Cobordism theories:
An algebraic journey from topology to geometry

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Master's thesis by:
J.T. Lyczak

Written under the supervision of:
Prof. dr. C.F. Faber

Second reader:
dr. J. Stienstra

Universiteit Utrecht

Abstract

Cobordism theories have been studied for a long time in various forms and guises. Many of these theories were shown to have a very nice structure and became important tools in the study of differentiable manifolds. When Quillen [45] gave an axiomatic description for the theory of complex cobordism, it became possible to define an equivalent tool for the language of schemes. Levine and Moore [35] translated Quillen's axioms into algebraic geometry and defined algebraic cobordism. This theory has strong relations with the Chow group and K-theory, just like cobordism theories in algebraic topology relate to homology and K-theory of vector bundles. Algebraic cobordism was given a geometric interpretation by Levine and Pandharipande in [36]. In [34] Lee and Pandharipande were able to extend this to a theory of schemes with bundles, similar to the theory Atiyah and Singer used in their proof of the Atiyah-Singer index theorem [3]. Using this theory Tzeng [61] was able to prove conjectures of Vainsencher [62] and Göttsche [20] about nodal curves on surfaces. This generalises the result of Fomin and Mikhalkin [15] that the number of nodal curves through a specific number of points on the projective plane is a polynomial in the degree of the curve, to an arbitrary surface.

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Contents

Introduction

In the first chapter we will look at classical bordism and cobordism theories in algebraic topology: their geometrical definition, their structure as generalised homology or cohomology theories and their axiomatic definition as given by Quillen.

The geometrical definition gives an immediate equivalence relation on manifolds of a fixed dimension: two manifolds are called bordant if they together form the boundary of another manifold, called a cobordism. Adding structure on a cobordism which restricts to a structure on the boundary gives many other examples of bordism theories. One could for example demand orientability of the tangent bundle or any other G-structure on this bundle. This gives rise to the notion of oriented bordism and complex bordism. Classes in both these theories are shown to be uniquely determined by certain characteristic classes in singular homology. Furthermore, such theories are shown to be given by homotopy classes of a spectrum and hence they are all examples of generalised homology theories. The spectrum associated to these homology theories, called the Thom-spectrum, also gives us generalised cohomology theories such as oriented cobordism and complex cobordism. The geometric interpretation is however quite lost. Quillen showed in [45] that it is possible to define complex cobordism over a manifold X as an equivalence relation on specific maps $Y \to X$. Apart from this geometric interpretation, Quillen gave axioms with make complex cobordism into the universal such theory with pushforwards, pullbacks, Chern classes and an exterior product.

In Chapter 2 we will look at theories on schemes with are not unlike cobordism theories. First we will examine the Chow group $\mathrm{CH}_*(X)$. This group is a quotient of the free abelian group on closed subschemes of X. The relations are given by a condition similar to cobordism in the first chapter. This graded group has an interesting structure, most notably it admits a theory of characteristic classes for vector bundles. If X is smooth of dimension n, we will see that $\mathrm{CH}^*(X) = \mathrm{CH}_{n-*}(X)$ even has the structure of a graded ring.

Secondly, we will look at the K-theory of locally free sheaves and quasi-coherent sheaves on schemes. These are quotients of the free abelian groups on the set of isomorphism classes of these types of sheaves and although the definition does not resemble any of the equivalence relations we have seen so far, most of the structure and theorems on the Chow group apply to these groups.

In the third chapter, we will follow Levine and Morel [35] in their construction of a theory of schemes whose axioms are a direct analogue of the axioms of complex cobordism as given by Quillen [45]. They define an oriented Borel-Moore homology theory as a theory which satisfies these conditions and show that the Chow group and a slightly altered version of K-theory are examples of such theories. Then a construction of algebraic cobordism Ω_* is given and it will appear to be the universal oriented Borel-Moore homology theory, in the sense that for any such theory A_* there is a unique morphism $\Omega_* \to A_*$ which respects the structure of pullbacks, pushforwards, products and Chern classes. Also algebraic cobordism of the point $\operatorname{Spec} k$ is examined. It will be identified as the Lazard ring, the ring classifying all commutative formal group laws of rank one. Using this identification we give a basis, which will show that each class is uniquely determined by its Chern numbers of the tangent sheaf.

Chapter 4 gives an alternate definition of algebraic cobordism: Levine and Pandharipande [36] gave a geometrical condition for two schemes of the same dimension to be equivalent in algebraic cobordism, called the double point relation. They went on to show that these relations are not only satisfied in algebraic cobordism, but also suffice to define it.

Additional structure on schemes may descend to a structure on the spaces occurring in the double point relation. The main example is given by schemes with a vector bundle of a fixed rank. The classes in the theory $\omega_{n,r}$, of schemes of dimension n and vector bundles of rank r, thus obtained can be shown to be uniquely determined by the combined Chern numbers of the tangent bundle and the given vector bundle.

In the last chapter, we will address a proof of Tzeng [61] for a conjecture of Vainsencher [62]. The projective space of all global sections of a line bundle on a surface has a subset parametrizing nodal curves with δ nodes. The Zariski closure of this set is called the Severi variety in the case of $\mathbb{P}^2$ and it was Fomin and Mikhalkin [15] who were able to prove that the degree of the Severi variety for curves of degree d with δ nodes was a polynomial in d of degree 2δ, for large enough d. A similar result for a general surface S was conjectured by Vainsencher [62]. He conjectured that the degree of the points parametrizing nodal curves with δ nodes in a line bundle $\mathcal{L}$ with enough global sections should be a polynomial in degree δ in the Chern numbers of the tangent sheaf of S and the line bundle $\mathcal{L}$.

Tzeng used Hilbert schemes of points as suggested by Göttsche. She used a result from Li and Wu [37] showing that the Hilbert schemes of points behave well on the spaces in the double point relation. Then she used algebraic cobordism of surfaces with line bundles to show the existence of these polynomials, as conjectured by Göttsche.

Notation

We will assume the reader is familiar with the basics of differentiable manifolds, algebraic topology and scheme theory. All our manifolds are assumed to be smooth, compact and hence are embeddable in some $\mathbb{R}^N$ by Whitney's theorem [64, Theorem 5]. Manifolds are allowed to have a boundary and those that do not will be called *closed*. For an oriented manifold M we will write $\overline{M}$ for M in the opposite orientation. We will write pt for a one-pointed topological space.

The categories of graded groups and graded rings with their respective homomorphisms will be denoted by $\mathbf{Ab}_*$ and $\mathbf{R}_*$ respectively.

We will use k to denote a field, which will be algebraically closed throughout. This is needed for several reasons, but most importantly for the construction of algebraic cobordism in Chapter 3. In Chapter 5 we will restrict ourselves to $k = \mathbb{C}$, so that we can use analytic arguments. Each scheme will be a separated scheme finite type over k. The categories of all schemes will be denoted by $\mathbf{Sch}_k$. A scheme will be called a variety if it is an irreducible and reduced scheme. A subscheme is a subvariety if it is a variety by itself. A product of schemes is always the fibred product over $\operatorname{Spec} k$.

We will write $\mathbf{Coh}(X)$ for the category of coherent sheaves on a scheme X. A divisor on a scheme or variety will always mean a Cartier divisor.

An l.c.i. morphism is a morphism that factors as a regular embedding and a smooth morphism. Here smooth is defined as in [26, III.10] together with the assumption of quasi-projectivity. This is first needed in the definition of oriented Borel-Moore homology theories in Chapter 3. So the quasi-projectivity condition could be dropped on smooth morphisms in Chapter 2.

We will write $\mathbf{Sm}_k$ for the full subcategory of $\mathbf{Sch}_k$ whose objects are smooth quasi-projective schemes over k.

Chapter 1

Complex cobordism

In this chapter we will look at the origin of bordism and cobordism theories. The geometric notion of cobordisms gives us unoriented bordism $\mathfrak{N}_*$ which is a graded ring. This ring turns out to be a polynomial ring in infinitely many generators. This theory is expanded to assign a graded ring to each compact manifold M which is an $\mathfrak{N}_*$-module. If we consider the case where M is a point we get back $\mathfrak{N}_*$. We can use these geometric ideas to construct variations of this theory, such as oriented bordism and complex bordism. We will see that these theories are examples of generalised homology theories as they are represented by the Thom-spectrum. This spectrum also gives us cohomology theories, which are named oriented cobordism and complex cobordism. Then we will give the axioms for a complex oriented cohomology theory and use a geometric interpretation of complex cobordism to prove that this is in fact the universal complex oriented cohomology theory.

The material in this chapter can be found in the book [58] by Stong or the articles [45] and [46] by Quillen.

1.1 Unoriented bordism

The notion of diffeomorphism of differentiable manifolds is for most applications too general and the problem of determining whether two manifolds are non-diffeomorphic is in general easier answered by examining the class of the spaces in question under a more restrictive equivalence relation. In all such applications one will want equivalence classes which are not too large, otherwise too much information is lost, but otherwise not too small, because then too little information is gained. One equivalence relation on all manifolds, that stood the test of time, is the notion of bordism.

Definition 1.1. Consider two *closed*, i.e. without boundary, compact n-dimensional manifolds M and N. We say they are *bordant* if there exists an $(n+1)$-dimensional manifold W, such that the boundary ∂W of W is diffeomorphic to $M \coprod N$. The manifold W is called a *cobordism* between M and N.
The set of bordism classes of dimension n is denoted by $\mathfrak{N}_n$ and the class of a manifold M is denoted by $[M]$.

Bordism is indeed an equivalence relation: symmetry is obvious, reflexivity follows from the cylinder $M \times I$, and transitivity follows from the technical collar lemma. This lemma says that any component M of the boundary of a manifold W has a neighbourhood diffeomorphic to $M \times [0, \epsilon)$, called the collar of M. So if two manifolds have a diffeomorphic component on the boundary, then we can glue these manifold along this boundary component to get a new smooth manifold. For more details see [28, Chapter 7].

Each such set $\mathfrak{N}_n$ is an abelian group with addition given by

$$[M] + [N] := [M \coprod N]$$

and the empty manifold of dimension n as unit. This is well-defined as $[M \coprod N] = [M' \coprod N]$ if M and M' are bordant, using the disjoint union of the cobordism between M and M' with the cylinder over N.

If we have two closed manifolds M and N of dimension m and n respectively, then we get an $m+n$-dimensional manifold $M \times N$. Note that this product respects the relation of bordism. Indeed, if M and M' are bordant by a cobordism W, then the boundary of $W \times N$ equals

$$\partial(W \times N) = \partial W \times N \coprod W \times \partial N = \partial W \times N = (M \coprod M') \times N = (M \times N) \coprod (M' \times N).$$

So we arrive at the following definition.

Definition 1.2. Let $\mathfrak{N}_*$ be the graded group

$$\bigoplus_{n=0}^{\infty} \mathfrak{N}_n.$$

Using the product given above, we get a commutative associative graded ring. This is called the *unoriented bordism ring.*

It is easy to see that for all $\alpha \in \mathfrak{N}_*$ the class 2α does in fact equal zero. This is because for each manifold M, the class $2[M]$ is represented by $M \coprod M$ which is a boundary for $M \times I$. This gives $\mathfrak{N}_*$ the natural structure of a $\mathbb{Z}_2$-algebra, each graded part is in particular a direct sum of a number of copies of $\mathbb{Z}_2$. This number is explicitly known.

Theorem 1.3. *[60, Théorème IV.12] The graded $\mathbb{Z}_2$-algebra $\mathfrak{N}_*$ is a polynomial algebra generated by a unique x_i of degree i for all i not of the form $2^t - 1$. So*

$$\mathfrak{N}_* = \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, x_9, \ldots].$$

This theorem agrees in degree 0 with the observation that one point cannot bound a 1-dimensional manifold. It also shows that the only closed manifold of dimension 1, namely S^1, is a boundary of a compact closed manifold of dimension 2, for example the closed disc.

In dimension 2 this theorem shows that there exists a surface which does not bound a 3-dimensional manifold. One such example is given by the real projective plane $\mathbb{RP}^2$. So each closed compact surface is either bordant to $\mathbb{RP}^2$ or it is the boundary of a compact 3-dimensional manifold.

We also get the following interesting result in the degree 3: $\mathfrak{N}_3 = 0$, which proves that every closed 3-dimensional manifold is the boundary of a 4-dimensional manifold.

To determine to which bordism class a manifold belongs, the Stiefel-Whitney numbers of the tangent bundle of M are of particular interest. In this case we will simply speak of the Stiefel-Whitney numbers of M. These numbers relate a manifold to its bordism class by the following theorem by Pontrjagin.

Theorem 1.4. *[44] If W is an $(n+1)$-dimensional manifold with boundary M, then all the Stiefel-Whitney numbers of M are zero.*

This allows one to show that two manifolds are not bordant, by producing a Stiefel-Whitney number which differs for these spaces. We do also have the converse statement, due to Thom [60, Théorème IV.3].

Theorem 1.5. *Two smooth closed manifolds of the same dimension are bordant if and only if their Stiefel-Whitney numbers are the same.*

One can for example show that all Stiefel-Whitney numbers of $\mathbb{RP}^{2t-1}$ are zero, and so each real projective space of odd dimension must be the boundary of a smooth manifold.

It is relatively easy to turn the graded algebra $\mathfrak{N}_n$ into a relative theory.

Definition 1.6. Let X be a smooth closed manifold. Two maps $M \to X$ and $N \to X$ for M and N smooth closed manifolds of dimension n are said to be *bordant over* X, if there exists a cobordism over X such that the boundary maps restrict to $M \coprod N \to X$. That is, there is an $(n+1)$-dimensional manifold $W \to X$ over X, such that the boundary of W equals $M \coprod N$ and the map $W \to X$ restricts to the structure map of M and N on the boundary.
Let $\mathfrak{N}_n(X)$ denote the free group generated by classes of n-dimensional manifolds over X, modulo bordism over X. A group structure can be defined as before, making

$$\mathfrak{N}_*(X) = \bigoplus_{n=0}^{\infty} \mathfrak{N}_n(X)$$

into a graded group, called the *unoriented bordism ring over* X.

As each element in this group has order 2 we have a natural structure of a $\mathbb{Z}_2$-module and hence a vector space over the field of two elements. We can find a basis for a specific manifold once the homology is known.

Theorem 1.7. *[8, Theorem 1.6] Let X be of the homotopy type of a CW-complex. Then we have the following isomorphism of $\mathbb{Z}_2$-modules*

$$\mathfrak{N}_*(X) = H_*(X, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathfrak{N}_*.$$

Note that we are back at the unoriented bordism ring if we take X to be a point.

Now by adding structure to a cobordism, such that this additional structure restricts to the boundary gives in general a new bordism group. For example we could consider oriented cobordisms, then we would say that two smooth closed oriented manifolds M and N of dimension n are bordant if $M \coprod \overline{N}$ is the boundary of some $(n+1)$-dimensional oriented manifold. We could even do all this over a fixed manifold X, giving us the graded group $\Omega_*^{\mathrm{or}}(X)$ called *oriented bordism.*

1.2 B-bordism

We can generalise the examples in the previous section. To that end we will assume that each n-dimensional manifold M is embedded in $\mathbb{R}^{n+r}$. Whitney [64, Theorem 5] proved that it is possible to take $r = n$, but for our purposes it will be convenient to let the dimension of the Euclidean space be arbitrary. Such an inclusion gives a normal bundle of rank r on M. Of course this normal bundle depends on the embedding of M in a Euclidean space. For example, we could simply look at the embedding

$$M \to \mathbb{R}^{n+r} \to \mathbb{R}^{n+r+t}$$

by adding t dimensions. Trivially, our new normal bundle is the direct sum of the old one and the trivial bundle on M of rank t. This is a typical example of the following theorem.

Theorem 1.8. *[43, Remark 9.13] Let M be embedded in $\mathbb{R}^{n+r}$ and $\mathbb{R}^{n+s}$. Then for large enough t we have that the normal bundles of the embeddings*

$$M \to \mathbb{R}^{n+r} \to \mathbb{R}^{n+r+(s+t)} \quad \text{and} \quad M \to \mathbb{R}^{n+s} \to \mathbb{R}^{n+s+(r+t)}$$

are isomorphic as bundles over M.

Put differently, for two normal bundles η_1 and η_2 for different embeddings there exist trivial bundles τ_1 and τ_2 such that the bundles $\eta_1 \oplus \tau_1$ and $\eta_2 \oplus \tau_2$ are isomorphic. One says that the normal bundles are stably isomorphic, which is in fact an equivalence relation on bundles on M.

Now note that in the case of oriented bordism one gets a map $M \to BSO(n+r)$, independent of the embedding in $\mathbb{R}^{n+r}$ for large enough r. These maps are easily seen to be compatible and we get a map $M \to BSO = \lim_{\to} BSO(r)$ which also does not depend on the embedding.

We can repeat this construction for any family of fibrations $\pi_r : B_r \to BO(r)$ for large enough r with maps $b_r : B_r \to B_{r+1}$, such that the following square commutes

$$\begin{tikzcd} B_r \arrow[r, "b_r"] \arrow[d, "\pi_r"] & B_{r+1} \arrow[d, "\pi_{r+1}"] \\ BO(r) \arrow[r] & BO(r+1) \end{tikzcd}$$

We will need compatible maps $M \to B_r$ for sufficiently large r. Such information is encoded by a single map $M \to B$, where we use the limit $B = \lim_{\to} B_r$. By naturality of direct limits, we get a map $B \to BO = \lim_{\to} BO(r)$ and we can show this is a fibration as well. So any lifting μ in

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ M \arrow[r] \arrow[ur, dashed, "\mu"] & BO \end{tikzcd}$$

belongs to the same homotopy class. Such a class is called a *B-structure* on M.

Note that if we have a B-structure on a manifold M, then we get a natural B-structure on a closed submanifold N by restricting the map to N. This allows us to define an inverse of a B-structure. So let $\mu : M \to B$ be a manifold with a B-structure. Define $i_0 : M \to M \times I$ by taking the second coordinate equal to 0. As a manifold with boundary, we have an embedding $M \times I \to \mathbb{R}^{n+r} \times \mathbb{R}_+$ such that the intersection of $M \times I$ with $\mathbb{R}^{n+r} \times \{0\}$ equals $\partial(M \times I) = M_0 \coprod M_1$, where we use the subscripts to distinguish the two copies of M. This gives us the following commutative square, where we get a diagonal since $B \to BO$ is a fibration

$$\begin{tikzcd} M \arrow[r, "\mu"] \arrow[d, "i_0"] & B \arrow[d, "\pi"] \\ M \times I \arrow[r] \arrow[ur, dashed, "\mu_{M \times I}"] & BO \end{tikzcd}$$

This gives a B-structure on $M \times I$. The two copies of M on the boundary now get their own B-structure, as they are closed submanifolds of $M \times I$. The B-structure μ_0 on M_0 is obviously the same as μ on M. However, μ_1 on M_1 can be a different B-structure. We call this the *inverse B-structure* and we denote it by $-\mu$. It is clear that taking the inverse of $-\mu$ gives back μ.

Now we say that two n-dimensional closed B-manifolds (M, μ) and (N, ν) are B-bordant if there is an $(n+1)$-dimensional B-manifold W, such that the boundary is isomorphic as a B-manifold to $(M \coprod N, \mu \coprod -\nu)$.

Using the notions of B-manifolds, i.e. manifolds with a B-structure, and B-cobordisms we can construct a bordism theory analogous to what we did before.

Definition 1.9. Let B be the limit of a family of fibrations $B_r \to BO(r)$. Then we define *B-bordism* Ω_*^B as the set of pairs $[M, \mu]$ of B-bordism classes graded by the dimension of M. It is fact a graded group with addition given by taking the disjoint union of two B-manifolds. The inverse is given by

$$-[M, \mu] = [M, -\mu].$$

For any manifold X we can consider triples (M, μ, f), where (M, μ) is a B-manifold and f is a map $M \to X$. Such a triple is called an *X-structure* where we assume the fibrations to be understood. An *inverse of an X-structure* is constructed by taking the inverse B-structure.

A cobordism between two X-structures (M,μ,f) and (N,ν,g) is a B-manifold $W \to X$ with boundary $M \coprod N$, such that the X-structure on W restricts to the X-structure on M and to the inverse X-structure on N. The X-structures on N and M are said to be X-bordant.

Definition 1.10. We will write $\Omega_*^B(X)$ for the set of X-bordant classes of X-structures. This is once again a graded group, called *B-bordism over X*.

A quite common case of such a family of fibrations occurs if we have topological subgroups $G_r \to O_r$, such that G_r embeds naturally in G_{r+1}. Then we can simply take $B_r = BG_r$ with limit BG. In this case we will write $\Omega_*^G(X)$ for $\Omega_*^{BG}(X)$.

One can check that if we take $B_r = BSO(r)$ we do arrive at oriented bordism briefly described in the previous section. Also, if one would take $B_r = BO(r)$ then it is easily seen that the above discussion is empty, and we simply have the unoriented bordism theory. We will later look at the interesting case of the fibrations given by the inclusion $U_n \to O_{2n}$.

1.3 The Thom-spectrum and B-cobordism

We saw that unoriented bordism is simply a base extension of homology. This proves that unoriented bordism is a generalised homology theory, in the sense that the Eilenberg-MacLane axioms are still satisfied except for the dimension axiom. This is even more generally the case.

Theorem 1.11. *Every B-bordism theory is a generalised homology theory.*

The proof is by explicitly showing that Ω_*^B is represented by a spectrum, see Appendix B.4. For that purpose we will need the following construction.

Definition 1.12. Let M be a compact manifold with a bundle ξ of rank r. Any embedding of the total space $T(\xi)$ gives ξ an inner product, so we can take $D^\circ(\xi)$ to be the vectors in the fibres of ξ which have length strictly less than 1. This is a fibre space on M with fibre the open unit disc. The one-point compactification of the total space is called the Thom-space of ξ and denoted by $M(\xi)$.
One can view this as the closed disc bundle $D(\xi)$, i.e. we include the unit vectors in each fibre, where we identify all vectors of unit length.

Note that the map $B_r \to BO(r)$ gives a rank r bundle on B_r, which we will denote by β_r. The normal bundle of M in $\mathbb{R}^{n+r}$ is the pullback of β_r along the maps $M \to B_r$ belonging to the B-structure of M. The corresponding Thom-space of β_r will be denoted by MB_r.

One can check that the construction of the Thom-space is natural in the sense that $M(\eta \times \xi) = M(\eta) \wedge M(\xi)$ and that bundle maps become maps of the respective Thom-spaces, see for example [59, Definition 12.27 & Proposition 12.28].

As for any embedding $M \to \mathbb{R}^{n+r}$ we can embed the total space of the normal bundle η in $\mathbb{R}^{n+r}$, such that the zero-section $M \to E(\eta)$ composed with $E(\eta) \to \mathbb{R}^{n+r}$ equals the given embedding of M. This embedding of the total space looks like a thickened version of M. [28, Section 4.5].

So in particular we get an open subset $E(\eta) \subseteq \mathbb{R}^{n+r}$. After taking the one-point compactification we get a reversed map

$$S^{n+r} \to M(\eta).$$

If we compose this map with the natural $M(\eta) \to MB_r$, we see that a B-manifold M determines a homotopy class in $\pi_{n+r}(MB_r)$. We can extend this to manifolds with an X-structure $f : M \to X$, by considering the following Cartesian square

$$\begin{array}{ccc} T(\eta) & \xrightarrow{(T(\eta), f\circ\rho)} & T(\beta_r) \times X \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow \\ M & \xrightarrow{(\mu_r, f)} & B_r \times X \end{array}$$

Since we can view $T(\beta_r) \times X$ as the total space of the direct sum of the bundles β_r over B_r and the trivial bundle $\underline{\mathbb{R}}^0$ over X, we get a bundle map from η to $\beta_r \times \underline{\mathbb{R}}^0$ on $B_r \times X$, and hence by naturality of the Thom-spaces a map

$$M(\eta) \to M(\beta_r \times \underline{\mathbb{R}}^0) = M(\beta_r) \wedge M(\underline{\mathbb{R}}^0) = MB_r \wedge X_+.$$

Here we used that the one-point compactification of an already compact space is constructed by adding a disjoint point. We will still use the notation X_+ for such spaces.

So now we get a map $S^{n+r} \to M(\eta) \to MB_r \wedge X_+$ and hence a class in

$$\pi_{n+k}(MB_r \wedge X_+).$$

Obviously this class depends on the chosen embedding. So let us examine how this class relates to the class of the normal bundle $\eta \oplus \underline{\mathbb{R}}$ given by the embedding $M \to \mathbb{R}^{n+r} \to \mathbb{R}^{n+r} \times \mathbb{R}$. We will see that it is simply the suspension of the map $S^{n+r} \to MB_r \wedge X_+$ composed with a natural map. This natural map will be the corresponding map

$$\vartheta : M(\beta_r \oplus \underline{\mathbb{R}}) \to M(\beta_{r+1}) = MB_{r+1}.$$

of Thom-spaces of the bundle map in the square

$$\begin{tikzcd}
T(\beta_r \oplus \underline{\mathbb{R}}) \arrow[r] \arrow[d] & T(\beta_{r+1}) \arrow[d] \\
B_r \arrow[r] & B_{r+1}
\end{tikzcd}$$

classifying the bundle $\beta_r \oplus \underline{\mathbb{R}}$ on B_r of rank $r+1$. We can simplify the first Thom-space $M(\beta_r \oplus \underline{\mathbb{R}})$ even further by noting that the bundle $\beta_r \oplus \underline{\mathbb{R}}$ is the same as the product of the bundle β_r on B_r and the trivial bundle $\underline{\mathbb{R}}$ of rank 1 over a point pt. So this gives

$$M(\beta_r \oplus \underline{\mathbb{R}}) = M(\beta_r \times \underline{\mathbb{R}}_*) = M(\beta_r) \wedge M(\underline{\mathbb{R}}_*) = MB_r \wedge S^1 = \Sigma MB_r$$

and we get a map

$$\vartheta : \Sigma MB_r \to MB_{r+1}.$$

These maps give the family of spaces MB_r a very important structure.

Theorem 1.13. *The Thom-space of a bundle over a CW-complex is also a CW-complex. We can choose representatives for B_r which are CW-complexes, such that B_r is a subcomplex of B_{r+1}. This gives $\vartheta : \Sigma M_r \to M_{r+1}$ also the structure of a CW-subcomplex. Hence, the maps ϑ turn MB_r into a spectrum MB, called the Thom-spectrum.*

Using these spectrum maps, we get the following commutative diagram

$$\begin{tikzcd}
\Sigma S^{n+r} \arrow[r] \arrow[d, "\cong"] & \Sigma M(\eta) \arrow[r] \arrow[d, "\cong"] & \Sigma MB_r \wedge X_+ \arrow[d, "\vartheta\wedge\mathrm{Id}"] \\
S^{n+r+1} \arrow[r] & M(\eta \oplus \underline{\mathbb{R}}) \arrow[r] & MB_{r+1} \wedge X_+
\end{tikzcd}$$

which gives a natural map

$$\pi_{n+r}(MB_r \wedge X_+) \to \pi_{n+r+1}(MB_{r+1} \wedge X_+).$$

Using these maps we can take the direct limit and get an element in

$$\lim_{r\to\infty} \pi_{n+r}(MB_r \wedge X_+).$$

This is exactly the homotopy of the spectrum $MB \wedge X_+$

$$\pi_n(MB \wedge X_+) := [\Sigma^n S^0, MB \wedge X_+]$$

so this object has a natural group structure. One can check that this construction is well-defined on B-bordant classes, respects the group structures of the bordism and the homotopy groups and is even invertible.

Theorem 1.14. *The construction above gives an isomorphism of graded groups*

$$\Omega_*^B(X) \cong \pi_*(MB \wedge X_+).$$

By Theorem B.14 we immediately get a proof for Theorem 1.11.

Of course, for each generalised homology theory, there is also a generalised cohomology theory corresponding to the same spectrum.

Definition 1.15. Let B_r be a compatible family of fibrations as before. We define B-cobordism Ω_B^* as the generalised cohomology theory coming from the Thom-spectrum, i.e. for a CW-complex X we have

$$\Omega_B^*(X) = [\Sigma^{-*} S^0 \wedge X_+, MB].$$

The coefficient groups of these homology and cohomology theories are

$$\Omega_*^B = [\Sigma^* S^0, MB] \quad \text{and} \quad \Omega_B^* = [\Sigma^{-*} S^0, MB].$$

They satisfy the following obvious relation $\Omega_n^B = \Omega_B^{-n}$ for all n.

If the B_r have some geometrical interpretation, then for any B-bordism class we have a manifold with such geometrical structure on its stable normal bundle. The geometry of B-cobordism theories is however lost. We will see that cobordism classes in the case of $B = BU$ do however have a geometric interpretation.

1.4 Complex cobordism

We will now specialize to the case $B = BU$. This limit comes from the family of inclusions

$$\begin{tikzcd}
U_n \arrow[r, equal] \arrow[d] & U_n \arrow[r] \arrow[d] & U_{n+1} \arrow[d] \\
O_{2n} \arrow[r] & O_{2n+1} \arrow[r] & O_{2n+2}
\end{tikzcd}$$

which give fibrations $BU(n) \to BO(2n)$. By the limiting process, we do not lose information by leaving out the odd terms.

The corresponding homology and cohomology theories are called complex bordism and complex cobordism. One can show that complex bordism can be seen as classes of smooth closed manifolds with an almost complex structure on the stable normal bundle. The classes of course come from the boundary of cobordisms with an almost complex structure on its stable normal bundle. Like with unoriented bordism, the class of a manifold can be uniquely determined by computing the relevant characteristic classes.

Theorem 1.16. *The class of a smooth closed manifold M of dimension n with an almost complex stable normal bundle in Ω_*^U is uniquely determined by its Chern numbers*

$$c_1(\tau)^{d_1}c_2(\tau)^{d_2}\dots c_n(\tau)^{d_n}[M]$$

for all tuples $(d_1, d_2, \dots, d_n)$ such that $\sum id_i = n$.

To view complex cobordism as classes of maps as done by Quillen in [46], we will need the following notion.

Definition 1.17. Let $f : Y \to X$ be a map of manifolds.
Suppose that Y is connected, then $\dim f(y) - \dim y$ is constant for all $y \in Y$. If it is even we can define a *complex orientation* on a map f as follows: a complex orientation on f is a factorization of f as

$$Y \xrightarrow{i} T(\xi) \xrightarrow{\rho} X,$$

where ξ is a complex bundle over X with projection ρ of its total space to X and i is an embedding of Y in this total space, together with a complex structure on the normal bundle η of Y in $T(\xi)$. We say two complex orientations $Y \xrightarrow{i_1} T(\xi_1) \longrightarrow X$ and $Y \xrightarrow{i_2} T(\xi_2) \longrightarrow X$ are equivalent if there exists a complex bundle ξ on X, such that ξ_1 and ξ_2 are subbundles of ξ, such that the composite embeddings $Y \to T(\xi_i) \to T(\xi)$ come from a map $\bar{i} : Y \times I \to T(\xi)$, such that $\bar{i}_t : Y \to T(\xi), y \mapsto \bar{i}(y, t)$ is an embedding for all t, which coincides with i_1 and i_2 for respectively $t = 0$ and $t = 1$.

If the relative dimension is odd, then we define a complex orientation on f as a complex orientation of the map

$$Y \to X \to X \times \mathbb{R}$$

where we embedded X in $X \times \mathbb{R}$ by putting the second coordinate equal to zero.

A complex orientation on $f : Y \to X$ for general Y is given by a complex orientation on each component of Y. Equivalence of complex orientation in the last cases follows from defined equivalence for maps of pure even dimension.

There are several ways to construct a map with a complex orientation from other such maps. Of course, for two maps $Y_1 \to X$ and $Y_2 \to X$ with a given complex orientation we can define a natural complex orientation on $Y_1 \coprod Y_2 \to X$. Another general construction exists when a map $g : X' \to X$ is transversal to a map $f : Y \to X$ with a given complex orientation. In this case, there is a canonical complex orientation on the base extension of f by g. That is, we construct a complex orientation on $X' \times_X Y \to X'$ simply by picking a representative $Y \to T(\xi) \to X$, in the case of pure even dimension, and consider the composition of Cartesian squares

$$\begin{tikzcd}
X' \times_X Y \arrow[r] \arrow[d, "i'"] & Y \arrow[d, "i"] \\
T(\xi') \arrow[r] \arrow[d, "\rho'"] & T(\xi) \arrow[d, "\rho"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

It is easily checked that the normal bundle of $X' \times_X Y$ in $T(\xi')$ inherits a complex structure from the normal bundle of Y in $T(\xi)$, which is well-defined on classes of complex orientations. The case of pure odd dimension is handled similarly.

Now consider an oriented cobordism $f : W \to X$ with boundary $M \coprod \overline{N}$. By a standard result of differential topology [28, page 156] there exists a map $W \to \mathbb{R}$, such that the image equals I and the inverse images of 0 and 1 are respectively M and N. This gives a map $\tilde{f} : W \to X \times I$ such that the inclusion $X \to X \times I$ at any constant $c \in I$ is obviously transversal to $\tilde{f}$. This allows one to pull back the complex-orientation on $\tilde{f}$ to complex orientations on $M \to X$ and $N \to Y$. In this case we say that these maps are *cobordant*.

Theorem 1.18. *[46, Proposition 1.2] Let X be a manifold. Complex cobordism $\Omega_U^d(X)$ over X is isomorphic to cobordism classes of complex-oriented maps of pure dimension $-d$.*

Quillen needs the assumption that these maps are proper, but since we are dealing with compact Hausdorff spaces, any map is proper.

In this new formulation, one can give complex cobordism a new interpretation of pushforwards, pullbacks and even a ring structure.

Definition 1.19 ([46, Section 1])**.** We can define the following structure on $\Omega_U^n(X)$ in terms of cobordism classes of complex-oriented maps. To that end let $f : M \to X$ and $f' : M' \to X'$ be maps of compact closed manifolds with a given complex orientation.

(i) Let $g : Y \to X$ be any map. There exists a map g' homotopic to g which is transversal to f. So we get a well-defined pullback of g by
$$g^* : \Omega_U^*(X) \to \Omega_U^*(Y), \quad [M \to X] \mapsto [M \times_X Y \to Y]$$
using the pushout square of g' and f.

(ii) If $h : X \to Y$ is a complex-oriented map of pure dimension d, then we get a well-defined pushforward
$$h_* : \Omega_U^*(X) \to \Omega_U^{*-d}(Y), \quad [M \to X] \mapsto [M \to X \to Y].$$

(iii) We can add two cobordism classes $[M \to X]$ and $[N \to X]$ by taking the disjoint union of M and N, and endowing it with the natural complex orientation. This sum has an inverse by representing the complex-orientation of a map $[M \to X]$ by a factorization $M \to T(\xi) \to \mathbb{C}^s \times X \to X$ where we use that X is compact to embed ξ in a trivial complex bundle over X, see for example Swan's theorem in [51, Theorem 1.6.5]. Now we define the *inverse* as the same factorization $M \to \mathbb{C}^s \times X \to X$ with the same complex structure on the normal bundle of M in $\mathbb{C}^s \times X$, but with the complex structure on $\mathbb{C}^s \times X$ given by $J(z_1, z_2, \ldots, z_{n-1}, z_n) = (iz_1, iz_2, \ldots, iz_{n-1}, -iz_n)$.

(iv) Also two classes $[M \to X] \in \Omega_U^d(X)$ and $[M' \to X'] \in \Omega_U^e(X')$ give a natural class
$$[M \to X] \times [M' \to X'] = [M \times M' \to X \times X'] \in \Omega_U^{d+e}(X \times X').$$
This is called the *exterior product*.

