Anti-de Sitter black holes in gauged supergravity

Supergravity flow, thermodynamics and phase transitions

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Anti-de Sitter zwarte gaten in geijkte superzwaartekracht
Superzwaartekracht flow, thermodynamica en faseovergangen

(met een samenvatting in het Nederlands)

Proefschrift

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Contents

Publications

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- **Chapter 3**
  A. Gnecchi and C. Toldo, "On the U(1) non-BPS first order flow in gauged supergravity", arXiv: 1211.1966, JHEP 1303 (2013) p. 088.
- **Chapter 4**
  C. Toldo and S. Vandoren, "Static nonextremal AdS black hole solutions", arXiv: 1207.3014, JHEP 1209 (2012), p. 048.
- **Chapter 5**
  A. Gnecchi, K. Hristov, D. Klemm, C. Toldo and O. Vaughan, "Rotating black holes in gauged supergravity", arXiv:1311.1795, JHEP 1401 (2014), p. 127.
- **Chapter 6**
  K. Hristov, C. Toldo and S. Vandoren, "Phase transitions of magnetic AdS_4 black holes with scalar hair", arXiv: 1304.5187, Phys. Rev. D88 (2013), p. 026019

  and unpublished material.
- **Chapter 7**
  K. Hristov, C. Toldo and S. Vandoren, "On BPS bounds in $d = 4, \mathcal{N} = 2$ gauged supergravity", arXiv: 1110.2688, JHEP 1112 (2011) 014.
- **Chapter 8**
  A. Gnecchi and C. Toldo, "First order flow for nonextremal AdS_4 black holes and mass from holographic renormalization", in preparation.

Other publications to which the author has contributed:

- K. Hristov, C. Toldo and S. Vandoren, "Black branes in AdS: BPS bound and asymptotic charges", arXiv: 1201.6592, Fortsch.Phys. 60 (2012) 1057-1063.

To Women in Physics

Introduction

Black holes

Black holes are regions of spacetime from which not even light can escape. Once you fall inside a black hole there is no hope to come back, and a surface called event horizon marks the point of no return. The presence of an event horizon distinguishes black holes from other celestial objects.

It is now common belief that black holes do indeed exist in our universe as final state of the evolution of very heavy stars, collapsed under their own self-gravitational force. Recent astrophysical measurements provided evidence for the existence of a supermassive black hole in the galactic center of our galaxy, the Milky Way [1–3]. These measurements also show that black holes are ubiquitous in our Universe, hence they are not just an intellectual possibility: they exist in Nature.

John Mitchell first in 1783 realized that very massive celestial objects could deflect light and drag it back due to gravitational effects. Such objects will therefore appear black. In 1916, a gravitational object with these characteristics was shown by Karl Schwarzschild to be possible within the solid framework of General Relativity (GR). Schwarzschild's solution [4] exhibits an event horizon that shields a curvature singularity, where the effects of gravity are dramatic.

Black holes are within the most elementary objects from the point of view of General Relativity. They are specified by a finite and small set of parameters. Once a star collapses to form a black hole, the gravitational field around it sweeps away all details about the star that formed it, except mass, spin and charge. In this sense, the black hole is pretty much like a structureless elementary particle, like an electron.

Recent developments have made it clear that a black hole is so "simple" not because it is like an elementary particle, but rather because it is a statistical ensemble. Indeed a statistical ensemble is specified by a small number of parameters such as energy, temperature, charge. In this respect, the simplicity of a black hole is no different than the simplicity of a thermal ensemble.

However, a black hole possesses huge entropy[1], meaning that it is composed by

[1]For instance, the black hole whose presence at the center of the galaxy was suggested by recent observations has mass $M = 3.7 \times 10^6 M_\odot$ and entropy $S = 5 \times 10^{66} J/K$ [5].

very many microstates. A theory valid in the regime of very strong gravity, namely quantum gravity, should be able to describe such microstates. Therefore black hole entropy provides us with important quantitative information about the degrees of freedom of quantum gravity. For this reason a black hole is a theoretical laboratory for such a theory, a sort of "hydrogen atom" for quantum gravity. But before delving into the intricacies of quantum gravity let us first introduce the concept of black hole thermodynamics.

Black hole thermodynamics

One of the remarkable properties of black holes is that we can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is surprising because a priori there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

The correspondence between laws of black hole mechanics and laws of thermodynamics dates back forty years ago [6] and it is summarized in this table:

	Laws of Thermodynamics	**Laws of Black hole Mechanics**
zeroth law	Temperature T is constant at equilibrium	Surface gravity κ is constant on the event horizon
first law	$dE = TdS + \mu dQ + \omega dJ$	$dM = \frac{\kappa}{8\pi} dA + \mu dQ + \omega dJ$
second law	Entropy S of an isolated system never decreases: $\delta S \geq 0$	Area of the event horizon A never decreases: $\delta A \geq 0$

Here A is the area of the event horizon, and κ is the surface gravity, which can be thought of roughly as the acceleration at the horizon, μ is the chemical potential conjugate to the charge Q, and ω is the angular speed conjugate to the angular momentum J.

Bekenstein first proposed that a black hole must possess an entropy proportional to the area of the event horizon [7]. A decisive step forward took place when Hawking showed that in the semiclassical approximation black holes are not quite black: they emit radiation [8] at a thermal spectrum with temperature

$$T = \frac{\hbar\kappa}{2\pi}\,. \tag{1}$$

Using this in the first law, one can conclude therefore that a black hole possesses entropy equal to

$$S_{B-H} = \frac{Ac^3}{4\hbar G_N}\,, \tag{2}$$

where the speed of light c, the Planck constant $\hbar$ and the Newton's constant G_N appear.

On the one hand, Hawking's discovery of black hole radiation gave a conclusive positive answer to the question whether there is any physical significance to the analogy between laws of mechanics and laws of thermodynamics. On the other hand, it raised many important conceptual issues.

First of all, Hawking radiation is exactly thermal and contains no information about the states inside the black hole. This leads to the problem of information loss, since particles carrying information can fall inside the black hole, but what comes out is just featureless radiation. This loss of information is in contrast with unitary evolution of quantum mechanics, and it constitutes a paradox that is still a source of debate nowadays [9].

Another puzzle is the interpretation of black hole entropy. According to Boltzmann's principle, thermodynamic entropy has a statistical interpretation in terms of microscopic configurations with the same macroscopic properties. Thus, S_{B-H} measures the multiplicity of black hole microstates. Any candidate theory for a consistent theory of quantum gravity has to be able to identify these microstates composing a black hole, and their subsequent counting has to reproduce the thermodynamic entropy of the black hole. This task still represents a challenge: indeed theoretical physicists have not been able yet to perform the microstate counting in the simple case of a Schwarzschild black hole. Nevertheless, the matching between macroscopic and microscopic entropy was successfully achieved in the realm of string theory for very specific classes of black holes, as explained in the next subsection.

Black holes and quantum gravity

A theory that unifies quantum mechanics and gravitation is needed when the gravitational force becomes strong, such as in the interior of a black hole, or to describe what happened just after the Big Bang.

It has been a long standing challenge for theoretical physics to construct a theory that reconciles the principles of Quantum Mechanics with those of General Relativity. Almost a century after these two groundbreaking ideas, constructing a quantum theory of gravity is to be considered still work in progress.

Superstring theory is so far one of the leading candidates for a quantum theory of gravity. For its consistency, superstring theory requires the existence of additional 6 space dimensions, for a total of 10 spacetime dimensions. The idea underlying string theory is to replace the point-like nature of fundamental particles with that of strings. The elementary particles that we observe, like electrons, quarks and gravitons, are believed to be oscillations of these strings.

String theory also contains D-branes, extended membranes of various spatial dimensions on which strings can end. When D-branes are wrapped around the compact directions, they appear to the four-dimensional observer as charged, localized objects, and superimposing them yields a configuration that has the properties of a black hole. The counting of degeneracies of strings on D-branes gives a statistical interpretation to the thermodynamic entropy. This program was successfully carried out in [10–12], achieving the matching between microscopic and macroscopic entropy for extremal (zero-temperature) and near extremal black holes.

So far, the black holes that are best understood in this scenario are those that arise as solutions to theories of gravity with a certain amount of supersymmetry, namely supergravity theories. In particular, supergravity is the low energy effective theory for string theory, and allows for extremal supersymmetric black hole solutions. These configurations are stable, and this allows the counting to be more easily performed. Supergravity black holes are typically supported by scalar fields, and these exhibit an interesting flow mechanism called the attractor mechanism. This mechanism states that as one moves towards the horizon of the black hole, the scalar fields flow to specific values dependent just on the electromagnetic charges, losing all memory of their initial values far away from the horizon. The attractor mechanism is at the heart of the recent progress in string theory in reproducing the thermodynamic black hole entropy by microstate counting.

A mentioned before, despite this amount of success, the microstate explanation of the entropy-area law for more general and/or realistic black holes such as the Schwarzschild or the Kerr configurations still represents an open problem. This is true also regarding for instance black holes with nonzero temperature, or those with different asymptotic behavior, for instance black holes in theories with a nonzero cosmological constant.

AdS Black holes and the AdS/CFT correspondence

A fresh way to understand the characteristics of quantum gravity is offered by the holographic principle. A hologram is a two-dimensional object able to produce a three dimensional image. In the same spirit, the holographic principle, that has its roots in the ideas of [13, 14], states that the description of quantum gravity in a volume of spacetime is encoded in a set of laws on the boundary surface of that volume. More explicitly, the AdS/CFT correspondence asserts the duality between string theory on d-dimensional in Anti-de Sitter (AdS) spacetime and conformal field theories in $d-1$ dimensions. The most robust example of this AdS/CFT correspondence is provided by string theory in $AdS_5 \times S^5$, conjectured to be equivalent to $\mathcal{N}=4$ super-Yang Mills in four spacetime dimensions [15].

Remarkably, black hole entropy scales as the area of the event horizon rather than the volume: it shows a holographic behavior. The power of the AdS/CFT conjecture itself lies in allowing an easier dual description of the system: quantities like black hole entropy might have a clearer interpretation at the level of the dual field theory, rather than on the gravity side [16]. Moreover, the AdS/CFT correspondence provides us with another tool to tackle other black hole fundamental questions like for instance the information paradox.

The holographic correspondence can also be used in the opposite direction. The idea in this case is to use the duality to study interesting aspects of quantum field theories which cannot be approached by perturbative techniques. Phenomena like superconductivity or heavy ions collisions are modeled by strongly coupled field theories, which are very difficult to solve. The dual picture, involving processes and phase transitions of AdS black holes can be more tractable. Many fruitful attempts in this direction have been made so far, see for instance [17, 18].

For this reason, the recent years witnessed renewed attention towards Anti-de Sitter black holes. Nevertheless, the interest in AdS black holes dates back to the 1980s, when it was discovered that they display a rich thermodynamics [19]. Indeed due to the stabilizing effect of a negative cosmological constant a Schwarzschild AdS black hole can be in equilibrium with thermal radiation at a fixed temperature, while the corresponding asymptotically flat configuration cannot (unless one puts the black hole in a finite box). In this sense Anti-de Sitter spacetime acts like a box of finite volume and it allows for multiple branches of black holes with positive specific heat.

AdS black holes can therefore be considered as a toy models for the study of black hole thermodynamics: the aim in studying this simplified scenario is to extract information about the universal behavior of black hole thermodynamics [20], valid also in more general scenarios.

Motivation and Content of this thesis

To recap, let us summarize the motivations to study Anti-de Sitter black holes in three main points:

- Anti-de Sitter black holes for their characteristics of stability can be considered the ideal playground for studying black hole thermodynamics.
- They provide interesting gravitational backgrounds to describe the physics of strongly coupled field theories via AdS/CFT, with a view towards possible connections with condensed matter and many body physics.
- The AdS/CFT correspondence can give important insights on the microscopic interpretation of black hole entropy and other fundamental black hole issues.

We will focus here on asymptotically Anti-de Sitter black hole solutions arising from a specific theory, namely four-dimensional $\mathcal{N} = 2$ gauged supergravity.

Gauged supergravity is the low energy effective theory describing string compactifications in presence of fluxes. These compactifications typically result in four-dimensional theories where a scalar potential is present[2]. For this reason gauged supergravity theories can support an effective negative cosmological constant, hence they can have an AdS ground state. The potential that usually encounters in these

[2]Gauged supergravity potentials may support an effective cosmological constant, provide mass terms for the fields (moduli stabilization) and describe scenarios of spontaneous supersymmetry breaking. There is renewed interest for phenomenological models arising from flux compactifications (for a review see [21]), however we will not address these concepts here.

models is unbounded from below, nevertheless Anti-de Sitter solutions can still be stable [22–24], and positive mass theorems hold [25, 26]. We focus on $\mathcal{N}=2$ supergravity, namely a theory with 8 real supercharges: in this case the amount of supersymmetry is enough to allow the existence of charged BPS black hole solutions, and at the same time the theory is not too restrictive. The ungauged version of the $\mathcal{N}=2$ theory has been extensively studied, furthermore the techniques of [12] for the microstate counting regarding four-dimensional black holes were developed exactly in this framework.

