrm{rs}} \times \mathbb{C})/\widehat{\mathcal{L}^{\geq} G} \to \widehat{\mathcal{L}G}^{\mathrm{rs}}/\widehat{\mathcal{L}^{\geq} G} \tag{1.1}$$

is a morphism of (ind-)schemes whose fibers are free $\mathcal{O}_B$-modules of rank 1, i.e. a line bundle.

We now lift the $\mathcal{L}G(\mathcal{C}^0/B)^{\mathrm{red}}$-action on $\widehat{\mathcal{L}G}^{\mathrm{rs}}/\widehat{\mathcal{L}^{\geq} G}$ to the domain of (1.1) using the trivial character: $f \cdot [(g, z)] = [(fg, z)]$ for local sections f of $\mathcal{L}G(\mathcal{C}^0/B)$ and (g, z) of $\widehat{\mathcal{L}G} \times \mathbb{C}$. Since χ is invariant under $Z(G)$, this action is indeed well defined. Hence, we have a morphism of B schemes:

Definition 1.3.9.

$$\begin{array}{rc} \mathcal{L} := & \mathcal{L}G(\mathcal{C}^0/B)^{\mathrm{red}} \backslash \widehat{\mathcal{L}G}^{\mathrm{rs}} \times \mathbb{C}/\widehat{\mathcal{L}^{\geq} G} \\ & \big\downarrow \\ & \mathcal{L}G(\mathcal{C}^0/B)^{\mathrm{red}} \backslash \widehat{\mathcal{L}G}^{\mathrm{rs}}/\widehat{\mathcal{L}^{\geq} G} = \mathcal{M}_G^{\mathrm{rs}} \end{array}$$

Over any affine $U \subseteq B$, $\widehat{\mathcal{L}G}^{\mathrm{rs}}(U) \times \mathbb{C} \to \widehat{\mathcal{L}G}^{\mathrm{rs}}(U)$ is a line bundle. Since the left action of $\mathcal{L}^{\geq} G(U)$ on $\widehat{\mathcal{L}G}^{\mathrm{rs}}(U) \times \mathbb{C}$ is free, and the right action on their quotient by $\mathcal{L}G(\mathcal{C}^0/B)(U)^{\mathrm{red}}$ as well, $\mathcal{L}$ is a line bundle over $\mathcal{M}_G^{\mathrm{rs}}(\mathcal{C}_U/U)$. Hence $\mathcal{L}$ is a line bundle over all of $\mathcal{M}_G^{\mathrm{rs}}$. This line bundle is the canonical positive generator of $\mathrm{Pic}_{\mathcal{C}/B}\, \mathcal{M}_G^{\mathrm{rs}}(\mathcal{C}/B)$. A section over some affine $U \subseteq B$ is a function $\theta : \widehat{\mathcal{L}G}(\mathcal{C}_U/U)^{\mathrm{rs}} \to \mathbb{C}$ that satisfies

$$\theta(fgh) = \theta(g)\chi(h), \quad f \in \mathcal{L}G(\mathcal{C}_U^0/U),\ g \in \widehat{\mathcal{L}G}(\mathcal{C}_U/U)^{\mathrm{rs}}, \quad h \in \widehat{L^{\geq} G}(\mathcal{C}_U/U).$$

Remark 1.3.10. Any element in $\mathrm{Pic}_{\mathcal{C}/B}(\mathcal{M}_G^{rs})$ is obtained as a power of $\mathcal{L}$. For an integer $\ell \in \mathbb{Z}$, we can define $\mathcal{L}^\ell$ in a way analogous to $\mathcal{L}$: if in the definition of $\mathcal{L}$ above, we use $(e^\chi)^\ell = e^{\ell\chi}$ instead of the character e^χ of $\widehat{\mathcal{L}^{\geq}G}$, the resulting line bundle is $\mathcal{L}^\ell$.

By the previous remark, the determinant bundle on $\mathcal{M}_G^{rs}(\mathcal{C}/B)$ is of course also given by a power of $\mathcal{L}$; we give this number explicitly.

Definition 1.3.11. Consider the projection $\pi_1 : \mathcal{C}_g = \mathcal{M}_G^{rs} \times_B \mathcal{C} \to \mathcal{M}_G^{rs}$ and define

$$\mathcal{L}_{\det} := \det R^\bullet \pi_{1*} \mathrm{Ad}_{P/\mathcal{C}_G} = \bigotimes_{i \geq 0} (\det R^i \pi_{1*} \mathrm{Ad}_{P/\mathcal{C}_G})^{(-1)^i},$$

where $\mathcal{P}$ was the universal G-bundle on $\mathcal{C}_G$.

This line bundle is called the Knudson-Mumford determinant of $\mathrm{Ad}_{\mathcal{P}/\mathcal{C}_G}$ over $\mathcal{M}_G^{rs}$. Since stable bundles do not admit sections, we have that $\pi_{1*}\mathrm{Ad}_{\mathcal{P}/\mathcal{C}_G} = 0$, and as a consequence

$$\mathcal{L}_{\det} = (\det R^1 \pi_{1*} \mathrm{Ad}_{\mathcal{P}/\mathcal{C}_G})^* = (\det \theta_{\mathcal{M}_G^{rs}}/B)^* = \omega_{\mathcal{M}_G^{rs}/B}.$$

The degree such a Knudson-Mumford determinant of a vector bundle, associated to $\mathcal{P}$ though a representation ρ, is given by the Dynkin index of ρ (see [16][3]). In the case of the adjoint representation this is minus twice the *dual Coxeter number* $\check{h}$, so

$$\omega_{\mathcal{M}_G^{rs}/B} = \mathcal{L}_{\det} = \mathcal{L}^{-2\check{h}}. \tag{1.2}$$

[3]Notice that our definition of Knudson-Mumford determinant is dual to the one in [16].

Chapter 2

Generalized θ-functions and Hitchin's connection

In this chapter, G *is a simple simply connected complex Lie group with dual Coxeter number* $\check{h}$*. We will assume an integer* $\ell > 0$*, called* level*, to be fixed.*

For a family of curves $\mathcal{C}/B$, we will construct a vector bundle Θ_ℓ over B called the Verlinde bundle. It comes with a natural flat projective holomorphic connection, nowadays called the Hitchin connection, which was first introduced in Hitchin's paper [14].

2.1 Generalized θ-functions

We recall the setting of the previous chapter: for a family of curves $\mathcal{C}/B$: $\mathcal{C} \to B$ we have a quasi-projective morphism $\mathcal{M}_G^{rs} \to B$, the family of moduli spaces of regularly stable G-bundles on $\mathcal{C}$, together with an open embedding over B into a projective morphism $\mathcal{M}_G^{ss} \to B$, with complement of codimension ≥ 2. In this chapter, we write $\mathcal{M} = \mathcal{M}_G^{rs}$ and $\overline{\mathcal{M}} = \mathcal{M}_G^{ss}$ for short.

Definition 2.1.1. The sheaf of generalized θ-functions (of level ℓ for the family $\mathcal{C} \to B$ with respect to the group G) is defined to be the $\mathcal{O}_B$-module

$$\Theta_\ell := (\mathcal{C}/B)_* \mathcal{L}^\ell.$$

A was stated in Chapter 1, line bundles on $\mathcal{M}$ extend uniquely to $\overline{\mathcal{M}}$; we will denote the extension of $\mathcal{L}^\ell$ to $\overline{\mathcal{M}}$ also by $\mathcal{L}^\ell$. Local sections of $\mathcal{L}^\ell$ over $\mathcal{M}$ also extend uniquely to $\overline{\mathcal{M}}$, and so $(\mathcal{M}/B)_*\mathcal{L}^\ell \simeq (\overline{\mathcal{M}}/B)_*\mathcal{L}^\ell$. Since $\mathcal{L}^\ell$ is coherent and $(\overline{\mathcal{M}}/B)$ proper, these sheaves are coherent $\mathcal{O}_B$-modules. In particular:

Corollary 2.1.2. *Θ_ℓ is coherent.*

2.2 The Hitchin connection

2.2.1 (Projective) connections

We briefly recall some general notions, and refer the reader to the appendix for details. Let X be a smooth complex scheme, Y a smooth scheme of finite type over X and $\mathcal{F}$ a coherent sheaf on Y. We denote by $\mathcal{D}^k_{Y/X}\mathcal{F}$ the sheaf of differential operators of order at most k on $\mathcal{F}$ over X. If $X = \text{Spec } \mathbb{C}$ we just write $\mathcal{D}^k_Y\mathcal{F}$ for $\mathcal{D}^k_{Y/X}\mathcal{F}$. From its definition it is clear that $\mathcal{D}^k_{Y/X}\mathcal{F} \subseteq \mathcal{D}^{k+1}_{Y/X}\mathcal{F}$; the corresponding quotients sequences are called *symbol sequences*, and the quotient maps are denoted σ^k and called *symbol maps*. For $k = 1$ this gives

$$0 \longrightarrow \mathrm{End}_{\mathcal{O}_X}\mathcal{F} \longrightarrow \mathcal{D}^1_{Y/X}\mathcal{F} \xrightarrow{\sigma^1(\mathcal{F})} \theta_{Y/X} \otimes \mathrm{End}_{\mathcal{O}_X}\mathcal{F} \longrightarrow 0\ .$$

By corollary A.1.6 from the appendix, $\sigma^1(\mathcal{F})$ is surjective iff $\mathcal{F}$ is locally free - assume this is the case. The corresponding exact sequence has a subsequence

$$0 \longrightarrow \mathrm{End}_{\mathcal{O}_Y}\mathcal{F} \longrightarrow \mathcal{A}_{Y/X}\mathcal{F} \xrightarrow{\sigma^1(\mathcal{F})} \theta_{Y/X}\ , \tag{2.1}$$

where $\mathcal{A}_{Y/X}\mathcal{F}$ is the preimage of $\theta_{Y/X} \otimes \text{id}$ under $\sigma^1(\mathcal{F})$. We call this exact sequence the *Atiyah* sequence and its extension class is called the *Atiyah class* and is denoted $c_A(\mathcal{F})$.

Note that $\mathcal{A}_{Y/X}\mathcal{F}$ has a natural Lie algebra structure, for it is a subalgebra of $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$. A *holomorphic connection over* X on $\mathcal{F}$ is a global right inverse of $\sigma^1(\mathcal{F})$ - clearly, this can only exist if $\sigma_1(\mathcal{F})$ is surjective. Such a holomorphic connection is called *flat* if it is an algebra morphism.
Assume that (2.1) is right-exact. Instead of the Atiyah sequence, one can consider its projectivization:

$$0 \longrightarrow \mathrm{End}^0_{\mathcal{O}_Y}\mathcal{F}/\mathcal{O}_Y \longrightarrow \mathcal{A}^0_{Y/X}\mathcal{F} \xrightarrow{\mathbb{P}\sigma^1(\mathcal{F})} \theta_{Y/X} \longrightarrow 0\ ,$$

where $\mathrm{End}^0_{\mathcal{O}_Y}\mathcal{F} = (\mathrm{End}_{\mathcal{O}_Y}\mathcal{F})/\mathcal{O}_Y$ and $\mathcal{A}^0_{Y/X}\mathcal{F} = (\mathcal{A}_{Y/X}\mathcal{F})/\mathcal{O}_Y$. A *projective holomorphic connection over* Y/X *on a coherent sheaf* $\mathcal{F}$ is a right inverse of $\mathbb{P}\sigma^1(\mathcal{F})$. Note that since $\mathcal{O}_Y$ is an ideal in $\mathcal{A}_{Y/X}(\mathcal{F})$, the quotient receives an algebra structure. We say that a projective holomorphic connection over X is *flat* if it is an algebra morphism. Clearly, a holomorphic connection determines a projective holomorphic connection. If the latter is flat, the former is called *projectively flat*, which in particular is the case if the holomorphic connection itself is flat. However, not every projective holomorphic connection comes from a right inverse of $\sigma^1(\mathcal{F})$.

2.2.2 Connections from heat operators

The following approach comes from [27] and is applied to our situation: the fibration $\mathcal{M} \to B$ and the line bundle $\mathcal{L}^\ell$ over $\mathcal{M}$. For a direct image sheaf such as Θ_ℓ, there is a special class of (projective) connections, namely those coming from heat operators on the sheaf which was pushed down. Morally speaking, heat operators are first order differential operators on the direct image, whose endomorphism part acts as fiberwise differential operators. This is made precise by the following:

Definition 2.2.1. Let $\mathcal{D}^k_{\mathcal{M}}\mathcal{L}^\ell$ be the sheaf of differential operators of order at most k on $\mathcal{L}^\ell$ over $\mathcal{M}$, and $\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell$ the subsheaf of differential operators on $\mathcal{L}^\ell$ of order at most k over $\mathcal{M}/B$.. We define

$$\mathcal{W}^k_{\mathcal{M}/B}(\mathcal{L}^\ell) := \mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell + \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell$$

to be the *sheaf of heat operators of order at most* k *on* $\mathcal{L}^\ell$ *with respect to* $\mathcal{M}/B$.

For brevity, we write $\mathcal{W}^k_\ell$ for $\mathcal{W}^k_{\mathcal{M}/B}\mathcal{L}^\ell$. Since $\mathcal{L}^\ell$ is locally free,

$$\mathcal{W}^k_\ell/\mathcal{W}^{k-1}_\ell = \mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell/\mathcal{D}^{k-1}_{\mathcal{M}/B}\mathcal{L}^\ell \simeq \mathrm{Sym}_k\theta_{\mathcal{M}/B} \otimes \mathrm{End}_{\mathcal{O}_{\mathcal{M}}}\mathcal{L}^\ell = \mathrm{Sym}_k\theta_{\mathcal{M}/B},$$

the isomorphism being given by $\sigma^k(\mathcal{L}^\ell)$. In other words, we have an exact sequence

$$0 \longrightarrow \mathcal{W}^{k-1}_\ell \longrightarrow \mathcal{W}^k_\ell \xrightarrow{w_k} \mathrm{Sym}^k\theta_{\mathcal{M}/B} \longrightarrow 0. \tag{2.2}$$

On the other hand, by taking the quotient of $\mathcal{W}^k_\ell$ with respect to $\mathcal{D}^k_{\mathcal{M}/B}(\mathcal{L}^\ell)$, we find

$$\mathcal{W}^k_\ell/\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell = \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell/(\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \cap \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell) = \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell/\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell \simeq (\mathcal{M}/B)^*\theta_B$$

and the associated exact sequence

$$0 \longrightarrow \mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow \mathcal{W}^k_\ell \xrightarrow{\sigma_n} (\mathcal{M}/B)^*\theta_B \longrightarrow 0\,. \tag{2.3}$$

The map σ_n is called the *normal symbol*. Apply $(\mathcal{M}/B)_*$ to (2.3) and obtain the following exact sequence:

$$0 \longrightarrow (\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow (\mathcal{M}/B)_*\mathcal{W}^k_\ell \xrightarrow{(\mathcal{M}/B)_*\sigma_n} \theta_B \longrightarrow 0\,, \tag{2.4}$$

where we used that

$$(\mathcal{M}/B)_*(\mathcal{M}/B)^*\theta_B = (\mathcal{M}/B)_*\mathcal{O}_\mathcal{M} \otimes_{\mathcal{O}_B} \theta_B = (\overline{\mathcal{M}}/B)_*\mathcal{O}_{\overline{\mathcal{M}}} \otimes_{\mathcal{O}_B} \theta_B = \theta_B.$$

Lemma 2.2.2. *We have a natural inclusion* $(\mathcal{M}/B)_*\mathcal{W}^k_\ell \subseteq \mathcal{A}_{\mathcal{M}/B}\Theta_\ell$*, such that* $(\mathcal{M}/B)_*\sigma_n$ *coincides with* $\sigma_1(\Theta_\ell)$*.*

Proof. Clearly, $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell$ commutes with the $\mathcal{O}_B$-action when acting on Θ_ℓ, so $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \subseteq \mathrm{End}_{\mathcal{O}_B}\Theta_\ell \subseteq \mathcal{D}^1_B\Theta_\ell$. Furthermore, if s, f, H are local sections of Θ_ℓ, $\mathcal{O}_B$ and $(\mathcal{M}/B)_*\mathcal{D}^1_\mathcal{M}\mathcal{L}^\ell$ respectively, then we have that

$$H(fs) = [H, f]s + fH(s) = \sigma^1(\Theta_\ell)(H)(f)s + fH(s)$$

by definition of $\sigma^1(\Theta_\ell)$, so that H acts as a first order differential operator on Θ_ℓ. Hence $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \subseteq \mathcal{D}^1_B\Theta_\ell$ and consequently $(\mathcal{M}/B)_*\mathcal{W}^k_\ell \subseteq \mathcal{D}^1_\mathcal{M}\Theta_\ell$. By interpreting a local section f of $\mathcal{O}_B$ as a local section of the sheaf $(\mathcal{M}/B)_*\mathcal{D}^1_\mathcal{M}\mathcal{L}^\ell$, we also have that $[H, f] = (\mathcal{M}/B)_*\sigma_n(H)(f)$, so that σ_n equals $\sigma_1(\Theta_\ell)$ on $(\mathcal{M}/B)_*\mathcal{D}^1_\mathcal{M}\mathcal{L}^\ell$. However, since both σ_n and $\sigma_1(\Theta_\ell)$ vanish on $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell$ so that they coincide on $(\mathcal{M}/B)_*\mathcal{W}^k_\ell$. Finally,

$$(\mathcal{M}/B)_*\mathcal{W}^k_\ell = ((\mathcal{M}/B)_*\sigma_n)^{-1}\theta_B \subseteq \sigma^1(\Theta_\ell)^{-1}(\theta_B) = \mathcal{A}_{\mathcal{M}/B}\Theta_\ell.$$ □

As an immediate consequence we obtain the following.

Corollary 2.2.3. *A (local) section of* $(\mathcal{M}/B)_*\sigma_n$ *provides a (local) holomorphic connection on* Θ_ℓ*.*

Such local sections, if they exist, need not be unique, as one can see from (2.4): they are affine with respect to sections of $\mathrm{Hom}_{\mathcal{O}_B}(\theta_B, (\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell)$.

Theorem 2.2.4. *Suppose* $k \geq 0$*, then* $\pi_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell = \mathcal{O}_B$*, so that if* $(\mathcal{M}/B)_*\mathcal{W}^k_\ell \to \theta_B$ *has sections, they must be unique up to* $\mathcal{O}_B$*.*

Proof. We will prove the statement by induction on k. For $k = 0$, the statement is true because $(\mathcal{M}/B)_* \mathcal{D}^0_{\mathcal{M}/B}\mathcal{L}^\ell = (\mathcal{M}/B)_*\mathcal{O}_\mathcal{M} = (\overline{\mathcal{M}}/B)_*\mathcal{O}_{\overline{\mathcal{M}}} = \mathcal{O}_B$, so suppose it is true for a $k \geq 0$. Consider $k+1$-th symbol sequence of $\mathcal{L}^\ell$

$$0 \longrightarrow \mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow \mathcal{D}^{k+1}_{\mathcal{M}/B}\mathcal{L}^\ell \xrightarrow{\sigma^k} \mathrm{Sym}_{k+1}\theta_{\mathcal{M}/B} \longrightarrow 0$$

and its quotient with respect to $\mathcal{D}^{k-1}_{\mathcal{M}/B}\mathcal{L}^\ell$,

$$0 \longrightarrow \mathrm{Sym}_k\theta_{\mathcal{M}/B} \longrightarrow \mathcal{D}^{k+1}_{\mathcal{M}/B}\mathcal{L}^\ell/\mathcal{D}^{k-1}_{\mathcal{M}/B}\mathcal{L}^\ell \xrightarrow{\overline{\sigma}^k} 0,$$

where the identification with $\mathrm{Sym}_k\theta_{\mathcal{M}/B}$ was made by σ^k. These give rise to connecting homomorphisms

$$(\mathcal{M}/B)_*\mathrm{Sym}_{k+1}\theta_{\mathcal{M}/B} \xrightarrow{\delta_{k+1}} R^1(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \;,$$

$$(\mathcal{M}/B)_*\mathrm{Sym}_{k+1}\theta_{\mathcal{M}/B} \xrightarrow{\widetilde{\delta}_{k+1}} R^1(\mathcal{M}/B)_*\mathrm{Sym}_k\theta_{\mathcal{M}/B}$$

By theorem A.2.1 from the appendix, $\widetilde{\delta}_{k+1}$ is given by cupping with $(k+1)c_A(\mathcal{L}^\ell) - kc_A(\omega_{\mathcal{M}/B})$. For a line bundle, the Atiyah class coincides with the first Chern class, so $c_A(\mathcal{L}^\ell) = c_1(\mathcal{L}^\ell) = \ell c_1(\mathcal{L})$. By formula (1.2) from the first chapter, $\omega_{\mathcal{M}/B} = \mathcal{L}^{-2\check{h}}$, so $c_A(\omega_{\mathcal{M}/B}) = -2\check{h}c_1(\mathcal{L})$. It follows that $\widetilde{\delta}_{k+1}$ is given by cupping with

$$(k+1)\ell c_1(\mathcal{L}) + k2\check{h}c_1(\mathcal{L}) = (\ell(k+1) + 2\check{h}k)c_1(\mathcal{L}).$$

With $\ell, \check{h} > 0$, we see that this is a nonzero multiple of $c_1(\mathcal{L})$. Since moreover it is known that, fiber wise over B, the first Chern class of $\mathcal{L}$ is represented by a Kähler form, $\widetilde{\delta}_{k+1}$ is injective. However, $\tilde{\delta}_{k+1} = R^1\sigma^k \circ \delta_{k+1}$, so δ_{k+1} must also be injective. From the exact sequence

$$0 \longrightarrow (\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow (\mathcal{M}/B)_*\mathcal{D}^{k+1}_{\mathcal{M}/B}\mathcal{L}^\ell \xrightarrow{(\mathcal{M}/B)_*\sigma^k}$$

$$(\mathcal{M}/B)_*\mathrm{Sym}_{k+1}\theta_{\mathcal{M}/B} \xrightarrow{\delta_{k+1}} R^1(\mathcal{M}/B)_*(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \;,$$

we can then conclude that $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \simeq (\mathcal{M}/B)_*\mathcal{D}^{k+1}_{\mathcal{M}/B}\mathcal{L}^\ell$. □

The previous theorem makes the following notions interesting:

Definition 2.2.5. We call $(\mathcal{M}/B)_*\mathcal{W}_\ell^k/\mathcal{O}_B$ the sheaf of *projective heat operators of order at most* k, and $\mathbb{P}(\mathcal{M}/B)_*\sigma_n$ the induced normal symbol map.

We thus arrive at an interesting uniqueness result, which was also noted in [24]:

Corollary 2.2.6. *For every* $k \geq 0$*: if* $\mathbb{P}(\mathcal{M}/B)_*\sigma_n : \mathcal{W}_\ell^k/\mathcal{O}_B \to \theta_B$ *has a (local) section, then it is unique and gives a (local) projective holomorphic connection on* Θ_ℓ*. Hence there can be at most one projective holomorphic connection that comes from a heat operator.*

Note that since $\mathcal{W}_\ell^k/\mathcal{O}_B \subseteq \mathcal{W}_\ell^{k+1}/\mathcal{O}_B$, and the map $\mathcal{W}_\ell^{k+1}/\mathcal{O}_B \to \theta_B$ extends $\mathcal{W}_\ell^k/\mathcal{O}_B \to \theta_B$, it follows that if $\mathcal{W}_\ell^k/\mathcal{O}_B \to \theta_B$ has a section, then $\mathcal{W}_\ell^{k+n}/\mathcal{O}_B \to \theta_B$ has a section for every $n \geq 0$.

2.2.3 Hitchin's connection

Below we will show the existence of a projective holomorphic connection under the following assumption: *suppose* $\mathcal{C}/B$ *is miniversal.* Consider the following setting: the Cartesian diagram

$$\begin{array}{ccc} \mathcal{Y} := \mathcal{M} \times_B \mathcal{C} & \xrightarrow{\pi_2} & \mathcal{C} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \mathcal{M} & \longrightarrow & B, \end{array}$$

the universal principal G-bundle $P/\mathcal{Y} : P \to \mathcal{Y}$ and its adjoint bundle $\mathrm{Ad}_{P/\mathcal{Y}}$. In Chapter 1 we saw that

$$\theta_{\mathcal{M}/B} = R^1\pi_{1*}\mathrm{Ad}_{P/\mathcal{Y}}, \quad \Omega_{\mathcal{M}/B} = \pi_{1*}\mathrm{Ad}^*_{P/\mathcal{Y}} \otimes \pi_2^*\Omega_{\mathcal{C}/B}.$$

Since we assumed the family $\mathcal{C}/B$ to be miniversal, the Kodaira-Spencer map is an isomorphism $\theta_B = R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B}$. We let tr be as in remark 1.3.6. By AdG invariance, it defines an morphism (of $\mathcal{O}_\mathcal{Y}$-modules) $\mathrm{Ad}_{P/\mathcal{Y}} \simeq \mathrm{Ad}^*_{P/\mathcal{Y}}$. Since tr is nondegenerate, this map is an isomorphism. Use this and the natural contraction between $\theta_{\mathcal{C}/B}$ and $\Omega_{\mathcal{C}/B}$ to produce the following map:

$$\begin{array}{ccccc} (\mathcal{M}/B)^*R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B} & & & & \\ \otimes & \longrightarrow & R^1\pi_{1*}\mathrm{Ad}^*_{P/\mathcal{Y}} & \longrightarrow & R^1\pi_{1*}\mathrm{Ad}_{P/\mathcal{Y}} \\ \pi_{1*}(\mathrm{Ad}^*_{P/\mathcal{Y}} \otimes \pi_2^*\Omega_{\mathcal{C}/B}) & & & & \| \\ \| & & & & \\ e(\mathcal{M}/B)^*\theta_B \otimes \Omega_{\mathcal{M}/B} & & & & \theta_{\mathcal{M}/B}, \end{array}$$

In [14], Hitchin makes the important observation that this map is nondegenerate, and therefore induces an injective map

$$H : (\mathcal{M}/B)^*\theta_B \to \mathrm{Sym}_2\theta_{\mathcal{M}/B}.$$

Remark 2.2.7. Suppose our family is pointed, i.e $(\mathcal{C}/B, p)$ is an N-pointed family of curves as defined before, for some choice of p. Then

$$\theta_B = R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B}(-p) \subseteq R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B} \tag{2.5}$$

The map $(\mathcal{M}/B)^*R^1(\mathcal{C}/B)_*\theta_{\mathcal{C}/B} \to \mathrm{Sym}_2\theta_{\mathcal{M}/B}$ remains well defined, so by taking the composition of the inclusion (2.5) and this map, we still have an injective map $(\mathcal{M}/B)^*\theta_B \to \mathrm{Sym}_2\theta_{\mathcal{M}/B}$.

By the previous remark, we can assume H to be defined and injective, regardless whether $\mathcal{C}/B$ is pointed or not. We pull the second symbol sequence of $\mathcal{L}^\ell$ over B back along H, and obtain an extension of $(\mathcal{M}/B)^*\theta_B$ by $\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell & \longrightarrow & \mathcal{D}^2_{\mathcal{M}/B}\mathcal{L}^\ell & \xrightarrow{\sigma^2} & \mathrm{Sym}_2\theta_{\mathcal{M}/B} & \longrightarrow & 0\,. \\ & & \| & & \uparrow & & \uparrow H & & \\ 0 & \longrightarrow & \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell & \longrightarrow & (\sigma^2)^{-1}(\mathrm{im}\, H) & \xrightarrow{H^*\sigma^2} & (\mathcal{M}/B)^*\theta_B & \longrightarrow & 0 \end{array}$$

As we saw previously (for $k = 1$), the extension class defined by the upper sequence in $\mathrm{Ext}^1_{\mathcal{O}_B}((\mathcal{M}/B)^*\theta_B, \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)$ was given by cupping with $(2\ell + 2\check{h})c_1(\mathcal{L})$, so the lower one is given by cupping with $(2\ell + 2\check{h})H^*c_1(\mathcal{L})$. We now compare this with the extension class that

$$0 \longrightarrow \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell \xrightarrow{\sigma_n} (\mathcal{M}/B)^*\theta_B \longrightarrow 0\,. \tag{2.6}$$

determines in $\mathrm{Ext}^1_{\mathcal{O}_B}(((\mathcal{M}/B)^*\theta_B, \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell))$ - denote this by $e_n(\mathcal{L}^\ell)$.

Theorem 2.2.8. $e_n(\mathcal{L}^\ell) = \ell\ H^* c_A(\mathcal{L})$.

This result was first obtained by Hitchin in [14] - though in a somewhat different form - using complex differential geometric methods. More recently, in [24] Sun and Tsai gave a different proof using sheaf cohomology and Beilinson-Schechtman's *trace complex* introduced in [5].

As a consequence of theorem 2.2.8,

$$0 \longrightarrow \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow (\sigma^2)^{-1}(\operatorname{im} H) \xrightarrow{\frac{2\ell+2\check{h}}{\ell} H^*\sigma^2} (\mathcal{M}/B)^*\theta_B \longrightarrow 0$$

defines the same class in $\operatorname{Ext}^1_{\mathcal{O}_B}((\mathcal{M}/B)^*\theta_B, \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)$ as (2.6). By subtracting them, we obtain an extension of $(\mathcal{M}/B)_*\theta_B$ by $\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell$ that is trivial: consider the maps

$$0 \longrightarrow (\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)^{\oplus 2} \xrightarrow{i_\oplus} (\sigma^2)^{-1}(\operatorname{im} H) \oplus \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell \xrightarrow{p_\oplus} ((\mathcal{M}/B)^*\theta_B)^{\oplus 2} \longrightarrow 0$$

where $i_\oplus$ is the direct sum of the given inclusions, and $p_\oplus = (-\frac{2\ell+2\check{h}}{\ell}\sigma_n) \oplus \sigma_n$. Now let $\Delta((\mathcal{M}/B)^*\theta_B)$ denote the diagonal in $(\mathcal{M}/B)^*\theta_B \oplus (\mathcal{M}/B)^*\theta_B$, and $\Delta^a(\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)$ the anti-diagonal in $\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell \oplus \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell$. Then define

$$\widetilde{\mathcal{W}} := \frac{p_\oplus^{-1}\Delta((\mathcal{M}/B)^*\theta_B)}{\Delta^a(\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)}.$$

We observe that the kernel of the natural map

$$(\sigma^2)^{-1}(\operatorname{im} H) \oplus \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell \subseteq \mathcal{D}^2_{\mathcal{M}/B}\mathcal{L}^\ell \oplus \mathcal{D}^1_{\mathcal{M}}\mathcal{L}^\ell \to \mathcal{W}^2_\ell$$

is precisely $\Delta^a(\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)$, so that we have a natural injection $\widetilde{W} \hookrightarrow \mathcal{W}^2_\ell$. Identifying

$$\begin{aligned} \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell &= (\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell)^{\oplus 2}/\Delta^a(\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell) \\ (\mathcal{M}/B)^*\theta_B &= \Delta((\mathcal{M}/B)^*\theta_B) \end{aligned}$$

we have that $i_\oplus, p_\oplus$ define maps $i : \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell \to \widetilde{\mathcal{W}}$ and $p : \widetilde{\mathcal{W}} \to (\mathcal{M}/B) * \theta_B$ which are injective, surjective, respectively.

This construction gives us an extension which is embedded in the second symbol sequence of the heat operators of $\mathcal{L}^\ell$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^2_{\mathcal{M}/B}\mathcal{L}^\ell & \longrightarrow & \mathcal{W}^2_\ell & \longrightarrow & \pi^*\theta_B & \longrightarrow & 0\,. \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & \mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell & \xrightarrow{i_\oplus} & \widetilde{\mathcal{W}} & \xrightarrow{p_\oplus} & \pi^*\theta_B & \longrightarrow & 0 \end{array}$$

By construction, the extension class of the lower sequence is trivial, so that $(\mathcal{M}/B)_*$ is right exact when applied to the lower sequence, and we find an embedded short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathcal{M}/B)_*\mathcal{D}^2_{\mathcal{M}/B}\mathcal{L}^\ell & \longrightarrow & (\mathcal{M}/B)_*\mathcal{W}^2_\ell & \xrightarrow{(\mathcal{M}/B)_*\sigma_n} & \theta_B & & \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & (\mathcal{M}/B)_*\mathcal{D}^1_{\mathcal{M}/B}\mathcal{L}^\ell & \longrightarrow & (\mathcal{M}/B)_*\widetilde{\mathcal{W}} & \xrightarrow{(\mathcal{M}/B)_*p_\oplus} & \theta_B & \longrightarrow & 0. \end{array} \tag{2.7}$$

It follows that $(\mathcal{M}/B)_*p$ locally has sections, and that such local sections provide local sections of $(\mathcal{M}/B)_*\sigma_n$. In particular, $(\mathcal{M}/B)_*\sigma_n$ is surjective.

Corollary 2.2.9 (local Hitchin connection)**.** *The normal map*

$$(\mathcal{M}/B)_*\sigma_n : (\mathcal{M}/B)_*\mathcal{W}^2_\ell \to \theta_B$$

is surjective so that

$$0 \longrightarrow (\mathcal{M}/B)_*\mathcal{D}^2_{\mathcal{M}/B}\mathcal{L}^\ell \longrightarrow (\mathcal{M}/B)_*\mathcal{W}^2_\ell \xrightarrow{(\mathcal{C}/B)_*\sigma_n} \theta_B \longrightarrow 0$$

is exact. As a consequence, Θ_ℓ *locally has a connection given by heat operators, and in particular,* Θ_ℓ *is locally free.*

In corollary 2.2.6 we saw that, under certain conditions, $(\mathcal{M}/B)_*\mathcal{D}^k_{\mathcal{M}/B}\mathcal{L}^\ell \simeq \mathcal{O}_B$, so the last corollary implies that

$$\mathbb{P}(\mathcal{M}/B)_*\sigma_n : \mathcal{W}^2_\ell/\mathcal{O}_B \to \theta_B$$

is an isomorphism, and hence has a unique global section.

Corollary 2.2.10 (Hitchin projective connection)**.** *We have that* Θ^ℓ *has a flat projective holomorphic connection, that is given by*

$$0 \longrightarrow \mathrm{End}^0_{\mathcal{O}_B}\Theta_\ell \longrightarrow \mathcal{A}^0_{\mathcal{M}/B}\,\Theta^\ell \underset{\curvearrowleft\,\nabla^H}{\longrightarrow} \theta_B \longrightarrow 0\,,$$

where $\nabla^H := \mathbb{P}((\mathcal{M}/B)_*\sigma_n)^{-1}$*. It is uniquely determined by the property that it can be locally given by a heat operator.*

It just remains to explain the flatness, so let X, Y be local vectorfields on B. Then, at least locally, there are local sections $W_X, W_Y, W_{[X,Y]}$ of $(\mathcal{M}/B)_*\sigma_n$ that, respectively, lift $X, Y, [X, Y]$. By direct computation one can verify that $[W_X, W_Y]$ is a local section of $(\mathcal{M}/B)_*\mathcal{W}^3_\ell$ and that $(\mathcal{M}/B)_*\sigma_n([\nabla^H_X, \nabla^H_Y]) = [X, Y]$. Hence the normal symbol of $[W_X, W_Y] - W_{[X,Y]}$ vanishes, so that it is in fact a local section of $(\mathcal{M}/B)_*\mathcal{D}^3_{\mathcal{M}/B}\mathcal{L}^\ell$. However, $(\mathcal{M}/B)_*\mathcal{D}^3_{\mathcal{M}/B}\mathcal{L}^\ell = \mathcal{O}_B$ by theorem 2.2.4, so that the image of $[W_X, W_Y] - W_{[X,Y]}$ in $\mathrm{End}^0_{\mathcal{O}_B}\Theta_\ell$ vanishes.

Chapter 3

Algebraic data of G-bundles over pointed curves

Fix a simple and simply connected complex Lie group G. In this chapter we investigate certain algebraic data determined by the following geometric data: a family of pointed curves over a smooth complex base space, that is endowed with a principal G-bundle. More precisely, we will consider the following: a smooth finite dimensional complex manifold B, an N-pointed family of curves $(\mathcal{C}/B, p)$, $N \geq 1$, and a principal G-bundle $P/\mathcal{C} : P \to \mathcal{C}$. To this we shall associate certain $\mathcal{O}_B$-modules.

3.1 The data of an affine family of pointed curves

We recall the natural affine open cover of $\mathcal{C}$ that was introduced in Chapter 1: let $\widehat{\mathcal{C}}_p$ be the completion of p in $\mathcal{C}$, and let $\mathcal{C}^0$ be given by $\mathcal{C}\backslash p$. The intersection $\mathcal{C}^0 \cap \widehat{\mathcal{C}}_p$ is denoted $\widehat{\mathcal{C}}_p^0$. We have that $\widehat{\mathcal{C}}_p, \widehat{\mathcal{C}}_p^0, \mathcal{C}^0$ are B-schemes, and that

$$\mathcal{O} := (\widehat{\mathcal{C}}_p/B)_* \mathcal{O}_{\widehat{\mathcal{C}}_p}, \quad \mathcal{A} := (\mathcal{C}^0/B)_* \mathcal{O}_{\mathcal{C}^0}, \quad \mathcal{K} := (\widehat{\mathcal{C}}_p^0/B)_* \mathcal{O}_{\widehat{\mathcal{C}}_p^0},$$

are quasi-coherent $\mathcal{O}_B$-algebras. The restriction map $\mathcal{C}^0 \to \widehat{\mathcal{C}}_p^0$ induces an injective map of sheaves $\mathcal{A} \hookrightarrow \mathcal{K}$, so that we can identify $\mathcal{A}$ with an $\mathcal{O}_B$-subalgebra of $\mathcal{K}$. Similarly, the restriction $\widehat{\mathcal{C}}_p^0 \to \widehat{\mathcal{C}}_p^0$ allows us to identify $\mathcal{O}$ with an $\mathcal{O}_B$-subalgebra of $\mathcal{K}$. There is another $\mathcal{O}_B$-subalgebra of $\mathcal{K}$ that we will often use:

$$\mathcal{K}^0 := \text{Ker } d_{\mathcal{K}/\mathcal{O}_B} \subseteq \mathcal{K}.$$

As $\mathcal{O}_B$-algebra, $\mathcal{O}$ has precisely N maximal ideals, which correspond to the points $p_1, \ldots, p_N$ - *let* $\mathfrak{m}_i$ *be the maximal ideal corresponding to* p_i. We denote the intersection of these by

$$\mathfrak{m} := \cap_i \mathfrak{m}_i.$$

Clearly, the inclusion $\mathcal{K}^0 \subseteq \mathcal{O}$ induces an isomorphism $\mathcal{K}^0 = \mathcal{O}/\mathfrak{m}$.

For a small affine open $U \subseteq B$, we have formal coordinates $(t_1, \ldots, t_N)$ along $p = (p_1, \ldots, p_n)$ over U. Denoting $\mathcal{O}_B(U)$ by R, we have that

$$\mathcal{O}(U) = \bigoplus_{i=1}^{N} R[[t_i]] \subseteq \mathcal{K}(U) = \bigoplus_i R((t_i))$$

and also $\mathcal{A}(U) \subseteq \bigoplus_i R((t_i))$. Furthermore,

$$\mathfrak{m}_i = t_i\mathcal{O}(U), \quad \mathfrak{m} = \sum_i t_i\mathcal{O}(U), \mathcal{K}^0 = \oplus_i R,$$

where in the last sum the i-th summand corresponds p_i. Hence, up to isomorphism, all the information of $(\mathcal{C}/B, p)$, restricted to U, is encoded by the subalgebra $\mathcal{A}(U)$ of $\sum_i R((t_i))$.

Lemma 3.1.1. *The map sending* $(\mathcal{C}/B, p)$ *to the quasi-coherent* $\mathcal{O}_B$*-algebras* $\mathcal{O}, \mathcal{A} \subseteq \mathcal{K}$ *as above is functorial. It has a left inverse if we forget the ordering of* $p = (p_1, \ldots, p_N)$.

Proof. The functoriality is obvious from the definition. For the left inverse, let $(\mathcal{C}/B, p)$ be a family of N-pointed curves giving rise to quasi coherent $\mathcal{O}_B$-algebras $\mathcal{O}, \mathcal{A} \subseteq \mathcal{K}$ as above. Then, continuing with the notation introduced above, we have that $\mathcal{C}^0 = \textbf{Spec } \mathcal{A}$, $\widehat{\mathcal{C}}_p = \textbf{Spec } \mathcal{O}$ and $\widehat{\mathcal{C}}_p^0 = \textbf{Spec } \mathcal{K}$. Moreover, the maps $\widehat{\mathcal{C}}_p^0 \to \widehat{\mathcal{C}}_p$, $\widehat{\mathcal{C}}_p^0 \to \mathcal{C}^0$ are the ones determined by the inclusions $\mathcal{O}, \mathcal{A} \subseteq \mathcal{K}$. Hence $\mathcal{C} = \textbf{Spec } \mathcal{O} \times_{\textbf{Spec } \mathcal{K}} \textbf{Spec } \mathcal{A}$. The points $p_1, \ldots, p_N$ are determined by the maximal $\mathcal{O}_B$ ideals of $\mathcal{O}$, though without ordering. □

Example 3.1.2. Suppose $B = \text{Spec } R$. Let $f(x) \in R[x]$ be of the form $(x - r_1) \cdots (x - r_{2g+1})$, $r_i \in R$ and $g > 0$, such that there are no r_i, r_j that are simultaneously zero on Spec R. So f is family of polynomials over B with distinct roots. Then $A = R[x,y]/(y^2 - f(x))$ defines an affine family $\mathcal{C}^0 :=$ Spec A of plane curves over B, whose regular functions are given by $R[x] + R[x]y$. Let $\widetilde{\mathcal{C}}$ be the closure of this curve in $\mathbb{P}^2 \times B$, then it consists of $\mathcal{C}^0$ and a single (R-) point p at infinity. This point is singular, for every curve in the family $\widetilde{\mathcal{C}}/B$. Let $\mathcal{C}$ be the normalization of $\widetilde{\mathcal{C}}$, then $\mathcal{C}$ is smooth and has a closed R point determined by p - we also call it p. Near p, we can find a parameter t such that $x = t^{-2}$ and $y = t^{-1-2g}u(t)$, with $u(t)$ a root of $t^{2(2g+1)}f(t^{-1})$ in $R[[t]]$. In this case $\mathcal{O}(B) = R[[t]] \subseteq \mathcal{K}(B) = R((t))$ and $\mathcal{A}(B) = R[x] + R[x]y = R[t^{-2}] + t^{-1-2g}u(t)$.

From the lemma above, we see that a pointed family of curves over B corresponds to a certain triple of $\mathcal{O}_B$-algebras. Therefore, the cohomology data of the former must also be expressible in the latter. In the next subsection, we will show how this is done.

3.1.1 Cohomology of an affine family of pointed curves

We will first study the cohomology of a single pointed curve in greater detail; after that we treat the relative case in lesser detail. The reason for this approach is the following: the arguments used for the relative case are local variations of the ones used in the absolute case, but the notation used in the relative case is more involved than in the absolute case. Hopefully, after reading the part concerned with the absolute case, the reader can fill in the details for the relative case. In the rest of this chapter, we will repeatedly use this approach.

So, let $(C/\operatorname{Spec}\mathbb{C}, p)$ be an N-pointed curve. In this case, we will identify $\mathcal{O}, \mathcal{K}, \mathcal{A}, \mathcal{K}^0$ with their global sections, so that they are algebras over $\mathbb{C}$. Consider the singular cohomology of the pair (C, C^0), $C^0 = C \backslash p$ with complex coefficients and focus on the following part of the long exact sequence of the pair (C, C^0):

$$H^1(C, C^0; \mathbb{C}) \longrightarrow H^1(C; \mathbb{C}) \longrightarrow H^1(C^0; \mathbb{C}) \longrightarrow H^2(C, C^0; \mathbb{C}) \longrightarrow$$
$$\longrightarrow H^2(C; \mathbb{C}) \longrightarrow H^2(C^0; \mathbb{C}).$$

Using the Thom isomorphism $H^k(C, C^0; \mathbb{C}) = H^{k-2}(p; \mathbb{C})$, we see that

$$H^k(C, C^0; \mathbb{C}) = \begin{cases} 0 & k \geq 2 \\ \mathcal{K}^0 & k = 0. \end{cases}$$

Using the labeling of the points, we can identify $\mathcal{K}^0$ with $\mathbb{C}^N$.

By Poincare duality, we have that $H^2(C;\mathbb{C}) \simeq H_0(C;\mathbb{C}) \simeq \mathbb{C}$ and by retracting C^0 to a 1 skeleton we see that $H^2(C^0;\mathbb{C}) = 0$. Thus, the exact sequence above simplifies to

$$0 \longrightarrow H^1(C;\mathbb{C}) \longrightarrow H^1(C^0;\mathbb{C}) \xrightarrow{w} \mathcal{K}^0 \xrightarrow{\sum_{i=1}^N} \mathbb{C} \longrightarrow 0\,. \tag{3.1}$$

In terms of De Rham representatives, the map $\vec{w}$ can be computed explicitly: take an element of $H^1(C^0;\mathbb{C})$ and represent it by a closed smooth 1-form α. Also, choose small open neighborhoods V_i, U_i of p_i diffeomorphic to B^2 such that $\overline{V}_i \subseteq U_i$, and denote $U = \bigcup_i U_i, V = \bigcup_i U_i$. We call a C^∞-function $\varphi : C \to [0,1]$ a *smooth cutoff function at* $p_1, \ldots, p_N$, *subordinate to* U, V if φ is equal to 1 on $C\backslash U$, and if φ is identically 0 on V - choose such U, V, φ. Then $\varphi\alpha$ has the same periods as α on $C\backslash U$, so that it represents the same class in $H^1(C\backslash U;\mathbb{C}) = H^1(C^0;\mathbb{C})$. However, since φ vanishes near p, $d(\varphi\alpha)$ is defined over C and therefore defines an element in $H^2(U, \partial U;\mathbb{C}) \simeq H^2(B^2, S^1;\mathbb{C})^N$. But $H_2(B^2, S^1;\mathbb{C})$ has a canonical generator given by B^2, so $H^2(B^2, S^1;\mathbb{C}) = H_2(B^2, S^1;\mathbb{C})^*$ can be canonically identified with $\mathbb{C}$ by the integration paring. Choose diffeomorphisms $\kappa_i : U_i \to B^2$ such that $\kappa_i(p_i) = 0$, denote $\alpha_i = (\kappa_i)^*\alpha$ and $\varphi_i = (\kappa_i)^*\varphi$. Then the i-th component of $w(\alpha)$ is

$$\int_{U_i} d(\varphi\alpha) = \int_{B^2} d(\varphi_i\alpha_i) = \int_{S^1} \varphi\alpha_i = \int_{S^1} \alpha_i,$$

i.e. $w(\alpha)_i$ is the winding number of α at p_i. Note that

$$\sum_i w(\alpha)_i = \int_{\partial(C\backslash U)} d(\alpha) = 0,$$

so that $w(\alpha)$ lies in the hyperplane of $\mathbb{C}^N$ with zero component sum.