(v) Pulling back via the diagonal $\Delta : X \to X \times X$ and precomposing with the exterior product, gives a map
$$\Omega_U^d(X) \otimes \Omega_U^e(X) \xrightarrow{\times} \Omega_U^{d+e}(X \times X) \xrightarrow{\Delta^*} \Omega_U^{d+e}(X)$$
which, together with the natural map
$$\Omega_U^d(pt) \otimes \Omega_U^e(X) \xrightarrow{\times} \Omega_U^{d+e}(pt \times X) \cong \Omega_U^{d+e}(X),$$
turns $\Omega_U^*(X)$ into a graded Ω_U^*-algebra.
The identity for this algebra is given by the class of the identity map on X, which we will denote, by abuse of notation, by $\mathrm{Id_X}$ as well.

Quillen [46] noticed that the following properties are also satisfied by these data on complex cobordism.

Theorem 1.20. *Complex cobordism $\Omega_U^*(X)$ satisfies the following properties.*

(i) *Consider a map $g : Y \to Z$, then g^* depends only on the homotopy class of g.*

(ii) *Let $f : X \to Z$ be complex oriented, then for any $g : Y \to Z$ transverse to f we get a Cartesian square*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

If we give f' the complex orientation coming from f we have

$$g^* \circ f_* = f'_* \circ g'^*.$$

(iii) *If two composable morphisms $h : X \to Y$ and $g : Y \to Z$ have complex orientations, the given complex orientation on $g \circ h$ gives the following identity of maps*

$$g_* \circ h_* = (g \circ h)_*.$$

Note that the first statement does in fact say that pullbacks are well-defined. It also shows that two homotopy equivalent spaces give isomorphic complex cobordism rings. In particular, we get that complex cobordism of the total space of any vector bundle equals complex cobordism of the base space.

1.5 Euler-classes and the formal group law

Let ξ be a real vector bundle of rank r on a manifold X. A general section of ξ which is not identically zero determines a submanifold Z of X of codimension r, in particular it determines a class $[Z \to X]$ in complex cobordism. Using the structure of complex cobordism we can describe this class in the case of a complex structure.

Definition 1.21. Let X be a manifold with a complex vector bundle ξ of rank r. Let $s : X \to T(\xi)$ denote the zero-section of ξ. Then s has a natural complex orientation and we define the *Euler-class* of ξ by

$$e(\xi) := s^* s_*(\mathrm{Id_X}) \in \Omega_U^{2r}(X).$$

Note that if we follow through our definitions we see that this class is the zero-locus of any map $X \to T(\xi)$ which is transverse to the zero-section inclusion $X \to T(\xi)$. If we can take this map to be a section, then we get indeed the class of the zero-locus of a general section of ξ.

These classes are very useful to express complex cobordism of any complex projective space, or even complex projective bundles.

Theorem 1.22. *[9](Conner and Floyd) Let ξ be a complex vector bundle of rank $q+1$ on a manifold X. The total space of the bundle $\pi : \mathbb{P}(\xi) \to X$ of lines in ξ with fibre $\mathbb{CP}^q$ has a natural line bundle $\mathcal{O}(1)$, where each fibre is the one-dimensional complex vector space of functionals on the line in the total space of ξ. Let λ denote the Euler-class of this line bundle.*
The ring homomorphism

$$\pi^* : \Omega_U^*(X) \to \Omega_U^*(\mathbb{P}(\xi))$$

makes $\Omega_U^*(\mathbb{P}(\xi))$ into an $\Omega_U^*(X)$-module. This module is in fact free with basis $1, \lambda, \lambda^2, \ldots, \lambda^q$.

One can show that the Euler-class of the direct sum of vector bundles is simply the product of the respective Euler-classes. This corresponds the fact that the zero-locus of a general section of $\xi \oplus \xi'$ is the intersection of the zero-loci of the two separate bundles. Note that this also agrees with the correct codimensions of the respective submanifolds.

The Euler-class of a tensor product is a little more complicated, but for one-dimensional vector bundles it is governed by a formal group law in the sense of Definition A.1.

Theorem 1.23 ([46, Proposition 2.7]). *There exist unique classes $a_{i,j} \in \Omega_U^{2(1-i-j)}$ such that the formal power series*

$$F(u,v) = \sum_{i,j\geq 0} a_{i,j} u^i v^i$$

satisfies

$$e(\lambda_1 \otimes \lambda_2) = F(e(\lambda_1), e(\lambda_2))$$

for all one-dimensional vector bundles λ_1 and λ_2 on a manifold X.
The map $\mathbb{L}^* \to \Omega^{2*}$ classifying this formal group law is in fact an isomorphism, and hence F is the universal formal group law.

Note that the sum is always finite as complex cobordism is concentrated in degree $\leq \dim X$ by Theorem 1.18.

Proof. We will sketch the proof of the existence of the formal group law. This is done by considering the line bundle

$$\operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1)$$

on $\mathbb{CP}^n \times \mathbb{CP}^n$. By Theorem 1.22 we get that

$$\Omega_U^n(\mathbb{CP}^n \times \mathbb{CP}^n) = \mathbb{Z}[\lambda_1, \lambda_2]/(\lambda_1^n, \lambda_2^n)$$

where λ_i is the Euler-class of the ith factor. So by the same theorem there exist unique $a_{i,j}^n$ such that

$$e(\operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1)) = \sum_{0\leq i,j\leq n} a_{i,j}^n \lambda_1^i \lambda_2^j.$$

One can now prove that the coefficients $a_{i,j}^n$ do in fact stabilize if $n \to \infty$, which gives us a formal power series.

As every line bundle is the pullback of $\mathcal{O}(1)$ on some $\mathbb{CP}^n$ we only need to check unity, commutativity and associativity for the line bundles $\mathcal{O}(1)$, $\operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1)$ and $\operatorname{pr}_1^* \mathcal{O}(1) \otimes \operatorname{pr}_2^* \mathcal{O}(1) \otimes \operatorname{pr}_3^* \mathcal{O}(1)$ on respectively $\mathbb{CP}^n$, $\mathbb{CP}^n \times \mathbb{CP}^n$ and $\mathbb{CP}^n \times \mathbb{CP}^n \times \mathbb{CP}^n$ for large enough n.

The proof for the universality was given by Quillen in [46]. □

We know the structure of the Lazard ring by Theorem A.4 and hence we also know the structure of Ω_U^*. It is possible to give explicit generators of degree $-2i$ for all natural i in terms of smooth manifolds, but it is far from trivial. We do however have the following useful theorem.

Theorem 1.24 ([46, Theorem 6.5]). *Any product of complex projective spaces $\mathbb{CP}^k$ is not null-cobordant for any $n \geq 0$ and hence they generate Ω_U^* over the rationals, that is*

$$\Omega_U^* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^k \mid k \geq 1]$$

where the degree of $\mathbb{CP}^k$ is obviously equal to $-2k$.

Quillen did not only prove that the formal group law on complex cobordism is universal, but that complex cobordism is itself universal as a cohomology theory with some additional structure.

Theorem 1.25 ([46, Proposition 1.10])**.** *Consider a functor h from the category of smooth compact manifolds to the category of graded rings, with the additional information of pushforwards for complex oriented maps, such that the properties of Theorem 1.20 and Theorem 1.22 are satisfied. Then we have a unique natural transformation*

$$\vartheta_h(X) : \Omega_U^*(X) \to h^*(X)$$

which commutes with the complex-oriented pushforwards, such that $\vartheta_h(pt) : \mathbb{L}^* \to h^*(pt)$ determines the Euler-classes in h^* of tensor products of line bundles in terms of the Euler-classes of the respective line bundles when defined analogously to Definition 1.21.

Such a theory as described by the conditions on h is called a *complex oriented cohomology theory.* Hence complex cobordism is the universal such theory.

Chapter 2

Invariant graded groups for schemes

In algebraic topology the theories of homology, cohomology and cobordism assign a graded group to each topological space. These groups contain in general much information about this specific space. The more structure a space has, e.g. a C^∞-manifold or a CW-complex, the more structure can be defined on the group and more results can be derived. This can give even more information about the space.

However, for the study of a scheme X we would like to consider other such theories. The theories we represent below are uniquely suited for this purpose, as they express the structures of the closed subvarieties of X and of the quasi-coherent sheaves on X as a graded group.

2.1 Chow group

An object widely used in algebraic geometry is the Chow group, which has many applications in enumerative geometry [13]. This group shows similarities in its definition and its properties to both complex cobordism and cohomology in algebraic topology. The group is a quotient of the free abelian group generated by classes of subvarieties, whose dimensions provide a grading. The quotient comes from an equivalence relation on subvarieties of the same dimension.

Definition 2.1. Let X be any scheme. A *k-cycle* on X is an element of the free abelian group

$$\mathcal{Z}_k(X)$$

generated by subvarieties of X of dimension k.
We will also be interested in the group of all cycles on X given by

$$\mathcal{Z}_*(X) = \bigoplus_{k \geq 0} \mathcal{Z}_k(X).$$

Any closed subscheme Z of X defines a cycle on X by

$$\sum \eta_i Z_i$$

where Z_i are the irreducible components of Z and η_i is the multiplicity of Z_i in Z, i.e. the length of the local ring $\mathcal{O}_{Z_i,Z}$.

We can now define the notion of rational equivalence on these classes, which shows an obvious similarity to cobordisms.

Definition 2.2. Let W be a subscheme of $X \times \mathbb{P}^1$ of dimension $k+1$, not contained in a single fibre over $\mathbb{P}^1$. The fibres over the two points, say 0 and ∞, of the composition $W \to X \times \mathbb{P}^1 \to \mathbb{P}^1$ are naturally subschemes of X and define elements W_0 and W_∞ in $\mathcal{Z}_k(X)$. Let $\mathrm{Rat}_k(X)$ be the subgroup of $\mathcal{Z}_k(X)$ generated by the elements $W_0 - W_\infty$ for all such subschemes W.
The *Chow group* is defined as

$$\mathrm{CH}_*(X) = \bigoplus_{k \geq 0} \mathcal{Z}_k(X)/\operatorname{Rat}_k(X).$$

The relation induced on $\mathcal{Z}_*$ by this quotient is called *rational equivalence*. We will denote the class of a subscheme Z in $\mathrm{CH}_*(X)$ by $[Z]$.

For certain types of morphisms, it is possible to construct pullbacks or pushforwards.
Let $f : X \to Y$ be a morphism of schemes and Z a subvariety of X. If f reduces the dimension of Z, that is $\dim f(Z) < \dim Z$, then we define $f_*(Z) = 0 \in \mathcal{Z}_*(X)$. If f respects the dimension of Z, that is $\dim f(Z) = \dim Z$, then the function field of Z is a finite field extension of $f(Z)$ and we define

$$f_*(Z) = [K(Z) : K(f(Z))]\overline{f(Z)} \in \mathcal{Z}_*(Y).$$

This extends to a homomorphism $f_* : \mathcal{Z}_*(X) \to \mathcal{Z}_*(Y)$.

For a flat morphism $g : X \to Y$ of relative dimension d, we have an obvious pullback by $g^*[Z] = [g^{-1}Z]$ for $[Z] \in \mathcal{Z}_*(Y)$. Note that it increases the dimension by d.

Lemma 2.3 ([18, Section 1.4 & 1.7])**.** *Let $f : X \to Y$ be a proper morphism and $g : X \to Y$ a flat morphism of relative dimension d. The pushforward of f respects rational equivalence and hence descends to a graded group homomorphism of Chow groups*

$$f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y).$$

These map are functorial, in the sense that for two composable proper morphisms f and f' we have that $(f \circ f')_* = f_* \circ f'_*$. The pullback

$$g^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+d}(X)$$

is well-defined and is a homomorphism of graded groups as well. These pullbacks, just as the pushforwards, behave well under composition.

These two types of maps do behave well together, as one can see in the following lemma.

Lemma 2.4 ([18, Proposition 1.7.1])**.** *Let $f : X \to Z$ be a flat morphism of relative dimension d and $g : Y \to Z$ a proper morphism. Consider the following Cartesian square:*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

We have the identity $g^* f_* = f'_* g'^*$ of maps $\mathrm{CH}_*(X) \to \mathrm{CH}_{*+d}(Y)$.

We are able to define pullbacks for a more general class of morphisms. We will first need Theorem 3.3a from [18] which is the algebraic analogue of the fact that the cohomology of any topological space X and $X \times \mathbb{R}^1$ are isomorphic, as the spaces are homotopic.

Theorem 2.5 (Homotopy invariance)**.** *Consider the projection $\mathrm{pr}_1 : X \times \mathbb{A}^r \to X$ for any scheme X. The pullback induces an isomorphism of graded groups*

$$\mathrm{pr}_1^* : \mathrm{CH}_*(X) \to \mathrm{CH}_{*+r}(X \times \mathbb{A}^r).$$

Even for a vector bundle $\mathcal{E}$ of rank r over X, which is locally of the form $U \times \mathbb{A}^r$, we have an isomorphism $\pi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_{*+r}(\mathcal{E})$ induced by the projection π. Pulling back via π is in fact defined, because the projection is a flat morphism.

Note that this gives us that the Chow group for any affine space $\mathbb{A}^n$ is concentrated in degree n, where it equals $\mathbb{Z}$ generated by the class of $\mathbb{A}^n$.

The isomorphism described by the homotopy invariance property completes the following construction.

Definition 2.6. Let $i : X \to Y$ be a regular embedding of codimension d and Z a subvariety of Y of dimension k. Define the inverse image W of Z under i, that is we have the following Cartesian square

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "j'"] & Z \arrow[d, "j"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

The pullback $\mathcal{N} = (j')^*\mathcal{N}_{X/Y}$ of the normal bundle of X in Y is a vector bundle of rank d on W. Now let $\mathcal{J}_X \subseteq \mathcal{O}_Y$ and $\mathcal{J}_W \subseteq \mathcal{O}_Z$ be the ideal sheaves of X in Y, and W in Z. Then by the construction of W we have a surjection $j^*\mathcal{J}_X \to \mathcal{J}_W$, which gives a surjective map of sheaves of algebras

$$\bigoplus_{n\geq 0} j^*\left(\mathcal{J}_X^n/\mathcal{J}_X^{n+1}\right) \longrightarrow \bigoplus_{n\geq 0} \mathcal{J}_W^n/\mathcal{J}_W^{n+1}.$$

This gives a closed embedding

$$\mathbf{Spec}\left(\bigoplus_{n\geq 0} \mathcal{J}_W^n/\mathcal{J}_W^{n+1}\right) \longrightarrow \mathbf{Spec}\left(\bigoplus_{n\geq 0} j^*\left(\mathcal{J}_X^n/\mathcal{J}_X^{n+1}\right)\right).$$

of the normal cone C_ZW of W in Z in $\mathcal{N}$. As C_ZW is of pure dimension k, it determines an element $[C_ZW] \in \mathrm{CH}_k(\mathcal{N})$. By the previous theorem, this corresponds to a unique class in $\mathrm{CH}_{k-d}(W)$. As closed immersions are stable under base change, we get that $j' : W \to X$ is a closed immersion as well, and a subvariety of W also defines a subvariety of X of the same dimension. So we get a class in $\mathcal{Z}_{k-d}(X)$ which we will denote by $i^![Z]$.
Now if $f = p \circ i : X \to P \to Y$ is an l.c.i. morphism, with p smooth of dimension d and i a regular embedding of codimension e, then we have a map

$$f^! : \mathcal{Z}_k(Y) \to \mathcal{Z}_{k+d-e}(X), \quad Z \mapsto p^*i^!(Z).$$

The pullback via p does exist, because smooth morphisms are by definition of constant relative dimension. These morphisms $i^!$ and $f^!$ are called *Gysin morphisms.*

The properties of the map $f^!$ are very similar to those of the pullback of flat morphisms.

Theorem 2.7. *Let $f = p \circ i$ be an l.c.i. morphism as in the previous definition.*

(i) The construction of $i^!$ respects rational equivalence.

(ii) Each l.c.i. morphism f gives a map of Chow groups

$$f^! : \mathrm{CH}_k(Y) \to \mathrm{CH}_{k+d-e}(X).$$

(iii) This map is independent of the factorization into a smooth and a regular embedding.

(iv) The construction of the Gysin morphisms for regular embeddings is functorial.

(v) Consider the Cartesian square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "i'"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Y \arrow[r, "i"] & Z \end{tikzcd}$$

where i, and hence i', is a regular embedding of codimension d.
If p and p' are proper, then

$$i^! \circ p_* = (p')_* \circ (i')^! : \mathrm{CH}_*(X) \to \mathrm{CH}_{*-d}(Y).$$

If p is flat, then so is p', and we get

$$(i')^! \circ p^* = (p')^* \circ i^! : \mathrm{CH}_*(Z) \to \mathrm{CH}_{*+e-d}(X \times_Z Y).$$

(vi) *The Gysin morphisms for l.c.i. morphisms are functorial and they commute with the push-forwards of proper morphisms in the sense of Lemma 2.4.*

(vii) *If f is both flat and a regular embedding, then the two notions of pullback coincide:*

$$f^! = f^*.$$

Proof. The second statement immediately follows from the first, which is proven in Proposition 5.2 in [18]. The statements for regular embeddings can be found in Theorems 6.2 and 6.5 in this book as well. The assertions for l.c.i. morphisms together with the last one, are collected in Proposition 6.6 in [18]. □

2.2 Chern class operators

We can also construct morphisms $\mathrm{CH}_*(X) \to \mathrm{CH}_{*-1}(X)$ for each line bundle $\mathcal{L}$ on X. The sheaf $\mathcal{L}$ restricts to a line bundle on any subvariety $Z \subseteq X$ of dimension k and as Z is a subvariety, this gives a divisor D of Z, see Theorem C.1. Now D is a $k-1$-dimensional subvariety of X. We now get the following definition and statement from Section 2.5 in [18].

Definition 2.8. Denote the class of D in $\mathrm{CH}_{k-1}(X)$ by $c_1(\mathcal{L}) \cap [Z]$. This construction respects rational equivalence and extends linearly to a group homomorphism

$$\mathrm{CH}_*(X) \to \mathrm{CH}_{*-1}(X)$$

denoted by $c_1(\mathcal{L})\cap\_$ or if we do not want to specify the argument, $\tilde{c}_1(\mathcal{L})$. This map is called the *first Chern class operator* of $\mathcal{L}$.

We will see that we can extend this to homomorphisms

$$c_i(\mathcal{E})\cap\_ : \mathrm{CH}_*(X) \to \mathrm{CH}_{*-i}(X)$$

for each $0 \le i \le r$, where $\mathcal{E}$ is a locally free sheaf of rank r on X. Before presenting the proof we will state the following properties of the Chern class operators.

Theorem 2.9. *Let X be a scheme with a vector bundle $\mathcal{E}$ of rank r. There exist unique homomorphisms*

$$c_i(\mathcal{E})\cap\_ : \mathrm{CH}_*(X) \to \mathrm{CH}_{*-i}(X)$$

for $0 \le i \le r$, such that for vector bundles $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ on X, a morphism of schemes $f : Y \to X$ and α a cycle on X the following conditions are satisfied.

(i) *The Chern class operators of $\mathcal{E}$ and $\mathcal{E}'$ commute:*

$$c_i(\mathcal{E}) \cap (c_j(\mathcal{E}') \cap \alpha)) = c_j(\mathcal{E}') \cap (c_i(\mathcal{E}) \cap \alpha)).$$

(ii) *Let f be a proper morphism, then the equality*

$$f_* (c_i(f^*\mathcal{E}) \cap \alpha) = c_i(\mathcal{E}) \cap f_*(\alpha)$$

holds.

(iii) If f is a flat morphism of constant relative dimension, then we have

$$c_i(f^*\mathcal{E}) \cap f^*(\alpha) = f^*\left(c_i(\mathcal{E}) \cap \alpha\right).$$

Now define the Chern class operator polynomial by

$$\tilde{c}_t(\mathcal{E}) = 1 + \tilde{c}_1(\mathcal{E})t + \ldots + \tilde{c}_r(\mathcal{E})t^r.$$

(iv) Suppose that

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is a short exact sequence of locally free sheaves on X, then the equality

$$\tilde{c}(\mathcal{E}) = \tilde{c}(\mathcal{E}')\tilde{c}(\mathcal{E}'')$$

holds as operators on $\mathrm{CH}_*(X)$.

(v) Consider line bundles $\mathcal{L}$ and $\mathcal{M}$ on X. The homomorphism $\tilde{c}_1(\mathcal{L})$ coincides with the one given above. Also, by the linearity of the correspondence of divisors and line bundles we have

$$\tilde{c}_1(\mathcal{L} \otimes \mathcal{M}) = \tilde{c}_1(\mathcal{L}) + \tilde{c}_1(\mathcal{M})$$

and

$$\tilde{c}_1(\mathcal{L}^\vee) = -\tilde{c}_1(\mathcal{L}).$$

Using these properties we can determine the Chern class operators of a vector bundle $\mathcal{E}$ if it would admit a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}$$

such that the quotients of consecutive terms are line bundles. Because then we could calculate the Chern class operator polynomial by expanding the equality

$$\tilde{c}_t(\mathcal{E}) = \prod_{i=1}^{r} \tilde{c}_t(\mathcal{E}_i/\mathcal{E}_{i-1})$$

which follows from repetitive applications of the additivity property for short exact sequences.

By the following principle we can actually assume the existence of such a filtration for general vector bundles.

Theorem 2.10 (Splitting principle)**.** *Let X be a scheme and $\mathcal{E} \to X$ a vector bundle of rank r on X. There exists a smooth scheme $\tilde{X}$ together with a flat morphism $\phi : \tilde{X} \to X$, such that*

$$\phi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(\tilde{X})$$

is injective and the bundle $\phi^*\mathcal{E}$ on $\tilde{X}$ admits a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_r = \phi^*\mathcal{E}$$

such that the respective quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are line bundles on $\tilde{X}$.

Proof. We prove the existence of a space $\tilde{X}$, such that $\phi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(\tilde{X})$ is injective and $\phi^*\mathcal{E}$ has a one-dimensional subbundle. The proof is then completed by induction on the rank of $\mathcal{E}$ for general spaces, as it trivially holds for line bundles.
Consider the projective bundle $\mathbb{P}(\mathcal{E}) \to X$ with flat projection ϕ. Then ϕ^* is injective by Corollary 3.1 in [18] and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ is the sought subbundle of $\phi^*\mathcal{E}$. □

Now note that this principle and properties *(iii)-(v)* in Theorem 2.9 do suffice to prove the uniqueness of the Chern class operators.

We already saw in Theorem 2.5 how the Chow group of an affine bundle relates to that of the base space. There is also an important result for projective bundles, which is easily expressed in Chern class operators.

Theorem 2.11 (Projective bundle property, see [18, Theorem 3.3b]). *Let $\mathcal{E}$ be a vector bundle of rank $q+1$ on a scheme X. Consider the corresponding projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ and the following homomorphisms*

$$c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^i \cap \pi^* \_ : \mathrm{CH}_{*+i-q}(X) \to \mathrm{CH}_*(\mathbb{P}(\mathcal{E}))$$

for all $0 \le i \le q$. These homomorphisms sum to an isomorphism

$$\bigoplus_{i=0}^{q} \mathrm{CH}_{*+i-q}(X) \to \mathrm{CH}_*(\mathbb{P}(\mathcal{E})).$$

Using this theorem we can show the existence of the Chern class operators following Grothendieck's approach [21].

Proof of Theorem 2.9. Write q for the dimension of $\mathbb{P}(\mathcal{E})$ over X. So q equals $r-1$, where r is the rank of the bundle $\mathcal{E}$ on X.

Now let $\alpha \in \mathrm{CH}_*(X)$ be any class. Then we have that $\pi^*\alpha \in \mathrm{CH}_{*+q}(\mathbb{P}((E)))$ and hence

$$c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^q \cap \pi^*\alpha \in \mathrm{CH}_*(\mathbb{P}((E))).$$

Hence for each $0 \le i \le q$ there exists a unique $\beta_i \in \mathrm{CH}_{*+i-q}(X)$ such that

$$c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^q \cap \pi^*\alpha = \sum_{i=0}^{q} c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^i \cap \pi^*\beta_i.$$

Now we can define $c_0(\mathcal{E}) \cap \alpha = \alpha$ and

$$c_i(\mathcal{E}) \cap \alpha = (-1)^{i+1}\beta_{q+1-i}.$$

It is easily seen that these maps define homomorphisms

$$\tilde{c}_i(\mathcal{E}) : \mathrm{CH}_*(X) \to \mathrm{CH}_{*-i}(X)$$

which are uniquely determined by the identity

$$\sum_{i=0}^{r} (-1)^i \tilde{c}_1(\mathcal{O}(1))^{r-i} \circ \pi^* \circ \tilde{c}_i(\mathcal{E}) = 0$$

of endomorphisms of $\mathrm{CH}_*(\mathbb{P}(\mathcal{E}))$.

The proofs of the properties in Theorem 2.9 follow similarly to those of Chern classes in [21]. We will only prove properties *(iv)* and *(v)* as the other statements are proven in a similar manner.

(iv) Note that it is clear from Definition 2.8 that this property holds for line bundle.

Let us write $\mathcal{E}'$ for the pullback $f^*\mathcal{E}$. By the naturality of the projective bundle construction we have an isomorphism of $f^*(\mathbb{P}(\mathcal{E})) = \mathbb{P}(\mathcal{E}')$ and hence a map $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$ by the top map $\tilde{f}$ in

$$\begin{array}{ccc} f^*\mathbb{P}(\mathcal{E}) & \xrightarrow{\tilde{f}} & \mathbb{P}(\mathcal{E}) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

which satisfies $\mathcal{O}_{\mathbb{P}(\mathcal{E}')} = \tilde{f}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}$. Now we get that

$$\begin{aligned} 0 =& \tilde{f}^*\left(\sum_{i=0}^{r}(-1)^i\tilde{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{r-i}\circ\pi^*\circ\tilde{c}_i(\mathcal{E})(\alpha)\right) \\ =& \sum_{i=0}^{r}(-1)^i\tilde{f}^*\circ\tilde{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^{r-i}\circ\pi^*\circ\tilde{c}_i(\mathcal{E})(\alpha) \\ =& \sum_{i=0}^{r}(-1)^i\tilde{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1))^{r-i}\circ\tilde{f}^*\circ\pi^*\circ\tilde{c}_i(\mathcal{E})(\alpha) \\ =& \sum_{i=0}^{r}(-1)^i\tilde{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1))^{r-i}\circ\pi'^*\circ f^*\circ\tilde{c}_i(\mathcal{E})(\alpha). \end{aligned}$$

By the definition of the Chern class operators we see that $f^*\circ\tilde{c}_i(\mathcal{E})(\alpha) = f^*(c_i(\mathcal{E})\cap\alpha)$ equals $c_i(f^*\mathcal{E})\cap f^*\alpha$.

(v) Consider the line bundle $\mathcal{E} = \mathcal{O}_X(D)$ then $\pi : \mathbb{P}(\mathcal{E}) \to X$ is an isomorphism. Under this correspondence we see that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ equals $\mathcal{O}_X(D)$. Now $\tilde{c}_1(\mathcal{E})$ is uniquely determined by

$$0 = \tilde{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))\circ\pi^* - \pi^*\circ\tilde{c}_1(\mathcal{E})$$

and the result follows as π and hence π^* is an isomorphism. □

2.3 Properties of the Chow group

So far, the Chow group shares some properties with cobordism theories in algebraic topology, such as the homotopy invariance and projective bundle property. The similarity extends to the following properties. The first is the existence of an product on Chow groups.

Theorem 2.12 ([17, Proposition 1.10])**.** *Let X and Y be schemes. There is a well-defined map*

$$\times : \mathrm{CH}_i(X)\otimes\mathrm{CH}_j(Y)\to\mathrm{CH}_{i+j}(X\times Y)$$

which maps $[W]\otimes[Z]\mapsto[W\times Z]$.

This product is called the *exterior product* of the Chow group.

Note that $W\times Z$ is irreducible and reduced, just like W and Z since we are working over an algebraically closed field [19, Lemma 4.23]. So $W\times Z$ is indeed a subvariety of $X\times Y$.

The second property is the algebraic analogue of excision in cohomology stated in [18] as Proposition 1.8.

Theorem 2.13 (Excision)**.** *Let Z be a closed subscheme of a scheme X. Let $j : Z \to X$ be the closed immersion and $i : U \to X$ be the open immersion of the complement. Then we have that the following sequence*

$$\mathrm{CH}_k(Z)\xrightarrow{j_*}\mathrm{CH}_k(X)\xrightarrow{i^*}\mathrm{CH}_k(U)\to 0$$

is exact for all k.

Like homology of a CW-complex is generated by classes of the cells, we have a similar result for Chow groups, coming from [18, Examples 1.9.1 & 19.1.11]

Theorem 2.14 (Cellular decomposition)**.** *Consider a scheme X which has a filtration*

$$\emptyset = X_0\subset X_1\subset\ldots\subset X_k = X$$

such that $X_i\backslash X_{i-1}$ has irreducible components $U_{i,\alpha}$ which are affine spaces $\mathbb{A}^{n_{i,\alpha}}$. If the closures $V_{i,\alpha} = \overline{U_{i,\alpha}}$ of all such components are smooth, then the classes of these closures $[V_{i,\alpha}]$ form a basis for $\mathrm{CH}_*(X)$.

This shows that the Chow group of a projective space $\mathbb{P}^n$ consists in degree d of $\mathbb{Z}$ generated by the class of a projective subspace of dimension d.

Together with the homotopy invariance property, one sees that the exterior product is an isomorphism if at least one of the spaces X and Y admits such a cellular decomposition.

2.4 Chow ring

Consider two subvarieties V and W of dimension k and l of a scheme X of dimension n. The intersection $V \cap W$ is in general a subvariety of dimension $k+l-n$ and one might like to use this to define a product on Chow groups. This is not always possible, but we will show it is possible if X is a non-singular quasi-projective variety. In this case we even get a graded ring, if we grade subvarieties not by their dimension, but by their codimension instead.

Definition 2.15. Let X be a smooth variety of pure dimension n. Then define the graded group

$$\mathrm{CH}^*(X) = \mathrm{CH}_{n-*}(X)$$

called the *Chow ring* of X. The Chow ring for non-connected smooth schemes is defined as the direct sum of the Chow ring of the components.

We would now like to define the intersection product on $\mathrm{CH}^*(X)$. So let V and W be two subvarieties of X. If they meet in a nice enough manner, we would like to define $[V]\cdot[W]$ as the class of the set-theoretic intersection. This product respects the new grading, as for general intersections we would expect $\operatorname{codim} V + \operatorname{codim} W = \operatorname{codim} V \cap W$. In general we would like to define the product $[V]\cdot[W]$ as a sum of the irreducible components of $V\cap W$ taking into account some kind of multiplicities. For this to work we need that the irreducible components Z_i of $V \cap W$ are in fact of the right dimension.

Definition 2.16. Let V and W be subvarieties of X. We say that they intersect properly if $\operatorname{codim} V + \operatorname{codim} W = \operatorname{codim} V \cap W$.

So if V and W intersect properly we would like to define some coefficients $i(V,W;Z_i)$ such that

$$[V]\cdot[W] = \sum_i i(V,W;Z_i)[Z_i]$$

extends to a product on $\mathrm{CH}^*(X)$. The definition of these local intersection multiplicities comes from Serre [52] and is given by

$$i(V,W;Z_i) = \sum_{j\geq 0} (-1)^j \operatorname{length}_{\mathcal{O}_{Z_i,X}} \operatorname{Tor}_j^{\mathcal{O}_{Z_i,X}}(\mathcal{O}_{Z_i,X}/\mathcal{I}_V, \mathcal{O}_{Z_i,X}/\mathcal{I}_W).$$

This definition does in fact respect rational equivalence if X lies in $\mathbf{Sm}_k$.

Now there are several ways to proceed: one could do a lot of work to prove Chow's moving lemma, which says that one can move cycles within their rational equivalence class to get proper intersections, see for example [49]. A cleaner way is to make use of the work we did in defining the Gysin morphisms and note that the following product coincides with the one given above, in the case of proper intersection.

Definition 2.17. Let α and β be classes in $\mathrm{CH}^*(X)$ for a scheme $X \in \mathbf{Sm}_k$. Then we define the intersection product of these classes as

$$\alpha\cdot\beta = \delta^!(\alpha\times\beta),$$

which is well-defined as the diagonal embedding $\delta : X \to X\times X$ of a smooth scheme is regular.

Note that each morphism between two smooth quasi-projective varieties is in fact an l.c.i. morphism, so we have pullbacks for all morphisms in $\mathbf{Sm}_k$. By abuse of notation we will often write f^* for the pullback via a general morphism f.

Theorem 2.18 ([18, Proposition 8.3]). *The defined product makes* $\mathrm{CH}^*(X)$ *into a ring with unit given by the class* $[X] \in \mathrm{CH}^0(X)$ *for all* $X \in \boldsymbol{Sm}_k$.
This makes CH^* *a covariant functor from* $\boldsymbol{Sm}_k$ *to the category of graded commutative rings, with the morphisms* $\mathrm{CH}^*(f)$ *given by* f^* *as defined above, using the intersection product. In particular, for any morphisms* $f : X \to Y$ *and* $g : Y \to Z$*, we have*

$$(g \circ f)^* = f^* \circ g^*.$$

Note that unlike the pullback for general morphisms in $\mathbf{Sm}_k$, the pushforward for proper morphism is not a ring homomorphism. One can see that the pushforward of the inclusion of a point in a quasi-projective smooth variety X of positive dimension does not map the multiplicative unit of the point to the multiplicative unit of $\mathrm{CH}^*(X)$.

This product allows us to give a nice relation between pullbacks and pushforwards in the Chow ring.

Lemma 2.19 (Projection formula [18, Proposition 8.1.1b]). *Let* $f : X \to Y$ *be a proper morphism. Then for all classes* $\alpha \in \mathrm{CH}^*(X)$ *and* $\beta \in \mathrm{CH}^*(Y)$ *we have*

$$f_*(\alpha \cdot f^*\beta) = f_*(\alpha) \cdot \beta.$$

Of course, all the properties for the Chow group are also satisfied by the Chow ring, most importantly the excision property, homotopy invariance, the cellular decomposition property, the splitting principle, the projective bundle property and the existence of the degree map for projective varieties.

In particular we can determine the ring structure on $\mathrm{CH}^*(\mathbb{P}^n)$. The Chow ring of a projective space is a free abelian group with one generator in each dimension, generated by any linear subspace of the correct codimension. Since the intersection of two general linear subspaces is also a linear subspace of the expected dimension, we see that

$$\mathrm{CH}^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{n+1})$$

where h is the class of any hyperplane.

2.5 Chern classes and numbers

Let X be a general scheme of dimension n with a vector bundle of rank r. Using the Chern class operators $\tilde{c}_i(\mathcal{E})$, one can define classes $c_i(\mathcal{E}) \in \mathrm{CH}_{n-i}(X)$ by evaluating on the class $[X] \in \mathrm{CH}_n(X)$. These classes do in general possess less structure than the Chern class operators. However, given the intersection pairing one can retrieve the information of the Chern class operators in terms of these classes.

Definition 2.20. Let $X \in \mathbf{Sm}_k$ be a smooth quasi-projective scheme with a vector bundle $\mathcal{E}$ of rank r. Define

$$c_i(\mathcal{E}) := c_i(\mathcal{E}) \cap [X] \in \mathrm{CH}^i(X)$$

for all $0 \leq i \leq r$. These classes are called the *Chern classes* of $\mathcal{E}$.