The aim of this thesis is twofold. On the one hand, we would like to study the properties of black hole solutions in supergravity, in particular for models that allow an embedding in string theory. We work in the setup of Fayet-Iliopoulos gauged $\mathcal{N}=2$ supergravity, which allows for Anti-de Sitter asymptotics. Since black holes are the best test ground for the fundamental principles of quantum gravity, the knowledge of the possible black hole solutions in a given theory can be regarded as a first step in the program of solving this theory. Therefore we study the properties of the supergravity flow and the techniques to generate exact solutions. We provide explicit examples of the newly found extremal and nonextremal solutions, and we compute conserved quantities for these configurations.

On the other hand, we are interested in the thermodynamic properties of the AdS configurations, and their phase diagram. The phase transitions on the gravity side might have interesting field theory counterparts. Therefore we elaborate on the phase transitions of a class of static black hole configurations, and via the AdS/CFT correspondence we give an interpretation of these transitions in the dual field theory, which falls in the class of ABJM [27] models. We also show some preliminary results about phase transitions for the recently found [28, 29] rotating AdS configurations. We calculate the thermodynamic quantities for the solutions and we compare different techniques for computing the mass for Anti-de Sitter black holes. Lastly, we identify the BPS bound for the supergravity theories taken in consideration.

Let us mention that solutions allowing an embedding in string theory can also be studied in their own right as string theory ground states, and one can try to understand black microscopic degeneracy along the lines of [10, 12, 16]. The successful techniques applied previously for the microstate counting of asymptotically flat black holes here cannot be applied directly, since the scalars and in particular the dilaton are fixed at infinity. This represent a very interesting subject that lies a bit outside the scope of this work, nevertheless some remarks on the topic are made throughout the body and in the conclusions of the thesis.

In the following we briefly discuss the content of the single chapters.

Chapter 1 offers a fair amount of review of gauged $\mathcal{N}=2$ four-dimensional supergravity. The Lagrangian and the supersymmetry variations in the general case, along with (some details of) the gauging procedure are discussed there. However, a complete description is beyond the scope of this thesis. We focus here on highlighting the differences with respect to the ungauged case, and we include all relevant details about the theory that we will take into consideration in the rest of the thesis, namely abelian Fayet-Iliopoulos gauged supergravity in absence of hypermultiplets.

Chapter 2 is devoted to the description of the supersymmetric solutions of gauged supergravity in presence of Fayet-Ilioupoulos gauging and vector multiplets, in absence of hypermultiplets. We will restrict ourselves to the case of electric gaugings. We will first describe the scalarless AdS solutions found by Romans in minimal gauged supergravity. Then we will deal with the BPS electric solutions, that contain naked singularities, and the magnetic ones firstly found in [30], that represent genuine Anti-de Sitter black holes. We will elaborate more on the latter class of solutions, giving details about the near-horizon geometry and the entropy as a function of the gauging and black hole charges.

Subsequently, in Chapter 3 we will describe the first branch of deformed solutions obtained from the supersymmetric ones. We will explain how to obtain extremal-non BPS black holes from the BPS ones, by means of first order equations. This procedure is based on a specific rotation of the black hole charges: this yields inequivalent squarings of the one-dimensional reduced action, therefore a first order flow that differs from the supersymmetric one.

Chapter 4 involves thermal Anti-de Sitter solutions, found by solving analytically the full system of Einstein, Maxwell's and scalar equations of motion. The solution generating technique is based on a minimal modification of the BPS ansatz, and will allow us to find a nonextremal deformation of electric and magnetic solutions. We furthermore will show that the product of the areas of the event horizons does not depend on the mass of the solutions, but just one the cosmological constant and on the black hole electromagnetic charges.

Chapter 5 contains a further generalization of the gauged supergravity Anti-de Sitter black hole configurations. We will find solutions with nonvanishing angular momentum, electric charges and NUT charges, for a total of 7 independent parameters. These solutions are found for the simplest prepotential. The characterization of these solutions represents a further step towards the identification of the most general black hole solution in this theory.

The thermodynamics of the static magnetic solutions will be developed in Chapter 6. In this part we will compute the thermodynamic quantities of the configurations and we will find that the first Law and the Smarr relations are satisfied. We will then show that a phase transition between small and large AdS black holes arises in the canonical ensemble, for a suitable value of magnetic charge. Finally we give an interpretation of this phase transition in the dual field theory, via AdS/CFT. We will discuss some preliminary results of the study of the thermodynamics for the recently found rotating configurations described in Chapter 5.

Chapter 7 is devoted to the analysis of the superalgebra structure of the magnetic configurations. It turns out that electric and magnetic configurations satisfy two different BPS bounds, arising from the corresponding two different superalgebras. We will treat explicitly the simple case of minimal gauged supergravity, and we will provide an expression for the BPS bound saturated by the electric and magnetic BPS solutions. We elaborate further on the quantities appearing in the superalgebra, also in the case of nonminimal couplings.

Finally, in Chapter 8 we illustrate the computation of the mass of an AdS black hole configuration by means of holographic renormalization techniques. We will also present a first order flow for nonextremal configurations coming from the squaring of the action tailored appositely on the nonextremal AdS black holes.

Chapter 1

$\mathcal{N} = 2$ gauged supergravity

This chapter is devoted to review the main features of (gauged) $\mathcal{N} = 2$ supergravity in four dimensions. We expose the details that will be relevant for the development of the subsequent chapters. However, the review is far from complete. Original references on gauged $\mathcal{N} = 2$ supergravity coupled to matter consist of [31–33], while a standard review reference of the topic is for instance [34].

Here we first recap the features of ungauged supergravity coupled to vector and hypermultiplets. We then describe the procedure of gauging, which allows in particular for a scalar potential and therefore for solutions with more general asymptotics, such as Anti-de Sitter black holes.

After describing the procedure of gauging, we focus on the case of minimal gauged supergravity (no vector multiplets, just gravity supermultiplets) and abelian Fayet-Iliopoulos gauged supergravity with vector multiplets (and no hypermultiplets). These are the two models of interest for the computations presented in the thesis. We will always set the hypermultiplets to zero, so we will be brief in treating the details about quaternionic geometry (a thorough description can be found, for instance, in [34]).

Let us finally mention that for some specific choice of prepotential the gauged supergravity models that we treat have uplift in string theory. We will mention it later on when we will deal with specific solutions to those models.

Throughout this thesis we will adopt Planck units, namely $8\pi G_N = \hbar = c = k_B = 1$. Spacetime indices are denoted with greek lowercase letters that run from 0 to 3, namely $\mu, \nu, \rho .. = 0, 1, 2, 3$. Further conventions and notations can be found in Appendix A.

For those who are not familiar with the concept of supersymmetry and supergravity, let us mention some introductory material reviewing the basic concepts of supersymmetry [35, 36], along with the standard and/or comprehensive textbook references on the subject [37, 38].

1.1 Supergravity field content

In this chapter we review the field content of $\mathcal{N} = 2$ four dimensional supergravity. The multiplet containing the graviton field is called the gravity multiplet and for $\mathcal{N} = 2$ theories it consists of the graviton $g_{\mu\nu}$, a doublet of gravitini $\psi_{\mu A}$ $(A = 1, 2)$ with positive chirality and a gauge field A^0_μ called graviphoton [39, 40]. The R-symmetry group of the theory transforms the supersymmetry charges into each other, and acts on the index A. In $\mathcal{N} = 2$ theories in four dimensions the R-symmetry group is $SU(2) \times U(1)$.

Spinors of positive and negative chirality obey respectively

$$\gamma^5\psi = \psi\,, \qquad \gamma^5\psi = -\psi\,. \tag{1.1}$$

Negative chirality fermions are given by $\psi^A_\mu \equiv (\psi_{\mu A})*$. ψ^A and ψ_A are the chiral components of Majorana spinors.

The number of on shell bosonic degrees of freedom (2 for the graviton and 2 for the gauge field) matches the number of on shell fermionic ones (2 for each gravitino).

In supergravity theories, the gravity multiplet can be coupled to matter field, which sit in supersymmetry multiplets. Matter content of our interest are vector multiplets and hypermultiplets. Hypermultiplets are set to zero in all examples that we treat explicitly in this thesis.

1.1.1 Vector multiplets

Lagrangian and supersymmetry variation

We start now reviewing the properties of the vector multiplets. They consist of a complex scalar z, a doublet of chiral fermions λ^A called gaugini and a gauge field A_μ. Also in this case the fermionic degrees of freedom match the bosonic ones.

We couple a number n_V of vector multiplets to the gravity multiplet. The indices are $\Lambda = 0, ...n_v$ and $i = 1, .., n_V$. The bosonic part of the Lagrangian has this form:

$$\mathscr{L}_{vector} = \frac{R}{2} + g_{i\bar{k}}(z,\bar{z})\partial_\mu z^i \partial^\mu \bar{z}^{\bar{k}} + I_{\Lambda\Sigma}(z,\bar{z})F^\Lambda_{\mu\nu}F^{\mu\nu,\Sigma} + \frac{1}{2}R_{\Lambda\Sigma}(z,\bar{z})\epsilon^{\mu\nu\rho\sigma}F^\Sigma_{\mu\nu}F^\Sigma_{\rho\sigma}\,, \tag{1.2}$$

where R is the Ricci scalar, $g_{i\bar{k}}$ is the metric on the target manifold and $R_{\Lambda\Sigma}$ and $I_{\Lambda\Sigma}$, the matrices appearing in the vector kinetic terms[1], are given respectively as the imaginary and real part of a quantity $\mathcal{N}_{\Lambda\Sigma}$ called the period matrix

$$I_{\Lambda\Sigma} = \mathrm{Im}(\mathcal{N}_{\Lambda\Sigma})\,, \qquad R_{\Lambda\Sigma} = \mathrm{Re}(\mathcal{N}_{\Lambda\Sigma})\,. \tag{1.3}$$

$\mathcal{N} = 2$ supersymmetry constrains the quantities $\mathcal{N}_{\Lambda\Sigma}$ and $g_{i\bar{k}}$, in particular they need to be related to each other (more details will follow). The action coming from the full Lagrangian (including fermions) is invariant under supersymmetry variations, which

[1] In our conventions $F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)$.

transform bosons into fermions and vice versa. Since we are interested in bosonic configurations, and set the fermions to zero, the bosonic supersymmetry variations are identically zero. The supersymmetry variations of the fermions are instead given by

$$\begin{aligned}\delta_\varepsilon \psi_{\mu A} &= (\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab})\varepsilon_A + \frac{i}{2}\mathcal{A}_\mu \varepsilon_A + T^-_{\mu\nu}\gamma^\nu \epsilon_{AB}\varepsilon^B\,, &(1.4)\\ \delta_\varepsilon \lambda^{iA} &= i\partial_\mu z^i \gamma^\mu \varepsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B\,, &(1.5)\end{aligned}$$

up to higher order in fermions. The quantities $T^-_{\mu\nu}$, $\mathcal{A}_\mu$ and $G^{-i}_{\mu\nu}$ will be defined in the next subsection.

Here the parameter ε_A is an arbitrary doublet of Majorana spinors, therefore carrying 8 independent parameters. The full action is invariant under arbitrary supersymmetry variations, therefore we say that the theory is $\mathcal{N} = 2$ (eight real supercharges in four dimensions). A solution can break part of the supersymmetry. In that case, if computed on the solution, the equations

$$\begin{aligned}\delta_{\varepsilon_S}\psi_{\mu A} &= 0\,, &(1.6)\\ \delta_{\varepsilon_S}\lambda^{iA} &= 0\,, &(1.7)\end{aligned}$$

are solved by a specific parameter ε_S. The number of independent components of ε_S gives the amount of supersymmetry preserved by the solution. The equations (1.6),(1.7) are called Killing spinor equations, or BPS (or supersymmetry) equations of the solution.