To compute the cohomology of C^0, can also use a holomorphic resolution: since C^0 is affine, $H^{0,1}(C^0) = H^1(C^0, \mathcal{O}_{C^0}) = 0$, so $H^1(C^0;\mathbb{C}) = H^{1,0}(C^0)$. Therefore the complex

$$0 \longrightarrow \mathcal{O}(C^0) \xrightarrow{d} \omega(C^0) \longrightarrow 0\,,$$

computes the cohomology of C^0. Here $\mathcal{O}(X)$ and $\omega(X)$ denote, respectively, the holomorphic functions and holomorphic 1-forms of a complex variety X.

Hence, for the first cohomology group we have

$$H^1(C^0;\mathbb{C}) = \omega(C^0)/\, d\mathcal{O}(C^0). \tag{3.2}$$

Thus if we represent an element of $H^1(C^0;\mathbb{C})$ by a holomorphic 1-form on C^0, w sends it to the vector of $2\pi i$ times residues at the points. The image of $H^{1,0}(C;\mathbb{C}) = \omega(C)$ in $H^1(C^0;\mathbb{C}) = \omega(C^0)/\, d\mathcal{O}(C^0)$ can now also be understood as those holomorphic 1-forms on C^0 that extend to C, and consequently, the nontrivial elements of $H^{0,1}(C;\mathbb{C})$ are represented by holomorphic 1-forms on C^0 that do *not* extend to C.

Consider the subset V of elements of $\omega(C^0)$ that represent the image of $H^1(C;\mathbb{C})$. By identifying elements of $\omega(C^0)$ with their restrictions to $\widehat{\mathcal{C}}^0_p$, we can consider the subset $d^{-1}V \subseteq \mathcal{K}$, i.e. the elements of $\mathcal{K}$ whose differential is the restriction of a holomorphic differential on C^0, that corresponds to an element in $H^1(C;\mathbb{C})$. Since a differential on $\widehat{\mathcal{C}}^0_p$ can always be integrated locally, the image of $d(d^{-1}V)$ in $\omega(C^0)/\mathcal{O}(C^0)$ coincides with that of $H^1(C;\mathbb{C})$; we identify these two. The kernel of the map $d : (d^{-1}V) \to H^1(C;\mathbb{C})$ is $\mathcal{A} + \mathcal{K}^0 \subseteq \mathcal{K}$, so that $H^1(C;\mathbb{C}) = d^{-1}(V)/(\mathcal{A} + \mathcal{K}^0)$ where the isomorphism is induced by d.

We now generalize the discussion above to the relative case: let $(\mathcal{C}/B, p)$ be an N-pointed family of curves, and $\mathcal{O}, \mathcal{K}, \mathcal{A}, \mathcal{K}^0, \mathfrak{m}$ the corresponding $\mathcal{O}_B$-modules. Below we will use the following notation: $H^k(X;\mathcal{F})$ denotes the singular cohomology of a topological space X with coefficients in a sheaf $\mathcal{F}$ over X. Using this, we have for a small enough $U \subseteq B$ the following analogue of (3.1):

$$0 \longrightarrow H^1(\mathcal{C}_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B)_U \longrightarrow H^1(\mathcal{C}^0_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B)_U \xrightarrow{w} \longrightarrow \mathcal{K}^0(U) \longrightarrow \mathcal{O}_B(U) \longrightarrow 0.$$

Here $H^1(\mathcal{C}_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B)_U$ is short for $H^1(\mathcal{C}_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B)/H^1(U;\mathcal{O}_B)$, and $H^1(\mathcal{C}^0_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B)_U$ has a similar meaning. Assuming that U is simply connected, the exact sequence simplifies to

$$0 \longrightarrow H^1(\mathcal{C}_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B) \longrightarrow H^1(\mathcal{C}^0_U;(\mathcal{C}_U/U)^{-1}\mathcal{O}_B) \xrightarrow{w} \longrightarrow \mathcal{K}^0(U) \longrightarrow \mathcal{O}_B(U) \longrightarrow 0.$$

The assignments that maps U to the $\mathcal{O}_B(U)$-modules above are presheaves. By sheafification we obtain

$$0 \longrightarrow R^1(\mathcal{C}/B)_*(\mathcal{C}/B)^{-1}\mathcal{O}_B \longrightarrow R^1(\mathcal{C}^0/B)_*(\mathcal{C}^0/B)^{-1}\mathcal{O}_B \xrightarrow{w} \longrightarrow (\mathcal{C}/B)_*\mathcal{K}^0 \longrightarrow \mathcal{O}_B \longrightarrow 0. \quad (3.3)$$

The relative version of (3.2) reads

$$R^1(\mathcal{C}^0/B)_*(\mathcal{C}^0/B)^{-1}\mathcal{O}_B = \frac{(\mathcal{C}^0/B)_*\omega_{\mathcal{C}^0/B}}{d_{\mathcal{C}^0/B}\mathcal{A}},$$

where we recall that $\mathcal{A}$ was defined as $(\mathcal{C}^0/B)_*\mathcal{O}_{\mathcal{C}^0}$.

Definition 3.1.3. Define

$$\mathcal{H} := R^1(\mathcal{C}/B)_*(\mathcal{C}/B)^{-1}\mathcal{O}_B$$

and let $\mathcal{B}$ be the maximal subsheaf of $\mathcal{K}$ consisting of sections $b \in \mathcal{K}(U)$ such that $d_{\widehat{\mathcal{C}}^0_p/B} b$ is the restriction of an element of $\mathcal{H}(U) \subseteq \omega_{\mathcal{C}^0/B}(\mathcal{C}^0_U)/ d_{\mathcal{C}^0/B}\mathcal{A}(U)$ to $\widehat{\mathcal{C}}^0_p|_U$, for any $U \subseteq B$.

Lemma 3.1.4. *The sheaf $\mathcal{H}$ is a locally free $\mathcal{O}_B$-module of finite rank with a natural flat connection and the sheaf $\mathcal{B}$ is a quasi-coherent $\mathcal{O}_B$-module. Furthermore, the differential $d : \mathcal{K} \to \omega_{\widehat{\mathcal{C}}^0_p/B}$ induces an isomorphism $\mathcal{H} \simeq \mathcal{B}/(\mathcal{A} + \mathcal{K}^0)$.*

Proof. For a small contractible $U \subseteq B$ containing a (closed) point b, $\mathcal{C}_b$ is a retract of $\mathcal{C}_U$. Hence we have a *natural* isomorphism

$$\mathcal{H}(U) = H^1(\mathcal{C}_U; (\mathcal{C}_U/U)^{-1}\mathcal{O}_B) = H^1(\mathcal{C}_b; \mathbb{C}) \otimes \mathcal{O}_B.$$

This clearly shows that $\mathcal{H}$ is locally free and of finite rank, and the naturality of the isomorphism makes the flat structure determined by the isomorphism is independent of the choices made.

Locally over B, $\omega_{\widehat{\mathcal{C}}^0_p/B}$ is of the form $(\sum_i \mathcal{O}_B((t_i)))\{ dt_1, \ldots, dt_N\}$, so that locally, a section of $\omega_{\widehat{\mathcal{C}}^0_p/B}$ has a primitive in $\mathcal{K}$. Thus

$$0 \longrightarrow \mathcal{A} + \mathcal{K}^0 \longrightarrow \mathcal{B} \xrightarrow{d_{\widehat{\mathcal{C}}^0_p/B}} \mathcal{H} \longrightarrow 0$$

is exact. From this we see that $\mathcal{B}$, being an extension of quasicoherent sheaves, is quasicoherent. □

The $\mathcal{O}_B$-module $\mathcal{K}$ has a natural presymplectic structure, as we will see in the next subsection.

3.1.2 (Pre)symplectic structure: the absolute case

We return, for the moment, a single N pointed curve $(C/\operatorname{Spec}\ \mathbb{C}, p)$, and let the $\mathbb{C}$-algebras $\mathcal{K}, \mathcal{O}, \mathcal{A}, \mathcal{K}^0$ and $\mathfrak{m}$ be as before.

Definition 3.1.5. Let $(\cdot,\cdot) : \mathcal{K} \otimes_{\mathbb{C}} \mathcal{K} \to \mathbb{C}$ be defined by

$$f \otimes g \mapsto \sum_i \operatorname{Res}_{p_i}(g\ df).$$

In terms of the coordinates t_i at p_i introduced before,

$$(t_n^k, t_m^l) = \sum_i \operatorname{Res}_i(t_m^l t_n^{k-1}\ dt) = \delta_{n,m}\delta_{l+k,0}.$$

If $f, g \in \mathcal{K}$, then $\operatorname{Res}_{p_i}(f\ dg) = \operatorname{Res}_{p_i}(\ d(fg)) - \operatorname{Res}_{p_i}(g\ df) = -\operatorname{Res}_{p_i}(g\ df)$, so that $(\cdot,\cdot)$ is antisymmetric. Furthermore, by the formula above, we see that if dg is nonzero, there is always an $f \in \mathcal{K}$ such that $\sum_i \operatorname{Res}_{p_i}(f\ dg)$ is nonzero, so that the kernel of $(\cdot,\cdot)$ coincides with that of d, i.e. $\mathcal{K}^0$. We can therefore conclude that $(\cdot,\cdot)$ is a degenerate presymplectic form on $\mathcal{K}$ with kernel $\mathcal{K}^0$.

Lemma 3.1.6. *If $f_1, f_2 \in \mathcal{B}$ and $c(f_1), c(f_2)$ are the elements that f_1, f_2 define in $H^1(C;\mathbb{C})$ (via $d_{\widehat{\mathcal{C}}_p^0/B} : \mathcal{B}/(\mathcal{A} + \mathcal{K}^0) \simeq \mathcal{H}$), respectively, then*

$$\int_C c(f_1) \wedge c(f_2) = -2\pi i (f_1, f_2).$$

Proof. We can find De Rham representatives as follows: let U_j, V_j be small open neighborhoods of p_j such that the following holds: f_i is defined on $U \backslash p$ for $U = (\bigcup_j U_j)$, $p_j \in \overline{V}_j \subseteq U_j$, U_j is diffeomorphic to B^2 and $\overline{U}_i \cap \overline{U}_j = \emptyset$ if $i \neq j$. Choose diffeomorphisms $\kappa_j : U_j \to B^2$ such that $\kappa_j(p_j) = 0$, and subsequently a cutoff function $\phi : C \to [0,1]$ near p, subordinate to U, V, where $V = \bigcup_j V_j$. The differentials df_i on $\widehat{\mathcal{C}}_p^0$ are by assumption restrictions of differentials, say α_i, on C^0. We then have that

$$\widetilde{\alpha}_i := \phi\alpha_i + f_i\ d\phi$$

represents $c(f_i)$: first observe that this is a well defined closed 1-form on C. Second, since on $C \backslash U$ we have that $\widetilde{\alpha}_i = \alpha_i$, the periods of $\widetilde{\alpha}_i, \alpha_i$ on C are the same, so that these forms define the same cohomology class. Note that on U, $\widetilde{\alpha_i} = \phi\, df_i + f_i\, d\phi = d(f_i\phi)$. The rest now follows from Stokes' theorem:

$$\begin{aligned} \int_C \widetilde{\alpha}_1 \wedge \widetilde{\alpha_2} &= \int_{C\setminus U} \alpha_1 \wedge \alpha_2 + \int_U d(f_1\phi) \wedge \widetilde{\alpha}_2 = 0 + \int_U d(f_1\phi\widetilde{\alpha}_2) \\ &= \int_{\partial U} f_1\phi(\phi\, df_2 + f_2\, d\phi) = \int_{\partial U} f_1\, df_2 = 2\pi i \sum_i \mathrm{Res}_{p_i}(f_1\, df_2) \\ &= -2\pi i (f_1, f_2). \end{aligned}$$

In the first we used that the wedge product of the forms of type $(1,0)$ vanishes and in the fourth step that $\phi = 1$ on $\overline{C\setminus U}$. □

For any $\mathcal{F} \subseteq \mathcal{K}$, we write $\mathcal{F}^\perp$ for the annihilator of $\mathcal{F}$ in $\mathcal{K}$ with respect to $(\cdot,\cdot)$.

Lemma 3.1.7. *The following holds*

- $\mathcal{B} = \mathcal{A}^\perp$
- $(\cdot,\cdot)$ *is well-defined and nondegenerate on* $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{K}^0)$ *and corresponds to the integration paring under the isomorphism* $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{K}^0) = H^1(C;\mathbb{C})$, *which is induced by* d.
- *the isomorphism* $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{K}^0) \simeq H^1(C;\mathbb{C})$ *restricts to an isomorphism* $\mathcal{B} \cap \mathfrak{m} = \omega(C)$.

Proof. If $f \in \mathcal{B}$ and $g \in \mathcal{A}$, then $df \in \omega(C^0)$ and therefore $g\, df \in \omega(C^0)$ as well. But $(f,g) = \sum_i \mathrm{Res}_{p_i}\, g\, df = 0$ by the residue theorem, so $f \in \mathcal{A}^\perp$. The residue theorem also tells us that if $\sum_i \mathrm{Res}_{p_i}\, df = 0$, then $df \in \omega(\mathcal{C}^0)$. Thus, if $f \in \mathcal{A}^\perp$ and $g = 1 \in \mathcal{A}$, then $0 = (f,g) = \sum_i \mathrm{Res}_{p_i}\, df$, so $df \in \omega(C^0)$ and $f \in \mathcal{B}$. Obviously, $(\cdot,\cdot)$ is well defined on $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{K}^0)$, and since it corresponds to the integration pairing by the previous lemma, it is nondegenerate. Finally, if $f \in \mathcal{B}\cap\mathcal{O}$, then df is regular near p, so $df \in \omega(C)$. Conversely, if $\alpha \in \omega(C^0)$ is regular near p, then so is a primitive in $\mathcal{K}$, hence $d(\mathcal{B}\cap\mathcal{O}) = d(\mathcal{B}\cap\mathfrak{m}) = \omega(C)$. But d is injective on $\mathfrak{m}$, so $d : \mathcal{B}\cap\mathfrak{m} \to \omega(C)$ is an isomorphism. □

We summarize what we have derived above: to a pointed curve (C,p), we associated a vector space $H^1(C;\mathbb{C})$, a symplectic form on $H^1(C;\mathbb{C})$ and a Lagrangian subspace $(\mathcal{B}\cap\mathfrak{m})/(\mathcal{A}+\mathcal{K}^0)$. The relative version of this is precisely the input data of Chapter 4. Before going to the relative case, we will first give the relation with the presymplectic structure of $\mathcal{K}$.

Proposition 3.1.8. *Let $\mathcal{K}, \mathcal{K}^0, \mathcal{O}, \mathfrak{m}, \mathcal{A}, (\cdot,\cdot)$ be as before, and define $\mathcal{F}^+ = \mathcal{A}^\perp \cap \mathfrak{m}$. Let $\mathcal{A}^-$ be a complement for $\mathcal{K}^0$ in $\mathcal{A}$. There exist linear subspaces $\mathcal{F}^- \subseteq \mathcal{A}^\perp$ and $\mathcal{A}^+ \subseteq \mathfrak{m}$ such that*

- $\mathcal{K} = \mathcal{A}^- \oplus \mathcal{F}^- \oplus \mathcal{K}^0 \oplus \mathcal{F}^+ \oplus \mathcal{A}^+$ *and* $\mathfrak{m} = \mathcal{F}^+ \oplus \mathcal{A}^+$,
- $(\mathcal{A}^\pm, \mathcal{F}^\pm) = 0$,
- $(\cdot,\cdot)$ *restricts to a perfect pairing between* $\mathcal{F}^-, \mathcal{F}^+$ *and between* $\mathcal{A}^-, \mathcal{A}^+$,
- *the isomorphism* $\mathcal{A}^\perp/(\mathcal{A}^- + \mathcal{K}^0) = H^1(C;\mathbb{C})$ *identifies* $\mathcal{F}^-$ *with* $H^{0,1}(C) = H^1(C, \mathcal{O}_C)$ *and* $\mathcal{F}^+$ *with* $H^{1,0}(C) = H^0(C, \omega_C)$,
- $\mathcal{K}^{\leq 0} := \mathcal{A} + \mathcal{F}^- + \mathcal{K}^0$ *is independent of the choice of* $\mathcal{F}^-$.

Proof. Take for $\mathcal{F}^-$ a lift of $H^{0,1}(C) \subseteq H^1(C;\mathbb{C})$ to $\mathcal{A}^\perp$ under the isomorphism $\mathcal{A}^\perp/(\mathcal{A} + \mathcal{K}^0) \to H^1(C;\mathbb{C})$, such that $\mathcal{F}^- \cap (\mathcal{A} + \mathcal{K}^0) = 0$ and $\mathcal{F}^- \cap \mathcal{F}^+ = 0$. We review how $\mathcal{F}^-$ is identified with $H^{0,1}(C)$: let f be in $\mathcal{F}^-$ and let U, V, ϕ be as in the proof of lemma 3.1.6. Also, let $\alpha \in \omega(C^0)$ be the differential whose restriction to $\widehat{\mathcal{C}}_p^0$ is df. Then the De Rham class of f is represented by $\phi\alpha + f\,d\phi$. By the way $\mathcal{F}^-$ was chosen, the $(1,0)$ part of this expression is exact; the $(0,1)$ part equals $f\bar\partial\phi$. Under the standard isomorphism $H^1(C, \mathcal{O}_C) \simeq H^{0,1}(C)$, $f\bar\partial\phi$ corresponds to $[f] \in H^1(C, \mathcal{O}_C) = \mathcal{K}/(\mathcal{A} + \mathcal{O})$. So $\mathcal{F}^-$ maps surjectively onto $\mathcal{K}/(\mathcal{A}+\mathcal{O})$, and consequently $\mathcal{K} = \mathcal{A}+\mathcal{F}^-+\mathcal{O} = \mathcal{A}^-+\mathcal{F}^-+\mathcal{K}^0 + \mathfrak{m}$. Since in addition, all summands intersect trivially, $\mathcal{K} = \mathcal{A}^- \oplus \mathcal{F}^- \oplus \mathcal{K}^0 \oplus \mathfrak{m}$.
Take $\mathcal{A}^+$ to be $(\mathcal{F}^-)^\perp \cap \mathfrak{m}$, then we have that

$$\mathcal{F}^+ \cap \mathcal{A}^+ = \mathfrak{m} \cap \mathcal{A}^\perp \cap (\mathcal{F}^-)^\perp = 0,$$

for $d(\mathfrak{m} \cap \mathcal{A}^\perp \cap (\mathcal{F}^-)^\perp) = H^{1,0}(C) \cap H^{0,1}(C) = 0$. Hence the map $\mathcal{F}^+ \to \mathfrak{m}/\mathcal{A}^+$ is injective. However, $(\cdot,\cdot)$ is well defined and nondegenerate on $\mathcal{F}^- \otimes \mathfrak{m}/\mathcal{A}^+$, so $\mathcal{F}^+ \to \mathfrak{m}/\mathcal{A}^+$ must be surjective as well. It follows that $\mathfrak{m} = \mathcal{F}^+ \oplus \mathcal{A}^+$, and as a result of that, $\mathcal{K} = \mathcal{A}^- \oplus \mathcal{F}^- \oplus \mathcal{K}^0 \oplus \mathcal{F}^+ \oplus \mathcal{A}^+$.

The second, third and forth assertion now follow from the definitions (and lemma 3.1.7 for the fourth one). For the final assertion we remark that $\mathcal{K}^{\leq 0} = d^{-1}H^{0,1}(C)$. □

3.1.3 (Pre)symplectic structure: the relative case

We now return to the relative setting: let $(\mathcal{C}/B, p)$ be a family of N-pointed curves and $\mathcal{K}, \mathcal{O}, \mathfrak{m}, \mathcal{K}^0, \mathcal{A}, \mathcal{H}$ the corresponding $\mathcal{O}_B$-modules.

Definition 3.1.9. Define $(\cdot,\cdot) : \mathcal{K} \otimes_{\mathcal{O}_B} \mathcal{K} \to \mathcal{O}_B$ to be the map that for local sections f, g of $\mathcal{K}$ is given by

$$f \otimes g \mapsto \mathrm{Res}_p(g\ d_{\mathcal{C}/B} f).$$

If, locally over B, $t_1, \ldots, t_N$ are coordinates on $\mathcal{C}$ along p, then

$$(t_i^k t_j^l) = \frac{1}{2\pi i} \mathrm{Res}_p(t_j^l t_i^{k-1}\ dt) = \delta_{i,j}\delta_{l+k,0}.$$

The discussion above for the absolute case can, locally over B, be repeated for the current relative setting. We give the resulting statements.

Corollary 3.1.10. *The following holds:*

- *The kernel of* $(\cdot,\cdot)$ *is* $\mathcal{K}^0$,
- *for every* $U \subseteq B$, $b \in U$ *and* $f_1, g_1 \in \mathcal{K}(U)$,

  $$\int_{\mathcal{C}_b} c(f_1) \wedge c(f_2) = -2\pi i (f_1, f_2),$$

  where $c(f_i)$ *denotes the class of* f_i *in* $\mathcal{H}(U)$
- $\mathcal{B} = \mathcal{A}^\perp$,
- $(\cdot,\cdot)$ *is well defined and nondegenerate on* $\mathcal{A}^\perp/(\mathcal{A} + \mathcal{K}^0)$ *and corresponds to* $-2\pi i$ *times the integration paring under the isomorphism* $\mathcal{A}^\perp/(\mathcal{A}+\mathcal{K}^0) \simeq \mathcal{H}$ *induced by* $d_{\mathcal{C}/B}$,
- $d_{\mathcal{C}/B}$ *induces an isomorphism* $\mathcal{B} \cap \mathfrak{m} \simeq (\mathcal{C}/B)_* \omega_{\mathcal{C}/B}$.

The proofs of lemmas 3.1.6 and 3.1.7 can *locally* be adapted to the relative setting - we will leave the details to the reader. The relative analogue of proposition 3.1.8 is the following:

Proposition 3.1.11. *Let the* $\mathcal{O}_B$*-modules* $\mathcal{K}, \mathcal{K}^0, \mathcal{O}, \mathfrak{m}, \mathcal{A}$ *and the* $\mathcal{O}_B$*-bilinear form* $(\cdot,\cdot)$ *be as before, and define* $\mathcal{F}^+ = \mathcal{A}^\perp \cap \mathfrak{m}$. *Locally over* B*, one can choose a complement* $\mathcal{A}^-$ *for* $\mathcal{A} \cap \mathcal{K}^0$ *in* $\mathcal{A}$ *such that there exist* $\mathcal{O}_B$*-submodules* $\mathcal{F}^- \subseteq \mathcal{A}^\perp$ *and* $\mathcal{A}^+ \subseteq \mathfrak{m}$ *such that*

- $\mathcal{K} = \mathcal{A}^- \oplus \mathcal{F}^- \oplus \mathcal{K}^0 \oplus \mathcal{F}^+ \oplus \mathcal{A}^+$ *and* $\mathfrak{m} = \mathcal{F}^+ \oplus \mathcal{A}^+$,
- $(\mathcal{A}^\pm, \mathcal{F}^\pm) = 0$,
- $(\cdot,\cdot)$ *restricts to a perfect pairing between* $\mathcal{F}^-, \mathcal{F}^+$ *and between* $\mathcal{A}^-, \mathcal{A}^+$,
- *the isomorphism* $\mathcal{A}^\perp/(\mathcal{A}^- + \mathcal{K}^0) = \mathcal{H}$ *induced by* $\mathrm{d}_{\mathcal{C}/\mathrm{B}}$ *identifies* $\mathcal{F}^-$ *with* $\mathrm{R}^1(\mathcal{C}/\mathrm{B})_*\mathcal{O}_\mathcal{C}$ *and* $\mathcal{F}^+$ *with* $(\mathcal{C}/\mathrm{B})_*\omega_{\mathcal{C}/\mathrm{B}}$,
- $\mathcal{K}^{\leq 0} := \mathcal{A} + \mathcal{F}^- + \mathcal{K}^0 \subseteq \mathcal{K}$ *is independent of the choice of* $\mathcal{F}^-$ *and is therefore defined over all of* B.

We emphasize that in the relative case, $\mathcal{A}^\pm$, $\mathcal{F}^-$ can only be assumed to exist *locally* over B: since they are not chosen naturally, local choices of these sheaves will in general not glue to global ones. Before proceeding to the next section, we recall the following identities:

$$\begin{aligned}
(\widehat{\mathcal{C}}_p/\mathrm{B})_*\Omega_{\mathcal{C}/\mathrm{B}} &= \Omega_{\mathcal{O}/\mathcal{O}_\mathrm{B}},\\
(\widehat{\mathcal{C}}_p^0/\mathrm{B})_*\Omega_{\mathcal{C}/\mathrm{B}} &= \Omega_{\mathcal{K}/\mathcal{O}_\mathrm{B}},\\
(\mathcal{C}^0/\mathrm{B})_*\Omega_{\mathcal{C}/\mathrm{B}} &= \Omega_{\mathcal{A}/\mathcal{O}_\mathrm{B}}.
\end{aligned}$$

3.2 The data of an affine family of pointed curves endowed with a labeled G-bundle

We will extend the situation of the previous section, i.e. families of pointed curves, to principal G-bundles over them. This will be done first for the absolute case: we start by letting $(C/\operatorname{Spec}\mathbb{C}, p)$ be a pointed curve and $\widehat{\mathcal{C}}_p$, $\widehat{\mathcal{C}}_p^0$, C^0, $\mathcal{O}$, $\mathcal{K}$, $\mathcal{A}$, $\mathfrak{m}$, $\mathcal{K}^0$ be as before.

We recall that we fixed a simple and simply connected complex Lie group G with Lie algebra $\mathfrak{g}$, and let $P/C : P \to C$ be a principal G-bundle. Define

$$\mathcal{O}_\mathfrak{g} := \mathrm{H}^0(\widehat{\mathcal{C}}_p, \mathrm{Ad}_{P/C}), \quad \mathcal{K}_\mathfrak{g} := \mathrm{H}^0(\widehat{\mathcal{C}}_p^0, \mathrm{Ad}_{P/C}), \quad \mathcal{A}_\mathfrak{g} := \mathrm{H}^0(C^0, \mathrm{Ad}_{P/C}).$$

Note that $\mathcal{O}_\mathfrak{g}$ is an $\mathcal{O}$-module, $\mathcal{K}_\mathfrak{g}$ is a $\mathcal{K}$-module and $\mathcal{A}_\mathfrak{g}$ is a $\mathcal{A}$-module. By the first of these observations, we can define another subalgebra of $\mathcal{K}_\mathfrak{g}$:

$$\mathfrak{m}_\mathfrak{g} := \mathcal{O}_\mathfrak{g}\mathfrak{m}.$$

Restriction to $\widehat{\mathcal{C}}_p^0$ gives inclusions $\mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g} \subseteq \mathcal{K}_\mathfrak{g}$. By construction, $\mathrm{Ad}_{P/C}$ inherits a Lie product, which in turn defines a Lie product on $\mathcal{K}_\mathfrak{g}$ over $\mathcal{K}$. Clearly, $\mathcal{A}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}$ are subalgebras of $\mathcal{K}_\mathfrak{g}$.

Note that P/C is trivial over $\widehat{\mathcal{C}}_p$ and choose a trivialization - we can then identify $P_{\widehat{\mathcal{C}}_p}/\widehat{\mathcal{C}}_p$ with $G \times \widehat{\mathcal{C}}_p/\widehat{\mathcal{C}}_p$, and correspondingly, identify the restriction of $\mathrm{Ad}_{P/C}$ to $\widehat{\mathcal{C}}_p$ with $\mathfrak{g} \otimes \mathcal{O}$. Hence $\mathcal{O}_\mathfrak{g} \simeq \mathfrak{g} \otimes \mathcal{O}$ and $\mathcal{K}_\mathfrak{g} \simeq \mathfrak{g} \otimes \mathcal{K}$. We let $\widetilde{\mathcal{K}}^0_\mathfrak{g}$ be the subalgebra of $\mathcal{O}_\mathfrak{g}$ corresponding to $\mathfrak{g} \otimes \mathcal{K}^0 \subseteq \mathfrak{g} \otimes \mathcal{O}$ with respect to the trivialization chosen, so

$$\mathcal{O}_\mathfrak{g} = \widetilde{\mathcal{K}}^0_\mathfrak{g} \otimes_{\mathcal{K}^0} \mathcal{O}, \quad \mathcal{K}_\mathfrak{g} = \widetilde{\mathcal{K}}^0_\mathfrak{g} \otimes_{\mathcal{K}^0} \mathcal{K}$$

as Lie algebras over $\mathcal{O}$, $\mathcal{K}$, respectively. Since C^0 is affine, P is also trivial over C^0, as we saw in Chapter 1. Choose a trivialization $P_{C^0}/C^0 \simeq G \times C^0/C^0$ and let $\mathfrak{a}$ be the subalgebra of $\mathcal{A}_\mathfrak{g}$ corresponding to $\mathfrak{g} \otimes 1$ under the identification $\mathrm{Ad}_{P_{C^0}/C^0} \simeq \mathfrak{g} \times C^0/C^0$. So $\mathcal{A}_\mathfrak{g} = \mathfrak{a} \otimes \mathcal{A}$ as $\mathcal{A}$-Lie algebras and $\mathfrak{a} \simeq \mathfrak{g}$. Since $\mathfrak{a}$ provides a trivialization of $\mathrm{Ad}_{P/C}$ over C^0, the restriction of this to $\widehat{C}^0_p$ provides a trivialization of $\mathrm{Ad}_{P/C}$ over $\widehat{C}^0_p$. Hence, regarding $\mathfrak{a}$ a subalgebra of $\mathcal{K}_\mathfrak{f}\mathfrak{g}$, we have that $\mathfrak{a}\mathcal{K} = \mathcal{K}_\mathfrak{g}$.

Lemma 3.2.1. *Up to isomorphism,* $\mathrm{Ad}_{P/C}$ *and its Lie product can be reconstructed from* $\widetilde{\mathcal{K}}^0_\mathfrak{g}$ *and* $\mathfrak{a} \subseteq \widetilde{\mathcal{K}}^0_\mathfrak{g} \otimes_{\mathcal{K}^0} \mathcal{K}$ *as above.*

Proof. Both $\widetilde{\mathcal{K}}^0_\mathfrak{g}$ and $\mathfrak{a}$ provide a trivialization of $\mathrm{Ad}_{P/C}$ over $\widehat{\mathcal{C}}_p$. Hence they differ by a transition function $g : \widehat{\mathcal{C}}^0_p \to \mathrm{Ad}(G)$. The $\mathrm{Ad}G$-bundle

$$(\mathfrak{g} \times C^0) \times_{\widehat{C}^0_p} (\mathfrak{g} \times \widehat{C}_p), \tag{3.4}$$

obtained by gluing the trivial bundles on $C^0, \widehat{C}_p$ over $\widehat{C}^0_p$ by means of g, is isomorphic to $\mathrm{Ad}_{P/C}$. Furthermore, the Lie products on $\mathcal{A}_\mathfrak{g}$, $\mathcal{O}_\mathfrak{g}$ define Lie products on (3.4) over $C^0, \widehat{C}_p$. The restrictions of these products to $\mathfrak{g} \times \widehat{C}^0_p$ are the same. Moreover, since $\mathrm{Ad}G$ leaves the product of $\mathfrak{g}$ invariant, we also have that g leaves the product on $\mathfrak{g} \times \widehat{C}^0_p$ invariant. Hence the product on (3.4) over $C^0, \widehat{C}_p$ glue over $\widehat{C}^0_p$. This product is identified with the one on Ad_{PC} under the isomorphism that identifies (3.4) with the latter. □

Remark 3.2.2. Since P/C is trivial on $C^0, \widehat{C}_p$, it is obtained by gluing $G \times C^0$, $G \times \widehat{C}_p$ over $\widehat{C}^0_p$ by means of a transition function $\widetilde{g} : \widehat{C}^0_p \to G$. The adjoint bundle is then obtained by the transition function $\widetilde{\mathrm{Ad}(g)}$. Hence, we can take g in the proof above to be equal to $\mathrm{Ad}\widetilde{g}$. Since $Z(G) = \mathrm{Ker}(G \to \mathrm{Ad}G)$, $\widetilde{g}$ is in general not uniquely determined by g. We also note that the class of g in $LG(C^0)\backslash LG(\widehat{C}^0_p)/L^{\geq 0}G(\widehat{C}^0_p)$ corresponds to the isomorphism type of P/C in the moduli space of G-bundles over C.

Assume that P/C has a flat connection ∇. This determines a flat connection on $\mathrm{Ad}_{P/C}$, which we also denote ∇. Since the latter comes from a connection on P/C, it preserves $[\cdot,\cdot]$. Given ∇, there is a canonical choice for $\widetilde{\mathcal{K}}^0_{\mathfrak{g}}$:

$$\mathcal{K}^0_{\mathfrak{g}} := \mathrm{Ker}\,\nabla \subseteq \mathcal{O}_{\mathfrak{g}}.$$

Lemma 3.2.3. *For any choice of* $\mathfrak{a}$ *as before,* $\mathrm{Ad}_{P/C}$, ∇ *can be reconstructed from* $\mathcal{K}^0_{\mathfrak{g}}$, $\mathfrak{a} \subseteq \mathcal{K}^0_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{K}$ *up to isomorphism.*

Proof. We already saw how P/C can be reconstructed from $\mathcal{K}^0_{\mathfrak{g}}, \mathfrak{a}$. Let $\nabla^{\widehat{\mathcal{C}}_p}$ be the unique connection on $\mathrm{Ad}_{P_{\widehat{\mathcal{C}}_p}/\widehat{\mathcal{C}}_p}$ that has $\mathcal{K}^0_{\mathfrak{g}}$ as its flat sections and let $\nabla^{\widehat{\mathcal{C}}^0_p}$ be its restriction to $\widehat{\mathcal{C}}^0_p$. Then $\nabla^{\widehat{\mathcal{C}}^0_p}$ has the property that if X is the restriction of a vectorfield on C^0, then $\nabla^{\widehat{\mathcal{C}}^0_p}_X$ preserves $\mathcal{A}_{\mathfrak{g}}$. This means that $\nabla^{\widehat{\mathcal{C}}^0_p}$ extends uniquely to a connection over both $\widehat{\mathcal{C}}_p$ and C^0, and therefore defines a connection on $\mathrm{Ad}_{P/C}$, which by construction coincides with ∇. □

Lemma 3.2.4. *One can choose a trivialization of* P_{C^0}/C^0 *such that the corresponding* $\mathfrak{a}$ *is a subalgebra of* $\mathcal{K}^0_{\mathfrak{g}}$ *iff* P/C *is trivial and* ∇ *has no monodromy. If* P/C *is regularly stable, then* $\mathfrak{a} \cap \mathcal{K}^0_{\mathfrak{g}} = 0$ *for any trivialization of* P_{C^0}/C^0.

Proof. In case P/C is trivial and ∇ has no monodromy, then there is a trivialization of P_{C^0}/C^0 coming from a flat trivialization of P/C - this has the property that $\mathfrak{a} = \mathfrak{a} \cap \mathcal{K}^0_{\mathfrak{g}}$. Conversely, if $\mathfrak{a} \cap \mathcal{K}^0_{\mathfrak{g}} = \mathfrak{a}$, then $\mathfrak{a}$ consists of flat sections that extend to p, so that $\mathrm{Ad}_{P/C}$ has a global basis of flat sections. Hence we can take g as in the proof of lemma 3.2.1 to be the map $\widehat{C}^0_p \to \{1\}$. Let $\widetilde{g}$ be the transition function corresponding to the trivialization of P/C over $C^0, \widehat{C}_p$ chosen, then $\mathrm{Ad}\widetilde{g} = g$. Hence $\widetilde{g}$ takes values in $Z(G)$. Since G is a simple algebraic group, $Z(G)$ is discrete and therefore g must be constant, and therefore extends to C^0. By changing the trivialization over C^0 accordingly, we can assume g to be constant equal to 1, and hence P/C trivial. Since the trivialization was flat, P/C admits a global flat trivialization so that ∇ has trivial monodromy.

Finally, if $X \in \mathfrak{a} \cap \mathcal{K}^0_{\mathfrak{g}}$, then $X \in H^0(C, \mathrm{Ad}_{P/C})$, which is the space of infinitesimal automorphisms of P/C, which trivial if P/C is regularly stable. □

We proceed to the relative setting: let$(\mathcal{C}/B, p)$ be a family of pointed curves and $P/\mathcal{C} : P \to \mathcal{C}$ a principal G-bundle. Furthermore, let $\widehat{\mathcal{C}}_p/B, \widehat{\mathcal{C}}^0_p/B, \mathcal{C}^0/B$ and their structure sheaves $\mathcal{O}, \mathcal{K}, \mathcal{A}$ be as before.

Definition 3.2.5. We define the following $\mathcal{O}_B$-modules

$$\mathcal{O}_{\mathfrak{g}} := (\widehat{\mathcal{C}}_p/B)_* \mathrm{Ad}_{P/\mathcal{C}}, \quad \mathcal{K}_{\mathfrak{g}} := (\widehat{\mathcal{C}}^0_p/B)_* \mathrm{Ad}_{P/\mathcal{C}}, \quad \mathcal{A}_{\mathfrak{g}} := (\mathcal{C}^0/B)_* \mathrm{Ad}_{P/\mathcal{C}}.$$

Clearly, $\mathcal{O}_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g}}$, $\mathcal{A}_{\mathfrak{g}}$ are also modules over $\mathcal{O}$, $\mathcal{K}$, $\mathcal{A}$ respectively, and we define

$$\mathfrak{m}_{\mathfrak{g}} := \mathcal{O}_{\mathfrak{g}} \mathfrak{m}.$$

Suppose that $P/\mathcal{C}$ has a flat connection ∇ over $\mathcal{C}/B$, i.e. an algebra morphism $\theta_{\mathcal{C}/B} \to (P/\mathcal{C})^G_* \theta_{P/B}$ which is a section of the derivative of $P/\mathcal{C}$. Given ∇ we define the $\mathcal{K}^0$-module

$$\mathcal{K}^0_{\mathfrak{g}} := \mathrm{Ker}\, \nabla \subseteq \mathcal{K}_{\mathfrak{g}}.$$

As in the absolute setting, the Lie bracket of $\mathfrak{g}$ determines a $\mathcal{K}$-linear Lie bracket $[\cdot, \cdot]$ on $\mathcal{K}_{\mathfrak{g}}$ and $\mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}}$ are subalgebras of $\mathcal{K}_{\mathfrak{g}}$. Furthermore, $[\cdot, \cdot]$ is flat for ∇, so that $\mathcal{K}^0_{\mathfrak{g}}$ is a subalgebra of $\mathcal{K}_{\mathfrak{g}}$, and $\mathcal{O}_{\mathfrak{g}} = \mathcal{O} \otimes_{\mathcal{K}^0} \mathcal{K}^0_{\mathfrak{g}}$, $\mathcal{K}_{\mathfrak{g}} = \mathcal{K} \otimes_{\mathcal{K}^0} \mathcal{K}^0_{\mathfrak{g}}$ as $\mathcal{O}, \mathcal{K}$ Lie algebras, respectively.

Locally over B, we can assume $\mathcal{C}^0$ to be affine, so that $P_{\mathcal{C}^0}/\mathcal{C}^0$ is trivial - let $U \subseteq B$ such so that this is the case. If we choose a trivialization of $P_U/\mathcal{C}^0_U$, we obtain, in a similar way as above, a $\mathcal{O}_U$ subalgebra $\mathfrak{a}$ of $\mathcal{K}_{\mathfrak{g}}|_U$ that is isomorphic to $\mathfrak{g} \otimes \mathcal{O}_U$ and such that $\mathcal{A}_{\mathfrak{g}}|_U = \mathfrak{a} \otimes \mathcal{A}_U$ as $\mathcal{A}_U$-algebra. Lemmas 3.2.3 and 3.2.4, and their proofs, can be adapted to this relative setting, provided we work *locally* over B; we will just give the statement.

Corollary 3.2.6. *We can reconstruct* $\mathrm{Ad}_{P/\mathcal{C}}$*, the Lie product on* $\mathrm{Ad}_{P/\mathcal{C}}$ *and* ∇ *from* $\mathcal{K}^0_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}} \subseteq \mathcal{K}_{\mathfrak{g}}$*, up to isomorphism. Furthermore, let* $\mathfrak{a}$ *be as above for a local trivialization (local over* B*) of* $P_{\mathcal{C}^0}/\mathcal{C}^0$*. Then, still locally over* B*,*

- *one can choose* $\mathfrak{a} \subseteq \mathcal{K}_{\mathfrak{g}}$ *as above such that* $\mathfrak{a} \subseteq \mathcal{K}^0_{\mathfrak{g}}$ *iff* $P/\mathcal{C}$ *is trivial and* ∇ *has no monodromy,*

- *if* $P/\mathcal{C}$ *is regularly stable, then* $\mathfrak{a} \cap \mathcal{K}^0_{\mathfrak{g}} = 0$ *for any (local) choice of* $\mathfrak{a}$*.*

For the second to last point, we recall that since ∇ lifts sections of $\theta_{\mathcal{C}/B}$, parallel transport is only defined for paths in a fiber of $\mathcal{C}/B$.

3.2.1 The cohomology of a family of G-bundles over a family of pointed curves

There is a presymplectic structure on $\mathcal{K}_{\mathfrak{g}}$ that has properties similar to that of $\mathcal{K}$. In order to define it, we need the following: let tr be a nontrivial symmetric bilinear G-invariant form on $\mathfrak{g}$. We recall that symmetric bilinear G-invariant forms on $\mathfrak{g}$ are proportional, and that the Killing form is one of them. Since G is simple, the Killing form is nondegenerate, and hence tr as well. Later on, we will assume tr to have a specific normalization, but for the moment we do not need this.

Since $\mathcal{K}^0_{\mathfrak{g}}$ can locally over B be identified with $\mathfrak{g} \otimes \mathcal{K}^0$, we can transfer tr to a $\mathcal{K}^0$ bilinear form on $\mathcal{K}^0_{\mathfrak{g}}$ - one can check that this does not depend on the choice of isomorphism. This extends $\mathcal{K}$ linearly to $\mathcal{K}_{\mathfrak{g}} = \mathcal{K}^0_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{K}$, and we also denote the resulting bilinear form by tr. By construction, it is nondegenerate and invariant under the adjoint action of $\mathcal{K}_{\mathfrak{g}}$ on itself. Moreover, it is flat with respect to ∇.

We will describe the (pre)symplectic structure first for the absolute case, and we will do this using the notation introduced above. In particular, P/C is a principal G-bundle with flat connection ∇ over a pointed curve (C, p). We will also need the following: *let $\mathcal{G}$ be the sheaf of flat local sections of* $\mathrm{Ad}_{P/C}$ *over* C, so $\mathcal{K}^0_{\mathfrak{g}} = H^0(\widehat{C}^0_p, \mathcal{G})$. Below we will denote by $H^i(X; \mathcal{F})$ the i-th cohomology group of the topological space X with coefficients in the sheaf $\mathcal{F}$ over X; if X is a scheme and $\mathcal{F}$ and $\mathcal{O}_X$-module, then $H^i(X, \mathcal{F})$ denotes the coherent cohomology group.

We consider the following part of the exact sequence of the pair C, C^0 with coefficients in $\mathcal{G}$:

$$H^1(C, C^0; \mathcal{G}) \longrightarrow H^1(C; \mathcal{G}) \longrightarrow H^1(C^0; \mathcal{G}) \longrightarrow$$
$$\longrightarrow H^2(C, C^0; \mathcal{G}) \longrightarrow H^2(C; \mathcal{G}) \longrightarrow H^2(C^0; \mathcal{G}) \longrightarrow 0.$$

Using the Thom isomorphism, we have that $H^i(C, C^0; \mathcal{G}) = H^{i-2}(p; \mathcal{G})$. This vanishes for $i \neq 2$, and in the remaining case we can identify it with $\mathcal{K}^0_{\mathfrak{g}}$. Furthermore, observe that C^0 has the homotopy type of a 1-skeleton, so that $H^2(C^0; \mathcal{G}) = 0$. By Poincare duality, $H^2(C; \mathcal{G}) = H_0(C; \mathcal{G})^*$, which in turn can be identified with $H^0(C; \mathcal{G})$. The exact sequence above therefore reduces to

$$0 \longrightarrow H^1(C; \mathcal{G}) \longrightarrow H^1(C^0; \mathcal{G}) \xrightarrow{w_{\mathfrak{g}}} \mathcal{K}^0_{\mathfrak{g}} \longrightarrow H^0(C; \mathcal{G}) \longrightarrow 0. \quad (3.5)$$

The cohomology groups above can be computed using the De Rham resolution with differential ∇, e.g $H^1(C^0; \mathcal{G})$ is given by

$$\{x \in \mathcal{E}^1_{C^0}(\mathcal{G}) \mid \nabla x = 0\} / \nabla \mathcal{E}^0_{C^0}(\mathcal{G}).$$

Since $H^2(C^0;\mathcal{G}) = 0$, this is just equal to $\mathcal{E}^1_{C^0}(\mathcal{G})/\nabla\mathcal{E}^0_{C^0}(\mathcal{G})$. The $(0,1)$ part of $H^1(C^0;\mathcal{G})$ with respect to the Hodge decomposition vanishes, for

$$\mathcal{E}^{0,1}_{C^0}(\mathcal{G})/\nabla^{0,1}\mathcal{E}^{0,0}_{C^0}(\mathcal{G}) = \mathcal{E}^{0,1}_{C^0}(\mathrm{Ad}_{P/C})/\overline{\partial}\mathcal{E}^{0,0}_{C^0}(\mathrm{Ad}_{P/C}) = H^1(C^0, \mathrm{Ad}_{P/C}).$$

Here the left term is the $(0,1)$ part of the complex De Rham cohomology group, the middle part is the first Dolbeault cohomology group of the sheaf $\mathrm{Ad}_{P/C}$ and the last term is the sheaf cohomology of $\mathrm{Ad}_{P/C}$ over C^0. We also used that $\mathcal{E}^{p,q}(\mathcal{G}) = \mathcal{E}^{p,q}(\mathrm{Ad}_{P/C})$. Since C^0 is affine and $\mathrm{Ad}_{P/C}$ coherent, $H^1(C^0,\mathcal{G})$ vanishes. It therefore follows that

$$\begin{aligned} H^1(C^0;\mathcal{G}) &= \mathcal{E}^{1,0}_{C^0}(\mathcal{G})/\nabla^{1,0}\mathcal{E}^{0,0}_{C^0}(\mathcal{G}) = \mathcal{E}^{1,0}_{C^0}(\mathrm{Ad}_{P/C})/\nabla^{1,0}\mathcal{E}^{0,0}_{C^0}(\mathrm{Ad}_{P/C}) \\ &= \frac{H^0(C^0, \omega_{C^0} \otimes \mathrm{Ad}_{P/C})}{\nabla H^0(C^0, \mathrm{Ad}_{P/C})} = \frac{\omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{g}}}{\nabla \mathcal{A}_{\mathfrak{g}}}. \end{aligned}$$

In the last step we used that $\omega_{C^0} \otimes_{\mathcal{O}_{C^0}} \mathrm{Ad}_{P/C} = \mathfrak{a} \otimes_{\mathbb{C}} \omega_{C^0}$, where $\mathfrak{a}$ is as in lemma 3.2.3.