Now Example 8.1.6 in [18] shows that for any class $\alpha \in \mathrm{CH}^*(X)$ we have

$$c_i(\mathcal{E})\alpha = (c_i(\mathcal{E}) \cap [X])\, \alpha = (c_i(\mathcal{E}) \cap \alpha)\, [X] = c_i(\mathcal{E}) \cap \alpha.$$

So for non-singular X the Chern classes give the same information as the more general Chern class operators.

We can use this to give a more natural formulation of the projective bundle property in Theorem 2.11.

Theorem 2.21 (Projective bundle property)**.** *Let $\mathcal{E}$ be a vector bundle of rank $q+1$ on $X \in \boldsymbol{Sm}_k$ and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ be the corresponding projective bundle with projection $\pi : \mathbb{P} \to X$. Let ξ be the class in $\mathrm{CH}^1(\mathbb{P})$ corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Then $\mathrm{CH}^*(\mathbb{P})$ is a free $\mathrm{CH}^*(X)$-module, via the map*

$$\pi^* : \mathrm{CH}^*(X) \to \mathrm{CH}^*(\mathbb{P}),$$

with basis $1, \xi, \xi^2, \ldots, \xi^q$.

Now consider any class $\alpha \in \mathrm{CH}_0(X)$ for a proper scheme X. As X is proper, so is the structure map $\pi_X : X \to \operatorname{Spec} k$. So we can push α to $\mathrm{CH}_*(k)$, which is concentrated in degree 0 where it equals $\mathbb{Z}$. So we get a class $\pi_{X*}\alpha \in \mathbb{Z}$. We can express this number in a different manner: α is the formal sum of classes of (reduced) points. Since rational equivalence in dimension zero reduces to linear equivalence on curves in $X \times \mathbb{P}^1$, we get a well-defined natural homomorphism of groups.

Definition 2.22. Let X be a projective scheme of dimension n. Then we have a group homomorphism

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}, \quad \sum n_i[P_i] \mapsto \sum n_i \operatorname{length} \mathcal{O}_{P_i,X}.$$

We can of course extend this homomorphism to the whole of $\mathrm{CH}_*(X)$ by mapping all classes of a different degree to zero. This map is more commonly denoted by

$$\int_X : \mathrm{CH}_*(X) \to \mathbb{Z}, \quad \alpha \mapsto \deg(\alpha_0),$$

where α_0 denotes the degree 0 part of α. The number $\int_X \alpha$ is called the *degree of the Chern class* α.

That is, we count the number of reduced points in the formal sum. We can use this map to define interesting invariants for proper schemes.

Let $\mathcal{E}$ be any rank r vector bundle on a projective scheme $X \in \mathbf{Sm}_k$ of dimension n. Now consider a polynomial p with integer coefficients in n variables $x_1, x_2, \ldots, x_n$ where each x_i is of degree i. This polynomial defines a class in $\mathrm{CH}^*(X)$ by mapping x_i to $c_i(\mathcal{E})$. So if p is homogeneous of degree d, then this defines a homogeneous class in $\mathrm{CH}^d(X)$. In particular, we can extract the degree n part.

Definition 2.23. Let $X \in \mathbf{Sm}_k$ be a projective scheme of dimension n. Let $\mathcal{P}$ be the set of weighted polynomials p of degree n, as above. The *Chern numbers of a vector bundle* $\mathcal{E}$ are defined by

$$\int_X p\left(c_1(\mathcal{E}), c_2(\mathcal{E}), \ldots, c_n(\mathcal{E})\right)$$

for all $p \in \mathcal{P}$. For $\mathcal{E} = \mathcal{T}_X$ we talk about the *Chern numbers of* X.

As an example we will compute the Chern numbers of a projective curve C of genus g. In this case, the only interesting polynomial is x_1. So we compute

$$\int_C c_1(\mathcal{T}_C) = \int_C -[K] = -\deg K = 2 - 2g,$$

where K is a canonical divisor on C.

2.6 K-theory

We will now consider K-theory, which consists of two specific examples of the general Grothendieck group construction [51, Theorem 1.1.3]. Most of the material in this section comes from [39], which gives a thorough treatise on the subject. The more specialized results come from [22].

Definition 2.24. Let $X \in \mathbf{Sch}_k$ be a scheme. Consider the free abelian group $\mathfrak{J}$ generated by isomorphism classes of coherent sheaves on X. For a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$$

of coherent sheaves on X, we define an element $\mathcal{E} - \mathcal{F} - \mathcal{G}$ of $\mathfrak{J}$. Let $\mathfrak{J}'$ be the subgroup generated by these elements for all short exact sequences. Define *K-theory of coherent sheaves on X*

$$K_\bullet(X) = \mathfrak{J}/\mathfrak{J}'.$$

The class of a sheaf $\mathcal{E}$ is denoted by $\gamma_\bullet(\mathcal{E})$.
Let us now replace the isomorphism classes of the objects of the category of coherent sheaves with those of the full subcategory of locally free sheaves. Define $\mathfrak{L}$ as the free abelian group on these classes and let $\mathfrak{L}'$ be the subgroup generated by the relations associated to the exact sequences of locally free sheaves. We define *K-theory of locally free sheaves on X* by

$$K^\bullet(X) = \mathfrak{L}/\mathfrak{L}'.$$

In this group we will use $\gamma^\bullet(\mathcal{E})$ for the image of a locally free sheaf $\mathcal{E}$ in this group. If no confusion can arise we will also use $[\mathcal{E}]$ to denote the class of a sheaf in either of the K-theories.

Clearly, for all X we have a group homomorphism

$$K^\bullet(X) \to K_\bullet(X).$$

This map is in general not an isomorphism, but it does relate the structures of the two groups. Let us first examine these structures.

Theorem 2.25. *Let $X \in \boldsymbol{Sch}_k$ be a scheme.*

(i) The group $K^\bullet(X)$ is a ring with multiplication given by

$$\gamma^\bullet(\mathcal{E}_1)\gamma^\bullet(\mathcal{E}_2) = \gamma^\bullet(\mathcal{E}_1 \otimes \mathcal{E}_2)$$

for all locally free sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$. The identity is given by $\gamma^\bullet(\mathcal{O}_X)$.

(ii) The group $K_\bullet(X)$ is a $K^\bullet(X)$-module, with the module structure given by

$$\gamma^\bullet(\mathcal{E})\gamma_\bullet(\mathcal{F}) = \gamma_\bullet(\mathcal{E} \otimes \mathcal{F}).$$

The proof of the above theorem is straightforward, using the fact that locally free sheaves are flat. It also follows that the map $K^\bullet(X) \to K_\bullet(X)$ is even a homomorphism of $K^\bullet(X)$-modules.

One important property of K-theory is the following result coming from [39, Corollary 1.12].

Theorem 2.26 (Excision for K-theory)**.** *Let U be the open complement of a closed subscheme Z of an $X \in \boldsymbol{Sch}_k$. Then we have an exact sequence*

$$K_\bullet(Z) \to K_\bullet(X) \to K_\bullet(U) \to 0,$$

where the maps are given by extending a generator by zero outside of Z and restricting to U.

In particular, $K_\bullet$ allows for both pullbacks and pushforwards for certain classes of morphisms.

If one has a construction to make a new sheaf out of a given one, this does not necessarily descend to a morphism in either K-theory. The construction should map a coherent sheaf to a coherent sheaf, or a locally free sheaf to a locally free sheaf, but it should also be exact to easily extend to a morphism on either $K_\bullet$ or $K^\bullet$. Such examples are given by pushforwards for closed immersions and pullbacks for open immersions as in the above theorem. Let us examine the possibilities of extending pullbacks and direct images in this manner.

The pullback of a general morphism is exact, although it does not always carry a coherent sheaf to another coherent sheaf. This is however the case for Noetherian schemes, which gives us a theory of general pullbacks on $K^\bullet$ and $K_\bullet$.

The direct image f_* of a locally free sheaf is not always locally free either. This is however the case for coherent sheaves, but unfortunately this functor is not exact. It is however left-exact and gives rise to the higher image functors $R^k f_*$. A well-known result says that the sheaves $R^k f_*(\mathcal{F})$ are coherent if f is projective and $\mathcal{F}$ is coherent [26, Theorem III.8.8(b)]. We can use this to define well-defined pushforwards for projective morphisms.

Theorem 2.27. *Let $f : X \to Y$ be a morphism of schemes in $\boldsymbol{Sch}_k$.*

(i) The pullback of f on locally free sheaves defines a ring homomorphism

$$f^! : K^\bullet(Y) \to K^\bullet(X), \quad [\mathcal{E}] \mapsto [f^*\mathcal{E}].$$

(ii) If f is projective we have a group homomorphism given by

$$f_! : K_\bullet(X) \to K_\bullet(Y), \quad [\mathcal{F}] \mapsto \sum_{i=0}^{\infty} (-1)^i [R^i f_* \mathcal{F}].$$

(iii) For all f we have a well-defined map

$$f^* : K_\bullet(Y) \to K_\bullet(X), \quad [\mathcal{F}] \mapsto [f^*\mathcal{F}]$$

which is also a group homomorphism.

These morphisms behave well under composition. So if $g : Y \to Z$ is another morphism then $f^* \circ g^* = (g \circ f)^*$, $f^! \circ g^! = (g \circ f)^!$, and if f and g are both projective then so is their composition $g \circ f$ and $g_! \circ f_! = (g \circ f)_!$.

Proof. The only non-trivial part of the theorem is the statement that $g_! \circ f_! = (g \circ f)_!$. We will prove this following Manin [39] using the Grothendieck spectral sequence. Consider the composition of functors

$$\mathbf{Coh}(X) \xrightarrow{f_*} \mathbf{Coh}(Y) \xrightarrow{g_*} \mathbf{Coh}(Z).$$

We will prove that f_* maps an injective object in $\mathbf{Coh}(X)$ to a g_*-acyclic object.

So let us prove that

$$(R^p g_*)\, f_* \mathcal{I} = 0$$

for $\mathcal{I}$ an injective sheaf on X and $p > 0$. We know that the sheaf $(R^p g_*)\, f_* \mathcal{I}$ is the sheaf associated to the presheaf mapping an open $U \subseteq Z$ to

$$H^p(g^{-1}(U), f_* \mathcal{I}\big|_{g^{-1}(U)}).$$

As injectivity of sheaves is preserved under taking the direct image and restricting to an open subset, and injective sheaves are acyclic, we get that these cohomology groups do indeed vanish.

So for any $\mathcal{F} \in \mathbf{Coh}(X)$ there exists a spectral sequence given by

$$E_2^{pq} = R^p g_*(R^q f_* \mathcal{F}) \Longrightarrow R^n (g \circ f)_* \mathcal{F}.$$

Let

$$0 = F_n^0 \subseteq F_n^1 \subseteq \ldots \subseteq F_n^n = R^n (g \circ f)_* \mathcal{F}$$

be the filtration of the limit, so by definition

$$F^{p+1}_{p+q}/F^{p}_{p+q} = E^{pq}_{\infty}.$$

Now we find

$$\begin{aligned}
(g \circ f)_! \mathcal{F} &= \sum_{n=0}^{\infty} (-1)^n [R^n (g \circ f)_* \mathcal{F}] \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{n} \left([F^{p+1}_n] - [F^p_n] \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{n} [F^{p+1}_n / F^p_n] \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{n} [E^{p,n-p}_{\infty}] \\
&= \sum_{n=0}^{\infty} (-1)^n [E^n_{\infty}].
\end{aligned}$$

On the other hand

$$\begin{aligned}
(g_! \circ f_!) \mathcal{F} &= \sum_{p=0}^{\infty} (-1)^p \sum_{q=0}^{\infty} (-1)^q [R^q g_* \left(R^p f_* \mathcal{F} \right)] \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{p=0}^{n} [E^{p,n-p}_2] \\
&= \sum_{n=0}^{\infty} (-1)^n [E^n_2].
\end{aligned}$$

The boundary homomorphism δ_a splits $E^{\bullet}_a$ in bounded complexes, whose terms are counted with alternating signs in the above sum. The fact that the alternating sum of the classes of the terms of a complex equals the alternating sum of the classes of the homology of this complex, implies that

$$\sum_{n=0}^{\infty} (-1)^n [E^n_2] = \sum_{n=0}^{\infty} (-1)^n [E^n_3] = \ldots = \sum_{n=0}^{\infty} (-1)^n [E^n_{\infty}],$$

proving the claim. $\square$

Especially the maps $f^!$ and $f_!$ satisfy a nice relation.

Lemma 2.28 (Projection formula)**.** *Let $f : X \to Y$ be a projective morphism between $X, Y \in$ **Sch**$_k$. For any classes $\alpha \in K_{\bullet}(X)$ and $\beta \in K^{\bullet}(Y)$ we have the following formula:*

$$f_! \left(\alpha \cdot f^!(\beta) \right) = f_!(\alpha) \cdot \beta.$$

The first multiplication comes from the module structure of $K_{\bullet}(X)$ and the second is the multiplication in the ring $K^{\bullet}(X)$.

Proof. Clearly both sides are linear in both α and β, so it is enough to consider $\alpha = \gamma_{\bullet}(\mathcal{F})$ and

$\beta = \gamma^\bullet(\mathcal{E})$ for $\mathcal{F}$ a coherent sheaf on X and $\mathcal{E}$ a locally free sheaf on Y. In this case we have

$$\begin{aligned} f_!\left(\alpha \cdot f^!(\beta)\right) &= \sum_{i=0}^{\infty}(-1)^i[R^i f_*(\mathcal{F} \otimes f^*\mathcal{E})] \\ &= \sum_{i=0}^{\infty}(-1)^i[R^i f_*(\mathcal{F}) \otimes \mathcal{E})] \\ &= \sum_{i=0}^{\infty}(-1)^i[R^i f_*(\mathcal{F})] \cdot \gamma^\bullet(\mathcal{E}) \\ &= \left(\sum_{i=0}^{\infty}(-1)^i[R^i f_*(\mathcal{F})]\right) \cdot \gamma^\bullet(\mathcal{E}) = f_!(\alpha) \cdot \beta. \end{aligned}$$

Here we used that $R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) = R^i f_*(\mathcal{F}) \otimes \mathcal{E}$. This follows from the projection formula for a coherent sheaf $\mathcal{F}$ and a locally free sheaf $\mathcal{E}$, see for example [26, Exercise II.5.1(d)],

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) = f_*(\mathcal{F}) \otimes \mathcal{E}.$$

So the right derived functors in $\mathcal{F}$ of both sides coincide. Furthermore we have $R^i\left(f_*(\_) \otimes \mathcal{E}\right) = R^i f_*(\_) \otimes \mathcal{E}$ since tensor multiplication with $\mathcal{E}$ is exact, as $\mathcal{E}$ is locally free. □

2.7 Properties of K-theory

Now that we have defined two K-theories together with several multiplications, pullbacks and pushforwards which give group, ring and module structures on these objects, we can examine additional properties.

We already saw that the excision property holds in $K_\bullet$. Note that maps in that theorem are simply the pushforward of the open immersion of U in X, and the pullback of the closed immersion of Z in X. The latter is defined as any closed immersion is projective.

K-theory does also satisfy the projective bundle property, best expressed in terms of the $K^\bullet(X)$-module structure on $K_\bullet(X)$.

Theorem 2.29 ([22, IX Corollaire 3.2])**.** *Let $\mathcal{E}$ be a vector bundle of rank $q+1$ on $X \in \boldsymbol{Sch}_k$ and let $\mathbb{P} = \mathbb{P}(\mathcal{E})$ be the corresponding projective bundle with projection $\pi : \mathbb{P} \to X$. Let ξ be the class in $K^\bullet(\mathbb{P})$ corresponding to the invertible sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Then $K_\bullet(\mathbb{P})$ is a free $K^\bullet(X)$-module, by the map*

$$\pi^* : K_\bullet(X) \to K_\bullet(\mathbb{P}),$$

with basis $1, \xi, \xi^2, \ldots, \xi^q$. So the map

$$(K_\bullet(X))^{\oplus q+1} \to K_\bullet(\mathbb{P}), \quad (x_i)_{0 \le i \le q} \mapsto \sum_{i=0}^{q} \pi^*(x_i)\xi^i$$

is an isomorphism of $K^\bullet(X)$-modules.

One can also show that $K^\bullet(\mathbb{P})$ is a free $K^\bullet(X)$-module with basis $1, \xi, \xi^2, \ldots, \xi^q$. See for example [22, VI Théorème 1.1]. In this case we even know that $K^\bullet(\mathbb{P})$ is a $K^\bullet(X)$-algebra generated by a single element ξ.

We have seen some obvious similarities of K-theory with the Chow group and ring, but also between the two introduced versions of K-theory. The relation between $K^\bullet(X)$ and $K_\bullet(X)$ is even more clear when X is a regular quasi-projective scheme.

Theorem 2.30. *Let $X \in \boldsymbol{Sm}_k$ be a smooth quasi-projective scheme. Then the map*

$$K^\bullet(X) \to K_\bullet(X)$$

is an isomorphism of groups.

Proof. A proof can be found in [26, Exercise III.6.11]. The main idea is to construct an inverse using a locally free resolution

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

for a coherent sheaf $\mathcal{F}$ on X, by defining

$$K_\bullet(X) \to K^\bullet(X), \quad \gamma_\bullet(\mathcal{F}) \mapsto \sum_{i=0}^{n} (-1)^i \gamma^\bullet(\mathcal{E}_i).$$

After one shows this is well-defined, one immediately sees that both compositions equal the identity. Note that as X is a scheme over an algebraically closed field k, being smooth or regular are equivalent. So we can use the regularity assumption in proving the existence of such a locally free resolution. □

Chapter 3

Algebraic cobordism

In [35] Levine and Morel define the notion of oriented Borel-Moore homology theories, an algebraic version of cobordism theory as a homology theory on a reasonable full subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ which contains $\mathbf{Sm}_k$. This basically consists of assigning to each object a graded group, together with the following structure: pushforwards for projective morphisms, pullbacks for local complete intersection morphisms and Chern class operators for vector bundles. The axioms will also imply the existence of a formal group law, which relates the first Chern class operator of the tensor product of two line bundles to the Chern class operators of the respective line bundles.

Such a theory gives after restricting and re-indexing a cohomology theory on $\mathbf{Sm}_k$, which is a direct analogue of the axiomatic framework for complex oriented cohomology theory as considered by Quillen in [46]. From such a theory on $\mathbf{Sm}_k$, we can get back to the oriented Borel-Moore homology theory on $\mathbf{Sm}_k$, but not necessarily to the one on $\mathcal{V}$.

Levine and Morel construct an oriented Borel-Moore homology on $\mathbf{Sch}_k$, called algebraic cobordism Ω_*, which is the universal such theory. By the above discussion we immediately get a universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$.

3.1 Notation and preliminary definitions

We will define several theories on subcategories of $\mathbf{Sch}_k$. We will need some conditions on these subcategories.

Definition 3.1. A full subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ is called an *admissible* subcategory if it satisfies the following conditions.

(i) The empty scheme and $\operatorname{Spec} k$ are in $\mathcal{V}$.

(ii) If we have a scheme X in $\mathcal{V}$ and a smooth, so also quasi-projective, morphism $Y \to X$ then we must have that Y lies in $\mathcal{V}$ as well.

(iii) If X and Y are in $\mathcal{V}$, then so are their disjoint union $X \coprod Y$ and their product $X \times Y$.

Note that any admissible subcategory $\mathcal{V}$ contains $\mathbf{Sm}_k$.

Definition 3.2. An admissible subcategory $\mathcal{V}$ in $\mathbf{Sch}_k$ is called *l.c.i.-closed* if also satisfies the following two axioms.

(i) If $Y \to X$ is an l.c.i. morphism and X lies in $\mathcal{V}$, then so does Y.

(ii) For any regular embedding $Z \to X$ in $\mathcal{V}$, we have that the blow up of X in Z also lies in $\mathcal{V}$.

Sometimes we restrict a subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ to the subcategory $\mathcal{V}'$ which has the same objects, but whose morphisms are exactly the projective ones.

We will also need the notion of nice intersections of closed subschemes, which generalises to arbitrary pairs of morphisms to the same scheme.

Definition 3.3. Two morphisms $X, Y \to Z$ in an admissible subcategory $\mathcal{V}$ are called *transverse* in $\mathcal{V}$ when

(i) the fibred product $X \times_Z Y$ lies in $\mathcal{V}$;

(ii) the $\mathcal{O}_Z$-modules
$$\mathrm{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{O}_Y) = 0$$
are trivial for all $q > 0$.

Subschemes of Z are transverse if the corresponding closed embeddings are transverse.

3.2 Oriented Borel-Moore homology theories

The following definition comes from [35, Definition 5.1.3].

Definition 3.4. An *oriented Borel-Moore homology theory* A_* on some l.c.i.-closed admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ consists of

(D1) a functor
$$A_* : \mathcal{V}' \to \mathbf{Ab}_*, \qquad X \mapsto A_*(X)$$
such that $A_*(\emptyset) = 0$ and the natural map
$$A_*(X) \oplus A_*(Y) \to A_*(X \coprod Y)$$
is an isomorphism, i.e. A_* is additive.

(D2) a homomorphism of graded groups
$$f^* : A_*(X) \to A_{*+d}(Y)$$
for any l.c.i. morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d.

(D3) an associative commutative unital bilinear graded pairing
$$\begin{gathered} A_*(X) \otimes A_*(Y) \to A_*(X \times Y) \\ u \otimes v \mapsto u \times v \end{gathered}$$
for each two spaces X and Y in $\mathcal{V}$. The unit element is denoted by $1 \in A_0(\operatorname{Spec} k)$.

So we have the additional structure of projective pushforwards, pullbacks for l.c.i. morphisms and the so called exterior product. We will need three axioms to ensure these structure go well together. We will also need three more axioms called the *projective bundle property*, the *extended homotopy property* and the *cellular decomposition property.*

(BM1) For l.c.i. morphisms $g : Z \to Y$ en $f : Y \to X$ in $\mathcal{V}$ of pure relative dimension d and e, we have
$$(f \circ g)^* = f^* \circ g^* : A_*(X) \to A_{*+d+e}(X).$$
In addition $\mathrm{Id_X}^* = \mathrm{Id_{A_*(X)}}$.

(BM2) For a projective $f : X \to Z$ and an l.c.i. morphism $g : Y \to Z$ which are transverse in $\mathcal{V}$ we get the Cartesian square
$$\begin{tikzcd} X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

in which f' is projective and g' is an l.c.i. morphism, since both projective and l.c.i. morphisms are stable under base extension. We have equality of the following maps

$$g^* f_* = f'_* g'^*.$$

(BM3) Consider two morphisms $f : X' \to X$ and $g : Y' \to Y$ in $\mathcal{V}$.
If both morphisms are projective, then

$$(f \times g)_*(u' \times v') = f_*(u') \times g_*(v')$$

for all $u' \in A_*(X')$ and $v' \in A_*(Y')$.
If f and g are both l.c.i. morphisms, then

$$(f \times g)^*(u \times v) = f^*(u) \times g^*(v)$$

for all $u \in A_*(X)$ and $v \in A_*(Y)$.

Let $\mathcal{L}$ be a line bundle on some $X \in \mathcal{V}$. Let $s : X \to \mathcal{L}$ be the zero-section and define the *first Chern class operator* of $\mathcal{L}$ by

$$\begin{aligned} \tilde{c}_1(\mathcal{L}) : A_*(X) &\to A_{*-1}(X) \\ u &\mapsto s^* s_*(u). \end{aligned}$$

(PB) Let $\mathcal{E}$ be a locally free sheaf on $X \in \mathcal{V}$ of rank $q+1$ and let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the associated projective bundle. Define for $i \in \{0, 1, \ldots, q\}$ the map $\xi^{(i)}$ as the composition

$$A_{*+i-q}(X) \xrightarrow{\pi^*} A_{*+i}(\mathbb{P}(\mathcal{E})) \xrightarrow{\tilde{c}_1(O(1)_{\mathcal{E}})^i} A_*(\mathbb{P}(\mathcal{E}))$$

These maps sum to a homomorphism

$$\bigoplus_{i=0}^{q} A_{*+i-q}(X) \to A_*(\mathbb{P}(\mathcal{E}))$$

which is in fact an isomorphism.

(EH) For any vector bundle $\mathcal{E}$ of rank r over $X \in \mathcal{V}$, and any $\mathcal{E}$-torsor $p : V \to X$, we have an isomorphism

$$p^* : A_*(X) \to A_{*+r}(V).$$

(CD) Consider

$$W = \underbrace{\mathbb{P}^N \times \ldots \times \mathbb{P}^N}_{r}$$

for integers r and N. Let $p_i : W \to \mathbb{P}^N$ be the projection on the ith factor and $X_0, \ldots, X_N$ the usual homogeneous coordinates on $\mathbb{P}^N$. For non-negative integers $n_1, \ldots, n_r$ we have a subscheme $i : E \to W$ defined by $\prod_{i=1}^{r} p_i^*(X_N)^{n_i} = 0$. The pushforward

$$i_* : A_*(E) \to A_*(W)$$

is injective.

A morphism of oriented Borel-Moore homology theories is a natural transformation of the functors, which commutes with pullbacks and exterior products.

We will see that the projective bundle property is a generalisation of the property by the same name we saw for the Chow group, the Chow ring, and K-theory. The cellular decomposition property does not resemble Theorem 2.14, but in Theorem 3.8 we will show that the statements are most definitely related. The extended homotopy property obviously implies the homotopy invariance properties stated in Chapter 2. This statement is clearly stronger and can be used to prove a general splitting principle in any oriented Borel-Moore homology theory, see Remark 4.1.2 in [35]. Together with the projective bundle property this allows one to extend the first Chern class operators of line bundles to general Chern class operators for vector bundles of any rank, similar to the proof of Theorem 2.9. Most of the properties of Chern classes stated in that theorem, also hold in any oriented Borel-Moore homology theory. The main exception is property *(v)*, which does not hold in every oriented Borel-Moore homology theory.

In complex cobordism we have a similar situation with the Euler classes. Quillen [45] recognized that the Euler classes of tensor products of line bundles are given by the universal formal group law. This also holds in the algebro-geometric case, by the important result from [35, Proposition 5.2.6].

Theorem 3.5. *Let A_* be any oriented Borel-Moore homology theory on $\mathcal{V}$. There exists a unique commutative formal group law of rank one*

$$F_A(u,v) = u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j$$

on A_* with $a_{i,j} \in A_{i+j-1}(k)$ such that

$$\tilde{c}_1(\mathcal{L}\otimes\mathcal{M}) = F_A(\tilde{c}_1(\mathcal{L}), \tilde{c}_1(\mathcal{M}))$$

for two line bundles over the same base X. This is well-defined as $\tilde{c}_1(\mathcal{L})^n = 0$ for $n > \dim X$.

So each oriented Borel-Moore homology theory A_* determines a unique graded ring homomorphism $\mathbb{L}_* \to A_*(k)$.

Chow group as an oriented Borel-Moore homology

We will now show that the Chow group is an example of an oriented Borel-Moore homology theory on $\mathbf{Sch}_k$ with the structure of pullbacks and pushforwards defined as in Section 2.1. Clearly we have pushforwards for any projective morphism, as we have defined pushforwards for all proper morphisms. We also saw a definition of pullbacks for l.c.i. morphisms and a bilinear graded pairing

$$\mathrm{CH}_*(X)\otimes\mathrm{CH}_*(Y)\to\mathrm{CH}_*(X\times Y)$$

which is associative and commutative, given by the exterior product, $[V]\otimes[W]\mapsto[V\times W]$. The unit of this product is given by the class of the unique point $[\operatorname{Spec} k]\in\mathrm{CH}_0(\operatorname{Spec} k)$. The axioms (BM1) and (BM2) were proven in Theorem 2.4. To prove that the exterior product respects the pullbacks and pushforwards of this theory, i.e. axiom (BM3), we have the following result.

Theorem 3.6. *Let $f : X \to Y$ and $g : W \to Z$ be morphisms of schemes in $\boldsymbol{Sch}_k$ and let $f\times g$ be the product map $X\times W\to Y\times Z$.*

(i) If both f and g are proper, then the same holds for $f\times g$ and we have

$$(f\times g)_*(\alpha\times\beta) = f_*\alpha\times g_*\beta$$

for any classes $\alpha\in\mathrm{CH}_*(X)$ and $\beta\in\mathrm{CH}_*(W)$.

(ii) If f and g are l.c.i. morphisms, then so is $f\times g$ and for cycle classes $\gamma\in\mathrm{CH}_*(Y)$ and $\delta\in\mathrm{CH}_*(Z)$ we have

$$(f\times g)^*(\gamma\times\delta) = f^*\gamma\times g^*\delta.$$

Proof. The first statement is proved in Proposition 1.10 in [18]. There one can also find a proof for the second statement in the case of flat morphisms. By Remark 5.1.2 in [35] we have that regular embeddings are closed under composition and exterior product. The second statement follows from these two results. □

This immediately proves the axiom (BM3) for the Chow group. Now we get a first Chern class operator for a line bundle $\mathcal{L}$ on an $X \in \mathbf{Sch}_k$, given by

$$\tilde{c}_1(\mathcal{L}) = s^* \circ s_*$$

where $s : X \to \mathcal{L}$ is the zero-section of the line bundle. We will show this coincides with the defined $c_1(\mathcal{L})\cap\_$ which we saw before.

So let $[Z] \in \mathrm{CH}_k(X)$ be the class of a subvariety Z of dimension k of X. Let us explicitly calculate $s^*s_*[Z]$. If we consider X as a closed subscheme of the total space of the line bundle, then we see that Z is a closed subvariety under this identification. Let us write L for the total space of $\mathcal{L}$.

By the definition of pushforwards for closed embeddings, which are projective, we see that $s_*[Z]$ is the unique class in $\mathrm{CH}_k(L)$ which is just the inclusion of Z in L, using the closed embeddings $Z \xrightarrow{j} X \xrightarrow{s} L$. So Z defines a closed subvariety of L, which we will also denote by Z. Now pushing forward via s, which is also a regular embedding, we get by the definition of such pushforwards a Cartesian square

$$\begin{tikzcd} Z \arrow[r, "\cong"] \arrow[d, "j'"] & Z \arrow[d, "j"] \\ X \arrow[r, "s"] & L \end{tikzcd}$$

and a degree k class by embedding Z in the total space of the normal bundle of X in L restricted to Z. The normal bundle of X in L is simply $\mathcal{L}$, which restricts to $\mathcal{L}|_Z$ on Z with total space $L|_Z$. Now the normal cone of Z in Z, which is in fact Z, defines a subvariety of $L|_Z$ of degree k as the zero-subscheme of the restriction of $\mathcal{L}$ to Z. Its class in $\mathrm{CH}_k(L|_Z)$ corresponds to a class in $\mathrm{CH}_{k-1}(Z)$ by the isomorphism in Theorem 2.5. We know this class explicitly: as Z is a variety the line bundle $\mathcal{L}|_Z$ corresponds to a Cartier divisor. This divisor on Z immediately gives a cycle on X. It is clear that this class equals the class of the restriction to Z of the divisor on X corresponding to $\mathcal{L}$.

It is left to prove that the three geometric axioms (PB), (EH) and (CD) do in fact hold. The projective bundle property comes directly from Theorem 2.11. The extended homotopy axiom is a further generalisation of the homotopy invariance property and follows from the following theorem.

Theorem 3.7. *Let $\mathcal{E}$ be a locally free sheaf on $X \in \mathbf{Sch}_k$ of rank r. Then if $\pi : V \to X$ is an $\mathcal{E}$-torsor, then π^* is an isomorphism of Chow groups*

$$\pi^* : \mathrm{CH}_*(X) \to \mathrm{CH}_{*+r}(V).$$

The proof of this theorem follows from the result in Chapter 3 of [35], see in particular Remark 3.6.4.

The last axiom does not look like the cellular decomposition property we saw before for the Chow group. We will show that this axiom is however implied by a weaker version of the cellular decomposition property in Theorem 2.14.

Theorem 3.8. *The axiom (CD) of oriented Borel-Moore homology functors for the Chow group is implied by the statement:*
consider a scheme $X \in \mathbf{Sch}_k$ which has a filtration

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_k = X$$

such that $X_i \backslash X_{i-1}$ has irreducible components $U_{i,\alpha}$ which are affine spaces $\mathbb{A}^{n_{i,\alpha}}$. If the closures $V_{i,\alpha} = \overline{U_{i,\alpha}}$ of all such components are smooth, then $\mathrm{CH}_*(X)$ is generated by the classes $[V_{i,\alpha}]$.

Proof. Let E be the subscheme of W as in the axiom (CD). Write $X_j^{(i)} = p_i^*(X_j)$ for the jth variable on the ith factor of W. Then we have that E_{red} is the union of the divisors

$$E_i = \mathbb{P}^N \times \ldots \times \mathbb{P}^{N-1} \times \ldots \times \mathbb{P}^N,$$

where the closed embedding $\mathbb{P}^{N-1} \to \mathbb{P}^N$ at the ith coordinate is given by the vanishing of the last variable $X_n^{(i)}$. Let us fist say something about the Chow group of spaces of these forms.

Consider a space $X = \mathbb{P}^{M_1} \times \ldots \times \mathbb{P}^{M_t}$. Then repeated application of the projective bundle property shows that $\mathrm{CH}_*(X)$ is the free abelian group on the classes

$$[\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_t}]$$

for all inclusions $\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_t} \to \mathbb{P}^{M_1} \times \ldots \times \mathbb{P}^{M_t}$ defined by the vanishing of $M_i - m_i$ distinct coordinates on the ith factor. The classes of these subvarieties $\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_t}$ are clearly independent of the choice of these coordinates by the definition of rational equivalence.

Since each $\mathbb{P}^M$ satisfies the condition of the cellular decomposition axiom, by the decomposition

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \ldots \subseteq \mathbb{P}^M,$$

we can define a decomposition of E. Indeed, consider

$$\prod_{i=1}^{r} \mathbb{P}^0 = \coprod_{\sum m_i = 0} \prod_i \mathbb{P}^{m_i} \subseteq \coprod_{\sum m_i = 1} \prod_i \mathbb{P}^{m_i} \subseteq \ldots \subseteq \coprod_{\sum m_i = N^2-1} \prod_i \mathbb{P}^{m_i} = E$$

where each union is taken over all products of projective subspaces of $\mathbb{P}^N$ such that the total dimension is fixed. So in particular, each occurrence of m_i lies between 0 and N. This decomposition does indeed satisfy the necessary conditions and it is easily checked that the closure of each irreducible component of the complement of consecutive parts is exactly

$$\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_t}$$

where $0 \leq m_i \leq N$ for each i, but also $m_i \leq N-1$ for at least one i. So $\mathrm{CH}_*(E)$ is the free abelian group on the classes of these subvarieties. On the other hand we know that $\mathrm{CH}_*(W)$ is the free abelian group on these classes and the additional class

$$\mathbb{P}^N \times \ldots \times \mathbb{P}^N.$$

This proves that the map $\mathrm{CH}_*(E) \to \mathrm{CH}_*(W)$ is indeed injective. $\square$

So we conclude the following theorem where we calculate the formal group law, whose existence is guaranteed by Theorem 3.5.