Special Kähler geometry

Supersymmetry requires the scalar manifold $\mathcal{SM}$ to be to be a special Kähler space [32]. The Kähler property implies that the metric on the target space is derived from a real function $\mathcal{K}(z,\overline{z})$ called Kähler potential of the theory:

$$g_{i\overline{k}} = \partial_i \partial_{\overline{k}}\mathcal{K}\,. \tag{1.8}$$

The function $\mathcal{K}$ is not unique, as the metric $g_{i\overline{k}}$ is invariant under so called Kähler transformations:

$$\mathcal{K} \to \mathcal{K}' = \mathcal{K} + f(z) + \overline{f}(\overline{z})\,. \tag{1.9}$$

The special Kähler property requires the existence of a symplectic bundle, with sections X^Λ and F_Λ, which are holomorphic functions of the scalar fields z^i. Inequivalent definitions of Special Kähler geometry exist. We consider the definition n.2 of [41] (other useful references are for instance [42, 43]). The Kähler potential is given in terms of these sections as

$$\mathcal{K} = -ln[i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)]\,. \tag{1.10}$$

We call $V \equiv (X^\Lambda, F_\Lambda)$. We introduce now the notation $\langle , \rangle$ to indicate the symplectic product between two vectors a and b:

$$\langle a,b\rangle \equiv a^T \Omega b\,, \qquad \Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \tag{1.11}$$

so that (1.10) reads:

$$\mathcal{K} = -log[i\langle \overline{V}, V\rangle]\,. \tag{1.12}$$

A further requirement for special Kähler geometry is [41]

$$\langle V, \partial_i V\rangle = 0\,. \tag{1.13}$$

The constraints (1.10) and (1.13) can be equivalently formulated in terms of the covariantly holomorphic sections $\mathcal{V} \equiv (L^\Lambda, M_\Lambda)$ defined in this way:

$$L^\Lambda \equiv e^{\mathcal{K}/2} X^\Lambda\,, \qquad M_\Lambda \equiv e^{\mathcal{K}/2} F_\Lambda\,. \tag{1.14}$$

Moreover, we define

$$U_i \equiv D_i\mathcal{V} \equiv (\partial_i + \frac{1}{2}\partial_i\mathcal{K})\mathcal{V}\,, \tag{1.15}$$

$$\overline{U}_{\bar\imath} \equiv D_{\bar\imath}\overline{\mathcal{V}} \equiv (\partial_{\bar\imath} + \frac{1}{2}\partial_{\bar\imath}\mathcal{K})\overline{\mathcal{V}}\,, \tag{1.16}$$

and we define upper and lower parts as $(f_i^\Lambda, h_{i,\Lambda}) \equiv U_i$. In this case the constraints are [41]:

$$\langle \mathcal{V}, \overline{\mathcal{V}}\rangle = i\,, \tag{1.17}$$

$$D_{\overline{m}}\mathcal{V} = 0\,, \tag{1.18}$$

$$\langle \mathcal{V}, U_i\rangle = 0\,, \tag{1.19}$$

$$\langle U_j, U_k\rangle = 0\,. \tag{1.20}$$

We can furthermore derive the following identities:

$$\langle \overline{\mathcal{V}}, U_i\rangle = 0\,, \tag{1.21}$$

$$\langle U_i, \overline{U}_{\bar k}\rangle = i g_{i\bar k}\,. \tag{1.22}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$, appearing in the kinetic terms or the vectors, is defined by the following properties

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma \qquad \overline{h}_{\Lambda,\bar\imath} = \mathcal{N}_{\Lambda\Sigma}\overline{f}^\Sigma_{\bar\imath} \tag{1.23}$$

It can be shown [41] that the $(n_V+1)\times(n_V+1)$ matrix $(\overline{f}^\Lambda_{\bar\imath}, L^\Lambda)$ is invertible, so that $\mathcal{N}$ can be derived as

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} \overline{h}_{\Lambda,\bar\imath} \\ M_\Lambda \end{pmatrix} \cdot \begin{pmatrix} \overline{f}^\Sigma_{\bar\imath} \\ L^\Sigma \end{pmatrix}^{-1} \tag{1.24}$$

Moreover, $\mathcal{N}_{\Lambda\Sigma}$ is symmetric [41] and its imaginary part $I_{\Lambda\Sigma}$ (1.3) is invertible with inverse $I^{\Lambda\Sigma}$ and negative definite, providing a positive sign for the kinetic terms of the vectors (see Lagrangian (1.2)).

From the previous relations one can derive a set of useful identities [43], among which we quote

$$\begin{aligned} L^{\Lambda} I_{\Lambda\Sigma} \overline{L}^{\Sigma} &= -\frac{1}{2}, &(1.25)\\ f_{i}^{\Lambda} I_{\Lambda\Sigma} \overline{f}_{\overline{m}}^{\Sigma} &= -\frac{1}{2} g_{i\overline{m}}, &(1.26)\\ f_{i}^{\Lambda} g^{i\overline{m}} \overline{f}_{\overline{m}}^{\Sigma} &= -\frac{1}{2} I^{\Lambda\Sigma} - \overline{L}^{\Lambda} L^{\Sigma}. &(1.27) \end{aligned}$$

The Kähler $U(1)$ connection is defined as

$$\mathcal{A}_{\mu} = -\frac{i}{2}\left(\partial_{i}\mathcal{K}\partial_{\mu}z^{i} - \partial_{\overline{\imath}}\mathcal{K}\partial_{\mu}z^{\overline{\imath}}\right). \tag{1.28}$$

and it appears in the supercovariant supersymmetry variation (1.4). Lastly, let us define the quantities $T^{-}_{\mu\nu}$ and $G^{i-}_{\mu\nu}$ appearing in (1.4):

$$T^{-}_{\mu\nu} = 2iL^{\Lambda} I_{\Lambda\Sigma} F^{\Sigma-}_{\mu\nu}, \qquad G^{i-}_{\mu\nu} = -g^{i\overline{n}} \overline{f}^{\Lambda}_{\overline{n}} I_{\Lambda\Sigma} F^{\Sigma-}_{\mu\nu}, \tag{1.29}$$

where with the superscript $^{-}$ we denoted the anti-self-dual field strength. The self-dual (+) and anti-self-dual (−) parts of a 2-tensor $O_{\mu\nu}$ are defined as

$$O^{\pm}_{\mu\nu} = \frac{1}{2}\left(O_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}O^{\rho\sigma}\right). \tag{1.30}$$

and are related by complex conjugacy.

Remark

In certain cases, that include the explicit examples treated in this thesis, the holomorphic sections X^{Λ} and F_{Λ} can be specified in terms of a single holomorphic function $F(X^{\Lambda})$, called the prepotential of the model taken into consideration. In applications to supersymmetry, F is given as $F(X) = \frac{1}{2}X^{\Lambda}F_{\Lambda}$, and is a homogeneous function of second degree. We then have $F_{\Lambda} = \partial_{X^{\Lambda}}F$ and $z^{i} = X^{i}/X^{0}$.

1.1.2 Duality

Before explaining the procedure of gauging, let us mention that the equations of motion derived by the Lagrangian (1.2) are invariant under the so-called electromagnetic (U) duality group of the theory [44].

Let's consider the Lagrangian (1.2). The Bianchi identities and the Maxwell's equation read:

$$D^{\mu}(\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma,\Lambda})=0 \qquad D^{\mu}(\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}_{\Lambda})=0 \tag{1.31}$$

where

$$G_{\mu\nu,\Lambda}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{\delta\mathscr{L}}{\delta F^{\Lambda}_{\rho\sigma}}\,. \tag{1.32}$$

We are interested in transformations that leave the Bianchi, Maxwell's and the scalar equation of motion invariant. It turns out [44] that such transformations act on the scalars as diffeomorphisms γ of the scalar manifold. Due to supersymmetry, these induce a transformation on the field strengths of the form

$$\begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix}'=\mathcal{S}_{\gamma}\begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix}, \tag{1.33}$$

where the matrix S_{γ}

$$\mathcal{S}_{\gamma}=\begin{pmatrix} A_{\gamma} & B_{\gamma} \\ C_{\gamma} & D_{\gamma} \end{pmatrix} \tag{1.34}$$

is symplectic: $\mathcal{S}_{\gamma}^{T}\Omega\mathcal{S}_{\gamma}=\Omega$. The sections transform as

$$\begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix}'=\mathcal{S}_{\gamma}\begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix}+c_{\gamma}(z)\begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix}, \tag{1.35}$$

where $c_{\gamma}(z)$ corresponds to the Kähler transformation (1.9) induced by γ. The period matrix $\mathcal{N}$ also transforms [43]:

$$\mathcal{N}'_{\Lambda\Sigma}=[(C_{\gamma}+D_{\gamma}\mathcal{N})(A_{\gamma}+B_{\gamma}\mathcal{N})^{-1}]_{\Lambda\Sigma}\,. \tag{1.36}$$

Configurations related by a transformation S_{γ} are solutions to the same equations of motion. The duality transformations described above are a symmetry of the equations of motion but generically they are not a symmetry of the Lagrangian (1.2).

We stress that the duality group is a subgroup of the full symplectic group. Within the symplectic group there can be transformations (1.35) that change the theory in consideration, and in particular they can change the prepotential[2]. We call the latter simply "symplectic transformations". An example of symplectic transformation is the one that relates these two different prepotentials (see for instance [45])

$$F_{stu}=\frac{X^1X^2X^3}{X^0} \qquad \leftrightarrow \qquad F_{\text{squareroot}}=-2i\sqrt{X^0X^1X^2X^3}\,. \tag{1.37}$$

[2]Sometimes they transform the sections in such a way that a prepotential does not even exist. This is not a problem since in principle a description in terms of prepotential is not needed.

Applying one of these transformations to a solution of the theory with prepotential F_{stu} generates a solution in the theory $F_{\text{squareroot}}$ and vice versa.

For future reference, we mention that the transformation in Appendix B is a symplectic one. An example of duality transformation is instead given in Appendix D.

1.1.3 Hypermultiplets

Bosonic lagrangian

A hypermultiplet of four-dimensional $\mathcal{N}=2$ supergravity consists of two chiral spinors ζ_α (hyperini) and four real scalars denoted by q^u. With n_H hypermultiplets, we have $4n_H$ scalar fields, q^u with $u=1,\ldots,4n_H$ and $2n_H$ fermions: ζ_α with $\alpha=1,\ldots,2n_H$.

The bosonic Lagrangian for the hypermultiplet sector is:

$$\mathscr{L}_{hyper} = h_{uv}\partial_\mu h^u \partial^\mu h^v \,. \tag{1.38}$$

This part is added to the Lagrangian (1.2) in order to have the full ungauged $\mathcal{N}=2$ theory coupled to vector multiplets and hypermultiplets. In this theory the hypermultiplets do not directly couple to vector multiplet fields and they minimally couple to the graviton multiplet. The supersymmetry variation of the fermions has this form:

$$\delta_\varepsilon \zeta_\alpha = i\mathcal{U}_u^{B\beta}\partial_\mu q^u \gamma^\mu \epsilon_{AB}\mathbb{C}_{\alpha\beta}\varepsilon^A \,. \tag{1.39}$$

We will shortly introduce the quantities appearing in this equation.

Quaternionic geometry

Supersymmetry in this case requires the hyperscalars manifold $\mathcal{HM}$ to be a quaternionic space [46]. A quaternionic manifold is a $4n_H$ dimensional real manifold endowed with a metric h_{uv}, and a quaternionic structure that we denote by J^x, with $x=1,2,3$. J^x satisfy the quaternionic algebra:

$$J^x J^y = -\delta^{xy} + \epsilon^{xyz} J^z \,, \tag{1.40}$$

and the metric is hermitean with respect to each J^x. We can introduce a triplet of quaternionic two-forms $K^x_{uv} = h_{uw}(J^x)^w{}_v$. They are covariantly constant with respect to the $SU(2)$ bundle connection ω^x:

$$DK^x \equiv dK^x - \epsilon^{xyz}\omega^y K^z = 0 \,. \tag{1.41}$$

The $SU(2)$ connection ω^x defines the $SU(2)$ curvature $\Omega^x \equiv d\omega^x - \frac{1}{2}\epsilon^{xyz}\omega^y\omega^z$, and in the quaternionic case we have the relation [34]:

$$\Omega^x = \lambda K^x \,, \tag{1.42}$$

with λ a nonvanishing real number (if $\lambda=0$ we have instead a manifold of Hyper-Kähler type, that features in the rigid $\mathcal{N}=2$ hypermultiplet case). Local supersymmetry requires $\lambda=-1$. The Ricci scalar curvature of the quaternionic manifold is then constant and given by $R=-8n_H(n_H+2)$, therefore it is always negative [46].

We can furthermore decompose the metric h_{uv} in terms of the quaternionic vielbein $\mathcal{U}_u^{A\alpha}$, whose indices are $A,B=1,2,3$ and $\alpha,\beta=1,\dots,2n_H$:

$$h_{uv}=\mathcal{U}_u^{A\alpha}\mathcal{U}_v^{A\alpha}\mathbb{C}_{\alpha\beta}\epsilon_{AB}\,, \tag{1.43}$$

where $\mathbb{C}_{\alpha\beta}=-\mathbb{C}_{\beta\alpha}$ is the symplectic tensor (previously denoted by Ω) and $\epsilon_{AB}=-\epsilon_{BA}$ is the $SU(2)$ tensor. These are the quantities appearing in the supersymmetry variation (1.39). The vielbein with lower indices is defined as $\mathcal{U}_{A\alpha}\equiv\epsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta}$.

1.2 Gauging procedure

We have seen so far that the $\mathcal{N}=2$ supergravity vector and hypermultiplet sectors contain both a nonlinear sigma model. Holomorphic isometries of the scalar manifold can be lifted to symplectic transformations on the electromagnetic field strengths, as in 1.1.2. In particular, the isometries whose image $\mathcal{S}_\gamma$ in the symplectic group (1.34) is block-diagonal, $B=C=0$, are global symmetries of Lagrangian [44].

It is then possible to make some of these symmetries local while preserving supersymmetry, hence having a so-called theory of gauged supergravity. In particular, for the gauging we identify the gauge group G as a subgroup of the isometry group of the product space $\mathcal{SM}\times\mathcal{HM}$.

If G is nonabelian, supersymmetry requires it to be a subgroup of the isometries of $\mathcal{SM}$, since the sections must belong to the adjoint representation. In the abelian case the special manifold is not required to have any isometries and, if the hypermultiplets are charged with respect to n_V+1 $U(1)$s, then the quaternionic manifold should have (at least) n_V+1 isometries.

The gauging has the effect of replacing ordinary derivatives with covariant ones. Furthermore the ungauged Lagrangian is supplemented by other terms, namely a scalar potential V_g and fermion-fermion bilinears with scalar field dependent coefficients.

The full resulting bosonic Lagrangian of gauged $\mathcal{N}=2$ supergravity will be given in section 1.2.4. Let us here review the procedure of the gauging [31–34, 47], first considering the isometries of the vector multiplets scalar manifold and then those of the hypermultiplet sector (let us also mention [48], [49] as [50] as additional references for this part).

Remark

As said before, with reference to formula (1.34), the isometries with $B=C=0$ are exact symmetries of the Lagrangian.