The map $w_{\mathfrak{g}}$ in (3.5) has a description similar to the one for map the w introduced before: take an element of $H^1(C^0;\mathcal{G})$ and represent is by some element α of $H^0(C^0, \omega_{C^0} \otimes \mathrm{Ad}_{P/C})$. Its restriction to $\widehat{\mathcal{C}}^0_p$ can be written as $X^i\alpha_i$ for certain $X_i \in \mathcal{K}^0_{\mathfrak{g}}$, $\alpha_i \in \omega_{\mathcal{K}}$. Identifying $\mathcal{K}^0_{\mathfrak{g}}$ with $\oplus_i \mathcal{K}^0_{\mathfrak{g}} \cap H^0(\widehat{C}^0_{p_i}; \mathrm{Ad}_{P/C})$, we have that the i-th component of $w_{\mathfrak{g}}(\alpha)$ with respect to this direct sum is given by

$$w_{\mathfrak{g}}(\alpha)_i = \frac{1}{2\pi i} \mathrm{Res}_{p_i}(\alpha_j) X^j. \tag{3.6}$$

3.2.2 (Pre)symplectic structure: the absolute case

Using the $\mathcal{K}$-bilinear form tr on $\mathcal{K}_{\mathfrak{g}}$ introduced before, we define the following.

Definition 3.2.7. Let $(\cdot,\cdot)_{\mathfrak{g}} : \mathcal{K}_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{K}_{\mathfrak{g}} \to \mathbb{C}$ be the $\mathcal{O}_B$ bilinear form on $\mathcal{K}_{\mathfrak{g}}$ that satisfies

$$(x, y)_{\mathfrak{g}} = \sum_i \mathrm{Res}_{p_i} \mathrm{tr}(y, \nabla x)$$

for local sections x, y of $\mathcal{K}_{\mathfrak{g}}$.

If $f, g \in \mathcal{K}$ and $X, Y \in \mathcal{K}^0_{\mathfrak{g}}$ then

$$(X \otimes f, Y \otimes g) = \mathrm{tr}(X, Y)(f, g).$$

From this, we immediately see that $(\cdot, \cdot)_{\mathfrak{g}}$ is $\mathbb{C}$ linear, antisymmetric, and has kernel $\mathcal{K}^0_{\mathfrak{g}}$. Below we will construct a "nice" decomposition of $\mathcal{K}_{\mathfrak{g}}$, similar to the one for $\mathcal{K}$ as in proposition 3.1.8.

We start by noting that restriction to $\widehat{\mathcal{C}}^0_p$ identifies $H^0(C^0, \omega_C \otimes \mathrm{Ad}_{P/C})$ with a subset of $H^0(\widehat{\mathcal{C}}^0_p, \omega_C \otimes \mathrm{Ad}_{P/C}) = \omega_{\mathcal{K}/\mathcal{K}^0} \otimes_{\mathcal{K}^0} \mathcal{K}_{\mathfrak{g}}$. *Define $\mathcal{B}_{\mathfrak{g}}$ to be the maximal subset of $\mathcal{K}_{\mathfrak{g}}$ such that $\nabla(\mathcal{B}_{\mathfrak{g}})$ lies in the image of* $H^1(C; \mathcal{G}) \subseteq H^1(C^0; \mathcal{G})$. We have that $\nabla(\mathcal{B}_{\mathfrak{g}}) = H^0(C^0, \omega_C \otimes \mathrm{Ad}_{P/C})$, because over $\widehat{\mathcal{C}}^0_p$ we can solve the system of first order equations given by $\nabla f = \alpha$, $\alpha \in H^0(C^0, \omega_C \otimes \mathrm{Ad}_{P/C})$ and $f \in \mathcal{K}_{\mathfrak{g}}$. Therefore, the sequence

$$0 \longrightarrow \mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}} \longrightarrow \mathcal{B}_{\mathfrak{g}} \xrightarrow{\nabla} H^1(C; \mathcal{G}) \longrightarrow 0$$

is exact. We can now formulate the analogues of Lemmas 3.1.6 and 3.1.7 :

Lemma 3.2.8. *Suppose $f_1, f_2 \in \mathcal{B}_{\mathfrak{g}}$ and let $c(f_1), c(f_2)$ be De Rham representatives for the classes they define in $H^1(C; \mathcal{G})$. Then*

$$\int_C \mathrm{tr}(c(f_1) \wedge c(f_2)) = -2\pi i (f_1, f_2)_{\mathfrak{g}}.$$

Proof. Let U, V, ϕ be as in the proof of lemma 3.1.6. Without loss of generality, we can assume that $\mathrm{Ad}_{P/C}$, ∇ are trivial over U, and we let X_i be a basis of flat sections of $\mathrm{Ad}_{P/C}$ over U. Furthermore, let α_1, α_2 be the elements in $H^0(C^0; \omega_C \otimes \mathrm{Ad}_{P/C})$ whose restriction to $\widehat{\mathcal{C}}^0_p$ equals $\nabla f_1, \nabla f_2$, respectively. Then $\widetilde{\alpha}_i = \phi \alpha_i + f_i \, d\phi$ represents $c(f_i)$ in $H^1(C; \mathrm{Ad}_{P/C})$. Over U we have that $\widetilde{\alpha}_i = \phi \nabla f_i + f_i \, d\phi = \nabla(\phi f_i)$ so that

$$\int_C \mathrm{tr}(c(f_1) \wedge c(f_2)) = \int_{C \setminus U} \mathrm{tr}(\widetilde{\alpha}_1 \wedge \widetilde{\alpha}_2) + \int_U \mathrm{tr}(\widetilde{\alpha}_1 \wedge \widetilde{\alpha}_2) = \int_U \mathrm{tr}(\nabla(\phi f_1) \wedge \nabla(\phi f_2)),$$

where we used that over $C \setminus U$, $\widetilde{\alpha}_i = \alpha_i$ so $\mathrm{tr}(\alpha_1 \wedge \alpha_2)$ is of type $(2, 0)$.

On U we can write $f_i = f_i^j X_j$ for certain $f_i^j \in \mathcal{K}$. Using that the X_j's are flat, we conclude that

$$\begin{aligned}
\int_C \mathrm{tr}(c(f_1) \wedge c(f_2)) &= \int_U \mathrm{tr}(X_i \, d(\phi f_1^i) \wedge X_j \, d(\phi f_2^j)) \\
&= \mathrm{tr}(X_i, X_j) \int_U d(\phi f_1^i) \wedge d(\phi f_2^j) \\
&= \mathrm{tr}(X_i, X_j) \sum_k 2\pi i \, \mathrm{Res}_{p_k}(f_1^i, f_2^j) = -2\pi i (f_1, f_2).
\end{aligned}$$

To derive the second to last step, one can use Stokes's theorem in a similar way as at the end of lemma 3.1.6. □

Lemma 3.2.9. *The following holds*

- $\mathcal{B}_\mathfrak{g} = \mathcal{A}_\mathfrak{g}^\perp$,
- $(\cdot,\cdot)_\mathfrak{g}$ *is a well defined and symplectic form on* $A_\mathfrak{g}^\perp/(\mathcal{A}_\mathfrak{g}+\mathcal{K}_\mathfrak{g}^0)$ *and corresponds to the* tr*-integration pairing under the isomorphism*

$$\mathcal{A}_\mathfrak{g}^\perp/(\mathcal{A}_\mathfrak{g}+\mathcal{K}_\mathfrak{g}^0) = H^1(C;\mathcal{G})$$

that is induced by ∇,

- *this isomorphism restricts to an isomorphism* $\mathcal{B}_\mathfrak{g}\cap\mathfrak{m}_\mathfrak{g} = H^0(C,\omega_C\otimes \mathrm{Ad}_{P/C})$.

Proof. If $f\in\mathcal{B}_\mathfrak{g}$ and $g\in\mathcal{A}_\mathfrak{g}$, then $\nabla f\in H^0(C^0,\omega_C\otimes\mathrm{Ad}_{P/C})$, and therefore $\mathrm{tr}(g\nabla f)\in\omega(C^0)$ as well. The sum of the residues of this differential at p vanishes, so $(f,g)=0$ - hence $\mathcal{B}_\mathfrak{g}\subseteq\mathcal{A}_\mathfrak{g}^\perp$. Conversely, suppose that $f\in\mathcal{A}_\mathfrak{g}^\perp$, then for all $g\in\mathcal{A}_\mathfrak{g}$, $0=(f,g)=\sum_i \mathrm{Res}_{p_i}\mathrm{tr}(g\nabla f)$, so $\mathrm{tr}(g\nabla f)$ lies in $\omega(\mathcal{C}^0)$ (for all $g\in\mathcal{A}_\mathfrak{g}$) by the residue theorem. If we let g run over an orthogonal basis $\{Y_i\}$ of $\mathfrak{a}$, with $\mathfrak{a}$ as before for some choice of trivialization of P_{C^0}/C^0 and write $\nabla f = Y_i\alpha^i$ for some meromorphic 1-form α^i, we see that $\alpha^i\in\omega(C^0)$ for all i. Hence $\nabla f\in H^0(C^0,\omega_C\otimes\mathrm{Ad}_{P/C})$, so $f\in\mathcal{B}_\mathfrak{g}$ and $\mathcal{A}^\perp\subseteq B$.

Obviously, $(\cdot,\cdot)_\mathfrak{g}$ is well defined on $\mathcal{A}_\mathfrak{g}^\perp/(\mathcal{A}_\mathfrak{g}+\mathcal{K}_\mathfrak{g}^0)$, and since it corresponds to the tr-integration pairing by the previous lemma, it is nondegenerate. Finally, if $f\in\mathcal{B}_\mathfrak{g}\cap\mathfrak{m}_\mathfrak{g}$, then ∇f is regular near p, so $\nabla f\in H^0(C,\omega_C\otimes\mathrm{Ad}_{P/C})$. Conversely, if $Y_i\alpha^i\in H^0(C,\omega_C\otimes\mathrm{Ad}_{P/C})$ is regular near p, then so is it primitive in $\mathcal{K}_\mathfrak{g}$, and hence $\mathcal{B}_\mathfrak{g}\cap\mathcal{O}_\mathfrak{g}\to H^0(C,\omega_C\otimes\mathrm{Ad}_{P/C})$ is surjective; its kernel is obviously $\mathcal{B}_\mathfrak{g}\cap\mathcal{K}_\mathfrak{g}^0$. □

Using these lemmas, we can formulate the "orthogonal" decomposition result for the absolute case:

Proposition 3.2.10. *Let* $\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathfrak{m}_\mathfrak{g},\mathcal{K}_\mathfrak{g}^0,\mathcal{A}_\mathfrak{g},(\cdot,\cdot)_\mathfrak{g}$ *be as before, and define* $\mathcal{F}_\mathfrak{g}^+ = \mathcal{A}_\mathfrak{g}^\perp\cap\mathfrak{m}_\mathfrak{g}$. *Also, let* $\mathcal{A}_\mathfrak{g}^-$ *be a complement for* $\mathcal{K}_\mathfrak{g}^0\cap\mathcal{A}_\mathfrak{g}$ *in* $\mathcal{A}_\mathfrak{g}$. *There exist linear subspaces* $\mathcal{F}_\mathfrak{g}^-\subseteq\mathcal{A}_\mathfrak{g}^\perp$ *and* $\mathcal{A}_\mathfrak{g}^+\subseteq\mathfrak{m}_\mathfrak{g}$ *such that*

- $\mathcal{K}_\mathfrak{g} = \mathcal{A}_\mathfrak{g}^-\oplus\mathcal{F}_\mathfrak{g}^-\oplus\mathcal{K}_\mathfrak{g}^0\oplus\mathcal{F}_\mathfrak{g}^+\oplus\mathcal{A}_\mathfrak{g}^+$ *and* $\mathfrak{m}_\mathfrak{g} = \mathcal{F}_\mathfrak{g}^+\oplus\mathcal{A}_\mathfrak{g}^+$,
- $(\mathcal{A}_\mathfrak{g}^\pm,\mathcal{F}_\mathfrak{g}^\pm) = 0$,

- $(\cdot,\cdot)_{\mathfrak{g}}$ *restricts to a perfect pairing between* $\mathcal{F}_{\mathfrak{g}}^-, \mathcal{F}_{\mathfrak{g}}^+$ *and between* $\mathcal{A}_{\mathfrak{g}}^-, \mathcal{A}_{\mathfrak{g}}^+$,
- *the isomorphism* $\mathcal{A}_{\mathfrak{g}}^{\perp}/(\mathcal{A}_{\mathfrak{g}}^- + \mathcal{K}_{\mathfrak{g}}^0) \simeq H^1(C;\mathcal{G})$ *identifies* $\mathcal{F}_{\mathfrak{g}}^-$ *and* $\mathcal{F}_{\mathfrak{g}}^+$ *with* $H^1(C, \mathrm{Ad}_{P/C})$ *and* $H^0(C, \omega_C \otimes \mathrm{Ad}_{P/C})$, *respectively,*
- $\mathcal{K}_{\mathfrak{g}}^{\leq 0} := \mathcal{A}_{\mathfrak{g}} + \mathcal{F}_{\mathfrak{g}}^- + \mathcal{K}_{\mathfrak{g}}^0$ *is independent of the choice of* $\mathcal{F}_{\mathfrak{g}}^-$.

Proof. Take for $\mathcal{F}_{\mathfrak{g}}^-$ a lift of $H^{0,1}(\mathcal{C};\mathcal{G}) \subseteq H^1(C;\mathcal{G})$ to $\mathcal{A}_{\mathfrak{g}}^{\perp}$ such that $\mathcal{F}_{\mathfrak{g}}^- \cap (\mathcal{A}_{\mathfrak{g}} + \mathcal{K}_{\mathfrak{g}}^0) = 0$ and $\mathcal{F}_{\mathfrak{g}}^- \cap \mathcal{F}_{\mathfrak{g}}^+ = 0$. We review how $\mathcal{F}_{\mathfrak{g}}^-$ is identified with $H^{0,1}(\mathcal{C};\mathcal{G})$: let f be in $\mathcal{F}_{\mathfrak{g}}^-$, let U, V, ϕ be as in the proof of lemma 3.1.6, and let α be the element of $H^0(C^0, \omega_C \otimes \mathrm{Ad}_{P/C})$ corresponding to f. Then the De Rham class of f in $H^1(C;\mathcal{G})$ is represented by $\phi\alpha + f\, d\phi$. By the way $\mathcal{F}_{\mathfrak{g}}^-$ was chosen, the $(1,0)$ part of this expression is exact; the $(0,1)$ part equals $f\bar{\partial}\phi$. Under the standard isomorphism $H^1(C, \mathrm{Ad}_{P/C}) \simeq H^{0,1}(C;\mathcal{G})$, $f\bar{\partial}\phi$ corresponds to $[f] \in H^1(C, \mathrm{Ad}_{P/C}) = \mathcal{K}_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{O}_{\mathfrak{g}})$. So $\mathcal{F}_{\mathfrak{g}}^-$ maps surjectively onto $\mathcal{K}_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{O}_{\mathfrak{g}})$, an consequently $\mathcal{K}_{\mathfrak{g}} = \mathcal{A}_{\mathfrak{g}} + \mathcal{F}_{\mathfrak{g}}^- + \mathcal{O}_{\mathfrak{g}} = \mathcal{A}_{\mathfrak{g}}^- + \mathcal{F}_{\mathfrak{g}}^- + \mathcal{K}_{\mathfrak{g}}^0 + \mathfrak{m}_{\mathfrak{g}}$. Since in addition, all summands intersect trivially, $\mathcal{K}_{\mathfrak{g}} = \mathcal{A}_{\mathfrak{g}}^- \oplus \mathcal{F}_{\mathfrak{g}}^- \oplus \mathcal{K}_{\mathfrak{g}}^0 \oplus \mathfrak{m}_{\mathfrak{g}}$.
Take $\mathcal{A}_{\mathfrak{g}}^+$ to be $(\mathcal{F}_{\mathfrak{g}}^-)^{\perp} \cap \mathfrak{m}_{\mathfrak{g}}$, then because $\mathcal{F}_{\mathfrak{g}}^+ \cap \mathcal{A}_{\mathfrak{g}}^+ = 0$, the map $\mathcal{F}_{\mathfrak{g}}^+ \to \mathfrak{m}_{\mathfrak{g}}/((\mathcal{F}_{\mathfrak{g}}^-)^{\perp} \cap \mathfrak{m}_{\mathfrak{g}})$ is injective. However, $(\cdot,\cdot)$ is well defined and nondegenerate on $\mathcal{F}_{\mathfrak{g}}^- \times \mathfrak{m}_{\mathfrak{g}}/\mathcal{A}_{\mathfrak{g}}^+$, so $\mathcal{F}_{\mathfrak{g}}^+ \to \mathfrak{m}_{\mathfrak{g}}/((\mathcal{F}_{\mathfrak{g}}^-)^{\perp} \cap \mathfrak{m}_{\mathfrak{g}})$ must be surjective as well. It follows that $\mathfrak{m}_{\mathfrak{g}} = \mathcal{F}_{\mathfrak{g}}^+ \oplus \mathcal{A}_{\mathfrak{g}}^+$, and as a result of that, $\mathcal{K}_{\mathfrak{g}} = \mathcal{A}_{\mathfrak{g}}^- \oplus \mathcal{F}_{\mathfrak{g}}^- \oplus \mathcal{K}_{\mathfrak{g}}^0 \oplus \mathcal{F}_{\mathfrak{g}}^+ \oplus \mathcal{A}_{\mathfrak{g}}^+$. For the final assertion we remark that $\mathcal{K}_{\mathfrak{g}}^{\leq 0} = \nabla^{-1} H^{0,1}(C;\mathcal{G})$. □

3.2.3 (Pre)symplectic structure: the relative case

We continue the discussion for a family of pointed curves $(\mathcal{C}/B, p)$, endowed with a principal G-bundle $(P, \mathcal{C})$, and assume that P has a flat connection ∇ over $\mathcal{C}/B$. Again, we *denote the sheaf of flat local sections of* $\mathrm{Ad}_{P/\mathcal{C}}$ *over* $\mathcal{C}$ *by* $\mathcal{G}$.

Analogous to 3.5, the long exact sequence of the relative cohomology over B of the pair $\mathcal{C}, \mathcal{C}^0$ with coefficients in $\mathcal{G}$ reduces to

$$0 \longrightarrow R^1(\mathcal{C}/B)_*\mathcal{G} \longrightarrow R^1(\mathcal{C}^0/B)_*\mathcal{G} \xrightarrow{w_{\mathfrak{g}}} \mathcal{K}_{\mathfrak{g}}^0 \longrightarrow (\mathcal{C}/B)_*\mathcal{G} \longrightarrow 0,$$

where this is now an exact sequence of $\mathcal{O}_B$-modules. Note that if P/C is regularly stable, $(\mathcal{C}/B)_*\mathcal{G} \subseteq (\mathcal{C}/B)_*\mathrm{Ad}_{P/\mathcal{C}} = 0$, since the last term is the sheaf of infinitesimal automorphisms. Furthermore, we have that

$$R^1(\mathcal{C}^0/B)_*\mathcal{G} = \frac{(\mathcal{C}^0/B)_*\omega_{\mathcal{C}^0/B} \otimes \mathrm{Ad}_{P/C}}{\nabla(\mathcal{C}^0/B)_*\mathrm{Ad}_{P/C}} = \frac{\omega_{\mathcal{A}/\mathcal{O}_B} \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{g}}}{\nabla \mathcal{A}_{\mathfrak{g}}}.$$

Equation (3.6) for $w_{\mathfrak{g}}$ still holds, provided we now interpret $\mathcal{K}_{\mathfrak{g}}, \mathcal{K}^0_{\mathfrak{g}}$ as $\mathcal{O}_B$-modules: if $\alpha_i X^i$ is a local section of $R^1(\mathcal{C}^0/B)_*\mathcal{G}$, where α_i, X^i are local section of $R^1(\mathcal{C}^0/B)_*(\mathcal{C}^0/B)^{-1}\mathcal{O}_B$ and $(\mathcal{C}^0/B)_*\mathcal{G}$ respectively, then

$$w_{\mathfrak{g}}(\alpha_i X^i)_j = \mathrm{Res}_{p_j}(\alpha_i)X^i.$$

Definition 3.2.11. Let $\mathcal{H}_{\mathfrak{g}}$ be short for $R^1(\mathcal{C}/B)_*\mathcal{G}$. Furthermore, let $\widetilde{\mathcal{B}}_{\mathfrak{g}}$ be the maximal subsheaf of $(\mathcal{C}^0/B)_*(\omega_{\mathcal{C}/B} \otimes \mathrm{Ad}_{P/\mathcal{C}})$ with image in $R^1(\mathcal{C}^0/B)_*\mathrm{Ad}_{P/\mathcal{C}}$ equal to $\mathcal{H}_{\mathfrak{g}}$, and let $\mathcal{B}_{\mathfrak{g}}$ be the maximal $\mathcal{K}^0$-submodule of $\mathcal{K}_{\mathfrak{g}}$ such that $\nabla\mathcal{B}_{\mathfrak{g}} \subseteq \widetilde{\mathcal{B}}_{\mathfrak{g}}$, where we identified $\widetilde{\mathcal{B}}_{\mathfrak{g}}$ with a submodule of $\mathcal{K}_{\mathfrak{g}} \otimes \omega_{\mathcal{C}/B}$.

The map $\nabla : \mathcal{K}_{\mathfrak{g}} \to \mathcal{K}_{\mathfrak{g}} \otimes \Omega_{\mathcal{K}/\mathcal{O}_B}$ is surjective, so that by definition, $\nabla : \mathcal{B}_{\mathfrak{g}} \to \widetilde{\mathcal{B}}_{\mathfrak{g}}$ is also surjective. Hence $\mathcal{B}_{\mathfrak{g}}$ is mapped surjectively to $\mathcal{H}_{\mathfrak{g}}$, so that the following sequence is exact:

$$0 \longrightarrow \mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}} \longrightarrow \mathcal{B}_{\mathfrak{g}} \xrightarrow{\nabla} \mathcal{H}_{\mathfrak{g}} \longrightarrow 0.$$

Note that for a closed $b \in B$, the fiber of $\mathcal{H}_b$ is just $H^1(\mathcal{C}_b; \mathcal{G})$. Moreover, the pairing that $(\cdot,\cdot)_{\mathfrak{g}}$ induces on $\mathcal{H}_b$ is the tr-integration pairing from lemma 3.2.8, where the integration is over $\mathcal{C}_b$.

We have the following analogue of lemma 3.2.9:

Lemma 3.2.12. *The following holds:*

- $\mathcal{B}_{\mathfrak{g}} = \mathcal{A}^{\perp}_{\mathfrak{g}}$,
- $(\cdot,\cdot)_{\mathfrak{g}}$ *is a well defined and symplectic form on* $\mathcal{A}^{\perp}_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}})$ *and corresponds to the* tr*-fiber-integration pairing under* $\mathcal{A}^{\perp}_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}}) \simeq \mathcal{H}_{\mathfrak{g}}$,
- *this isomorphism identifies* $\mathcal{B}_{\mathfrak{g}} \cap \mathfrak{m}_{\mathfrak{g}}$ *with* $(\mathcal{C}/B)_*\omega_{\mathcal{C}/B} \otimes \mathrm{Ad}_{P/\mathcal{C}}$.

The relative version of proposition 3.2.10 reads:

Proposition 3.2.13. *Let* $\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}, \mathcal{K}^0_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}}, (\cdot,\cdot)_{\mathfrak{g}}$ *be as before, and define* $\mathcal{F}^+_{\mathfrak{g}} = \mathcal{A}^{\perp}_{\mathfrak{g}} \cap \mathfrak{m}_{\mathfrak{g}}$. Locally over B, *there are* $\mathcal{O}_B$*-modules* $\mathcal{A}^-_{\mathfrak{g}} \subseteq \mathcal{A}_{\mathfrak{g}}$, $\mathcal{F}^-_{\mathfrak{g}} \subseteq \mathcal{A}^{\perp}_{\mathfrak{g}}$ *and* $\mathcal{A}^+_{\mathfrak{g}} \subseteq \mathfrak{m}_{\mathfrak{g}}$ *such that* $\mathcal{A}^-_{\mathfrak{g}}$ *is a complement for* $\mathcal{K}^0_{\mathfrak{g}} \cap \mathcal{A}_{\mathfrak{g}}$ *in* $\mathcal{A}_{\mathfrak{g}}$ *and*

- $\mathcal{K}_{\mathfrak{g}} = \mathcal{A}^-_{\mathfrak{g}} \oplus \mathcal{F}^-_{\mathfrak{g}} \oplus \mathcal{K}^0_{\mathfrak{g}} \oplus \mathcal{F}^+_{\mathfrak{g}} \oplus \mathcal{A}^+_{\mathfrak{g}}$ *and* $\mathfrak{m}_{\mathfrak{g}} = \mathcal{F}^+_{\mathfrak{g}} \oplus \mathcal{A}^+_{\mathfrak{g}}$,
- $(\mathcal{A}^{\pm}_{\mathfrak{g}}, \mathcal{F}^{\pm}_{\mathfrak{g}}) = 0$,
- $(\cdot,\cdot)_{\mathfrak{g}}$ *restricts to a perfect pairing between* $\mathcal{F}^-_{\mathfrak{g}}, \mathcal{F}^+_{\mathfrak{g}}$ *and between* $\mathcal{A}^-_{\mathfrak{g}}, \mathcal{A}^+_{\mathfrak{g}}$,

- *the isomorphism* $\mathcal{A}_{\mathfrak{g}}^{\perp}/(\mathcal{A}_{\mathfrak{g}}^{-}+\mathcal{K}_{\mathfrak{g}}^{0}) \simeq \mathcal{H}_{\mathfrak{g}}$ *identifies* $\mathcal{F}_{\mathfrak{g}}^{-}$ *with* $R^1(\mathcal{C}/B)_*\mathrm{Ad}_{P/C}$ *and* $\mathcal{F}^{+}$ *with* $(\mathcal{C}/B)_*(\omega_{\mathcal{C}/B}\otimes \mathrm{Ad}_{P/\mathcal{C}})$*,*

- $\mathcal{K}_{\mathfrak{g}}^{\leq 0} := \mathcal{A}_{\mathfrak{g}} + \mathcal{F}_{\mathfrak{g}}^{-} + \mathcal{K}_{\mathfrak{g}}^{0}$ *is independent of the choice of* $\mathcal{F}_{\mathfrak{g}}^{-}$*.*

The proof is an adaptation of the proof of proposition 3.2.10, where one considers $\mathcal{C}_U/U$ instead of C/ Spec , where U is such that $\mathcal{C}^0$ affine; again, details are left to the reader. We note that for regularly stable bundles, $\mathcal{A}_{\mathfrak{g}} \cap \mathcal{O}_{\mathfrak{g}} = 0$, so that we can take $\mathcal{A}_{\mathfrak{g}}^{-} = \mathcal{A}_{\mathfrak{g}}$.

3.2.4 G-symmetry for the case of a trivial bundle with trivial connection

In Chapter 5, we will need a certain G-symmetry of $\mathcal{K}_{\mathfrak{g}}$ in order to construct the WZW connection. More specifically:

Assumption 3.2.14 (triviality assumption). We assume that there is a G-action on $\mathcal{K}_{\mathfrak{g}}$ that commutes with ∇, leaves $[\cdot,\cdot]$ invariant, preserves $\mathcal{A}_{\mathfrak{g}}$ and acts without fixed points.

Lemma 3.2.15. *Suppose* $P/\mathcal{C}$ *is trivial and* ∇ *has trivial monodromy. Then, locally over* B*,* $\mathcal{K}_{\mathfrak{g}}$ *has a* G*-action that satisfies (*3.2.14*), and preserves* tr*. Moreover, one can locally identify* $\mathcal{K}_{\mathfrak{g}}$ *with* $\mathcal{K}\otimes\mathfrak{g}$*, such that* $1\otimes\mathfrak{g}$ *consists of flat sections.*

Proof. By corollary 3.2.6, locally over B there exists a subalgebra $\mathfrak{a}$ of $\mathcal{A}_{\mathfrak{g}} \cap \mathcal{K}_{\mathfrak{g}}^{0}$, isomorphic to $\mathfrak{g}$ and such that $\mathcal{A}_{\mathfrak{g}} = \mathfrak{a}\mathcal{A}$ - choose such an $\mathfrak{a}$ and an isomorphism $\mathfrak{a} \simeq \mathfrak{g}$. Then by via the adjoint action, G acts $\mathcal{K}$ linearly on $\mathfrak{g}\otimes\mathcal{K} \simeq \mathfrak{a}\otimes\mathcal{K} = \mathcal{K}_{\mathfrak{g}}$ and preserves $\mathcal{A}_{\mathfrak{g}}$. The Lie bracket of $\mathcal{K}_{\mathfrak{g}}$ is preserved by this action since the adjoint action of G on $\mathfrak{g}$ preserves the product; the same goes for tr. Furthermore, $\mathfrak{a} \subseteq \mathcal{K}^0$, so $\mathfrak{a}\mathcal{K}^0 = \mathcal{K}_{\mathfrak{g}}^{0}$. Since this is preserved by G, it follows that ∇ commutes with the G-action by $\mathcal{K}$ linearity. Finally, there are no fixed points in $\mathfrak{g}$ under G (recall that G is a simple algebraic group), so there are also no fixed points in $\mathcal{K}_{\mathfrak{g}}$ under G. □

We now investigate the relation of proposition 3.2.13 with the G-action. First, note that because $\mathcal{K}_{\mathfrak{g}}^{0}$ is preserved under G, and that the action is $\mathcal{K}$ linear, $\mathfrak{m}\mathcal{K}_{\mathfrak{g}}^{0} = \mathfrak{m}_{\mathfrak{g}}$ and $\mathcal{O}\mathcal{K}_{\mathfrak{g}}^{0} = \mathcal{O}_{\mathfrak{g}}$ are also preserved. In fact, it is possible to preserve the entire decomposition of $\mathcal{K}_{\mathfrak{g}}$ of proposition 3.2.13:

Lemma 3.2.16. *Assume that locally over* B*,* $\mathcal{K}_{\mathfrak{g}}$ *has a* G*-action as described in (3.2.14). Then* G *acts symplectically on* $\mathcal{K}_{\mathfrak{g}}$ *and preserves* $\mathcal{F}^+_{\mathfrak{g}}$*. In addition, one can, locally over* B*, choose* $A^{\pm}_{\mathfrak{g}}$*,* $B_{\mathfrak{g}}$ *and* $F^+_{\mathfrak{g}}$ *as in proposition 3.2.13 to be* G*-invariant.*

Proof. Since G preserves $\mathfrak{m}_{\mathfrak{g}}$ and $\mathcal{A}_{\mathfrak{g}}$, $\mathcal{F}^+_{\mathfrak{g}}$ is clearly also G-invariant. Furthermore, since the G-action commutes with ∇, we have that

$$\begin{aligned}(g\cdot X_1 f_1, g\cdot X_2 f_2)_{\mathfrak{g}} &= \operatorname{Res}_p \operatorname{tr}(g\cdot X_2 f_2 \nabla g\cdot X_1 f_1)\\ &= \operatorname{tr}(g\cdot X_2, g\cdot X_1)\operatorname{Res}_p(f_2\, d_{\mathcal{C}/B} f_1)\\ &= \operatorname{tr}(X_2, X_1)\operatorname{Res}_p(f_2\, d_{\mathcal{C}/B} f_1) = (X_1 f_1, X_2 f_2)_{\mathfrak{g}}\end{aligned}$$

for any $X_i \in \mathcal{K}^0_{\mathfrak{g}}$ and $f_i \in \mathcal{K}$. Here we used the G invariance of tr. Since $\mathcal{K}_{\mathfrak{g}}$ is generated by $= \mathcal{K}\mathcal{K}^0_{\mathfrak{g}}$, it follows that G preserves $(\cdot,\cdot)_{\mathfrak{g}}$. As a consequence, orthogonal complements of invariant subspaces are preserved, and in particular, $\mathcal{A}^{\perp}_{\mathfrak{g}}$ is G-invariant. In the proof of 3.2.13, $\mathcal{A}^-_{\mathfrak{g}}$ was chosen as a complement for $\mathcal{K}^0_{\mathfrak{g}} \cap \mathcal{A}_{\mathfrak{g}}$ in $\mathcal{A}_{\mathfrak{g}}$, $\mathcal{F}^-$ as a complement for $\mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}} + \mathcal{F}^+_{\mathfrak{g}}$ in $\mathcal{A}^{\perp}_{\mathfrak{g}}$ and $\mathcal{A}^+_{\mathfrak{g}}$ as complement for $\mathcal{F}^+_{\mathfrak{g}}$ in $\mathfrak{m}_{\mathfrak{g}}$. Since G is reductive, one can choose a complement for G-invariant subspace to be G-invariant. Hence $\mathcal{F}^-_{\mathfrak{g}}, \mathcal{A}^{\pm}_{\mathfrak{g}}$ can be chosen to be G-invariant. □

3.3 Derivations

We finish this chapter with the relations between the presymplectic structure described above and the derivations associated to the pointed family $(\mathcal{C}/B, p)$. Besides $\theta_{\mathcal{C}/B}$, these are

- $\theta_{\widehat{\mathcal{C}}^0_p/B} = \theta_{\mathcal{K}/\mathcal{O}_B}$,
- $\theta_{\widehat{\mathcal{C}}_p/B} = \theta_{\mathcal{O}/\mathcal{O}_B}$,
- $(\mathcal{C}^0/B)_*\theta_{\mathcal{C}^0/B} = \theta_{\mathcal{A}/\mathcal{O}_B}$.

Restriction of $\widehat{\mathcal{C}}_p$, $\mathcal{C}^0$, to $\widehat{\mathcal{C}}^0_p$ gives, respectively, injective algebra morphisms $\theta_{\mathcal{O}/\mathcal{O}_B} \hookrightarrow \theta_{\mathcal{K}/\mathcal{O}_B}$, $\theta_{\mathcal{A}/\mathcal{O}_B} \hookrightarrow \theta_{\mathcal{K}/\mathcal{O}_B}$, and we will often identify $\theta_{\mathcal{O}/\mathcal{O}_B}, \theta_{\mathcal{A}/\mathcal{O}_B}$ with their images in $\theta_{\mathcal{K}/\mathcal{O}_B}$. Furthermore, we will sometimes also consider the following sheaves of $\mathcal{O}_B$-algebras:

Definition 3.3.1. We denote the subsheaves of, respectively, $\theta_{\mathcal{K}/\mathbb{C}}, \theta_{\mathcal{O}/\mathbb{C}}, \theta_{\mathcal{A}/\mathbb{C}}$ that preserve $\mathcal{O}_B$ by $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}, \theta_{\mathcal{O},\mathcal{O}_B/\mathbb{C}}, \theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}$.

Again, restriction to $\widehat{\mathcal{C}}_p^0$ identifies $\theta_{\mathcal{O},\mathcal{O}_B/\mathbb{C}}$ and $\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}$ with subalgebras of $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$. By definition, we have inclusions

$$\theta_{\mathcal{K}/\mathcal{O}_B} \subseteq \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}, \quad \theta_{\mathcal{O}/\mathcal{O}_B} \subseteq \theta_{\mathcal{O},\mathcal{O}_B/\mathbb{C}}, \quad \theta_{\mathcal{A}/\mathcal{O}_B} \subseteq \theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}.$$

Lemma 3.3.2. *For local sections* x, y, z *of* $\mathcal{K}_{\mathfrak{g}}$*,* e, f, g *of* $\mathcal{K}$ *and* D *of* $\theta_{\mathcal{K}/\mathcal{O}_B}$ *we have the following properties:*

$$(ef, g) = (e, fg) + (f, eg) \tag{3.7}$$
$$(De, f) = -(e, Df) = (Df, e) \tag{3.8}$$
$$(eDf, g) = (eDg, f) \tag{3.9}$$
$$0 = ([x, y], z)_{\mathfrak{g}} + ([y, z], x)_{\mathfrak{g}} + ([z, y], x)_{\mathfrak{g}} \tag{3.10}$$
$$(\nabla_D x, y)_{\mathfrak{g}} = -(x, \nabla_D y)_{\mathfrak{g}} = (\nabla_D y, x)_{\mathfrak{g}} \tag{3.11}$$
$$([x, \nabla_D y], z)_{\mathfrak{g}} = (y, [x, \nabla_D z])_{\mathfrak{g}} = -([x, \nabla_D z], y)_{\mathfrak{g}} \tag{3.12}$$

Proof. The first statement is the equality

$$\sum_i \mathrm{Res}_{p_i}(g\ d_{\mathcal{C}/B}(ef)) = \sum_i \mathrm{Res}_{p_i}(eg\ d_{\mathcal{C}/B} f) + \mathrm{Res}_{p_i}(gf\ d_{\mathcal{C}/B} e).$$

For (3.8), let L_D denote the Lie derivative with respect to D and ι_D contraction of a 1-form with D, so $L_D = d_{\mathcal{C}/B}\iota_D + \iota_D\, d_{\mathcal{C}/B}$ when acting on differential forms. Observe that

$$f\ d_{\mathcal{C}/B}(De) + Df\ d_{\mathcal{C}/B} e = f\ d_{\mathcal{C}/B} L_D(e) + L_D(f)\ d_{\mathcal{C}/B} e = L_D(e\ d_{\mathcal{C}/B} f).$$

and that because $\Omega^{\wedge 2}_{\mathcal{C}/B} = 0$,

$$L_D(e\ d_{\mathcal{C}/B} f) = (\ d_{\mathcal{C}/B}\iota_D + \iota_D\ d_{\mathcal{C}/B})(e\ d_{\mathcal{C}/B} f) = \ d_{\mathcal{C}/B}\iota_D(e\ d_{\mathcal{C}/B} f)$$

is exact. It follows that

$$\begin{aligned}(De, f) + (e, Df) &= \sum_i \mathrm{Res}_{p_i}(f\ d_{\mathcal{C}/B} De) + \sum_i \mathrm{Res}_{p_i}(Df\ d_{\mathcal{C}/B} e) \\ &= \sum_i \mathrm{Res}_{p_i}(L_D(e\ d_{\mathcal{C}/B} f) = 0.\end{aligned}$$

Relation (3.9) simply follows from (3.8) by noting that eD is also again an element of $\theta_{K,R/\mathbb{C}}$.

To prove (3.10)-(3.12), we can, without loss of generality, assume that x, y, z are respectively of the form Xe, Yf, Zg, where $X, Y, Z \in \mathcal{K}^0_{\mathfrak{g}}$. Then

$$\begin{aligned}([Xe, Yf], Zg)_{\mathfrak{g}} &= \mathrm{tr}([X, Y], Z)(ef, g) \\ &= -\mathrm{tr}(X, [Z, Y])(e, fg) - \mathrm{tr}(Y, [X, Z])(f, eg) \\ &= -(Xe, [Zg, Yf])_{\mathfrak{g}} - (Yf, [Xe, Zg])_{\mathfrak{g}} \\ &= -([Yf, Zg], Xe)_{\mathfrak{g}} - ([Zg, Xe], Yf)_{\mathfrak{g}}\end{aligned}$$

proves the (3.10),

$$(\nabla_D Xe, Yf)_{\mathfrak{g}} = \mathrm{tr}(X, Y)(De, f) = -\mathrm{tr}(X, Y)(e, Df) = -(Xe, \nabla_D Yf),$$

proves (3.12) and

$$\begin{aligned}([Xe, \nabla_D Yf], Zg)_{\mathfrak{g}} &= \mathrm{tr}([X, Y], Z)(eD(f), g) = \mathrm{tr}(Y, [X, Z])(f, eDg) \\ &= (Yf, [Xe, \nabla_D Zg])_{\mathfrak{g}}\end{aligned}$$

proves the sixth equation. □

Finally, we have the following useful result.

Lemma 3.3.3. *If* D *is a local section of* $\in \theta_{\mathcal{A}/\mathcal{O}_B}$, f *a local section of* $\mathcal{A}^\perp$ *and* x *of* $\mathcal{A}^\perp_{\mathfrak{g}}$*, then* Df *is a local section of* $\mathcal{A}$ *and* $\nabla_D x$ *of* $\mathcal{A}_{\mathfrak{g}}$*. If* ∇ *extends to a connection over* B*, so* ∇ *is also defined on* $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$*, then for* D *a local section of* $\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}$*,* D *maps* $\mathcal{A}$ *to itself and* ∇_D *maps* $\mathcal{A}_{\mathfrak{g}}$ *to itself.*

Proof. By definition of $\mathcal{B} = \mathcal{A}^\perp$, f has the property that $d_{\hat{\mathcal{C}}^0_p/B} f$ extends to a local section of $(\mathcal{C}^0/B)_*\omega_{\mathcal{C}/B}$, so that Df, being the contraction between sections of $\theta_{\mathcal{C}^0/B}$ and $(\mathcal{C}^0/B)_*\omega_{\mathcal{C}/B}$, is a section of $(\mathcal{C}^0/B)_*\mathcal{O}_{\mathcal{C}} = \mathcal{A}$.

A similar argument goes for $\mathcal{B}_{\mathfrak{g}} = \mathcal{A}^\perp_{\mathfrak{g}}$: $\nabla_D x$ is the contraction between D, ∇x, and since the first lies $\theta_{\mathcal{C}^0/B}$, and the second in $(\mathcal{C}^0/B)_*\mathrm{Ad}_{P/C}$, the contraction is a local section of $(\mathcal{C}^0/B)_*\mathrm{Ad}_{P/C} = \mathcal{A}_{\mathfrak{g}}$.

The final assertion follows form the observation that is f is a local section of $\mathcal{A}$ (or $\mathcal{A}_{\mathfrak{g}}$), then locally over B, both D (or ∇_D) and f are defined on $\mathcal{C}^0$, so that Df (or $\nabla_D f$) is also defined on $\mathcal{C}^0$, and hence is a local section of $\mathcal{A}$ (or $\mathcal{A}_{\mathfrak{g}}$). □

Chapter 4

The Fock representation associated to a variation of Hodge structure

In the previous chapter, we have extracted certain algebraic data from a geometric object: a family of pointed curves, endowed with a G-bundle with connection. This data will serve as input for this chapter and the next. Morally speaking, in this chapter we will only use the cohomological data associated to the geometry, whereas in the next one, we will use a little more. Both outputs will be compared in the last chapter where, as we will see, the results of this chapter can be interpreted as the "graded" version of the next chapter, with respect to a certain natural filtration. We start by, basically, redoing the first part of [10].

4.1 The Fock module $\mathbb{F}(\mathcal{H}, \mathcal{F})$.

We will work with the following setting: let B be a smooth finite dimensional complex manifold with sheaf of holomorphic functions $\mathcal{O}_B$. Furthermore, we assume given a locally free $\mathcal{O}_B$-module $\mathcal{H}$ of finite rank, a $\mathcal{O}_B$-bilinear symplectic form $(\cdot,\cdot)$, and a Lagrangian $\mathcal{O}_B$-submodule $\mathcal{F}$ of $\mathcal{H}$. The reader can take $B, \mathcal{F}, \mathcal{H}, (\cdot,\cdot)$ as in the previous chapter as example in mind. Below we will often shorten the assumption 'x is a local section of an $\mathcal{O}_B$-module $\mathcal{M}$' to $x \in \mathcal{M}$ - we trust that no confusion will arise.

Since $(\cdot,\cdot)$ is nondegenerate,

$$\mathcal{H} \to \mathcal{H}^* := \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{H}, \mathcal{O}_B) : a \mapsto (a, \cdot)$$

is an isomorphism; we call the pair $(\mathcal{H}, (\cdot,\cdot))$ a symplectic $\mathcal{O}_B$-module. There is a Lie algebra naturally associated to the symplectic $\mathcal{O}_B$-module $(\mathcal{H}, (\cdot,\cdot))$:

Definition 4.1.1. Define $\widehat{\mathcal{H}} := \mathcal{H} \oplus \mathcal{O}_B$ and endow it with the product

$$[a \oplus r, b \oplus r'] = 0 \oplus (a, b).$$

The global section $0 \oplus 1$ is denoted $\hbar$.

One easily checks that $(\widehat{\mathcal{H}}, [\cdot,\cdot])$ is a Lie algebra with center $\mathcal{O}_B\hbar$; it is called *the Heisenberg algebra*. If we regard $\mathcal{H}$ as Lie algebra with trivial product, than $\widehat{\mathcal{H}}$ is a central extension of $\mathcal{H}$ by $\mathcal{O}_B$ with central generator $\hbar$.

Identify the isotropic submodule $\mathcal{F} \subseteq \mathcal{H}$ with its image in $\widehat{\mathcal{H}}$, and observe that it is an abelian subalgebra, as is the trivial central extension $\widehat{\mathcal{F}} := \mathcal{F} + \hbar\mathcal{O}_B$. The latter has a character χ, given by the projection to $\mathcal{O}_B$ along $\mathcal{F}$. One can use χ, or any multiple of χ, to define an action of $\widehat{\mathcal{F}}$ on $\mathcal{O}_B$.

Definition 4.1.2. To the tuple $(\mathcal{H}, (\cdot,\cdot), \mathcal{F}, \ell)$, with $\ell \in \mathbb{Z}$ and the rest as above, we associate the Verma module

$$\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell := U\widehat{\mathcal{H}} \otimes_{U\widehat{\mathcal{F}}} \mathcal{O}_B,$$

where $\widehat{\mathcal{F}}$ acts on $\mathcal{O}_B$ by $\ell\chi$. We call this the *Fock module associated to* $(\mathcal{H}, (\cdot,\cdot), \mathcal{F})$ *of level* ℓ and write f_ℓ for the (global) generator $1 \otimes 1$.

From now on, we will assume ℓ *in* $\mathbb{Z}$ *to be fixed.* Note that $\hbar$ acts on $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$ as multiplication by ℓ, and that ℓ is in fact characterized by this property.

If $\mathcal{F}'$ is a Lagrangian subspace complementary to $\mathcal{F}$, we have that the inclusion $\mathrm{Sym}_\bullet \mathcal{F}' \to \mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$ is an isomorphism: surjectivity follows by commuting all factors $\hbar\mathcal{O}_B$ and $\mathcal{F}$ to the right, were they act via multiplication and 0 respectively. Injectivity follows from the vanishing of $(\cdot,\cdot)$ on $\mathcal{F}'$. We write $\vec{x} \circ f_0$ for the image of $\vec{x} \in \mathrm{Sym}_\bullet \mathcal{F}'$ in $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$

4.1.1 The representation of $\mathfrak{sp}(\mathcal{H})$ on $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$

There is a canonical representation of $\mathfrak{sp}(\mathcal{H})$ on $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$, as we will show next. By tensoring an isomorphism $\mathcal{H} \to \mathcal{H}^*$ with $\mathcal{H}$, we obtain another isomorphism $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}^* = \mathrm{End}(\mathcal{H})$. In case of the one induced by the symplectic form, we call the resulting map E:

$$E : \mathcal{H} \otimes \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}^* : a \otimes b \longmapsto (x \mapsto a(b, x)) \ .$$

The natural involution on $\mathcal{H} \otimes \mathcal{H}$ corresponds to an involution on $\mathrm{End}(\mathcal{H})$ under this isomorphism, and we denote it by $(\cdot)^\top$, so

$$(E_{a\otimes b})^\top = E_{b\otimes a}.$$

Since $(E_{a\otimes b}x, y) + (x, E_{a\otimes b}y) = (b, x)(a, y) + (b, y)(x, a)$, we see that $E_{a\otimes b}$ is symplectic iff $a = b$. It follows that E restricts to an isomorphism $\mathrm{Sym}_2\mathcal{H} \to \mathfrak{sp}(\mathcal{H})$ and that $\mathfrak{sp}(\mathcal{H})$ is the fixedpoint set of $(\cdot)^\top$.