Theorem 3.9. *The Chow group on $\boldsymbol{Sch}_k$ is an oriented Borel-Moore homology theory. The formal group law is the additive one, i.e. it is given by*

$$F_{\mathrm{CH}_*}(u, v) = u + v.$$

Proof. The axioms have been verified. The formal group law is uniquely determined by the property

$$\tilde{c}_1(\mathcal{L} \otimes \mathcal{M}) = \tilde{c}_1(\mathcal{L}) + \tilde{c}_1(\mathcal{M})$$

given in Theorem 2.9. $\square$

K-theory as an oriented Borel-Moore homology

Now let us turn our attention to K-theory. Although we do have a lot of structure which looks like what we need for an oriented Borel-Moore homology theory, we do not have a grading on $K_\bullet$. One way to fix this problem is to formally add a grading variable as done in Example 1.1.5 and Example 2.2.5 in [35].

So define for any $X \in \mathbf{Sch}_k$ the following graded group

$$K_\bullet[\beta, \beta^{-1}](X) := K_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}],$$

called *graded K-theory.* where we use the natural $\mathbb{Z}$-module structure on $K_\bullet(X)$ coming from the abelian group structure. A basis for this group is given by the elements of the form $[\mathcal{F}]\beta^k$ for $\mathcal{F}$ a coherent sheaf on X and k any integer. The integer k is called the degree of the element, which makes $K_\bullet[\beta, \beta^{-1}](X)$ into a graded group.

We will need an extension of the pullbacks and pushforwards.

Definition 3.10. We define the following data for an oriented Borel-Moore homology theory on $K_\bullet[\beta, \beta^{-1}]$:

(i) Let $f : X \to Y$ be a morphism and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X and Y, respectively. If f is projective, then we define the pushforward for $K_\bullet[\beta, \beta^{-1}]$ on basis elements by

$$f_* \left([\mathcal{F}]\beta^k\right) = \left(f_![\mathcal{F}]\right)\beta^k.$$

For f a smooth morphism of dimension d we define

$$f^* \left([\mathcal{G}]\beta^l\right) = \left(f^*[\mathcal{G}]\right)\beta^{l+d}.$$

(ii) The product is given by mapping $[\mathcal{F}]\beta^k \otimes [\mathcal{G}]\beta^l$ in $\mathrm{CH}_k(X) \otimes \mathrm{CH}_l(Y)$ to

$$[\mathcal{F} \boxtimes \mathcal{G}]\beta^{k+l} \in \mathrm{CH}_{k+l}(X \times Y).$$

Here we write $\mathcal{F} \boxtimes \mathcal{G}$ for the tensor product of the pullbacks of the sheaves via the respective projections. Note that this gives a coherent sheaf, as the pullback of a coherent sheaf is coherent for Noetherian schemes and coherence is preserved under taking the tensor product.

From the properties of the pullbacks and pushforwards for K-theory we immediately get that these maps are functorial as well. We now prove that the defined theory also satisfies (BM2) and (BM3).

Theorem 3.11. *Consider the Cartesian square*

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where f is projective and g is flat of relative dimension d. Then by stability of these properties under base extension we find that f' is projective and g' is flat of relative dimension d. Then we have the equality of maps

$$g^* f_* = g'^* f'_*.$$

Proof. Let $\mathcal{F}$ be any coherent sheaf on X. Then we find that

$$\begin{aligned} g^* f_* \left([\mathcal{F}]\beta^k \right) =& g^* \left(\sum_{m=0}^{\infty} (-1)^m [R^m f_* \mathcal{F}]\beta^k \right) \\ =& \sum_{m=0}^{\infty} (-1)^m [g^* R^m f_* \mathcal{F}]\beta^{k+d} \\ =& \sum_{m=0}^{\infty} (-1)^m [R^m f'_* \left(g'^* \mathcal{F} \right)]\beta^{k+d} \\ =& f'_* \left([g'^* \mathcal{F}]\beta^{k+d} \right) \\ =& f'_* g'^* \left([\mathcal{F}]\beta^k \right). \end{aligned}$$

where we used that higher direct image functors commute with flat base extension, see for example [26, Proposition III.9.3]. □

Theorem 3.12. *Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms of schemes in $\boldsymbol{Sch}_k$ and let $f \times g$ be the product map $X' \times Y' \to X \times Y$.*

(i) If both f and g are proper, then we have

$$(f \times g)_* \circ \times = \times \circ (f_* \times g_*).$$

(ii) If f and g are smooth morphisms of relative dimensions d and e, then the equality

$$(f \times g)^* \circ \times = \times \circ (f^* \times g^*).$$

holds for maps

$$K_\bullet[\beta, \beta^{-1}]_*(X \times Y) \to K_\bullet[\beta, \beta^{-1}]_{*+d+e}(X' \times Y').$$

Proof. Let $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{G}$ and $\mathcal{G}'$ be coherent sheaves on X, X', Y and Y' respectively.

(i) For the first part we will need the identity

$$R^m(f \times g)[\mathcal{F}' \otimes \mathcal{G}'] = \bigoplus_{i+j=m} R^i f_* \mathcal{F}' \boxtimes R^j g_* \mathcal{G}$$

derived in Result 6.8.7.1 from [23] part III. One can now write

$$\begin{aligned} (f \times g)_*([\mathcal{F}']\beta^k \times [\mathcal{G}']\beta^l) =& \sum_{m=0}^{\infty} (-1)^m [R^m (f \times g)_* (\mathcal{F}' \boxtimes \mathcal{G}')]\beta^{k+l} \\ =& \sum_{m=0}^{\infty} (-1)^m \sum_{i+j=m} [R^i f_* \mathcal{F}' \boxtimes R^j g_* \mathcal{G}']\beta^{k+l} \\ =& \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \left([R^i f_* \mathcal{F}']\beta^k \times [R^j g_* \mathcal{G}']\beta^l \right) \\ =& \left(\sum_{i=0}^{\infty} (-1)^i [R^i f_* \mathcal{F}']\beta^k \right) \times \left(\sum_{j=0}^{\infty} (-1)^j [R^j g_* \mathcal{G}']\beta^l \right) \\ =& f_*[\mathcal{F}']\beta^k \times g_*[\mathcal{G}']\beta^l. \end{aligned}$$

(ii) Consider the following diagram

$$\begin{tikzcd}
X' \times Y' \arrow[rr, "\mathrm{pr}_2"] \arrow[dr, "f\times g"] \arrow[dd, "\mathrm{pr}_1"] & & Y' \arrow[dr, "g"] & \\
& X \times Y \arrow[rr, "\mathrm{pr}_2"] \arrow[dd, "\mathrm{pr}_1"] & & Y \\
X' \arrow[dr, "f"] & & & \\
& X & &
\end{tikzcd}$$

Following the maps around this diagram, we find

$$\begin{aligned}
(f \times g)^*([\mathcal{F}]\beta^k \times [\mathcal{G}]\beta^l) =& (f \times g)^* \left([\mathcal{F} \boxtimes \mathcal{G}]\beta^{k+l}\right) \\
=& (f \times g)^* \left([\mathrm{pr}_1^* \mathcal{F} \otimes \mathrm{pr}_2^* \mathcal{G}]\beta^{k+l}\right) \\
=& [(f \times g)^* (\mathrm{pr}_1^* \mathcal{F} \otimes \mathrm{pr}_2^* \mathcal{G})]\beta^{k+l+d+e} \\
=& [(f \times g)^* \mathrm{pr}_1^* \mathcal{F} \otimes (f \times g)^* \mathrm{pr}_2^* \mathcal{G}]\beta^{k+l+d+e} \\
=& [\mathrm{pr}_1^* f^* \mathcal{F} \otimes \mathrm{pr}_2^* g^* \mathcal{G}]\beta^{k+l+d+e} \\
=& [f^* \mathcal{F} \boxtimes g^* \mathcal{G}]\beta^{k+l+d+e}.
\end{aligned}$$

Hence

$$(f \times g)^*([\mathcal{F}]\beta^k \times [\mathcal{G}]\beta^l) = f^*[\mathcal{F}]\beta^k \times g^*[\mathcal{G}]\beta^l$$

which concludes the proof. □

Before examining the validity of the three geometric axioms we will look at the theory of Chern classes in graded K-theory, as computed in [35].

Theorem 3.13. *The first Chern class operators as defined in Definition 3.4 for the theory $K_\bullet[\beta, \beta^{-1}]$ for a line bundle $\mathcal{L}$ on any $X \in \boldsymbol{Sch}_k$ are given by*

$$\tilde{c}_1(\mathcal{L}) : K_\bullet[\beta, \beta^{-1}]_*(X) \to K_\bullet[\beta, \beta^{-1}]_{*-1}(X), \qquad u \mapsto (1 - [\mathcal{L}^\vee])\, \beta^{-1} \cdot u.$$

Here we use the $K^\bullet$-module structure on $K_\bullet$ to express the first Chern class operators in terms of multiplication by classes of locally free sheaves.

Proof. Let us write L for the total space of the line bundle $\mathcal{L}$. Consider the image $s(X)$ of the zero-section s. It is clear that $s(X)$ is an effective divisor on L corresponding to the line bundle $\pi^*\mathcal{L}$. This gives the following exact sequence

$$0 \to \pi^*\mathcal{L}^\vee \to \mathcal{O}_{s(X)} \to \mathcal{O}_L \to 0.$$

This gives on classes in K-theory:

$$[s_*\mathcal{O}_X] = [\mathcal{O}_{s(X)}] = [O_L] + [\pi^*\mathcal{L}^\vee].$$

If we push this identity forward to K-theory of X we get

$$[s^*s_*\mathcal{O}_X] = [O_X] + [\mathcal{L}^\vee]$$

since $\pi \circ s$ equals the identity map on X. If we now tensor with any locally free sheaf $\mathcal{M}$, which is exact so well-defined on classes of coherent sheaves, we get

$$\tilde{c}_1(\mathcal{L})[\mathcal{M}]\beta^k = s^*s_* \left([\mathcal{M}]\beta^k\right) = [s^*s_*\mathcal{M}]\beta^{k-1} = ([M] + [\mathcal{L}^\vee \otimes \mathcal{M}])\, \beta^{k-1}.$$

So indeed

$$\tilde{c}_1(\mathcal{L})u = (1 - [\mathcal{L}^\vee])\, \beta^{-1} \cdot u.$$ □

Using these Chern classes and Theorem 2.29 one can show that K-theory does indeed satisfy the projective bundle property. First of all one can multiply the basis $\mathcal{O}_{\mathbb{P}}(i)$ for $0 \leq i \leq q$ given in Theorem 2.29 with the invertible $\mathcal{O}_{\mathbb{P}}(-q)$ to find that the classes of $\mathcal{O}_{\mathbb{P}}(i)$ for $-q \leq i \leq 0$ also form a basis. Now note that $\tilde{c}_1(\mathcal{O}_{\mathcal{P}}(1))^i$ for $0 \leq i \leq q$ also form a basis as they arise from multiplying the vector of basis vectors $\mathcal{O}_{\mathbb{P}}(i)$ for $-q \leq i \leq 0$ by an upper triangular matrix with only invertible elements on the diagonal. For the other two axioms we have the following two results.

Theorem 3.14. *The theory $K_\bullet[\beta, \beta^{-1}]$ satisfies the extended homotopy axiom of oriented Borel-Moore homology theories.*

The proof follows through the same steps of the proof for the extended homotopy axiom in the Chow group, see Remark 3.6.4 in [35].

Now we address the statement of cellular decomposition. We will first prove that this graded K-theory satisfies a stronger statement, similar to the traditional cellular decomposition in the Chow group.

Theorem 3.15. *Let X be a scheme which admits a filtration*

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_k = X$$

such that $X_i \backslash X_{i-1}$ has irreducible components $U_{i,\alpha}$ which are affine spaces $\mathbb{A}^{n_{i,\alpha}}$. If the closures $V_{i,\alpha} = \overline{U_{i,\alpha}}$ of all such components are smooth, then the map

$$\bigoplus K_\bullet[\beta, \beta^{-1}]_*(V_{i,\alpha}) \to K_\bullet[\beta, \beta^{-1}]_*(X)$$

is a surjection.

Proof. By the excision property of K-theory we have that for the closed subscheme $X_{\mathrm{red}} \to X$ the graded groups $K_\bullet[\beta, \beta^{-1}]_*(X)$ and $K_\bullet[\beta, \beta^{-1}]_*(X_{\mathrm{red}})$ are isomorphic, since the complement is empty.
So we can assume that X is reduced and we proceed by Noetherian induction. So we can assume that

$$\bigoplus_{i \leq k-1} K_\bullet[\beta, \beta^{-1}]_*(V_{i,\alpha}) \to K_\bullet[\beta, \beta^{-1}]_*(X_{k-1})$$

is surjective.

Write k_α for the open inclusion $U_{N,\alpha} \to V_{N,\alpha}$ and consider the open and closed embeddings

$$V_{N,\alpha} \xrightarrow{i_\alpha} X \xleftarrow{j_\alpha} U_{N,\alpha}.$$

For two points that map to the same element of X, we have that the three local rings of these points are identical. So the higher Tor functors in the definition of transverse morphisms vanish and so these morphisms are transverse. The fibre product is simply $U_{N,\alpha}$ and note that i_α and j_α are trivially a projective morphism, and a smooth and quasi-projective morphism. So we may apply axiom (BM2) to the Cartesian square

$$\begin{tikzcd} U_{N,\alpha} \arrow[r, equal] \arrow[d, "k_\alpha"] & U_{N,\alpha} \arrow[d, "j_\alpha"] \\ V_{N,\alpha} \arrow[r, "i_\alpha"] & X \end{tikzcd}$$

This gives that

$$k_\alpha^* = j_\alpha^* i_{\alpha *}.$$

This identity fits in the following diagram, where the exact rows follow from Theorem 2.26

$$\begin{tikzcd}
K_\bullet[\beta,\beta^{-1}]_*(X_{k-1}) \arrow[r] & K_\bullet[\beta,\beta^{-1}]_*(X) \arrow[r, "\bigoplus j_\alpha^*"] & K_\bullet[\beta,\beta^{-1}]_*(\coprod_\alpha U_{N,\alpha}) \arrow[r] & 0 \\
\bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(X_{k-1}\cap V_{N,\alpha}) \arrow[r] & \bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(V_{N,\alpha}) \arrow[r, "\bigoplus k_\alpha^*"] \arrow[u, "\bigoplus i_{\alpha *}"] & \bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(U_{N,\alpha}) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Now any element of $K_\bullet[\beta,\beta^{-1}]_*(X)$ maps to an element of $\bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(U_{N,\alpha})$, which comes from $\bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(V_{N,\alpha})$ in the lower row. When we map this element back to $K_\bullet[\beta,\beta^{-1}]_*(X)$ we arrive at the element we started with modulo an element coming from $K_\bullet[\beta,\beta^{-1}]_*(X_{k-1})$. So the map

$$K_\bullet[\beta,\beta^{-1}]_*(X_{k-1}) \oplus \bigoplus_\alpha K_\bullet[\beta,\beta^{-1}]_*(V_{N,\alpha}) \to K_\bullet[\beta,\beta^{-1}]_*(X)$$

is surjective and by induction we are done. □

From this result we can conclude the axiom (CD) for an oriented Borel-Moore homology theory.

Theorem 3.16. *The graded K-theory $K_\bullet[\beta,\beta^{-1}]$ satisfies the cellular decomposition axiom from Definition 3.4.*

Proof. There is a general proof which shows that a functor which is an oriented Borel-Moore homology theory except possibly for (CD) axiom, but does have the classical decomposition property as in Theorem 3.15, does necessarily have to satisfy this last axiom as well. This proof is similar to that of Theorem 3.2 and can be found in [35, Lemma 5.2.10]. □

This gives us the following result.

Theorem 3.17. *The theory $K_\bullet[\beta,\beta^{-1}]$ with the defined pullbacks, pushforwards and product is an oriented Borel-Moore homology. The formal group law is given by*

$$F_K(u,v) = u + v - \beta uv.$$

So the formal group law is in fact periodic, and hence multiplicative.

Proof. We already checked all the necessary axioms. We only address the verification of the formal group law. This follows directly from the fact, that

$$\begin{aligned}
\tilde{c}_1(\mathcal{L}\otimes\mathcal{M}) =&(1-[\mathcal{L}^\vee\otimes\mathcal{M}^\vee])\beta^{-1}\\
=&(1-[\mathcal{L}^\vee][\mathcal{M}^\vee])\beta^{-1}\\
=&(1-[\mathcal{L}^\vee])\beta^{-1} + (1-[\mathcal{M}^\vee])\beta^{-1} - \beta^{-1}\left(1-[\mathcal{L}^\vee]-[\mathcal{M}^\vee]+[\mathcal{L}^\vee][\mathcal{M}^\vee]\right)\\
=&(1-[\mathcal{L}^\vee])\beta^{-1} + (1-[\mathcal{M}^\vee])\beta^{-1} - \beta\left((1-[\mathcal{L}^\vee])\beta^{-1}\cdot(1-[\mathcal{M}^\vee])\beta^{-1}\right)\\
=&\tilde{c}_1(\mathcal{L}) + \tilde{c}_1(\mathcal{M}) - \beta\tilde{c}_1(\mathcal{L})\tilde{c}_1(\mathcal{M}).
\end{aligned}$$

This completes the proof. □

We will later see that these oriented Borel-Moore homology functors are not just any examples of such theories. We will see that they are in fact the universal ones with their given formal group laws.

3.3 Algebraic cobordism

We will now look at algebraic cobordism as defined by Levine and Morel in [35].

Universal oriented Borel-Moore homology

We will later look at the definition, but first we will state the most important theorem on algebraic cobordism.

Theorem 3.18 ([35, Theorem 7.1.3]). *There exists an oriented Borel-Moore homology theory Ω_* which is universal, in the sense that for other such homology theories A_* there exists a unique morphism of oriented Borel-Moore homology theories*

$$\vartheta_{A_*} : \Omega_* \to A_*.$$

This oriented Borel-Moore homology theory is called algebraic cobordism.

Let $\mathcal{M}_*(X)$ be the set of projective morphisms $f : Y \to X$ with $Y \in \mathbf{Sm}_k$, modulo isomorphisms over X. Give this set a monoid structure by disjoint union of the domains of the morphisms and grade it by the dimension of the components of the domain. Let $\mathcal{M}_*(X)^+$ denote its graded group completion. Now since any projective morphism $f : Y \to X$ with $Y \in \mathbf{Sm}_k$ gives an element $f_*1_{\Omega_*(Y)}$ in $\Omega_*(X)$ denoted by $[f : Y \to X] \in \Omega_*(X)$, we get an obvious map $\mathcal{M}_*(X)^+ \to \Omega_*(X)$. Note that for any projective morphism $j : X \to W$ and object $[f : Y \to X] \in \Omega_*(X)$, the pushforward is given by composing

$$j_*[Y \to X] = j_* \circ f_*1_{\Omega_*(Y)} = (j \circ f)_*1_{\Omega_*(Y)} = [j \circ f : Y \to W]$$

and that the class $[X \to X]$ is in fact the identity $1_{\Omega_*(X)}$ in $\Omega_0(X)$.

Theorem 3.19 (Lemma 2.5.11 in [35]). *The map*

$$\mathcal{M}_*(X)^+ \to \Omega_*(X)$$

is a homomorphism of graded groups and it is in fact surjective.

Hence algebraic cobordism is generated by projective morphisms over the space considered.

Algebraic cobordism over a point

As Ω_* is an oriented Borel-Moore homology theory, there must exist a morphism of graded groups $\mathbb{L}_* \to \Omega_*(k)$ which relates the Chern classes of products of line bundles to the Chern classes of the respective line bundles. The coefficients of the associated formal group law for this universal oriented Borel-Moore homology theory lie in algebraic cobordism over a point. This gives $\Omega_*(k)$ a formal group law of rank one, and this pair is in fact the universal one.

Theorem 3.20 ([35, Theorem 4.3.7]]). *The group homomorphism*

$$\mathbb{L}_* \to \Omega_*(k)$$

belonging to the structure of the universal oriented Borel-Moore homology theory, is an isomorphism of graded groups.

A direct consequence is that the formal group law of any oriented Borel-Moore homology theory A_* is determined by the unique morphism $\vartheta_{A_*} : \Omega_* \to A_*$ over a point:

$$\vartheta_{A_*}(k) : \mathbb{L}_* \cong \Omega_*(k) \to A_*(k).$$

Apparently algebraic cobordism over a point is interesting by itself. In the rational case we can even give an explicit basis. To that end let us recall the following notion.

Definition 3.21. Let n be a fixed non-negative integer. A *partition* of size n is a multi-set of positive integers, i.e. an object uniquely determined by its elements and their multiplicities, such that the sum of the elements counted with their respective multiplicities is n. The set of all partitions of size n is denoted by $\mathcal{P}_n$.
If λ is a partition then the size of λ is defined as the sum of its elements, and the length of the partition $\ell(\lambda)$ is the sum of the multiplicities of the distinct elements in λ.

For any partition λ of n with elements $\lambda_1, \ldots, \lambda_{\ell(\lambda)}$ we can define the following space over k

$$\mathbb{P}^\lambda = \mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_{\ell(\lambda)}}.$$

This defines an element $[\mathbb{P}^\lambda \to \operatorname{Spec} k]$ in $\Omega_n(k)$ and hence a map $\phi : \mathcal{P}_n \to \Omega_n(k)$. By Theorem 3.20 and Theorem A.4 we see that $\Omega_n(k)$ is a polynomial ring. The group homomorphism ϕ is injective, which is best expressed in the following manner.

Theorem 3.22 ([36, Corollary 3])**.** *For any non-negative n we have*

$$\Omega_n(k) \otimes \mathbb{Q} = \sum_{\lambda \in \mathcal{P}_n} \mathbb{Q}[\mathbb{P}^\lambda].$$

So $\Omega_*(k) \otimes \mathbb{Q}$ is a polynomial ring in the classes $[\mathbb{P}^l]$ of degree l. Note that we did not lose any information by considering $\Omega_*(k)$ as a $\mathbb{Q}$-module instead of an abelian group, since there is no torsion.

To express the class of any n-dimensional smooth projective k-scheme in this basis indexed by $\mathcal{P}_n$ we have the following theorem. It uses polynomials of degree n in n variables $x_1, \ldots, x_n$ where each x_i is of degree i. The $\mathbb{Q}$-vector space of these polynomials is denoted by $\mathcal{C}_n$.

Theorem 3.23 ([35, Remark 4.3.4])**.** *The pairing of $\mathbb{Q}$-vector spaces*

$$\rho : \Omega_n(k) \otimes \mathbb{Q} \times \mathcal{C}_n \to \mathbb{Q}$$

given by computing the Chern numbers of a space $[X] \in \Omega_n(k) \otimes \mathbb{Q}$ and a polynomial $\Theta(x_1, x_2, \ldots, x_n) = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$ of graded degree n is well-defined. Symbolically we have

$$\rho([X], \Theta) = \int_X \Theta(c_1(T_X), c_2(T_X), \ldots, c_n(T_X)).$$

Furthermore, this pairing is non-degenerate. In particular, the class of any smooth projective k-scheme in algebraic cobordism is uniquely determined by its Chern numbers.

Note that $\Omega_n(k) \otimes \mathbb{Q}$ and $\mathcal{C}_n$ do in fact have the same dimension as vector spaces over $\mathbb{Q}$. We have a bijection of basis elements, given by mapping any partition

$$\lambda = \{\underbrace{1, \ldots, 1}_{c_1}, \underbrace{2, \ldots, 2}_{c_2}, \ldots, \underbrace{k, \ldots, k}_{c_k}, \ldots\}$$

to the polynomial $x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n}$ of graded degree n.

Specific formal group laws

Let R_* be a graded ring with a formal group law, corresponding to a morphism $\mathbb{L}_* \to R_*$. Now note that for any oriented Borel-Moore homology theory A_* we get another such theory by taking the tensor product of the graded $\mathbb{L}_*$-modules A_* and R_*. Note that the morphism defining the formal group law factors through R_*, by which we can control this group law. In particular we can define an oriented Borel-Moore homology theory with any specified formal group law: let $S_* \subseteq R_*$ be the subring generated by the coefficients of the formal group law. Then any ring homomorphism $S_* \to T_*$ will give T_* the formal group law coming from R_*. Note that these natural rings associated to the additive and periodic formal group laws are $\mathbb{Z}$ and $\mathbb{Z}[\beta, \beta^{-1}]$ respectively.

We have actually already seen the universal oriented Borel-Moore homology theories with the additive and the periodic formal group laws as proven in Theorem 7.1.4 in [35].

Theorem 3.24. *The Chow group is the universal oriented Borel-Moore homology theory with an additive formal group law.*
So the natural morphism of oriented Borel-Moore homology theories

$$\mathbb{Z} \otimes_{\mathbb{L}_*} \Omega_* \to \mathrm{CH}_* .$$

is in fact an isomorphism.

We have a similar result identifying graded K-theory as the universal oriented Borel-Moore homology theory with a periodic formal group law, at least on an appropriate full subcategory of $\mathbf{Sch}_k$.

Theorem 3.25. *The unique morphism*

$$\mathbb{Z}[\beta, \beta^{-1}] \otimes_{\mathbb{L}_*} \Omega_* \to K_\bullet[\beta, \beta^{-1}]_*.$$

of oriented Borel-Moore homology theories with a periodic formal group law is an isomorphism over $\boldsymbol{Sm}_k$.

So for any oriented Borel-Moore homology theory A_* on $\mathbf{Sm}_k$ with a periodic formal group law, there exists a unique morphism

$$K_\bullet[\beta, \beta^{-1}]_* \to A_*$$

of oriented Borel-Moore homology theories.

3.4 Oriented Borel-Moore functors of geometric type

Levine and Morel defined Ω_* in the first place as a less structured universal homology theory and proved that it does in fact have the structure of an oriented Borel-Moore homology theory.

Oriented Borel-Moore functors with product

Let us consider Definition 2.1.11 from [35].

Definition 3.26. An *oriented Borel-Moore functor with product* A_* on some admissible subcategory $\mathcal{V}$ of $\mathbf{Sch}_k$ consists of

(D1) an additive functor
$$A_* : \mathcal{V}' \to \mathbf{Ab}_*, \qquad X \mapsto A_*(X);$$

(D2) a homomorphism of graded groups
$$f^* : A_*(X) \to A_{*+d}(Y)$$
for any smooth morphism $f : Y \to X$ in $\mathcal{V}$ of relative dimension d;

(D3) a homomorphism of graded abelian groups
$$\tilde{c}_1(\mathcal{L}) : A_*(X) \to A_{*-1}(X)$$
for every line bundle $\mathcal{L}$ over $X \in \mathcal{V}$;

(D4) an associative commutative unital bilinear graded pairing
$$\begin{aligned} A_*(X) \otimes A_*(Y) &\to A_*(X \times Y) \\ (u, v) &\mapsto u \times v \end{aligned}$$
for each two spaces X and Y in $\mathcal{V}$. The unit element is denoted by $1 \in A_0(\operatorname{Spec} k)$.

The following conditions are required to hold:

(A1) For smooth morphisms $g : Z \to Y$ and $f : Y \to X$ in $\mathcal{V}$ of pure relative dimension d and e, we have
$$(f \circ g)^* = f^* \circ g^* : A_*(X) \to A_{*+d+e}(X).$$
In addition $\mathrm{Id_X}^* = \mathrm{Id_{A_*(X)}}$.

(A2) For a projective $f : X \to Z$ and a smooth morphism $g : Y \to Z$ which are transverse in $\mathcal{V}$ we get the Cartesian square

$$\begin{tikzcd} X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

in which f' is projective and g' is smooth, since projective and smooth morphisms are stable under base extension. We have equality of the following maps

$$g^* f_* = f'_* g'^*.$$

(A3) Consider a projective morphism $f : Y \to X$ in $\mathcal{V}$. Then we have for all line bundles $\mathcal{L}$ over X that

$$f_* \circ \tilde{c}_1(f^*\mathcal{L}) = \tilde{c}_1(\mathcal{L}) \circ f_*.$$

(A4) Consider a smooth morphism $f : Y \to X$ in $\mathcal{V}$ of pure relative dimension. Then we have for all line bundles $\mathcal{L}$ over X that

$$\tilde{c}_1(f^*\mathcal{L}) \circ f^* = f^* \circ \tilde{c}_1(f^*\mathcal{L}).$$

(A5) The Chern class operators depend only on the isomorphism class of the line bundle and commute. So for two line bundles $\mathcal{L}$ and $\mathcal{M}$ on X we have

$$\tilde{c}_1(\mathcal{L}) \circ \tilde{c}_1(\mathcal{M}) = \tilde{c}_1(\mathcal{M}) \circ \tilde{c}_1(\mathcal{L}).$$

And if $\mathcal{L} \cong \mathcal{M}$ then $\tilde{c}_1(\mathcal{L}) = \tilde{c}_1(\mathcal{M})$.

(A6) For two projective morphisms f and g in $\mathcal{V}$

$$\times \circ (f_* \times g_*) = (f \times g)_* \circ \times.$$

(A7) For two smooth morphisms f and g in $\mathcal{V}$ of pure relative dimension

$$\times \circ (f^* \times g^*) = (f \times g)^* \circ \times.$$

(A8) Let X and Y be in $\mathcal{V}$ and pick $\alpha \in A_*(X)$, $\beta \in A_*(Y)$ and a line bundle $\mathcal{L}$ on X. Then

$$\tilde{c}_1(\mathcal{L})(\alpha) \times \beta = \tilde{c}_1(p_1^*\mathcal{L})(\alpha \times \beta).$$

It is clear that any oriented Borel-Moore homology theory on $\mathcal{V}$ defines an oriented Borel-Moore functor with product on $\mathcal{V}$. Note that we saw a few of these properties for the Chow group in Chapter 2.

Oriented Borel-Moore functors of geometric type

Unlike for oriented Borel-Moore homology theories, Chern classes of oriented Borel-Moore functors with product need not be governed by a formal group law. To guarantee this structure we need one more axiom.

Definition 3.27 ([35, Definition 2.1.12])**.** Let R_* be a graded ring. An *oriented Borel-Moore R_*-functor* is an oriented Borel-Moore functor with product together with a graded ring homomorphism

$$R_* \to A_*(k).$$

We will in particular be interested in $\mathbb{L}_*$-functors, since this gives a formal group law $F_A \in A_*(k)[[u,v]]$. The following definition [35, Definition 2.2.1] makes sure that this group law does what we want it to and reflects some geometric properties.

Definition 3.28. An *oriented Borel-Moore functor of geometric type* is an oriented Borel-Moore $\mathbb{L}_*$-functor which satisfies the following three axioms.

(Dim) For any $Y \in \mathbf{Sm}_k$ and a family $(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ of line bundles on Y with $n > \dim_k(Y)$, it holds that
$$\tilde{c}_1(\mathcal{L}_1) \circ \ldots \circ \tilde{c}_1(\mathcal{L}_n) = 0 \in A_*(Y).$$

(Sect) Let $Y \in \mathbf{Sm}_k$ have a line bundle $\mathcal{L}$ with a section s transverse to the zero-section and let $i : Z \to Y$ be the immersion of the closed zero-subscheme of s. Then we have
$$\tilde{c}_1(\mathcal{L})(1_Y) = i_*(1_Z).$$

(FGL) For two line bundles $\mathcal{L}$ and $\mathcal{M}$ over the same smooth base Y, we have
$$F_A(\tilde{c}_1(\mathcal{L}), \tilde{c}_1(\mathcal{M}))(1_Y) = \tilde{c}_1(\mathcal{L} \otimes \mathcal{M})(1_Y) \in A_*(Y).$$

Note that the axiom (FGL) is generalisation of Grothendieck's axiom that $\mathrm{Pic}(X) \to A_1(X)$ is a group homomorphism in [21].

As before a morphism of two theories is a natural transformation of the underlying functors, which respects the pullbacks and Chern class operators.

As announced these functors generalise the previously defined homology theories.

Theorem 3.29 (Theorem 4.1.10, Theorem 5.2.6 in [35])**.** *Every oriented Borel-Moore homology theory on $\mathcal{V}$ defines an oriented Borel-Moore functor of geometric type.*

So as before we have two interesting examples. The Chow group and the graded K-theory are examples of oriented Borel-Moore functors of geometric type on $\mathbf{Sch}_k$. The formal group laws are the same as for these theories as oriented Borel-Moore homology theories.

Universal oriented Borel-Moore functor of geometric type

Now we can construct the universal oriented Borel-Moore functor of geometric type as done in Section 2.4 in [35].

Definition 3.30. For any $X \in \mathcal{V}$ let $\mathcal{C}_*(X)$ the free abelian group generated by ordered sequences of the form
$$(f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r)$$
with f a projective morphism, $Y \in \mathbf{Sm}_k$ irreducible over k, and $\mathcal{L}_i$ line bundles over Y. We identify these elements if the morphisms are isomorphic over X, and the line bundles are isomorphic via this isomorphism or simply a reordering. These elements modulo these conditions are called *cobordism cycles*. The grading is given by $\dim_k(Y) - r$. We define $\underline{\Omega}_*(X)$ as the quotient of $\mathcal{C}_*(X)$ by the subgroup generated by the following two relations.

(i) For any smooth equi-dimensional morphism $\pi : Z \to Y$ and line bundles $\mathcal{M}_1, \ldots, \mathcal{M}_s$ on Z with $s > \dim_k Z$ and any line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on Y we have
$$(f : Y \to X, \pi^*(\mathcal{M}_1), \ldots, \pi^*(\mathcal{M}_s), \mathcal{L}_1, \ldots, \mathcal{L}_r) = 0.$$

(ii) For any cobordism cycle $(f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r)$ over X, and section $s : Y \to \mathcal{L}$ of a line bundle with corresponding divisor $i : D \to Y$ we have
$$(f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r, \mathcal{L}) = (f \circ i : D \to X, i^*\mathcal{L}_1, \ldots, i^*\mathcal{L}_r).$$

The graded group $\underline{\Omega}_*$ is called algebraic *pre-cobordism.*

A theory of pushforwards and Chern classes on algebraic pre-cobordism is easily constructed as follows.

Definition 3.31. For any projective morphism $g : X \to X'$ we define a map $g_* : \underline{\Omega}_*(X) \to \underline{\Omega}_*(X')$ on generators as

$$g_*(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r) = (g \circ f : Y \to X', \mathcal{L}_1, \dots, \mathcal{L}_r).$$

For any line bundle $\mathcal{L}$ on X we get a map $\tilde{c}_1(\mathcal{L}) : \underline{\Omega}_*(X) \to \underline{\Omega}_{*-1}(X)$ on cobordism cycles as

$$\tilde{c}_1(\mathcal{L})(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r) = (f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r, f^*\mathcal{L}).$$

These definitions are easily checked to be well-defined. They also give us the following result.

Lemma 3.32. *Algebraic pre-cobordism satisfies the axioms (Dim) and (Sect).*

To construct a theory which also satisfies *(FGL)* we need to make sure a formal group law exists, which we can do by the following construction.

Definition 3.33. Define Ω_* as the quotient of $\underline{\Omega}_* \otimes_{\mathbb{Z}} \mathbb{L}_*$ by the subgroup generated by all relations

$$F_{\mathbb{L}}(\tilde{c}_1(\mathcal{L}), \tilde{c}_1(\mathcal{M}))(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r) = \tilde{c}_1(\mathcal{L} \otimes \mathcal{M})(f : Y \to X, \mathcal{L}_1, \dots, \mathcal{L}_r).$$

Here $F_{\mathbb{L}}$ is the formal group law on $\underline{\Omega}_* \otimes_{\mathbb{Z}} \mathbb{L}_*$ given by the obvious map $\mathbb{L}_* \to \underline{\Omega}_* \otimes_{\mathbb{Z}} \mathbb{L}_*$.
This construction immediately implies the following result.