We can relax this constraint and allow for $C \neq 0$. Under these more general transformations, the Lagrangian is invariant up to a total derivative, which does not affect the equations of motion. We can gauge this class of isometries, too. We define as "electric gauging" the procedure for which we gauge generic isometries with $B = 0$. In case of nonzero C, the procedure involves the introduction of additional c-dependent terms in the Lagrangian [31] (see also [49]).

Gauging isometries with $B \neq 0$ involves instead so-called magnetic gaugings. Some details about the magnetic gauging procedure, and an example of theory with magnetic gaugings is given in Appendix B.

Apart from that, throughout this thesis we restrict ourselves to the case for which $B = C = 0$.

1.2.1 Vector multiplets

Let us consider isometries on the scalar manifold $g_{i\bar{k}}$, acting on the scalars in the following way:

$$\delta_G z^i = \alpha^\Lambda k^i_\Lambda \,, \tag{1.44}$$

where α^Λ are the constant parameters of the transformation, and k^Λ_i are the Killing vectors. Isometries should respect the complex structure of the manifold, therefore we require that k^Λ_i should be holomorphic.

These transformations leave the metric invariant. We recall here that for a Special Kähler manifold the metric can be derived from the Kähler potential $g_{m\bar{n}} = \partial_m \partial_{\bar{n}} \mathcal{K}$. Therefore the isometries should leave the Kähler potential invariant up to Kähler transformations (1.9):

$$\delta_G \mathcal{K} = \mathcal{L}_\Lambda \mathcal{K} = k^i_\Lambda \partial_i \mathcal{K} + k^{\bar{\imath}}_\Lambda \partial_{\bar{\imath}} \mathcal{K} = \alpha^\Lambda (r_\Lambda(z) + \bar{r}_\Lambda(\bar{z})) \,. \tag{1.45}$$

The holomorphic Killing vector fields must span a Lie algebra with commutation relations

$$[k_\Lambda, k_\Sigma] = f^\Gamma_{\Lambda\Sigma} k_\Gamma \,, \tag{1.46}$$

where the structure constants $f^\Gamma_{\Lambda\Sigma}$ are those of the Lie group we want to gauge.

The fact that the isometry group should be embedded into the symplectic group requires the holomorphic sections V to transform as [34]:

$$\delta_G V = \alpha^\Lambda (T_\Lambda V + r_\Lambda(z) V) \,. \tag{1.47}$$

where T_Σ is an infinitesimal symplectic transformation, and $r_\Lambda(z)$ induces the Kähler transformation (1.45) on the Kähler potential.

In particular, the matrix $T \in Sp(2n+2, \mathbb{R})$ is of the form:

$$T_\Lambda = \begin{pmatrix} \alpha_\Lambda & 0 \\ 0 & \alpha^T_\Lambda \end{pmatrix} \,, \tag{1.48}$$

where $a_\Lambda = -(f_\Lambda)^\Gamma_\Sigma$.

Furthermore, the closure of the gauge transformations on the Kähler potential, namely (1.45), implies[50]

$$k_\Lambda^i \partial_i r_\Gamma - k_\Gamma^i \partial_i r_\Lambda = f_{\Lambda\Gamma}^{\Sigma} r_\Sigma \,. \tag{1.49}$$

As a consequence of the gauging, the Lagrangian of the ungauged theory gets modified by the replacement of ordinary derivatives with covariant derivatives:

$$\partial_\mu z^i \to \nabla_\mu z^i = \partial_\mu z^i + g k_\Lambda^i A_\mu^\Lambda \,. \tag{1.50}$$

The gauge field transforms as $\delta_G A_\mu^\Lambda = -D_\mu \alpha^\Lambda / g$. Notice that the additional terms in the covariant derivative introduce couplings with the electric vector field, but not for the magnetic counterparts (the magnetic vector fields defined from (1.32)), hence the name "electric gauging". The full nonabelian field strengths

$$F_{\mu\nu}^\Lambda = \frac{1}{2}(\partial_\mu A_\nu^\Lambda - \partial_\nu A_\mu^\Lambda) + \frac{g}{2} f_{\Sigma\Gamma}^{\Lambda} A_\mu^\Sigma A_\nu^\Gamma \tag{1.51}$$

appear in the Lagrangian. In order for supersymmetry to be preserved, the introduction of fermion-fermion bilinears with scalar dependent coefficients and of a scalar potential is necessary. The full Lagrangian and the supersymmetry variations are given in section 1.2.4.

1.2.2 Hypermultiplets

Similar to the vector multiplet case, also isometries of the hypermultiplet scalar manifold can be gauged. We start by assuming that the metric admits Killing vectors k_Λ^u satisfying

$$[\widetilde{k}_\Lambda, \widetilde{k}_\Sigma] = f_{\Lambda\Sigma}^{\Gamma} \widetilde{k}_\Gamma \,, \tag{1.52}$$

with the same structure constants as in equation (1.46). The metric and the ungauged sigma model are invariant under the global transformations

$$\delta_G q^v = \widetilde{k}_\Lambda^v \alpha^\Lambda \,. \tag{1.53}$$

In order to make this global invariance local, we replace as usual the standard derivatives with covariant ones:

$$D_\mu q^u \equiv \partial_\mu q^u + g A_\mu^\Lambda k_\Lambda^u \,. \tag{1.54}$$

Further details about the relation between these Killing vectors and the Hyper Kähler structure can be found for instance in [34]. Gaugings these isometries introduce additional fermionic terms and a scalar potential that couples scalars in the vector multiplets with the hyperscalars.

1.2.3 Moment maps

For the vector multiplets, the metric on the Kähler manifold is given by the derivative of the Kähler potential. In the same way, also the holomorphic Killing vectors can be obtained by derivatives of suitable prepotentials.

The holomorphic Killing vectors preserve the metric on the scalar manifold, as well as the complex structure of the manifold. This can be shown to imply the existence of real, scalar functions $\mathcal{P}_\Lambda$ called *moment maps*.

They satisfy

$$ik^i_\Lambda g_{i\bar{l}} = \partial_{\bar{l}}\mathcal{P}_\Lambda\,, \qquad -ik^{\bar{l}}_\Lambda g_{i\bar{l}} = \partial_i\mathcal{P}_\Lambda\,. \tag{1.55}$$

Recalling eq. (1.8) and the fact that k_Λ are holomorphic, we have

$$ik^i_\Lambda g_{i\bar{l}} = \partial_{\bar{l}}(ik^m_\Lambda\partial_m\mathcal{K})\,,$$

therefore

$$\mathcal{P}_\Lambda = ik^m_\Lambda\partial_m\mathcal{K} - ir_\Lambda(z) = \tag{1.56}$$

$$= -i\overline{k}^{\overline{n}}_\Lambda\partial_{\overline{n}}\mathcal{K} + i\overline{r}_\Lambda(\overline{z})\,, \tag{1.57}$$

where r_Λ was defined in (1.45). One can then prove [34] the relation

$$k^i_\Lambda g_{i\bar{l}}k^{\bar{l}}_\Sigma - k^i_\Sigma g_{i\bar{l}}k^{\bar{l}}_\Lambda = if^\Gamma_{\Lambda\Sigma}P_\Gamma\,, \tag{1.58}$$

called equivariance condition.

These moment maps appear in the U(1) Kähler connection (1.28), that due to the gauging gets modified and gives

$$\mathcal{A}_\mu = -\frac{i}{2}\left(\partial_i\mathcal{K}\partial_\mu z^i - \partial_{\bar{\imath}}\mathcal{K}\partial_\mu z^{\bar{\imath}}\right) + gA^\Lambda_\mu P_\Lambda\,. \tag{1.59}$$

Even though the quaternionic spaces are not complex, we can still define quaternionic moment maps [51], in analogy with what we did for the Special Kähler manifolds [47]. Indeed to each Killing vector of the quaternionic manifold we can associate a triplet $\mathcal{P}^x_\Lambda(q)$ of 0-form prepotentials. The corresponding property (in analogy with (1.55)) in this case is

$$K^x_{uv}k^v_\Lambda = D_u\mathcal{P}^x_\Lambda \quad \text{where} \quad D_u\mathcal{P}^x_\Lambda \equiv \partial_u\mathcal{P}^x_\Lambda + \epsilon^{xyz}\omega^x\mathcal{P}^z_\Lambda\,, \tag{1.60}$$

with equivariance condition:

$$K^x_{uv}k^u_\Lambda k^v_\Sigma + \frac{1}{2}\epsilon^{xyz}\mathcal{P}^y_\Lambda\mathcal{P}^z_\Sigma = \frac{1}{2}f^\Delta_{\Lambda\Sigma}\mathcal{P}^x_\Delta\,. \tag{1.61}$$

The presence of the gauging modifies the $SU(2)$ connection in such a way that

$${\omega_{\mu}}_A{}^B = \partial_\mu q^u {\omega_u}_A{}^B + \frac{i}{2}gA^\Lambda{\sigma^x}_A{}^B\mathcal{P}^x_\Lambda\,. \tag{1.62}$$

Such modified connection appears, for instance, in the fermion supersymmetry variations.

In a theory with only abelian vectors, $n_H=0$, the moment maps can still be nonzero constants:

$$\mathcal{P}^x_\Lambda=\xi^x_\Lambda(\text{constant}), \qquad \epsilon^{xyz}\xi^y_\Lambda\xi^z_\Sigma=0\,. \tag{1.63}$$

The last condition is given by eq. (1.61) with $f^\Delta_{\Lambda\Sigma}=0$ (abelian gauge group). The constants ξ^x_Λ are called *Fayet-Iliopoulos parameters*. In this case one gauges a $U(1)$ subgroup of the $SU(2)$ factor in the R-symmetry (more details will be provided later in sections 1.3 and 1.4). One can also gauge the full $SU(2)$, resulting in Fayet-Iliopoulos nonabelian gaugings [52, 53].

1.2.4 Full bosonic Lagrangian - gauged case

At the end of the day, we can write down the full bosonic action of the on-shell Lagrangian of four-dimensional gauged $\mathcal{N}=2$ supergravity, in presence of n_V vector multiplets and n_H hypermultiplets. It has this form:

$$\begin{aligned}\mathscr{L}=\frac{R}{2}+g_{i\bar{k}}\nabla_\mu z^i\nabla^\mu\bar{z}^{\bar{k}}+h_{uv}\nabla_\mu q^u\nabla^\mu q^v+I_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\mu\nu,\Sigma}+\\+\frac{1}{2}R_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\mu\nu}-V_g(z,\bar{z},q)\,.\end{aligned} \tag{1.64}$$

The scalar potential couples the scalars of the vector multiplets and those of the hypermultiplets, and has this form:

$$V(z,\bar{z},q)=g^2\Big[(g_{i\bar{k}}k^i_\Lambda k^{\bar{k}}_\Sigma+4h_{uv}k^u_\Lambda k^v_\Sigma)\bar{L}^\Lambda L^\Sigma+(g_{i\bar{k}}f^\Lambda_i\bar{f}^\Sigma_{\bar{k}}-3\bar{L}^\Lambda L^\Sigma)P^x_\Lambda P^x_\Sigma\Big]\,. \tag{1.65}$$

The full Lagrangian is supersymmetric and the fermionic variations read:

$$\begin{aligned}\delta_\varepsilon\psi_{\mu A} &= \widetilde{\mathcal{D}}\varepsilon_A\equiv(\partial_\mu-\frac{1}{4}\omega^{ab}_\mu\gamma_{ab})\varepsilon_A+\frac{i}{2}\mathcal{A}_\mu\varepsilon_A & (1.66)\\ &+ \omega_{\mu A}{}^B\varepsilon_B+T^-_{\mu\nu}\gamma^\nu\epsilon_{AB}\varepsilon^B+igS_{AB}\gamma^\mu\varepsilon^B\,, & (1.67)\\ \delta_\varepsilon\lambda^{iA} &= i\nabla_\mu z^i\gamma^\mu\varepsilon^A+G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B+gW^{iAB}\varepsilon_B\,, & (1.68)\\ \delta_\varepsilon\zeta_\alpha &= i\mathcal{U}^{B\beta}_u\nabla_\mu q^u\gamma^\mu\epsilon_{AB}\mathbb{C}_{\alpha\beta}\varepsilon^A+gN^A_\alpha\varepsilon_A\,, & (1.69)\end{aligned}$$

up to terms of higher order in the fermionic fields. The matrices S_{AB}, W^{iAB} and N^A_σ are called respectively gravitino, gaugino and hyperino mass matrices, and are given by the following expressions:

$$\begin{aligned}S_{AB} &= \frac{i}{2}P^x_\Lambda L^\Lambda\sigma^x_{AB}\,, & (1.70)\\ W^{iAB} &= k^i_\Lambda\bar{L}^\Lambda\epsilon^{AB}+ig^{i\bar{k}}\bar{f}^\Lambda_{\bar{k}}P^x_\Lambda\sigma^{AB}_x\,, & (1.71)\\ N^A_\alpha &= 2\mathcal{U}^A_{\alpha u}\tilde{k}^u_\Lambda\bar{L}^\Lambda\,. & (1.72)\end{aligned}$$

We can rewrite the scalar potential in terms of the matrices just introduced, as

$$V = -6 S^{AB} S_{AB} + \frac{1}{2} g_{i\bar{k}} W^{iAB} W^{\bar{k}}_{AB} + N^{A}_{\alpha} N^{\alpha}_{A} \,. \tag{1.73}$$

Further details about notation and conventions can be found in Appendix A.