We now introduce a special class of local bases of $\mathcal{H}$, in which E takes a very nice form.

Definition 4.1.3. We call a set of local sections $\{e_i\}_{|i|=1,\ldots,n} \subset \mathcal{H}$ a *local quasisymplectic basis* of $\mathcal{H}$ over $\mathcal{O}_B$, if locally it is a basis for $\mathcal{H}$ over $\mathcal{O}_B$ and $(e_i, e_j) = i\delta_{i,-j}$.

Given such $\{e_i\}$ and a $x \in \mathcal{H}$, we have that $x = \sum_i \frac{(x, e_{-i})}{i} e_i$. Using this, one readily verifies that

$$E^{-1}(\alpha) = \sum_{|i|=1}^{n} \frac{-1}{i} \alpha e_i \otimes e_{-i} = \sum_{|i|,|j|=1}^{n} \frac{(\alpha e_i, e_j)}{ij} e_{-i} \otimes e_{-j}.$$

If we regard $\mathrm{Hom}(\mathcal{H}/\mathcal{F}, \mathcal{F})$ as a submodule of $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$, then E restricts to an isomorphism of $\mathrm{Sym}_2\mathcal{F}$ to $\mathrm{Hom}(\mathcal{H}/\mathcal{F}, \mathcal{F})$: suppose $a \otimes a \in \mathrm{Sym}_2\mathcal{F}$, then $E_{a\otimes a}$ obviously has its image in $\mathcal{F}$ and has kernel equal to $\mathcal{F}$, so that it factors over $\mathcal{H}/\mathcal{F} \simeq \mathcal{F}'$. Conversely, if $E_{a\otimes b}$ maps to $\mathcal{F}$ and has kernel $\mathcal{F}$, then $b \in \mathcal{F}$ and $(a, \mathcal{F}) = 0$, so $a \in \mathcal{F}$. By choosing a Lagrangian submodule $\mathcal{F}' \subseteq \mathcal{H}$ complementary to $\mathcal{F}$, we can identify $\mathcal{H}/\mathcal{F}$ with $\mathcal{F}'$, so that E restricts to an isomorphism $\mathrm{Sym}_2\mathcal{F} = \mathrm{Hom}(\mathcal{F}', \mathcal{F})$. Similarly, E restricts to an isomorphism $\mathrm{Sym}_2\mathcal{F}' \to \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$. We now define

$$\tau : \mathrm{End}(\mathcal{H}) \to U\widehat{\mathcal{H}}[\hbar^{-1}] : \alpha \mapsto \frac{1}{2\hbar} E^{-1}(\alpha).$$

Lemma 4.1.4. *The restriction of τ to $\mathfrak{sp}(\mathcal{H})$ is a homomorphism of Lie algebras. Furthermore, if $\alpha \in \mathfrak{sp}(\mathcal{H})$, then* $\mathrm{ad}\,\tau(\alpha)$ *acts on* $U\widehat{\mathcal{H}}[\hbar^{-1}]$ *as* α.

Proof. It suffices to check these statements for α, β of the form $E_{a\otimes a}$, $E_{b\otimes b}$, respectively. We start with the last statement; let $x \in \mathcal{H}$, then

$$\operatorname{ad}\tau(E_{a\otimes a})(x) = \frac{1}{2\hbar}[a\circ a, x] = \frac{1}{2\hbar}(a\circ[a,x]+[a,x]\circ a) = a(a,x) = E_{a\otimes a}x.$$

For the first statement we have that

$$\begin{aligned} & 4\hbar^2[\tau(E_{a\otimes a}), \tau(E_{b\otimes b})] = [a\circ a, b\circ b] \\ = \; & [a,b]\circ a\circ b + a\circ[a,b]\circ b + b\circ[a,b]\circ a + b\circ a\circ[a,b] \\ = \; & 2\hbar(a,b)(a\circ b + b\circ a) = (a,b)\tau(E_{a\otimes b+b\otimes a}) \end{aligned}$$

and

$$\begin{aligned} [E_{a\otimes a}, E_{b\otimes b}](x) &= (b,x)(a,b)a - (a,x)(b,a)b = (a,b)((b,x)a + (a,x)b) \\ &= (a,b)E_{a\otimes b+b\otimes a}(x) \end{aligned}$$

so that $\tau([E_{a\otimes a}, E_{b\otimes b}]) = (a,b)\tau(E_{a\otimes b+b\otimes a})$ □

The lemma above shows that, if we let ρ denote right multiplication of $U\widehat{\mathcal{H}}$ on $\mathbb{F}(\mathcal{H},\mathcal{F})_\ell$, then $\rho\circ\tau$ is a representation of $\mathfrak{sp}(\mathcal{H})$ on $\mathbb{F}(\mathcal{H},\mathcal{F})_\ell$. For $E(a\otimes a) = \alpha$, $\vec{x} \in U\widehat{\mathcal{H}}$, we can give an explicit formula for the action:

$$\begin{aligned} 2\hbar\rho(\tau(\alpha))(\vec{x}\circ f_\ell) = a\circ a\circ\vec{x}\circ f_\ell &= \operatorname{ad}[a\circ a, \vec{x}]\circ f_\ell + \vec{x}\circ a\circ a\circ f_\ell \\ &= 2\hbar\alpha(\vec{f})\circ f_\ell + 2\hbar\vec{f}\circ\tau(\alpha)\circ f_\ell. \end{aligned}$$

We now make a small digression here to discuss the notion of *normal ordering*.

4.1.2 Intermezzo: normal ordering

Suppose $\mathcal{M}$ is an $\mathcal{O}_B$-module and $\{\mathcal{M}_i\}_{i\in I}$ are submodules of $\mathcal{M}$ such that $\mathcal{M} = \bigoplus_{i\in I}\mathcal{M}_i$. In case $\mathcal{M}$ has a topology, we relax the latter requirement to the condition that $\bigoplus_{i\in I}\mathcal{M}_i$ is dense in $\mathcal{M}$. We also assume that a partial order $\mathcal{P}$ on I is given. The normal ordering associated to $(\{\mathcal{M}_i\}_{i\in I}, \mathcal{P})$ is the map

$$\mathcal{M}^{\otimes 2} \to \mathcal{M}^{\otimes 2} : \mathcal{M}_i\otimes\mathcal{M}_j \ni a\otimes b = \begin{cases} b\otimes a & j < i \\ a\otimes b & \text{otherwise} \end{cases}$$

We call it nontrivial if it is not the identity. The common notation for a normal ordering is $x \mapsto : x :$ (and the colons are called the *normal ordering symbol*), but the reader should be aware that the meaning of this depends on the data mentioned above. This can lead to very different normal orderings; we analyze the possibilities.

The set $\mathcal{NO}(\mathcal{M})$ of pairs $(\{\mathcal{M}_i\}_{i\in I}, \mathcal{P})$ forms a partial order, where we declare that $(\{\mathcal{M}_i\}_{i\in I}, \mathcal{P}) \le (\{\mathcal{M}'_i\}_{i\in I'}, \mathcal{P}')$ iff there exists an order preserving map $\nu : I' \to I$ such that $\mathcal{M}'_{i'} \subseteq \mathcal{M}_{\nu(i)}$. The minimal element is $(\mathcal{M}, \emptyset)$ and leads to a normal ordering map which is the identity. In fact, any pair $(\{\mathcal{M}_i\}_{i\in I}, \emptyset)$ leads to a trivial normal order. The simplest nontrivial orderings are obtained by a decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, i.e. $I = \{1,2\}$ and $1 < 2$: these form the smallest elements in our partial order with a nontrivial normal ordering map. The maximal elements on the other hand are of the form $(\{\mathcal{M}_i\}_{i\in I}, \mathcal{P})$ such that every $\mathcal{M}_i$ is free of rank 1 and $\mathcal{P}$ is a total order - this corresponds to giving a totally ordered basis $\{v_i\}$ of $\mathcal{M}$.

We return to the representation τ. Suppose $\mathcal{F}'$ is a maximal isotropic subspace of $\mathcal{H}$ complementary to $\mathcal{F}$. Then the decomposition $\mathcal{F}' \oplus \mathcal{F}$ defines a normal ordering, which shall be denoted $n_{\mathcal{F}',\mathcal{F}} : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$. Furthermore, let ϕ denote the composition of the natural maps $\mathcal{H}^{\otimes 2} \to U\mathcal{H} \to U\widehat{\mathcal{H}}[\hbar^{-1}]$, and write $\phi_{\mathcal{F}',\mathcal{F}} := \phi \circ n_{\mathcal{F}',\mathcal{F}}$. We define

$$\tau_{\mathcal{F}'} := \frac{1}{2\hbar}\phi_{\mathcal{F}',\mathcal{F}} \circ E^{-1} : \mathrm{End}(\mathcal{H}) \to U\widehat{\mathcal{H}}[\hbar^{-1}].$$

Remark 4.1.5. If $(\{\mathcal{H}_i\}_{i\in I}, \mathcal{P}) \in \mathcal{NO}(\mathcal{H})$ defines a normal ordering map n' and $(\{\mathcal{H}_i\}_{i\in I}, \mathcal{P}) > \mathcal{F}' \oplus \mathcal{F}$, then $\frac{1}{2\hbar}\phi \circ n' \circ E^{-1} = \tau_{\mathcal{F}'}$. This follows from the fact that for $a, b \in \mathcal{F}$ (or $a, b \in \mathcal{F}'$), $a \circ b = b \circ a$, so that within isotropic subspaces, the ordering has no effect when composed with τ. Thus we can freely refine the normal order defined by $\mathcal{F}' \oplus \mathcal{F}$ used in the definition of $\tau_{\mathcal{F}'}$ - the result will stay the same.

Lemma 4.1.6. $\tau_{\mathcal{F}'}(\alpha) = \tau(\alpha) + \frac{1}{2}\,\mathrm{tr}(\alpha)^{\mathcal{F}}$*, where* $\mathrm{tr}(\cdot)^{\mathcal{F}}$ *denotes the trace of the* $\mathrm{End}(\mathcal{F})$ *component with respect to the decomposition* $\mathcal{H} = \mathcal{F}' \oplus \mathcal{F}$*.*

Proof. It suffices to check this for elements of the form $E((a + a') \otimes (b + b'))$,

$a, b \in \mathcal{F}$ and $a', b' \in \mathcal{F}'$. On one the hand one has

$$\begin{aligned} &\tau_{\mathcal{F}'}(E_{(a+a')\otimes(b+b')}) = \frac{1}{2\hbar}\phi_{\mathcal{F}',\mathcal{F}}((a+a')\otimes(b+b')) \\ = \;& \tau(E_{(a+a')\otimes(b+b')}) + \frac{1}{2\hbar}(-a\circ b' + b'\circ a) \\ = \;& \tau(E_{(a+a')\otimes(b+b')}) + \frac{1}{2\hbar}[b', a] = \tau(E_{(a+a')\otimes(b+b')}) + \frac{1}{2}(b', a). \end{aligned}$$

whereas on the other

$$\begin{aligned} \operatorname{tr}(E_{(a+a')\otimes(b+b')})^{\mathcal{F}} &= \operatorname{tr}((b,\cdot)a + (b',\cdot)a + (b,\cdot)a' + (b',\cdot)a')^{\mathcal{F}} \\ &= \operatorname{tr}((b',\cdot)a) = (b', a). \end{aligned}$$

□

One should be aware that $\operatorname{tr}(\alpha)^{\mathcal{F}}$ depends on the choice of $\mathcal{F}'$, for it is equal to $\operatorname{tr}(\pi_{\mathcal{F}}\circ\alpha\circ\pi_{\mathcal{F}})$, where $\pi_{\mathcal{F}}$ is the projection of $\mathcal{H}$ on $\mathcal{F}$ *along* $\mathcal{F}'$. From this and the previous lemma, it follows that $\tau_{\mathcal{F}'}$ depends on the choice of $\mathcal{F}'$, but $\operatorname{ad}\tau_{\mathcal{F}'}$ does not. Another direct consequence of the lemma is the following.

Corollary 4.1.7. *If* $\alpha, \beta \in \operatorname{End}(\mathcal{H})$, *then*

$$[\tau_{\mathcal{F}'}(\alpha), \tau_{\mathcal{F}'}(\beta)] = \tau_{\mathcal{F}'}([\alpha,\beta]) - \frac{1}{2}\operatorname{tr}([\alpha,\beta])^{\mathcal{F}} = \tau_{\mathcal{F}'}([\alpha,\beta]) - \frac{1}{2}\operatorname{tr}(\alpha[\pi_{\mathcal{F}'},\beta]),$$

where $\pi_{\mathcal{F}'}$ *is the projection* $\mathcal{H} = \mathcal{F}\oplus\mathcal{F}' \to \mathcal{F}'$.

Proof. Since $\tau(\cdot)$ and $\tau_{\mathcal{F}'}(\cdot)$ differ by a central term, we have that

$$[\tau_{\mathcal{F}'}(\alpha), \tau_{\mathcal{F}'}(\beta)] = [\tau(\alpha), \tau(\beta)] = \tau([\alpha,\beta]) = \tau_{\mathcal{F}'}([\alpha,\beta]) - \frac{1}{2}\operatorname{tr}([\alpha,\beta])^{\mathcal{F}}.$$

Therefore, it remains to show that $\operatorname{tr}([\alpha,\beta])^{\mathcal{F}} = \operatorname{tr}(\alpha[\pi_{F'},\beta])$. We can decompose $\alpha, \beta : \mathcal{H}\to\mathcal{H}$ according to the decomposition $\mathcal{H} = \mathcal{F}\oplus\mathcal{F}'$:

$$\alpha = \alpha^{\mathcal{F}\mathcal{F}} + \alpha^{\mathcal{F}\mathcal{F}'} + \alpha^{\mathcal{F}'\mathcal{F}} + \alpha^{\mathcal{F}'\mathcal{F}'}, \quad \beta = \beta^{\mathcal{F}\mathcal{F}} + \beta^{\mathcal{F}\mathcal{F}'} + \beta^{\mathcal{F}'\mathcal{F}} + \beta^{\mathcal{F}'\mathcal{F}'},$$

where $\alpha^{PQ}, \beta^{PQ} \in \operatorname{Hom}(P, Q)$ for $P, Q \in \{\mathcal{F}, \mathcal{F}'\}$. In terms of this, we have that

$$\begin{aligned} \operatorname{tr}([\alpha,\beta])^{\mathcal{F}} &= \operatorname{tr}(\alpha^{\mathcal{F}\mathcal{F}}\beta^{\mathcal{F}\mathcal{F}} + \alpha^{\mathcal{F}'\mathcal{F}}\beta^{\mathcal{F}\mathcal{F}'} - \beta^{\mathcal{F}\mathcal{F}}\alpha^{\mathcal{F}\mathcal{F}} - \beta^{\mathcal{F}'\mathcal{F}}\alpha^{\mathcal{F}\mathcal{F}'}) \\ &= \operatorname{tr}(\alpha^{\mathcal{F}'\mathcal{F}}\beta^{\mathcal{F}\mathcal{F}'} - \beta^{\mathcal{F}'\mathcal{F}}\alpha^{\mathcal{F}\mathcal{F}'}). \end{aligned}$$

Furthermore, since $[\pi_{\mathcal{F}'}, \beta^{\mathcal{F}\mathcal{F}}] = [\pi_{\mathcal{F}'}, \beta^{\mathcal{F}'\mathcal{F}'}] = 0$, we also have that

$$\begin{aligned} \operatorname{tr}(\alpha[\pi_{\mathcal{F}'}, \beta]) &= \operatorname{tr}(\alpha[\pi_{\mathcal{F}'}, \beta^{\mathcal{F}'\mathcal{F}} + \beta^{\mathcal{F}\mathcal{F}'}]) = \operatorname{tr}(\alpha\beta^{\mathcal{F}\mathcal{F}'} - \alpha\beta^{\mathcal{F}'\mathcal{F}}) \\ &= \operatorname{tr}(\alpha^{\mathcal{F}'\mathcal{F}}\beta^{\mathcal{F}\mathcal{F}'} - \alpha^{\mathcal{F}\mathcal{F}'}\beta^{\mathcal{F}'\mathcal{F}}) = \operatorname{tr}(\alpha^{\mathcal{F}'\mathcal{F}}\beta^{\mathcal{F}\mathcal{F}'} - \beta^{\mathcal{F}'\mathcal{F}}\alpha^{\mathcal{F}\mathcal{F}'}). \end{aligned}$$

□

Before finishing this section, we prove two formulas for $\tau_{\mathcal{F}'}$ that will be used later on.

Lemma 4.1.8. *Let $\mathcal{F}, \mathcal{F}', \tau_{\mathcal{F}'}$ be as above and let $\alpha, \beta \in \mathfrak{sp}(\mathcal{H})$ be such that $\alpha\mathcal{F} = 0 = \beta\mathcal{F}'$. Then*

$$\tau_{\mathcal{F}'}(\alpha\beta) = -\tau_{\mathcal{F}'}(\beta\alpha).$$

Proof. Choose a basis $\mathcal{O}_B$-basis $e_1, \dots, e_n$ of $\mathcal{F}$ and let $e_{-1}, \dots, e_{-n}$ be a basis of $\mathcal{F}'$ such that $(e_i, e_{-j}) = i\delta_{i,j}$, so $\{e_i\}$ is a quasisymplectic basis of $\mathcal{H}$. Then

$$\begin{aligned} 2\hbar\tau_{\mathcal{F}'}(\alpha\beta) &= \phi_{\mathcal{F}',\mathcal{F}}\left(\sum_{|i|,|j|=1}^{n} \frac{(\alpha\beta e_i, e_j)}{ij} e_{-i} \otimes e_{-j}\right) \\ &= \phi_{\mathcal{F}',\mathcal{F}}\left(\sum_{|i|,|j|=1}^{n} \frac{-(\beta e_i, \alpha e_j)}{ij} e_{-i} \otimes e_{-j}\right). \end{aligned}$$

Since $\alpha\mathcal{F} = \beta\mathcal{F}' = 0$, $(\beta e_i, \alpha e_j)$ can only be nonzero if $i > 0$ and $j < 0$. In that case, $\phi_{\mathcal{F}',\mathcal{F}}(e_{-i} \otimes e_{-j}) = e_{-i} \circ e_{-j}$ so that

$$2\hbar\tau_{\mathcal{F}'}(\alpha\beta) = \sum_{|i|,|j|=1}^{n} \frac{-(\beta e_i, \alpha e_j)}{ij} e_{-i} \circ e_{-j}.$$

For $\tau_{\mathcal{F}'}(\beta\alpha)$ we have exactly the converse:

$$\begin{aligned} 2\hbar\tau_{\mathcal{F}'}(\beta\alpha) &= \phi_{\mathcal{F}',\mathcal{F}}\left(\sum_{|i|,|j|=1}^{n} \frac{-(\alpha e_i, \beta e_j)}{ij} e_{-i} \otimes e_{-j}\right) \\ &= \sum_{|i|,|j|=1}^{n} \frac{-(\alpha e_i, \beta e_j)}{ij} e_{-j} \circ e_{-i} = \sum_{|i|,|j|=1}^{n} \frac{-(\alpha e_j, \beta e_i)}{ij} e_{-i} \circ e_{-j} \\ &= \sum_{|i|,|j|=1}^{n} \frac{(\beta e_i, \alpha e_j)}{ij} e_{-i} \circ e_{-j}, \end{aligned}$$

which equals $-2\hbar\tau_{\mathcal{F}'}(\alpha\beta)$. □

Lemma 4.1.9. *Let $\mathcal{F}, \mathcal{F}'$ be as above and let $\mathrm{Hom}(\mathcal{F}', \mathcal{F}), \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ be the corresponding direct summands of $\mathrm{End}(\mathcal{H})$. If $\alpha \in \mathfrak{sp}(\mathcal{H}) \cap \mathrm{Hom}(\mathcal{F}, \mathcal{F}')$ and $\beta \in \mathfrak{sp}(\mathcal{H}) \cap \mathrm{Hom}(\mathcal{F}', \mathcal{F})$, then*

$$\mathrm{tr}(\alpha\beta)^{\mathcal{F}} = 0, \quad \mathrm{tr}(\beta\alpha)^{\mathcal{F}} = \mathrm{tr}(\alpha\beta)$$

Proof. Simply observe that $\alpha\beta$ lies in the direct summand $\mathrm{Hom}(\mathcal{F}', \mathcal{F}')$ of $\mathrm{End}(\mathcal{H})$, and $\beta\alpha$ in $\mathrm{Hom}(\mathcal{F}, \mathcal{F})$. □

4.2 A unitary structure on $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$.

In this section we will describe a natural unitary structure on $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$, of which the canonical connection has scalar curvature.

Convention. For any $\mathcal{O}_B$-module $\mathcal{M}$, we write $\mathcal{M}_\infty$ for $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{M}$, where $\mathcal{E}$ denotes the sheaf of C^∞-functions on B; as an exception to this, we write $\mathbb{F}_\infty(\mathcal{H}, \mathcal{F})_\ell$ for $\mathcal{E} \otimes_{\mathcal{O}_B} \mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$.

We observe that the constructions above for the tuple $(\mathcal{O}_B, \mathcal{H}, (\cdot,\cdot), \mathcal{F})$ can also be done for the tuple $(\mathcal{E}, \mathcal{H}_\infty, (\cdot,\cdot), \mathcal{F}_\infty)$, where we also write $(\cdot,\cdot)$ for the C^∞-extension of the $\mathcal{O}_B$-linear symplectic form on $\mathcal{H}$, and that

$$\mathbb{F}_\infty(\mathcal{H}, \mathcal{F})_\ell = \mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell.$$

Of course, τ induces a representation of $\mathfrak{sp}(\mathcal{H})$ on $\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell$.

We now assume the following additional data: a real structure on $\mathcal{H}_\infty$ given by a conjugation map $a \mapsto \overline{a}$, and a C^∞-connection on $\mathcal{H}_\infty$ over $\mathcal{B}$. The following compatibility conditions are assumed: $(\cdot,\cdot)$ and ∇ are real with respect to $a \mapsto \overline{a}$, $(\cdot,\cdot)$ is flat for ∇ and

$$\langle\cdot,\cdot\rangle : a \otimes b \mapsto -i(a, \overline{b})$$

is negative definite on $\mathcal{F}_\infty$. Finally, ∇ is compatible with the $\mathcal{O}_B$-structure: $\nabla^{0,1}$ is $\mathcal{O}_B$-linear and $\nabla\mathcal{H} = 0$.

Remark 4.2.1. Though the following can be applied to any suitable variation of Hodge structure, the idea goal is to apply it to the sheaves $\mathcal{H}_\mathfrak{g}$, $\mathcal{F}^+_\mathfrak{g}$ from Chapter 3 - the reader can keep these in mind as example.

From the assumptions it follows that $\overline{\mathcal{F}}_\infty$ is a Lagrangian complement for $\mathcal{F}_\infty$ in $\mathcal{H}_\infty$ and that $\langle\cdot,\cdot,\rangle$ is positive definite on $\overline{\mathcal{F}}_\infty$. Thus $\langle\cdot,\cdot\rangle$ is a positive definite Hermitian form on $\overline{\mathcal{F}}_\infty$, that extends to one on $\mathrm{Sym}_\bullet\overline{\mathcal{F}}_\infty$, and carries over to $\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell$ using the above introduced isomorphism $\mathrm{Sym}_\bullet\overline{\mathcal{F}}_\infty \to \mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell$.

Definition 4.2.2. Denote

- the right multiplication of $a \in \bigotimes \mathcal{H}_\infty$ on $\mathcal{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)$ by ρ,
- the adjoint of an element $\alpha \in \mathrm{End}\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell$ with respect to $\langle \cdot, \cdot \rangle$ by $\dagger$.

Lemma 4.2.3. *If* $a \in \mathcal{H}_\infty$*, then* $\rho(a)^\dagger = i\rho(\overline{a})$*. Moreover, if* $a_1, \dots, a_k \in \mathcal{H}_\infty$*, then*

$$\rho(a_1 \otimes \cdots \otimes a_k)^\dagger = i^k \rho(\overline{a_k} \otimes \cdots \otimes \overline{a_1}).$$

In particular, if $s \in \mathrm{Sym}_2\mathcal{F}_\infty + \mathrm{Sym}_2\overline{\mathcal{F}}_\infty$*, then* $\rho(s+\overline{s})$ *is a unitary transformation of* $\mathbb{F}_\infty(\mathcal{H}, \mathcal{F})_\ell$*.*

With a little abuse of notation we write $(a_1 \otimes \cdots \otimes a_k)^\dagger = i^k \overline{a}_k \otimes \cdots \otimes \overline{a}_1$.

Proof. The pairing $\langle \cdot, \cdot \rangle$ is characterized by the following formula: for $x_i, y_j \in \overline{\mathcal{EF}}$, we have that

$$\langle x_1 \circ \cdots \circ x_n, y_1 \circ \cdots \circ y_n \rangle = \begin{cases} \sum_{\sigma \in S_n} \prod_{i=1}^n \langle x_i, y_{\sigma(i)} \rangle & n = m \\ 0 & n \neq m. \end{cases}$$

We are therefore to prove that

$$\langle a_1 \circ \cdots \circ a_k \circ x_1 \circ \cdots \circ x_n, y_1 \circ \cdots \circ y_m \rangle = \langle x_1 \circ \cdots \circ x_n, i^n \overline{a}_k \circ \cdots \circ \overline{a}_1 y_1 \circ \cdots \circ y_n \rangle.$$

Note that it suffices to prove this for $k = 1$ - one then just “moves the a_i’s one by one to the other entry of the hermitian form”. We can also assume that $a = a_1 \in \mathcal{F}_\infty$, because the statement for $a \in \overline{\mathcal{F}}_\infty$ is just the statement for a in reverse order. The rest now follows by direct computation:

$$\rho(a) x_1 \circ \cdots \circ x_k = \sum_{i=1}^n (a, x_i) x_1 \circ \cdots \circ \widehat{x}_i \circ \cdots \circ x_k = \sum_{i=1}^n -i \langle x_i, \overline{a} \rangle x_1 \circ \cdots \circ \widehat{x}_i \circ \cdots \circ x_k.$$

From this it is also clear that we can restrict ourselves to the case $m = n - 1$.

$$\begin{aligned} &\langle a \circ x_1 \circ \cdots \circ x_n, y_1, \circ \cdots \circ y_{n-1} \rangle \\ &\quad = -i \sum_{\sigma \in S_n} \langle x_{\sigma(1)}, a \rangle \langle x_{\sigma(2)}, y_1 \rangle \cdots \langle x_{\sigma(n)}, y_{n-1} \rangle \\ &\quad = \langle x_1 \circ \cdots \circ x_n, i\overline{a} \circ y_1 \circ \cdots \circ y_{n-1} \rangle. \end{aligned}$$

□

4.3 The connection $\nabla^{\mathbb{F}}$

Below, we will denote the (p,q) forms on B by $\mathcal{E}^{p,q}$, and similarly denote the sections of an $\mathcal{O}_B$-module $\mathcal{M}$ with values in $\mathcal{E}^{p,q}$ by $\mathcal{E}^{p,q}(\mathcal{M})$.

Consider the map

$$\sigma : \mathcal{F} \to \mathcal{E}^{1,0}(\overline{\mathcal{F}}_\infty),$$

defined as the composition of $\nabla : \mathcal{F} \to \mathcal{E}^{1,0}(\mathcal{H})$ *with the projection* $\mathcal{E}^{1,0}(\mathcal{H}) \to \mathcal{E}^{1,0}(\overline{\mathcal{F}}_\infty)$. Note that we used that $\nabla^{0,1}\mathcal{F} = 0$. If $f \in \mathcal{O}_B$ and $a \in \mathcal{F}$, then $\nabla(fa) = f\nabla(a) + (\,df)a$. Since the last term is an element of $\mathcal{E}^{1,0}(\mathcal{F})$, it follows that $\sigma(fa) = f\sigma(a)$. Hence σ is an element of $\mathcal{E}^{1,0}(\mathrm{Hom}(\mathcal{F}, \overline{\mathcal{F}}))$; it is called the *second fundamental form of* ∇ *with respect to the decomposition* $\mathcal{H}_\infty = \mathcal{F}_\infty \oplus \overline{\mathcal{F}}_\infty$. We observe that since, respectively, $\mathcal{F}_\infty$ is isotropic and $(\cdot,\cdot)$ is flat,

$$(\sigma(a), b) = (\nabla(a), b) = \,d(a,b) - (a, \nabla b) = -(a, \sigma(b))$$

for all $a, b \in \mathcal{F}_\infty$. This means that σ lies $\mathcal{E}^{1,0}(\mathfrak{sp}(\mathcal{H})) \cap \mathcal{E}^{1,0}(\mathrm{Hom}(\mathcal{F}, \overline{\mathcal{F}}_\infty))$, so that

$$s := E^{-1} \circ \sigma$$

is an element of $\mathcal{E}^{1,0}(\mathrm{Sym}_2\mathcal{F})$. Similarly, $\overline{s} = E^{-1} \circ \overline{\sigma} \in \mathcal{E}^{0,1}(\mathrm{Sym}_2\overline{\mathcal{F}}_\infty)$. By Lemma 4.2.3, $\rho(s+\overline{s})$ acts unitary on $\mathbb{F}_\infty(H,F)_\ell$ with respect $\langle\cdot,\cdot\rangle$.

Next, consider the $\mathrm{Hom}(\overline{\mathcal{F}}_\infty, \overline{\mathcal{F}}_\infty)$ part of ∇, i.e. the composition of $\nabla : \overline{\mathcal{F}}_\infty \to \mathcal{E}^1(\mathcal{H})$ and the projection $\mathcal{E}^1(\mathcal{H}) \to \mathcal{E}^1(\overline{\mathcal{F}_\infty})$. This is a connection on $\overline{\mathcal{F}}_\infty$ and we denote it by $\nabla^{\overline{\mathcal{F}}_\infty}$. Observe that for all $a, b \in \overline{\mathcal{F}}_\infty$

$$\begin{aligned} d\langle a,b\rangle &= -i(\nabla^{\overline{\mathcal{E}\mathcal{F}}} a, \overline{b}) - i(a, \overline{\nabla^{\overline{\mathcal{E}\mathcal{F}}} b}) = -i(\nabla a, \overline{b}) - i(a, \overline{\nabla b}) \\ &= -i(\nabla a, \overline{b}) - i(a, \nabla\overline{b}) = -i\,d(a,\overline{b}) = \,d\langle a,b\rangle, \end{aligned}$$

so $\nabla^{\overline{\mathcal{F}}_\infty}$ is unitary with respect to $\langle\cdot,\cdot\rangle$. It follows that under the identification $\mathrm{Sym}_\bullet\overline{\mathcal{F}}_\infty \simeq \mathbb{F}_\infty(\mathcal{H},\mathcal{F})_\ell$, $\nabla^{\overline{\mathcal{F}}}$ becomes a unitary connection on $\mathbb{F}_\infty(\mathcal{H},\mathcal{F})_\ell$ with the property that $\nabla^{\overline{\mathcal{F}}_\infty}\hbar = 0$.

Definition 4.3.1.

$$\nabla^{\mathbb{F}} := \nabla^{\overline{\mathcal{F}}_\infty} + \rho(s+\overline{s}).$$

A direct consequence of lemma 4.2.3 is the following.

Corollary 4.3.2. *The connection $\nabla^{\mathbb{F}}$ on $\mathbb{F}_\infty(\mathcal{H},\mathcal{F})_\ell$ is unitary.*

We give another description of $\nabla^{\mathbb{F}}$: suppose $\vec{x} = x_1 \otimes \cdots \otimes x_n \in \mathrm{Sym}_\bullet \overline{\mathcal{F}}_\infty$, then using that $\mathrm{ad}(s)(x_i) = \overline{\sigma}(x_i)$ and that $\nabla^{\overline{\mathcal{F}}_\infty} + \overline{\sigma}$ equals ∇ when acting on $\overline{\mathcal{F}}_\infty$, it follows that in $\mathbb{F}_\infty(\mathcal{H},\mathcal{F})_\ell$

$$\begin{aligned}
\nabla^{\mathbb{F}}(\vec{x}\circ f_\ell) &= \nabla^{\overline{\mathcal{E}\mathcal{F}}}(\vec{x})\circ f_\ell + s\circ\vec{x}\circ f_\ell + \overline{s}\circ\vec{x}\circ f_\ell \\
&= (\nabla^{\overline{\mathcal{E}\mathcal{F}}} + \mathrm{ad}\, s)(\vec{x})\circ f_\ell + \vec{x}\circ s\circ f_\ell + \overline{s}\circ\vec{x}\circ\circ f_\ell \\
&= \nabla(\vec{x})\circ f_\ell + \overline{s}\circ\vec{x}\circ f_\ell \\
&= (\nabla + \rho(\overline{s})(\vec{x}\circ f_\ell).
\end{aligned}$$

Hence we have proven the following.

Corollary 4.3.3. $\nabla^{\mathbb{F}} = \nabla + \rho(\overline{s})$.

We use this observation to compute the curvature of $\nabla^{\mathbb{F}}$.

Theorem 4.3.4. $(\nabla^{\mathbb{F}})^2 = \frac{1}{2}\,\mathrm{tr}(\sigma\wedge\sigma^\dagger)$*, so in particular,* $\nabla^{\mathbb{F}}$ *is projectively flat.*

Proof. First observe that $\rho(\overline{s}) = 2\hbar\tau(\sigma)$, and that

$$(\nabla^{\mathbb{F}})^2 = (\nabla + \tau(\sigma))^2 = \nabla^2 + \nabla\tau(\sigma) + \tau(\sigma)\nabla + \tau(\sigma)\wedge\tau(\sigma) = \nabla(\tau(\sigma)).$$

This uses the fact that ∇ is flat and that $[\tau(\sigma(v)),\tau(\sigma(w))] = \tau([\sigma(v),\sigma(w)]) = 0$ for all derivations v, w, because σ takes it values in $\mathrm{Hom}(\mathcal{F}_\infty,\overline{\mathcal{F}}_\infty)$. Since $(\cdot,\cdot)$ is flat for ∇, τ is flat for ∇ so the curvature equals $\tau(\nabla\sigma)$. It therefore suffices to show that $\tau(\nabla\sigma)\circ\vec{x}\circ f_\ell = \frac{1}{2}\,\mathrm{tr}(\sigma\wedge\overline{\sigma})\vec{x}\circ f_\ell$, and in this, we can assume that $\vec{x}$ lies in the image of $\mathrm{Sym}_\bullet\overline{\mathcal{F}}_\infty$ in $U\widehat{\mathcal{H}}$.

We start by expanding ∇ on a local basis of $\mathcal{H}_\infty$ that is adapted to the decomposition $\mathcal{H}_\infty = \mathcal{F}_\infty \oplus \overline{\mathcal{F}}_\infty$: choose a local basis of $\mathcal{F}_\infty$, and use the conjugates as basis of $\overline{\mathcal{F}}_\infty$. On this basis, ∇ takes the form $d + \omega$, where connection matrix ω is of the form

$$\omega = \begin{pmatrix} A & \overline{\sigma} \\ \sigma & \overline{A} \end{pmatrix};$$

here A is the $\mathrm{End}(\mathcal{F}_\infty)$ part of ω, and its $\mathrm{End}(\overline{\mathcal{F}}_\infty)$ part is $\overline{A}$, because ∇ is real. With respect to this local basis we have

$$\begin{aligned}
\nabla(\sigma) &= d\sigma + \omega\wedge\sigma + \sigma\wedge\omega \\
&= d\sigma + \overline{A}\wedge\sigma + \overline{\sigma}\wedge\sigma + \sigma\wedge A + \sigma\wedge\overline{\sigma}
\end{aligned}$$

We can simplify this using that ∇ is flat: its curvature is $d\omega + \omega \wedge \omega = 0$, and the $\mathrm{Hom}(\mathcal{F}_\infty, \overline{\mathcal{F}}_\infty)$ part of this reads $d\sigma + \overline{A} \wedge \sigma + \sigma \wedge A = 0$. Using this, we find that $\nabla(\sigma) = \overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma}$. Lemma 4.1.6 tells us that

$$(\nabla^{\mathbb{F}})^2 = \tau(\overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma}) = \tau_{\mathcal{E}\overline{\mathcal{F}}}(\overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma}) + \frac{1}{2}\,\mathrm{tr}(\overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma})^{\mathcal{E}\mathcal{F}}.$$

We use Lemma 4.1.9 to simplify the last term: $\mathrm{tr}(\overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma})^{\mathcal{F}_\infty} = -\,\mathrm{tr}(\overline{\sigma} \wedge \sigma) = \mathrm{tr}(\sigma \wedge \overline{\sigma})$, where in the last step we used the invariance property of the trace and the anti-commutativity of 1-forms. It therefore suffices to show that $\tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma + \sigma \wedge \overline{\sigma}) \circ \vec{x} \circ f_\ell = 0$ for any $\vec{x} \in \mathrm{Sym}_\bullet \overline{\mathcal{F}}_\infty$. One has that

$$\tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma) \circ \vec{x} \circ f_\ell \quad = \quad [\tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma), \vec{x}] \circ f_\ell + \vec{x} \circ \tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma) \circ f_\ell$$

The last term vanishes because $\tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma)$ takes its values in $\overline{\mathcal{F}}_\infty \circ \mathcal{F}_\infty$ and the first term vanishes because $[\tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma), \vec{x}] = [\tau(\overline{\sigma} \wedge \sigma), \vec{x}] = \overline{\sigma} \wedge \sigma(\vec{x}) = 0$. The result now follows from the fact that $\tau_{\overline{\mathcal{F}}_\infty}(\sigma \wedge \overline{\sigma}) = \tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma)$. To see this, decompose σ on a local basis $dx^1, \cdots, dx^k$ of 1-forms on B: $\sigma = dx^i \sigma_i$, where $\sigma_i \in \mathfrak{sp}(\mathcal{H}_\infty)$ and $\sigma_i \overline{\mathcal{F}}_\infty = 0$. By conjugation, $\overline{\sigma} = dx^i \overline{\sigma}_i$ and $\overline{\sigma}_i \mathcal{F}_\infty = 0$. We now apply Lemma 4.1.8:

$$\begin{aligned} \tau_{\overline{\mathcal{F}}_\infty}(\sigma \wedge \overline{\sigma}) &= dx^i \wedge dx^j \tau_{\overline{\mathcal{F}}_\infty}(\sigma_i \overline{\sigma}_j) = -\,dx^i \wedge dx^j \tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma}_j \sigma_i) \\ &= dx^j \wedge dx^i \tau_{\overline{F}}(\overline{\sigma}_j \sigma_i) = \tau_{\overline{\mathcal{F}}_\infty}(\overline{\sigma} \wedge \sigma). \end{aligned}$$

□

Remark 4.3.5. Assume that we are given a $U\widehat{\mathcal{H}}$-submodule $\mathcal{Z}$ of $\mathbb{F}(\mathcal{H}, \mathcal{F})_\ell$ such that $\mathcal{Z}_\infty$ is preserved by ∇ as submodule of $\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell$. Then $\nabla^{\mathbb{F}} = \nabla + \rho(\overline{s})$ obviously also preserves Z_∞. It follows that $\nabla^{\mathbb{F}}$ is a connection on $\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell / \mathcal{Z}_\infty$, whose curvature is of course still $\frac{1}{2}\,\mathrm{tr}(\sigma \wedge \overline{\sigma})$.

Denote the orthocomplement of $\mathcal{Z}_\infty$ with respect to $\langle \cdot, \cdot \rangle$ by $\mathcal{Z}_\infty^\perp$, then $\nabla^{\mathbb{F}}$ preserves $\mathcal{Z}_\infty^\perp$: if $\vec{y} \in \mathcal{Z}_\infty$ and $x \in \mathcal{Z}_\infty^\perp$ then

$$\langle \nabla^{\mathbb{F}} x, y \rangle = d\langle x, y \rangle - \langle x, \nabla^{F} y \rangle = 0.$$

By identifying $\mathcal{Z}_\infty^\perp$ with $\mathbb{F}(\mathcal{H}_\infty, \mathcal{F}_\infty)_\ell / \mathcal{Z}_\infty$, the latter obtains an innerproduct with respect to which $\nabla^{\mathbb{F}}$ is unitary.

We summarize the main results of the chapter in a theorem.

Theorem 4.3.6. *Assume the following given:*

- *a finite dimensional complex manifold* B *with structure sheaf* $\mathcal{O}_B$ *and smooth function sheaf* $\mathcal{E}$*,*

- *a free* $\mathcal{O}_B$*-module of finite rank* $\mathcal{H}$ *with symplectic form* $(\cdot,\cdot)$ *and a Lagrangian submodule* $\mathcal{F}$*,*

- *a flat connection* ∇ *and real structure* $a \mapsto \overline{a}$ *on* $\mathcal{H}_\infty$ *such that* $(\cdot,\cdot)$ *is real and flat,* ∇ *is real,* $a \otimes b \mapsto -i(a, \overline{b})$ *is negative definite on* $\mathcal{F}_\infty$ *and* $\overline{\mathcal{F}}_\infty$ *is a Lagrangian complement for* $\mathcal{F}_\infty$ *in* $\mathcal{H}_\infty$*.*

To this we associated

- *an* $\mathcal{O}_B$*-Lie algebra* $\widehat{\mathcal{H}}$*,*

- *an* $U\widehat{\mathcal{H}}$*-module* $\mathbb{F}(\mathcal{H},\mathcal{F})_\ell$ *with a representation* τ *of* $\mathfrak{sp}(\mathcal{H})$*,*

- *a positive definite hermitian form* $\langle\cdot,\cdot\rangle$ *on the* $\mathcal{E}$*-module* $\mathbb{F}_\infty(\mathcal{H},\mathcal{F})_\ell$*,*

- *a connection* $\nabla^{\mathbb{F}}$ *on* $\mathbb{F}(\mathcal{H}_\infty,\mathcal{F}_\infty)_\ell$ *that is unitary with respect to* $\langle\cdot,\cdot\rangle$ *and whose curvature is a scalar, namely* $\frac{1}{2}\operatorname{tr}(\sigma \wedge \sigma^\dagger)$*.*

For any $U\widehat{\mathcal{H}}$*-submodule* $\mathcal{Z}_\ell \subseteq \mathbb{F}(H, F)_\ell$ *that is preserved by* ∇*,* $\nabla^{\mathbb{F}}$ *descends to a connection on* $\mathbb{F}(\mathcal{H}_\infty,\mathcal{F}_\infty)_\ell/\mathcal{Z}_\infty$ *that acts unitary with respect to the inherited innerproduct. These associations are functorial.*

The last assertion follows from the fact that all constructions depend naturally on the input data; no choices were made.

Chapter 5

Conformal blocks and the WZW connection

We continue to assume an integer $\ell > 0$ (level) to be fixed. This chapter consists of two parts: the first is more or less the infinite dimensional analogue of the previous chapter, whereas the second one is a variation of the first, in that we assume an additional Lie algebra structure present on the presymplectic module. We try to present it in such a way that it best resembles the previous chapter. As we did there, we assume the data from Chapter 3 given. In particular, B is a smooth complex variety and $\mathcal{K}^0$, $\mathcal{K}$, $\mathcal{O}$, $\mathfrak{m}$, $\mathcal{A}$ are $\mathcal{O}_B$-module as in Chapter 3. Locally over B, $\mathcal{K}^0, \mathfrak{m} \subseteq \mathcal{O} \subseteq \mathcal{K}$ are of the form $\bigoplus_{i=1}^N \mathcal{O}_B$, $\sum_i t_i \mathcal{O}$, $\bigoplus_{i=1}^N \mathcal{O}_B[[t_i]]$, $\bigoplus_{i=1}^N \mathcal{O}_B((t_i))$, respectively, for certain local sections $t_1, \ldots, t_N$ of $\mathcal{O}$. We call the latter local coordinates of $\mathcal{O}$. Moreover, we assume that $\mathcal{K}$ has a presymplectic form $(\cdot, \cdot)$ with kernel $\mathcal{K}^0$ that satisfies the properties of proposition 3.1.11 and lemmas 3.3.2,3.3.3.

Remark 5.0.7. Below, we will endow $\mathcal{O}_B$-modules with a topology. By this we mean the following: a topology on an $\mathcal{O}_B$-module $\mathcal{F}$ is a topology on $\mathcal{F}(U)$ for every open $U \subseteq B$, such that the restriction maps are continuous and moreover, the addition on $\mathcal{F}(U)$ and multiplication by $\mathcal{O}_B(U)$ are continuous (for every open $U \subseteq B$).

5.1 The Fock module $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$

We define a filtration $\{F^k\mathcal{K}\}_{K\in\mathbb{Z}}$ on $\mathcal{K}$ as follows: let $F^k\mathcal{K} := \mathfrak{m}^k$ for $k > 0$, $F^0\mathcal{K} = \mathcal{O}$ and for $k < 0$ we define $F^k\mathcal{K}$ to be the maximal subsheaf of $\mathcal{K}$ such that $\mathfrak{m}^{-k}F^k\mathcal{K} \subseteq \mathcal{O}$. The ideal $\mathfrak{m}$ of $\mathcal{O}$ also gives $\mathcal{K}$ a compatible topology, called the $\mathfrak{m}$-adic topology. It is characterized by the property that it is invariant under addition of $\mathcal{K}$ and that a sequence $(a_n)_{n\in\mathbb{N}} \in \mathcal{K}$ converges to 0 iff for every k, $a_n \in F^k\mathcal{K}$ for n large enough; $\mathcal{K}$ is complete with respect to this topology.

The presymplectic form $(\cdot,\cdot)$ is continuous as map from $\mathcal{K} \times \mathcal{K} \to \mathcal{O}_B$, where $\mathcal{O}_B$ has the discrete topology. This amounts to the following property: for every $f \in \mathcal{K}$ we have that $(f, F^k\mathcal{K}) = 0$ for k large enough.

Definition 5.1.1. We define $\mathcal{K}^{\text{red}} := \mathcal{K}/\mathcal{K}^0$ as $\mathcal{O}_B$-module and give it the topology it inherits from $\mathcal{K}$.

Note that since $\mathcal{K}^0$ is not an ideal, $\mathcal{K}^{\text{red}}$ does not inherit the commutative algebra structure of $\mathcal{K}$. On the other hand, $(\cdot,\cdot)$ clearly descends to a symplectic form on $\mathcal{K}^{\text{red}}$ - we also denote it by $(\cdot,\cdot)$ and note that it is continuous. *The image of $\mathcal{O}$ in $\mathcal{K}^{\text{red}}$ is denoted $\mathcal{O}^{\text{red}}$*, and since $\mathcal{K}^0 \subseteq \mathcal{O}$, we can - and will - identify it with $\mathcal{O}/\mathcal{K}^0 = \mathfrak{m}$. It is maximal isotropic with respect to $(\cdot,\cdot)$.

In the previous chapter, we defined a Fock module for a triple $(\mathcal{H}, \mathcal{F}, (\cdot,\cdot))$, that by construction had an action of the symplectic algebra of $\mathcal{H}$. Here we will do something similar for the triple $(\mathcal{K}^{\text{red}}, \mathcal{O}^{\text{red}}, (\cdot,\cdot))$ and a subsheaf of $\mathfrak{sp}(\mathcal{K}^{\text{red}})$.