Theorem 3.34. *Let A_* be any oriented Borel-Moore functor of geometric type. There is a unique morphism*

$$\Omega_* \to A_*,$$

so Ω_* is the universal oriented Borel-Moore functor of geometric type.

We would like to consider Ω_* also as an oriented Borel-Moore homology theory, so in particular we would need to define pullbacks for all l.c.i. morphisms. This takes the better part of [35] and follows the construction of the Chow group in [18].

Theorem 3.35 (Theorem 4.1.11, Theorem 7.1.1 in [35])**.** *With the underlying structure of projective pushforwards, smooth pullbacks and first Chern class operators, there is only one structure for Ω_* which makes it into an oriented Borel-Moore homology theory.*
Viewed as such, Ω_* is the universal homology theory described in Theorem 3.18.

3.5 Oriented cohomology theories

Instead of considering homology theories one would like to study cohomology theories similar to the one presented in Section 1.5. The following cohomology theory is a direct algebraic analogue of the axioms in [46] which define complex oriented cobordism as the universal theory satisfying some axioms, expressing the functorial and geometric nature of the theory.

Definition 3.36 ([35, Definition 1.1.2])**.** An *oriented cohomology theory* consists of

(D1) an additive functor

$$A^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{R}^*.$$

(D2) a homomorphism of graded $A^*(X)$-modules

$$f_* : A^*(Y) \to A^{*+d}(X)$$

for any projective morphism $f : Y \to X$ of relative codimension d. The $A^*(X)$-module structure on $A^*(Y)$ is given by the homomorphism $f^* : A^*(X) \to A^*(Y)$.

The following four axioms must be satisfied.

(A1) For two composable projective morphisms $g : Z \to Y$ en $f : Y \to X$ in $\mathcal{V}$ of pure relative dimension respectively d and e we have

$$(f \circ g)_* = g_* \circ f_* : A^*(X) \to A^{*+d+e}(X).$$

In addition $\mathrm{Id}_{\mathrm{X}*} = \mathrm{Id}_{\mathrm{A}_*(\mathrm{X})}$.

(A2) For a smooth $f : X \to Z$ of relative dimension d and a morphism $g : Y \to Z$ which are transverse in $\mathcal{V}$ we get the Cartesian square

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

We have equality of the following maps

$$g^* f_* = f'_* g'^*.$$

(PB) Consider some $X \in \mathbf{Sm}_k$ with $\mathcal{E}$ some rank $q+1$ vector bundle. Let $\mathcal{O}(1) \to \mathbb{P}(\mathcal{E})$ be the canonical bundle on $\mathbb{P}(\mathcal{E})$ with zero-section $s : \mathbb{P}(\mathcal{E}) \to \mathcal{O}(1)$. Using the multiplicative unit $1 \in A^0(\mathbb{P}(\mathcal{E}))$ we can define an element $\xi \in A^1(\mathbb{P}(\mathcal{E}))$ by $\xi = s^* s_*(1)$. Then $A^*(\mathbb{P}(\mathcal{E}))$ is a free $A^*(X)$-module with basis $1, \xi, \xi^2, \ldots, \xi^q$.

(EH) For any vector bundle $\mathcal{E}$ of rank r over $X \in \mathcal{V}$, and any $\mathcal{E}$-torsor $p : V \to X$, we have an isomorphism
$$p^* : A^*(X) \to A^*(V).$$

Note that axiom (D2) is simply the projection formula we saw in Lemma 2.19 for the Chow ring and in Lemma 2.28 for sheaves and K-theory.

Consider a homology theory A_* with smooth pullbacks and projective pushforwards. Note that for schemes X of pure dimension n we can define a graded group by $A^*(X) = A_{n-*}(X)$ by inverting the grading on the given graded group. If X is smooth we can extend this definition by applying this definition to all connected components and taking the direct sum. It is easily seen that the exterior product translates to a similar structure on A^*. Now for any homomorphism of $A_*(X) \to A_{*+d}(Y)$ for X and Y of dimensions n and m we get a corresponding morphism

$$A^*(X) \to A^{*+m-n-d}(Y).$$

In particular for any l.c.i. morphism $f : X \to Y$ in an oriented Borel-Moore homology theory, we get a graded group homomorphism

$$f_* : A^*(X) \to A^*(Y).$$

We also get a graded group homomorphism which increases the degree by d

$$f^* : A^*(X) \to A^{*+d}(Y)$$

for a projective morphism of relative dimension d.

It is important to note that the diagonal embedding $\delta : X \to X \times X$ is a regular embedding as X is smooth. If we compose the pullback of this map with the exterior product we get a product

$$A^*(X) \otimes A^*(X) \to A^*(X \times X) \xrightarrow{\delta^*} A^*(X),$$

which gives $A^*(X)$ the structure of a graded ring.

That each cohomology theory gives a homology theory by the inverse process is obvious. The question is how much of the additional structure of the respective homology and cohomology is preserved under this correspondence. That question is answered by the following theorem.

Theorem 3.37 (Theorem 5.2.1 in [35])**.** *The above correspondence is an equivalence of the category of oriented Borel-Moore homology theories on* $\boldsymbol{Sm}_k$ *and the category of oriented cohomology theories on* $\boldsymbol{Sm}_k$*.*

Chapter 4

Algebraic cobordism by double point relations

The construction given in Chapter 3 for algebraic cobordism is heavily involved and lacks the geometric description of its topological counterpart. Although a naive form of cobordantness does hold in Ω_*, these relations do not satisfy to define the theory.

The article [36] solves this problem by providing a geometric definition for an oriented Borel-Moore functor of geometric type ω_*. After proving that $\omega_* \cong \Omega_*$ as oriented Borel-Moore $\mathbb{L}_*$-functors, Levine and Pandharipande conclude that ω_* is the universal such theory and has a unique structure of an oriented Borel-Moore homology theory.

4.1 Naive cobordism theory

Mimicking the definition of cobordisms in topology one may be interested in the fibres of a morphism $W \to \mathbb{P}^1 \times X \to \mathbb{P}^1$. Obviously these fibres W_ζ for $\zeta \in \mathbb{P}^1$ define classes $[W_\zeta \to X] \in \Omega_*(X)$. One might wonder if these classes are invariant under choice of the fibre. This is examined in Section 2.3 of [35].

Definition 4.1. Let $f : W \to \mathbb{P}^1 \times X$ be a projective morphism with both W and X in $\mathbf{Sm}_k$, such that f is transverse, in the sense of Definition 3.3, to the inclusions $\{0, \infty\} \times X \to \mathbb{P}^1 \times X$. Such an f is called a *geometric cobordism*. Two fibres of a geometric cobordism are called *naively algebraically cobordant*.

Let us first show that naively algebraically cobordant spaces over X define the same class in $\Omega_*(X)$.

Theorem 4.2 ([35, Lemma 2.3.3])**.** *Let $f : W \to \mathbb{P}^1 \times X$ be a geometric cobordism and let $f_0 : W_0 \to X$ and $f_\infty : W_\infty \to X$ be the fibres of the composition $W \to \mathbb{P}^1 \times X \to \mathbb{P}^1$ over 0 and ∞. Then the equality*

$$[W_0 \to X] = [W_\infty \to X]$$

holds in $\Omega_*(X)$.

Proof. Clearly the maps $i_0 : W_0 \to W$ and $i_\infty : W_\infty \to W$ are the inclusions of zero-subschemes of sections of $(\mathrm{pr}_2 \circ f)^* \mathcal{O}_{\mathbb{P}^1}$. Hence by (Sect) we get

$$[i_0 : W_0 \to W] = i_{0*} 1_{\Omega_*(W)} = \tilde{c_1}\left((\mathrm{pr}_2 \circ f)^* \mathcal{O}_{\mathbb{P}^1}\right) = i_{\infty *} 1_{\Omega_*(W)} = [i_\infty : W_0 \to W]. \qquad \square$$

So one might attempt to examine the quotient of $\mathcal{M}(X)$ by the subgroup generated by all the elements $[W_0 \to X] - [W_\infty \to X]$ for naively algebraically cobordant pairs W_0 and W_∞. Let us call this quotient $\Omega_*^{\mathrm{naive}}(X)$, where the grading is given as before which is respected by geometric cobordism relations. In general the relations given by the geometric cobordisms are not enough and Ω_*^{naive} is not equal to Ω_*, not even over an algebraically closed point as shown in [35] in Remark 1.2.9. Levine and Morel show that two curves over k in $\Omega_1^{\text{naive}}(X)$ are equivalent if and only if they have the same genus and number of connected components. However $\Omega_1(k) = \mathbb{L}_1 = \mathbb{Z}$ is generated by the class of $\mathbb{P}^1$. Indeed, by Theorem 3.23 we have that an irreducible curve C of genus g must be in the class $[C] = (1-g)[\mathbb{P}^1]$, as all relevant Chern numbers agree. This is clearly impossible in $\Omega_1^{\text{naive}}(X)$.

4.2 Double point relations

We will now give a correct geometric definition for algebraic cobordism. All the definitions, statements and proofs in this section come from Levine and Pandharipande [36].

As before we will work with $\mathcal{M}(X)$ which is the set of isomorphism classes of projective morphisms $f : Y \to X$ over X with Y irreducible. Clearly this is a monoid under the disjoint union of domains, so let $\mathcal{M}(X)^+$ be its group completion. It is clear that $\mathcal{M}(X)^+$ is the free abelian group on isomorphism classes of projective morphisms $f : Y \to X$ for Y irreducible. The image of such a map in $\mathcal{M}(X)^+$ is denoted by $[f : Y \to X]$. By this construction we can define a grading on $\mathcal{M}(X)^+$ by letting $\mathcal{M}_n(X)^+$ be the subgroup generated by morphisms $[f : Y \to X]$ with Y irreducible and of dimension n.

Definition 4.3. For $Y \in \mathbf{Sm}_k$ of pure dimension we say that a morphism

$$\pi : Y \to \mathbb{P}^1$$

is a *double point degeneration* over $\zeta \in \mathbb{P}^1$ if

$$\pi^{-1}(\zeta) = A \coprod_D B$$

for two smooth divisors A and B which intersect transversely at $D = A \cap B$ called the *double point locus* of π over $\zeta \in \mathbb{P}^1$.

Whenever we have a double point degeneration we get some interesting bundles on the double point locus. Using the normal bundles $\mathcal{N}_{A/D}$ and $\mathcal{N}_{B/D}$ of D in respectively A and B, we get bundles $\mathcal{O}_D \oplus \mathcal{N}_{A/D}$ and $\mathcal{O}_D \oplus \mathcal{N}_{B/D}$ on D. Since $A + B$ is rationally equivalent to any other divisor $\pi^{-1}\zeta$, not meeting D we see that $\mathcal{O}_Y(A+B)$ restricted to D is trivial. This implies that

$$\mathcal{N}_{A/D} \otimes \mathcal{N}_{B/D} \cong \mathcal{O}_A(D)|_D \otimes \mathcal{O}_B(D)|_D \cong (\mathcal{O}_Y(B) \otimes \mathcal{O}_Y(A))\,|_D \cong \mathcal{O}_D,$$

where we used that A and B intersect transversely in the penultimate equivalence. Now we get an isomorphism of bundles: $(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \otimes \mathcal{N}_{B/D} \cong \mathcal{N}_{B/D} \oplus \mathcal{O}_D$. Hence $\mathcal{O}_D \oplus \mathcal{N}_{A/D}$ and $\mathcal{O}_D \oplus \mathcal{N}_{B/D}$ have isomorphic associated projective bundles

$$\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \qquad \text{and} \qquad \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{B/D})$$

over D. We will write $\mathbb{P}(\pi) \to D$ for the isomorphism class of these bundles.

Definition 4.4. For any $Y \in \mathbf{Sm}_k$ of pure dimension and every projective morphism

$$\pi : Y \to X \times \mathbb{P}^1$$

such that the composition

$$\pi_2 : Y \to X \times \mathbb{P}^1 \to \mathbb{P}^1$$

is a double point degeneration over $0 \in \mathbb{P}^1$, we define the associated *double point relation* over X by

$$[Y_\zeta \to X] + [\mathbb{P}(\pi_2)] - [A \to X] - [B \to X].$$

Here $\mathbb{P}(\pi_2)$, A and B are given by the double point degeneration and Y_ζ is $\pi_2^{-1}(\zeta)$ for any regular value ζ of $\mathbb{P}^1$. We define $\mathcal{R}(X) \subset \mathcal{M}_*(X)^+$ as the subgroup generated by all double point relations. Since all double point relations are homogeneous we get a grading on this subgroup $\mathcal{R}_*(X)$.

Clearly the relation does not only depend on π. There is also the choice of the ζ, but modulo double point relations, any such choice is equivalent.

Lemma 4.5. *Let $Y \in \boldsymbol{Sm}_k$ be of pure dimension and*

$$\pi : Y \to X \times \mathbb{P}^1$$

a projective morphism, such that $\pi_2 = p_2 \circ \pi : Y \to \mathbb{P}^1$ is smooth over 0 and $\infty \in \mathbb{P}^1$. Then

$$[Y_0 \to X] - [Y_\infty \to X] \in \mathcal{R}_*(X).$$

Proof. By definition with $A = X_0$ and $B = \emptyset$ the morphism π_2 is a double point degeneration over $0 \in \mathbb{P}^1$. The associated double point relation is exactly

$$[Y_0 \to X] - [Y_\infty \to X] \in \mathcal{R}_*(X). \qquad \square$$

This implies that $[Y_0 \to X] = [Y_\infty \to X]$ in $\mathcal{M}_*(X)^+/\mathcal{R}_*(X)$. We saw that this property holds in $\Omega_*(X)$ in Theorem 4.2. This is not a coincidence, so let us examine $\mathcal{M}_*(X)^+/\mathcal{R}_*(X)$.

Definition 4.6. Define *double point cobordism theory* as

$$\omega_*(X) = \mathcal{M}_*(X)^+/\mathcal{R}_*(X).$$

The grading comes from the grading on $\mathcal{M}_*(X)^+$, which is respected by double point relations.

Now comes the main result of [36].

Theorem 4.7. *The graded groups $\omega_*(X) \cong \Omega_*(X)$ are naturally isomorphic as oriented Borel-Moore homology theories.*

We will outline the proof. For this we will need some relations in Ω_* given in [35]. These relations are associated to a strict normal crossing divisor, using the coefficients of the formal group law. We will only require the case of a strict normal crossing divisor of degree 2, for which we will need the formal power series $F^{1,1} \in \Omega_*(k)[[u,v]]$ given by

$$F^{1,1}(u,v) = \sum_{i,j\geq 1} a_{i,j} u^{i-1} v^{j-1},$$

if $F(u,v) = u + v + \sum_{i,j\geq 1} a_{i,j} u^i v^j \in \Omega_*(k)[[u,v]]$ is the formal group law of Ω_*.

Definition 4.8. Let E_1 and E_2 be two smooth transverse intersecting divisors on $Y \in \mathbf{Sm}_k$. Define the sum $E = E_1 + E_2$ and the inclusion of the intersection $i_D : D \to Y$. Then we can define an element for E in $\Omega_*(Y)$ by

$$[E \to Y] := [E_1 \to Y] + [E_2 \to Y] + i_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(E_1)), \tilde{c}_1(\mathcal{O}_D(E_2)))(1_D)\right)$$

where $\mathcal{O}_D(E_i)$ denotes the restriction of $\mathcal{O}_Y(E_i)$ to D.

This class depends only on the divisor class of E.

Theorem 4.9. *Let E be the divisor as above on a $Y \in \boldsymbol{Sm}_k$. Then we have that*

$$[E \to Y] = \tilde{c}_1(\mathcal{O}_Y(E))(1_Y).$$

In particular, if F is a smooth divisor which is linear equivalent to E, then we have

$$[E \to Y] = [F \to Y].$$

This theorem is the basis for many properties and theorems concerning algebraic cobordism. Its proof can be found at the end of Section 3.1 in [35].

Proof of Theorem 4.7. $\underline{\omega_*(X) \to \Omega_*(X)}$:
We will first find a map $\omega_*(X) \to \Omega_*(X)$ of graded groups. We have the map

$$\mathcal{M}_*(X)^+ \to \Omega_*(X)$$

given in Theorem 3.19. We will prove that $\mathcal{R}_*(X)$ maps to zero, i.e. Ω_* respects double point relations.
So let $Y \to X \times \mathbb{P}^1$ be a double point degeneration of X, with fibres $A + B$ over 0 and Y_ζ over $\zeta \in \mathbb{P}^1$. Then the previous theorem gives us

$$[A \to Y] + [B \to Y] + i_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(A)), \tilde{c}_1(\mathcal{O}_D(B)))(1_D)\right) = [Y_\zeta \to Y].$$

Pushing forward to X by $Y \to X \times \mathbb{P}^1 \to X$ shows that it is enough to prove that

$$[A \to X] + [B \to X] + j_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(A)), \tilde{c}_1(\mathcal{O}_D(B)))(1_D)\right) = [Y_\zeta \to X]$$

where j_D is the composition $D \to Y \to X$. So we want to show that

$$j_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(A)), \tilde{c}_1(\mathcal{O}_D(B)))(1_D)\right) + [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \to X] = 0.$$

We will prove the more general result

$$F^{1,1}(\tilde{c}_1(\mathcal{O}_D(A)), \tilde{c}_1(\mathcal{O}_D(B)))(1_D) + [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \to D] = 0.$$

Let us write $\mathbb{P}$ for $\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D})$. Now consider the surjection

$$\mathcal{O}_D \oplus \mathcal{N}_{A/D} \to \mathcal{N}_{A/D}.$$

This gives us a closed embedding $s : D = \mathbb{P}(\mathcal{N}_{A/D}) \to \mathbb{P} = \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D})$, since $\mathcal{N}_{A/D}$ is locally free. Obviously s is a section of the projection map $\mathbb{P}_D \to D$. Denote the image of s by D as well, now the normal bundle of D in $\mathbb{P}$ is $\mathcal{N}_{A/D}$.
We will look at the deformation to the normal cone of $s : D \to \mathbb{P}$. This space W is the blow up of $D \times 0$ in $\mathbb{P} \times \mathbb{P}^1$. It comes with a map $W \to \mathbb{P}^1$, such that over all points ζ in $\mathbb{P}^1 \backslash \{0\}$ the fibre W_ζ is simply $\mathbb{P}$. However, above $0 \in \mathbb{P}^1$, the fibre W_0 consists of the blow up of $\mathbb{P}$ in D and of $\tilde{\mathbb{P}} = \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{D/\mathbb{P}}) = \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{D/\mathbb{A}})$. However, the blow up of $\mathbb{P}$ in D is simply $\mathbb{P}$ and $\tilde{\mathbb{P}} \to D$ is given by the projection $\mathcal{O}_D \oplus \mathcal{N}_{\mathbb{P}/D} \to \mathcal{N}_{\mathbb{P}/D}$.
Now Theorem 4.9 gives that

$$[W_\infty \to W] = [W_0 \to W] = [\mathbb{P} \to W] + [\tilde{\mathbb{P}} \to W] + i_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(\mathbb{P})), \tilde{c}_1(\mathcal{O}_D(\tilde{\mathbb{P}})))(1_D)\right).$$

If we push forward via the morphism

$$\rho : W \to \mathbb{P} \times \mathbb{P}^1 \to \mathbb{P} \xrightarrow{s} D$$

we see that W_∞, $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are all isomorphic as D-schemes. So we get

$$[\mathbb{P} \to D] = [\mathbb{P} \to D] + [\mathbb{P} \to D] + \rho_* i_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(\mathbb{P})), \tilde{c}_1(\mathcal{O}_D(\tilde{\mathbb{P}})))(1_D)\right).$$

Clearly $i \circ \rho = \mathrm{Id_D}$, so

$$\rho_* i_{D*}\left(F^{1,1}(\tilde{c}_1(\mathcal{O}_D(\mathbb{P})), \tilde{c}_1(\mathcal{O}_D(\tilde{\mathbb{P}})))(1_D)\right) = F^{1,1}(\tilde{c}_1(\mathcal{O}_D(\mathbb{P})), \tilde{c}_1(\mathcal{O}_D(\tilde{\mathbb{P}})))(1_D).$$

Clearly $\mathcal{O}_W(\mathbb{P})$ restricts to $\mathcal{N}_{D/\mathbb{P}}$ on D and similar to $\mathcal{O}_D(A) \otimes \mathcal{O}_D(B) \cong \mathcal{N}_{D/A} \otimes \mathcal{N}_{D/B} \cong \mathcal{O}_D$ we have $\mathcal{O}_D(\mathbb{P}) \otimes \mathcal{O}_D(\tilde{\mathbb{P}}) \cong \mathcal{O}_D$. So we even get

$$\mathcal{O}_D(\tilde{\mathbb{P}}) \cong \mathcal{O}_D(\mathbb{P})^\vee \cong \mathcal{O}_D(A)^\vee \cong \mathcal{O}_D(B).$$

Now we conclude

$$F^{1,1}(\tilde{c}_1(\mathcal{O}_D(A)), \tilde{c}_1(\mathcal{O}_D(B)))(1_D) + [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \to D] = 0$$

and the map $\mathcal{M}(X)^+ \to \Omega_*(X)$ descends to a surjective map

$$\nu(X) : \omega_*(X) \to \Omega_*(X)$$

mapping $[f : Y \to X]^\omega$ to $[f : Y \to X]^\Omega := f_* 1_Y$.

$\underline{\Omega_*(X) \to \omega_*(X)}$:
Now we construct a morphism in the other direction which will be an inverse for ν. In [36] Levine and Pandharipande prove that ω_* allows for the structure of smooth pullbacks, first Chern classes and a formal group law making it an oriented Borel-Moore functor of geometric type. So by Theorem 3.18 we get a functor of oriented Borel-Moore $\mathbb{L}_*$-functors

$$\vartheta : \Omega_* \to \omega_*.$$

This morphism satisfies

$$\vartheta(X)\left([f : Y \to X]^\Omega\right) = \vartheta(X)\left(f_*^\Omega 1_Y^\Omega\right) = f_*^\omega\left(\vartheta(X) 1_Y^\Omega\right) = f_*^\omega 1_X^\omega = [f : Y \to X]^\omega,$$

where we used superscripts to denote the theory in which we are considering a basis element, pullback or identity. So $\vartheta \circ \nu = \mathrm{Id}_\omega$ and $\nu(X)$ is surjective for all $X \in \mathbf{Sch}_k$ and we have the isomorphism of oriented Borel-Moore functors of geometric type

$$\omega_* \cong \Omega_*. \qquad \square$$

This immediately gives that all the results in section 3 for the algebraically constructed theory Ω_*, also hold for this geometric theory.

4.3 Algebraic cobordism with vector bundles

Many variations have been made on cobordism theories in algebraic topology. We saw a few of them in Chapter 1. Another such theory was used by Atiyah and Singer in 1963 for their proof of the Atiyah-Singer index theorem [3]. They considered cobordism classes of compact smooth oriented manifolds together with a given vector bundle.

A similar theory of spaces with vector bundles can be defined in algebraic geometry, as done in [34] or even [38]. In the first case we directly get the analogue of the theory used by Atiyah and Singer. The definition of this theory follows quite naturally from the definition of algebraic cobordism in terms of double point relations.

Definition 4.10. Let $\mathcal{M}_{n,r}(X)$ be the set of pairs $(f : Y \to X, \mathcal{E})$ consisting of a projective map f between $X \in \mathbf{Sch}_k$ and $Y \in \mathbf{Sm}_k$, and $\mathcal{E}$ a vector bundle on Y of rank r, up to isomorphism.
Like $\mathcal{M}_n(X)$ the set $\mathcal{M}_{n,r}(X)$ has an obvious monoid structure given by taking the disjoint unions of the domains of the projective morphisms. The bundle of the sum will be the two bundles of the terms on the respective components.
The group completion of $\mathcal{M}_{n,r}(X)$ is denoted by $\mathcal{M}^+_{n,r}$ and the image of $(f : Y \to X, \mathcal{E})$ by $[f : Y \to X, \mathcal{E}]$.

The double point relations of spaces over X translate directly to the double point relation of spaces with bundles over X.

Definition 4.11. Let $Y \in \mathbf{Sm}_k$ be of pure dimension $n+1$ and $\mathcal{E}$ be a rank r vector bundle on Y. Then any projective map $g : Y \to X \times \mathbb{P}^1$ such that the map

$$\pi : Y \to Y \times \mathbb{P}^1 \to \mathbb{P}^1$$

is a double point degeneration over some $\zeta \in \mathbb{P}^1$, gives an element

$$[f : Y_\zeta \to X, \mathcal{E}_{Y_\zeta}] - [f : A \to X, \mathcal{E}_A] - [f : B \to X, \mathcal{E}_B] + [f : \mathbb{P}(\pi) \to X, \mathcal{E}_{\mathbb{P}(\pi)}]$$

of $\mathcal{M}^+_{n,r}(X)$. Here A, B, $\mathbb{P}(\pi)$ and Y_ζ are the known spaces related to the double point degeneration and $\mathcal{E}_Z$ denotes the pullback of the bundle $\mathcal{E}$ over the obvious maps $Z \to Y$.
The set of all these relations is denoted by $\mathcal{R}_{n,r}(X) \subseteq \mathcal{M}^+_{n,r}$.

As before this gives an interesting theory [34].

Definition 4.12. *Algebraic cobordism with bundles* for an $X \in \mathbf{Sch}_k$ is defined by

$$\omega_{n,r}(X) = \mathcal{M}_{n,r}(X)/\mathcal{R}_{n,r}(X).$$

Obviously

$$\omega_{*,r}(X) = \bigoplus_{n=0}^{\infty} \omega_{n,r}(X)$$

coincides with $\omega_*(X)$ for $r = 0$. If $X = \operatorname{Spec} k$ the known $\mathbb{Q}$-basis for $\omega_{n,0}(X)$ given by products of projective spaces extends to a similar basis for $\omega_{n,r}(X)$ by considering specific bundles on these spaces. Basis elements are most easily expressed by a correspondence with special bijections.

Definition 4.13. Consider pairs (λ, μ) such that

(i) λ is a partition of n;

(ii) μ is a subpartition of λ of length $\ell(\mu) \leq r$.

Such a pair is called a *partition of size n and of type r*. Let $\mathcal{P}_{n,r}$ denote the set of all partitions of size n and type r.

Each partition (λ, μ) of size n and type r gives an element of $\omega_{n,r}(k)$ as follows: let $\mathbb{P}^\lambda$ denote the product of projective spaces over k with dimensions exactly the parts of λ. So

$$\mathbb{P}^\lambda = \mathbb{P}^{\lambda_1} \times \ldots \times \mathbb{P}^{\lambda_{\ell(\lambda)}}.$$

Now for every part m of the subpartition μ we can define a line bundle $\mathcal{L}_m$ on $\mathbb{P}^\lambda$ by pulling back $\mathcal{O}_{\mathbb{P}^m}(1)$ via the projection on the correct factor. For the other factors which are not in the subpartition we can take the pullback of structure sheaf. So we have a map

$$\phi : \mathcal{P}_{n,r} \to \omega_{n,r}(k)$$

by taking the direct sum of the above line bundles, written symbolically as

$$(\lambda, \mu) \mapsto [\mathbb{P}^\lambda, \mathcal{O}^{r-\ell(\mu)} \oplus \bigoplus_{m \in \mu} \mathcal{L}_m].$$

These elements are in fact the ones we were looking for.

Theorem 4.14 ([34, Theorem 1])**.** *The elements $\phi(\lambda, \mu)$ for all $(\lambda, \mu) \in \mathcal{P}_{n,r}$ form a basis for $\omega_{n,r}(k) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space.*

In general $\omega_{*,r}(X)$ has more natural structure. For example, it has the structure of an $\omega_*(k)$-module, given on generators by

$$[W \to \operatorname{Spec} k] \cdot [Y \to X, \mathcal{E}] = [W \times Y \to X, \operatorname{pr}_2^* \mathcal{E}]$$

where $\operatorname{pr}_2 : W \times Y \to Y$ is the projection. Over this ring there is an even simpler basis if $X = \operatorname{Spec} k$.

Theorem 4.15 ([34, Theorem 2]). *The graded group $\omega_{*,r}(k)$ is a free $\omega_*(k)$-module. A basis is given by $\phi(\lambda, \lambda)$ for all partitions λ with $\ell(\lambda) \leq r$.*

Algebraic cobordism with bundles over any space X is well understood in terms of algebraic cobordism $\omega_*(X)$ and algebraic cobordism with bundles over a point $\omega_{*,r}(k)$ as shown by the next theorem [34, Theorem 3].

Theorem 4.16. *Algebraic cobordism with bundles is, as a $\omega_*(k)$-module, ordinary algebraic cobordism extended by the scalars of algebraic cobordism with bundles over a point, i.e. for any $X \in \boldsymbol{Sch}_k$*

$$\omega_{*,r}(X) = \omega_*(X) \otimes_{\omega_*(k)} \omega_{*,r}(k).$$

An isomorphism is given on basis elements by

$$[f : Y \to X] \otimes \phi(\lambda, \lambda) \mapsto [f \circ \mathrm{pr}_2 : Y \times \mathbb{P}^1 \to X, \mathcal{O}^{r-\ell(\lambda)} \oplus \bigoplus_{m \in \lambda} \mathrm{pr}_1^* \mathcal{L}_m]$$

where pr_i denote the projections from $Y \times \mathbb{P}^1$ to the respective factors.

It is also shown in [36] that the classes of algebraic cobordism with bundles over k are still uniquely determined by Chern numbers as in Theorem 3.23.

We will need the $\mathbb{Q}$-vector space of polynomials of degree n in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_r$, where x_i and y_i are of degree i. We will write $\mathcal{C}_{n,r}$ for this vector space. One can identify monomials in $\mathcal{C}_{n,r}$ with Chern numbers of the tangent sheaf $\mathcal{T}_X$ and the vector bundle $\mathcal{E}$ for a class $[X, \mathcal{E}] \in \omega_{n,r}(k)$ by letting x_i be the ith Chern class of the tangent bundle $\mathcal{T}_X$ and y_i the ith Chern class of the vector bundle $\mathcal{E}$. Note that if X is of dimension n and $\mathcal{E}$ has rank r, then these are precisely the interesting Chern numbers for X and $\mathcal{E}$. In Theorem 4 in [34] it is shown that these numbers uniquely determine the cobordism class.

Theorem 4.17. *The pairing*

$$\omega_{n,r}(k) \otimes \mathbb{Q} \times \mathcal{C}_{n,r} \to \mathbb{Q}$$

defined by evaluating each monomial as the Chern number on each class $[X, \mathcal{E}]$ is well-defined. The pairing is also non-degenerate. So each class $[X, \mathcal{E}]$ is uniquely determined by the combined Chern classes of $\mathcal{T}_X$ and $\mathcal{E}$.

An easy combinatorial argument similar to the one in 3.3 shows that the number of basis element of $\omega_{2,1} \otimes \mathbb{Q}$ given in Theorem 4.14 is equal to the dimension of $\mathcal{C}_{n,r}$. Just map

$$x_1^{e_1} \ldots x_n^{e_n} y_1^{f_1} \ldots y_r^{f_r}$$

to the element of $\mathcal{P}_{n,r}$ where i occurs $e_i + f_i$ in the partition and f_i times in the subpartition. Here we take $f_i = 0$ for $i > r$.

In the next chapter we will make use of this theorem in the specific case of surfaces with a line bundle. For completeness we will state the explicit basis elements of $\omega_{2,1}$.

Corollary 4.18. *The $\mathbb{Q}$-vector space $\omega_{2,1} \otimes \mathbb{Q}$ is four-dimensional. A class $[S, \mathcal{L}]$ of a surface S with a line bundle $\mathcal{L}$ is uniquely determined by the following four Chern numbers:*

$$c_1(\mathcal{L})^2, \quad c_1(\mathcal{L})c_1(\mathcal{T}_S), \quad c_1(\mathcal{T}_S)^2 \quad \textit{and} \quad c_2(\mathcal{T}_S).$$

One could also use the line bundle and the dual of the tangent sheaf, in which case only $c_1(\mathcal{L})c_1(\mathcal{T}_S)$ changes by a sign. The dual of $\mathcal{T}_S$ is the locally free sheaf of rank 2 of differential forms which we will denote by K_S.

Sometimes we will simply identify a locally free sheaf with its first Chern class and write $\mathcal{L}^2$, $\mathcal{L}K_S$ and K_S^2 for $c_1(\mathcal{L})^2$, $c_1(\mathcal{L})c_1(K_S)$ and $c_1(K_S)^2$.

Chapter 5

Applications of algebraic cobordism to nodal curves

Because the class of a k-scheme in rational algebraic cobordism is uniquely determined by Chern classes, see Corollary 4.18, this theory seems well suited for applications in enumerative geometry. One such application is the proof by Tzeng in [61] of a conjecture of Göttsche [20] about nodal curves on surfaces. Nodal curves are one-dimensional schemes with only nodes as singularities. These curves are of special interest as every smooth curve is isomorphic to the normalization of a nodal plane curve [26, Corollary IV.3.11]. Important results on these kind of curves were already obtained by Steiner, Cayley and Roberts ([57], [7], [50]) in the 19th century. Enriques and Severi were able to put these results in to context in [14] and [54] by defining a projective scheme parametrizing all curves in the plane of a fixed degree and number of nodes. This scheme comes with an embedding into a projective space and its degree is a way of counting nodal plane curves. Ran [47], [48] obtained some interesting results for special cases of degree and number of nodes. In [25] Harris and Pandharipande were able to compute these degrees for up to three nodes and in [6] Caporaso and Harris found a way to compute all these degrees by use of a recursive formula. In 1994 Di Francesco and Itzykson [11] conjectured that these numbers of curves are in fact polynomials in the degree of the plane curves in consideration. Vainsencher [62] confirmed his conjecture up to six nodes and even generalised this conjecture to the counting of nodal curves on arbitrary surfaces. His results were extended by Kleiman and Piene [29] to seven and eight nodes. Fomin and Mikhalkin were able to prove the conjecture for the projective plane in 2009 [15]. A proof for the general case was found by Tzeng [61] using algebraic cobordism and an approach via Hilbert scheme of points as suggested by Göttsche [20].

5.1 Nodal plane curves

We will first look at nodal curves in the projective plane $\mathbb{P}^2$. So let us first recall some facts about general plane curves. For more information on plane curves and nodal curves one can consult Fulton [16], [17], which are the main source for this section.

Each plane curve defines an effective divisor on $\mathbb{P}^2$ and so corresponds to an invertible sheaf on the projective plane with a non-zero global section. Since each line bundle on $\mathbb{P}^2$ is of the form $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$, we see that each plane curve is the zero-subscheme of a homogeneous polynomial in three variables of degree d. We can define the *degree of a curve* as this $d \geq 0$.

So we can write each plane curve by an equation

$$\sum_{i+j+k=d} a_{i,j,k} X^i Y^j Z^k = 0$$

with not all coefficients equal to zero. So the $\binom{d-1}{2}$-tuple $(a_{i,j,k})$ defines a curve and these equations form a $\binom{d+2}{2}$-dimensional vector space without the origin. It is clear that scaled tuples define the same curve, so we get a bijection between plane curves of degree d and closed points in $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(d))) \cong \mathbb{P}^{\frac{d(d+3)}{2}}$. This projective space of global sections of a line bundle $\mathcal{L}$ is called the *linear system* of $\mathcal{L}$ and denoted by $|\mathcal{L}|$.