Let us comment here on the fact that the gauging breaks the electromagnetic duality that a generic Einstein-Maxwell theory in four dimensions possesses. Indeed, due to the presence of a scalar potential, the gauged theory does not necessarily admit the same duality group of the ungauged theory. The symmetries of the equations of motion depend on the specific form of the potential and one needs to analyze them case by case.

1.3 Comparison of ungauged and gauged minimal supergravity

In this section we deal with minimal supergravity. Minimal supergravity is characterized by the gravity supermultiplet, in absence of matter supermultiplets. In this framework we show some basic differences between the gauged and the ungauged version of $\mathcal{N} = 2$ supergravity.

The $\mathcal{N} = 2$ supergravity multiplet in four dimensions consist of the graviton, two gravitini and the graviphoton. In the ungauged case, the bosonic Lagrangian obtained by setting the two gravitini to zero is

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \,, \tag{1.74}$$

This corresponds to simple Einstein-Maxwell theory. The gravitino variation, whose vanishing is sufficient in order to have bosonic BPS configurations, consists in:

$$\delta_{\varepsilon} \psi_{\mu A} = \left(\partial_{\mu} - \frac{1}{4} \omega^{ab}_{\mu} \gamma_{ab} \right) \varepsilon_A + \frac{1}{4} F_{\mu\nu} \gamma^{\nu} \epsilon_{AB} \varepsilon^{B} \,. \tag{1.75}$$

The theory possess a global symmetry group named R-symmetry that acts non-trivially on the two supercharges and consequently rotates the gravitino doublet $(\psi_{\mu 1}, \psi_{\mu 2})$. In the case of 4d $\mathcal{N} = 2$ theories the R-symmetry group is $U(1) \times SU(2)$. Furthermore, the theory is endowed with $U(1)$ gauge symmetry.

What happens in the gauged theory is that a subgroup of the global R-symmetry is gauged[3]. The gauged version of minimal supergravity is achieved by promoting a $U(1)$ subgroup of $SU(2)$ factor of the R-symmetry group to a local symmetry, i.e. gauging a $U(1)$ factor. In this way the gravitini become charged under the $U(1)$

[3] In particular, whenever $\mathcal{P}_{\Lambda} \neq 0$ the gravitino is charged under the $U(1)$ factor and whenever $P^{x}_{\Lambda} \neq 0$ the gravitino is charged with respect to the $SU(2)$ factor of the $SU(2) \times U(1)$ R-symmetry group.

group, and the charge is the gauge coupling constant g. The bosonic action gets modified accordingly, and supersymmetry requires the presence of an additional term, proportional to g^2:

$$\mathcal{L} = \frac{R}{2} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + 3g^2 \,, \tag{1.76}$$

This corresponds to the cosmological Einstein-Maxwell theory with cosmological constant $\Lambda = -3g^2$. The full action is invariant under supersymmetry variations of the fields. In particular the gravitini ones are modified due to the gauging and assume the following form:

$$\delta_\varepsilon \psi_{\mu A} = \left(\partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \varepsilon_A + \frac{1}{4} F_{\mu\nu} \gamma^\nu \epsilon_{AB} \varepsilon^B + \frac{i}{2} g A_\mu \sigma^a{}_A{}^B \varepsilon_B - \frac{1}{2} g \sigma^a_{AB} \varepsilon^B \,, \tag{1.77}$$

where $a = 1, 2$, or 3 corresponds the three different ways to embed the $U(1)$ subgroup in the $SU(2)$ factor of the R-symmetry group.

One observation is in order here: the fixed negative value $\Lambda = -3g^2$ excludes de Sitter vacua as solutions of minimal supergravity models. Solution of the equations of motion with nonvanishing gauge coupling constant will instead be asymptotically Anti-de Sitter, in contrast to solutions of the ungauged theory (1.74), that will be asymptotically flat.

Including additional matter allows for more general solutions. In particular, also de Sitter vacua are allowed but they always break supersymmetry.

1.4 Abelian Fayet-Iliopoulos gauged supergravity in absence of hypermultiplets

As mentioned before, this model is characterized by constant quaternionic moment maps $\mathcal{P}^x_\Lambda = const = \xi^x_\Lambda$ called Fayet-Iliopoulos (FI) terms.

In the case of $U(1)^p$ gauge fields with non vanishing FI terms $\xi^x_p = (0, \xi_p, 0)$ the gauge field $A_\mu \equiv A^\Lambda_\mu \xi_\Lambda$ gauges a $U(1)$ subgroup of the $SU(2)$ R-symmetry group. This linear combination of fields indeed appears in the coupling to the gravitini.

As the gauge group is abelian, the only charged fields are the gravitini and gaugini, meanwhile the vector multiplet scalars are neutral. The bosonic Lagrangian for the $U(1)$ Fayet-Iliopoulos gauged supergravity in absence of hypermultiplets is:

$$\mathscr{L} = \frac{R}{2} + g_{i\bar{l}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{l}} + I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{2} R_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} F^\Sigma_{\rho\sigma} - V_g \,, \tag{1.78}$$

where the indices i, j, l for the scalar fields run from 1 to n_V and the symplectic indices are $\Lambda, \Sigma = 1, \ldots, n_V + 1$. The only difference with respect to the bosonic Lagrangian of the ungauged theory is the scalar potential V_g, generated by the FI terms

$$V_g = g^2 (g^{i\bar{k}} f^\Lambda_i \bar{f}^\Sigma_{\bar{k}} - 3 \bar{L}^\Lambda L^\Sigma) \xi_\Lambda \xi_\Sigma \,. \tag{1.79}$$

The potential be rewritten as

$$V_g = g^{i\bar{J}} D_i\mathcal{L}\overline{D}_{\bar{J}}\overline{\mathcal{L}} - 3|\mathcal{L}|^2 \qquad \text{where} \quad D_i\mathcal{L} \equiv \partial_i\mathcal{L} + \frac{1}{2}\partial_i K\,\mathcal{L}\,, \tag{1.80}$$

in terms of a superpotential

$$\mathcal{L} = L^\Lambda g_\Lambda\,. \tag{1.81}$$

We have defined

$$g_\Lambda \equiv g\xi_\Lambda\,, \tag{1.82}$$

and L^Λ are the covariantly holomorphic introduced in 1.14 (upper part of $\mathcal{V}$).

The supersymmetry variations of gravitini and gaugini are respectively given by

$$\delta_\varepsilon\psi_{\mu A} = \nabla_\mu\varepsilon_A + 2iF^{\Lambda\,-}_{\mu\nu} I_{\Lambda\Sigma}L^\Sigma\gamma^\nu\epsilon_{AB}\varepsilon^B - \frac{g}{2}\mathcal{L}\sigma^2_{AB}\gamma_\mu\varepsilon^B\,, \tag{1.83}$$

$$\delta_\varepsilon\lambda^{iA} = i\partial_\mu z^i\gamma^\mu\varepsilon^A - g^{i\bar{k}}\bar{f}^\Lambda_{\bar{k}} I_{\Lambda\Sigma}F^{\Sigma\,-}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B + i\overline{D}^{i}\overline{\mathcal{L}}\sigma^{2,AB}\varepsilon_B\,, \tag{1.84}$$

up to higher order terms in the fermionic fields. Considering the supersymmetry variations of gravitini and gaugini is sufficient if we search for purely bosonic BPS configurations in this model, and this would be the case in all chapters of this thesis. The supercovariant derivative of the supersymmetry parameter ε reads:

$$\nabla_\mu\varepsilon_A = (\partial_\mu - \frac{1}{4}\omega^{ab}_\mu\gamma_{ab})\varepsilon_A + \frac{1}{4}\left(\mathcal{K}_i\partial_\mu z^i - \mathcal{K}_{\bar{\imath}}\partial_\mu\bar{z}^{\bar{\imath}}\right)\varepsilon_A + \frac{i}{2}g_\Lambda A^\Lambda_\mu{\sigma^2}_A{}^B\varepsilon_B\,, \tag{1.85}$$

and similarly for the gravitini. The fact that only the Pauli matrix σ^2 appears in the supersymmetry variation is due to the fact that we have chosen the quaternionic moment maps to be aligned in the y-direction, namely $P^\Lambda_x = \delta^{x2}\xi_\Lambda$. We can do this without loss of generality since, prior to gauging, we use the $SU(2)$ R-symmetry to rotate them.

We treated here the case of electric gauging. Let us anticipate that we can restore symplectic covariance by allowing also magnetic gaugings, introducing a vector

$$\mathcal{G} = (g^\Lambda, g_\Lambda)\,. \tag{1.86}$$

The potential gets a contribution from these magnetic terms g^Λ and it can be written as (1.80) with $\mathcal{L} = \langle\mathcal{G},\mathcal{L}\rangle$. The case of magnetic gaugings is less understood [49, 54–59]. We will elaborate on the meaning of magnetic gaugings in Appendix B, where also a solution of the FI theory with magnetic gauging is described.

Coming back to our case, the introduction of electric gaugings provides electric charges for the fermions in the theory. Indeed gravitini and gaugini possess electric charge $\pm g_\Lambda = \pm g\xi_\Lambda$. The two gravitini have opposite charge due to the fact that they correspond to two different eigenstates of σ^2, and the same holds for the gaugini. In a theory with multiple nonlocal charges, consistency of the path integral formulation requires the Dirac quantization condition

$$g_\Lambda p^\Lambda = g\xi_\Lambda p^\Lambda = n\,, \qquad n \in \mathbb{Z} \tag{1.87}$$

to be satisfied. It involves the electric charges of the gravitini and gaugini g_Λ and the black hole magnetic charges p^Λ. This is a crucial point for the later chapters.

Chapter 2

Static BPS black hole solutions in AdS_4

In this chapter we focus on static spherically symmetric BPS solutions of $\mathcal{N}=2$ four-dimensional gauged supergravity models that contain AdS_4 vacua. We work in gauged $\mathcal{N}=2$ supergravity in presence of abelian gaugings with Fayet-Iliopoulos terms without hypermultiplets. In this framework fermions are charged under the gauge group. We restrict ourselves to solutions of theories with electric gaugings only, such that gravitini and gaugini have electric charge.

The scalars, if present, are neutral under the gauge group; for this reason this is a tractable model and provides a good starting point for the study of black holes in gauged supergravity. For numerical solutions in presence of other gaugings, see for instance [60]. The discussion of the stationary and rotating configurations is postponed to Chapter 5.

We first describe the minimal, scalarless case. Minimal solutions represent a "prototype" for the genuine BPS hairy solutions of our interest and were first found by Romans [61]. They preserve 1/2 and 1/4 of supersymmetries but they present naked singularities. Solutions with constant scalars present similar features and were studied in [62] [63]: they are also nakedly singular.

It turns out that genuine black hole solutions with finite nonzero area of the event horizon require both radially varying scalar fields and a reduced amount of preserved supersymmetry (the solution must preserve one quarter of supersymmetries, instead of half). Indeed, black holes preserving half of the supersymmetries are singular also in case of nontrivial scalar profiles, as found by Duff and Liu [64]. Genuine 1/4-BPS black hole solutions with radially dependent scalars were eventually found by Cacciatori and Klemm [30] in 2009. These configurations represent the starting point of the analysis of the thesis, hence they are treated in more detail in the last section of this chapter.

Throughout this chapter we deal with solutions with spherical horizon topology. Nevertheless, unlike the case for asymptotically flat static black holes, the topology of the event horizon of asymptotically AdS_4 black holes is not unique. The horizon can be a Riemann surface of any genus, as explained in [65]. Therefore in gauged supergravity BPS configurations with different horizon topology (i.e. black branes, toroidal or hyperbolic black holes) exist. We will comment on this later on.

2.1 Romans' solutions

Romans [61] first analyzed the supersymmetry properties of Anti-de Sitter black holes in minimal $\mathcal{N}=2$ gauged supergravity. This setting corresponds to the presence of the gravity multiplet alone: graviton $g_{\mu\nu}$, two gravitini ψ^1_μ, ψ^2_μ, and graviphoton A_μ.

The configurations found by Romans are solution to the equations of motion of the Lagrangian (1.76)[1]. The supersymmetry variations of the gravitini read

$$\delta_\epsilon \psi_\mu = \widetilde{\mathcal{D}}_\mu \epsilon = \left(\partial_\mu - \frac{1}{4}\omega^{ab}_\mu \gamma_{ab} - \frac{i}{2} g \gamma_\mu + i\frac{g}{2} A_\mu \sigma^2 + \frac{1}{4} F_{\lambda\tau}\gamma^{\lambda\tau}\gamma_\mu \sigma^2 \right) \epsilon \,. \tag{2.1}$$

BPS solutions are obtained requiring the vanishing of the supersymmetry variations computed on the solution, for a suitable spinorial parameter ϵ called Killing spinor of the solution. The various configurations we describe preserve different amount of supersymmetry and therefore have a different Killing spinor. While here we are interested in a recap of the general characteristics of these minimal solutions, we will treat the details of Killing spinors and BPS bounds in Chapter 7.