Definition 5.1.2. Define

$$\widehat{\mathcal{K}} := \mathcal{K} \oplus \mathcal{O}_B,\ \widehat{\mathcal{O}} := \mathcal{O} \oplus \mathcal{O}_B,\ \widehat{\mathcal{K}}^{\text{red}} := \widehat{\mathcal{K}}/\mathcal{K}^0,\ \widehat{\mathcal{O}}^{\text{red}} := \widehat{\mathcal{O}}/\mathcal{K}^0,$$

and give $\widehat{\mathcal{K}}$ an $\mathcal{O}_B$-Lie algebra structure by defining the product as

$$[a \oplus r, a' \oplus r'] = 0 \oplus (a, a'),$$

for local sections a, a' of $\mathcal{K}$. It is commonly called the *oscillator algebra*, and the element $0 \oplus 1$ is denoted $\hbar$.

If we regard $\mathcal{K}$ as Lie algebra with trivial product, then $\widehat{\mathcal{K}}$ is the central extension of $\mathcal{K}$ by $\mathcal{O}_B$, determined by the cocycle $(\cdot,\cdot)$.

Corollary 5.1.3. *$\widehat{\mathcal{O}}$ is a subalgebra of $\widehat{\mathcal{K}}$, the Lie bracket of $\widehat{\mathcal{K}}$ descends to $\widehat{\mathrm{K}}^{\mathrm{red}}$, and with respect to this product, $\widehat{\mathrm{O}}^{\mathrm{red}}$ is a subalgebra.*

The first claim is a consequence of the fact that $\mathcal{O}$ is an isotropic subspace, the second follows from the observation that $[\mathcal{K}^0, \mathcal{K}] = 0$, hereby identifying $\mathcal{K}$ with its image in $\widehat{\mathcal{K}}$.

The subalgebra $\widehat{\mathcal{O}} = \mathcal{O} + \hbar\mathcal{O}_B$ has a character χ given by the projection on the second summand, and this character descends to $\widehat{\mathcal{O}}^{\mathrm{red}}$.

Definition 5.1.4. For $\ell \in \mathbb{Z}$, define

$$\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell := \mathrm{U}\widehat{\mathcal{K}} \otimes_{\mathrm{U}\widehat{\mathcal{O}}} \mathcal{O}_B,$$

where $\widehat{\mathcal{O}}$ acts on $\mathcal{O}_B$ via $\ell\chi$. We call this the *Fock module associated to the tuple* $(\mathcal{O}_B, \mathcal{K}, \mathcal{O}, (\cdot,\cdot))$ *of level* ℓ. The $\mathrm{U}\widehat{\mathcal{K}}$ generator $1 \otimes 1$ is denoted v_ℓ.

We will assume ℓ to be fixed. As a direct consequence of this definition, we have the following:

Corollary 5.1.5. *For any submodule $\mathcal{V} \subseteq \mathcal{O}$, χ is well defined on $\widehat{\mathcal{O}}/\mathcal{V}$ and*

$$\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell = \mathrm{U}\widehat{\mathcal{K}}/(\mathcal{V}) \otimes_{\mathrm{U}\widehat{\mathcal{O}}/(\mathcal{V})} \mathcal{O}_B,$$

where $\widehat{\mathcal{O}}$ acts on $\mathcal{O}_B$ by $\ell\chi$. Here $(\mathcal{V})$ denotes the two sided ideal generated by $\mathcal{V}$. In particular

- *for $\mathcal{V} = \mathcal{K}^0$ we get that $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell = \mathrm{U}\widehat{\mathcal{K}}^{\mathrm{red}} \otimes_{\mathrm{U}\widehat{\mathcal{O}}^{\mathrm{red}}} \mathcal{O}_B$,*
- *for $\mathcal{V} = \mathcal{O}$ and $\mathcal{O}' \subseteq \mathcal{K}$ a Lagrangian submodule complementary to $\mathcal{O}$:*

  $$\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell = \mathrm{U}\widehat{\mathcal{K}}/(\mathcal{O}) \otimes_{\mathrm{U}\widehat{\mathcal{O}}/(\mathcal{O})} \mathcal{O}_B = \mathrm{Sym}_\bullet \mathcal{O}',$$

  where the last isomorphism in induced by $\mathcal{O}' \subseteq \mathcal{K} \hookrightarrow \widehat{\mathcal{K}}$.

5.1.1 The representation of $\mathfrak{sp}(\mathcal{K}^{\mathrm{red}})$ on $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$

Since $(\cdot,\cdot)$ is continuous in both variables, it follows that if $a \otimes a'$ is a local section of $\mathcal{K} \otimes_{\mathcal{O}_B} \mathcal{K}$, then $x \mapsto (a, x)a'$ is a continuous endomorphism of $\mathcal{K}$.

Definition 5.1.6. For topological $\mathcal{O}_B$-modules $\mathcal{M}, \mathcal{M}'$, we denote by

$$\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{M}, \mathcal{M}') \subseteq \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{M}, \mathcal{M}')$$

the *continuous* $\mathcal{O}_B$-linear maps from $\mathcal{M}$ to $\mathcal{M}'$.

With $\mathcal{M}, \mathcal{M}'$ as in the definition, the continuity property can be characterized by the following: if $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}$ and $\alpha \in \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{M}, \mathcal{M}')$, then α is continuous iff $\lim_{n\to\infty} a_n = a$ implies that $\lim_{n\to\infty} \alpha(a_n) = \alpha(a)$.

As was remarked above, the presymplectic form is a continuous. Hence we have a map

$$\mathcal{K}^{\otimes 2} \to \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}, \mathcal{K}) : a \otimes a' \mapsto (a', \cdot)a.$$

Clearly, its kernel is $\mathcal{K} \otimes \mathcal{K}^0$, so that it factors over $\mathcal{K} \otimes \mathcal{K}^{\mathrm{red}}$ and descends to an injective map $\mathcal{K} \otimes \mathcal{K}^{\mathrm{red}} \to \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K})$, which in turn descends to an injective map

$$E : (\mathcal{K}^{\mathrm{red}})^{\otimes 2} \to \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}}).$$

This map is not surjective, as we will see later. We introduce a special kind of local basis for $\mathcal{K}^{\mathrm{red}}$ over $\mathcal{O}_B$ in order to facilitate computations that we will perform below.

Definition 5.1.7. We call a set of local sections $\{e_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{K}^{\mathrm{red}}$ a *local topological quasisymplectic basis* (of $\mathcal{K}^{\mathrm{red}}$ over $\mathcal{O}_B$) if, locally over B,

- its $\mathcal{O}_B$ span is dense in $\mathcal{K}^{\mathrm{red}}$ and if the e_i's are linearly independent over $\mathcal{O}_B$,
- $(e_i, e_{-j}) = i\delta_{i,j}$,
- $\overline{\sum_{j\geq i} e_i \mathcal{O}_B}$ is a neighborhood basis of 0.

The last property can be reformulated more explicitly: for every k there is a N_k such that $e_i \in F^k\mathcal{K}$ for every $i \geq N_k$, and there is an M_k such that $F^k\mathcal{K}$ is contained in $\overline{\sum_{i\geq M_k} e_i \mathcal{O}_B}$. From this it is immediately clear that the filtration on $\mathcal{K}$ determines the topology.

The next example shows that such a local basis indeed exists.

Example 5.1.8. Let $t_1, \dots, t_n$ be coordinates of $\mathcal{O}$ over some open $U \subseteq B$, and take a bijection $m = (m_1, m_2) : \mathbb{N}^{>0} \to \mathbb{N}^{>0} \times \{1, \dots, n\}$. Define

$$e_i = t_{m_2(i)}^{m_1(i)}, \quad e_{-i} = \frac{i}{-m_1(i)} t_{m_2(i)}^{-m_1(i)}, \ i \in \mathbb{N}^{>0},$$

and identify them with their image in $\mathcal{K}^{\mathrm{red}}(U)$. Then $\{e_i\}_{i\neq 0}$ is a basis of $\mathcal{K}^{\mathrm{red}}(U) \simeq \bigoplus_i \mathcal{O}_B(U)((t_i))/\mathcal{O}_B(U)$ and $(e_i, e_j) = i\delta_{i,-j}$. Finally, using that

$$F^k\mathcal{K}(U) = \overline{\sum_{i_1+\dots+i_N \geq k} t^{i_1} \cdots t^{i_N} \mathcal{O}_B(U)}$$

one readily checks that the third property also holds, so that $\{e_i\}_{i\neq 0}$ is a quasisymplectic topological basis over U.

Choose a topological quasisymplectic basis $\{e_i\}_{i\neq 0}$ over some open part $U \subseteq B$. For every k, $F^k\mathcal{K}(U)$ is contained in the closure of the span of the e_i's for $i > M_k$, for a certain M_k, so for any $x \in \mathcal{K}^{red}(U)$, $x = \sum_{j\geq j_{min}, j\neq 0}^{\infty} x_j e_j$ for certain $x_j \in \mathcal{O}_B(U)$, $j_{min} \in \mathbb{Z}$. In fact, the x_j's can are given explicitly by the formula $(x, e_-j) = jx_j$:

$$\begin{aligned}\sum_{|i|=1}^{\infty} \frac{(x, e_{-i})}{i} e_i &= \sum_{|i|=1}^{\infty} \frac{(\lim_{N\to\infty} \sum_{j=j_{min}}^{N} x_j e_j, e_{-i})}{i} e_i \\ = \sum_{|i|=1}^{\infty} \lim_{N\to\infty} \sum_{j=j_{min}}^{N} \frac{x_j (e_j, e_{-i})}{i} e_i &= \sum_{|i|=1}^{\infty} \lim_{N\to\infty} \sum_{j=j_{min}}^{N} x_i \delta_{i,j} e_i \\ = \sum_{|i|=1}^{\infty} x_i e_i = x,\end{aligned}$$

where the last term converges since $x_j = 0$ for $j < j_{min}$. Note that here we had to use the continuity of $(\cdot,\cdot)$. This formula

$$x = \sum_{|i|=1}^{\infty} \frac{(x, e_{-i})}{i} e_i \tag{5.1}$$

will be extremely useful in the rest of this chapter. *In what follows, we will allow ourselves some sloppiness in the notation and write* $\sum_i$ *instead of* $\sum_{|i|=1}^{\infty}$, *when denoting a linear combination of* e_i*'s.*

We return to our discussion of the map E; suppose $\alpha \in \mathrm{Hom}^{ct}_{\mathcal{O}_B}(\mathcal{K}^{red}, \mathcal{K}^{red})$, then

$$\alpha(x) = \alpha(\lim_{N\to\infty} \sum_{|i|=1}^{N} x_i e_i) = \lim_{N\to\infty} \sum_{|i|=1}^{N} x_i \alpha(e_i) = \sum_i \frac{(x, e_{-i})}{i} \alpha(e_i).$$

If $\sum_i \frac{-1}{i} \alpha(e_{-i}) \otimes e_i$ would converge in $(\mathcal{K}^{red})^{\otimes 2}$, then its image under E would equal α. However, this series need not converge in $(\mathcal{K}^{red})^{\otimes 2}$ since, for example, $\sum_{i>0} e_{-i} \otimes e_i$ does not converge, because $\sum_{i>0} e_{-i}$ does not converge in $\mathcal{K}^{red}$. We remedy this by the following: let $\widetilde{E}$ be the map

$$\mathrm{Hom}^{ct}_{\mathcal{O}_B}(\mathcal{K}^{red}, \mathcal{K}^{red}) \to \varprojlim_k (\mathcal{K}^{red})^{\otimes 2} / (F^k\mathcal{K}^{red} \otimes \mathcal{K}^{red} + \mathcal{K}^{red} \otimes F^k\mathcal{K}^{red})$$

that sends $\alpha \in \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$ to $\sum_i \frac{-1}{i}\alpha(e_{-i}) \otimes e_i$. To see that this is well defined, note that $\sum_i \frac{-1}{i}\alpha(e_{-i}) \otimes e_i$ will always converge in $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$ because, for every k, either $\alpha(e_i) \in F^k\mathcal{K}^{\mathrm{red}}$ or $e_{-i} \in F^k\mathcal{K}^{\mathrm{red}}$, provided $|i|$ is big enough. Using this we can define

$$\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}} := \mathrm{im}\, \widetilde{E}.$$

Lemma 5.1.9. *If* $\sum_i a_i \otimes b_i \in \overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$, *then* $x \mapsto \sum_i (b_i, x) a_i$ *is an element of* $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$. *The hereby defined extension of* E *to* $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$ *is an isomorphism of* $\mathcal{O}_B$*-modules onto* $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$.

We also denote this extension by E.

Proof. Choose a quasisymplectic topological basis $\{e_i\}_{i \neq 0}$ of $\mathcal{K}^{\mathrm{red}}$. By definition, any element of $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$ is of the form $\sum_i \frac{-1}{i}\alpha(e_{-i}) \otimes e_i$, so

$$E_{\sum_i \frac{-1}{i}\alpha(e_{-i})\otimes e_i} = \sum_i \frac{-1}{i}\alpha(e_{-i})(e_i, \cdot) = \sum_i \frac{1}{i}\alpha(e_i)(\cdot, e_{-i}) = \alpha.$$

This shows that the first claim holds, but also that E is surjective. However, since $(\cdot, \cdot)$ is nondegenerate on $\mathcal{K}^{\mathrm{red}}$, it readily follows that E is also injective. □

This extended version of the map E can be considered as the infinite dimensional version of its namesake in the previous chapter. We would proceed in a similar way to the discussion of the previous chapter, i.e. let $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$ act on $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$. To that end, we introduce a completion of a universal enveloping algebra.

5.1.2 Intermezzo: filtration and completion of a universal enveloping algebra.

Let L be a filtered Lie algebra over $\mathcal{O}_B$, whose Lie algebra structure is compatible with the filtration, in that $[F^nL, F^mL] \subseteq F^{n+m}L$ for all n, m. Its universal enveloping algebra UL inherits two filtrations: the left filtration $F^n_l UL := F^nL \circ UL$ and the right filtration $F^n_r UL := UL \circ F^nL$. If L is commutative, $F^n_l UL = F^n_r UL$, but in general they will differ. *We will always use the right filtration, and simply denote it by* F^nUL. It has the following relation with the Lie product:

$$[F^nUL, F^mUL] \subseteq F^{\min(n,m)}UL.$$

However, for a fixed section $\vec{x}$ of $F^n UL$, there is an m such that $[F^k UL, \vec{x}] \subseteq F^{k+m} UL$. Therefore,

$$\overline{U}L := \varprojlim_k UL/F_r^k UL,$$

the completion of UL with respect to the filtration, inherits the Lie product of UL. As to the behavior with respect to morphisms, if $f : L \to L'$ is a morphism of filtered $\mathcal{O}_B$ Lie algebras, so in particular $f(F^k L) \subseteq F^k L'$, then by construction, f induces a morphism $\overline{f} : \overline{U}L \to \overline{U}L'$. This map is compatible with the filtrations: $\overline{f}(F^k \overline{U}L) \subseteq F^k \overline{U}L'$.

We apply his to the filtered Lie algebras $\mathcal{K}^{red}$, $\widehat{\mathcal{K}}^{red}$: this gives us filtered Lie algebras $\overline{U}\mathcal{K}^{red}$ and $\overline{U}\widehat{\mathcal{K}}^{red}$, respectively. The projection of the latter on its first summand, i.e. $\widehat{\mathcal{K}}^{red} = \mathcal{K}^{red} \oplus \mathcal{O}_B \to \mathcal{K}$ is compatible with the filtrations, so it induces a map $\overline{U}\widehat{\mathcal{K}}^{red} \to \overline{U}\mathcal{K}^{red}$. Similarly, we have an induced map $\overline{U}\widehat{\mathcal{O}}^{red} \hookrightarrow \overline{U}\widehat{\mathcal{K}}^{red}$. The completion of $U\widehat{\mathcal{K}}^{red}$ induces a completion of $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$:

$$\overline{\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell} = \overline{U}\widehat{\mathcal{K}}^{red} \otimes_{\overline{U}\widehat{\mathcal{O}}^{red}} \mathcal{O}_B,$$

but since the image of $F^k U\widehat{\mathcal{K}}^{red} \circ \nu_\ell$ in $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$ vanishes for $k \geq 1$, $\overline{\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell} = \mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$. This implies that the action of $U\widehat{\mathcal{K}}^{red}$ on $\mathbb{F}(\mathcal{K}, \mathcal{O})_\ell$ extends to $\overline{U}\widehat{\mathcal{K}}^{red}$.

We resume our discussion of the map E. Under the obvious map $(\mathcal{K}^{red})^{\otimes 2} \to U\widehat{\mathcal{K}}^{red}$, $\mathcal{K}^{red} \otimes F^k \mathcal{K}^{red}$ maps to $F^k U\widehat{\mathcal{K}}^{red}$, but $F^k \mathcal{K}^{red} \otimes \mathcal{K}^{red}$ does not: for example, we have that $\sum_{i>N} e_{-i} \circ e_i$ converges in $\overline{U}\widehat{\mathcal{K}}^{red}$, because $e_i \in F^k U\widehat{\mathcal{K}}^{red}$ for i big enough. If however $\sum_{i>N} e_i \circ e_{-i}$ would also converge, then $\sum_{i>N}^M e_i \circ e_{-i} - e_{-i} \circ e_i = \sum_{i>N}^M i\hbar$ would converge, which is not the case. Therefore, the map $\mathcal{K}^{red} \otimes \mathcal{K}^{red} \to U\widehat{\mathcal{K}}^{red}$ does not extend to the completions. This problem is remedied by a normal ordering: *choose a $\mathcal{O}' \subseteq \mathcal{K}^{red}$ that is maximal isotropic and complementary to $\mathcal{O}^{red}$*. Using the notation from the previous chapter, we denote the normal ordering associated to the decomposition $\mathcal{K}^{red} = \mathcal{O}' \oplus \mathcal{O}^{red}$ by $n_{\mathcal{O}', \mathcal{O}^{red}}$. Then for $k > 0$, the image of

$$n_{\mathcal{O}', \mathcal{O}^{red}}(F^k \mathcal{K}^{red} \otimes \widehat{\mathcal{K}}^{red}) \subseteq \mathcal{O}' \oplus F^k \mathcal{K}^{red} + F^k \mathcal{K}^{red} \oplus \mathcal{O}^{red}$$

does lie in $F^k U\widehat{\mathcal{K}}^{red}$ since $[F^k \mathcal{K}^{red}, \mathcal{O}^{red}] = 0$. We have therefore shown that the composition of the natural map $\phi : (\mathcal{K}^{red})^{\otimes 2} \to U\widehat{\mathcal{K}}^{red}$ with $n_{\mathcal{O}', \mathcal{O}^{red}}$ extends to a map

$$\phi_{\mathcal{O}', \mathcal{O}^{red}} : \overline{(\mathcal{K}^{red})^{\otimes 2}} \to \overline{U}\widehat{\mathcal{K}}^{red}.$$

Following the lines of the previous chapter, we define:

Definition 5.1.10.

$$\tau_{\mathcal{O}'} := \frac{1}{2\hbar}\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}} \circ E^{-1} : \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}}) \to \overline{U}\widehat{\mathcal{K}}^{\mathrm{red}}[\hbar^{-1}].$$

The kernel of this map has a nice description. To identify it, let $\overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}}$ and $\overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$ denote the closure of $\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}$ and $\bigwedge^2 \mathcal{K}^{\mathrm{red}}$ in $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$, respectively, so that $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}} = \overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}} \oplus \overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$. This decomposition induces a decomposition

$$E(\overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}}) \oplus E(\overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}})$$

of $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$. The restriction of $\tau_{\mathcal{O}'}$ to $\overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$ vanishes: first note that $\mathcal{K}^{\mathrm{red}} \wedge \mathcal{K}^{\mathrm{red}} = \mathcal{O}^{\mathrm{red}} \wedge \mathcal{O}^{\mathrm{red}} + O' \wedge O' + \mathcal{O}^{\mathrm{red}} \wedge O'$. Since $n_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is the identity on the first two summands, their image under $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is locally generated by expressions of the form $a \circ b - b \circ a = (a,b)\hbar = 0$ and $a' \circ b' - b' \circ a' = (a',b')\hbar = 0$ for local section a, b of $\in \mathcal{O}^{\mathrm{red}}$ and a', b' of $\mathcal{O}'$. The third summand is also mapped to 0, for $n_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}(a \otimes a' - a' \otimes a) = a' \otimes a - a' \otimes a = 0$. Thus, $\mathrm{Ker}\,\tau_{\mathcal{O}'} \supseteq \overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$, and as we will see in a minute, this inclusion is in fact an equality.

The restriction of $\tau_{\mathcal{O}'}$ to $\overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}}$ is more interesting.

Definition 5.1.11. Let $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$ denote the *continuous* symplectic $\mathcal{O}_B$-linear maps $\mathcal{K}^{\mathrm{red}} \to \mathcal{K}^{\mathrm{red}}$, so $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}) \subseteq \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$.

Let $\{e_i\}_{i\neq 0}$ be a local topological quasisymplectic basis for $\mathcal{K}^{\mathrm{red}}$ and α a local section of $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$, then

$$E^{-1}(\alpha) = \sum_i \frac{-1}{i}\alpha(e_i) \otimes e_{-i} = \sum_{i,j} \frac{(\alpha e_i, e_j)}{ij} e_{-i} \otimes e_{-j}.$$

This expression is symmetric iff $(\alpha e_i, e_j) = (\alpha e_j, e_i) = -(e_i, \alpha e_j)$, i.e. iff α is a local section of $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$. Hence, E *restricts to an isomorphism* $\overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}} \to \mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$.

Corollary 5.1.12. $\tau_{\mathcal{O}'}$ *restricts to an injective map* $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}) \to \overline{U}\widehat{\mathcal{K}}^{\mathrm{red}}[\hbar^{-1}]$.

Note that from this result and the inclusion $\mathrm{Ker}\,\tau_{\mathcal{O}'} \supseteq \overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$ it follows that $\mathrm{Ker}\,\tau_{\mathcal{O}'} = \overline{\bigwedge^2 \mathcal{K}^{\mathrm{red}}}$.

Proof. It remains to show that $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is injective on $\overline{\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}}$. Since $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is the identity on $\mathrm{Sym}_2 \mathcal{O}^{\mathrm{red}}$, $\mathrm{Sym}_2 \mathcal{O}'$, it is certainly injective on $\overline{\mathrm{Sym}_2 \mathcal{O}^{\mathrm{red}}} + \overline{\mathrm{Sym}_2 \mathcal{O}'}$. Thus, we are left to show that $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is injective on (sums of) expressions of the form $a \otimes b + b \otimes a$, $a \in \mathcal{O}'$ and $b \in \mathcal{O}^{\mathrm{red}}$. However, $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ maps such $a \otimes b + b \otimes a$ to $2a \circ b$, , . From this it readily follows that $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}$ is injective on all of $\mathrm{Sym}_2 \mathcal{K}^{\mathrm{red}}$. □

We continue to investigate this restriction of $\tau_{\mathcal{O}'}$. Suppose $\mathcal{V}$ is an isotropic $\mathcal{O}_B$-submodule of $\mathcal{K}$. If α is a local section of $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$ that maps $\mathcal{V}^\perp$ to $\mathcal{V}$, then we can say the following about $\tau_{\mathcal{O}'}$:

Lemma 5.1.13. *Let $\mathcal{V} \subseteq \mathcal{K}^{\mathrm{red}}$ be an isotropic $\mathcal{O}_B$-submodule, and α a local section of* $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}, \mathcal{K}^{\mathrm{red}})$ *such that $\alpha(\mathcal{V}^\perp) \subseteq \mathcal{V}$. Then $E^{-1} \in \overline{\mathcal{V} \oplus \mathcal{K} + \mathcal{K} \oplus \mathcal{V}}$.*

Proof. Let $\check{\mathcal{V}}$ be a complement for $\mathcal{V}^\perp$, and let $\mathcal{W}$ a complement for $\mathcal{V}$ in $\mathcal{V}^\perp$ such that $(\mathcal{V}, \mathcal{W}) = 0$. Then $\mathcal{K} = \mathcal{V} \oplus \mathcal{W} \oplus \check{\mathcal{V}}$ and $(\cdot, \cdot)$ restricts to a nondegenerate pairing between $\mathcal{V}$ and $\mathcal{V}^\perp$. Now use that $E^{-1}\alpha = \sum_i \frac{-1}{i} \alpha e_i \otimes e_{-i}$ for any local topological quasisymplectic basis $\{e_i\}_{i \neq 0}$. Choose one such that $\{e_i\}_{i \in I}$ is a basis for $\mathcal{V}$, $\{e_{-i}\}_{i \in I}$ a basis for $\check{\mathcal{V}}$ and $\{e_i\}_{\pm i \neq I}$ a basis for $\mathcal{W}$ for certain $I \subseteq \mathbb{Z} \backslash \{0\}$. Then clearly, $\sum_{\pm i \in I} \alpha e_i \otimes e_{-i}$ lies in $\overline{\mathcal{K} \otimes \mathcal{V}}$. Moreover, if $i \neq I$, then $\alpha e_i \in \alpha \mathcal{V}^\perp \subseteq \mathcal{V}$, so that $\sum_{\pm i \neq I} \alpha e_i \otimes e_{-i}$ lies in $\overline{\mathcal{V} \otimes \mathcal{K}}$. □

Below, we will use $\tau_{\mathcal{O}'}$ to construct a central extension of $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$. The crucial step for this is the next theorem.

Theorem 5.1.14. *If α, β are local sections of $\mathfrak{sp}(\mathcal{K}^{\mathrm{red}})$ then* $\mathrm{ad}\, \tau_{\mathcal{O}'}(\alpha)$ *acts on $\overline{\mathrm{U}}\widehat{\mathrm{K}}$ as α and*

$$[\tau_{\mathcal{O}'}(\alpha), \tau_{\mathcal{O}'}(\beta)] = \tau_{\mathcal{O}'}([\alpha, \beta]) + \frac{1}{2} \mathrm{tr}(\alpha[\pi_{\mathcal{O}'}, \beta]),$$

where $\pi_{\mathcal{O}'}$ is the projection of $\mathcal{O}' \oplus \mathcal{O}^{\mathrm{red}}$ on its first summand.

Proof. Let $\{e_i\}_{i \neq 0}$ be a local quasisymplectic topological basis of K. We have that

$$2\hbar \tau_{\mathcal{O}'}(\alpha) = \sum_i \phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}\left(\frac{\alpha(e_i)}{i} \otimes e_{-i}\right) = \lim_{N \to \infty} \left(\sum_{i=-N}^{N} \frac{\alpha(e_i)}{i} \circ e_{-i} - c_{\alpha,N} \hbar \right)$$

for certain local sections $c_{\alpha,N}$ of $\mathcal{O}_B$. Therefore, if x is a local section of $\widehat{\mathrm{K}}^{\mathrm{red}}$, then using the continuity of α and the Lie product (which is equivalent with the continuity of the presymplectic form), it follows that

$$[2\hbar\tau_{\mathcal{O}'}(\alpha), x] = -\lim_{N\to\infty}\sum_{i=-N}^{N}[\frac{\alpha(e_i)}{i}\circ e_{-i}, x]$$
$$= \sum_{i=-\infty}^{\infty}\hbar(\frac{(e_{-i}, x)}{i}\alpha(e_i) + \frac{(\alpha(e_i), x)}{i}e_{-i})$$
$$= \hbar\sum_i \frac{(x, e_{-i})}{i}\alpha(e_i) + \hbar\sum_i \frac{(\alpha x, e_{-i})}{i}e_i = 2\hbar\alpha(x).$$

Using this, we have that

$$[\tau_{\mathcal{O}'}(\alpha), \tau_{\mathcal{O}'}(\beta)] = \sum_i \frac{-1}{2\hbar i}\,\mathrm{ad}\,\tau_{\mathcal{O}'}(\alpha)\phi_{\mathcal{O}',\mathcal{O}}(\beta(e_i)\otimes e_{-i})$$
$$= \sum_i \frac{-1}{2\hbar i}\,\mathrm{ad}\,\tau_{\mathcal{O}'}(\alpha)(\beta(e_i)\otimes e_{-i}) = \sum_i \frac{-1}{2\hbar i}(\alpha\beta e_i\circ e_{-i} + \beta e_i\circ\alpha e_{-i}),$$

where we used that $\phi_{\mathcal{O}',\mathcal{O}}(\beta(e_i)\otimes e_{-i})$ differs from $\beta e_i\circ e_{-i}$ by sum of commutators, i.e. by something central. We now split the remaining expression in its normally ordered part and the correction terms. In order to do that, we first note that if x, y are local section of $\mathcal{K}^{\mathrm{red}}$, then

$$\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}(x\otimes y) - x\circ y = -[\pi_{\mathcal{O}^{\mathrm{red}}}x, \pi_{\mathcal{O}'}y] = -(x, \pi_{\mathcal{O}'}y)\hbar,$$

where $\pi_{\mathcal{O}'}, \pi_{\mathcal{O}^{\mathrm{red}}}$ denote the projection of $\mathcal{K}^{\mathrm{red}} = \mathcal{O}'\oplus\mathcal{O}^{\mathrm{red}}$ on the first and second summand, respectively. Using this we conclude that $[\tau_{\mathcal{O}'}(\alpha), \tau_{\mathcal{O}'}(\beta)]$ equals

$$\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}\left(\sum_i \frac{\alpha\beta e_i\otimes e_{-i} + \beta e_i\otimes\alpha e_{-i}}{-2\hbar i}\right)$$
$$+\sum_i \frac{-1}{2i}\left((\alpha\beta e_i, \pi_{\mathcal{O}'}e_{-i}) + (\beta e_i, \pi_{\mathcal{O}'}\alpha e_{-i})\right). \quad (5.2)$$

Rewriting

$$\sum_i \frac{1}{i}\beta(e_i)\otimes\alpha e_{-i} = \sum_{i,j}\frac{(\alpha e_{-i}, e_{-j})}{ij}\beta e_i\otimes e_j$$
$$= \sum_{i,j}\frac{1}{ij}\beta((\alpha e_{-j}, e_{-i})e_i)\otimes e_j = \sum_j \frac{1}{j}\beta\alpha e_{-j}\otimes e_j$$
$$= -\sum_i \frac{1}{i}\beta\alpha e_i\otimes e_{-i},$$

we see that the first two terms of (5.2) add up to

$$\phi_{\mathcal{O}',\mathcal{O}}\sum_i \frac{1}{2\hbar i}\,(\beta\alpha-\alpha\beta)e_i\otimes e_{-i} = \frac{1}{2\hbar}\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}E^{-1}([\alpha,\beta]) = \tau_{\mathcal{O}'}([\alpha,\beta]).$$

For the remaining two terms of (5.2) we have that

$$\begin{aligned} & \sum_i \frac{1}{2i}\,((\alpha\beta e_i, \pi_{\mathcal{O}'}e_{-i}) + (\beta e_i, \pi_{\mathcal{O}'}\alpha e_{-i})) \\ = \quad & \sum_i \frac{-1}{2i}\,((e_i, \beta\alpha\pi_{\mathcal{O}'}e_{-i}) - (e_i, \beta\pi_{\mathcal{O}'}\alpha(e_{-i}))) \\ = \quad & \sum_i \frac{1}{2i}(e_i, (\beta\pi_{\mathcal{O}'}\alpha - \beta\alpha\pi_{\mathcal{O}'})e_{-i}) = \frac{1}{2}\,\mathrm{tr}(\beta\pi_{\mathcal{O}'}\alpha - \beta\alpha\pi_{\mathcal{O}'}) \\ = \quad & \frac{1}{2}\,\mathrm{tr}(\alpha\beta\pi_{cO'} - \alpha\pi_{\mathcal{O}'}\beta) = \frac{1}{2}\,\mathrm{tr}(\alpha[\pi_{\mathcal{O}'}, \beta]). \end{aligned}$$

In the last step we used the cyclic invariance of the trace. □

One should compare this result with lemma (4.1.7).

Example 5.1.15. Let $\{e_i\}_{i\neq 0}$ be a topological quasisymplectic basis over some open $U\subseteq B$. In the rest of this example, we restrict ourselves to U. Define

$$T_k : \mathcal{K}^{\mathrm{red}} \to \mathcal{K}^{\mathrm{red}} : e_i \mapsto \begin{cases} ie_{i+k} & i+k\neq 0 \\ 0 & i+k=0 \end{cases}.$$

and call these the *translation operators associated to the topological quasisymplectic basis* $\{e_i\}_{i\neq 0}$. These T_k are sections of $\mathfrak{sp}^{ct}(\mathcal{K}^{\mathrm{red}})$, for

$$(T_k e_i, e_j) = i(i+k)\delta_{i+j+k,0} = (e_i, T_k e_j)$$

and for any n, we have that $T_k e_i \in F^n\mathcal{K}^{red}$, provided i is large enough. Furthermore, we have that

$$E^{-1}(T_k) = \sum_{i\neq 0} \frac{-1}{i} T_k e_i \otimes e_{-i} = -\sum_{i\neq 0, i\neq -k} e_{i+k} \otimes e_{-i}.$$

If $k \neq 0$, then $e_{i+k} \circ e_{-i} = e_{-i} \circ e_{i+k}$ so that

$$\tau_{\mathcal{O}'}(T_k) = \frac{-1}{2\hbar} \sum_{i\neq 0,-k} e_{i+k} \circ e_{-i}, \tag{5.3}$$

independent of the choice of $\mathcal{O}'$. For $k = 0$ this relation holds trivially, so that (5.3) holds for all k. We shall now illustrate the last result of theorem 5.1.14 for the T_k's:

$$[\tau_{\mathcal{O}'} T_k, \tau_{\mathcal{O}'} T_l] = \tau_{\mathcal{O}'}[T_k, T_l] + \frac{1}{2}\operatorname{tr}(T_k[\pi_{\mathcal{O}'}, T_l]).$$

To this end, we first compute both terms on the right hand side of this equation. From the definition it is clear that $T_k T_l e_i = i(i+l)e_{i+k+l}$ for all $i \neq -l, -k-l$, and 0 otherwise. Using this, a small computation shows that $[T_k, T_l]e_i = i(l-k)e_{i+k+l} = (l-k)T_{k+l}e_i$, and as a consequence $[T_k, T_l] = (l-k)T_{k+l}$. Next we compute the trace:

$$\begin{aligned}
\operatorname{tr}(T_k[\pi_{\mathcal{O}'}, T_l]) &= \sum_i \frac{1}{i}((T_k\pi_{\mathcal{O}'}T_l - T_kT_l\pi_{\mathcal{O}'})e_i, e_{-i}) \\
&= \sum_{i\neq -l,-l-k, i+l<0} \frac{i(i+l)}{i}(e_{i+k+l}\,, e_{-i}) - \sum_{i\neq -l,-l-k, i<0} \frac{i(i+l)}{i}(e_{i+k+l}\,, e_{-i}) \\
&= \delta_{k+l,0} \sum_{i<-k, i\neq 0} (i+k)(i+k+l) - \delta_{k+l,0} \sum_{i<0}(i+l)(i+k+l) \\
&= \delta_{k+l,0}\frac{k^3-k}{6}.
\end{aligned}$$

Combining these results, we find that

$$[\tau_{\mathcal{O}'} T_k, \tau_{\mathcal{O}'} T_l] = -(k-l)\tau_{\mathcal{O}'} T_{k+l} + \delta_{k+l,0}\frac{k^3-k}{12}. \tag{5.4}$$

This relation is known as the *Virasoro relation*, and an algebra whose generators $\{\tau_{\mathcal{O}'} T_k\}_{k\in\mathbb{Z}}$ satisfy it, is known as the Virasoro algebra - it has many incarnations, and we will later choose a specific one.

Theorem 5.1.14 allows us to define a central extension of $\mathfrak{sp}(\mathcal{K}^{\mathrm{red}})$:

Definition 5.1.16. Let $\widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}$ be the subalgebra of $\overline{\mathrm{U}}\widehat{\mathcal{K}}^{\mathrm{red}}$ generated by the image of $\tau_{\mathcal{O}'}$.

Corollary 5.1.17. *The following holds:*

- $\widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}$ *does not depend on the choice of* $\mathcal{O}'$*,*
- *the image of* $\mathcal{O}_{\mathrm{B}}$ *is contained in* $\widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}$*,*
- $\tau_{\mathcal{O}'}$ *factors to an* $\mathcal{O}_{\mathrm{B}}$*-module isomorphism* $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}) \to \widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}/\mathcal{O}_{\mathrm{B}}$ *and*

$$0 \longrightarrow \mathcal{O}_{\mathrm{B}} \longrightarrow \widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})} \longrightarrow \mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}) \longrightarrow 0. \tag{5.5}$$

  is a central extension of Lie algebras.

The image of the global section 1 of $\mathcal{O}_{\mathrm{B}}$ in $\widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}$ is denoted c.

Proof. By the Theorem 5.1.14 it is clear that $\operatorname{im}\tau_{\mathcal{O}'} \subseteq \widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})} \subseteq \operatorname{im}\tau_{\mathcal{O}'} + \mathcal{O}_{\mathrm{B}}$, and from Example 5.1.15 we see that $\mathcal{O}_{\mathrm{B}} \subseteq \widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})}$. All but the first assertion follows from this and, again, Theorem 5.1.14.

If $\mathcal{O}''$ is another Lagrangian submodule of $\mathcal{K}^{\mathrm{red}}$ complementary to $\mathcal{O}^{\mathrm{red}}$, then $\phi_{\mathcal{O}',\mathcal{O}^{\mathrm{red}}}x$ and $\phi_{\mathcal{O}'',\mathcal{O}^{\mathrm{red}}}x$ differ by a sum of commutators, i.e. by an element of $\hbar\mathcal{O}_{\mathrm{B}}$. Hence $\tau_{\mathcal{O}'} \equiv \tau_{\mathcal{O}''} \bmod \mathcal{O}_{\mathrm{B}}$, which proves the first assertion. □

Thus, the map $\tau_{\mathcal{O}'}$ is section of $\widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})} \to \mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$ as $\mathcal{O}_{\mathrm{B}}$-modules, but depends on the choice of $\mathcal{O}'$, and is therefore noncanonical.

5.1.3 An incarnation of the Virasoro algebra

We recall that $\theta_{\mathcal{K}/\mathcal{O}_{\mathrm{B}}}$ denotes the derivations of $\mathcal{K}$ over $\mathcal{O}_{\mathrm{B}}$. Since we assumed that the properties of lemma 3.3.2 holds, $\theta_{\mathcal{K}/\mathcal{O}_{\mathrm{B}}}$ preserves the presymplectic form, i.e. $\theta_{\mathcal{K}/\mathcal{O}_{\mathrm{B}}} \subseteq \mathfrak{sp}(\mathcal{K})$. Since $\theta_{\mathcal{K}/\mathcal{O}_{\mathrm{B}}}$ annihilates $\mathcal{K}^0$, every such derivation descends to a well defined map $\mathcal{K}^{\mathrm{red}} \to \mathcal{K}$, which in turn descends to a map $\mathcal{K}^{\mathrm{red}} \to \mathcal{K}^{\mathrm{red}}$. The latter is of course symplectic. Moreover, one easily checks that it acts continuously, and that if it is trivial, than the derivation as map from $\mathcal{K} \to \mathcal{K}$ is trivial. Hence we can identify $\theta_{\mathcal{K}/\mathcal{O}_{\mathrm{B}}}$ with a subalgebra of $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})$.

Definition 5.1.18. Define $\widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B}$ as the subalgebra of $\overline{U}\widehat{\mathcal{K}}^{\mathrm{red}}[\hbar^{-1}]$ generated by the image of $\theta_{\mathcal{K}/\mathcal{O}_B}$ under $\tau_{\mathcal{O}'}$.

We could alternatively have defined $\widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B}$ as the pullback

$$\begin{array}{ccc} \widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B} & \dashrightarrow & \theta_{\mathcal{K}/\mathcal{O}_B} \\ \downarrow & & \downarrow \\ \widehat{\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}})} & \longrightarrow & \mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}). \end{array}$$

From this and corollary 5.1.17, it follows that this algebra is independent of the choice of $\mathcal{O}'$.

Example 5.1.19. In this example we assume that $N = 1$. Locally over B, we choose a coordinate $t = t_1$, and restrict ourselves to the part of B over which t is defined. So $\mathcal{K} = \mathcal{O}_B((t))$ and $\theta_{\mathcal{K}/\mathcal{O}_B} = \mathcal{K}\frac{\partial}{\partial t}$. If we define

$$e_i = t^i, \quad T_k = t^{k+1}\frac{\partial}{\partial t}, \quad i,k \in \mathbb{Z}, i \neq 0,$$

then e_i is a quasi topological basis of $\mathcal{K}^{\mathrm{red}}$ and the e_i's and T_k's satisfy the criteria of example 5.1.15. Thus for $N = 1$, $\widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B}$ is generated by c, $\{\tau_{\mathcal{O}'}t^{k+1}\frac{\partial}{\partial t}\}_{k\in\mathbb{Z}}$ as $\mathcal{O}_B$-module, and the algebra structure is given by

$$[\tau_{\mathcal{O}'}t^{k+1}\frac{\partial}{\partial t}, \tau_{\mathcal{O}'}t^{l+1}\frac{\partial}{\partial t}] = -(k-l)\tau_{\mathcal{O}'}t^{k+l+1}\frac{\partial}{\partial t} + \delta_{k,-l}\frac{k^3-k}{12}c.$$

5.2 The Segal-Sugawara representation

Until now, we have considered $\mathcal{K}$ (and in the previous chapter $\mathcal{H}$) as a presymplectic (symplectic) $\mathcal{O}_B$-module, without any additional algebraic structure. In this section we will extend the considerations above to the setting where an additional Lie product is present. *This setting is the output of Chapter 3; more precisely, we fix a simple complex algebraic group* G *with Lie algebra* $\mathfrak{g}$ *and let* $\mathcal{K}_{\mathfrak{g}}, \mathcal{K}^0_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}}, (\cdot,\cdot)_{\mathfrak{g}}$ *be as in Chapter 3.* For the readers convenience, we recall that

- $\mathcal{K}^0_{\mathfrak{g}}$ is locally over B of the form $\mathcal{K}^0 \otimes \mathfrak{g}$ (as $\mathcal{K}^0$ Lie algebra),
- $\mathcal{K}_{\mathfrak{g}} = \mathcal{K}^0_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{K}$, $\mathcal{O}_{\mathfrak{g}} = \mathcal{O}_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{K}^0_{\mathfrak{g}}$ and $\mathfrak{m}_{\mathfrak{g}} = \mathfrak{m} \otimes_{\mathcal{K}^0} \mathcal{K}^0_{\mathfrak{g}}$ as $\mathcal{K}$, $\mathcal{O}$ and $\mathfrak{m}$ Lie algebras, respectively,

- $\mathcal{A}_\mathfrak{g}$ is a subalgebra of $\mathcal{K}_\mathfrak{g}$ that is locally of the form $\mathcal{A} \otimes \mathfrak{g}$ as $\mathcal{A}$ algebra,
- $(\cdot,\cdot)_\mathfrak{g}$ is an $\mathcal{O}_B$-bilinear and ad-invariant presymplectic form on $\mathcal{K}_\mathfrak{g}$ with kernel $\mathcal{K}^0_\mathfrak{g}$,
- $\mathcal{A}_\mathfrak{g}$ is an isotropic and $\mathcal{O}_\mathfrak{g}$ a maximal isotropic submodule of $\mathcal{K}_\mathfrak{g}$,
- $\mathcal{K}_\mathfrak{g}$ has a connection ∇ as $\mathcal{K}$-module over $\mathcal{O}_B$.

The $\mathcal{K}$-module structure gives $\mathcal{K}_\mathfrak{g}$ a natural filtration and topology. The former is given by

$$F^k\mathcal{K}_\mathfrak{g} := F^k\mathcal{K} \otimes_{\mathcal{K}^0} \mathcal{K}^0_\mathfrak{g} \subseteq \mathcal{K}_\mathfrak{g},$$

and the latter is the one coming from the given topology on $\mathcal{K}$ and the indiscreet topology on $\mathfrak{g}$, under a local identification $\mathcal{K}_\mathfrak{g} = \mathcal{K} \otimes \mathfrak{g}$. More explicitly, a sequence (x_n) in $\mathcal{K}_\mathfrak{g}$ converges to 0 iff for every k there exists an N_k such that $x_n \in \mathfrak{m}^k \otimes \mathcal{K}^0_\mathfrak{g}$ for all $n \geq N_k$. This, and translation invariance, also determine the topology. Since $\mathcal{K}$ is complete, so is $\mathcal{K}_\mathfrak{g}$.

Lemma 5.2.1. *The Lie bracket on $\mathcal{K}_\mathfrak{g}$ is continuous and compatible with the filtration, in the sense that*

$$[F^k\mathcal{K}_\mathfrak{g}, F^l\mathcal{K}_\mathfrak{g}] \subseteq F^{k+l}\mathcal{K}_\mathfrak{g}.$$

Furthermore, for any local section D of $\theta_{\mathcal{K}/\mathcal{O}_B}$, ∇_D is continuous and compatible with the filtration, in that if D lies in $F^k\theta_{\mathcal{K}/\mathcal{O}_B}$, then for all l

$$\nabla_D F^l\mathcal{K}_\mathfrak{g} \subseteq F^{k+l}\mathcal{K}_\mathfrak{g}.$$

The proof is straightforward.