Now consider a point P on a curve C of degree d defined by $F(X,Y,Z) = 0$, without loss of generality we can assume that this point is $[0 : 0 : 1]$. On the affine plane defined by $Z \neq 0$ the curve is given by $f(x,y) = F(x,y,1) = 0$. Let us write f as the sum of its homogeneous parts

$$f = f_c + f_{c+1} + \ldots + f_d$$

with $f_c \neq 0$. Since we are working over $\mathbb{C}$ we can factor f_c into the product of c linear functions in x and y:

$$f_c = l_1 \cdot \ldots \cdot l_c.$$

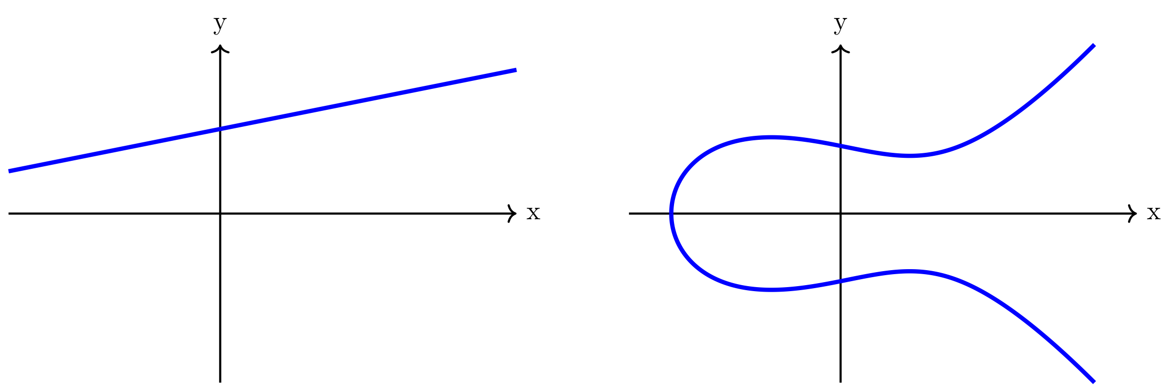
Each linear equation l_i describes a line through P and these lines are called the *tangents* of C at P.

If C is smooth at P then c will equal 1 and there is exactly one tangent. However, if P is a singular point then we must have that $c > 1$ and there are more tangents, which could even coincide. Note that this also shows that a curve given by $F(X,Y,Z)$ is singular at P precisely if the derivatives

$$\frac{dF}{dX}, \ \frac{dF}{dY} \quad \text{and} \quad \frac{dF}{dZ}$$

vanish at P.

Let us look at the examples of plane curves in Figure 5.1. The curve on the left is of degree 1 and is simply a line. It is clear that this curve is smooth at all points. A quick calculation shows that the curve on the right is also smooth.



(a) A degree 1 curve defined by $X + 5Z - 5Y = 0$

(b) A degree 3 curve defined by $ZY^2 - X^3 + 2XZ^2 - 4Z^3 = 0$

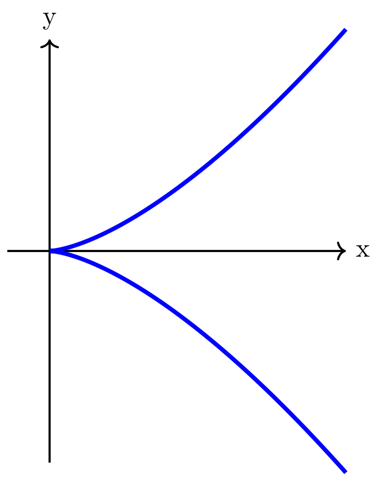
Figure 5.1: Examples of plane curves

Now let us look at the two examples of singular curves in Figure 5.2. The first has a singularity at the origin where it has two tangents in the definition we gave above. These tangents even coincide. The second curve has three distinct tangents at the origin and hence is indeed a singular curve.

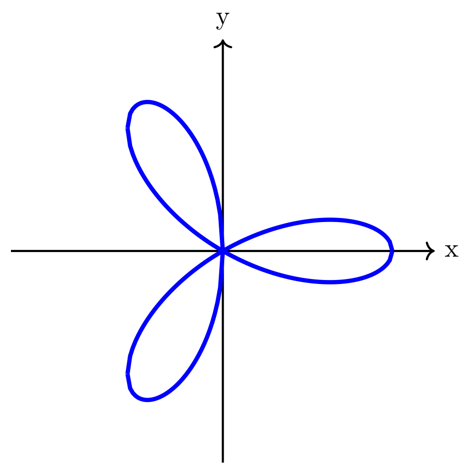
So the best thing that could happen in a singular point, is that there are two tangents which are distinct.

Definition 5.1. A point on a curve is called a *node* if the curve has two distinct tangents at this point. A curve whose only singularities are nodes is called a *nodal curve*.

Two examples of nodal curves are shown in Figure 5.3.

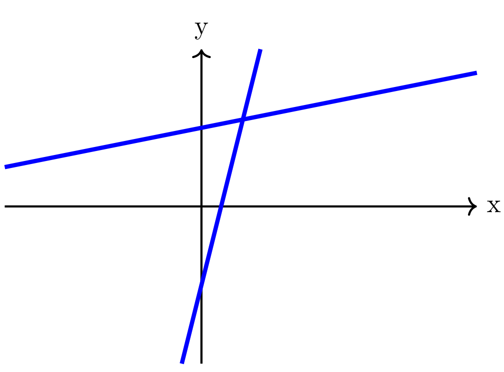


(a) A degree 3 curve defined by $ZX^2 - Y^3 = 0$

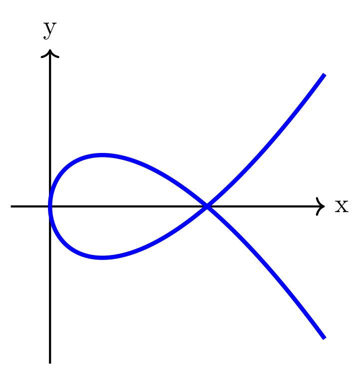


(b) A degree 4 curve defined by $(X^2 + Y^2)^2 - ZX^3 + 3XY^2Z = 0$

Figure 5.2: Examples of singular plane curves



(a) A nodal curve of degree 2



(b) A nodal curve of degree 3

Figure 5.3: Examples of nodal plane curves

5.2 Severi varieties

One way to study nodal plane curves is by examining the Severi variety.

Definition 5.2. Let $\mathcal{V}_{d,\delta}$ denote the Zariski closure of the points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ corresponding to nodal plane curves of degree d with exactly δ nodes. This closed set in its reduced scheme structure is called the *Severi variety of type* d, δ.

Let us examine the Severi variety for small d. The linear system of degree 1 curves is isomorphic to $\mathbb{P}^2$. The dimension reflects that there is a unique line through two general points on the plane. In this case all curves are smooth, so we have $\mathcal{V}_{1,0} = \mathbb{P}^2$ and $\mathcal{V}_{1,\delta}$ is empty for all $\delta \geq 1$.

We will now look at the conics, so $d = 2$. A degree 2 curve is either a smooth conic or an intersecting pair of lines. So the Severi variety is empty for $\delta > 1$. Let us write the equation

$$G(X,Y,Z) = AX^2 + BXY + CXZ + DY^2 + EYZ + FZ^2$$

of a conic. It is smooth if and only if the derivative

$$\begin{aligned}\frac{dF}{dX} =& 2AX + BY + CZ \\ \frac{dF}{dY} =& BX + 2DY + EZ \\ \frac{dF}{dZ} =& CX + EY + 2FZ\end{aligned}$$

are not all zero in a point on the curve. We know that if such a point does exist then

$$\det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix} = 0. \tag{5.1}$$

On the other hand, if $2AX + BY + CZ$, $BX + 2DY + EZ$ and $CX + EY + 2FZ$ are all zero then

$$2G(X,Y,Z) = (2AX + BY + CZ)X + (BX + 2DY + EZ)Y + (CX + EY + 2FZ)Z$$

is also zero. So $\mathcal{V}_{2,1}$ is determined by a single equation in $\mathbb{P}^5$ and hence has dimension 4. This proves that the general curve in $\mathbb{P}^5$ is smooth, so $\dim \mathcal{V}_{2,0} = 5$.

One can show that the $\mathcal{V}_{d,0}$ equals the $\mathbb{P}^{\frac{d(d+3)}{2}}$ as we will see in Theorem 5.3, which means that the general curve of degree d is smooth. So in particular for $d = 3$ we find $\mathcal{V}_{3,0} = \mathbb{P}^9$. We can also see that the maximal number of nodes occurs if the curve consists of 3 lines, in which case we have 3 nodes. We can also have two nodes if the curve is the union of a line and a conic, and one node if it is an irreducible nodal cubic curve such as in Figure 5.3b. Since an irreducible nodal curve of degree 3 cannot have more than $\frac{(d-1)(d-2)}{2} = 1$ node this completely describes the cubic nodal curves. Here we use that the genus, which is non-negative, of a nodal plane curve of degree d with δ nodes is equal to

$$\frac{(d-1)(d-2)}{2} - \delta$$

see for example Proposition 8.3.5 in [16].

For $\delta = 3$ we have to select three lines, each of which comes from a two-dimensional space. So the dimension of $\mathcal{V}_{3,3}$ equals 6. For two nodes we need to choose a general line and conic, so $\dim \mathcal{V}_{3,2}$ equals $2 + 5 = 7$. We already stated that the dimension of the Severi variety of type $3, 0$ is 9 and hence equals the entire space of curves of degree 3, so $\mathcal{V}_{3,1}$ is a proper closed subset of $\mathbb{P}^9$ and its dimension is strictly less than 9. One would expect this dimension to be 8 and this is indeed the case as proven by Severi in 1921 [54].

Theorem 5.3. *The Severi variety of type d, δ is non-empty for $0 \leq \delta \leq \frac{d(d-1)}{2}$ and has codimension δ.*

This also shows that there are nodal curves with all expected number of nodes and it proves the claim that a general degree d curve is smooth.

Now note that in general the Severi variety is not irreducible: fix the degree d and the number of nodes δ. Now for any positive integers d_i such that $\sum d_i = d$ and $\sum_{i \neq j} d_i d_j = \delta$ we will look at the curves whose irreducible components are smooth of degree d_i. These curves are indeed of degree d and have δ nodes. The Zariski closure of these points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ is indeed of codimension δ as each component of degree d_i comes from a $\frac{d_i(d_i+3)}{2}$-dimensional space:

$$\sum_i \frac{d_i(d_i+3)}{2} = \frac{1}{2}\left(\sum_i d_i\right)^2 - \sum_{i \neq j} d_i d_j + \frac{3}{2}\sum_i d_i = \frac{d(d+3)}{2} - \delta.$$

For example, the sequences $(4,1,1)$ and $(3,3)$ for $d = 6$ and $\delta = 9$ produce two distinct closed subsets of codimension 6 in $\mathcal{V}_{6,9}$, from which we conclude that this Severi variety is not irreducible. Severi also stated in 1921 that there is always a single component of the Severi variety which contains the irreducible curves of the given degree and the correct number of nodes. His proof was however wrong and it was not before 1985 before Harris gave a correct proof in [24].

Theorem 5.4. *The irreducible nodal curves with δ nodes and of degree d lie in a single irreducible component of the Severi variety.*

5.3 Severi degree

The Severi variety comes with an embedding into a projective space so one could study properties of this embedding, such as the Hilbert polynomial. This seems to be rather hard in general, but we can say something about the degree of the Severi variety.

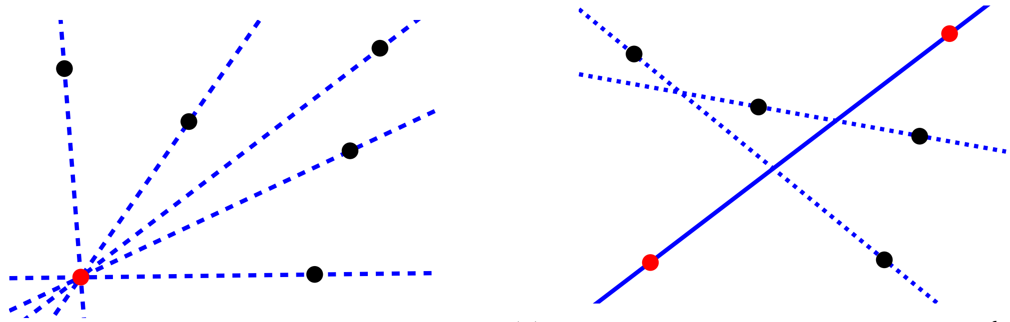
Definition 5.5. The degree of the Severi variety is called the *Severi degree* and is denoted by $\mathcal{N}_{d,\delta}$.

We know by Theorem 5.3 that the Severi variety $\mathcal{V}_{d,\delta}$ is of codimension δ. To compute its degree one can intersect it with a δ-dimensional linear space and compute the length of the zero-dimensional scheme thus obtained. In general, such a linear space will yield a space of reduced points, and we can simply count the number of points in the intersection.

So let us consider what it means to have a δ-dimensional linear subspace in $\mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(d)))$. Notice that in $\mathbb{P}^2 \cong |\mathcal{O}(1)|$ with coordinates U, V and W, a line $UX + VY + WZ$ consists exactly of points representing lines on the projective plane passing through the point $[X : Y : Z]$. Note that all the points in a linear system representing curves of degree d through a fixed point, make up a linear subspace of codimension 1. Since the intersection of two linear subspaces of codimension q and r is in general a linear subspace of codimension $q + r$, we see that the curves through k fixed points in the plane form a k-codimensional linear subspace in $\mathbb{P}^{\frac{d(d+3)}{2}}$. This gives us the following geometric interpretation of the Severi degree.

Lemma 5.6. *The degree $\mathcal{N}_{d,\delta}$ of the Severi degree $\mathcal{V}_{d,\delta}$ equals the number of nodal plane curves of degree d with δ nodes passing through $\frac{d(d+3)}{2} - \delta$ general points in the plane.*

Let us use this to compute some Severi degrees for small d. In the case $d = 1$ we only have smooth curves and the degree of $\mathcal{V}_{1,0}$ is obviously 1, as $\mathcal{N}_{d,0} = 1$ for all d. This shows that there is a smooth conic passing through five general points, since the Severi variety $\mathcal{V}_{2,0}$ is the projective space of dimension 5. For $\mathcal{N}_{2,1}$ we need to count the ways to draw a pair of lines through four general points. Since the points are general each line will pass through exactly two points and through each point will pass exactly one line. So pick a point and look at the possible lines passing through this point. This line is obviously determined by the choice of a second point and this choice also determines the other line. So we have $\mathcal{N}_{2,1} = 3$. Notice that we could have obtained this result from the fact that $\mathcal{V}_{2,1}$ is a hypersurface in $\mathbb{P}^5$ defined by the single equation (5.1) of degree 3.



(a) For one point there are five possible lines

(b) After choosing such a line, there are $\mathcal{N}_{2,1} = 3$ ways to pick the other lines

Figure 5.4: The computation of $\mathcal{N}_{3,3}$

Let us consider $d = 3$. As before we have $\mathcal{N}_{3,0} = 1$ and we will now examine the cases $1 \leq \delta \leq 3$. For $\delta = 3$ we want to know the number of line triples passing through six points. As before, a first point determines five possible lines. Of the remaining four points we know that there are 3 ways to pick a line pair. This gives us that the degree of the Severi variety $\mathcal{V}_{3,3}$ equals 15. For two nodes, we will need to count the number of combinations of a line and a conic passing through seven points. As the points are general there will be exactly five points on the conic and two on the line. Now any choice of two points determines such a line, as there is a unique line passing through these two points and a unique conic passing through the remaining 5 points. The number of ways to choose two points out of these seven equals $\mathcal{N}_{3,2} = \binom{7}{2} = 21$.

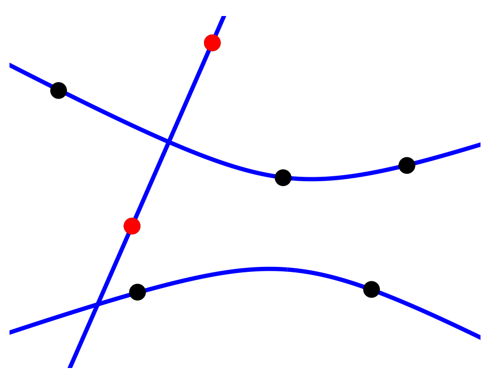


Figure 5.5: The computation of $\mathcal{N}_{3,2}$:
Any two points determine a line and a conic

We are left with the case of an irreducible cubic curve and these are not as well understand as lines and conics. We will see that in fact there are twelve irreducible conics with one node through 8 general points. This was already known by Steiner [57] who showed the following general result in 1854.

Theorem 5.7. *The number of nodal curves of degree d with exactly one node through $\frac{d(d+3)}{2} - 1$ points equals*

$$3(d-1)^2.$$

So instead of looking at a fixed degree we get a nice result by fixing the number of nodes.

The next step is the case of exactly two nodes. This problem was completely solved by Cayley in 1866, [7]. We will state his result in terms of Severi degrees.

Theorem 5.8. *Let d be a positive integer. The Severi degree of type $d, 2$ equals*

$$\mathcal{N}_{d,2} = \frac{3}{2}(d-1)(d-2)(3d^2-3d-11).$$

Roberts [50] was even able to find a similar result for three nodes.

Theorem 5.9. *The Severi degree $\mathcal{N}_{d,3}$ equals*

$$\frac{9}{2}d^6 - 27d^5 + \frac{9}{2}d^4 + \frac{423}{2}d^3 - 229d^2 - \frac{829}{2}d + 525$$

for $d \geq 3$.

So the Severi degree for fixed δ is still a polynomial in d of degree 2δ, although this polynomial gives the wrong number for small d. So one might conjecture that such a polynomial exists for all δ, which agrees with $\mathcal{N}_{d,\delta}$ for large enough d. This was done by Di Francesco and Itzykson in [11]. Their conjecture was confirmed by Vainsencher in [62] up to six nodes and Kleiman and Piene extended his results up to eight nodes in [29]. But it was not until 2009 that Fomin and Mikhalkin [15] found a proof for the general case.

Theorem 5.10. *There exist polynomials $N_\delta(d)$ of degree 2δ, called Severi polynomials, such that for all δ*

$$N_\delta(d) = \mathcal{N}_{d,\delta}$$

for large enough d.

5.4 Nodal curves on surfaces

Vainsencher did not only show the existence of Severi polynomials for the projective plane for $\delta \leq 6$. He also generalised Di Francesco and Itzykson's conjecture to arbitrary surfaces, as follows.

Let S be a surface with a line bundle $\mathcal{L}$, and let $|\mathcal{L}|$ denote the linear system of $\mathcal{L}$. We are interested in effective divisors $C \in |\mathcal{L}|$ which have exactly δ nodes. Such divisors obviously need not exist, as $|\mathcal{L}|$ for example could be empty.

Let us first consider the case of smooth curves in $|\mathcal{L}|$. If $\mathcal{L}$ is very ample then the general curve in $|\mathcal{L}|$ is smooth. This follows from the fact that $\mathcal{L}$ defines an embedding of the surface S in a projective space. Divisors in $|\mathcal{L}|$ now correspond to intersections of S with a linear hyperplane. By Bertini's theorem [26, Theorem II.8.18] such a divisor is smooth in general, since we are working over an algebraically closed field.

In the case of $\delta > 0$ nodes, very ampleness may not be strong enough to ensure the existence of nodal curves with exactly δ nodes in the linear system. We will need an assumption on $\mathcal{L}$ coming from [4], ensuring that there are enough global sections.

Definition 5.11. Let $\mathcal{L}$ be a line bundle on a surface S. We say that $\mathcal{L}$ is *k-very ample* if for every zero-dimensional subscheme Z of length $k+1$, the restriction of global sections on S to Z is surjective, i.e. the map

$$H^0(S, \mathcal{L}) \to H^0(Z, \mathcal{L} \otimes \mathcal{O}_Z)$$

is surjective.

Note that if $\mathcal{L}$ is very ample, it is 1-very ample. In general, if $\mathcal{L}$ and $\mathcal{M}$ are very ample, then $\mathcal{L}^{\otimes l} \otimes \mathcal{M}^{\otimes m}$ is $(l+m)$-very ample. Also note that a k-very ample line bundle contains curves through any $k+1$ points. However, we are obviously not interested in divisors only containing specific points, but also in the local nature of the curve at these points. We have the following result by Göttsche from [20, Proposition 5.2].

Theorem 5.12. *Consider an integer $\delta \geq 1$ and S a smooth projective surface. Let $\mathcal{L}$ be a $(5\delta-1)$-very ample (5-very ample if $\delta = 1$) line bundle. Then any δ-dimensional linear subsystem $\mathcal{V} \subseteq |\mathcal{L}|$ contains finitely many points with at least δ singularities. In fact these singularities are all nodes.*

We will supply a proof for this theorem, when proving Theorem 5.15 in the next section. There, we will also find an explicit expression for this number of curves as an intersection number.

For this reason we will define the notion of the *number of δ-nodal curves* of an invertible sheaf $\mathcal{L}$, which is the number of curves with exactly δ nodes in a general δ-dimensional linear subspace of the linear system $|\mathcal{L}|$.

The number of δ-nodal curves of a line bundle $\mathcal{O}(d)$ on $\mathbb{P}^2$ can be viewed as the Severi degree $\mathcal{N}_{d,\delta}$. Theorem 5.10 on Severi degrees is generalised by the following statement, which was conjectured by Vainsencher [62] and proven by Kleiman and Piene [29] up to eight nodes.

Theorem 5.13. *Let $\delta \geq 0$ be an integer. There exists a polynomial P_δ of degree δ in four variables such that*

$$P_\delta(\mathcal{L}^2, \mathcal{L}K_S, K_S^2, c_2(K_S))$$

equals the number of δ-nodal curves in $\mathcal{L}$ if $\mathcal{L}$ is very ample if $\delta = 0$, 5-very ample for $\delta = 1$ or $(5\delta - 1)$-very ample for $\delta > 1$.
These polynomials are called the nodal polynomials.

We already saw that in the case of smooth curves with a very ample line bundle we have $P_0 = 1$. We will present the proof for $\delta > 0$ as given by Tzeng in [61].

5.5 Hilbert schemes

We will translate the problem into one of intersection numbers on Hilbert schemes of points, as done by Göttsche [20].

First we will recall the existence and properties of Hilbert schemes of points on a surface S from [42]. The central notion will be that of a flat family of subschemes of X of dimension 0 and length n, i.e. a flat surjective morphism $Z \to T$ such that each fibre Z_t for $t \in T$ consists of n closed points in X. Here we count points with their multiplicities, that is the length of the local ring. Now for any such family we get that for any morphism $S \to T$, the fibre product $Z \times_T S \to S$ is another such family. Now among these objects, there exists a universal one.

Theorem 5.14. *[42, Theorem 1.1 and Theorem 1.8] Let X be a projective scheme over k. There exists a projective scheme $X^{[n]}$ with a closed subscheme $Z_n(X) \subseteq X \times X^{[n]}$ over $X^{[n]}$, of subschemes of X of dimension 0 and length n, such that the following condition is satisfied:*
for any flat family $\vartheta : Z \to T$ of subschemes of X of dimension 0 and length n there exists a unique morphism $T \to X^{[n]}$ such that the family over T is the pullback of the universal family over $X^{[n]}$, i.e.

$$\begin{tikzcd} Z \cong T \times_{X^{[n]}} Z_n(X) \arrow[r] \arrow[d, "\vartheta"] & Z_n(X) \arrow[d, "q_n"] \\ T \arrow[r] & X^{[n]} \end{tikzcd}$$

Here q_n is the restriction of the second projection.

We say that $X^{[n]}$ is the Hilbert scheme of n points of X and $Z_n(X)$ is called a universal family over $X^{[n]}$. Furthermore, if S is a smooth projective surface, then $S^{[n]}$ is projective and smooth of dimension $2n$.

From now on let S be a smooth projective surface, together with a line bundle $\mathcal{L}$. The morphism $p_n : Z_n(S) \to S$ and the flat morphism $q_n : Z_n(S) \to S^{[n]}$ finite of degree n, allow us to define a sheaf $\mathcal{L}_n := (q_n)_*(p_n)^*\mathcal{L}$ on $S^{[n]}$.

The sheaf $(p_n)^*\mathcal{L}$ is invertible, since locally free sheaves remain locally free after being pulled back via any morphism. The morphism q_n is quasi-finite, since it obviously has finite fibres. It is also projective, since the composition $Z_n(S) \to S^{[n]} \to \operatorname{Spec} k$ is projective and $S^{[n]}$ is separated over k, see for example [26, Corollary II.4.8(e)]. Now by Théorème 8.11.1 in [23, Troisiéme partie] we get that q_n is finite. The fibres of q_n are zero-dimensional, so its higher image functors all vanish. So we get by Corollaire 7.9.10 in [23, Seconde partie] that $(q_n)_*(p_n)^*\mathcal{L}$ is locally free on $S^{[n]}$. Furthermore, its rank equals the Euler-characteristic of $(q_n)_*(p_n)^*\mathcal{L}$ which is simply the dimension of the global sections of the fibres. Evidently this equals dr.

Now in our case we are interested in the subschemes of S which are disjoint unions of spaces of the form $\operatorname{Spec}(\mathcal{O}_{S,x_i}/m^2_{S,x_i})$, for disjoint points $x_1,\dots,x_\delta$. These subschemes are parametrized by a locally closed subscheme $S^\delta_{2,0} \subset S^{[3\delta]}$. Let S^δ_2 be its closure in its reduced scheme structure, then S^δ_2 is clearly birational to $S^{[\delta]}$.

We will be interested in the intersection number

$$d_\delta(\mathcal{L}) := \int_{S^\delta_2} c_{2\delta}(\mathcal{L}_{3\delta}).$$

Theorem 5.15 (Proposition 5.2 in [20]). *Let S be a smooth projective surface and $\delta \geq 1$ any integer. If a line bundle $\mathcal{L}$ on S is $(5\delta-1)$-very ample (5-very ample if $\delta = 1$), then the finite number of curves in a general δ-dimensional linear subsystem as described in Theorem 5.12 equals $d_\delta(\mathcal{L})$.*

For the proof we will need a geometric interpretation of Chern classes as given by [18, Example 14.3.2].

Lemma 5.16. *Fix a positive integer p. Let X be a variety of dimension n with a locally free sheaf $\mathcal{F}$ of rank r which is generated by global sections. Consider global sections $s_1, s_2, \dots, s_N$ with $N \geq r-p+1$. The set*

$$\Omega = \{x \in X \mid \dim \operatorname{Span}\big(s_1(x), s_2(x), \dots, s_{r-p+1}(x)\big) \leq r-p\}$$

has a natural scheme structure of codimension p, for a general choice of such sections. The corresponding cycle $[\Omega]$ represents $\int_X c_p(\mathcal{F})$.

We will only be interested in the cases that either $n = p$, so that Ω is a zero-dimensional subscheme, or $n < p$, in which case Ω is empty.

Proof. Note that any global section s of $\mathcal{L}$ gives a global section on $S^{[k]}$ by the map $H^0(S,\mathcal{L}) \otimes \mathcal{O}_{S^{[k]}} \to \mathcal{L}_k$. Such a section vanishes on a reduced $Z \subseteq S$ of dimension 0 and length k in $S^{[k]}$ precisely if the original section vanishes at all $x_i \in Z$. By abuse of notation, we will write s for this section as well. A germ of this new section contains exactly the same information as the stalks of $\mathcal{L}$ restricted to the corresponding subscheme Z of length k. Now if $\mathcal{L}$ is $k-1$-very ample, then the global sections of $\mathcal{L}$ map surjectively onto global sections of $\mathcal{L}_Z$. This is equivalent to saying that the locally free sheaf $\mathcal{L}_k$ is globally generated.

Fix a δ-dimensional linear subsystem $\mathcal{V}$ of $|\mathcal{L}|$ with basis $s_1, s_2, \dots, s_{\delta+1}$.

We will first prove Theorem 5.12.
Let us apply Lemma 5.16 to the sections $s_1,\dots,s_{\delta+1}$ of $\mathcal{L}_{3\delta}$ restricted to S^δ_2 with $p = 2\delta$. In this case we get that

$$\Omega = \{Z \in S^\delta_2 \mid \dim \operatorname{Span}\big(s_1(Z), s_2(Z), \dots, s_{\delta+1}(Z)\big) \leq \delta\}$$

represents $c_{2\delta}(\mathcal{L}_{3\delta})$. Now when does $\dim \operatorname{Span}\big(s_1(Z), s_2(Z), \dots, s_{\delta+1}(Z)\big) \leq \delta$ hold? Precisely if for some $Z \in S^\delta_2$ there exists a $D \in \mathcal{V}$ with corresponding non-zero section s of both $\mathcal{L}$ and $\mathcal{L}_{3\delta}$ such that $s(Z) = 0$, i.e. Z is a subscheme of D.

If we repeat this for $S_2^\delta \backslash S_{2,0}^\delta$ we see that

$$\{Z \in S_2^\delta \backslash S_{2,0}^\delta \mid \dim \operatorname{Span}\big(s_1(Z), s_2(Z), \ldots, s_{\delta+1}(Z)\big) \leq \delta\}$$

represents the class $c_{2\delta}(\mathcal{L}_{3\delta})$ on $S_2^\delta \backslash S_{2,0}^\delta$. However $\dim S_2^\delta \backslash S_{2,0}^\delta < \dim S_{2,0}^\delta = 2\delta$, so $\int_{S_2^\delta \backslash S_{2,0}^\delta} c_{2\delta}(\mathcal{L}_{3\delta}) = 0$. Hence Ω is contained in $S_{2,0}^\delta$ and all $Z \in \Omega$ are exactly the curves with at least δ singularities.

We will now show that none of these curves has more than δ singularities.
Let us apply the lemma to the sections $s_1, \ldots, s_{\delta+1}$ of $\mathcal{L}_{3\delta+3}$ on $S_2^{\delta+1}$ with $p = 2\delta + 3$. Note that the conditions of the lemma are satisfies, since $3\delta + 2 = 5$ for $\delta = 1$ and $3\delta + 2 < 5\delta - 1$ for $\delta > 1$.
In this case we have a representative of $c_{2\delta+3}(\mathcal{L}_{3\delta+3})$ given by

$$\{Z \in S_2^{\delta+1} \mid \dim \operatorname{Span}\big(s_1(Z), s_2(Z), \ldots, s_{\delta+1}(Z)\big) \leq \delta\}$$

consisting of the curves in $\mathcal{V}$ with $\delta+1$ singularities. As $S_2^{\delta+1}$ is of dimension $2\delta+2$, we have that $\int_{S_2^{\delta+1}} c_{2\delta+3}(\mathcal{L}_{3\delta+3}) = 0$. So the described set is empty, hence there are no divisors in $\mathcal{V}$ with more than δ singularities.

We are left to prove that all these δ singularities in Ω are indeed nodes.
Consider the following spaces: $S_{3,0}^\delta \subset S^{[5\delta]}$ are the points corresponding to the subschemes of S which are a disjoint union of δ non-reduced points of the form $\operatorname{Spec}\big(\mathcal{O}_{S,x_i}/(m_{S,x_i}^3 + xy)\big)$. As before, this space is a locally closed subset in $S^{[5\delta]}$. It is easily checked that it is smooth of dimension 4δ, as we get two dimensions for each point and one dimension for the choice of each x and y, since such elements determine the same non-reduced point up to scaling. Let S_3^δ be the closure of $S_{3,0}^\delta$. Then if a divisor $D \in \mathcal{V}$ has exactly δ singularities and these singularities are described by a point in $S_3^\delta \backslash S_{3,0}^\delta$, then at least one of these singularities is not a node, otherwise the point would lie in $S_{3,0}^\delta$ by definition. Now we apply Lemma 5.16 once more, to the case of $\mathcal{L}_{5\delta}$ on $S_3^\delta \backslash S_{3,0}^\delta$, with its reduced structure and $p = 4\delta$. This gives that $\int_{S_3^\delta \backslash S_{3,0}^\delta} c_{4\delta}(\mathcal{L}_{5\delta})$ is represented by

$$\{Z \in S_3^\delta \backslash S_{3,0}^\delta \mid \dim \operatorname{Span}\big(s_1(Z), s_2(Z), \ldots, s_{\delta+1}(Z)\big) \leq \delta\}$$

consisting of the curves in $\mathcal{V}$ with δ singularities of which at least one is not a node. As $\dim S_3^\delta \backslash S_{3,0}^\delta < \dim S_{3,0}^\delta = 4\delta$, this set is empty and hence for all $Z \in S_3^\delta \backslash S_{3,0}^\delta$ there is no $D \in \mathcal{V}$ such that $Z \subseteq D$. So these curves with exactly δ singularities have exactly δ nodes. $\square$

5.6 Degeneration of Hilbert schemes

Vainsencher conjectured in [62] that the number of curves with exactly δ nodes in a δ-dimensional linear subsystem of a very ample enough line bundle $\mathcal{L}$ on a smooth projective surface S only depends on the Chern numbers $\mathcal{L}^2$, $\mathcal{L}K_S$, K_S^2 and $c_2(K_S)$ of S and $\mathcal{L}$. This number apparently only depends on the class of $[S, \mathcal{L}]$ in $\omega_{2,1}(k) \otimes \mathbb{Q}$. Now that we know that the relevant degrees can be expressed in terms of Chern classes of line bundles on Hilbert schemes, we are interested how these Hilbert schemes vary along the fibres of a double point degeneration.

We will also need the notion of a DM-stack , which is quite involved. For our applications it will suffice to think of a DM-stack as a generalisation of a scheme, which allows for morphisms, fibres, line bundles and intersection theory as schemes do. For more information one can look at [32] which heavily depends on [30], or the fundamental paper [10].

We know what the Hilbert scheme of the smooth fibres should be. So let us consider the fibre over 0. We will need a relative Hilbert scheme $(S/D)^{[n]}$ of points on a surface S with a divisor D. This space is a proper separated DM-stack as proven in [65]. A good description of this space,

both by construction and as a moduli space, can be found in Section 1.3 in [53]. As before one can think of this space parametrizing n-tuples of points in S, but points in D will carry the additional information of a direction in the normal bundle of D in S.
As before we have a set $(S/D)_2^\delta \subset (S/D)^{[3\delta]}$ which is the Zariski closure of the set parametrizing singular points in S relative to D.

Li and Wu [37] used the notion of relative Hilbert schemes to define a family of Hilbert schemes associated to a double point degeneration $\pi : X \to \mathbb{P}^1$ with special fibre $A \coprod_D B$ with intersection D over $0 \in \mathbb{P}^1$. In this case X must be of pure dimension 3, if we want the fibres to be surfaces.

First we let $U \subseteq X$ be the subset consisting of 0 and the points in X with smooth fibres. Then we have that $X_U := X \times_{\mathbb{P}^1} U \to U$ is family of smooth surfaces, except for the fibre over $0 \in U$. Note that $\mathcal{L}$ restricts to a line bundle on X_U, which we will also denote by $\mathcal{L}$.

The following object constructed over U in [37] by Li and Wu shows that Hilbert schemes of points are well behaved in double point degeneration.