We take into consideration static, spherically symmetric charged black holes. The nonvanishing components of the vector field (graviphoton) have this form

$$A_t = \frac{Q_e}{r} \,, \qquad A_\phi = -Q_m \cos\theta \,, \tag{2.2}$$

and the field strengths are

$$F_{tr} = \frac{Q_e}{2r^2} \,, \qquad F_{\theta\phi} = \frac{Q_m}{2} \sin\theta \,. \tag{2.3}$$

The most general static and spherically symmetric solution of Einstein's equation with a negative cosmological constant and electromagnetic field is characterized by:

$$\mathrm{d}s^2 = U^2(r)\,\mathrm{d}t^2 - U^{-2}(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \,, \tag{2.4}$$

with

$$U^2(r) = 1 - \frac{2M}{r} + \frac{Q_e^2 + Q_m^2}{4r^2} + g^2 r^2 \,. \tag{2.5}$$

where M is the mass of the black hole, Q_e and Q_m are respectively the electric and magnetic charge of the black hole configuration:

$$\int_{S^2} F \equiv -4\pi Q_m \,, \qquad \int_{S^2} G \equiv -4\pi Q_e \,, \tag{2.6}$$

[1] Notations and conventions for this section are explained in Appendix A, and further details about these minimal supergravity solutions can be also found in Chapter 7. Our conventions in the definition of the charge are slightly different with respect to the ones that are commonly used. In particular, the electromagnetic charges are twice those defined in [61].

where G is defined as in (1.32). The cosmological constant Λ has this value:

$$\Lambda = -3g^2 \,. \tag{2.7}$$

The solution (2.5),(2.2) is asymptotically ($r \to \infty$) Anti-de Sitter, and it represents a genuine black hole for a range of parameters M, Q_e, Q_m, g. This was studied in [61], along with the extremality and supersymmetry properties of these configurations.

It turns out that in the class of (2.5),(2.2), there are two configurations preserving a fraction of supersymmetry [61]. We are going to describe them in the next two subsections.

2.1.1 The electric Reissner–Nordström configuration

The first one is the so-called AdS "electric Reissner–Nordström (RN)" solution, for which the magnetic charge Q_m is set to zero and the mass is equal to half the (absolute value of the) electric charge[2]

$$\text{Electric Reissner–Nordström:} \quad Q_m = 0 \,, \qquad M = |Q_e|/2 \,. \tag{2.8}$$

The Killing spinor ϵ_{AdS} for this solutions is subject to one projection, therefore the solution preserves one half of the supersymmetries (it is 1/2-BPS). The warp factor U has the form:

$$U^2 = \left(1 - \frac{Q_e}{2r}\right)^2 + g^2 r^2 \,. \tag{2.9}$$

It is evident that the function $U^2(r)$ has no real zeros, so there is no horizon shielding the $r = 0$ singularity[3]. Asymptotically ($r \to \infty$) the solution approaches AdS_4 spacetime with cosmological constant $\Lambda = -3g^2$ in standard conventions. The condition $M = |Q_e|/2$ is reminiscent of the BPS bound saturated in ungauged theories by the extremal supersymmetric Reissner–Nordström (asymptotically flat) configuration. Indeed the flat configuration can be simply retrieved by taking the limit $g \to 0$.

2.1.2 The cosmic dyon solution

The second supersymmetric solution is the so-called "cosmic dyon", for which the vanishing of the supersymmetry variation implies having zero mass M but nonzero fixed magnetic charge $Q_m = \pm 1/g$. The electric charge Q_e can instead be arbitrary.

$$\text{Cosmic dyon:} \quad M = 0 \,, \qquad Q_m = \pm\frac{1}{g} \,, \qquad Q_e \quad \text{arbitrary} \,. \tag{2.10}$$

[2]We stress once again that our conventions differ from the ones usually adopted, in which the BPS bound is $M = |Q_e|$. This is due to the fact that our electromagnetic charges are twice the ones defined in other literature, such as [61].

[3]In AdS spacetime, supersymmetry does not seem to provide a cosmic censorship, contrary to most cases in asymptotically flat spacetime [66]. Whether cosmic censorship in AdS_4 can be violated is still an open problem, see e.g. [67].

The warp factor in this case is

$$U^2 = \left(gr + \frac{1}{2gr}\right)^2 + \frac{Q_e^2}{4r^2}\,. \tag{2.11}$$

Again, there is a naked singularity at $r=0$. The Killing spinor for this configuration was constructed in [61] and it is subject to two projections: the solution is therefore 1/4 BPS (it preserves two out of eight initial supercharges). The form of the Killing spinor is very different with respect to the one of the electric solution. This fact is closely related to the superalgebra underlying the two configurations. The BPS bound for the cosmic dyon is (see also Chapter 7)

$$M = 0\,. \tag{2.12}$$

Given the expression for the magnetic charge (2.10) and metric (2.11) we see that the cosmic dyon does not have a flat limit. Indeed this configuration has no counterpart in the ungauged ($g=0$) theory.

We recall that in this theory the gravitini have electric charge g. Therefore for the cosmic dyon solution the Dirac quantization condition (1.87) is satisfied with $n=\pm 1$, namely

$$gQ_m = \pm 1\,. \tag{2.13}$$

It seems then that supersymmetry is compatible with the Dirac quantization, but picks out just two (± 1) of the possible integer values (also $n=0$, if we take into account the the electric configuration).

The two kind of supersymmetric configurations that we have just described represent the prototype of the static, spherically symmetric BPS solutions found so far. Indeed we will see in the next section that we can find the corresponding two classes of black holes also in presence of nontrivial scalars.

It turns out that we can find genuine black hole solutions if we dress Roman's cosmic dyon with scalars [30]. The electric 1/2-BPS configurations instead, despite the presence of nontrivial scalars, they still remain nakedly singular [62, 64].

Before describing the solutions with nontrivial scalar profiles, let us mention that generalizations of these solutions to nonspherical topologies of the event horizon were found in [68] and correspond to genuine static BPS solutions. Furthermore, solutions including NUT charge were found in [69]. Solutions with constant moduli have been constructed in [63] and they are closely related to Roman's ones. They present the same supersymmetry properties; furthermore they are affected by naked singularities too.

2.2 Solutions with nontrivial scalar profile

Before reviewing and describing the various BPS black hole solutions known so far let us introduce notations and general framework.

The configurations with nonconstant scalars are solutions to the equations of motion of the Lagrangian (1.78), that we rewrite here:

$$\mathscr{L} = \frac{R}{2} + g_{i\bar{l}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{l}} + I_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{2} R_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, \epsilon^{\mu\nu\rho\sigma} F^\Sigma_{\rho\sigma} - V_g\,, \tag{2.14}$$

with $\Lambda, \Sigma = 0, 1..., n_V$, $i, j = 1, ..., n_V$. The scalar potential is given by

$$V_g = g^{i\bar{j}}\, D_i \mathcal{L} \overline{D}_{\bar{j}} \overline{\mathcal{L}} - 3|\mathcal{L}|^2\,, \qquad \text{where} \quad D_i \mathcal{L} \equiv \partial_i \mathcal{L} + \frac{1}{2}\partial_i K\,\mathcal{L}\,, \tag{2.15}$$

and

$$\mathcal{L} = e^{K/2}\left(X^\Lambda g_\Lambda\right)\,. \tag{2.16}$$

The supersymmetry variation of gravitini is

$$\delta_\varepsilon \psi_{\mu A} = \nabla_\mu \varepsilon_A + 2i F^{\Lambda\,-}_{\mu\nu} I_{\Lambda\Sigma} L^\Sigma \gamma^\nu \epsilon_{AB} \varepsilon^B - \frac{g}{2} \mathcal{L} \sigma^2_{AB} \gamma_\mu \varepsilon^B\,, \tag{2.17}$$

and that of gaugini is

$$\delta_\varepsilon \lambda^{iA} = i\partial_\mu z^i \gamma^\mu \varepsilon^A - g^{i\bar{k}} \bar{f}^\Lambda_{\bar{k}} I_{\Lambda\Sigma} F^{\Sigma\,-}_{\mu\nu} \gamma^{\mu\nu} \epsilon^{AB}\,.\varepsilon_B + i\overline{D}^i \overline{\mathcal{L}} \sigma^{2,AB} \varepsilon_B\,. \tag{2.18}$$

For a BPS configuration the supersymmetry variations (2.17),(2.18) vanish.

The appropriate metric ansatz that captures static and spherically symmetric black holes solutions, interpolating between asymptotic AdS_4 space and near horizon $AdS_2 \times S^2$ geometry, contains two warp factors and can always be cast in the following form:

$$ds^2 = U^2(r) dt^2 - \frac{dr^2}{U^2(r)} - h^2(r) d\Omega^2\,. \tag{2.19}$$

Furthermore, the scalar fields z^i depend just on the radial coordinate. The reader may have noticed that, with respect to the minimal case, we allow for a more general form of the warp factors $h(r)$ and $U(r)$. Indeed the presence of scalars with radial profile gives a nontrivial contribution to the stress-energy tensor in the Einstein's equations. Therefore we cannot directly integrate Einstein's equations and get (2.4) and (2.5) as in the minimal case, but we will get more general functions (not necessarily polynomials in r).

The Maxwell's equations can be directly solved, and the non-zero components of the field strengths have this form:

$$F^\Lambda_{tr} = \frac{I^{\Lambda\Sigma}}{2h^2(r)} (R_{\Sigma\Gamma} p^\Gamma - q_\Sigma)\,, \qquad F^\Lambda_{\theta\phi} = \frac{p^\Lambda}{2} \sin\theta\,. \tag{2.20}$$

where q_Λ and p^Λ, $\Lambda = 0, ..., n_V$ are the electric and magnetic charges respectively, defined as

$$-\frac{1}{4\pi}\int_{S^2} F^\Lambda \equiv p^\Lambda\,, \qquad -\frac{1}{4\pi}\int_{S^2} G_\Lambda \equiv q_\Lambda\,, \tag{2.21}$$

where G_Λ is defined in (1.32).

In what follows we will review the two branches of BPS solutions: purely electric [64] and magnetic [30]. Just the latter correspond to genuine black hole with nonzero horizon area, the first being nakedly singular.

2.3 Duff-Liu 1/2 BPS electric solution

In [64] the Killing spinor for Anti-de Sitter BPS solutions of $\mathcal{N}=8$ supergravity was found, but the construction is more general and applies to $\mathcal{N}=2,4$ too. It turns out that the Killing spinor has the same form of the one found by Romans for electric configuration in absence of scalars. The solution we describe here therefore will preserve 1/2 of the supersymmetries.

The purely electric BPS solution of [64], also studied in [62], is characterized by the warp factors:

$$U^2(r) = e^{\mathcal{K}}(1+g^2r^2e^{-2\mathcal{K}})\,, \qquad h^2(r) = e^{-\mathcal{K}}r^2\,, \tag{2.22}$$

so that the metric and sections are of this form

$$ds^2 = e^{\mathcal{K}}(1+g^2r^2e^{-2\mathcal{K}})dt^2 - \frac{dr^2}{(1+g^2r^2e^{-2\mathcal{K}})} - e^{-\mathcal{K}}r^2d\Omega_2^2\,. \tag{2.23}$$

where the function $\mathcal{K}$, for all solutions we present in this thesis, coincides with the Kähler potential[4] computed as:

$$e^{\mathcal{K}} = i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)\,. \tag{2.24}$$

The sections are characterized by:

$$\mathrm{Im}X^\Lambda = 0\,, \qquad 2\mathrm{Im}F_\Lambda = H_\Lambda = \xi_\Lambda + \frac{q_\Lambda}{r}\,, \tag{2.25}$$

where ξ_Λ are the FI parameters and q_Λ are the electric charges. This is a purely electric solution $p^\Lambda=0$, therefore from (2.20) the field strengths have the form:

$$F^\Lambda_{tr} = \frac{I^{\Lambda\Sigma}q_\Sigma}{2h^2(r)}\,, \qquad F_{\theta\phi} = 0\,. \tag{2.26}$$

[4]While the metric on the scalar manifold $g_{l\overline{m}} = \partial_l\partial_{\overline{m}}\mathcal{K}$ does not depend on the choice of the sections (Kähler gauge), the Kähler potential $\mathcal{K}$ depends on it via the formula (1.9). Therefore, for a different choice of sections with respect to ours, the function $\mathcal{K}$ appearing in (2.22) and (2.23) does not coincide with the Kähler potential.

2.3.1 Explicit solution for the $F = -2i\sqrt{X^0(X^1)^3}$ model

We illustrate the explicit example of one single vector multiplet, in the model with prepotential $F = -2i\sqrt{X^0(X^1)^3}$ [62, 64], for which a string/M-theory embedding exists. The relevant special Kähler quantities for this model are computed in Appendix A. There are two vector fields and there is one scalar field z defined as

$$z = \frac{X^1}{X^0}\,. \tag{2.27}$$

The black hole has two electric charges q_0, q_1. The holomorphic sections are characterized by

$$X_0 = \frac{1}{6\sqrt{3}}\sqrt{\frac{H_1^3}{H_0}}\,, \qquad X_1 = \frac{1}{2\sqrt{3}}\sqrt{H_0H_1}\,, \tag{2.28}$$

so that

$$e^{-\mathcal{K}} = \frac{2}{3\sqrt{3}}\sqrt{H_0H_1^3}\,. \tag{2.29}$$

The sections are purely real, and so is the real scalar field, given (2.27): by ratio of functions H_Λ

$$z = \frac{X^1}{X^0} = \frac{3H_0}{H_1} = 3\frac{\xi_0 r + q_0}{\xi_1 r + q_1}\,. \tag{2.30}$$

All these solutions correspond to naked singularities. The nonextremal generalization was also discovered in [64], and gives thermal black holes with finite nonzero area of the event horizon. The derivation and the explicit example will be treated in detail in Chapter 4.