5.2.1 $\widehat{\mathcal{K}}_\mathfrak{g}$ and $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$

We now follow the same steps as in the beginning of this chapter:

Definition 5.2.2. We define $\widehat{\mathcal{K}}_\mathfrak{g} := \mathcal{K}_\mathfrak{g} \oplus \mathcal{O}_B$, and endow it with the Lie product that satisfies

$$[x \oplus r, y \oplus s] = [x, y] \oplus 0 + 0 \oplus (x, y)_\mathfrak{g},$$

for local sections x, y of $\mathcal{K}_\mathfrak{g}$ and r, s of $\mathcal{O}_B$. The central element $0 \oplus 1$ is denoted $\hbar$. We give $\widehat{\mathcal{K}}_\mathfrak{g}$ the product topology and define a filtration by

$$F^k\widehat{\mathcal{K}}_\mathfrak{g} = \begin{cases} F^k\mathcal{K}_\mathfrak{g} \oplus 0 & k > 0 \\ F^k\mathcal{K}_\mathfrak{g} \oplus \mathcal{O}_B & k \leq 0. \end{cases}$$

Remark 5.2.3. We often identify $\mathcal{K}_\mathfrak{g}$ with $\mathcal{K}_\mathfrak{g} \oplus 0 \subseteq \widehat{\mathcal{K}}_\mathfrak{g}$. This makes the meaning of $[x, y]$ in $\widehat{\mathcal{K}}_\mathfrak{g}$ ambiguous, for local section x, y on $\mathcal{K}_\mathfrak{g}$. To avoid this, *we will denote the Lie bracket of* $\mathcal{K}_\mathfrak{g}$ *by* $[\cdot, \cdot]'$ *in what follows, and the one on* $\widehat{\mathcal{K}}_\mathfrak{g}$ *by* $[\cdot, \cdot]$*:*

$$[x, y] = [x, y]' + \hbar(x, y)_\mathfrak{g}.$$

The obvious projection $\widehat{\mathcal{K}}_\mathfrak{g} \to \mathcal{K}_\mathfrak{g}$ and injection $\mathcal{O}_B \to \widehat{\mathcal{K}}_\mathfrak{g}$ are algebra morphisms, where we let the Lie product on $\mathcal{O}_B$ be the trivial one. Since $\mathcal{O}_B c_\mathfrak{g}$ is central,

$$0 \longrightarrow \mathcal{O}_B \xrightarrow{f \mapsto f c_\mathfrak{g}} \widehat{\mathcal{K}}_\mathfrak{g} \longrightarrow \mathcal{K}_\mathfrak{g} \longrightarrow 0$$

is a central extension of $\mathcal{O}_B$-Lie algebras. Since $\mathcal{A}_\mathfrak{g}$ and $\mathfrak{m}_\mathfrak{g}$ are subalgebras of $\mathcal{K}_\mathfrak{g}$ that are both isotropic with respect to $(\cdot, \cdot)_\mathfrak{g}$, $\mathcal{A}_\mathfrak{g} \oplus 0$ and $\mathfrak{m}_\mathfrak{g} \oplus 0$ are subalgebras of $\widehat{\mathcal{K}}_\mathfrak{g}$, and we identify them with $\mathcal{A}_\mathfrak{g}, \mathfrak{m}_\mathfrak{g}$ respectively. Furthermore,

$$\widehat{\mathcal{O}}_\mathfrak{g} := \mathcal{O}_\mathfrak{g} \oplus \mathcal{O}_B \subseteq \widehat{\mathcal{K}}_\mathfrak{g}$$

is also a subalgebra of $\widehat{\mathcal{K}}_\mathfrak{g}$:

$$[\widehat{\mathcal{O}}_\mathfrak{g}, \widehat{\mathcal{O}}_\mathfrak{g}] \subseteq [\mathcal{O}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}] \oplus 0 + 0 \oplus (\mathcal{O}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\mathfrak{g} \subseteq \mathcal{O}_\mathfrak{g} \oplus 0 + 0 \subseteq \widehat{\mathcal{O}}_\mathfrak{g}.$$

Definition 5.2.4. Let χ be the character of $\widehat{\mathcal{O}}_\mathfrak{g}$ defined by projection on the second summand: $\chi : \widehat{\mathcal{O}}_\mathfrak{g} = \mathcal{O}_\mathfrak{g} \oplus \mathcal{O}_B \to \mathcal{O}_B$. Then we define

$$\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell := U\widehat{\mathcal{K}}_\mathfrak{g} \otimes_{U\widehat{\mathcal{O}}_\mathfrak{g}} \mathcal{O}_B,$$

where $\widehat{\mathcal{O}}_\mathfrak{g}$ acts on $\mathcal{O}_B$ by $\ell\chi$, $\ell \in \mathbb{Z}$. We call this Verma module the *Fock module associated to the triple* $\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, (\cdot, \cdot)_\mathfrak{g}$ of level ℓ.

Remark 5.2.5. Given a representation $\rho : \mathcal{K}^0_\mathfrak{g} \to \mathrm{End}_{\mathcal{O}_B}(\mathcal{M})$ of $\mathcal{K}^0_\mathfrak{g}$ on a coherent $\mathcal{O}_B$-module, the definition above can be generalized to the following:

$$\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell,\rho} := U\widehat{\mathcal{K}}_\mathfrak{g} \otimes_{U\widehat{\mathcal{O}}_\mathfrak{g}} \mathcal{M},$$

where $\widehat{\mathcal{O}}_{\mathfrak{g}}$ acts as follows on $\mathcal{M}$: $c_{\mathfrak{g}}$ acts by multiplication with ℓ and $\mathcal{O}_{\mathfrak{g}}$ acts by the composition of $\mathcal{O}_{\mathfrak{g}} \to \mathcal{O}_{\mathfrak{g}}/\mathfrak{m}_{\mathfrak{g}} = \mathcal{K}^0_{\mathfrak{g}}$ with ρ.

If we locally identify $\mathcal{K}^0_{\mathfrak{g}}$ with $\mathcal{O}_B^N \otimes \mathfrak{g}$, we can locally define such a ρ as follows: for an N-tuple of weights $\vec{\lambda}$ of $\mathfrak{g}$, let V_{λ_i} be the highest weight representation of weight λ_i. Then $\mathcal{M}_{\vec{\lambda}} = \mathcal{O}_B \otimes \bigotimes_i V_{\lambda_i}$ is an $\mathcal{K}^0$-linear representation of $\mathcal{K}^0_{\mathfrak{g}}$ under the obvious action. We write $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell,\vec{\lambda}}$ for the corresponding Fock module. We call this the *Fock module of level ℓ and weight $\vec{\lambda}$*.

For $\vec{\lambda} = 0$, ρ is the trivial representation on $\mathcal{O}_B$, so that $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell,0} = \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell}$. For this reason, we can also call $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell}$ the Fock module of weight 0 and level ℓ.

5.2.2 The $\mathcal{O}_B$-linear action of $\mathfrak{sp}^{ct}(\mathcal{K}^{red}_{\mathfrak{g}})^G$ on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell}$.

Below, we will introduce an analogue of $\tau_{\mathcal{O}'}$ for our current setting. Due to the addition Lie algebra structure, the analogy will not be complete for we will only be able to prove a part of the properties of $\tau_{\mathcal{O}'}$.

Similar to the notation used earlier, we denote

$$\mathcal{K}^{red}_{\mathfrak{g}} := \mathcal{K}_{\mathfrak{g}}/\mathcal{K}^0_{\mathfrak{g}} = \mathcal{K}^0_{\mathfrak{g}} \otimes_{\mathcal{K}^0_{\mathfrak{g}}} \mathcal{K}^{red}$$

and give it the topology it inherits from $\mathcal{K}_{\mathfrak{g}}$. By construction, $(\cdot,\cdot)_{\mathfrak{g}}$ descends to a *symplectic* form on $\mathcal{K}^{red}_{\mathfrak{g}}$. The Lie algebra structure of $\mathcal{K}_{\mathfrak{g}}$, however, does not pass over to $\mathcal{K}^{red}_{\mathfrak{g}}$, since $\mathcal{K}^0_{\mathfrak{g}}$ is not an ideal of $\mathcal{K}_{\mathfrak{g}}$.

Since $\mathcal{K}^{red}_{\mathfrak{g}}$ is a topological symplectic $\mathcal{O}_B$-module, we also have the notion of quasisymplectic basis for $\mathcal{K}^{red}_{\mathfrak{g}}$ - the definition is the same as definition 5.1.7, but with $\mathcal{K}^{red}$ replaced by $\mathcal{K}^{red}_{\mathfrak{g}}$.

Example 5.2.6. A semi-natural choice for a local topological quasisymplectic basis is obtained as follows: suppose $X_0, \ldots, X_{n-1}$ is a local basis of $\mathcal{K}^0_{\mathfrak{g}}$ over $\mathcal{K}^0$, that is orthonormal with respect to tr, and let $\{e_i\}_{i\neq 0}$ be a local topological quasisymplectic basis of $\mathcal{K}^{red}$. Furthermore, let $f = (f_1, f_2) : \mathbb{Z}\backslash\{0\} \to \{0, \ldots, n-1\} \times \mathbb{Z}\backslash\{0\}$ be the function that maps a positive integer i to (j, k), where $j + kn = i$ and $0 \leq j \leq n-1$, and a negative integer i to $(j, -k)$. More explicitly,

$$f_2(i) = \begin{cases} \lfloor i/n \rfloor & i > 0 \\ \lceil i/n \rceil & i < 0 \end{cases}, \quad f_1(i) = i - nf_2(i).$$

Then $\widetilde{E}_i := X_{f_1(i)} \otimes e_{f_2(i)}$ is a local topological basis of $\mathcal{K}^{red}_{\mathfrak{g}}$. Since

$$(X_{f_1(i)} \otimes e_{f_2(i)}, X_{f_1(j)} \otimes e_{f_2(j)})_{\mathfrak{g}} = \delta_{i,-j} f_2(i),$$

we can make it quasisymplectic by a minor modification: $E_i := \frac{i}{f_2(i)}\widetilde{E}_i$ is a local topological basis satisfying $(E_i, E_j)_{\mathfrak{g}} = i\delta_{i,-j}$.

Locally over B, $\mathcal{K}^{\mathrm{red}}$ is indistinguishable from $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$ as topological symplectic $\mathcal{O}_B$-module. By choosing local topological quasisymplectic bases for both, one can find an explicit local identification. As a consequence of this, the constructions we did for $\mathcal{K}^{\mathrm{red}}$ pass over to $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$. The first example of this is the following: choose a local topological quasisymplectic basis $\{E_i\}_{i\neq 0}$ for $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$, then we locally have that

$$x = \sum_{|i|=1}^{\infty} \frac{(x, E_{-i})}{i} E_i,$$

for any local section x of $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$. Another example is the following isomorphism:

$$E_{\mathfrak{g}} : \overline{(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})^{\otimes 2}} \longrightarrow \mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}, \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}).$$

Here

$$\overline{(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})^{\otimes 2}} \subseteq \varprojlim_k (\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})^{\otimes 2}/(F^k\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}} \otimes \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}} + \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}} \otimes F^k\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}).$$

is defined in a way analogous to $\overline{(\mathcal{K}^{\mathrm{red}})^{\otimes 2}}$. The map $E_{\mathfrak{g}}$ is characterized by the property that $E_{\mathfrak{g}}(x, y)(z) = (y, z)x$ for local sections x, y, z of $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$. In terms of the $\{E_i\}_{i\neq 0}$, the inverse of $E_{\mathfrak{g}}$ is locally given by

$$E_{\mathfrak{g}}^{-1}(\alpha) = \sum_i \frac{-1}{i}\alpha E_i \otimes E_{-i} = \sum_{i,j} \frac{(\alpha E_i, E_j)}{ij} E_{-i} \otimes E_{-j}$$

for any section α of $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}, \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})$. Lemma 5.1.13 has the following analogue:

Lemma 5.2.7. *Let $\mathcal{V} \subseteq \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$ be an isotropic $\mathcal{O}_B$-submodule and α a local section of $\mathrm{Hom}^{\mathrm{ct}}_{\mathcal{O}_B}(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}, \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})$ such that $\alpha(\mathcal{V}^{\perp}) \subseteq \mathcal{V}$. Then $E_{\mathfrak{g}}^{-1} \in \overline{\mathcal{V} \oplus \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}} + \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}} \oplus \mathcal{V}}$.*

Let $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}})$ denote *the continuous symplectic $\mathcal{O}_B$-endomorphisms of $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$*. Then as third example of the above mentioned identification, we have that $E_{\mathfrak{g}}$ restricts to an isomorphism

$$E_{\mathfrak{g}} : \overline{\mathrm{Sym}_2\, \mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}} \to \mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}).$$

We would like to derive a result for $E_{\mathfrak{g}}$, similar to theorem 5.1.14, namely that up to a factor, ad $E_{\mathfrak{g}}^{-1}(\alpha)$ acts on $U\widehat{\mathcal{K}}_{\mathfrak{g}}$ as α. Unfortunately, perhaps, there is no natural map from $(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}$ to $U\widehat{\mathcal{K}}_{\mathfrak{g}}$. One way to overcome this is to assume the following: *let $\mathcal{O}'_{\mathfrak{g}}$ be an isotropic complement for $\mathcal{O}_{\mathfrak{g}}$ in $\mathcal{K}_{\mathfrak{g}}$. Such an $\mathcal{O}'_{\mathfrak{g}}$ may not exist globally, but will always exist locally - in the following we will restrict ourselves to some open subset of* B *if necessary, and assume that such an $\mathcal{O}'_{\mathfrak{g}}$ exists.* This allows us to identify $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$ with $\mathcal{O}'_{\mathfrak{g}} \oplus \mathfrak{m}_{\mathfrak{g}}$ in $\mathcal{K}_{\mathfrak{g}}$ and lift an element of $(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}$ to $\mathcal{K}_{\mathfrak{g}}^{\otimes 2}$ - this can then be mapped to $U\widehat{\mathcal{K}}_{\mathfrak{g}}$ in the obvious way. In order to extend this composition $(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2} \to U\widehat{\mathcal{K}}_{\mathfrak{g}}$ to $\overline{(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}}$, we introduce a normal ordering and a completion of the universal enveloping algebra; we start with the latter.

Give $U\widehat{\mathcal{K}}_{\mathfrak{g}}$ the right-filtration it inherits from $\widehat{\mathcal{K}}_{\mathfrak{g}}$, i.e.

$$F^k U\widehat{\mathcal{K}}_{\mathfrak{g}} := U\widehat{\mathcal{K}}_{\mathfrak{g}} \circ F^k \mathcal{K}_{\mathfrak{g}}$$

and denote the completion with respect to it by $\overline{U}\widehat{\mathcal{K}}_{\mathfrak{g}}$. Then, as before, we have for the induced completion of the Fock module

$$\overline{\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell}} = \overline{U}\widehat{\mathcal{K}}_{\mathfrak{g}} \otimes_{\overline{U}\widehat{\mathcal{O}}_{\mathfrak{g}}} \mathcal{O}_B = U\widehat{\mathcal{K}}_{\mathfrak{g}} \otimes_{\overline{U}\widehat{\mathcal{O}}_{\mathfrak{g}}} \mathcal{O}_B = \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell},$$

because $F^k\widehat{\mathcal{K}}_{\mathfrak{g}}$ annihilates $\mathcal{O}_B$ for $k > 0$. By this remark, the action of $U\widehat{\mathcal{K}}_{\mathfrak{g}}$ on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell}$ extends to $\overline{U}\widehat{\mathcal{K}}_{\mathfrak{g}}$.

Our choice of $\mathcal{O}'_{\mathfrak{g}}$ identifies $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$ with $\mathcal{O}'_{\mathfrak{g}} \oplus \mathfrak{m}_{\mathfrak{g}}$, and this gives us a normal ordering map

$$n_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} : (\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2} \to (\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}$$

Denote the natural map $\mathcal{K}_{\mathfrak{g}}^{\otimes 2} \to U\widehat{\mathcal{K}}_{\mathfrak{g}}$ by ϕ, its composition with the lifting map $(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2} \to \mathcal{K}_{\mathfrak{g}}^{\otimes 2}$ by $\widetilde{\phi}$, and the composition of this with $n_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}}$ by $\widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}}$. It is easy to see that $\widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}}$ extends to a map

$$\widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} : \overline{(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}} \to \overline{U}\widehat{\mathcal{K}}_{\mathfrak{g}}.$$

With a modest amount of foresight, we define the following.

Definition 5.2.8.

$$\tau_{\mathcal{O}'_{\mathfrak{g}}} := \frac{1}{2\hbar + 2\check{h}} \widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} \circ E_{\mathfrak{g}}^{-1} : \mathfrak{sp}^{ct}(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}) \to \overline{U}\widehat{\mathcal{K}}_{\mathfrak{g}}[\frac{1}{\hbar + \check{h}}],$$

where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$.

This determines an $\mathcal{O}_B$-linear action of $\mathfrak{sp}^{ct}(\mathcal{K}_\mathfrak{g}^{red})$ on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ by combining $\tau_{\mathcal{O}'_\mathfrak{g}}$ with left multiplication. In the next subsection we will prove that it is an $\mathcal{O}_B$-algebra representation for a certain $\mathcal{O}_B$-subalgebra of $\mathfrak{sp}^{ct}(\mathcal{K}_\mathfrak{g}^{red})$.

5.2.3 The projective representation of $\theta_{\mathcal{K}/\mathcal{O}_B}$ on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$.

In order for $\tau_{\mathcal{O}'_\mathfrak{g}}$ to have properties similar to those of $\tau_{\mathcal{O}'}$, we must restrict its domain and moreover make the triviality assumption 3.2.14 stated in section 3.2.4: *we assume that $\mathcal{K}_\mathfrak{g}$ has a $\mathcal{K}$-linear G-action that acts symplectically, preserves $\mathcal{A}_\mathfrak{g}$, leaves the Lie bracket invariant, commutes with ∇ and acts without nontrivial fixed points.* We recall that these conditions are satisfied if the bundle used as "input" in Chapter 3 has a *global flat* section.

Remark 5.2.9. Below we will show that $\theta_{\mathcal{K}/\mathcal{O}_B}$ can be identified with a subalgebra of $\mathfrak{sp}^{ct}(\mathcal{K}_\mathfrak{g}^{red})$, and that the action of $\theta_{\mathcal{K}/\mathcal{O}_B}$ on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ given by left-multiplication with $\tau_{\mathcal{O}'_\mathfrak{g}}$ is a projective Lie algebra representation. Theorem 5.2.14 and 5.2.16 are the key results for this. The proofs of these, and the two preliminary results below, are rather technical in nature; the reader who wishes to skip these technicalities can just read the statements of theorem 5.2.14, 5.2.16 and then proceed to definition 5.2.17.

Note that since $\mathcal{K}_\mathfrak{g}^0$ is preserved by the G-action, the latter descends to an action on $\mathcal{K}_\mathfrak{g}^{red}$.

Lemma 5.2.10. *The above mentioned G-action is continuous, so that the induced action on $(\mathcal{K}_\mathfrak{g}^{red})^{\otimes 2}$ extends to $\overline{(\mathcal{K}_\mathfrak{g}^{red})^{\otimes 2}}$. Furthermore, $E_\mathfrak{g}$ is G-equivariant with respect to this action and the one induced on $\mathfrak{sp}^{ct}(\mathcal{K}_\mathfrak{g}^{red})$, and restricts to an isomorphism*

$$E_\mathfrak{g} : \overline{\mathrm{Sym}_2 \mathcal{K}_\mathfrak{g}^{red}}^G \to \mathfrak{sp}^{ct}(\mathcal{K}_\mathfrak{g}^{red})^G.$$

Here the superscript G denotes the G-invariant part.

Proof. The first statement follows from the $\mathcal{K}$-linearity of the action on $\mathcal{K}_\mathfrak{g}$, and facts that G preserves $\mathcal{K}_\mathfrak{g}^0$ - this assures that G preserves $F^k\mathcal{K}_\mathfrak{g}$. The G-equivariance basically follows from the G-invariance of $(\cdot,\cdot)_\mathfrak{g}$: for local sections x, y, z of $\mathcal{K}_\mathfrak{g}^{red}$ we have that

$$g(E_\mathfrak{g}(x \otimes y)z) = (y, z)_\mathfrak{g} gx = (gy, gz)_\mathfrak{g} gx = E_\mathfrak{g}(gx \otimes gy)gz,$$

so that $g \circ E_\mathfrak{g}(x \otimes y) = E_\mathfrak{g}(x \otimes y) \circ g$ iff $x \otimes y$ is G-invariant. By subsequently using the $\mathcal{K}$-linearity of the G-action, one readily derives the G-equivariance. The final statement is then obvious. □

Remark 5.2.11. If $\bigoplus_{i\in I}\mathcal{W}_i$ is a decomposition of $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$ in G-invariant submodules, then a local section s of $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}\otimes\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$ is G-invariant iff the projection of s on every summand $\mathcal{W}_i\otimes\mathcal{W}_j$, $i,j\in I$, is G-invariant. This statement extends to $\overline{\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}\otimes\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}}$.

In order to prove the desired results for $\tau_{\mathcal{O}'_{\mathfrak{g}}}$, we will *assume that* $\mathcal{O}'_{\mathfrak{g}}$ *is a G-invariant isotropic complement for* $\mathcal{O}_{\mathfrak{g}}$ *in* $\mathcal{K}_{\mathfrak{g}}$. The following two technical results will allow us to prove a result analogous to theorem 5.1.14.

Lemma 5.2.12. *The lifting map* $(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^{\otimes 2}\to\mathcal{K}_{\mathfrak{g}}^{\otimes 2}$ *is equivariant. Furthermore, if we let* $C:\mathcal{K}_{\mathfrak{g}}^{\otimes 2}\to\mathcal{K}_{\mathfrak{g}}$ *be the map that sends local sections* $x\otimes y$ *to* $[x,y]$*, then* $C((\mathcal{K}_{\mathfrak{g}}^{\otimes 2})^G)=0$.

Proof. The first claim follows from the fact that $\mathcal{O}'_f g$ and $\mathfrak{m}_{\mathfrak{g}}$ are preserved by G. For the second assertion, we use that G leaves $[\cdot,\cdot]$ invariant, and that C maps G-invariant sections to G-invariant sections. Hence if $x\otimes y$ is G-invariant, then $g^*[x,y]=[g^*x,g^*y]=C(g^*(x\otimes y))=[x,y]$, $g\in G$. Thus $C((\mathcal{K}_{\mathfrak{g}}^{\otimes 2})^G)\subseteq\mathcal{K}_{\mathfrak{g}}$ is G-invariant, and must therefore be 0. □

Corollary 5.2.13. *If* $\alpha\in\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^G$*, then* $\operatorname{ad}\tau_{\mathcal{O}'_{\mathfrak{g}}}(\alpha)$ *acts on* $\widehat{K_{\mathfrak{g}}}$ *in* $\overline{U}\widehat{K}_{\mathfrak{g}}[1/(\hbar+\check{h})]$ *as* $\frac{1}{2\hbar+2\check{h}}\operatorname{ad}(\widetilde{\phi}\circ E_{\mathfrak{g}}^{-1}(\alpha))$ *(i.e. we can forget the normal ordering).*

Proof. By the lemma above, $E_{\mathfrak{g}}^{-1}(\alpha)$ is G-invariant, and by the successive remark, so are its projections onto subspaces $\mathcal{O}'_{\mathfrak{g}}\otimes\mathcal{O}'_{\mathfrak{g}}$, $\mathcal{O}'_{\mathfrak{g}}\otimes\mathfrak{m}_{\mathfrak{g}}$, $\mathfrak{m}_{\mathfrak{g}}\otimes O'_{\mathfrak{g}}$ and $\mathfrak{m}_{\mathfrak{g}}\otimes\mathfrak{m}_{\mathfrak{g}}$;we call them $s_{\pm\pm}$ where $+$ refers to $\mathfrak{m}_{\mathfrak{g}}$ and $-$ to $\mathcal{O}'_{\mathfrak{g}}$. By definition, $n_{O'_{\mathfrak{g}},\mathfrak{m}_{\mathfrak{g}}}$ is the identity on s_{++},s_{--},s_{-+} and will "swap" the factors of s_{+-}. More concretely, if $s_{+-}=\sum_k x_k\otimes y_k$, for local sections x_k of $\mathfrak{m}_{\mathfrak{g}}$ and y_k of $\mathcal{O}'_{\mathfrak{g}}$, then $n_{O'_{\mathfrak{g}},\mathfrak{m}_{\mathfrak{g}}}(s_{+-})=\sum_k y_k\otimes x_k$. Hence in $\overline{U}\widehat{K}_{\mathfrak{g}}$ we have

$$\begin{aligned}[\widetilde{\phi}_{O'_{\mathfrak{g}},\mathfrak{m}_{\mathfrak{g}}}(s_{+-}),x]&=\sum_k[x_k\circ y_k-[x_k,y_k]'-\hbar(x_k,y_k),x]\\&=\sum_k[x_k\circ y_k-[x_k,y_k]',x].\end{aligned}$$

However, s_{+-} is G-invariant by the previous remark, so $\sum_k[x_k,y_k]=0$ by Lemma 5.2.12, and the assertion follows. □

We are now ready to formulate the analogue of theorem 5.1.14, albeit only for part of $\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})$. Recall that since $\nabla_{\theta_{\mathcal{K}/\mathcal{O}_B}}$ acts continuously and symplectically on $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$, we have that $\nabla_{\theta_{\mathcal{K}/\mathcal{O}_B}}\subseteq\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})$. Moreover we assumed that the G-action commutes with ∇, so that $\nabla_{\theta_{\mathcal{K}/\mathcal{O}_B}}\subseteq\mathfrak{sp}^{\mathrm{ct}}(\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}})^G$ as well.

Theorem 5.2.14. *For a local section* D *of* $\theta_{\mathcal{K}/\mathcal{O}_B}$, $\operatorname{ad}\tau_{O'_{\mathfrak{g}}}(\nabla_D) = \nabla_D$ *when acting on* $U\widehat{K}_{\mathfrak{g}}$, *where we convene that* $\nabla\hbar = 0$.

Proof. Choose a local topological quasisymplectic basis $\{E_i\}_{i\neq 0}$ of $\mathcal{K}_{\mathfrak{g}}^{\mathrm{red}}$ such that E_i lies in $\mathfrak{m}_{\mathfrak{g}}$ for $i > 0$, and in $\mathcal{O}'_{\mathfrak{g}}$ for $i < 0$. Then according to the previous corollary,

$$\begin{aligned}
(2\hbar + 2\check{h})[\tau_{O'_{\mathfrak{g}}}(\nabla_D), x] &= \sum_{ij} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{ij}[E_{-i} \circ E_{-j}, x] \\
&= \sum_{ij} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{ij}\hbar\left((E_{-i}, x)_{\mathfrak{g}} E_{-j} + (E_{-j}, x)_{\mathfrak{g}} E_{-i}\right) \\
&\qquad + \sum_{ij} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{ij}\left([E_{-i}, x]' \circ E_{-j} + E_{-i} \circ [E_{-j}, x]'\right),
\end{aligned}$$

where we recall that $[\cdot,\cdot]'$ denotes the Lie bracket of $\mathcal{K}_{\mathfrak{g}}$. For the first term of the last expression we have that

$$\begin{aligned}
&\sum_{ij} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{ij}\hbar\left((E_{-i}, x)_{\mathfrak{g}} E_{-j} + (E_{-j}, x)_{\mathfrak{g}} E_{-i}\right) \\
= \ &\hbar \sum_j \frac{1}{j}\Big(\sum_i \frac{(\nabla_D E_j, E_i)_{\mathfrak{g}}}{i} E_{-i}, x\Big) E_{-j} + \hbar \sum_i \frac{1}{i}\Big(\sum_j \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{j} E_{-j}, x\Big) E_{-i} \\
= \ &-2\hbar \sum_i \frac{(\nabla_D E_i, x)_{\mathfrak{g}}}{i} E_{-i} = 2\hbar \sum_i \frac{(\nabla_D x, E_{-i})_{\mathfrak{g}}}{-i} E_i = 2\hbar \nabla_D x.
\end{aligned}$$

The second term

$$\sum_{ij} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}}{ij}\left([E_{-i}, x]' \circ E_{-j} + E_{-i} \circ [E_{-j}, x]'\right) \tag{5.6}$$

can be rewritten as

$$\sum_{ijk} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-i}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-k} \circ E_{-j} + \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-i} \circ E_{-k}$$

which, by renaming indices, is equal to

$$\sum_{ijk} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-k} \circ E_{-i} + \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-i} \circ E_{-k}.$$

We rewrite this expression using a normal ordering. It is important here that *we can choose this normal order*, and we take it to be of the following type: let $t_1, \ldots, t_N$ be coordinates of $\mathcal{O}$ and take $\mathcal{O}''_{\mathfrak{g}}$ to be the subspace of $\mathcal{K}_{\mathfrak{g}}$ generated by $\mathcal{K}^0_{\mathfrak{g}} t_i^k$ for $k < 0$ and $i = 1, \ldots, N$. Then (without using anything special about our normal ordering yet)

$$\begin{aligned}
&\widetilde{\phi}_{\mathcal{O}''_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} \sum_{ijk} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-k} \otimes E_{-i} \\
&+\widetilde{\phi}_{\mathcal{O}''_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} \sum_{ijk} \frac{(\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-i} \otimes E_{-k} \\
= \;& \widetilde{\phi}_{\mathcal{O}''_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} \sum_{ijk} \frac{(\nabla_D E_k, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_i) + (\nabla_D E_i, E_j)_{\mathfrak{g}}([E_{-j}, x]', E_k)_{\mathfrak{g}}}{ijk} E_{-i} \otimes E_{-k} \\
= \;& \widetilde{\phi}_{\mathcal{O}''_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}} \sum_{ik} \frac{-([\nabla_D E_k, x]', E_i)_{\mathfrak{g}} - ([\nabla_D E_i, x]', E_k)_{\mathfrak{g}}}{ik} E_{-i} \otimes E_{-k}.
\end{aligned}$$

Since we assumed lemma 3.3.2 to hold, this vanishes by 3.12: for any sections x, y, z of $\mathcal{K}_{\mathfrak{g}}$, $([\nabla_D x, y], z) = -([\nabla_D z, y], x)$.

At this point exploit our choice of $\mathcal{O}''_{\mathfrak{g}}$ to bring us to the setting of [15] (Lecture 10): first, choose a local basis $X_0, \ldots, X_n$ of $\mathcal{K}^0_{\mathfrak{g}}$ over $\mathcal{K}^0$ that is orthonormal with respect to tr. Then $\{X_i t_j^k\}$ is basis of $\mathcal{K}_{\mathfrak{g}}$ that is compatible with the topology, in the sense that $X_i t_j^k \in F^k \mathcal{K}_{\mathfrak{g}}$ for all i, j, k. We can make a quasisymplectic topological basis from this in a way similar to example 5.2.6 - let $\{E_i\}_{i \neq 0}$ be the result. Then

$$\mathcal{O}''_{\mathfrak{g}} := \sum_{i<0} E_i \mathcal{O}_B = \sum_{i,jk<0}$$

is a complement for $\mathcal{O}_{\mathfrak{g}}$ in $\mathcal{K}_{\mathfrak{g}}$.

Because our problem is $\mathcal{O}_B$-linear in X and D, we can assume without loss of generality that $X = X' t_n^m$ and $D = t_p^{q+1} \frac{\partial}{\partial t_p}$. With these two assumptions and the special choice of quasisymplectic topological basis, (5.6) takes the form

$$\delta_{n,p} \sum_{\substack{\alpha \\ i+q=j}} [X_\alpha, X']' t_n^{m-i} \circ X_\alpha t_n^{-j} + X_\alpha t_n^{-i} \circ [X_\alpha, X']' t_n^{m-j}. \tag{5.7}$$

This expression differs from its normally ordered version in a very simple way: one just 'swaps' the factors of $[X_\alpha, X']' t_n^{m-i} \circ X_\alpha t_n^{-j}$ for those i, j for which $j > 0$ and $m - i \geq 0$. By straightforward computation one can check that this difference is

$$\delta_{n,p} \sum_{\alpha} [X_\alpha, [X_\alpha, X']] t_p^{q+1} \frac{\partial}{\partial t_p} t_n^m = 2\check{h} \nabla_D x.$$

□

The proof of the last theorem, which is an adaptation of the computation found in [15] (Lecture 10), suggests that it is perhaps possible to extend it to all of $\mathfrak{sp}(\mathcal{K}_\mathfrak{g}^{\mathrm{red}})^G$ in the following sense: ad $\tau_{\mathcal{O}'_\mathfrak{g}}(\alpha)$ acts as α, for $\alpha \in \mathfrak{sp}(\mathcal{K}_\mathfrak{g}^{\mathrm{red}})^G$. It is only in the last part that we need information beyond the quasisymplectic structure and the G-action.

Remark 5.2.15. The last theorem in particular shows that ad $\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D)$ does not depend on the choice of $\mathcal{O}'_\mathfrak{g}$, for any section D of $\theta_{\mathcal{K}/\mathcal{O}_B}$. In the proof corollary 5.2.13 we even saw that $\widetilde{\phi}(E_\mathfrak{g}^{-1}(\nabla_D))$ and $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E_\mathfrak{g}^{-1}(\nabla_D))$ are equal modulo $\hbar\mathcal{O}_B$. Hence $\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\theta_{\mathcal{K}/\mathcal{O}_B}})$ does not depend on the choice of $\mathcal{O}'_\mathfrak{g}$, up to $\mathcal{O}_B$. As we will soon see, this $\mathcal{O}_B$ submodule of $\overline{U}\widehat{\mathcal{K}}_\mathfrak{g}[\frac{1}{\hbar+\check{h}}]$ is in fact a subalgebra.

Theorem 5.2.16. *For local sections* D, D' *of* $\theta_{\mathcal{K}/\mathcal{O}_B}$, *we have that*

$$[\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D), \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{D'})] = \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{[D,D']}) + \frac{\hbar}{2\hbar + 2\check{h}} \mathrm{tr}(\nabla_D[\pi_{\mathcal{O}'_\mathfrak{g}}, \nabla_D]).$$

Proof. Using the last theorem, the proof becomes almost a copy of that of theorem 5.1.14. Choose a local topological quasisymplectic basis $\{E_i\}_{i \neq 0}$ of $\mathcal{K}_\mathfrak{g}^{\mathrm{red}}$ such that E_i is in $\mathfrak{m}_\mathfrak{g}$ for $i > 0$ and in $\mathcal{O}'_\mathfrak{g}$ for $i < 0$. Then

$$\begin{aligned} &2(c + \check{h})[\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D), \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{D'})] \\ &\quad = \sum_i \frac{-1}{i} \,\mathrm{ad}\, \tau_{O'_\mathfrak{g}}(\nabla_D)\ \widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(\nabla_{D'}(E_i) \otimes E_{-i}) \\ &\quad = \sum_i \frac{-1}{i} (\nabla_D \nabla_{D'} E_i \circ E_{-i} + \nabla_{D'} E_i \circ \nabla_D E_{-i}). \end{aligned}$$

This expression is G-invariant, as can be seen from the fact that it is proportional to $[\phi E_\mathfrak{g}^{-1}(\nabla_D), \phi E_\mathfrak{g}^{-1}(\nabla_{D'})]$.

Let $n_{\mathcal{O}'_\mathfrak{g},\mathcal{O}_\mathfrak{g}} : \mathcal{K}_\mathfrak{g}^{\otimes 2} \to \mathcal{K}_\mathfrak{g}^{\otimes 2}$ be the normal ordering map associated to the decomposition $\mathcal{K}_\mathfrak{g} = \mathcal{O}'_\mathfrak{g} \oplus \mathcal{O}_\mathfrak{g}$; we denote the composition with the map to $U\widehat{K}_\mathfrak{g}$ by $\phi_{\mathcal{O}'_\mathfrak{g},\mathcal{O}_\mathfrak{g}}$. If x, y are sections of $\mathcal{K}_\mathfrak{g}$, then

$$\phi_{\mathcal{O}'_\mathfrak{g},\mathcal{O}_\mathfrak{g}}(x \otimes y) - x \circ y = -[\pi_{\mathcal{O}_\mathfrak{g}} x, \pi_{\mathcal{O}'_\mathfrak{g}} y] = -[\pi_{\mathcal{O}_\mathfrak{g}} x, \pi_{\mathcal{O}'_\mathfrak{g}} y]' - (x, \pi_{\mathcal{O}'_\mathfrak{g}} y)_\mathfrak{g} \hbar,$$

where $\pi_{\mathcal{O}_\mathfrak{g}'}, \pi_{\mathcal{O}_\mathfrak{g}}$ denote the projection of $\mathcal{K}_\mathfrak{g} = \mathcal{O}_\mathfrak{g}' \oplus \mathcal{O}_\mathfrak{g}$ on the first and second summand, respectively. Since both summands are preserved by G, these projections are G equivariant. Hence, if $x \otimes y$ is G-invariant, then so is $\pi_{\mathcal{O}_\mathfrak{g}} x \otimes \pi_{\mathcal{O}_\mathfrak{g}'} y$. In that case, $[\pi_{O_\mathfrak{g}} x, \pi_{O_\mathfrak{g}'} y]' = 0$ and

$$x \circ y = \phi_{\mathcal{O}_\mathfrak{g}', \mathcal{O}_\mathfrak{g}}(x \otimes y) + (x, \pi_{\mathcal{O}_\mathfrak{g}'} y)_\mathfrak{g} \hbar.$$

Using this observation, we conclude that $2(\hbar + \check{h})[\tau_{\mathcal{O}_\mathfrak{g}'}(\nabla_D), \tau_{\mathcal{O}_\mathfrak{g}'}(\nabla_{D'})]$ equals

$$\begin{aligned} &\phi_{\mathcal{O}_\mathfrak{g}', \mathcal{O}_\mathfrak{g}} \sum_i \frac{\nabla_D \nabla_{D'} E_i \otimes E_{-i} + \nabla_{D'} E_i \otimes \nabla_D E_{-i}}{-i} \\ &+\hbar \sum_i \frac{(\nabla_D \nabla_{D'} E_i, \pi_{\mathcal{O}_\mathfrak{g}'} E_{-i})_\mathfrak{g} + (\nabla_{D'} E_i, \pi_{\mathcal{O}_\mathfrak{g}'} \nabla_D E_{-i})_\mathfrak{g}}{-i}. \end{aligned} \tag{5.8}$$

By noting that

$$\begin{aligned} \sum_i \frac{1}{i} \nabla_{D'} E_i \otimes \nabla_D E_{-i} = \sum_{i,j} \frac{(\nabla_D E_{-i}, E_{-j})_\mathfrak{g}}{ij} \nabla_{D'} E_i \otimes E_j &= \\ \sum_{i,j} \frac{1}{ij} \nabla_{D'}((\nabla_D E_{-j}, E_{-i})_\mathfrak{g} E_i) \otimes E_j &= \\ \sum_j \frac{1}{j} \nabla_{D'} \nabla_D E_{-j} \otimes E_j = -\sum_i \frac{1}{i} \nabla_{D'} \nabla_D E_i \otimes E_{-i}, \end{aligned}$$

we see that the first part of (5.8) equals

$$\phi_{\mathcal{O}_\mathfrak{g}', \mathcal{O}_\mathfrak{g}} \sum_i \frac{-1}{i} (\nabla_D \nabla_{D'} - \nabla_{D'} \nabla_D) E_i \otimes E_{-i} = 2(\hbar + \check{h}) \tau_{\mathcal{O}_\mathfrak{g}'}([\nabla_D, \nabla_{D'}]).$$

For the second part of 5.8 we have that

$$\begin{aligned}
\sum_i \frac{\hbar}{-i}\Big((\nabla_D\nabla_{D'}E_i, \pi_{\mathcal{O}'_\mathfrak{g}}E_{-i}) + (\nabla_{D'}E_i, \pi_{\mathcal{O}'_\mathfrak{g}}\nabla_D E_{-i})\Big) &= \\
\sum_i \frac{\hbar}{-i}\Big((E_i, \nabla_{D'}\nabla_D\pi_{\mathcal{O}'_\mathfrak{g}}E_{-i}) - (E_i, \nabla_{D'}\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_D E_{-i})\Big) &= \\
\sum_i \frac{\hbar}{-i}(E_i, (\nabla_{D'}\nabla_D\pi_{\mathcal{O}'_\mathfrak{g}} - \nabla_{D'}\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_D)E_{-i}) &= \\
\sum_i \frac{\hbar}{i}((\nabla_{D'}\nabla_D\pi_{\mathcal{O}'_\mathfrak{g}} - \nabla_{D'}\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_D)E_i, E_{-i}) &= \\
\hbar\,\mathrm{tr}(\nabla_{D'}\nabla_D\pi_{\mathcal{O}'_\mathfrak{g}} - \nabla_{D'}\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_D) = \hbar\,\mathrm{tr}(\nabla_D\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_{D'} - \nabla_D\pi_{\mathcal{O}'_\mathfrak{g}}\nabla_{D'}) &= \\
\hbar\,\mathrm{tr}(\nabla_D[\pi_{\mathcal{O}'_\mathfrak{g}}, \nabla_{D'}]). \quad \square &
\end{aligned}$$

This theorem allows us to define the following.

Definition 5.2.17. Define

$$\widehat{\theta}_{\mathcal{K}/\mathcal{O}_B} := \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\theta_{\mathcal{K}/\mathcal{O}_B}}) + \mathcal{O}_B \subseteq \overline{U}\widehat{\mathcal{K}}_\mathfrak{g}[\frac{1}{\hbar + \check{h}}].$$

By the previous theorem, it is a $\mathcal{O}_B$ subalgebra; we let it inherit the product and call it the *Virasoro algebra*. The central element $1 \in \overline{U}\widehat{\mathcal{K}}_\mathfrak{g}[\frac{1}{\hbar+\check{h}}]$ is denoted $c_\mathfrak{g}$

From this definition and the previous theorem it is clear that $\mathcal{O}_B \subseteq \widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}$ by means of $1 \mapsto c_\mathfrak{g}$, and moreover that $\tau_{\mathcal{O}'_\mathfrak{g}} : \theta_{\mathcal{K}/\mathcal{O}_B} \to \widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}/c_\mathfrak{g}\mathcal{O}_B$ is an isomorphism of Lie algebras. Note that the latter map does not depend on the choice of $\mathcal{O}'_\mathfrak{g}$, so that we have a natural isomorphism of $\mathcal{O}_B$-algebras $\widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}/\mathcal{O}_B \to \theta_{\mathcal{K}/\mathcal{O}_B}$. We conclude that $\widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}$ is a central extension of $\theta_{\mathcal{K}/\mathcal{O}_B}$ by $\mathcal{O}_B$:

$$0 \longrightarrow \mathcal{O}_B + 1 \mapsto c_\mathfrak{g} \xrightarrow{+} \widehat{\theta}_{\mathcal{K}/\mathcal{O}_B} \longrightarrow \theta_{\mathcal{K}/\mathcal{O}_B} \longrightarrow 0\,.$$

As the notation suggests, $\widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}$ is related to $\widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B}$, and in fact, they are isomorphic. In order to prove this, we first relate the cocyles occurring in the theorem above and theorem 5.1.14.

Lemma 5.2.18. *Let $\mathcal{O}'$ be an isotropic $\mathcal{K}^0$ submodule of $\mathcal{K}$ that is a complement for $\mathcal{O}$, and take $\mathcal{O}'_{\mathfrak{g}} = \mathcal{K}^0_{\mathfrak{g}} \otimes_{\mathcal{K}^0} \mathcal{O}'$ - it is clearly* G*-invariant. For any local sections* D, D' *of* $\theta_{\mathcal{K}/\mathcal{O}_B}$ *we have that*

$$\dim(\mathfrak{g})\,\mathrm{tr}(D[\pi_{\mathcal{O}'}, D']) = \mathrm{tr}(\nabla_D[\pi_{\mathcal{O}'_{\mathfrak{g}}}, \nabla_{D'}]).$$

Proof. Locally over B, identify $\mathcal{K}^0_{\mathfrak{g}}$ with $\mathcal{K}^0 \otimes \mathfrak{g}$ and choose a local topological quasisymplectic basis $\{e_i\}_{i\neq 0}$ of $\mathcal{K}^{\mathrm{red}}$ such that e_i is in $\mathfrak{m}$ for $i > 0$ and in $\mathcal{O}'$ if $i < 0$. Let $X_1, \ldots, X_{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, orthogonal with respect to tr, so $(X_a e_i, X_b e_j)_{\mathfrak{g}} = i\delta_{ab}\delta_{i,-j}$. Then $\{X_a, e_i\}$ is a local topological basis of $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$, that diagonalizes $(\cdot,\cdot)_{\mathfrak{g}}$. We compute:

$$\mathrm{tr}(D[\pi_{\mathcal{O}'}, D']) = \sum_i \frac{1}{i}(D[\pi_{\mathcal{O}'}, D']e_i, e_{-i})$$

and

$$\begin{aligned}
\mathrm{tr}(\nabla_D[\pi_{\mathcal{O}'_{\mathfrak{g}}}, \nabla_{D'}]) &= \sum_i \sum_{a=1}^{\dim \mathfrak{g}} \frac{1}{i}(\nabla_D[\pi_{\mathcal{O}'_{\mathfrak{g}}}, \nabla_{D'}]X_a e_i, X_a e_{-i})_{\mathfrak{g}} \\
&= \sum_i \sum_{a=1}^{\dim \mathfrak{g}} \frac{1}{i}\,\mathrm{tr}(X_a, X_a)(D[\pi_{\mathcal{O}'}, D']e_i, e_{-i}) \\
&= \dim(\mathfrak{g})\,\mathrm{tr}(D[\pi_{\mathcal{O}'}, D']). \quad \square
\end{aligned}$$

Now let $\mathcal{O}'$, $\mathcal{O}'_{\mathfrak{g}}$ be as in the previous lemma. Then we define a map $\widehat{\theta}^{\mathcal{K}}_{\mathcal{K}/\mathcal{O}_B} \to \widehat{\theta}_{\mathcal{K}/\mathcal{O}_B}$ by sending $\tau_{\mathcal{O}'}(D)$ to $\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_D)$ and c to $\frac{\hbar}{\dim \mathfrak{g}(\hbar+\check{h})}c_{\mathfrak{g}}$. By the previous lemma and theorem 5.2.16, this is an isomorphism of algebras. One can check that this isomorphism does not depend on the choice of $\mathcal{O}'$.

Definition 5.2.19 (Segal-Sugawara representation)**.** We let the Virasoro algebra act on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$ by left multiplication, and denote the action by $T_{\mathfrak{g}}$.

A direct consequence of the definition above, we have the following result.

Corollary 5.2.20. *The representation* $T_{\mathfrak{g}}$ *of factors to a projective representation of* $\theta_{\mathcal{K}/\mathcal{O}_B}$ *on* $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$.

We give an explicit formula for the composition of $T_{\mathfrak{g}} \circ \tau_{\mathcal{O}'_{\mathfrak{g}}} \circ \nabla$, with $\mathcal{O}'_{\mathfrak{g}}$ as before. It suffices to describe the action on a local section of $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$ of the form $\vec{x} \circ v_\ell$, where $\vec{x}$ is a section of $U\widehat{\mathcal{K}}_\mathfrak{g}$. One has that for a local section D of $\theta_{\mathcal{K}/\mathcal{O}_B}$:

$$\begin{aligned} T_\mathfrak{g}(\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D)) \circ \vec{x} \circ v_\ell &= \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D) \circ \vec{x} \circ v_\ell \\ &= [\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D), \vec{x}] \circ v_\ell + \vec{x} \circ \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D) \circ v_\ell \\ &= \nabla_D(\vec{x}) \circ v_\ell + \vec{x} \circ \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D) \circ v_\ell. \end{aligned}$$

This formula also gives the projective action of D on $\vec{x} \circ v_\ell$, provided we interpret the expression modulo $\mathcal{O}_B \vec{x} \circ v_\ell$.

Given a choice $\mathcal{O}'_\mathfrak{g}$ of G-invariant isotropic complement of $\mathcal{O}_\mathfrak{g}$ in $\mathcal{K}_\mathfrak{g}$, *we define* $T_{\mathcal{O}'_\mathfrak{g}}$ *to be the map* $\theta_{\mathcal{K}/\mathcal{O}_B} \to \mathrm{End}_{\mathcal{O}_B}\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ *given by left multiplication with* $\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_D)$. As we have seen, $T_{\mathcal{O}'_\mathfrak{g}}$ defines a representation of $\theta_{\mathcal{K}/\mathcal{O}_B}$ as $\mathcal{O}_B$-module, but not as Lie algebra. Its projectivization however, does define a projective Lie algebra representation and moreover does not depend on $\mathcal{O}'_\mathfrak{g}$; *we denote it by*

$$\mathbb{P}T_\mathfrak{g} : \theta_{\mathcal{K}/\mathcal{O}_B} \to \mathrm{End}_{\mathcal{O}_B}\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell / \mathcal{O}_B.$$

5.2.4 The horizontal extension

In this subsection we assume a choice of G-invariant isotropic complement $\mathcal{O}'_\mathfrak{g}$ for $\mathcal{O}_\mathfrak{g}$ in $\mathcal{K}_\mathfrak{g}$ given. The projective action of $\theta_{\mathcal{K}/\mathcal{O}_B}$ on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ can be extended to derivations of $\mathcal{K}$ that are not $\mathcal{O}_B$-linear, provided they preserve $\mathcal{O}_B$ - recall that we denoted these, the derivations of $\mathcal{K}$ over $\mathbb{C}$ that preserve $\mathcal{O}_B$, by $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$. In order to extend $\mathbb{P}T_\mathfrak{g}$ to $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$, we have to make an additional assumption:

assume that ∇ *extends to the "horizontal" (i.e. the* B*-) directions, so that* ∇ *is also defined on* $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$.