Theorem 5.17. *We have the following objects and properties:*

(i) There is a proper and separated DM-stack $\mathcal{X}^{[n]}$ over U with $\pi^{[n]} : \mathcal{X}^{[n]} \to U$ the corresponding projection, which has the following fibres

$$\left(\pi^{[n]}\right)^{-1}(0) = \coprod_{k=0}^{n} \left(A/D\right)^{[n-k]} \times \left(B/D\right)^{[k]} \quad \text{and} \quad \left(\pi^{[n]}\right)^{-1}(\zeta) = X_\zeta^{[n]} \quad \text{for } \zeta \in U\backslash\{0\}.$$

We will write $i_\zeta^{[n]}$ for the inclusion of a fibre of the second kind. We will denote the inclusion of one term of the special fibre by

$$i^{[n,k]} : \left(A/D\right)^{[n-k]} \times \left(B/D\right)^{[k]} \to \mathcal{X}^{[n]}.$$

We will need the projections $\pi_1^{[n,k]}$ and $\pi_2^{[n,k]}$ from $\left(A/D\right)^{[n-k]} \times \left(B/D\right)^{[k]}$ to the respective factors.

(ii) We have the obvious universal closed subschemes $Z_n(X_\zeta)$, $Z_k(A)$ and $Z_k(B)$ in $X_\zeta \times X_\zeta^{[n]}$, $A \times (A/D)^{[k]}$ and $B \times (B/D)^{[k]}$ respectively. We will write p and q for all projections to the base spaces and the corresponding Hilbert scheme, in particular we will suppress indices and base spaces for readability.
This allows us to define $\mathcal{L}_\zeta^{[n]} := q_* p^* \mathcal{L}_{X_\zeta}$ and similarly $\mathcal{L}_A^{[k]}$ and $\mathcal{L}_B^{[k]}$.

(iii) There is also a universal family $Z_n \subseteq X_U \times \mathcal{X}^{[n]}$ with projections $P : Z_n \to X_U$ and $Q : Z_n \to \mathcal{X}^{[n]}$. This gives us a family over U using the composition

$$Z_n \to X_U \to U$$

with fibres

$$Z_n(X_\zeta) \quad \text{over } \zeta \neq 0 \text{ and } \quad \left(\coprod_{k=0}^{n} Z_{n-k}(A) \times (B/D)^{[k]}\right) \coprod \left(\coprod_{k=0}^{n} (A/D)^{[n-k]} \times Z_k(B)\right) \quad \text{over } 0.$$

Let us write j_ζ for the inclusion of the fibre over ζ, and split the other fibres in the inclusions

$$j_A^{[n,k]} : Z_{n-k}(A) \times (B/D)^{[k]} \to Z_n \quad \text{and} \quad j_B^{[n,k]} : (A/D)^{[n-k]} \times Z_k(B) \to Z_n.$$

Similar to all other cases with such universal closed subschemes this gives a sheaf on $\mathcal{L}^{[n]}$:

$$\mathcal{L}^{[n]} := Q_* P^* \mathcal{L}.$$

So, as before we have that the defined sheaves on $X_\zeta^{[n]}$, $X_A^{[k]}$, $X_B^{[k]}$ and $\mathcal{X}^{[n]}$ are still locally free sheaves.

Lemma 5.18 ([61, Lemma 3.6]). *(i) The sheaves $\mathcal{L}_\zeta^{[n]}$ and $\mathcal{L}^{[n]}$ are locally free sheaves on $X_\zeta^{[n]}$ and $\mathcal{X}^{[n]}$ respectively, both of rank $3n$.*

(ii) Also, $\mathcal{L}_A^{[k]}$ is a locally free sheaf of rank $3k$ on $X_A^{[k]}$. A similar result holds for $X_B^{[k]}$.

We would like to apply Theorem 5.12 to the fibres of $\mathcal{X}^{[n]}$ together with the appropriate vector bundles. For the smooth fibres, we have two ways to produce a vector bundle: we can get a vector bundle of rank $3n$ from the line bundle $\mathcal{L}_{X_\zeta}$ on X_ζ after pulling it back and pushing it forward through the maps

$$X_\zeta \xleftarrow{p} Z_n(X_\zeta) \xrightarrow{q} X_\zeta^{[n]}$$

We could also have obtained a vector bundle on $X_\zeta^{[n]}$ of the same rank by restricting $\mathcal{L}^{[n]}$ to the fibre over $\zeta \in U$. This does in fact give the same vector bundle. We have a similar result for the vector bundles on the fibre over 0.

Lemma 5.19 ([61, Lemma 3.7]). *We have the following identities of locally free sheaves of rank $3n$ on the smooth and general fibre of $\mathcal{X}^{[n]} \to U$.*

(i) $(i_\zeta^{[n]})^*\mathcal{L}^{[n]} = \mathcal{L}_\zeta^{[n]}$;

(ii) $(i^{[n,k]})^*\mathcal{L}^{[n]} = (\pi_1^{[n,k]})^*\mathcal{L}_A^{[k]} \oplus (\pi_2^{[n,k]})^*\mathcal{L}_B^{[n-k]}$.

For the proof we will need the following lemma [61, Lemma 3.6].

Lemma 5.20. *Consider a Cartesian square*

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

with f a finite morphism, g a closed immersion and $\mathcal{F}$ a locally free sheaf on X, then we have that

$$f'_* g'^* \mathcal{F} = g^* f_* \mathcal{F}.$$

Proof. As f is finite, it is in particular affine. So it is enough to prove the statement in the case that Z is affine. We have

$$Z = \operatorname{Spec} R,\ X = \operatorname{Spec} S \text{ and } Y = \operatorname{Spec} R/I.$$

The lemma now follows from the identity

$$M \otimes_S (S \otimes_R R/I) \cong M \otimes_R R/I$$

of R/I-modules. Here we view the S-module M as an R-module via the map $R \to S$ coming from $f : \operatorname{Spec} R \to \operatorname{Spec} S$. □

We can now prove Lemma 5.19.

Proof. (i) We have the following commutative diagram

$$\begin{tikzcd} X_\zeta^{[n]} \arrow[r, "i_\zeta^{[n]}"] & \mathcal{X}^{[n]} \\ Z_n(X_\zeta) \arrow[u, "q"] \arrow[r, "j_\zeta"] \arrow[d, "p"] & Z_n \arrow[u, "Q"'] \arrow[d, "P"] \\ X_\zeta \arrow[r, "i_\zeta"] & X_U. \end{tikzcd}$$

By definition we have

$$(i_\zeta^{[n]})^* \mathcal{L}^{[n]} = (i_\zeta^{[n]})^* Q_* P^* \mathcal{L}.$$

Now we can apply Lemma 5.20 to the upper square in the diagram:

$$(i_\zeta^{[n]})^* Q_* P^* \mathcal{L} = q_* j_\zeta^* P^* \mathcal{L}.$$

Using the commutativity of the lower square and the definition of $\mathcal{L}_\zeta^{[n]}$ we get that

$$(i_\zeta^{[n]})^* \mathcal{L}^{[n]} = q_* \left(P \circ j_\zeta\right)^* \mathcal{L} = q_* (i_\zeta \circ p)^* \mathcal{L} = q_* p^* \mathcal{L}_{X_\zeta}.$$

(ii) Consider the following commutative diagram

$$\begin{tikzcd} (A/D)^{[n-k]} & (A/D)^{[n-k]} \times (B/D)^{[k]} \arrow[l, "\pi_1^{[n,k]}"] \arrow[r, "i^{[n,k]}"] & \mathcal{X}^{[n]} \\ Z_{n-k}(A) \arrow[u, "q"] \arrow[d, "p"] & Z_{n-k}(A) \times (B/D)^{[k]} \arrow[l, "\mathrm{pr}_1"] \arrow[u, "q\times \mathrm{Id}"'] \arrow[r, "j_A^{[n,k]}"] \arrow[d, dashed, "\tau_A"] & Z_n \arrow[u, "Q"'] \arrow[d, "P"] \\ A/D \arrow[r, "i_A"] & A \coprod_D B \arrow[r, "i_0"] & X_U. \end{tikzcd}$$

Here i_A is the obvious inclusion and τ_A is the composition of the three maps $j_A^{[n,k]}$, p and i_A in the lower left square.

By definition we have

$$(i^{[n,k]})^* \mathcal{L}^{[n]} = (i^{[n,k]})^* Q_* P^* \mathcal{L}.$$

Now consider the following commutative square, which comes from adding the upper right square in the above diagram to the similar one with A and B, and k and $n-k$ reversed:

$$\begin{tikzcd} (A/D)^{[n-k]} \times (B/D)^{[k]} \arrow[r, "i^{[n,k]}"] & \mathcal{X}^{[n]} \\ Z_{n-k}(A) \times (B/D)^{[k]} \coprod (A/D)^{[n-k]} \times Z_k(B) \arrow[u, "q\times \mathrm{Id} \coprod \mathrm{Id} \times q"] \arrow[r, "j_A^{[n,k]} \coprod j_B^{[n,k]}"] & Z_n \arrow[u, "Q"'] \end{tikzcd}$$

If we apply Lemma 5.20 to this diagram, we get

$$(i^{[n,k]})^* Q_* P^* \mathcal{L} = (q \times \mathrm{Id} \coprod \mathrm{Id} \times q)_* (j_A^{[n,k]} \coprod j_B^{[n,k]})^* P^* \mathcal{L}.$$

We can rewrite this as follows

$$\begin{aligned}(i^{[n,k]})^* \mathcal{L}^{[n]} &= (q \times \mathrm{Id} \coprod \mathrm{Id} \times q)_* \left(\tau_A^* i_0^* \coprod \tau_B^* i_0^*\right) \mathcal{L} \\ &= (q \times \mathrm{Id})_* \left(\tau_A^* i_0^* \mathcal{L}\right) \oplus (\mathrm{Id} \times q)_* \left(\tau_B^* i_0^* \mathcal{L}\right) \\ &= (q \times \mathrm{Id})_* \left(\tau_A^* i_0^* \mathcal{L}\right) \oplus (\mathrm{Id} \times q)_* \left(\tau_B^* i_0^* \mathcal{L}\right).\end{aligned}$$

Let us examine each of these terms independently:

$$\begin{aligned}(q \times \mathrm{Id})_* \left(\tau_A^* i_0^* \mathcal{L}\right) &= (q \times \mathrm{Id})_* \left(\mathrm{pr}_1^* \, p^* i_A^* i_0^* \mathcal{L}\right) \\ &= (q \times \mathrm{Id})_* \left(\mathrm{pr}_1^* \, p^* \mathcal{L}_A\right) \\ &= \pi_1^{[n,k|*} q_* p^* \mathcal{L}_A \\ &= \pi_1^{[n,k|*} \mathcal{L}_A^{[k]}\end{aligned}$$

where we used Lemma 5.20 in the upper left square. Similarly, we have

$$(\mathrm{Id} \times q)_* \left(\tau_B^* i_0^* P^* \mathcal{L}\right) = \pi_2^{[n,k|*} \mathcal{L}_A^{[n-k]}$$

and the result follows. □

Lemma 5.21. *There is a family of closed subschemes $\mathcal{S}^{[3\delta]} \subset \mathcal{X}^{[3\delta]}$, which has the fibres*

$$\mathcal{S}^{[3\delta]} \cap X_\zeta^{[3\delta]} = (X_\zeta)_2^\delta \qquad \textit{over } \zeta$$

and over $0 \in U$

$$\mathcal{S}^{[3\delta]} \cap \left((A/D)^{[d]} \times (B/D)^{[3\delta-d]}\right) = \begin{cases} \emptyset & \textit{if } 3 \nmid d; \\ (A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon} & \textit{if } d = 3\epsilon \textit{ for } \epsilon \in \mathbb{N}. \end{cases}$$

This statement with proof can be found in [61, Lemma 3.8]. The proof only works over the complex numbers, as it uses the analytical structure of the projective schemes.

5.7 Relative generating function

Using the objects in the previous section one can relate the following generating functions for the spaces in Definition 4.4.

Definition 5.22. Define for a surface S with a line bundle $\mathcal{L}$ the generating function

$$\phi(S, \mathcal{L}) = \sum_{k \geq 0} d_k(S, \mathcal{L}) x^k$$

for $d_k(S, \mathcal{L}) = \int_{S_2^k} c_{2k}(\mathcal{L}^{[3k]})$ for $k > 0$ and $d_0(S, \mathcal{L}) = 1$.
Similarly we have the relative notion for a surface S with a smooth divisor D. In this case we get

$$\phi(S/D, \mathcal{L}) = \sum_{k \geq 0} d_k(S/D, \mathcal{L}) x^k$$

where $d_k(S/D, \mathcal{L}) = \int_{(S/D)_2^k} c_{2k}(\mathcal{L}^{[3k]})$ for $k > 0$ as before and $d_0(S/D, \mathcal{L}) = 1$.

We can use these generating functions to relate the number of δ-nodal curves in a general δ-dimensional linear subsystem on the classes in a double point relation.

Theorem 5.23. *Let $\pi : X \to \mathbb{P}^1$ be a double point degeneration, with X_ζ a smooth fibre over some $\zeta \in \mathbb{P}^1$ and the fibre over 0 being $A \coprod_D B$ with transverse intersection on the smooth divisor D. Let $\mathbb{P}(\pi)$ be the projective space $\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D})$ as before. Then we have*

$$\phi(X_\zeta, \mathcal{L}_\zeta)\phi(\mathbb{P}(\pi), \mathcal{L}_{\mathbb{P}(\pi)}) = \phi(A, \mathcal{L}_A)\phi(B, \mathcal{L}_B).$$

We will only need the relative generating functions for the proof of this theorem, which is largely based on repeated application of the following lemma.

Lemma 5.24. *Let $\pi : X \to \mathbb{P}^1$ be a double point degeneration with fibres as in Theorem 5.23. In this case we have*

$$\phi(X_\zeta, \mathcal{L}_\zeta) = \phi(A/D, \mathcal{L}_A)\phi(B/D, \mathcal{L}_B).$$

Proof. Let $\mathcal{S}$ be as in Lemma 5.21 associated to the double point degeneration $\pi : X \to \mathbb{P}^1$. As $\mathcal{S}^{[3\delta]} \to U$ is a flat family we get that

$$\int_{\mathcal{S}^{[3\delta]}|_0} c_{2\delta}(\mathcal{L}^{[3\delta]}) = \int_{\mathcal{S}^{[3\delta]}|_\zeta} c_{2\delta}(\mathcal{L}^{[3\delta]}).$$

Recall that $\mathcal{S}^{[3\delta]}$ restricts to

$$\coprod_{\epsilon=0}^{\delta} (A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}$$

over 0 and to $(X_\zeta)_2^\delta$ over ζ. These spaces come with the restrictions of the line bundle $\mathcal{L}^{[3\delta]}$, which equals

$$\coprod_{\epsilon=0}^{\delta} (\pi_1^{[3\delta,3\epsilon]})^* \mathcal{L}_A^{[3\delta]} \oplus (\pi_2^{[3\delta,3\epsilon]})^* \mathcal{L}_B^{[3\delta-3\epsilon]}$$

and $(\mathcal{L}_\zeta)^{[3\delta]}$ respectively. This gives

$$\begin{aligned}
\int_{\mathcal{S}^{[3\delta]}|_0} c_{2\delta}(\mathcal{L}^{[3\delta]}) &= \int_{\coprod_{\epsilon=0}^{\delta}(A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}} c_{2\delta}(\mathcal{L}^{[3\delta]}) \\
&= \sum_{\epsilon=0}^{\delta} \int_{(A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}} c_{2\delta}\left(\mathcal{L}^{[3\delta]}|_{(A/D)_2^\delta \times (B/D)_2^{\delta-\epsilon}}\right) \\
&= \sum_{\epsilon=0}^{\delta} \int_{(A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}} c_{2\delta}\left((\pi_1^{[3\delta,3\epsilon]})^* \mathcal{L}_B^{[3\delta]} \oplus (\pi_2^{[3\delta,3\epsilon]})^* \mathcal{L}_A^{[3\delta-3\epsilon]}\right) \\
&= \sum_{\epsilon=0}^{\delta} \int_{(A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}} \sum_{j=0}^{2\delta} c_j\left((\pi_1^{[3\delta,3\epsilon]})^* \mathcal{L}_B^{[3\delta]}\right) c_{2\delta-j}\left((\pi_2^{[3\delta,3\epsilon]})^* \mathcal{L}_A^{[3\delta-3\epsilon]}\right).
\end{aligned}$$

Now note that in the inner sum we only get a non-zero term if $j = 2\epsilon$. So we get

$$\begin{aligned}
\int_{\mathcal{S}^{[3\delta]}|_0} c_{2\delta}(\mathcal{L}^{[3\delta]}) &= \sum_{\epsilon=0}^{\delta} \int_{(A/D)_2^\epsilon \times (B/D)_2^{\delta-\epsilon}} c_{2\epsilon}\left((\pi_1^{[3\delta,3\epsilon]})^* \mathcal{L}_B^{[3\delta]}\right) c_{2\delta-2\epsilon}\left((\pi_2^{[3\delta,3\epsilon]})^* \mathcal{L}_A^{[3\delta-3\epsilon]}\right) \\
&= \sum_{\epsilon=0}^{\delta} \int_{(A/D)_2^\epsilon} c_{2\epsilon}\left((\pi_1^{[3\delta,3\epsilon]})^* \mathcal{L}_B^{[3\delta]}\right) \int_{(B/D)_2^{\delta-\epsilon}} c_{2\delta-2\epsilon}\left((\pi_2^{[3\delta,3\epsilon]})^* \mathcal{L}_A^{[3\delta-3\epsilon]}\right) \\
&= \sum_{\epsilon=0}^{\delta} d_\epsilon(A/D, \mathcal{L}_A) d_{\delta-\epsilon}(B/D, \mathcal{L}_B).
\end{aligned}$$

For the general fibre we find

$$\int_{\mathcal{S}^{[3\delta]}|_\zeta} c_{2\delta}(\mathcal{L}^{[3\delta]}) = \int_{(X_\zeta)_2^\delta} c_{2\delta}\left((\mathcal{L}_\zeta)^{[3\delta]}\right) = d_\delta(X_\zeta, \mathcal{L}_\zeta).$$

Hence $d_\delta(X_\zeta, \mathcal{L}_\zeta) = \sum_{\epsilon=0}^{\delta} d_\epsilon(A/D, \mathcal{L}_A) d_{\delta-\epsilon}(B/D, \mathcal{L}_B)$, which proves that

$$\phi(X_\zeta, \mathcal{L}_\zeta) = \phi(A/D, \mathcal{L}_A)\phi(B/D, \mathcal{L}_B). \qquad \square$$

We can apply this lemma to get expressions for the relative generating function in terms of the general generating function and a relative generating function of a projective bundle.

Lemma 5.25. *Let C be a smooth curve on a smooth projective surface Y with a line bundle $\mathcal{M}$. Let $\mathcal{N}$ be the normal bundle of C in Y. Define $\mathbb{P}_\mathcal{N}$ as the bundle $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{N})$. The line bundle restricts to a line bundle over C, which we can pull back to a line bundle $\mathcal{M}_{\mathbb{P}_\mathcal{N}}$. Then we have*

$$\phi(Y, \mathcal{M}) = \phi(Y/C, \mathcal{M})\phi(\mathbb{P}_\mathcal{N}/C, \mathcal{M}_{\mathbb{P}_\mathcal{N}}).$$

Proof. Let $\mathcal{Y}$ be the blow up of $Y \times \mathbb{P}^1$ along the curve $C \times \{0\}$. Then the fibre of $\mathcal{Y} \to \mathbb{P}^1$ over any $\zeta \neq 0$ is the smooth surface Y, but the fibre over 0 is the union of the smooth Y and $\mathbb{P}_\mathcal{N}$ which intersect transversely in C. Applying Lemma 5.24 to this case immediately gives us

$$\phi(Y, \mathcal{M}) = \phi(Y/C, \mathcal{M})\phi(\mathbb{P}_\mathcal{N}/C, \mathcal{M}_{\mathbb{P}_\mathcal{N}}). \qquad \square$$

Now consider the case of smooth surfaces A and B intersecting transversely in a smooth divisor D as in the special fibre of a double point degeneration. This gives us two isomorphic projective bundles $\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \cong \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{B/D})$ over D. These bundles both contain a divisor isomorphic to D via the inclusion corresponding to the projections $\mathcal{O}_D \oplus \mathcal{N}_{A/D}, \mathcal{O}_D \oplus \mathcal{N}_{B/D} \to \mathcal{O}_D$. These divisors are however not respected by the canonical isomorphism of the projective bundles. By the last two lemmas we are interested in the relative geometry of the bundles over D and we will distinguish them by writing $\mathbb{P}_A$ for the first and $\mathbb{P}_B$ for the second projective bundle. Note that both bundles actually contain two naturally embedded copies of D, which we will denote by

$$D_A := \mathbb{P}(\mathcal{N}_{A/D}) \quad \text{and} \quad D'_A := \mathbb{P}(\mathcal{O}_D) \quad \text{in } \mathbb{P}_A$$

and

$$D_B := \mathbb{P}(\mathcal{O}_D) \quad \text{and} \quad D'_B := \mathbb{P}(\mathcal{N}_{B/D}) \quad \text{in } \mathbb{P}_B.$$

Now note that the isomorphism $\mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{A/D}) \cong \mathbb{P}(\mathcal{O}_D \oplus \mathcal{N}_{B/D})$ identifies the divisors D_A and D_B, and D'_A and D'_B. The line bundle $\mathcal{L}_D$ over D pulls back to different line bundles over $\mathbb{P}_A$ via the different embeddings of D. We will write $\mathcal{L}_{\mathbb{P}_A}$ and $\mathcal{L}'_{\mathbb{P}_A}$ to remember over which divisor the line bundle is being pulled back. Similarly we have line bundles $\mathcal{L}_{\mathbb{P}_B}$ and $\mathcal{L}'_{\mathbb{P}_B}$ over $\mathbb{P}_B$.

Lemma 5.26. *The irreducible components of the special fibre of a double point degeneration satisfy the following relation*

$$\phi(A, \mathcal{L}_A) = \phi(A/D, \mathcal{L}_A)\phi(\mathbb{P}_A/D_A, \mathcal{L}_{\mathbb{P}_A}) \quad \textit{and}$$

$$\phi(B, \mathcal{L}_B) = \phi(B/D, \mathcal{L}_B)\phi(\mathbb{P}_B/D'_B, \mathcal{L}'_{\mathbb{P}_B}).$$

One more application of Lemma 5.25 gives us a relation to get rid of relative generating functions of projective bundles.

Lemma 5.27. *With the notation as before, we have*

$$\phi(\mathbb{P}_A, \mathcal{L}_{\mathbb{P}_A}) = \phi(\mathbb{P}_A/D_A, \mathcal{L}_{\mathbb{P}_A})\phi(\mathbb{P}_A/D'_A, \mathcal{L}'_{\mathbb{P}_A}).$$

Proof. Create a family of surfaces by blowing up $\mathbb{P}_A \times \mathbb{P}^1$ in $D_A \times \{0\}$. The fibre over any $\zeta \neq 0$ is simply $\mathbb{P}_A$ with line bundle $\mathcal{L}_{\mathbb{P}_A}$. The special fibre over 0 is, just as in the first part of the proof of Theorem 4.7, the union of two copies $\mathbb{P}_A$ intersecting transversely at D. One of these D's is embedded as D_A, while the other one as a D'_A. This gives that

$$\phi(\mathbb{P}_A, \mathcal{L}_{\mathbb{P}_A}) = \phi(\mathbb{P}_A/D_A, \mathcal{L}_{\mathbb{P}_A})\phi(\mathbb{P}_A/D'_A, \mathcal{L}'_{\mathbb{P}_A}). \qquad \square$$

With these results we can now state the following important theorem.

Theorem 5.28. *The generating function ϕ induces a homomorphism of groups*

$$\phi : \omega_{2,1} \otimes \mathbb{Q} \to \mathbb{Q}[[x]]^*,\ [S, \mathcal{L}] \mapsto \phi(S, \mathcal{L})(x).$$

Proof. Note that each formal power series in the image of ϕ is indeed invertible as the constant coefficient equals 1.

Let us now show that the map $(S, \mathcal{L}) \to \phi(S, \mathcal{L})$ is additive. So consider two smooth projective surfaces with a line bundle $(S_1, \mathcal{L}_1)$ and $(S_2, \mathcal{L}_2)$. Their disjoint union is the surface $S_1 \coprod S_2$ with the line bundle $\mathcal{L}_1 \coprod \mathcal{L}_2$ which is the direct sum of the pushforwards of the $\mathcal{L}_i$ via the map $S_i \to S_1 \coprod S_2$. Now consider the trivial family of surfaces $S_1 \coprod S_2$ over $\mathbb{P}^1$. Then the fibre over any ζ is smooth, and the fibre over 0 can be decomposed in S_1 and S_2 with empty intersection. Now we can apply Lemma 5.24 to get

$$\phi(S_1 \coprod S_2, \mathcal{L}_1 \coprod \mathcal{L}_2) = \phi(S_1/\emptyset, \mathcal{L}_1)\phi(S_2/\emptyset, \mathcal{L}_2).$$

As the Hilbert scheme of points of a surface relative to the empty set equals the Hilbert scheme of points of the surface, we see that ϕ maps sums to products.

Now, we just need to show that the map is well-defined. So consider a double point degeneration with relation:

$$[X_\zeta, \mathcal{L}_\zeta] + [\mathbb{P}(\pi), \mathcal{L}_{\mathbb{P}(\pi)}] = [A, \mathcal{L}_A] + [B, \mathcal{L}_B].$$

Then we have the equations of the Lemmas 5.24, 5.26 and 5.27

$$\begin{aligned}
\phi(X_\zeta, \mathcal{L}_\zeta) =& \phi(A/D, \mathcal{L}_A)\phi(B/D, \mathcal{L}_B)\\
\phi(A/D, \mathcal{L}_A)\phi(\mathbb{P}_A/D_A, \mathcal{L}_{\mathbb{P}_A}) =& \phi(A, \mathcal{L}_A)\\
\phi(B/D, \mathcal{L}_B)\phi(\mathbb{P}_B/D'_B, \mathcal{L}'_{\mathbb{P}_B}) =& \phi(B, \mathcal{L}_B)\\
\phi(\mathbb{P}_A, \mathcal{L}_{\mathbb{P}_A}) =& \phi(\mathbb{P}_A/D_A, \mathcal{L}_{\mathbb{P}_A})\phi(\mathbb{P}_A/D'_A, \mathcal{L}'_{\mathbb{P}_A}).
\end{aligned}$$

Now note that the canonical isomorphism of $\mathbb{P}_A$ and $\mathbb{P}_B$ maps D'_A to D'_B, and hence

$$\phi(\mathbb{P}_B/D'_B, \mathcal{L}'_{\mathbb{P}_B}) = \phi(\mathbb{P}_A/D'_A, \mathcal{L}'_{\mathbb{P}_A})$$

so we conclude that

$$\phi(X_\zeta, \mathcal{L}_\zeta)\phi(\mathbb{P}(\pi), \mathcal{L}_{\mathbb{P}(\pi)}) = \phi(X_\zeta, \mathcal{L}_\zeta)\phi(\mathbb{P}_A, \mathcal{L}_{\mathbb{P}_A}) = \phi(A, \mathcal{L}_A)\phi(B, \mathcal{L}_B).$$

Hence, double point relations do indeed map to $1 \in \mathbb{Q}[[x]]^*$ and the map ϕ is well-defined. $\square$

This immediately gives a proof of the following theorem.

Theorem 5.29. *There are invertible power series A_1, A_2, A_3 and A_4, such that*

$$\phi(S, \mathcal{L}) = A_1^{\mathcal{L}^2} A_2^{\mathcal{L}K_S} A_3^{K_S^2} A_4^{c_2(K_S)}.$$

In particular d_δ is a polynomial of degree δ in $\mathcal{L}^2$, $\mathcal{L}K_S$, K_S^2 and $c_2(K_S)$.

Proof. We know that $\omega_{2,1}\otimes\mathbb{Q}$ is isomorphic to $\mathbb{Q}^4$, by mapping a pair $[S,\mathcal{L}]$ to $(\mathcal{L}^2,\mathcal{L}K_S,K_S,c_2(K_S))$. Using this isomorphism, we get a group homomorphism

$$\mathbb{Q}^4 \cong \omega_{2,1}\otimes\mathbb{Q} \xrightarrow{\phi} \mathbb{Q}[[x]]^*.$$

Let A_1, A_2, A_3 and A_4 be the images of the basis given by $\mathcal{L}^2$, $\mathcal{L}K_S$, K_S^2 and $c_2(K_S)$. This proves the first part.
For the second part: note that the coefficient of x^i in $A_1(x)^{\mathcal{L}^2}$ is a polynomial in $\mathcal{L}^2$ of degree i, since

$$(1+p(t))^\lambda = \sum_{k=0}^{\infty}\binom{\lambda}{k}p(t)^k$$

for all λ. Similar results hold for $A_2^{\mathcal{L}K_S}$, $A_3^{K_S^2}$ and $A_4^{c_2(K_S)}$, so after multiplying these power series we get a power series where the δth coefficient is a degree δ polynomial in $\mathcal{L}^2$, $\mathcal{L}K_S$, K_S^2 and $c_2(K_S)$. □

This theorem together with Theorem 5.15 immediately completes the proof of Theorem 5.13.

Note that we could have picked any basis for $\omega_{2,1}\otimes\mathbb{Q}$. In this basis, using the result from [62], we can calculate

$$\begin{aligned}
A_1(x) =& 1+3x-\frac{33}{2}x^2+\frac{343}{2}x^3-\frac{17565}{8}x^4+\frac{250197}{8}x^5-\frac{7610077}{16}x^6+\dots\\
A_2(x) =& 1+2x-\frac{35}{2}x^2+225x^3-\frac{26473}{8}x^4+\frac{208647}{4}x^5-\frac{13734387}{16}x^6+\dots\\
A_3(x) =& 1-3x^2+\frac{188}{3}x^3-\frac{4789}{4}x^4+22507x^5-\frac{15203611}{36}x^6+\dots\\
A_4(x) =& 1+x-3x^2+\frac{59}{3}x^3-\frac{1615}{12}x^4+\frac{1911}{4}x^5+\frac{445349}{36}x^6+\dots.
\end{aligned}$$

Which gives us for $\delta=1$ and $\delta=2$:

$$\begin{aligned}
P_1 &= 3\mathcal{L}^2+2\mathcal{L}K_S+c_2(K_S),\\
P_2 &= \frac{1}{2}\left(d_1(d_1-7)-6K_S^2-25\mathcal{L}K_S-21\mathcal{L}^2\right).
\end{aligned}$$

and significantly longer expressions as δ goes up.

Tzeng also proves in [61] that in the basis $(K_S^2,\mathcal{L}K_S,\chi(\mathcal{L}),\chi(\mathcal{O}_S))$ one can express two of the corresponding power series as quasi-modular forms. These quasi-modular forms come from the number of δ-nodal curves on a K3-surface, as determined by Yau and Zaslow in [66].

Now given a pair $(S,\mathcal{L})$ we can use these power series to calculate $\phi(S,\mathcal{L})$. The lowest coefficients $d_k(\mathcal{L})$ will in fact coincide with the number of δ-nodal curves in $|\mathcal{L}|$. Up to which coefficient this still holds depends of course on the line bundle $\mathcal{L}$. It is clear from the above theorems that if $\mathcal{L}$ is l-very ample, then up to the $\lfloor\frac{l+1}{5}\rfloor$th coefficient, we do find the correct degree. In fact, it was proven in [31], that in this case the coefficients do represent the numbers we are looking for, even up to the lth coefficient. The remaining coefficients still agree with $\int_{S_2^k}c_{2k}(\mathcal{L}^{[3k]})$, but lack this combinatorial interpretation.

If we look back to the line bundles $\mathcal{O}(d)$ on $\mathbb{P}^2$ we should get back the Severi polynomials as discovered by Steiner and Cayley. So let us compute the relevant Chern numbers. It is known that the $c_1(K_S)$ equals the divisor of the determinant line bundle of K_S, which equals $-(n+1)h$ on $\mathbb{P}^n$, where h is a hyperplane. So in our case we get that $K_S^2=9$. Now using Riemann-Roch for surfaces

$$12\chi(K_S) = K_S^2+c_2(K_S)$$

we get that $c_2(K_S) = 3$. For the line bundle $c_1(\mathcal{O}(d))$ we simply have that it equals dh for h a line in $\mathbb{P}^2$. So this gives

$$\mathcal{L}^2 = d^2,\ \mathcal{L}K_S = -3d,\ K_S^2 = 9 \text{ and } c_2(K_S) = 3.$$

So we get

$$\begin{aligned}
N_1 &= 3\mathcal{L}^2 + 2\mathcal{L}K_S + c_2(K_S) = 3d^2 - 6d + 3 = 3(d-1)^2,\\
N_2 &= \frac{1}{2}\left(d_1(d_1 - 7) - 6K_S^2 - 25\mathcal{L}K_S - 21\mathcal{L}^2\right)\\
&= \frac{1}{2}\left(3(d-1)^2(3(d-1)^2 - 7) - 54 + 75d - 21d^2\right)\\
&= \frac{3}{2}(d-1)(d-2)(3d^2 - 3d - 11),
\end{aligned}$$

which are exactly the results in Theorems 5.7 and 5.8 by Steiner and Cayley.

Appendices

Appendix A

Algebra

A.1 Formal group laws

Consider the following definition.

Definition A.1. Let A be a ring. A *commutative formal group law of rank one* in A is a formal power series

$$F(u,v) = \sum_{i,j\geq 0} a_{i,j} u^i v^j \in A[[u,v]]$$

such that

(i) $F(u,0) = u = F(0,u)$ in $A[[u]]$;

(ii) $F(u,v) = F(v,u)$ in $A[[u,v]]$;

(iii) $F(u,F(v,w)) = F(F(u,v),w)$ in $A[[u,v,w]]$.

We will generally talk about a formal group law, when we actually mean a commutative formal group law of rank one.

There is an obvious category $\mathbf{R_{FGL}}$ whose objects are rings with a commutative formal group law and where the maps are ring homomorphisms $A \to B$ which map each coefficient of F_A to the corresponding coefficient of F_B.

Lemma A.2. *The category* $\mathbf{R_{FGL}}$ *has an initial element* $(\mathbb{L}, F_{\mathbb{L}})$ *called the Lazard ring.*

Hence for each ring A, a formal group law on A is uniquely determined by a map $\mathbb{L} \to A$.

Proof. Let us define the polynomial ring on $\mathbb{Z}$ with countably many variables indexed by $i,j \in \mathbb{N} \cup \{0\}$:

$$\mathbb{L}' = \mathbb{Z}[A_{i,j}].$$

Now we have a formal power series $F' \in \mathbb{L}'[[u,v]]$ by

$$F'(u,v) = \sum_{i,j\geq 0} A_{i,j} u^i v^j.$$

Note that the three conditions on this formal power series to be a commutative formal group law of rank one are actually polynomial relations in the $A_{i,j}$. Write I for the ideal generated by these relations. We define the Lazard ring by

$$\mathbb{L} := \mathbb{L}'/I.$$

Denote the image of $A_{i,j}$ in $\mathbb{L}$ by $a_{i,j}$, then it is clear that

$$F_{\mathbb{L}}(u,v) = \sum_{i,j\geq 0} a_{i,j} u^i v^j$$

defines a commutative one-dimensional formal group law on $\mathbb{L}$.
Now let (A, F_A) be any commutative formal group law on a ring A. We can define a map

$$\mathbb{L}' \to A$$

by sending $A_{i.j}$ to the corresponding coefficients of F_A. This map clearly factors through $\mathbb{L}$ as all elements of I are mapped to 0, giving a map $\mathbb{L} \to A$ sending $F_{\mathbb{L}}$ to F_A. □

We will however be dealing with graded rings, in which case a formal group law is defined on the elements of degree 1 or -1. This is well-defined if one assumes that the coefficient $a_{i,j}$ is of degree $i+j-1$ or $1-i-j$ respectively. In both cases one can define a grading on $\mathbb{L}$, such that graded ring homomorphisms $\mathbb{L}_* \to R_*$ in the first case and $\mathbb{L}^* \to R^*$ in the second case classify all formal group laws on R_* and R^* respectively.