We anticipate from Chapter 7 that for these electric configurations the BPS bound $M = |Z|$, where M is the mass of the black hole and Z is the central charge, is saturated, in analogy with Roman's BPS electric solution. The appropriate prescription for the computation of M and Z will be given in due time.

2.4 Cacciatori-Klemm 1/4 BPS magnetic solution

We now consider the supersymmetric AdS solutions found in [30] and subsequently developed further in [45, 70]. Since these solutions represent the first example of genuine static spherically symmetric BPS black hole in the realm of gauged supergravity, we treat them in somewhat more detail. We present here the first order supersymmetry equations, the explicit solution for a particular model and some considerations on the entropy and the near-horizon geometry.

Let us mention that these solutions were recently generalized to dyonic ones [71], by means of duality. Indeed one can perform a duality transformation on the Cacciatori-Klemm solution, and generate new configurations with additional charges.

The BPS bound and the mass of the solutions described in this section deserves a separate discussion. There are some subtleties in determining the BPS bound and conserved charges, due to the presence of magnetic charges. We will elaborate on this later on in Chapter7.

Lastly, let us mention that these solutions can be uplifted in M-theory and the presence of magnetic charge yields topological charges of the 11-dimensional spacetime. Indeed in the uplift there is an explicit mixing between the angular variable φ and the angular coordinates on the 7-dimensional internal space [70].

2.4.1 BPS flow equations

The BPS solution was initially found in [30], solving the equations of [72] that describe all the timelike supersymmetric solutions of $\mathcal{N}=2$, D=4 gauged supergravity coupled to abelian vector multiplets. The explicit Killing spinor for these configurations was found in [45] and [70]. We limit ourelves to giving a short recap of the first order flow derived by the Killing spinor equations, in the conventions of [45]. We work in the FI-gauged theory with purely electric gaugings.

We start with the static and spherically symmetric ansatz for the metric (2.19) and the field strengths (2.20). We take the following projection relations for the Killing spinor of the solution, in the conventions of [45, 73]

$$\gamma^0 \varepsilon_A = \pm i e^{i\alpha} \epsilon_{AB} \varepsilon^B \qquad \gamma^1 \varepsilon_A = e^{i\alpha} \delta_{AB} \varepsilon^B\,, \tag{2.31}$$

where α is an arbitrary phase depending in principle on the radial variable (in all the explicit solutions we describe here the phase α turns out to be constant). The choice of sign in the second condition will lead to two distinguishable Killing spinor solutions, corresponding to the two different signs in (2.13).

Given the two independent projection relations (2.31), the solutions will preserve 1/4 of the supersymmetries, as in the case of the cosmic dyon of [61]. Inserting the Killing spinor ansatz in the supersymmetry variations and imposing the vanishing of the gaugino variation $\delta_\varepsilon \lambda^i = 0$ we come to the equation

$$U \partial_r z^i = e^{i\alpha} g^{i\bar{k}} [\pm \frac{1}{h^2} \overline{D}_{\bar{k}} \overline{\mathcal{Z}} - i \overline{D}_{\bar{k}} \overline{\mathcal{L}}]\,, \tag{2.32}$$

where we introduced the quantity $\mathcal{Z} = p^\Lambda M_\Lambda - q_\Lambda L^\Lambda$. Defining the symplectic vector of the black hole charges $Q = (p^\Lambda, q_\Lambda)$ and recalling the definition (1.14) of the symplectic vector $\mathcal{V}$ we have

$$\mathcal{Z} = \langle \mathcal{V}, Q \rangle\,. \tag{2.33}$$

The relevant equations coming from the vanishing of the gravitino supersymmetry variation $\delta_\varepsilon \psi_\mu = 0$ are

$$\partial_r U = \pm \frac{1}{h^2} \mathrm{Re}(e^{-i\alpha}\mathcal{Z}) + \mathrm{Im}(e^{-i\alpha}\mathcal{L}) \tag{2.34}$$

$$\frac{\partial_r h}{h} + \frac{\partial_r U}{U} = \frac{2}{U} \mathrm{Im}(e^{-i\alpha}\mathcal{L}) \tag{2.35}$$

$$0 = \mathrm{Re}(e^{-i\alpha}\mathcal{L}) \pm \frac{1}{h^2} \mathrm{Im}(e^{-i\alpha}\mathcal{Z}) \tag{2.36}$$

$$g_\Lambda A_t^\Lambda = \pm 2U \,\mathrm{Re}(e^{i\alpha}\mathcal{L}) \,, \tag{2.37}$$

$$g_\Lambda p^\Lambda = \mp 1 \,. \tag{2.38}$$

The $\pm$ sign correspond to the two different choices of sign in the Killing spinor projection relations (2.31). Once again supersymmetry picks out the integers ± 1 in the Dirac quantization condition. Flipping the sign of the black hole charges, or alternatively, of the Fayet-Iliopoulos parameters, turns one branch into the other.

Equation (2.38) is a particular instance of the more general relation found by [72] between curvature of the event horizon, and value of integer n appearing in the Dirac quantization condition (1.87). In case of supersymmetric configurations with more general horizon topologies, we have that

$$g_\Lambda p^\Lambda = \mp \kappa \,. \tag{2.39}$$

with κ the curvature of the event horizon. In other words, for a BPS spherical solution with $g_\Lambda p^\Lambda = -1$ there is a corresponding BPS black brane ($\kappa = 0$) configuration with $g_\Lambda p^\Lambda = 0$ and a corresponding solution with hyperbolic horizon with $g_\Lambda p^\Lambda = +1$. Same story if we flip the sign of the charges, according to (2.38).

Two remarks are in order here. The first one is that condition (2.38) fails to hold in the limit of vanishing gauging $g \to 0$. Like in the case of the cosmic dyon, also the BPS magnetic solution found in this section is solitonic, as it does not have a flat (ungauged) limit.

Secondly, in [45] it was shown that the supersymmetric first order flow (2.32) -(2.34) equations are driven by a superpotential W, which is a function of the flow equations of black holes in ungauged supergravity, where the superpotential is the absolute value of the central charge for supersymmetric configurations.

Despite this formal similarity, the physical picture is different: in ungauged supergravity, the asymptotic value of the scalar fields supporting the black hole configuration is not fixed. For an extremal black hole, their value at the horizon is instead fixed in terms of the electromagnetic charges only, hence the name attractor to the near horizon geometry.

In gauged supergravity supersymmetric solutions are subject to a "double attractor" condition, meaning that supersymmetry[5] fixes the value of the scalars both at

[5] In this precise case we are dealing with first order supersymmetry equations. Nevertheless the attractor mechanism at the horizon is independent of supersymmetry, and relies just on extremality.

asymptotic infinity and at the horizon. At infinity, the scalars extremize the potential V_g, while the value of the scalars at the horizon depends both on the black hole electromagnetic charges, and Fayet-Iliopoulos parameters. Thus, the attractor point cannot be reached from a generic point in moduli space because of the asymptotic constraint.

2.4.2 Solutions

Solutions to the equations (2.32), (2.34)-(2.37) subject to the constraint (2.38), and with $\alpha = -\pi/2$ were found in [30] and [45, 70]. The configurations are purely magnetic $q_\Lambda = 0$ and the sections have the following form

$$\mathrm{Re}(X^\Lambda) = H^\Lambda\,, \qquad \mathrm{Re}(F_\Lambda) = 0\,, \qquad H^\Lambda = a_\Lambda + \frac{b_\Lambda}{r}\,. \tag{2.40}$$

The warp factors are of the form

$$U(r) = e^{\mathcal{K}/2}\left(gr + \frac{c}{2gr}\right)\,, \qquad h(r) = e^{-\mathcal{K}/2} r\,, \tag{2.41}$$

where c is a real constant and $\mathcal{K}$ is the Kähler potential (see again footnote 3 for details):

$$e^{\mathcal{K}} = i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)\,. \tag{2.42}$$

The radius of the event horizon r_h is given by:

$$r_h = \sqrt{-\frac{c}{2g^2}}\,. \tag{2.43}$$

so the warp factor U^2 can also be rewritten in this way:

$$U^2 = e^{\mathcal{K}}\,\frac{g^2(r^2 - r_h^2)^2}{r^2}\,. \tag{2.44}$$

The presence of a horizon depends crucially on the parameter c. In certain cases c can be negative, allowing the presence of a horizon shielding the singularities, that are located at the poles of the function $e^{\mathcal{K}}$. This happens only for certain models. For instance, a prepotential of the form $F = -iX^0X^1$ do not allow for genuine black holes. The cubic prepotential $F = -2i\sqrt{X^0(X^1)^3}$ makes it possible and we are going to describe these BPS black hole solutions in the next section.

2.4.3 Explicit solution for the $F = -2i\sqrt{X^0(X^1)^3}$

Let us now describe an explicit magnetic black hole solution. The sections are purely real and harmonic functions[6]:

$$X_0 = a_0 + \frac{b_0}{r}\,, \qquad X_1 = a_1 + \frac{b_1}{r}\,, \qquad z = \frac{X_1}{X_0}\,, \tag{2.45}$$

whose parameters are constrained by the Killing spinor equations to be:

$$a_0 = \frac{1}{4\xi_0}\,, \qquad b_0 = -\frac{\xi_1 b_1}{\xi_0}\,, \qquad a_1 = \frac{3}{4\xi_1}\,. \tag{2.46}$$

Furthermore, the value of c is given by

$$c = 1 - \frac{32}{3}(g\xi_1 b_1)^2\,, \tag{2.47}$$

and the magnetic charges, already subject to the Dirac condition (2.38), have this expression:

$$p^0 = \mp\frac{2}{g\xi_0}\left(\frac{1}{8} + \frac{8(g\xi_1 b_1)^2}{3}\right)\,, \qquad p^1 = \mp\frac{2}{g\xi_1}\left(\frac{3}{8} - \frac{8(g\xi_1 b_1)^2}{3}\right)\,, \tag{2.48}$$

or, vice versa, b_1 can be written in function of one of the magnetic charges, for instance in function of p^1:

$$b_1 = -\frac{3\sqrt{1 \pm 4g\xi_1 p^1/3}}{8g\xi_1}\,. \tag{2.49}$$

The scalar field then assumes this form

$$z = \frac{\xi_0(4b_1\xi_1 + 3\xi_0 r)}{\xi_1(-4b_1\xi_1 + r)} = \frac{\xi_0(\mp\sqrt{9+12\xi_1 g p^1} + 6gr)}{\xi_1(\pm\sqrt{9+12\xi_1 g p^1} + 2gr)}\,. \tag{2.50}$$

The singularity is not situated at $r = 0$, and this point is not even a horizon. The singularities arise from poles of the function $e^{\mathcal{K}}$, that reads

$$e^{\mathcal{K}} = \frac{1}{8\sqrt{\left(\frac{3}{4\xi_1} + \frac{b_1}{r}\right)^3\left(\frac{1}{4\xi_0} - \frac{b_1\xi_1}{\xi_0 r}\right)}}\,. \tag{2.51}$$

Genuine singularities are then located at $r_{sing} = \pm 4\xi_1 b_1, \mp\frac{4}{3}\xi_1 b_1$. The horizon, instead, is at

$$r_h = \sqrt{\frac{16}{3}(\xi_1 b_1)^2 - \frac{1}{2g^2}}\,. \tag{2.52}$$

[6]Strictly speaking the functions H_Λ are not harmonic, in the sense that they do not satisfy $\nabla^2 H = 0$ with respect to the metric (2.19). Nevertheless we will keep on calling "harmonic" because of their form, that in these coordinates looks the same as that of their counterparts in ungauged supergravity.

The requirement that the horizon must shield the singularity, namely $r_h > r_{sing}$, sets some constraints on the value of $g\xi_1 b_1$. This requirement sets the constraint

$$|g\xi_1 b_1| > 3/8\,, \tag{2.53}$$

with $\xi_1 b_1 < 0$ for solution I (upper sign) and $\xi_1 b_1 > 0$ for solution II (lower sign).

The area of the event horizon, and therefore the entropy, can be readily computed for the magnetic solutions at hand. It depends just on the black hole and gauging charges and it reads

$$\begin{aligned} S &= \frac{A}{4} = \pi h^2(r_h) = \qquad (2.54) \\ &= \sqrt{\frac{3\left(9-3(3+4\xi_1 g p^1)^2+2(3+4\xi_1 g p^1)\sqrt{(3+4\xi_1 g p^1)(3+12\xi_1 g p^1)}\right)}{16\xi_0\xi_1^3 g^4}}\,. \end{aligned}$$

This messy expression for the entropy of the BPS black hole is very different from the one obtained in ungauged supergravity for asymptotically flat black holes. In that case the entropy can be recast in a neat expression, namely it coincides with the quartic invariant $\mathcal{I}_4(P,Q)$ made from the electric and magnetic charges [74–77].

It turns out that one can recast the expression (2.54) for the area in a more elegant expression that originates from the quartic invariant, too. This was found to be valid when the scalar manifold is homogeneous [78, 79]. We illustrate this in Appendix B.