Locally over B, choose coordinates $t_1, \dots, t_n$ for $\mathcal{O}$, so that we can identify $\mathcal{K}$ with $\bigoplus_i \mathcal{O}_B((t_i))$. This identifies $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$ with $\theta_{\mathcal{K}/\mathcal{O}_B} \oplus \theta_{\mathcal{O}_B/\mathbb{C}}$, and correspondingly identifies a local section D of $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$ with a local section $D_v \oplus D_h$ of $\theta_{\mathcal{K}/\mathcal{O}_B} \oplus \theta_{\mathcal{O}_B/\mathbb{C}}$. Here D_h is characterized by the property that $D_h(t_i) = 0$ for all i. We now extend $T_{\mathcal{O}'_\mathfrak{g}}$ to $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$ by letting $T_{\mathcal{O}'_\mathfrak{g}}(D)$ act as follows $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$: for a section $\vec{x} \circ v_\ell$ of $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ we define

$$\begin{aligned} T_{\mathcal{O}'_\mathfrak{g}}(D)\vec{x} \circ v_\ell &:= T_{\mathcal{O}'_\mathfrak{g}}(D_v)\vec{x} \circ v_\ell + \nabla_{D_h}(\vec{x}) \circ v_\ell \\ &= \nabla_D(\vec{x}) \circ v_\ell + \vec{x} \circ T_{\mathcal{O}'_\mathfrak{g}}(D_v) v_\ell. \end{aligned} \tag{5.9}$$

A priori, this definition depends on the choice of coordinates $t_1, \dots, t_N$, but appearances can be deceiving:

Lemma 5.2.21. *The expression defined in* (5.9) *does not depend on the choice of the local parameters* $t_1, \dots, t_N$.

Proof. Let $s_1, \dots, s_N$ be another set of local coordinates of $\mathcal{O}$. This also gives us an identification of $\theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$ with $\theta_{\mathcal{K}/\mathcal{O}_B} \oplus \theta_{\mathcal{O}_B/\mathbb{C}}$, with respect to which D decomposes as $\widetilde{D}_v + \widetilde{D}_v$. So $\widetilde{D}_v + \widetilde{D}_v = D_v + D_v = D$, $\widetilde{D}_v, D_v \in \theta_{\mathcal{K}/\mathcal{O}_B}$ and $D_h(t_i) = \widetilde{D}_h(s_i) = 0$ for all i.

Since $\mathfrak{m} = (t_1, \dots, t_N) = (s_1, \dots, s_N)$ as ideals in $\mathcal{O}$, $\mathfrak{m}/\mathfrak{m}^2$ is an $\mathcal{O}_B$-module that is freely generated by the images of $t_1, \dots, t_N$, but also by the images of $s_1, \dots, s_N$. Hence $t_i = a_i^j s_j \mod \mathfrak{m}^2$ for an invertible $\mathcal{O}_B$-valued matrix $[a_i^j]$. If we write $D_0 = \widetilde{D}_h - D_h = D_v - \widetilde{D}_v$, then clearly $D_0 \in \theta_{\mathcal{K}/\mathcal{O}_B}$. However,

$$D_0(t_i) \in \widetilde{D}_h(a_i^j s_j + \mathfrak{m}^2) \subseteq \widetilde{D}_h(a_i^j) s_j + \widetilde{D}_h(\mathfrak{m})\mathfrak{m} \subseteq \mathfrak{m},$$

so $D_0(\mathfrak{m}) \subseteq \mathfrak{m}$. As a consequence, $\nabla_{D_0}\mathfrak{m}_\mathfrak{g} \subseteq \mathfrak{m}_\mathfrak{g}$. Since $\mathfrak{m}_\mathfrak{g}$ is maximal isotropic as submodule of $\mathcal{K}_\mathfrak{g}^{\text{red}}$, $\mathfrak{m}_\mathfrak{g}^\perp = \mathfrak{m}_\mathfrak{g}$ and we can apply lemma 5.2.7: $E_\mathfrak{g}^{-1}(\nabla_{D_0})$ lies in $\overline{\mathfrak{m}_\mathfrak{g} \oplus \mathcal{K}_\mathfrak{g}^{\text{red}} + \mathcal{K}_\mathfrak{g}^{\text{red}} \oplus \mathfrak{m}_\mathfrak{g}}$. It follows that $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}} E_\mathfrak{g}^{-1}(\nabla_{D_0})$ is in $\mathcal{K}_\mathfrak{g} \circ \mathfrak{m}_\mathfrak{g}$ and hence $\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{D_0}) \circ v_\ell = 0$ (for every choice of $\mathcal{O}'_\mathfrak{g}$). □

We observe that $T_{\mathcal{O}'_\mathfrak{g}} : \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}} \to \mathrm{End}_\mathbb{C}\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ is independent of $\mathcal{O}'_\mathfrak{g}$ up to scalar. *We denote the projectivization by* $\mathbb{P}T_\mathfrak{g}$*:*

$$\mathbb{P}T_\mathfrak{g} : \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}} \to \mathrm{End}_\mathbb{C}\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell/\mathcal{O}_B.$$

Since this extends the previous definition of $\mathbb{P}T_\mathfrak{g}$*, no confusion can arise.*

5.3 Co-invariants

The data we have used up till now, did not use the geometric structure that was the input for Chapter 3 (for that was encoded in $\mathcal{A}$ and $\mathcal{A}_\mathfrak{g}$). In this section we will incorporate this data.

Definition 5.3.1. We call

$$\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}} := \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})/\mathcal{A}_\mathfrak{g} \circ \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$$

the module of co-invariants.

Below, we will define an action of $\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}} = \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}} \cap \theta_{\mathcal{A}/\mathbb{C}}$ on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell,\mathcal{A}_\mathfrak{g}}$, and start by first defining this locally. This allows us to choose $\mathcal{A}_\mathfrak{g}^\pm$, $\mathcal{F}_\mathfrak{g}^\pm$ as in proposition 3.2.13; by lemma 3.2.16 we can assume them to be G-invariant. With $\mathcal{O}'_\mathfrak{g} := \mathcal{A}_\mathfrak{g}^- + \mathcal{F}_\mathfrak{g}^-$ as G-invariant complement for $\mathcal{O}_\mathfrak{g}$, consider $T_{\mathcal{O}'_\mathfrak{g}} : \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}} \to \mathrm{End}_\mathbb{C}\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$. We investigate the restriction of $T_{\mathcal{O}'_\mathfrak{g}}$ to $\theta_{\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}}$, and after that to $\theta_{\mathcal{A}/\mathcal{O}_B}$

Lemma 5.3.2. *If* D *is a local section of* $\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}$*, then* $T_{\mathcal{O}'_\mathfrak{g}}(D)$ *preserves* $\mathcal{A}_\mathfrak{g} \circ \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$*, and hence is well defined on* $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell,\mathcal{A}_\mathfrak{g}}$.

Proof. By equation 5.9,

$$T_{\mathcal{O}'_\mathfrak{g}}(D)\mathcal{A}_\mathfrak{g} \circ \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell = \nabla_D(\mathcal{A}_\mathfrak{g}) \circ \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell + \mathcal{A}_\mathfrak{g} \circ T_{\mathcal{O}'_\mathfrak{g}}(D)\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell.$$

By lemma 3.3.3, $\nabla_D(\mathcal{A}_\mathfrak{g}) \subseteq \mathcal{A}_\mathfrak{g}$, so the result follows. □

Lemma 5.3.3. *For any local section* D *of* $\theta_{\mathcal{A}/\mathcal{O}_B}$ *and* $\mathcal{O}'_\mathfrak{g}$ *as above,* $T_{\mathcal{O}'_\mathfrak{g}}(D)$ *maps* $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ *to* $\mathcal{A} \circ \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$

Proof. Choose a local topological quasisymplectic basis $\{E_i\}_{i\neq 0}$ such that for a certain γ, $\{E_i\}_{i>\gamma}$ is a basis for $\mathcal{A}_\mathfrak{g}^+$, $\{E_i\}_{i<\gamma}$ for $\mathcal{A}_\mathfrak{g}^-$, $\{E_i\}_{i=1,\dots,\gamma}$ for $\mathcal{F}_\mathfrak{g}^+$ and $\{E_i\}_{i=-\gamma,\dots,-1}$ is a basis for $\mathcal{F}_\mathfrak{g}^-$. We have that $E_\mathfrak{g}^{-1}(\nabla_D) = \sum_i \frac{-1}{i}\nabla_D E_i \otimes E_{-i}$.

By lemma 3.3.3, $\nabla_{\theta_{\mathcal{A}/\mathcal{O}_B}}$ sends $\mathcal{A}_\mathfrak{g}^\perp$ to $\mathcal{A}_\mathfrak{g}$, so $\sum_{i\leq\gamma} \frac{-1}{i}\nabla_D E_i \otimes E_{-i}$ lies in $\overline{\mathcal{A}_\mathfrak{g}^- \otimes \mathcal{A}_\mathfrak{g}^+}$. Hence $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}} \sum_{i\leq\gamma} \frac{-1}{i}\nabla_D E_i \otimes E_{-i}$ lies in $\mathcal{A}_\mathfrak{g} \circ \mathcal{K}_\mathfrak{g}$. In the following, we identify $\mathcal{A}_\mathfrak{g}^\pm, \mathcal{F}_\mathfrak{g}^\pm$ with their images in $\mathcal{K}_\mathfrak{g}^{\mathrm{red}}$.

The remaining part of $E_\mathfrak{g}^{-1}(\nabla_D)$, i.e. $\sum_{i>\gamma} \frac{-1}{i}\nabla_D E_i \circ E_{-i}$, can by characterized as the $\mathcal{K}_\mathfrak{g}^{\mathrm{red}} \otimes \mathcal{A}_\mathfrak{g}^-$ part of $E_\mathfrak{g}^{-1}(\nabla_D)$. We decompose this in its $\mathcal{A}_\mathfrak{g}^- \otimes \mathcal{A}_\mathfrak{g}^-$, $\mathcal{F}_\mathfrak{g}^- \otimes \mathcal{A}_\mathfrak{g}^-$, $\mathcal{F}_\mathfrak{g}^+ \otimes \mathcal{A}^-$ and $\mathcal{A}_\mathfrak{g}^+ \otimes \mathcal{A}_\mathfrak{g}^-$ parts and denote these by $(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{A}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$, $(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{F}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$, $(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{F}_\mathfrak{g}^+,\mathcal{A}_\mathfrak{g}^-}$ and $(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{A}_\mathfrak{g}^+,\mathcal{A}_\mathfrak{g}^-}$, respectively. Clearly, $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{A}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$ lies in $\overline{\mathcal{A}_\mathfrak{g} \circ \mathcal{K}_\mathfrak{g}}$. Furthermore, since $\mathcal{F}_\mathfrak{g}^+ \subseteq \mathfrak{m}_\mathfrak{g}$ and $\mathcal{A}_\mathfrak{g}^- \subseteq \mathcal{O}'_\mathfrak{g}$, we also have that $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E \in_\mathfrak{g} (\nabla_D))^{\mathcal{F}_\mathfrak{g}^+,\mathcal{A}_\mathfrak{g}^-}$ lies in $\overline{\mathcal{A}_\mathfrak{g} \circ \mathcal{K}_\mathfrak{g}}$; the same goes for $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E \in_\mathfrak{g} (\nabla_D))^{\mathcal{A}_\mathfrak{g}^+,\mathcal{A}_\mathfrak{g}^-}$.

It remains to show that $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{F}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$ takes its values in $\overline{\mathcal{A}_\mathfrak{g} \circ \mathcal{K}_\mathfrak{g}}$. The normal ordering acts as the identity on $(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{F}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$ because $\mathcal{F}_\mathfrak{g}^-$ and $\mathcal{A}_\mathfrak{g}^-$ are both contained in $\mathcal{O}'_\mathfrak{g}$. Hence, $\widetilde{\phi}_{\mathcal{O}'_\mathfrak{g},\mathfrak{m}_\mathfrak{g}}(E_\mathfrak{g}^{-1}(\nabla_D))^{\mathcal{F}_\mathfrak{g}^-,\mathcal{A}_\mathfrak{g}^-}$ is of the form $\sum_i f_i \circ a_i$ for certain sections f_i of $\mathcal{F}_\mathfrak{g}^-$ and a_i of $\mathcal{A}_\mathfrak{g}^-$. It is also G-invariant: since ∇_D is G equivariant, $E_\mathfrak{g}^{-1}(\nabla_D)$ is G-invariant, and since moreover the summands $\mathcal{A}^{\pm}, \mathcal{F}_{\mathfrak{g}}^{\pm}$ are preserved by G, the $\mathcal{F}_{\mathfrak{g}}^{-} \otimes \mathcal{A}_{\mathfrak{g}}^{-}$ part of $E_{\mathfrak{g}}^{-1}(\nabla_D)$ is also G-invariant. It follows that

$$\sum_i f_i \circ a_i = \sum_i a_i \circ f_i + \sum_i [f_i, a_i]' + (f_i, a_i)\hbar.$$

By G-invariance, $\sum_i [f_i, a_i]'$ vanishes, and because $(\mathcal{F}_{\mathfrak{g}}^{-}, \mathcal{A}_{\mathfrak{g}}^{-}) = 0$, it follows that $\sum_i (f_i, a_i)$ also vanishes. Hence $\widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}}, \mathfrak{m}_{\mathfrak{g}}}(E_{\mathfrak{g}}^{-1}(\nabla_D))^{\mathcal{F}_{\mathfrak{g}}^{-}, \mathcal{A}_{\mathfrak{g}}^{-}} = \sum_i a_i \circ f_i$ also lies in $\overline{\mathcal{A}_{\mathfrak{g}} \circ \mathcal{K}_{\mathfrak{g}}}$. □

From the previous two lemmas it follows that $T_{\mathcal{O}'_{\mathfrak{g}}}$ descends to a well defined action of $\theta_{\mathcal{A}, \mathcal{O}_B/\mathbb{C}} / \theta_{\mathcal{A}/\mathcal{O}_B}$ on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}$. Since $\theta_{\mathcal{A}, \mathcal{O}_B/\mathbb{C}} / \theta_{\mathcal{A}/\mathcal{O}_B} = \theta_B$, so that we have defined a map

$$T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}} : \theta_B \to \mathrm{End}_{\mathbb{C}} \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}.$$

We now show that this map is a connection: let f be a local section of $\mathcal{O}_B$, $\vec{x} \circ v_\ell$ of $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}$ and $\overline{D}$ of θ_B. Lift $\overline{D}$ locally to $D \in \theta_{\mathcal{A}, \mathcal{O}_B/\mathbb{C}}$. Then

$$\begin{aligned} T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}(\overline{D}) f\vec{x} \circ v_\ell &= \nabla_D(f\vec{x}) \circ v_\ell + f\vec{x} \circ \tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_D) v_\ell \\ &= D(f)\vec{x} \circ v_\ell + f\nabla_D(\vec{x}) \circ v_\ell + \tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_D) v_\ell \\ &= \overline{D}(f)\vec{x} \circ v_\ell + f T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}(\overline{D})\vec{x} \circ v_\ell, \end{aligned}$$

so that $T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}$ lifts $\overline{D}$ to a first order differential operator on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}$ whose symbol is the identity.

Corollary 5.3.4. *For any choice of $\mathcal{O}'_{\mathfrak{g}}$ as above, $T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}$ is a holomorphic connection on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}$ over* B.

In order to define $T_{\mathcal{O}'_{\mathfrak{g}}}$, we had to choose $\mathcal{A}_{\mathfrak{g}}^{\pm}, \mathcal{F}_{\mathfrak{g}}^{\pm}$ as above, and to do that, we had to restrict the base space. However, as we saw before, $T_{\mathcal{O}'_{\mathfrak{g}}}$ depends on $T_{\mathcal{O}'_{\mathfrak{g}}}$ only up to scalar, so that the projectivization of $T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}$ is independent of $\mathcal{O}'_{\mathfrak{g}}$. We give this map a name:

Definition 5.3.5 (WZW projective holomorphic connection)**.** Define ∇^{WZW} as the map

$$\theta_B \to \mathrm{End}_{\mathbb{C}} \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}} / \mathcal{O}_B$$

that is locally defined by $T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}$, with $\mathcal{O}'_{\mathfrak{g}}$ equal to $\mathcal{F}_{\mathfrak{g}}^{-} + \mathcal{A}_{\mathfrak{g}}^{-}$ as above.

Clearly, ∇^{WZW} is a *projective* holomorphic connection on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell; \mathcal{A}_{\mathfrak{g}}}$. We emphasize that this is now defined *globally over* B.

5.3.1 Conformal blocks

The $U\widehat{\mathcal{K}}_{\mathfrak{g}}$-module $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$ has a unique maximal submodule, which we denote by $\mathcal{Z}_\ell$. We already saw this in Chapter 1: there we denoted $\widehat{\mathcal{K}}_{\mathfrak{g}}$ by $\widehat{\mathcal{L}\mathfrak{g}}$, $\widehat{\mathcal{O}}_{\mathfrak{g}}$ by $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ and $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$ by $\mathcal{H}(\widehat{\mathcal{L}\mathfrak{g}}, \widehat{\mathcal{L}^{\geq}\mathfrak{g}})_\ell$. As was noted there, it is locally of the following form: locally over B, choose coordinates $t_1, \dots, t_N$ of $\mathcal{O}$ and identify $\mathcal{K}_{\mathfrak{g}}$ with $\mathfrak{g} \otimes \bigoplus_i ((t_i))$. Then $\mathcal{Z}_\ell$ is of the form $\bigoplus_i U\widehat{\mathcal{K}}_{\mathfrak{g}} \circ (X_\theta t_i^{-1})^{\ell+1} \circ v_\ell$, where X_θ is a highest root vector of $\mathfrak{g}$ (relative some choice of positive roots).

Definition 5.3.6. We let $\mathcal{Z}_{\ell;\mathcal{A}_{\mathfrak{g}}}$ be the image of $\mathcal{Z}_\ell$ in $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}$. We call

$$\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell^* := \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}} / \mathcal{Z}_{\ell;\mathcal{A}_{\mathfrak{g}}}$$

the module of *conformal coblocks* and

$$\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell := \mathrm{Hom}_{\mathcal{O}_B}(\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}} / \mathcal{Z}_{\ell;\mathcal{A}_{\mathfrak{g}}}, \mathcal{O}_B)$$

the module of *conformal blocks*.

In the literature, this module of conformal blocks is sometimes also called the module of vacua.

Below, we will see that $\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell$ is locally free so that $\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell$ and $\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell^*$ indeed are each others $\mathcal{O}_B$-dual.

Lemma 5.3.7. *The submodule $\mathcal{Z}_\ell$ of $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$ is preserved by ∇^{WZW}, so that ∇^{WZW} descends to a projective connection on $\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell / (\mathcal{A}_{\mathfrak{g}} \circ \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell + \mathcal{Z}_\ell)$, and hence defines connections on $\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell^*$ and $\mathcal{B}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{A}_{\mathfrak{g}})_\ell$ - we denote both by ∇^{WZW}.*

Proof. It suffices to show this locally, so let ∇^{WZW} be represented by $T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}$, where $\mathcal{O}'_{\mathfrak{g}}$ is as above. Furthermore, let $t_1, \dots, t_N$ and X_θ be as in the beginning of this subsection so that $\mathcal{Z}_\ell$ is generated by $(X_\theta t_i^{-1})^{\ell+1} \circ v_\ell$, $i = 1, \dots, N$. For a local section D on θ_B, there is a (unique) lift $\widetilde{D}$ of D to $\theta_{\mathcal{A}, \mathcal{O}_B/\mathbb{C}}$ such that $\widetilde{D} t_i = 0$ for all i (so the "vertical part" of $\widetilde{D}$ is 0). We compute:

$$T_{\mathcal{O}'_{\mathfrak{g}}}^{\mathcal{A}_{\mathfrak{g}}}(D)\mathcal{Z}_\ell = \sum_i \nabla_{\widetilde{D}}(U\widehat{\mathcal{K}}^{\mathrm{red}}) \circ (X_\theta t_i^{-1})^{\ell+1} \circ v_\ell + \sum_i U\widehat{\mathcal{K}}^{\mathrm{red}} \circ \nabla_{\widetilde{D}}(X_\theta t_i^{-1})^{\ell+1} \circ v_\ell.$$

The first term is clearly contained in $\mathcal{Z}_\ell$, so it suffices to prove that $\nabla_{\widetilde{D}}(X_\theta t_i^{-1})$ vanishes for all i. By the Leibniz rule, this expression equals $t_i \nabla_{\widetilde{D}} X_\theta + X_\theta \widetilde{D}(t_i)$. By definition of $\widetilde{D}$, $\widetilde{D}(t_i) = 0$. Now recall that we made a local identification $\mathfrak{g} \otimes \mathcal{K} \simeq \mathcal{K}_{\mathfrak{g}}$. By lemma 3.2.15, we can assume $\mathfrak{g}$ to consist of *flat* sections, so that $\nabla_{\widetilde{D}} X_\theta = 0$ as well. □

One of the nice properties of the quotient with respect to $\mathcal{Z}_\ell$ is the following:

Theorem 5.3.8. *$\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})_\ell$ is a coherent $\mathcal{O}_B$-module.*

For a proof, we refer the reader to [19], Proposition 3.2.

Locally over B, ∇^{WZW} is represented by a holomorphic connection, so the $\mathcal{O}_B$-module $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})^*_\ell$ locally has a holomorphic connection. Since it is coherent, by the previous theorem we conclude that

Corollary 5.3.9. *The $\mathcal{O}_B$-module $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})^*_\ell$ is locally free and of finite rank. The same holds for $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})_\ell$.*

5.4 The length filtration

By construction, $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$, $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ and $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})^*_\ell$ inherit any filtration from $U\widehat{\mathcal{K}}_\mathfrak{g}$. Above, we used the "right filtration" for the latter, but here we will introduce a different one: the *length filtration*, of which let the k-th part be given by

$$FL^k U\widehat{K}_\mathfrak{g} := \mathrm{im}(\sum_{i \le k} \widehat{\mathcal{K}}_\mathfrak{g}^{\otimes i}) \subseteq U\widehat{\mathcal{K}}_\mathfrak{g}.$$

The corresponding filtration on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ is then defined as

$$FL^k \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell := FL^k U\widehat{\mathcal{K}}_\mathfrak{g} \circ v_\ell.$$

This in turn defines a filtration on any quotient of $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$, by letting the k-th piece be the image of $FL^k \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ - we also denote this induced filtration by FL^k, so e.g.

$$FL^k \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}} := \mathrm{im}(FL^k \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell) \subseteq \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$$

is the length filtration on $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$.

Chapter 6

Comparisons

Let G be a fixed simple complex algebraic group with Lie algebra $\mathfrak{g}$, and tr a positive definite ad-invariant bilinear form on $\mathfrak{g}$. We assume tr to be normalized such that the longest root has length 2 (see remark 1.3.6). Moreover, we let G_c be a compact real form of G and fix two positive integers $\ell, \mathfrak{l}$.

In this chapter we will compare the results of Chapters 2,4 and 5. In each of these chapters we associated to a certain geometric setting, a locally free $\mathcal{O}_B$-module of finite rank with a flat projective holomorphic connection, for a given complex variety B.

6.1 (Graded) conformal blocks

We start by fixing a pointed family of curves $(\mathcal{C}/B, p)$. Recall that we can regard this as a family of complex structures, parameterized by B, on a pointed compact oriented pointed surface (Σ, p). We assume the latter to by equipped with a G_c-bundle P_c/Σ, together with a flat C^∞-connection $\nabla^{P_c/\Sigma}$. One can "multiply" all this by B; the result is a principal G_c-bundle

$$\widetilde{P}_c/\Sigma \times B : \widetilde{P}_c := P_c \times B \to \Sigma \times B$$

over $\Sigma \times B$ that maps to P_c/Σ by contraction in the B directions. We let $\nabla^{\widetilde{P}_c/\Sigma\times B}$ be the connection obtained by pulling $\nabla^{P_c/\Sigma}$ back to $\widetilde{P}_c/\Sigma \times B$; it is clearly a flat connection over $\Sigma \times B$. Let $P := \widetilde{P}_c \times_{G_c} G$ be the complexification of $\widetilde{P}_c$, and $P/\Sigma \times B$ the corresponding principal G-bundle; the corresponding complexification of $\nabla^{\widetilde{P}_c/\Sigma\times B}$ is denoted $\nabla^{P/\Sigma\times B}$. As we saw before, the complex structure $\mathcal{C}/B$ on $\Sigma \times B/B$ lifts to $P/\Sigma \times B$ via $\nabla^{P/\Sigma\times B}$: a local section s is holomorphic iff $(\nabla^{P/\Sigma\times B})^{0,1}s = 0$. We denote $P/\Sigma \times B$ endowed with this complex structure by $P/\mathcal{C}$. Furthermore, for simplicity we will write $\nabla^{P/\mathcal{C}}$ for be the flat connection $\nabla^{P/\Sigma\times B}$ on $P/\mathcal{C}$, whereas the connection on the adjoint bundle $\mathrm{Ad}_{P/\mathcal{C}}$ inherited from $\nabla^{P/\mathcal{C}}$ is denoted by ∇.

Remark 6.1.1. Note that if P_c/Σ has a *global* flat section with respect to $\nabla^{P_c/\Sigma}$, then $P/\mathcal{C}$ has a global flat section with respect to $\nabla^{P/\mathcal{C}}$. We will later assume this to be the case.

6.1.1 Graded conformal blocks

We now show how this setting, i.e. $(\mathcal{C}/B, p)$, $P/\mathcal{C}$ and ∇, gives rise to the data used as input in Chapter 4. Part of this was already defined in Chapter 3:

$$\mathcal{H}_{\mathfrak{g}} = R^1(\mathcal{C}/B)_*\mathcal{G}, \quad \mathcal{F}_{\mathfrak{g}} = (\mathcal{C}/B)_*(\omega_{\mathcal{C}/B} \otimes \mathcal{G}),$$

where $\mathcal{G}$ denotes the sheaf of flat local sections of $\mathrm{Ad}_{P/\mathcal{C}}$. The sheaf $\mathcal{H}_{\mathfrak{g}}$ came with a symplectic form $(\cdot,\cdot)$ given by the pairing that $(\cdot,\cdot)_{\mathfrak{g}}$ induces via the isomorphism $\nabla : \mathcal{A}_{\mathfrak{g}}^{\perp}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{K}_{\mathfrak{g}}^0) \to \mathcal{H}_{\mathfrak{g}}$. With respect to this symplectic structure, $\mathcal{F}_{\mathfrak{g}}$ is a Lagrangian submodule of $\mathcal{H}_{\mathfrak{g}}$, as we have seen before. However, since $P/\mathcal{C}$ and $\nabla^{P/\mathcal{C}}$ were obtained by trivial extension in the B direction, $\mathcal{H}_{\mathfrak{g}}$ trivializes: we have that

$$\mathcal{H}_{\mathfrak{g}} = H^1(\Sigma; \mathcal{G}_{P_c}) \otimes_{\underline{\mathbb{R}}} \mathcal{O}_B, \tag{6.1}$$

where $\mathcal{G}_{P_c}$ denotes the flat sections of $\mathrm{Ad}_{P_c/\Sigma}$ over Σ. This determines a flat connection $\nabla^{\mathcal{H}_{\mathfrak{g}}}$ on $\mathcal{H}_{\mathfrak{g}}$ that takes the form $\mathrm{id} \otimes d$ under the identification 6.1. We remark that such a connection is called a Gauss-Manin connection. It is compatible with the $\mathcal{O}_B$-module structure, in that $[(\nabla^{\mathcal{H}_{\mathfrak{g}}})^{0,1}, \mathcal{O}_B] = 0$. Furthermore, we have seen that the symplectic structure on $\mathcal{H}$ is given by the tr-integration pairing; it follows the symplectic structure on $\mathcal{H}$ is induced from the tr-integration pairing on $H^1(\Sigma; \mathcal{G}_{P_c})$. From this, in turn, it immediately follows that $(\cdot,\cdot)$ is flat with respect to ∇.

We recall some notation from Chapter 4: the sheaf of (p, q) forms on B with values in $\mathcal{M}$ is denoted $\mathcal{E}^{p,q}(\mathcal{M})$. Furthermore, we let $\mathcal{E}$ denote the sheaf of smooth functions on B, and for any $\mathcal{O}_B$-module $\mathcal{M}$ we denote the associated $\mathcal{E}$-module $\mathcal{E} \otimes_{\mathcal{O}_B} \mathcal{M}$ by $\mathcal{M}_\infty$. As an exception, we write $\mathbb{F}_\infty(\mathcal{H}_{\mathfrak{g}}, \mathcal{F}_{\mathfrak{g}})_l$,

$\mathbb{F}_\infty(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_\ell$ instead of the more cumbersome $(\mathbb{F}(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_l)_\infty$, $(\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_\ell)_\infty$. Using this notation, we have e.g. that $\mathcal{H}_{\mathfrak{g},\infty} = H^1(\Sigma;\mathcal{G}_{P_c})\otimes\mathcal{E}$ is a vector bundle on B with flat connection $\nabla^{\mathcal{H}_\mathfrak{g}}$, and holomorphic subbundle $\mathcal{F}_\mathfrak{g}$. The identification (6.1) shows that it has a natural real structure $a\mapsto\overline{a}$. Since the symplectic structure on $\mathcal{H}_\mathfrak{g}$ (and hence the one induced on $\mathcal{H}_{\mathfrak{g},\infty}$ as well) is induced from $H^1(\Sigma;\mathcal{G}_{P_c})$, we see that it is real; the same goes for ∇. Furthermore, we have that $\overline{\mathcal{F}}_{\mathfrak{g},\infty}$ is a Lagrangian complement for $\overline{\mathcal{F}}_{\mathfrak{g},\infty}$ in $\mathcal{H}_{\mathfrak{g},\infty}$: one just observes that in a fiber over a point $b\in B$, $(\overline{\mathcal{F}}_{\mathfrak{g},\infty})_b = H^{0,1}(\mathcal{C}_b;\mathcal{G})$, $(\overline{\mathcal{F}}_{\mathfrak{g},\infty})_b = H^{1,0}(\mathcal{C}_b;\mathcal{G})$ and $\mathcal{H}_{\mathfrak{g},\infty} = H^1(\mathcal{C}_b;\mathcal{G})$. Using this, one readily checks that

$$(a,b)\mapsto -i(a,\overline{b})$$

is a negative definite hermitian form on $\mathcal{F}_{\mathfrak{g},\infty}$ and a positive definite one on $\overline{\mathcal{F}}_{\mathfrak{g},\infty}$.

This brings us in the setting of Chapter 4: to the tuple $(\mathcal{H}_\mathfrak{g},\nabla,\mathcal{F}_\mathfrak{g},(\cdot,\cdot),\overline{(\cdot)})$ we have associated the Fock module $\mathbb{F}(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_l$ and an inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{F}_\infty(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_l$, such that the canonical connection (compatible with holomorphic and hermitian structure) has scalar curvature. We will now compare this with the results of Chapter 5: in Chapter 3 we associated to $(\mathcal{C}/B,p)$, $P/\mathcal{C}$ and ∇, the $\mathcal{O}_B$-modules $\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathcal{A}_\mathfrak{g}$ and a presymplectic form $(\cdot,\cdot)_\mathfrak{g}$ on $\mathcal{K}_\mathfrak{g}$. These satisfy the requirements needed to define $\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$, $\mathcal{Z}_\ell$ and $\mathcal{B}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathcal{A}_\mathfrak{g})^*_\ell$ in Chapter 5. For the former, we defined a filtration that we denoted $\{FL^k\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}\}_{k\in\mathbb{N}}$ and called the "length" filtration - let

$$\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$$

denote the graded quotient of $\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_\ell$ with respect to this filtration.

Lemma 6.1.2. *We have a natural isomorphism of $\mathcal{O}_B$-modules*

$$\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}\to\mathbb{F}(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_l.$$

Proof. Recall that the filtration of $\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ was induced by the filtration of $U\widehat{\mathcal{K}}_\mathfrak{g}$ that it received from the tensor algebra of $\widehat{\mathcal{K}}_\mathfrak{g}$. From this it is clear that the graded quotient of $U\widehat{\mathcal{K}}_\mathfrak{g}$ is commutative, and hence the image of

$$U\widehat{\mathcal{K}}_\mathfrak{g}\circ(X\circ Y - Y\circ X)\circ U\widehat{\mathcal{K}}_\mathfrak{g}\circ v_\ell \tag{6.2}$$

in $\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ vanishes, for any local sections X,Y of $\widehat{\mathcal{K}}_\mathfrak{g}$. If we also recall that $\mathcal{O}_\ell\circ v_\ell = 0$, $\mathcal{O}_B\hbar v_\ell = \mathcal{O}_B v_\ell$ and that $\mathcal{A}_\mathfrak{g}\circ\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ vanishes in

$\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$, then a PBW-type argument shows that $\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ can be naturally identified with $\mathrm{Sym}_\bullet(\mathcal{K}_\mathfrak{g}/(\mathcal{O}_\mathfrak{g}+\mathcal{A}_\mathfrak{g}))$ as $\mathcal{O}_B$-modules: using the commutativity relation (6.2), one "moves all factors in $\mathcal{A}_\mathfrak{g}$ to the left, and factors in $\widehat{\mathcal{O}}_\mathfrak{g}$ to the right".

The same trick can be applied to $\mathbb{F}(\mathcal{H}_\mathfrak{g}, \mathcal{F}_\mathfrak{g})_\mathfrak{l}$: using that $\widehat{\mathcal{F}_\mathfrak{g}} \circ f_\mathfrak{l} = \mathcal{O}_B f_\mathfrak{l}$ and that $[\widehat{\mathcal{F}_\mathfrak{g}}, \widehat{\mathcal{H}_\mathfrak{g}}] = \hbar\mathcal{O}_B$, we see by "moving all terms in $\widehat{\mathcal{F}_\mathfrak{g}}$ to the right" that

$$\begin{aligned}\mathbb{F}(\mathcal{H}_\mathfrak{g}, \mathcal{F}_\mathfrak{g})_\mathfrak{l} = \mathrm{Sym}_\bullet(\mathcal{H}_\mathfrak{g}/\mathcal{F}_\mathfrak{g}) &= \mathrm{Sym}_\bullet \mathcal{A}_\mathfrak{g}^\perp/((\mathcal{O}_\mathfrak{g} \cap \mathcal{A}_\mathfrak{g}^\perp) + \mathcal{A}_\mathfrak{g}) \\ &= \mathrm{Sym}_\bullet(\mathcal{K}_\mathfrak{g}/(\mathcal{O}_\mathfrak{g} + \mathcal{A}_\mathfrak{g})).\end{aligned}$$

Note that locally over B, we can choose a $\mathcal{F}_\mathfrak{g}^-$ as in proposition 3.2.13. Then the map $\bigotimes \mathcal{F}_\mathfrak{g}^- \to \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$ induces an isomorphism from $\mathrm{Sym}_\bullet \mathcal{F}_\mathfrak{g}^-$ to $\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$. However, $\nabla : \mathcal{F}_\mathfrak{g}^- \to \mathcal{H}_\mathfrak{g}$ gives an isomorphism $\mathrm{Sym}_\bullet \mathcal{F}_\mathfrak{g}^- \to \mathbb{F}(\mathcal{H}_\mathfrak{g}, \mathcal{F}_\mathfrak{g})_\ell$ so that by composition, we obtain an identification of $\mathcal{O}_B$-modules $\mathrm{gr}\,\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}} \simeq \mathbb{F}(\mathcal{H}_\mathfrak{g}, \mathcal{F}_\mathfrak{g})_\ell$. This is the same isomorphism as the one just defined, and therefore does not depend on the choice of $\mathcal{F}_\mathfrak{g}^-$. □

We recall that $\mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell$ has a maximal $U\widehat{\mathcal{K}}_\mathfrak{g}$ submodule $\mathcal{Z}_\ell$, and that $\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}$ denotes its image in $\mathbb{F}(\mathcal{K}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$. The module of graded conformal coblocks should of course be the graded quotient of $\mathbb{F}(\mathcal{K}, \mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}$ with respect to some filtration. Before make this precise, we first take a small detour to discuss some generalities about filtered modules, and show that "taking graded quotient commutes with dividing out by a submodule".

Let V be a module (over some ring or sheaf) with increasing filtration $\{V_k\}$ and submodule W. The latter naturally inherits a filtration that is given by $\{W_k = V_k \cap W\}$. Moreover, the quotient V/W also naturally inherits a filtration, in which the k-th piece is given by the image of V_k in V/W.

Lemma 6.1.3. *There is a natural isomorphism* $\mathrm{gr}\, V/W = \mathrm{gr}\, V/\,\mathrm{gr}\, W$, *where the right-hand side is the quotient in the category of graded modules.*

Proof. We compare the degree k summands: on the left hand side this is

$$(\mathrm{gr}\, V/W)_k = V_k + W/(V_{k-1}+W) = V_k/((V_{k-1}+W)\cap V_k) = V_k/(V_{k-1}+W_k),$$

whereas on the rights hand side this is given by

$$(\mathrm{gr}\, V/\,\mathrm{gr}\, W)_k = (\mathrm{gr}\, V)_k/(\mathrm{gr}\, W)_k = V_k/(V_{k-1} + W_k).$$

□

We now apply this to the submodule $\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}$ of $\mathbb{F}(\mathcal{K},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}$:

$$\operatorname{gr}(\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}) = \operatorname{gr}\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\operatorname{gr}\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}.$$

If we subsequently apply the natural isomorphism from lemma 6.1.2, we can conclude the following.

Corollary 6.1.4. *We have a natural isomorphism*

$$\operatorname{gr} B(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathcal{A}_\mathfrak{g})^*_\ell = \mathbb{F}(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_\iota/\operatorname{gr}\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}.$$

This result legitimates the following definition. Because of this result, *we call* $\mathbb{F}(\mathcal{H}_\mathfrak{g},\mathcal{F}_\mathfrak{g})_\iota/\operatorname{gr}\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}}$ *the sheaf of graded conformal coblocks*. This isomorphism gives a relation between ∇^{WZW} and $\nabla^{\mathbb{F}}$, provided ℓ, ι satisfy a certain condition. However, in order for the first one to be defined, we have to make more assumptions on our principal bundle and its connection: *assume that the assumption of remark* 6.1.1 *holds, so that* $(P/\mathcal{C},\nabla^{P/\mathcal{C}})$ *is (isomorphic to) the trivial* G*-bundle over* $\mathcal{C}/B$ *with corresponding connection.* For this setting, we have defined a projective holomorphic connection ∇^{WZW} on $\mathcal{B}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathcal{A}_\mathfrak{g})^*_\ell$ in Chapter 5. We recall how it acts on $\mathcal{B}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g},\mathcal{A}_\mathfrak{g})^*_\ell$. Of course, it suffices to do this locally, and hence we can choose coordinates $t_1,\dots,t_N$ of $\mathcal{O}$. Since $P/\mathcal{C}$ and $\nabla^{P/\mathcal{C}}$ were assumed trivial, $\mathrm{Ad}_{P/\mathcal{C}}$ has a global basis of flat sections. This ensures that we can identify $\mathcal{K}_\mathfrak{g}$ with $\mathcal{K}\otimes\mathfrak{g}$, where $1\otimes\mathfrak{g}$ consists of flat sections, and that $\mathcal{K}_\mathfrak{g}$ has a G-action that satisfies the conditions of 3.2.14 in Chapter 3. We define $\mathcal{O}'_\mathfrak{g} := \mathfrak{g}\otimes\bigoplus_i \mathcal{O}_B[t_i^{-1}]$, and note that it is a G-invariant isotropic complement for $\mathcal{O}_\mathfrak{g}$ in $\mathcal{K}_\mathfrak{g}$.

Now, let D be a local section of $\theta_B = \theta_{\mathcal{O}_B/\mathbb{C}}$ with a local lift $\widetilde{D}$ to $\theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}} \subseteq \theta_{\mathcal{K},\mathcal{O}_B/\mathbb{C}}$. Write $\widetilde{D} = \widetilde{D}_v + \widetilde{D}_h$ for $\widetilde{D}_h \in \theta_{\mathcal{A},\mathcal{O}_B/\mathbb{C}}$ such that $\widetilde{D}_h(t_i) = 0$ and $\widetilde{D}_v \in \theta_{\mathcal{K}/\mathcal{O}_B}$. Then, for a local section $\vec{x}\circ\nu_\ell$ of $\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\mathcal{Z}_\ell$ we have that $\nabla^{WZW}_{\widetilde{D}}\vec{x}\circ\nu_\ell$ is given by

$$\nabla_{\widetilde{D}_h}(\vec{x})\circ\nu_\ell + \tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\widetilde{D}_v})\circ\vec{x}\circ\nu_\ell = \nabla_{\widetilde{D}}(\vec{x})\circ\nu_\ell + \vec{x}\circ\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\widetilde{D}_v})\circ\nu_\ell.$$

Clearly, $\nabla_{\widetilde{D}}$ preserves the filtration (and $\mathcal{Z}_\ell$, as we saw in Chapter 5), so that it descends to $\operatorname{gr}(\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\mathcal{Z}_\ell)$. *We denote the induced map by* $\operatorname{gr}\nabla_{\widetilde{D}}$.

Before we discuss the second term, we introduce some notation: if $\vec{x}\circ\nu_\ell$ lies in $\mathrm{FL}^d\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_\ell$, then we write $[\vec{x}\circ\nu_\ell]_d$ for the image of $\vec{x}\circ\nu_\ell$ in $\operatorname{gr}(\mathbb{F}(\mathcal{K}_\mathfrak{g},\mathcal{O}_\mathfrak{g})_{\ell;\mathcal{A}_\mathfrak{g}}/\mathcal{Z}_{\ell;\mathcal{A}_\mathfrak{g}})_d$. We note that the image of $\vec{x}\circ\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\widetilde{D}_v})\circ\nu_\ell$ in the graded quotient is equal to $[\tau_{\mathcal{O}'_\mathfrak{g}}(\nabla_{\widetilde{D}_v})\circ\vec{x}\circ\nu_\ell]$. Inspired by this, we *define* *an endomorphism* $\operatorname{gr} T_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v})$ *of* $\operatorname{gr}(\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}/\mathcal{Z}_\ell)$*, homogeneous of degree* 2*, as follows. Any nontrivial section of* $\operatorname{gr}(\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}/\mathcal{Z}_\ell)_d$ *can be represented by some section* $\vec{x} \circ v_\ell$ *of* $\mathrm{FL}^d\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$*. It therefore suffices to define the action of* $\operatorname{gr} T_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v})$ *on* $[\vec{x} \circ v_\ell]_d$*:*

$$\operatorname{gr} T_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v})[\vec{x} \circ v_\ell]_d = [\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v}) \circ \vec{x} \circ v_\ell]_{d+2}.$$

Definition 6.1.5 (Graded WZW (projective) connection)**.** Let $\operatorname{gr} \nabla^{WZW}$ denote the projective holomorphic connection on $\operatorname{gr}(\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}/\mathcal{Z}_{\ell;\mathcal{A}_{\mathfrak{g}}})$ that is locally defined as follows: with the notation as above,

$$\operatorname{gr} \nabla^{WZW}_D = \operatorname{gr} \nabla_{\widetilde{D}_h} + \operatorname{gr} T_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v}).$$

A priori, it is not clear that this is indeed a (projective) connection, or even that it is well defined at all. However, by theorem 6.1.6, $\operatorname{gr} \nabla^{WZW}$ is indeed a well defined projective connection. Before we proceed with this theorem, a few comments on the name "Graded WZW connection" are in place.

In general, given a filtered vector bundle and connection on it that does not preserve the filtration, the connection does not pass over to the graded quotient of the vector bundle in a natural way. In this case, there is no such thing as "the graded connection". However, because of the way ∇^{WZW} is defined, there is at least an "obvious" way to define a (projective) connection on $\operatorname{gr}(\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}/\mathcal{Z}_{\ell;\mathcal{A}_{\mathfrak{g}}})$, which is definition 6.1.5. For this reason we, somewhat pretentiously, *call* $\operatorname{gr} \nabla^{WZW}$ *the graded WZW connection*. By the following theorem, this is at least an interesting object.