Definition A.3. Let $\mathbb{L}_*$ be the Lazard ring graded by letting $i+j-1$ be the degree of $a_{i,j}$. We could also use the opposite grading, such that the degree of $a_{i,j}$ is $1-i-j$. In that case we will write $\mathbb{L}^*$.
Note that $\mathbb{L}_0 = \mathbb{L}^0 = \mathbb{Z}$ is generated by $a_{0,1} = 1 = a_{1,0}$, and $\mathbb{L}_n = \mathbb{L}^{-n} = 0$ for $n < 0$.

The structure of the Lazard ring looks quite complicated, but it appears to be rather simple as proven by Lazard [33, Théorème II].

Theorem A.4. *The Lazard ring $\mathbb{L}_*$ is polynomial ring over $\mathbb{Z}$ with generators x_i with $\deg x_i = i$ for all non-negative integers i.*

Obviously $\mathbb{L}^*$ is a polynomial ring with a generator of degree $-i$ for each positive integer i.
There are some special examples of formal group laws. For example we have $(x,y) \mapsto x+y$. Another fundamental example, in which not all coefficients are zero, is given by

$$F(x,y) = x + y - \beta xy.$$

These two examples of formal group laws are important and have been given their own name.

Definition A.5. The formal group law

(i) $F(x,y) = x+y$ is called the *additive* group law;

(ii) $F(x,y) = x+y-\beta xy$ is called *multiplicative.*

A multiplicative formal group law on a ring R is given by a ring homomorphism $\mathbb{Z}[\beta] \to R$. In the case that $\beta \in R$ is invertible, we say that the formal group law is *periodic* and such a group law corresponds to a unique map $\mathbb{Z}[\beta, \beta^{-1}] \to R$.

Appendix B

Algebraic topology

We will recall some important notions from algebraic topology. We will assume the reader is familiar with homotopy theory, smooth manifolds and CW-complexes.

B.1 Bundles and classifying spaces

We will first look at bundles over manifolds.

Definition B.1. A *real vector bundle* ξ of rank r on a manifold X is a manifold $\pi : E \to X$ over X, such that

(i) each fibre $E_p = \pi^{-1}(p)$ has the structure of real vector space of dimension r;

(ii) X admits an open cover $\{U_\alpha\}$ such that there are diffeomorphisms

$$\rho_\alpha : E_U = \pi^{-1}(U) \to U \times \mathbb{R}^r;$$

(iii) the restriction of the isomorphisms ρ_α and ρ_β to $U_\alpha \cap U_\beta$ give an isomorphism

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\beta \cap U_\alpha) \times \mathbb{R}^r$$

and this isomorphism is given by a linear transformation on the second coordinate.

The manifold $X = B(\xi)$ is called the *base space* of the vector bundle and $E = T(\xi)$ is called the total space.

Instead of looking at real vector spaces, one could analogously define the notion of a complex vector bundle, which has a canonical structure of a real one.

Alternatively one could assume that the transition functions in the third condition are not only invertible, but even lie in $GL_r^+(\mathbb{R})$, $O_r(\mathbb{R})$ or $SO_r(\mathbb{R})$. This gives us the notion of oriented bundles, vector bundles with a metric and oriented bundles with a metric.

In particular, if we embed the total space of a vector bundle ξ in $\mathbb{R}^N$, then we have an inner product on each fibre and we may choose the transition functions in $O_r(\mathbb{R})$. If the vector bundle is also oriented, then we can choose them in $SO_r(\mathbb{R})$.

There are many examples of vector bundles: we have tangent bundles, cotangent bundles and normal bundles of embedded manifolds. We also have the trivial bundle $X \times \mathbb{R}^r$ of rank r, which we will denote by $\underline{\mathbb{R}}^r_X$, or $\underline{\mathbb{R}}^r$ if the base space is to be understood. Note that X is a trivial bundle of rank 0 over itself, sometimes denoted by $\underline{\mathbb{R}}^0$.

We can use the following construction to produce new vector bundles.

Theorem B.2. *Let ξ be a vector bundle over Y with total space E and let $f : X \to Y$ be smooth map of manifolds. Then there is a unique vector bundle $f^*\xi$ over X with total space f^*E, such that the fibre of $f^*\xi$ over x is the fibre of ξ over $f(x)$ and there exists a map $f^*E \to E$ which isomorphically maps the fibre over x to the fibre over $f(x)$ for which the following diagram commutes*

$$\begin{array}{ccc} f^*E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The bundle $f^*\xi$ is called the pullback of ξ via f.

In fact, the diagram in the above theorem is Cartesian.

We can use the notion of a pullback of a bundle to define maps between vector bundles $\xi : F \to X$ and $\eta : E \to Y$: a map $\tilde{f} : F \to E$ between total spaces of the bundles ξ and η is a *bundle map* if there exists a smooth map $f : X \to Y$ such that $\tilde{f}$ is the upper map in Theorem B.2.

Let G be either of the groups $GL_r^+(\mathbb{R})$, $O_r(\mathbb{R})$ or $SO_r(\mathbb{R})$. We will see that all vector bundles with transition functions in G have a close relation with homotopy theory.

Theorem B.3. *There exist a space $BG(r)$ of the homotopy type of a CW-complex, together with a bundle γ_r of rank r which has the following universal property:*
any vector bundle $E \to X$ of rank r with transition functions in G is isomorphic to the pullback of γ_r via a map

$$f : X \to BG(r),$$

which is unique up to homotopy.

The space $BG(r)$ is clearly unique up to homotopy and it is called the *classifying space* for G.

For more information on bundles and classifying spaces, see [41], [56] and [5].

B.2 Homology and cohomology

For homology we will need the notion of the category of pairs of topological spaces. The objects will simply be (X, A) where A is subset of X with the induced topology. Maps between such objects $(X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$. These maps do indeed behave as we would expect of morphisms, and we have a category.

We will be particularly interested in the cases where (X, A) consist of a CW-complex X and a subcomplex A. We will then say we have a CW-pair. Another interest is that of pairs which are of the homotopy type of a CW-pair: we say (X, A) is of the homotopy type of CW-pair if X is homotopy equivalent to a CW-complex Y, such that the homotopy equivalences restrict on A to a homotopy equivalence onto a subcomplex of Y. The most important result is that any manifold is of the homotopy type of a CW-complex [40].

Now we can define the notion of an unreduced homology theory, coming from [55].

Definition B.4. Let H_* assign a sequence of abelian groups $H_n(X, A)$ to each pair of spaces and $n \geq 0$, such that the following conditions are satisfied.

(Hom1) Each map $f : (X, A) \to (Y, B)$ gives a homomorphism of groups

$$f_* : H_*(X, A) \to H_*(Y, B)$$

which only depends on the homotopy class of f as a map of space pairs. The identity map is sent to the identity of each group.

(Hom2) For composable morphism f and g of space pairs, we have

$$f_* g_* = (f \circ g)_*.$$

(Hom3) Each pair of spaces gives a *connecting homomorphism*

$$\delta_* : H_*(X, A) \to H_{*-1}(A) := H_{*-1}(A, \emptyset)$$

which is natural in the sense that $\delta f_* = (f|_A)^* \delta$ for all $f : (X, A) \to (Y, B)$. It should furthermore make the following sequence into an exact one

$$\ldots \to H_{n+1}(X, A) \overset{\delta_{n+1}}{\to} H_n(A) \to H_n(X) \to H_n(X, A) \overset{\delta_n}{\to} H_{n-1}(A) \to \ldots$$

(Exc) If U is an open subset of X, such that its closure is contained in a subset $A \subseteq X$, the inclusion $i : (X \backslash U, A \backslash U) \to (X, A)$ induces an isomorphism of groups

$$i_* : H_*(X \backslash U, A \backslash U) \to H_*(X, A).$$

(Dim) The groups $H_n(pt)$ of a space consisting of a single point vanish for $n \neq 0$.

Then H_n is called an *ordinary (unreduced) homology theory.* The *coefficient group* of H_* is defined as $H_0(pt)$.

These axioms are often referred to as the Eilenberg-MacLane axioms. The (Exc) and (Dim) are known as the *dimension axiom* and the *excision axiom.*

For each abelian group G one can show that singular homology with coefficient in G is an ordinary homology theory with coefficient group G, see for example Section 2.2 in [27].

Similarly one can define the notion of an *ordinary cohomology theory* H^*, by looking at contravariant functor from the category of pairs of spaces to sequences of groups, see §3c in [55].

One can show that such a homology theory with a given coefficient group is uniquely determined on well-behaved spaces.

Theorem B.5 ([12])**.** *An ordinary unreduced homology theory is uniquely determined on pairs of space of the homotopy type of a CW-pair by its coefficient group.*

In some cases the condition of the dimension axiom is dropped and one arrives at the notion of an *generalised homology* or *cohomology theory.*

B.3 Characteristic classes

We can assign to vector bundles specific cohomology classes of the base space.

Theorem B.6. *There exists a unique way to associate to each vector bundle ξ of rank r over a closed compact manifold M cohomology classes*

$$w_k(\xi) \in H^i(M, \mathbb{Z}_2)$$

for all non-negative k, called the Stiefel-Whitney classes of ξ, such that

(i) the class $w_0(\xi)$ equals the unit element $1 \in H^0(M, \mathbb{Z}_2)$, and $w_k(\xi)$ is zero for $k > r$;

(ii) for a differentiable map $N \to M$ and a bundle ξ over M, we have that

$$w_k(f^* \xi) = f^*(w_k(\xi))$$

for all k;

(iii) for two bundles ξ and η over the same space, we have that

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_{k-i}(\xi) \cup w_i(\eta);$$

(iv) the first Stiefel-Whitney class of the tautological line bundle over $\mathbb{R}P^1$ is non-zero.

We can use the Stiefel-Whitney classes to find invariants for any smooth closed manifold M of dimension n with a bundle ξ. This is done as follows.

Considering the Stiefel-Whitney classes of ξ, we get cohomology classes of all relevant degrees. Using the cup product on these classes we get cohomology classes of the top degree. Now we have the following result.

Theorem B.7 (Poincare duality)**.** *Let M be a closed compact oriented manifold of dimension n, then we have natural isomorphisms*

$$H^k(M, \mathbb{Z}) \cong H_{n-k}(M, \mathbb{Z}).$$

If M is not oriented, we get an isomorphism for $\mathbb{Z}_2$-valued (co)homology

$$H^k(M, \mathbb{Z}_2) \cong H_{n-k}(M, \mathbb{Z}_2).$$

So the classes of top degree in the cohomology ring, define homology classes of degree 0. But by the dimension axiom we get that $H_0(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$ and we arrive at elements of $\mathbb{Z}_2$.

Definition B.8. Let $d_1, d_2, \ldots, d_n$ be non-negative integers, which satisfy

$$d_1 + 2d_2 + \ldots + nd_n = n.$$

The value of

$$w_1(\xi)^{d_1} w_2(\xi)^{d_2} \ldots w_n(\xi)^{d_n}$$

in $\mathbb{Z}_2$ is denoted by

$$w_1(\xi)^{d_1} w_2(\xi)^{d_2} \ldots w_n(\xi)^{d_n}[M]$$

and is called a *Stiefel-Whitney number* of ξ.

For more information, see [41]. In particular, Chapter 14 contains a similar exposition for characteristic classes for complex vector bundles in homology with integer coefficients. These Chern classes are very similar to Stiefel-Whitney classes and give rise to the notion of Chern numbers for complex vector bundles.

B.4 Spectra

This section contains the most important definitions and theorems from Chapter 8 in [59].

Definition B.9. A *spectrum* is a sequence of pointed CW-complexes (E_n), such that ΣE_n is a CW-subcomplex of E_{n+1} for all n.

Any CW-complex X gives a spectrum by considering $\Sigma^n X$. Any spectrum E and a pointed space give the spectrum $E \wedge X$ given by $(E_n \wedge X)$.

There is an obvious definition for functions between morphisms between spectra.

Definition B.10. A *function $f : E \to F$ of spectra* is a sequence of maps of CW-complexes $f_n : E_n \to F_n$ such that $\Sigma f_n : \Sigma E_n \to \Sigma F_n$ is the restriction of $f_{n+1} : \Sigma E_{n+1} \to \Sigma F_{n+1}$.

This is however not the notion of morphisms that make spectra into an interesting category.

Definition B.11. A *cofinal function* between two spectra $E \to F$ is a pair of a subspectra E' of E and a function $f : E' \to F$, such that every cell e in E has a suspension which lies in E'.

Two cofinal functions (E', f') and (E'', f'') are called equivalent if there is a cofinal function $(\tilde{E}, \tilde{f})$, such that $\tilde{E} \subseteq E' \cap E''$ and both f' and f'' restrict to $\tilde{f}$ on $\tilde{E}$.
An equivalence class of this relation is called a *map of spectra*.

There is a notion of homotopic maps of spectra, which allows us to define homotopy of a spectrum as follows.

Definition B.12. Let F be a spectrum. Then we define the *homotopy groups of* F as

$$\pi_n(F) = [\Sigma^n S^0, F]$$

where $[E, F]$ denotes the *homotopy classes of maps of spectra*.

As the domain $\Sigma^n S^0$ admits very few cofinal maps to any spectrum we have the following result.

Theorem B.13. *The homotopy of a spectrum equals the direct limit of the classical homotopy of the elements of the spectrum:*

$$\pi_n(F) = \lim_{\to} \pi_{n+k}(F_k, pt).$$

There is a strong relation between homology and cohomology theories and spectra. As shown by the following important theorem, coming from [63, 5.2 and 5.10] and [1, Remark 6.5].

Theorem B.14. *Any spectrum E makes*

$$X \mapsto \pi_n(E \wedge X_+)$$

into a generalised homology theory. On the other hand, each such theory is given by the above form for a certain spectrum.
Similarly, every generalised cohomology theory can be written as

$$X \mapsto [\Sigma^{-n} S^0, E]$$

for a spectrum E and each such theory is a generalised cohomology theory.

Appendix C

Algebraic geometry

We will give a short description of the notions line bundles, affine bundles, projective bundles, torsors, regular embeddings and l.c.i. morphisms in algebraic geometry.

C.1 Line bundles

Recall that a sheaf $\mathcal{F}$ on a scheme X which is locally isomorphic to $\mathcal{O}_X^r$ for some r, is called a *locally free sheaf of rank* r. If the rank is equal to 1, then we call $\mathcal{F}$ a *line bundle* or an *invertible sheaf* as their exists a sheaf $\mathcal{F}^\vee$ such that $\mathcal{F} \otimes \mathcal{F}^\vee = \mathcal{O}_X$. The tensor product of sheaves turns the set of isomorphism classes of invertible sheaves into an abelian group.

A *Cartier divisor* is a set of local section of the fraction field of the structure sheaf, such that the quotient of two sections restricted to their common domain is an invertible element of the structure sheaf. See [26, Section III.6] how to define an equivalence relation on Cartier divisors and how to turn the set of equivalence classes into a group.

Theorem C.1. *For varieties, as usual reduced and irreducible, there is an isomorphism between the group of isomorphism classes of line bundles and the group of equivalence classes of Cartier divisors.*

One can think of Cartier divisors as the transitions functions for the line bundle. Similarly one can use a Cartier divisor as gluing information to construct a locally free sheaf of rank one.

C.2 Affine bundles

One could wonder if there is an equivalent notion of vector bundles in the case of schemes. The following definition gives the objects we are looking for. The information in this section comes from Section 11.5 in [19].

Definition C.2. A *geometric vector bundle* of rank r over a scheme X, is an X-scheme $f : Y \to X$, such that X has an affine cover (U_i) which satisfies:

(i) there are isomorphisms $\theta_i : f^{-1}(U_i) \to \mathbb{A}^r_{U_i}$;

(ii) any two such isomorphisms induces on $\operatorname{Spec} V \subset U_i \cap U_j$ a morphism $\mathbb{A}^r_V \to \mathbb{A}^r_V$ coming from a linear map $A[x_1, \dots, x_r] \to A[x_1, \dots, x_r]$.

Exercise II.5.18 in [26] shows that for a locally free sheaf of rank r, we get a geometric vector bundle by taking the global spectrum of the graded symmetric algebra of $\mathcal{E}$:

$$\mathbb{V}(\mathcal{E}) := \mathbf{Spec}(\operatorname{Sym}(\mathcal{E})) = \mathbf{Spec}\left[\bigoplus_{i\geq 0} \left(\mathcal{E}^{\otimes i}/\langle x \otimes y - y \otimes x\rangle\right)\right].$$

One can easily define the sheaf of sections of a geometric vector bundle by considering all local sections of the map $Y \to X$. One can show that the sheaf of sections of $\mathbb{V}(\mathcal{E})$ is in fact $\mathcal{E}^\vee$ and we have constructed an equivalence between the categories of the locally free sheaves of rank r and geometric vector bundles of rank r.

We will pass freely between a geometric vector bundle and its sheaf of sections.

C.3 Torsors

Torsors are either sheaves or schemes over a scheme X, such that locally they have an action of a sheaf of groups or a group scheme over X. This action is assumed to be simply transitive on all opens. We will only need torsors under locally free sheaves and geometric vector bundles. Let us first look at the case of sheaves.

Definition C.3. Let $\mathcal{G}$ be a sheaf of abelian groups and $\mathcal{S}$ a sheaf of sets on a topological space X. An action of $\mathcal{G}$ on $\mathcal{S}$ is morphism of sheaves

$$\mathcal{G} \times \mathcal{S} \to \mathcal{S},$$

such that for every open $U \subseteq X$ we get an action of $\mathcal{G}(U)$ on $\mathcal{S}(U)$.
The sheaf $\mathcal{S}$ is also called a *$\mathcal{G}$-sheaf.*

Now we can define the notion of torsor under a locally free sheaf.

Definition C.4. Let $\mathcal{F}$ be a locally free sheaf on a scheme X. An *$\mathcal{F}$-torsor* $\mathcal{V}$ is an $\mathcal{F}$-sheaf of sets on X, such that

(i) for all open $U \subseteq X$ we have that the local action $\mathcal{F}(U) \times \mathcal{V}(U) \to \mathcal{V}(U)$ is simply transitive, i.e. for any two elements v_1 and v_2 in $\mathcal{V}(U)$ there is a unique $g \in \mathcal{G}(U)$ such that $gv_1 = v_2$;

(ii) there exists an open cover (U_α) of X, such that $\mathcal{V}(U_\alpha)$ is non-empty for all α.

For any U with $\mathcal{V}(U)$ non-empty, we get an isomorphism $\mathcal{V}(U) \to \mathcal{F}(U)$ by fixing some $v \in \mathcal{V}(U)$ and mapping any $w \in \mathcal{V}(U)$ to the unique $g_w \in \mathcal{F}(U)$ satisfying $g_w v = w$. So an $\mathcal{F}$-torsor locally agrees with the locally free sheaf and this gives $\mathcal{V}(U)$ the structure of a group. However, this structure depends on the choice of isomorphism and hence there is no canonical identity element. So the gluing is done, not by linear maps as in the case of locally free sheaves, but by affine linear maps which do not need to preserve the origin.

There is an obvious notion of maps of $\mathcal{F}$-torsors and hence of isomorphic torsors. Note that if an $\mathcal{F}$-torsor $\mathcal{V}$ has a global section, then by the above reasoning we get an isomorphism $\mathcal{V} \to \mathcal{F}$ which is an isomorphism of $\mathcal{F}$-torsors, where $\mathcal{F}$ has the obvious $\mathcal{F}$-torsor structure as it is a sheaf of groups.

Now let $\mathcal{U} = (U_\alpha)$ be a cover of X as in the definition of the torsor. Then we get a 1-cocycle of $\mathcal{V}$ on $\mathcal{U}$, as follows: pick elements v_α in all $\mathcal{V}(U_\alpha)$. Now pick $f_{\alpha\beta} \in \mathcal{F}(U_\alpha \cap U_\beta)$ as the unique element such that $f_{\alpha\beta}\left(v_\alpha|_{U_\alpha \cap U_\beta}\right) = v_\beta|_{U_\beta \cap U_\alpha}$. This is easily checked to be a cocycle and it can be shown that isomorphic torsors give equivalent 1-cocycles. On the other hand, each 1-cocycle gives the gluing information to construct an $\mathcal{F}$-torsor. This gives us the following theorem.

Theorem C.5. *There is a correspondence between classes in*

$$H^1(X, \mathcal{F})$$

and isomorphism classes of $\mathcal{F}$-torsors.

Now any $\mathcal{F}$-torsor restricted to some open U equals the sheaf of sections of the product $U \times \mathbb{A}^n \to U$. We can use the affine linear maps to glue these schemes to get a scheme $\mathbb{V}$ over X, which is trivially affine over X.

C.4 Local complete intersection morphisms

We will first need the definition of a regular embedding.

Definition C.6. A *regular embedding* of codimension d is an embedding $X \to Y$ with ideal sheaf $\mathfrak{I}$, such that the $\mathfrak{I}/\mathfrak{I}^2$ is a locally free sheaf of rank d on Y.
One can show that this is equivalent to $\mathfrak{I}$ being locally generated by a regular sequence, that is X is locally the zero-set of a regular sequence of functions, see page 36 in [2].

The bundle $\mathfrak{I}/\mathfrak{I}^2$ is called the *conormal sheaf* of X in Y.
Now we can describe an important class of morphisms.

Definition C.7. A *local complete intersection* morphism, or simply an l.c.i. morphism, is a morphism $f : X \to Y$ which factors as

$$X \xrightarrow{\varphi} P \xrightarrow{i} Y$$

where i is a closed regular embedding of codimension d and φ is smooth of dimension e. L.c.i. morphism have a well-defined relative dimension. In the notation of the case above the relative dimension would be $e - d$.

The most important case where we find l.c.i. morphism is in the following theorem.

Theorem C.8. *Let X be a smooth scheme. Then the diagonal embedding*

$$\delta : X \to X \times X$$

is an l.c.i. morphism.

For more information on regular embeddings and l.c.i. morphisms, the reader is referred to Appendix B.7 in [18].

C.5 Projective bundles

Whenever we have a geometric vector bundle over a scheme X, and hence a locally free sheaf, we could look at all the lines in a fibre through the origin, so that each fibre $\mathbb{A}^n$ turns into a projective space $\mathbb{P}^{n-1}$. We can do this for all fibres such that we get a scheme over X which locally looks like $\mathbb{P}_U^{r-1}$. We present the formal construction.

Definition C.9. Let $\mathcal{E}$ be a locally free sheaf of rank r on a scheme X. Then we define the scheme

$$\mathbb{P}(\mathcal{E}) = \mathbf{Proj}(\mathrm{Sym}(\mathcal{E}))$$

which comes with a natural map π to X. If $U \subset X$ is an open of X on which $\mathcal{E}|_U$ is isomorphic to $\mathcal{O}_U^r$, then we have an isomorphism

$$\pi^{-1}(U) = \mathbb{P}(\mathcal{E}|_U) \cong U \times \mathbb{P}^{r-1}.$$

We will give a geometric description of an important invertible sheaf on this bundle of projective spaces. For the details and an algebraic construction, see [26, Chapter II.7].

If we identify $\mathbb{P}(\mathcal{E})$ with the lines through the origin in the fibre of the geometric vector bundle associated with $\mathcal{E}$, we see that the pullback $\pi^*\mathcal{E}$ on $\mathbb{P}(\mathcal{E})$ contains a line bundle: Each fibre $\mathcal{E}_x$ over a point $\ell_x \in \mathbb{P}(\mathcal{E})$ which lies over $x \in X$ has a preferred line, namely $\ell_x \subset \mathcal{E}_x$. If we take this line as the fibre, we get an invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$, called the *canonical line bundle*.

Recall that for a closed embedding $X \to Y$ the notion of the conormal bundle: $\mathfrak{I}/\mathfrak{I}^2$, where $\mathfrak{I}$ is the ideal sheaf of X in Y. The dual $(\mathfrak{I}/\mathfrak{I}^2)^\vee$ is called the *normal bundle*. For a divisor D of Y, we can find it by restricting the line bundle $\mathcal{O}_Y(D)$ to D, see [26, Example IV.1.4.1]

Now consider two line bundles $\mathcal{L}$ and $\mathcal{M}$ on a scheme X. If we blow up the sheaf $\mathcal{L} \oplus \mathcal{M}$ we get a scheme $\mathbb{P}$ over X which locally looks like a product with $\mathbb{P}^1$. Note that we have inclusions

$$\mathbb{P}(\mathcal{L}) \to \mathbb{P} \leftarrow \mathbb{P}(\mathcal{M})$$

coming from the projections $\mathcal{L} \oplus \mathcal{M} \to \mathcal{L}, \mathcal{M}$. But $\mathbb{P}(\mathcal{L})$ and $\mathbb{P}(\mathcal{M})$ are simply X, so this gives two divisors in $\mathbb{P}$ isomorphic to X. One can show that the normal bundle of $\mathbb{P}(\mathcal{L})$ in $\mathbb{P}$ is simply $\mathcal{M}$ and vice versa.

Consider a scheme X with a subscheme Z with ideal sheaf $\mathfrak{I}$.

Definition C.10. The *blow up of X along Z* is the scheme

$$X_Z := \mathbf{Proj}(\bigoplus_{k \geq 0} \mathfrak{I}^k)$$

which comes with a morphism to X. This morphism is the identity away from Z and the fibre of Z is $\mathbb{P}(\mathfrak{I}/\mathfrak{I}^2)$, the projective bundle of the conormal sheaf.

We will most often use these kind of schemes in a specific setting, to create a family over $\mathbb{P}^1$ where most fibres are simply X but a specific fibre consists of X_Z and $\mathbb{P}(\mathfrak{I}/\mathfrak{I}^2 \oplus \mathcal{O}_Z)$ glued along $\mathbb{P}(\mathfrak{I}/\mathfrak{I}^2)$.

Definition C.11. The *deformation to the normal cone* of Z in X is the blow up of $X \times \mathbb{P}^1$ along $Z \times 0$. The morphism to $X \times \mathbb{P}^1$ composed with the projection to the second factor make the deformation to the normal cone into a family over $\mathbb{P}^1$ of deformations of X.

We will in particular be interested in the special fibres of such deformations.

For more information on blow ups and the deformation to the normal cone, one can consult [26, Section 2.7], [35, Section 2.5.1] and [18, Appendix B.6].

Bibliography

[1] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago press, 1995.

[2] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of Algebraic Curves: Volume II. With a contribution by Joseph Daniel Harris*, volume 268. Springer, 2011.

[3] M. F. Atiyah and I. M. Singer. The index of elliptic operators on compact manifolds. *Bulletin of the American Mathematical Society*, 69(3):422–433, 1963.

[4] M. Beltrametti and A. J. Sommese. Zero cycles and k-th order embeddings of smooth projective surfaces. In *Problems in the theory of surfaces and their classification, Symposia Math*, volume 32, pages 33–48, 1991.

[5] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Springer, 1973.

[6] L. Caporaso and J. Harris. Counting plane curves of any genus. *Inventiones mathematicae*, 131(2):345–392, 1998.

[7] A. Cayley. On the theory of involution. *Transactions of the Cambridge Philosophical Society*, 11:21–38, 1866.

[8] P. E. Conner and E. E. Floyd. *Differentiable periodic maps.* Springer, 1964.

[9] P. E. Conner and E. E. Floyd. *The relation of cobordism to K-theories.* Springer, 1966.

[10] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l'IHES*, 36(1):75–109, 1969.

[11] P. Di Francesco and C. Itzykson. Quantum intersection rings. In *The moduli space of curves*, pages 81–148. Springer, 1995.

[12] S. Eilenberg and N. E. Steenrod. Axiomatic approach to homology theory. *Proceedings of the National Academy of Sciences of the United States of America*, 31(4):117, 1945.

[13] D. Eisenbud and J. Harris. *3264 & All That Intersection Theory in Algebraic Geometry.* 2013.

[14] F. Enriques. Sui moduli duna classe di superficie e sul teorema desistenza per funzioni algebriche di due variabilis. *Atti Accad. Sci. Torino*, 47, 1912.

[15] S. Fomin and G. Mikhalkin. Labeled floor diagrams for plane curves. *arXiv preprint arXiv:0906.3828*, 2009.

[16] W. Fulton. Algebraic curves, An Introduction to Algebraic Geometry. available at: `http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf`.

[17] W. Fulton. On nodal curves. In *Algebraic geometry-open problems*, pages 146–155. Springer, 1983.

[18] W. Fulton. *Intersection Theory.* Ergebnisse Der Mathematik Und Ihrer Grenzgebiete, 3. Folge, Bd. 2. Springer-Verlag GmbH, 1998.

[19] U. Görtz and T. Wedhorn. *Algebraic Geometry: Part I: Schemes. With Examples and Exercises.* Advanced Lectures in Mathematics. Vieweg+Teubner Verlag, 2010.

[20] L. Göttsche. A conjectural generating function for numbers of curves on surfaces. *Communications in mathematical physics*, 196(3):523–533, 1998.

[21] A. Grothendieck. La théorie des classes de Chern. *Bulletin de la Société Mathématique de France*, 86:137–154, 1958.

[22] A. Grothendieck, P. Berthelot, and L. Illusie. *(SGA6) Séminaire de Géométrie Algébrique: Théorie des Intersections et Théorème de Riemann-Roch.* Number 225 in Lecture Notes in Math. Springer-Verlag, Heidelberg, 1971.

[23] A. Grothendieck and J. Dieudonné. Éléments de géométrie algébrique. *Publications mathématiques de l'I.H.É.S.*, (4 (Chapter 0, 1-7, and I, 1-10), 8 (II, 1-8), 11 (Chapter 0, 8-13, and III, 1-5), 17 (III, 6-7), 20 (Chapter 0, 14-23, and IV, 1), 24 (IV, 2-7), 28 (IV, 8-15), and 32 (IV, 16-21)), 1960-1967.

[24] J. Harris. On the Severi problem. *Inventiones mathematicae*, 84(3):445–461, 1986.

[25] J. Harris and R. Pandharipande. Severi degrees in cogenus 3. *arXiv preprint alg-geom/9504003*, 1995.

[26] R. Hartshorne. *Algebraic Geometry.* Number 52. Springer, 1977.

[27] A. Hatcher. Algebraic topology. *Cambridge UP, Cambridge*, 606, 2002.

[28] M. W. Hirsch. *Differential topology.* Springer, 1976.

[29] S. Kleiman and R. Piene. Enumerating singular curves on surfaces. *Contemporary Mathematics*, 241:209–238, 1999.

[30] D. Knutson. *Algebraic spaces.* Springer, 1971.

[31] M. Kool, V. Shende, and R. P. Thomas. A short proof of the Göttsche conjecture. *Geometry & Topology*, 15(1):397–406, 2011.

[32] G. Laumon and L. Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, Berlin, 2000.

[33] M. Lazard. Sur les groupes de Lie formels à un paramètre. *Bulletin de la Société Mathématique de France*, 83:251–274, 1955.

[34] Y.-P. Lee and R. Pandharipande. Algebraic cobordism of bundles on varieties. *arXiv preprint arXiv:1002.1500*, 2010.

[35] M. Levine and F. Morel. *Algebraic Cobordism.* Springer, 2007.

[36] M. Levine and R. Pandharipande. Algebraic cobordism revisited. *Inventiones mathematicae*, 176(1):63–130, 2009.

[37] J. Li and B. Wu. Good degeneration of Quot-schemes and coherent systems. *arXiv preprint arXiv:1110.0390*, 2011.

[38] C.-H. Liu, Y.-J. Tzeng, and S.-T. Yau. Algebraic cobordism of filtered vector bundles on varieties: Notes on a work of Lee and Pandharipande. *ArXiv e-prints math.AG 1104.0280*, 2011.

[39] Y. I. Manin. Lectures on the K-functor in algebraic geometry. *Russian Mathematical Surveys*, 24(5):1–89, 1969.

[40] J. Milnor. On spaces having the homotopy type of a CW-complex. *Transactions of the American Mathematical Society*, pages 272–280, 1959.

[41] J. W. Milnor and J. D. Stasheff. *Characteristic classes*, volume 76 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1974.

[42] H. Nakajima. *Lectures on Hilbert schemes of points on surfaces.* Number 18 in University Lecture Series. American Mathematical Soc., 1999.

[43] H. Osborn. *Vector bundles-Vol 1: Foundations and Stiefel-Whitney Classes.* Academic Press, 1983.

[44] L. S. Pontryagin. Characteristic cycles on differentiable manifolds. *Matematicheskii Sbornik*, 63(2):233–284, 1947.

[45] D. Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc*, 75(6):1293–1298, 1969.

[46] D. Quillen. Elementary proofs of some results of cobordism theory using Steenrod operations. *Advances in Mathematics*, 7(1):29–56, 1971.

[47] Z. Ran. The degree of a Severi variety. *Bulletin of the American Mathematical Society*, 17(1):125–128, 1987.

[48] Z. Ran. Enumerative geometry of singular plane curves. *Inventiones mathematicae*, 97(3):447–465, 1989.

[49] J. Roberts. Chow's moving lemma. *Appendix*, 2:53–82, 1970.

[50] S. Roberts. On a Simplified Method of obtaining the Order of Algebraical Conditions. *Proceedings of the London Mathematical Society*, 1(1):101–113, 1874.

[51] J. Rosenberg. *Algebraic K-theory and its applications*, volume 147. Springer, 1994.

[52] J.-P. Serre. *Algèbre locale, multiplicités: cours au Collège de France, 1957-1958*, volume 11. Springer, 1975.

[53] I. Setayesh. Relative Hilbert scheme of points. *arXiv preprint arXiv:1006.0011*, 2010.

[54] F. Severi. *Vorlesungen uber algebraische Geometrie.* Teubner, Leipzig, 1921.

[55] N. Steenrod and S. Eilenberg. Foundations of algebraic topology. *Princeton UP, Princeton*, 1952, 1952.

[56] N. E. Steenrod. *The topology of fibre bundles*, volume 14. Princeton University Press, 1951.

[57] J. Steiner. Allgemeine Eigenschaften der algebraischen Curven. *Journal für die reine und angewandte Mathematik*, 47:1–6, 1854.

[58] R. E. Stong. *Notes on cobordism theory*, volume 536. Princeton University Press Princeton, 1968.

[59] R. Switzer. *Algebraic Topology-Homotopy and Homology.* Springer-Verlag, Berlin, 1975.

[60] R. Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 28(1):17–86, 1954.

[61] Y.-J. Tzeng et al. A proof of the Göttsche-Yau-Zaslow formula. *Journal of Differential Geometry*, 90(3):439–472, 2012.

[62] I. Vainsencher. Enumeration of n-fold tangent hyperplanes to a surface. *Journal of Algebraic Geometry*, 4(4):503–526, 1995.

[63] G. W. Whitehead. Generalized homology theories. *Transactions of the American Mathematical Society*, pages 227–283, 1962.

[64] H. Whitney. The self-intersections of a smooth n-manifold in 2n-space. *Annals of Mathematics*, pages 220–246, 1944.

[65] B. Wu. The moduli stack of stable relative ideal sheaves. *arXiv preprint math/0701074*, 2007.

[66] S.-T. Yau and E. Zaslow. BPS states, string duality, and nodal curves on K3. *Nuclear Physics B*, 471(3):503–512, 1996.

List of Symbols

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