Theorem 6.1.6. *For* $l = \ell + \check{h}$*, the projectivization of* $\nabla^{\mathbb{F}}$ *coincides with* $\mathcal{E}$*-linear extension of* $\operatorname{gr} \nabla^{WZW}$*.*

Proof. Since P/C and ∇ are trivial (in the sense that there is a global flat section), $\mathcal{K}_{\mathfrak{g}}$ has a global G-action satisfying the conditions of assumption 3.2.14. Thus, locally over B, we choose $\mathcal{A}^{\pm}_{\mathfrak{g}}, \mathcal{F}^{\pm}_{\mathfrak{g}}$ as in proposition 3.2.13 and assume them to be G-invariant. Now, let $D, \widetilde{D}, \widetilde{D}_v$ and $\widetilde{D}_h$ be as above. A local section of $(\operatorname{gr} \mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_{\ell;\mathcal{A}_{\mathfrak{g}}}/\mathcal{Z}_\ell)_d$ can (locally) always be represented by a section $\vec{x} \circ v_\ell$ of $\mathrm{FL}^d\mathbb{F}(\mathcal{K}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}})_\ell$, and in this we can assume $\vec{x}$ to be in the image of $\bigotimes_\bullet \mathcal{F}^-_{\mathfrak{g}}$. We have that (up to a multiple of $\vec{x} \circ v_\ell$)

$$\operatorname{gr} \nabla^{WZW}_D[\vec{x} \circ v_\ell]_d = [\nabla_{\widetilde{D}}(\vec{x}) \circ v_\ell]_d + [\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_v}) \circ \vec{x} \circ v_\ell]_{d+2},$$

where $\mathcal{O}'_{\mathfrak{g}} = \mathcal{A}^-_{\mathfrak{g}} + \mathcal{F}^-_{\mathfrak{g}}$. On the other hand,

$$\nabla^{\mathbb{F}}_{D}[\vec{x} \circ \nu_\ell]_d = \nabla^{\mathcal{H}_{\mathfrak{g}}}_{D}[\vec{x} \circ \nu_\ell]_d + \rho(\overline{s}(D)) \circ [\vec{x} \circ \nu_\ell]_d.$$

Before we compare these two expression, we first introduce some more notation. Let $\vec{y}$ be a local section of $\mathsf{FL}^d U\widehat{\mathcal{K}}_{\mathfrak{g}}$. Then $\vec{y}$ is congruent to the image of some section $\widetilde{y}$ of $(\mathcal{F}^-_{\mathfrak{g}})^{\otimes d}$ modulo $\mathsf{FL}^{d-1}U\widehat{\mathcal{K}}_{\mathfrak{g}}$. Now, recall that ∇ induces a natural isomorphism $\mathcal{A}^\perp_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}}) = \mathcal{H}_{\mathfrak{g}}$. Via this isomorphism, $\widetilde{y}$ determines a section $[\vec{y}]_d$ of $U\widehat{\mathcal{H}}_{\mathfrak{g}}$, that is well defined in terms of $\vec{y}$ mod $\mathsf{FL}^{d-1}U\widehat{\mathcal{K}}_{\mathfrak{g}}$. This association $\vec{y} \mapsto [\vec{y}]_d$ has the property that $[\vec{y} \circ \vec{z}]_{d+e} = [\vec{y}]_d \circ [\vec{z}]_e$, for any section $\vec{z}$ of $\mathsf{FL}^e U\widehat{\mathcal{K}}_{\mathfrak{g}}$. Using this notation, the expressions above reduce to

$$\operatorname{gr} \nabla^{WZW}_{D}[\vec{x} \circ \nu_\ell]_d = [\nabla_{\widetilde{D}}(\vec{x})]_d \circ f_l + [\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_\nu}) \circ \vec{x} \circ \nu_\ell]_{d+2},$$

and

$$\nabla^{\mathbb{F}}_{D}[\vec{x} \circ \nu_\ell]_d = \nabla^{\mathcal{H}_{\mathfrak{g}}}_{D}[\vec{x}]_d \circ f_l + \rho(\overline{s}(D)) \circ [\vec{x}]_d \circ f_l.$$

By [12] lemma 3.7, we have that the action of $\nabla^{\mathcal{H}_{\mathfrak{g}}}_{D}$ on $\mathcal{H}_{\mathfrak{g}}$ is induced by $\nabla_{\widetilde{D}}$ on $\mathcal{K}_{\mathfrak{g}}$, so that

$$[\nabla_{\widetilde{D}}(\vec{x})]_d = \nabla^{\mathcal{H}_{\mathfrak{g}}}[\vec{x}]_d.$$

It therefore remains to prove that

$$[\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_\nu}) \circ \vec{x} \circ \nu_\ell]_{d+2} = \rho(\overline{s}(D)) \circ [\vec{x}]_d \circ f_l. \tag{6.3}$$

Since the problem is now $\mathcal{O}_B$-linear, we can restrict ourselves to the fiber over a point $b \in B$. This means that we can identify $\mathcal{H}_{\mathfrak{g}}$ with $H^1(\mathcal{C}_b; \mathcal{G})$, $\mathcal{F}_{\mathfrak{g}}$ with $H^{1,0}(\mathcal{C}_b; \mathcal{G})$ and $\overline{\mathcal{F}}_{\mathfrak{g},\infty} = H^{0,1}(\mathcal{C}_b; \mathcal{G})$ with $H^1(\mathcal{C}_b, \mathrm{Ad}_{P/\mathcal{C}_b})$. Moreover, we can choose $\mathcal{F}^-_{\mathfrak{g}}$ to be such that it maps to $\overline{\mathcal{F}}_{\mathfrak{g},\infty}$ under the map $\mathcal{A}^\perp_{\mathfrak{g}} \to \mathcal{A}^\perp_{\mathfrak{g}}/(\mathcal{A}_{\mathfrak{g}} + \mathcal{K}^0_{\mathfrak{g}}) = \mathcal{H}_{\mathfrak{g}}$.

We finish the proof with an explicit computation of both sides of equation (6.3). Choose a topological quasisymplectic basis $\{E_i\}_{i \neq 0}$ of $\mathcal{K}^{\mathrm{red}}_{\mathfrak{g}}$, that is adapted to $\mathcal{F}^\pm_{\mathfrak{g}}, \mathcal{A}^\pm_{\mathfrak{g}}$ in the following sense: for a certain $\gamma > 0$, $\{E_i\}_{i < -\gamma}$ is a basis of $\mathcal{A}^-_{\mathfrak{g}}$, $\{E_i\}_{i=-\gamma,\dots,-1}$ is a basis of $\mathcal{F}^-_{\mathfrak{g}}$, $\{E_i\}_{i=1\dots,\gamma}$ is a basis of $\mathcal{A}^+_{\mathfrak{g}}$ and $\{E_i\}_{i>\gamma}$

is a basis of $\mathcal{F}_{\mathfrak{g}}^{+}$. In terms of this, we have that

$$\begin{aligned}
&[\tau_{\mathcal{O}'_{\mathfrak{g}}}(\nabla_{\widetilde{D}_\nu}) \circ \vec{x} \circ \nu_\ell] \\
= \;& \frac{1}{2(\ell+\check{h})}\left[\widetilde{\phi}_{\mathcal{O}'_{\mathfrak{g}},\mathfrak{m}_{\mathfrak{g}}}\left(\sum_{i,j}\frac{(\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}}{ij}E_{-i}\otimes E_{-j}\right)\circ\vec{x}\circ\nu_\ell\right] \\
= \;& \frac{1}{2(\ell+\check{h})}\sum_{i,j}\frac{(\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}}{ij}[E_{-i}\circ E_{-j}\circ\vec{x}\circ\nu_\ell] \\
= \;& \frac{1}{2(\ell+\check{h})}\sum_{i,j=1}^{\gamma}\frac{(\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}}{ij}[E_{-i}\circ E_{-j}\circ\vec{x}\circ\nu_\ell] \\
= \;& \frac{1}{2(\ell+\check{h})}\sum_{i,j=1}^{\gamma}\frac{(\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}}{ij}[E_{-i}]\circ[E_{-j}]\circ[\vec{x}]\circ f_l.
\end{aligned}$$

The first equality is just the definition of $\tau_{\mathcal{O}'_{\mathfrak{g}}}$. In the second one, we used that normal ordering can be omitted due to the commutativity of the graded quotient. For the third equation, we use the special choice of the topological quasisymplectic basis, in combination with the observations that $[\mathcal{A}_{\mathfrak{g}}\circ\vec{x}\circ\nu_\ell] = 0$ and $[\mathcal{O}_{\mathfrak{g}}\circ\vec{x}\circ\nu_\ell] = [\vec{x}\circ\mathcal{O}_{\mathfrak{g}}\circ\nu_\ell] = 0$. However, $\{[E_i]\}_{|i|=1,\dots,\gamma}$ provides a quasisymplectic basis of $\mathcal{H}_{\mathfrak{g}}$ that is adapted to $\mathcal{F}_{\mathfrak{g}}$, $\overline{\mathcal{F}}_{\mathfrak{g}}$ in the sense that $[E_i] \in \mathcal{F}_{\mathfrak{g}}$ for $i > 0$ and $[E_i] \in \overline{\mathcal{F}}_{\mathfrak{g}}$ for $i < 0$. It follows that

$$\rho(\bar{s}(D))\circ[\vec{x}]_d\circ f_l = \frac{1}{2l}\sum_{i,j=1}^{\gamma}\frac{(\nabla_D^{\mathcal{H}_{\mathfrak{g}}}[E_i]_1,[E_j]_1)}{ij}[E_i]_1\circ[E_j]_1\circ[\vec{x}']_d\circ f_l.$$

Since by assumption $l = \ell + \check{h}$, we see that it suffices to prove that

$$(\nabla_D^{\mathcal{H}_{\mathfrak{g}}}[E_i]_1,[E_j]_1) = (\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}.$$

To see why this is the case, we first note that

$$(\nabla_D^{\mathcal{H}_{\mathfrak{g}}}[E_i]_1,[E_j]_1) = (\nabla_D^{\mathcal{H}_{\mathfrak{g}}}\nabla(E_i),\nabla(E_j)) = (\nabla(\nabla_{\widetilde{D}}E_i),\nabla(E_j)).$$

Since $(\cdot,\cdot)$ was defined as the pairing that $(\cdot,\cdot)_{\mathfrak{g}}$ induces via the isomorphism $\nabla : \mathcal{A}_{\mathfrak{g}}^{\perp}/\mathcal{A}_{\mathfrak{g}} \to \mathcal{H}_{\mathfrak{g}}$, we see that $(\nabla(\nabla_{\widetilde{D}}E_i),\nabla(E_j))$ indeed equals $(\nabla_{\widetilde{D}_\nu}E_i,E_j)_{\mathfrak{g}}$. □

As a direct consequence of the theorem above and corollary 4.3.2 we have

Corollary 6.1.7. *The graded WZW connection, i.e.* gr ∇^{WZW}*, is locally unitary.*

6.2 The Verlinde isomorphism

We conclude with a sketch of the Verlinde isomorphism, which identifies conformal blocks with θ-functions and the WZW-projective connection with the Hitchin projective connection. For details, we refer reader to [17].

Continue with the setting above; in particular assume $(\mathcal{C}/\mathrm{B}, \mathrm{p})$, $\mathrm{P}/\mathcal{C}$ and $\nabla^{\mathrm{P}/\mathcal{C}}$ given, but make an additional assumption on $\mathcal{C}/\mathrm{B}$: assume it to be miniversal. In this section we compare the results of section 5 with those of section 2. In the latter, we defined a morphism $\mathcal{M}/\mathcal{C} : \mathcal{M} \to \mathcal{C}$, a line bundle $\mathcal{L}$ on $\mathcal{M}$, and an $\mathcal{O}_\mathrm{B}$-module $\Theta_\ell := (\mathcal{M}/\mathcal{C})_* \mathcal{L}^\ell$. The latter is locally free and of finite rank, and has a natural flat projective connection (the Hitchin connection), which is unique with the property that is locally given by heat operators.

Theorem 6.2.1. *There is a natural isomorphism $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})_\ell \to \Theta_\ell$, called the Verlinde isomorphism, that identifies the WZW projective holomorphic connection on the right hand side with the Hitchin projective holomorphic connection on the left hand side.*

We will sketch the isomorphism: recall that for a given family of pointed curves $(\mathcal{C}/\mathrm{B}, \mathrm{p})$, we have (locally over B):

- loop groups $\mathcal{L}\mathrm{G}(\mathcal{C}^0/\mathrm{B})^{\mathrm{red}}, \mathcal{L}^{\geq}\mathrm{G} \subseteq \mathcal{L}\mathrm{G}$,
- a central extension $\widehat{\mathcal{L}\mathrm{G}}$ of $\mathcal{L}\mathrm{G}$ with subextension $\widehat{\mathcal{L}^{\geq}\mathrm{G}}$ of $\mathcal{L}^{\geq}\mathrm{G}$ and an injective homomorphism $\mathcal{L}\mathrm{G}(\mathcal{C}^0) \hookrightarrow \widehat{\mathcal{L}\mathrm{G}}$.

We denoted the Lie algebras of $\widehat{\mathcal{L}\mathrm{G}}$, $\widehat{\mathcal{L}^{\geq}\mathrm{G}}$ and $\widehat{\mathcal{L}\mathrm{G}}(\mathcal{C}^0/\mathrm{B})^{\mathrm{red}}$ by $\widehat{\mathcal{L}\mathfrak{g}}$, $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ and $\mathcal{L}\mathfrak{g}(\mathcal{C}^0/\mathrm{B})^{\mathrm{red}}$, respectively. If we trivialize P in a flat manner, we can identify, $\mathcal{K}_\mathfrak{g} = (\widehat{\mathcal{C}}^0_\mathrm{p}/\mathrm{B})_* \mathrm{Ad}_{\mathrm{P}/\mathcal{C}}$ with $(\widehat{\mathcal{C}}^0_\mathrm{p}/\mathrm{B})_*(\mathcal{O}_\mathcal{C} \otimes \mathfrak{g}) = \mathcal{K} \otimes \mathfrak{g} = \mathcal{L}\mathfrak{g}$. *We assume such a trivialization given*. The identification $\mathcal{K}_\mathfrak{g} = \mathcal{L}\mathfrak{g}$ induces identifications

$$\mathcal{O}_\mathfrak{g} = \mathcal{L}^{\geq}\mathfrak{g}, \quad \mathcal{A}_\mathfrak{g} = \mathcal{L}\mathfrak{g}(\mathcal{C}^0/\mathrm{B}), \quad \widehat{\mathcal{K}}_\mathfrak{g} = \widehat{\mathcal{L}\mathfrak{g}}_\mathfrak{g}, \quad \widehat{\mathcal{O}}_\mathfrak{g} = \widehat{\mathcal{L}^{\geq}\mathfrak{g}}.$$

We have seen that the family of moduli spaces of regularly stable G-bundles over $\mathcal{C}$ is (locally) given by

$$\mathcal{M} = \mathcal{L}\mathrm{G}(\mathcal{C}^0)^{\mathrm{red}} \backslash \mathcal{L}\mathrm{G}^{\mathrm{rs}} / \mathcal{L}^{\geq}\mathrm{G}.$$

On this, we defined a line bundle $\mathcal{L}^\ell$, whose total space is given by

$$\mathcal{L}\mathrm{G}(\mathcal{C}^0/\mathrm{B})^{\mathrm{red}} \backslash \widehat{\mathcal{L}\mathrm{G}}^{\mathrm{rs}} \times \mathbb{C} / \widehat{\mathcal{L}^{\geq}\mathrm{G}},$$

where $\mathcal{L}G(\mathcal{C}^0/B)^{red}$ acts trivially on $\mathbb{C}$ and $\widehat{\mathcal{L}^{\geq}G}$ via a character $e^{-\ell\chi}$. Now, let $\{U_i\}_{i\in I}$ be an open cover of $\mathcal{M}$ such that we have sections

$$g_i : \mathcal{M} \supseteq U_i \to \widehat{\mathcal{L}G}^{rs}.$$

Then $\mathcal{L}^\ell$ is trivialized over $\{U_i\}_{i\in I}$ by nonvanishing local sections

$$\sigma_i : U_i \to \mathcal{L}G(\mathcal{C}^0/B)^{red}\backslash\widehat{\mathcal{L}G}^{rs} \times \mathbb{C}/\widehat{\mathcal{L}^{\geq}G} : x \mapsto [g_i \times 1].$$

Over an overlap $U_i \cap U_j$, $g_i = f_{ij}g_jh_{ij}$ for certain sections f_{ij} of $\mathcal{L}G(\mathcal{C}^0/B)^{red}$ and h_{ij} of $\widehat{\mathcal{L}^{\geq}G}$, so $\sigma_i = e^{-\ell\chi}(h_{ij}) \cdot \sigma_j \cdot 1$. For a local section u of $\mathcal{B}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g}, \mathcal{A}_\mathfrak{g})_\ell$, we will define a local section θ_u of Θ_l using the following: the $U\widehat{\mathcal{K}}_\mathfrak{g}$-action on

$$\mathbb{F}(\widehat{\mathcal{L}\mathfrak{g}}, \widehat{\mathcal{L}^{\geq}\mathfrak{g}})_\ell = \mathbb{F}(\mathcal{K}_\mathfrak{g}, \mathcal{O}_\mathfrak{g})_\ell/\mathcal{Z}_\ell$$

integrates to an action of $\widehat{\mathcal{L}G}$. This has the property that $\mathcal{L}G(\mathcal{C}^0/B)$ acts trivially and $\widehat{\mathcal{L}^{\geq}\mathfrak{g}}$ acts via $e^{-\ell\chi}$. We define

$$\theta_u(U_i) = u(g_i \cdot v_\ell)\sigma_i.$$

This is well defined iff on $U_i \cap U_j$

$$u(g_i \cdot v_\ell)\sigma_i = u(g_j \cdot v_\ell)\sigma_j = u(f_{ij}g_ih_{ij} \cdot v_\ell)e^{\ell\chi}(h_{ij})\sigma_i = u(g_ih_{ij} \cdot v_\ell)e^{-\ell\chi}(h_{ij})\sigma_i,$$

where in the last step it was used that u is $\mathcal{L}G(\mathcal{C}^0/B)$ invariant. By construction, $\widehat{\mathcal{L}^{\geq}\mathfrak{g}} = \widehat{\mathcal{O}}_\mathfrak{g} = \mathcal{O}_\mathfrak{g} \oplus \mathcal{O}_B$ acts on v_ℓ via $\ell\chi$, which is ℓ times the projection on its second summand. The character $e^{\ell\chi}$ is the exponentiation of this action and hence $h_{ij} \cdot v_\ell = e^{\ell\chi}(h_{ij})v_\ell$ so that θ_u is well defined. This map $u \mapsto \theta_u$ defines the isomorphism claimed in the theorem. Under this isomorphism, the Segal-Sugawara tensor $\tau_{\mathcal{O}'}(\nabla_{\tilde{D}_v})$ acts as a second order differential operator on sections of $\mathcal{L}^\ell$, and hence, the WZW connection is (locally) represented by a heat operator. By the corollary 2.2.10, it must equal the Hitchin connection.

Appendix

A.1 Derivations, connections and differential operators

In this section, k will be an algebraically closed field of characteristic 0, S a finitely generated k-algebra, and R a finitely generated S-algebra. We remark that both R and S are Noetherian, and that for a Noetherian ring the notions of finitely generated, finitely presented and coherent modules coincide. References for this section are [8] and [9]. Define

$$\mathrm{Der}_S(R) := \{\partial \in \mathrm{End}_S(R) \mid \partial(ab) = \partial(a)b + a\partial(b)\}$$

to be the R-module of *derivations of* R *over* S. $\mathrm{Der}_S(R)$ is dual to the module of Kähler differentials J/J^2, where J is the kernel of the multiplication map $R \otimes_S R \to R$. If $x_1, \ldots, x_r$ are generators of R over S, then J is generated by the elements $x_i \otimes x_j - x_j \otimes x_i$, $i \neq j$. Hence J/J^2 is finitely generated, and as a consequence, $\mathrm{Der}_S(R)$ is finitely generated as well.

For a finitely generated R-module M define the R*-module of* R*-differential operators on* M *over* S *of order at most* i by

$$\begin{aligned} \mathcal{D}^i_{R/S}(M) &:= \mathrm{End}_R(M) \quad \text{for } i = 0 \\ \mathcal{D}^i_{R/S}(M) &:= \{D \in \mathrm{End}_S(M) \mid [D, \mathrm{End}_R(M)] \subseteq \mathcal{D}^{i-1}_{R/S}(M)\} \quad \text{for } i \geq 1. \end{aligned}$$

Clearly, $\mathcal{D}^i_{R/S}(M) \circ \mathcal{D}^j_{R/S}(M) \subseteq \mathcal{D}^{i+j}_{R/S}(M)$.

Lemma A.1.1. *There is a natural exact sequence*

$$0 \longrightarrow \mathrm{End}_R(M) \xrightarrow{i} \mathcal{D}^1_{R/S}(M) \xrightarrow{\sigma^1} \mathrm{Der}_S(R) \otimes_R \mathrm{End}_R(M)$$

called the first symbol sequence, where i *is the inclusion map.*

Proof. Let D be in $\mathcal{D}^1_{R/S}(M)$. Any element of R is a polynomial p in a set of generators, say $x_1, \dots, x_r$, of R over S. Let $\mu : R \to \mathrm{End}_R(M)$ be the multiplication map, then by the Leibniz rule

$$[D, \mu(p)] = \frac{\partial p}{\partial x^i}[D, \mu(x_i)],$$

with $\frac{\partial p}{\partial x^i}$ the formal derivative. Call $A_i = [D, \mu(x_i)]$ and define

$$\sigma^1(D) := \frac{\partial}{\partial x^i} \otimes A_i \in \mathrm{Der}_k(R) \otimes_R \mathrm{End}_R(M).$$

Of course, it is not yet clear that this is well defined. To see why this is the case, let $\rho(x_1, \dots, x_r)$ be a relation in R with coefficients in S, so that we must have $\frac{\partial \rho(x_1, \dots, x_r)}{\partial x^i} A_i = 0$. Since $\rho(x_1, \dots, x_r)$ vanishes in R, it follows that indeed

$$0 = [D, \mu(0)] = [D, \mu(\rho)] = \frac{\partial \rho}{\partial x^i} A_i.$$

Secondly, it is to be shown that $\sigma^1(D)$ is independent of the choice of generators, so let $y_1 \dots, y_s$ be another set of generators. We denote $[D, \mu(y_i)]$ by B_i and, by abbreviating $x_1, \dots, x_r$ to x and $y_1, \dots, y_s$ to y, express x in terms of y - we write $x(y)$ to express this. Any element in R can be represented by some $p(x) \in S[x_1, \dots, x_n]$, but can also be represented by $p(x(y)) \in S[y_1, \dots, y_s]$. By formally differentiating,

$$\frac{\partial p(x(y))}{\partial y_i} B_i = \frac{\partial p}{\partial x_j}\frac{\partial x_j}{\partial y_i} B_i$$

but also

$$\frac{\partial p(x)}{\partial x_j} A_j = \frac{\partial p}{\partial x_j} A_j = \frac{\partial p}{\partial x_j}[D, \mu(x_j(y))] = \frac{\partial p}{\partial x_j}\frac{\partial x_j}{\partial y_i}[D, \mu(y_i)] = \frac{\partial p}{\partial x_j}\frac{\partial x_j}{\partial y_i} B_i,$$

so that σ is indeed well defined. Finally, $\sigma^1(D)(r) = 0$ for every $r \in R$ iff D commutes with the R action on M, i.e. iff $D \in \mathrm{End}_R(M)$. □

Since M is finitely generated, so is $\mathrm{End}_R(M)$. Moreover, the image of σ is a submodule of $\mathrm{Der}_S(R) \otimes_R \mathrm{End}_R(M)$, and since is finitely generated, so is $\mathrm{im}(\sigma)$. Thus $\mathcal{D}^1_{R/S}(M)$, being an extension of the finitely generated R-modules $\mathrm{End}_R(M)$ and $\mathrm{im}(\sigma)$, is finitely generated.

We now specialize to the case $S = k$. In that case, there is an interesting criterion for σ to be surjective, i.e. for

$$0 \longrightarrow \mathrm{End}_R(M) \longrightarrow \mathcal{D}^1_{R/k}M \xrightarrow{\sigma^1} \mathrm{Der}_k(R) \otimes_R \mathrm{End}_R(M) \longrightarrow 0 \quad (A.4)$$

to be exact, provided R is nice enough. To simplify the proof, we first recall that a sequence of finitely generated R-modules is exact iff it is exact at all the maximal ideals. Since R is finitely generated over k, its localization at any maximal ideal is a local ring of finite dimension. Hence it suffices to prove the following.

Theorem A.1.2. *Suppose* R *is a regular local* k*-algebra of finite dimension. Then* (A.4) *is exact iff* M *is a free* R*-module.*

Proof. If M is free over R, say $M \simeq R^{\oplus r}$, any element $\partial \otimes A \in \mathrm{Der}_k(R) \otimes_R \mathrm{End}_R(M)$ is lifted by $A \circ \partial^{\oplus r}$. Hence in this case, (A.4) is exact.

Now suppose σ^1 is surjective. Denote the maximal ideal of R by $\mathfrak{m}$. If we complete R with respect to $\mathfrak{m}$, then the completion of σ^1 is also surjective. Moreover, if the completion of M is free, then M itself is also free. Hence we can assume R to be complete.

Let n be the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over k. Since R is a regular complete local k-algebra of dimension n, we know it is of the form $k[[x_1, \ldots, x_n]]$ for certain $x_1, \ldots, x_n \in R$. It then follows that $\mathrm{Der}_k(R) = R < \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} >$. Clearly,

- if $r \in R$, then $\partial r = 0$ for all $\partial \in \mathrm{Der}_k(R)$ iff $r \in k$,
- if $r \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$, then there is a $\partial \in \mathrm{Der}_k(R)$ such that $\partial r \in \mathfrak{m}^{k-1} \backslash \mathfrak{m}^k$ (for any $k \geq 1$).

Now let $m_1, \ldots, m_n$ be a set of generators of M over R, such their images in $M/\mathfrak{m}M$ are linearly independent over k. Suppose there is a relation between them, say $r_i m_i = 0$, $r_i \in R$. We can assume it to be of minimal $\mathfrak{m}$ degree in the following sense: for at least one of the i's, say $i = 0$, $r_i \in \mathfrak{m}^d \backslash \mathfrak{m}^{d+1}$ with d minimal. In case $d \geq 1$, then there is a $\partial \in \mathrm{Der}_k(R)$ such that $\partial r_0 \in \mathfrak{m}^{d-1} \backslash \mathfrak{m}^d$. We now assume σ to be surjective, so that there is a $D \in \mathcal{D}_{R/k}M$ such that $\sigma(D) = \partial$. Apply D to the relation:

$$0 = D(r_i m_i) = \partial(r_i) m_i + r_i D(m_i) = (\partial(r_i) + r_j A^j_i) m_i,$$

where $A_i^j \in R_m$ are such that $D(m_i) = A_i^j m_j$. We thus obtain a new relation with coefficients $(\partial(r_i) + r_j A_i^j) \in \mathfrak{m}^{d-1}$. However, since $r_j A_i^j = 0 \mod \mathfrak{m}^d$ but $r_0 \neq 0 \mod \mathfrak{m}^d$, we have obtained a relation of degree lower than d, contradicting the minimality.

Hence we can assume $d = 0$. In this case however, $r_i m_i$ projects to a nontrivial relation in $M/\mathfrak{m}M$ while the m_i's were assumed to be independent. The conclusion is that the m_i freely generated M, and that by the remark above, M is a free R-module. □

We return for the moment to the more general situation where R is finitely generated over S. Without proof, we remark that for higher order differential operators, we have similar results. In particular, for all $i \geq 0$, $\mathcal{D}^i_{R/S}(M)$ is finitely generated and we have an exact sequence

$$0 \longrightarrow \mathcal{D}^i_{R/S}(M) \longrightarrow \mathcal{D}^{i+1}_{R/S}(M) \xrightarrow{\sigma^{i+1}} \mathrm{Sym}_{i+1} \mathrm{Der}_S(R) \otimes_R \mathrm{End}_R(M) .$$

called the $i+1$-th symbol sequence. Here σ^{i+1} is defined inductively: for $D \in \mathcal{D}^{i+1}_{R/S}(M)$ and $x_1, \dots, x_n$ generators of R over S, $\sigma^{i+1}(D) = \frac{\partial}{\partial x_j} \otimes \sigma^i(D)$. Finally, if M is free, then σ^{i+1} is surjective.

A.1.1 Differential operators on $\mathcal{O}_S$-modules

We now discuss that sheafification of the discussion above. Suppose X is a Noetherian scheme over an algebraically closed field k, $\pi : Y \to X$ scheme of locally finite type, and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then we define $\theta_{Y/X}$ to be the subsheaf of $\mathcal{E}nd_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_Y$ of sections satisfying $\partial(fg) = \partial(f)g + f\partial(g)$, for local sections f, g of $\mathcal{O}_Y$ and ∂ of $\mathcal{E}nd_{\pi^{-1}\mathcal{O}_X}\mathcal{O}_Y$. If Y/X is smooth, then $\theta_{Y/X}$ is the sheaf of vector fields on Y over X, and if $X = \mathrm{Spec}\ \mathbb{C}$, we simply write θ_Y. We have the following relation with the notions above: if $X = \mathrm{Spec}\ R$, $Y = \mathrm{Spec}\ S$, then $\theta_{X/Y}$ is the sheafification of $\mathrm{Der}_S(R)$.

The definition of differential operators for sheaves is also analogous to the algebraic setting:

Definition A.1.3. Define $\mathcal{D}^0_{Y/X}(\mathcal{F}) := \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$. Starting from this, we inductively define $\mathcal{D}^i_{Y/X}(\mathcal{F})$ to be the subsheaf of $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$ consisting of $D \in \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})(U)$, such that $[D|_V, \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})(V)] \subseteq \mathcal{D}^{i-1}_{Y/X}(\mathcal{F})(V)$ for any open subsets $U \subseteq Y$ and $V \subseteq U$. We call $\mathcal{D}^i_{Y/X}(\mathcal{F})$ the *sheaf of differential operators on (the $\mathcal{O}_Y$-module) $\mathcal{F}$ over* X.

If $X = \text{Spec } R$, $Y = \text{Spec } S$ and $M = H^0(X, \mathcal{F})$, then $\mathcal{D}^i_{Y/X}(\mathcal{F})$ is the sheafification of $\mathcal{D}^i_{R/S}(M)$. Since $\mathcal{D}^i_{Y/X}(\mathcal{F}) \subset \mathcal{D}^{i+1}_{Y/X}(\mathcal{F})$ we can take a direct limit:

$$\mathcal{D}_{Y/X}(\mathcal{F}) := \varinjlim_i \mathcal{D}^i_{Y/X}(\mathcal{F}).$$

In inherits a filtration by construction, in which the i-th part is $\mathcal{D}^i_{Y/X}(\mathcal{F})$.

Since Y/X is of locally finite type, we can cover X by affine subsets $\{U_i = \text{Spec } S_i\}$ and Y by affine subsets $\{V_{ij} = \text{Spec } R_{ij}\}$ such that V_{ij} lies over U_i and R_{ij} is finitely generated over S_i as algebra. Therefore, $\theta_{X/Y}$ and $\mathcal{D}^i_{Y/X}(\mathcal{F})$ are locally (over Y) sheafifications of $\text{Der}_S(R)$, $\mathcal{D}^i_{R/S}M$, for certain (locally determined) R, S, M as before. Hence the properties of the latter derived above, extend to local properties of the former, and in particular

Corollary A.1.4. *For* $i \geq 0$ *there is a natural exact sequence called the* $i+1$ *symbol sequence:*

$$0 \longrightarrow \mathcal{D}^i_{Y/X}(\mathcal{F}) \longrightarrow \mathcal{D}^{i+1}_{Y/X}(\mathcal{F}) \xrightarrow{\sigma^{i+1}} \text{Sym}_{i+1}\theta_{Y/X} \otimes_{\mathcal{O}_Y} \text{End}_{\mathcal{O}_Y}(\mathcal{F}) \, .$$

Now assume $X = \text{Spec } k$, Y/X is smooth and $\mathcal{F}$ coherent. First recall that a sequence

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{D}^1_{Y/X} \xrightarrow{\sigma^1} \theta_{Y/X} \otimes_R \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F}) \longrightarrow 0 \qquad \text{(A.5)}$$

is exact iff it is locally exact, and that this is so, iff it is exact on the stalks at closed points. So $p \in Y$ be a closed point. Since Y/k is smooth and of finite dimension, $\mathcal{O}_{Y,p}$ satisfies the conditions of theorem A.1.2. So σ^1 is surjective at p iff $\mathcal{F}_p$ is a free $\mathcal{O}_{Y,p}$-module. We have therefore shown the following: (A.5) is exact iff $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module.

An often used subsequence of (A.5) is the following.

Definition A.1.5. Let $\mathcal{A}_{Y/X}(\mathcal{F})$ be the preimage of $\theta_{Y/X} \otimes \text{id}$ in $\mathcal{D}^1_{Y/X}(\mathcal{F})$, then we have an exact sequence

$$0 \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}) \longrightarrow \mathcal{A}_{Y/X} \xrightarrow{\sigma} \theta_{Y/X} \longrightarrow 0, \qquad \text{(A.6)}$$

where we identify $\theta_{Y/X}$ with $\theta_{Y/X} \otimes_R \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$ using $\text{id} \in \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$. This is called the *Atiyah sequence*. The class of it in $\text{Ext}(\theta_{Y/X}, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}))$ is denoted $c_A(\mathcal{F})$ and is called the *Atiyah class of* $\mathcal{F}$.

Clealy, if (A.5) is exact, then the Atiyah sequence is also exact. The converse, however, is also true: $\mathcal{D}^1_{Y/X}(\mathcal{F})$ is a left $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$-module by composition in $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$, $\theta_{Y/X} \otimes \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$ also has an obvious $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})$, and σ^1 commutes with these actions. Hence, if $\sigma^1(\mathcal{A}_{Y/X}(\mathcal{F})) = \theta_{Y/X} \otimes \mathrm{id}$, then

$$\begin{aligned}\sigma^1(\mathcal{D}^1_{Y/X}(\mathcal{F})) &= \sigma^1(\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})\mathcal{A}_{Y/X}(\mathcal{F})) = \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F})\sigma^1(\mathcal{A}_{Y/X}(\mathcal{F})) \\ &= \theta_{Y/X} \otimes_R \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{F}).\end{aligned}$$

Corollary A.1.6. *The sequence (A.6) is exact iff $\mathcal{F}$ is a locally free $\mathcal{O}_Y$-module.*

A global splitting of (A.6) is called a *holomorphic connection over* Y/X and a local splitting is called a *local holomorphic connection over* Y/X. In case $X =$ Spec $\mathbb{C}$, we just say (local) connection over Y. Clearly, the existence of a local holomorphic connection is equivalent with the local freeness of $\mathcal{F}$. If $k = \mathbb{C}$ and Y is a smooth variety, then it has a C^∞ structure, and we can tensor the Atiyah sequence with the sheaf of smooth functions - we call the result the smooth Atiyah sequence. This then will be exact iff the original Atiyah sequence is exact. Since the sheaf of smooth functions is fine, local splitting always glue to a global splitting, so that the smooth Atiyah sequence is exact iff there is a global splitting. We call such a splitting a C^∞ *connection* on $\mathcal{F}$.

Corollary A.1.7. *If Y is a smooth complex variety, then $\mathcal{F}$ is locally free iff it has a C^∞-connection.*

A.2 Extension classes associated to differential operators

We now assume Y to be a Kähler manifold, and $\mathcal{L}$ a line bundle on Y. The sheaf of differential operators of order at most k on $\mathcal{L}$ will be denoted $\mathcal{D}^k\mathcal{L}$. For every integer $k > 0$ we have the $(k+1)$-th symbol sequence

$$0 \longrightarrow \mathcal{D}^k\mathcal{L} \longrightarrow \mathcal{D}^{k+1}\mathcal{L} \xrightarrow{\sigma^{k+1}} \mathrm{Sym}_{k+1}\theta_Y \longrightarrow 0\ .$$

By reduction modulo $\mathcal{D}^{k-1}\mathcal{L}$ one finds

$$0 \longrightarrow \mathrm{Sym}_k\theta_Y \longrightarrow \mathcal{D}^{k+1}\mathcal{L}/\mathcal{D}^{k-1}\mathcal{L} \xrightarrow{\overline{\sigma}^{k+1}} \mathrm{Sym}_{k+1}\theta_Y \longrightarrow 0\ . \tag{A.7}$$

Theorem A.2.1. *The extension class defined by (A.7) is given by cupping with $kc_A(\mathcal{L}) - (k-1)c_A(\omega_Y)$, where c_A denotes is the Atiyah class.*

Proof. Choose C^∞ connections $\nabla^{\mathcal{L}}$, ∇ on $\mathcal{L}$ and θ_Y, respectively, and such that ∇ is torsion free. For line bundles, the construction of the connecting homomorphism via the Dolbeault resolution gives that the Atiyah classes are represented by the $(1,1)$ part of a curvature, so $c_A(\mathcal{L}) = \overline{\partial}\nabla^{\mathcal{L}}$ and $c_A(\omega_Y) = -\operatorname{tr}(\overline{\partial}\nabla)$; these identities will be used later on.

To compute the extension class defined by (A.7) we will use Dolbeault resolutions: we lift a local section of $\mathcal{S}ym_k\theta_Y$ to a C^∞ section of $\mathcal{D}^k\mathcal{L}/\mathcal{D}^{k-2}\mathcal{L}$, and subsequently take the $\overline{\partial}$ of this. The result will be a smooth section of $\mathcal{S}ym_{k-1}\theta_Y$ with values in the $(0,1)$ forms. The map thus defined, taking a local section of $\mathcal{S}ym_k\theta_Y$ to a $(0,1)$ form valued local section of $\mathcal{S}ym_{k-1}\theta_Y$, represents the connecting homomorphism. We will now compute this explicitly.
By induction on $k \geq 1$, we define C^∞ lifts for the symbol maps:

$$\begin{aligned}
&u_1 : \theta_Y \to \mathcal{E}^{0,0}\mathcal{D}^1\mathcal{L} : X \mapsto \nabla^{\mathcal{L}}_X \\
&\tilde{u}_{k+1} : \theta_Y^{\otimes(k+1)} \to \mathcal{E}^{0,0}\mathcal{D}^{k+1}\mathcal{L} : X \otimes X_1 \otimes \cdots \otimes X_k \\
&\quad \mapsto \nabla^{\mathcal{L}}_X \circ u_k(X_1 \otimes \cdots \otimes X_n) - \tilde{u}_k(\nabla_X(X_1 \otimes \cdots \otimes X_n)) \\
&u_k := \tilde{u}_k|_{\mathcal{S}ym_k\theta_Y}.
\end{aligned}$$

The extension class of (A.7) is given by the class that $\overline{\partial}u_k$ represents in

$$\mathrm{Ext}^1(\mathcal{S}ym^{k-1}\theta_Y, \mathcal{S}ym^k\theta_Y).$$

For local sections $X, X_1, \ldots, X_n$ of θ_Y one has

$$\begin{aligned}
&\overline{\partial}\tilde{u}_{k+1}(X, X_1, \ldots, X_n) \\
&\quad = (\overline{\partial}\nabla^{\mathcal{L}})(X) \circ \tilde{u}(X_1 \otimes \cdots \otimes X_n) + \nabla^{\mathcal{L}}_X \circ (\overline{\partial}\tilde{u})(X_1 \otimes \cdots \otimes X_n) \\
&\qquad = -(\overline{\partial}\tilde{u}_k)(\nabla_X(X_1 \otimes \cdots \otimes X_n)) - \tilde{u}_k\left(\overline{\partial}(\nabla_X(X_1 \otimes \cdots \otimes X_n))\right)
\end{aligned}$$

If we consider this module $\mathcal{E}^{0,1}\mathcal{D}^{k-1}\mathcal{L}$, we see that the last term vanishes and that $\nabla^{\mathcal{L}}_X \circ (\overline{\partial}\tilde{u})(X_1 \otimes \cdots \otimes X_n)$ simplifies to $X \circ (\overline{\partial}\tilde{u})(X_1 \otimes \cdots \otimes X_n)$, since $\nabla^{\mathcal{L}}_X \equiv X \bmod \mathcal{E}^{0,0}$. Therefore,

$$\begin{aligned}
\overline{\partial}\tilde{u}_{k+1}(X, X_1, \ldots, X_n) &\equiv c_A(\mathcal{L})(X)X_1 \otimes \cdots \otimes X_n + X \otimes (\overline{\partial}\tilde{u})(X_1 \otimes \cdots \otimes X_n) \\
&\quad -\tilde{u}_k\left(\overline{\partial}(\nabla_X(X_1 \otimes \cdots \otimes X_n))\right) \mod \mathcal{E}^{0,1}\mathcal{D}^{k-1}\mathcal{L} \\
&\equiv (c_A(\mathcal{L}) \otimes \mathrm{id} + \mathrm{id} \otimes \overline{\partial}\tilde{u})(X \otimes X_1 \otimes \cdots \otimes X_n) \\
&\quad -\overline{\partial}(\nabla_X(X_1 \otimes \cdots \otimes X_n)) \mod \mathcal{E}^{0,1}\mathcal{D}^{k-1}\mathcal{L}
\end{aligned}$$

To identify the last term, consider

$$\nabla(X \otimes_s X_1 \otimes_s \cdots \otimes_s X_n) \in \mathcal{E}^{0,0}\mathcal{S}ym_k\theta_Y \otimes \Omega_Y$$

and take the contraction tr_1 between Ω_Y and the first factor θ_Y, so

$$\mathrm{tr}_1 \nabla(X \otimes X_1 \otimes \cdots \otimes X_n) = \nabla_X(X_1 \otimes \cdots \otimes X_n) + \mathrm{tr}(\nabla X) X_1 \otimes \cdots X_n.$$

Since $\mathrm{tr}_1 \nabla(X \otimes X_1 \otimes \cdots \otimes X_n)$ is a C^∞ section of $\mathcal{S}ym_k\theta_Y$, the image of $\overline{\partial}\,\mathrm{tr}_1 \nabla(X \otimes X_1 \otimes \cdots \otimes X_n)$ in $H^1(Y, \mathcal{S}ym_k\theta_Y)$ is trivial, so that

$$-\overline{\partial}\nabla_X(X_1 \otimes \cdots \otimes X_n) \equiv \overline{\partial}\,\mathrm{tr}(\nabla X) X_1 \otimes \cdots X_n$$

on the level of cohomology. We will simplify this further using some standard differential geometry. If we denote the curvature of ∇ by R_∇, then $(\overline{\partial}\nabla)(X)$ is the $1,1$ part of $R_\nabla(X)$ - we denote this by $R_\nabla^{1,1}(X)$. Now let $h(\cdot,\cdot)$ be a local hermitian form on θ_Y, e_i be a local basis of vector fields that is holomorphic and satisfies $[e_i, e_j] = 0$ and $h(e_i, e_j) = \delta_{i,j}$. We locally have that

$$\overline{\partial}\,\mathrm{tr}(\nabla X) = \mathrm{tr}(\overline{\partial}\nabla X) = \sum_i h(\overline{\partial}\nabla_{e_i} X, e_i).$$

Since the problem is linear in X, we can without loss of generality assume that $[e_i, X] = 0$ for all i, so that the torsion-freeness of ∇ gives us $\nabla_{e_i} X = \nabla_X e_i$. It then follows that

$$\overline{\partial}\,\mathrm{tr}(\nabla X) = \sum_i h(\overline{\partial}\nabla_X e_i, e_i) = \mathrm{tr}(\overline{\partial}\nabla)(X) = c_A(\theta_Y^{\wedge top})(X) = -c_A(\omega_Y)(X).$$

Therefore, we have that in $H^1(Y, \mathcal{S}ym_k\theta_Y)$

$$\overline{\partial}(\nabla_X(X_1 \otimes \cdots X_n)) \equiv c_A(\omega_Y)(X) X_1 \otimes \cdots \otimes X_n,$$

and hence we have for the class of $\overline{\partial}\widetilde{u}_{k+1}(X, X_1, \ldots, X_n)$ in $H^1(Y, \mathcal{S}ym_k\theta_Y)$

$$\begin{aligned}[\overline{\partial}\widetilde{u}_{k+1}(X, X_1, \ldots, X_n)] \\ &= \left((c_A(\mathcal{L}) - c_A(\omega_Y)) \otimes \mathrm{id} + \mathrm{id} \otimes \overline{\partial}\widetilde{u}\right)(X \otimes X_1 \otimes \cdots \otimes X_n).\end{aligned}$$

Since $[\overline{\partial}\widetilde{u}_1(X)] = c_A(\mathcal{L})(X)$, by induction, $[\overline{\partial}\widetilde{u}_k]$ is given by cupping with the class

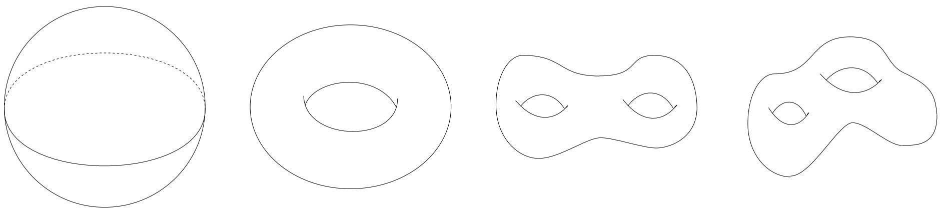
$$\begin{aligned}(c_A(\mathcal{L}) - c_A(\omega_Y)) \otimes \mathrm{id}_{\theta_Y^{k-1}} + \cdots + \mathrm{id}_{\theta_Y^{k-2}} \otimes (c_A(\mathcal{L}) \\ - c_A(\omega_Y)) \otimes \mathrm{id}_{\theta_Y} + \mathrm{id}_{\theta_Y^{k-1}} \otimes c_A(\mathcal{L}).\end{aligned}$$

We conclude that the class of $(A.7)$ is given by cupping the expression above with a section of $\mathcal{S}ym_k\theta_Y$; because of the symmetry this to cupping with the class $kc_A(\mathcal{L}) - (k-1)c_A(\omega_Y) \in H^1(Y, \Omega_Y)$. $\square$

Samenvatting

Dit proefschrift gaat over *conforme coblokken*, hun *gegradeerde quotient* en een unitaire structuur daarop. Om te weten wat dit nou precies betekend moet de lezer eigenlijk bij het begin van dit boekje beginnen; hieronder zal ik daarom alleen proberen om een idee te geven van wat voor dingen dit nu zijn. Daarbij heb ik de nodige details weggelaten waardoor het onderstaande in strikte zin niet meer helemaal klopt, maar hopelijk wel begrijpelijker is.

Om te beginnen moet ik eerst uitleggen wat een *compact gesloten Riemann oppervlak*[1] is. Dit gaat het makkelijkste door een paar voorbeelden te schetsen:



(a) 2-Sfeer (b) Torus (c) Geslacht 2 oppervlak (d) Nog een geslacht 2 oppervlak

Zo'n compact gesloten Riemann oppervlak is dus een oppervlak, dat overal glad is en geen rand heeft. De laatste twee oppervlakken kunnen op een continue manier in elkaar vervormd worden. Dat geldt niet voor ieder ander tweetal van de hierboven afgebeelde oppervlakken, aangezien ze een verschillend aantal *gaten* hebben (0, 1, 2 en 2 respectievelijk). Bij een continue vervorming mag er namelijk niet geknipt worden, zodat het aantal gaten al-

[1]Dit correspondeerd met wat in de text "curve" wordt genoemd

tijd constant blijft.

Vervolgens moet ik uitleggen wat een *vectorruimte* is: dat is een verzameling waarin je de elementen, die vectoren worden genoemd, met een getal kan vermenigvuldigen en onderling kan op tellen. Denk hierbij aan alle vectoren in het platte vlak die in een punt aangrijpen, waarbij optellen geschiedt door het kop-aan-staart leggen en vermenigvuldiging door de vector te schalen.

De ruimte van conformal coblokken is nu een vectorruimte die je op een natuurlijk manier kan produceren uit een compact gesloten Riemann oppervlak en geschikte *Lie groep* - wat dit laatste is leg ik verder niet uit. Deze constructie is zodanig dat als je een vector kiest uit deze ruimte van conforme blokken, behorende bij een Riemann oppervlak, en vervolgens het Riemann oppervlak op een continue manier vervormt, dat je de vector dan op een natuurlijke continue manier mee kan laten varieren. Dit laatste heet een *connectie*.

In dit proefschrift introduceer ik een variant op bovengenoemde conforme coblokken, genaamd *het gegradeerde quotient van conforme coblokken*, en een bijbehorende connectie. Hiervoor geeft ik een expliciete *compatibele unitaire structuur is* (een unitaire structuur is een lengte- en hoekbegrip voor vectoren, en compatibel betekend hier dat dit lengte- en hoekbegrip netjes meevariëert wanneer we het oppervlak vervormen). Zo'n explicite compatibele unitaire structuur voor conforme coblokken is tot op heden nog niet bekend; hopelijk helpt bovengenoemd resultaat voor de gegradeerde versie bij het vinden daarvan.

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Curriculum vitae

The author was born on July 22 1980, in the city of ’s-Hertogenbosch. He attended primary and secondary education in Schijndel until 1998. In that year, he started his studies Mathematics and Physics (TWIN program) at Utrecht University (UU) - both were completed in August 2003, the former cum laude. Starting from February 2004, he started his research as a PhD student at UU, under the guidance of prof. dr. E.J.N. Looijenga. This thesis is the result thereof.

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