Extremality constrains the near horizon (NH) geometry of the BPS magnetic solution to be of the form $AdS_2 \times S^2$. Namely:

$$ds^2_{NH} = \frac{r^2}{R_1^2}dt^2 - \frac{R_1^2}{r^2}dr^2 - R_2^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2.55}$$

with

$$R_1 = \frac{e^{-\mathcal{K}/2}}{2g}\,, \qquad R_2 = e^{-\mathcal{K}/2} r_h\,. \tag{2.56}$$

From the last equation we see that the radii of the AdS_2 and S^2 factors do not necessarily coincide. Actually, requiring $R_1 = R_2$ gives

$$r_h = \frac{1}{2g} \qquad \rightarrow \qquad |g\xi_1 b_1| = \frac{3}{8}\,, \tag{2.57}$$

that is excluded from the interval of allowed values (2.53). Therefore, the radii do not coincide, and we do not have full enhancement of supersymmetry at the horizon (since $AdS_2 \times S^2$ is fully supersymmetric only if $R_1 = R_2$ [80]). It was proven that the near horizon geometry is only 1/2 BPS [55, 60]. This is different with respect to the case of asymptotically flat extremal BPS black holes, where we have full supersymmetry enhancement at the horizon ($R_1 = R_2$).

Discussion

In this chapter we have discussed the known BPS black hole solutions of gauged supergravity in presence of Fayet-Iliopoulos gauging, in absence of hypermultiplets. We mainly focused on the magnetic solution presented first in [30]. This seed solution was recently studied in presence of generic prepotentials [79], while supersymmetric $AdS_2 \times S^2$ (near horizon) geometries were studied in [78] (and previously also in [81]).

As we mentioned earlier, dyonic solutions were recently found in [71] by means of duality rotation of the magnetic configuration. These solutions allow for a more general form for the scalar field, in particular in this case the scalars cannot be truncated to be real, but they are genuinely complex. It would be very interesting to see if those configurations exhaust the space of the possible supersymmetric solutions.

There exist also generalizations of the four-dimensional solutions in presence of nontrivial hypermultiplets. In that case the hyperscalars are charged and the BPS equations are considerably more difficult. Numerical solutions interpolating between AdS_4 and $AdS_2 \times S^2$ geometries were found recently in [60]. They preserve 1/4 of the supersymmetries too and they are dyonic. Let us mention furthermore that the analysis of the supersymmetry equations in presence of arbitrary gaugings, with both vector and hypermultiplets have been studied in [50].

As for the near horizon geometry analysis, in [82] it was pointed out that the near horizon geometry of non-BPS asymptotically flat black holes preserves some supersymmetries in the gauged theory. The supersymmetry properties of such near horizon geometry coincide with those of the Cacciatori-Klemm near horizon geometry. This can perhaps reveal deep connections between the entropy of Anti-de Sitter and asymptotically flat black holes but at the moment this conjecture is far from being understood.

Let us mention furthermore that the solutions presented in this chapter can be uplifted to M-theory along the lines of [83]. A detailed description of the uplift is presented in [60, 70].

Lastly, one might wonder if solutions analogue to the ones described in this chapter were found also in the realm of five-dimensional gauged supergravity. It turns out that static supersymmetric solutions of this kind in five dimensions correspond to magnetic black strings and were studied already in [84, 85]. Other five-dimensional supersymmetric solutions were found in [86–89] but they need angular momentum in order to develop a horizon.

Chapter 3

Extremal, non BPS deformation

Before the discovery of Anti-de Sitter genuine black hole BPS solutions, the analysis of AdS black holes in gauged supergravity focused on the extremal non-supersymmetric branch. The attractor mechanism and the flow equations for non-BPS black holes were studied in [90, 91], while the near horizon geometry of extremal black holes was investigated in [92]. Despite this analysis, explicit non-BPS black hole solutions with nontrivial scalar profiles in Anti-de Sitter were missing.

In this chapter we illustrate a technique that allows us to generate explicit extremal non-BPS solutions, starting from the BPS ones of [30]. The technique is based on the one already known for ungauged supergravity [93] and consist in a symplectic rotation of the charges of the BPS configuration. In the gauged case, the procedure allows for Anti-de Sitter solutions that break all supersymmetry, but can still be constructed from first order equations. The black hole obtained in this way have a nontrivial radial profile for the scalars and have finite, nonzero area of the event horizon. They were presented for the first time in [73, 94].

We will first explain the technique in detail, and then we will construct explicitly a non-BPS solution for the model with prepotential $F = -2i\sqrt{X^0(X^1)^3}$, that correspond to a deformation of the solution described in the previous chapter, Sec. 2.4.3. The solution is again magnetic, and for a suitable range of parameters corresponds to an extremal black hole. In particular, there is a difference with respect to the BPS case: for the non-supersymmetric configuration we obtain that the Dirac quantization condition (1.87)

$$g_\Lambda p^\Lambda = n \qquad n \in \mathbb{Z}\,. \tag{3.1}$$

can be satisfied with $n \in \mathbb{Z}\backslash\{0\}$. Thus, while supersymmetry fixes the value $n = \pm 1$, a tower of non-supersymmetric configurations with generic n can be constructed.

3.1 The original BPS and non-BPS squaring of the action: ungauged supergravity

It has been shown long ago [93] that, for ungauged supergravity theories, it is possible to obtain extremal black holes solutions by a suitable symplectic rotation of the charges of a BPS configuration, and thus derive a fake superpotential that drives the first order non supersymmetric flow.

More explicitly, such rotation acts linearly on the charges as a constant matrix $\mathcal{S} \in Sp(2n_V+2,\mathbb{R})$ [1], leaving the scalars unchanged[2]. It is a tool to achieve a different squaring of the action and thus get to a set of first order non-BPS equations, in the same way as for the ungauged supergravity case. There, the same rotation $\mathcal{S}$ was first introduced by Ceresole and Dall'Agata (see Sec. 3 of [93]). In particular, some non-BPS black holes can be derived by simply flipping some signs of the charges of the BPS solution.

We are going to show that the same conceptual idea also works for $\mathcal{N}=2$ supergravity with $U(1)$-gauging. This turns out to be particularly straightforward, since the only additional contribution to the Lagrangian is the gauging potential V_g.

To make the derivation and the references clearer, we first review the mechanism that leads to the BPS squaring of the action, starting from the one dimensional effective action derived in [45]. After reviewing the BPS squaring we will deal with the non-BPS one.

3.2 Squaring of the action - BPS case

The bosonic Lagrangian for the $U(1)$-gauged theory is again (1.78), namely

$$\mathscr{L} = \frac{R}{2} + g_{i\bar{l}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{l}} + I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{2} R_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} F^\Sigma_{\rho\sigma} - V_g\,, \tag{3.2}$$

with

$$V_g = g^{i\bar{j}} D_i\mathcal{L}\overline{D}_{\bar{j}}\overline{\mathcal{L}} - 3|\mathcal{L}|^2\,, \qquad \text{where} \quad D_i\mathcal{L} \equiv \partial_i\mathcal{L} + \frac{1}{2}\partial_i K\,\mathcal{L}\,, \tag{3.3}$$

and

$$\mathcal{L} = e^{K/2}\left(X^\Lambda g_\Lambda\right)\,. \tag{3.4}$$

The appropriate metric ansatz that captures static black holes solutions, interpolating between asymptotic AdS_4 space and near horizon $AdS_2 \times S^2$ geometry, contains two warp factors. We parametrize it in our canonical way

$$ds^2 \quad = \quad U^2(r)dt^2 - \frac{dr^2}{U^2(r)} - h^2(r)d\Omega^2_{(2)}\,. \tag{3.5}$$

[1] If we denote by n_V the number of abelian vector multiplets of the $N=2$ model, the theory has a total of n_V+1 abelian gauge fields, and the duality group G is a subset of $Sp(2n_V+2,\mathbb{R})$ [44].

[2] Given that it acts only on the charges, and not on the scalars, such transformation is neither a duality nor a symplectic reparameterization (with reference to the terminology used in Section 1.1.2).

The form of the vector fields is dictated by spherical symmetry, same as (2.20), and the scalar fields depend just on the radial variable. Upon this ansatz, the action for the Lagrangian (3.2) reduces (up to integration by part) to the form

$$\begin{aligned} S_{1d} &= \int dr \left\{ \left[-2hh'UU' - U^2 h'^2 + h^2 U^2 g_{i\bar{\jmath}} z^{i\prime} \bar{z}^{\bar{\jmath}\prime} + \frac{1}{h^2} V_{BH} - h^2 V_g \right] - 1 \right\} \\ &\quad + \int dr \frac{d}{dr} \left[2h' h U^2 + U' U h^2 \right] . \\ &= \int dr \; \mathcal{L}_{1d}^{eff} \; + \int dr \frac{d}{dr} \left[2h' h U^2 + U' U h^2 \right] \; , \end{aligned} \tag{3.6}$$

the boundary contribution is exactly canceled by the Gibbons-Hawking boundary term. The black hole potential V_{BH} is defined as

$$V_{BH} = |\mathcal{Z}|^2 + |D\mathcal{Z}|^2 \, , \tag{3.7}$$

with $\mathcal{Z} = \langle Q, \mathcal{V} \rangle$ as defined in (2.33). Alternatively, it can be written as

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q \; , \tag{3.8}$$

where $\mathcal{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a symplectic matrix whose elements are

$$A = I_{\Lambda\Sigma} + R_{\Lambda\Gamma} I^{\Gamma\Delta} R_{\Delta\Sigma} \, , \tag{3.9}$$

$$B = C^T = -R_{\Lambda\Sigma} I^{\Sigma\Gamma} \, , \tag{3.10}$$

$$D = I^{\Lambda\Sigma} \, . \tag{3.11}$$

One can explicitly check that is possible to rewrite (3.6) as the sum of squares. The expression was found in [45]:

$$\begin{aligned} S_{1d} &= \int dr \left\{ -\frac{1}{2h^2} \mathcal{E}^T \mathcal{M} \mathcal{E} - h^2 U^2 \left[(\alpha' + \mathcal{A}_r) + \frac{2}{U} \mathrm{Re}(e^{-i\alpha} \mathcal{L}) \right]^2 \right. \\ &\quad - h^2 U^2 \left[\left(\frac{h'}{h} + \frac{U'}{U} \right) - \frac{2}{U} \mathrm{Im}(e^{-i\alpha} \mathcal{L}) \right]^2 - (1 + \langle \mathcal{G}, Q \rangle) \\ &\quad \left. - 2 \frac{d}{dr} \left[h^2 U \, \mathrm{Im}(e^{-i\alpha} \mathcal{L}) + U \, \mathrm{Re}(e^{-i\alpha} \mathcal{Z}) \right] \right\} , \end{aligned} \tag{3.12}$$

where we introduced the symplectic vector

$$\mathcal{E}^T \equiv 2h^2 U^2 \left(\frac{1}{U} \mathrm{Im}(e^{-i\alpha} \mathcal{V}) \right)'^{\,T} - h^2 \mathcal{G}^T \Omega \mathcal{M}^{-1} + \frac{4}{U} (\alpha' + \mathcal{A}_r) \mathrm{Re}(e^{-i\alpha} \mathcal{V})^T + Q^T \, , \tag{3.13}$$

and the symplectic vector containing the Fayet-Iliopoulos parameters[3] $\mathcal{G} = (0, g_\Lambda)$.

The equations of motion following from this effective action are obtained by setting to zero the quantities that appear in each squared term:

$$
\begin{aligned}
\mathcal{E} &= 0 \,, & (3.14)\\
\frac{h'}{h} + \frac{U'}{U} &= \frac{2}{U}\,\mathrm{Im}(e^{-i\alpha}\mathcal{L}) \,, & (3.15)\\
(\alpha' + \mathcal{A}_r) &= -\frac{2}{U}\,\mathrm{Re}(e^{-i\alpha}\mathcal{L}) \,, & (3.16)
\end{aligned}
$$

supplemented by the constraint

$$
1 + \langle \mathcal{G}, Q \rangle = 1 + g_\Lambda p^\Lambda = 0 \,. \tag{3.17}
$$

Showing the relation between equations (3.14)-(3.17) coming from the squaring of the action and the BPS equations (2.32),(2.34)-(2.38) from the previous chapter requires some patient manipulation. In particular, (3.14) is a complex symplectic vector of equations whose information can be extracted by appropriate projections with all possible independent sections (in particular $\mathcal{V}, U_i, \mathcal{G}, Q$).

It was explicitly shown in [45] that the contraction gives the following equations:

$$
\begin{aligned}
\partial_r U &= \frac{1}{h^2}\mathrm{Re}(e^{-i\alpha}\mathcal{Z}) + \mathrm{Im}(e^{-i\alpha}\mathcal{L}) \,, & (3.18)\\
U \partial_r z^i &= e^{i\alpha} g^{i\bar{k}}[+\frac{1}{h^2}\overline{D}_{\bar{k}}\overline{\mathcal{Z}} - i\overline{D}_{\bar{k}}\overline{\mathcal{L}}] \,, & (3.19)\\
(\alpha' + \mathcal{A}_r) &= \frac{1}{U}\left(\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) - \mathrm{Re}(e^{-i\alpha}\mathcal{L})\right) . & (3.20)
\end{aligned}
$$

Compatibility of expressions (3.20) and (3.16) gives this constraint for the phase[4]

$$
0 = \mathrm{Re}(e^{-i\alpha}\mathcal{L}) + \frac{1}{h^2}\mathrm{Im}(e^{-i\alpha}\mathcal{Z}) \,. \tag{3.21}
$$

The system of equations (3.15),(3.17)-(3.19),(3.21) coming from the squaring of the 1d action, supplemented by the constraint

$$
g_\Lambda A_t^\Lambda = 2U\,\mathrm{Re}(e^{i\alpha}\mathcal{L}) \tag{3.22}
$$

is equivalent to the BPS equations (2.32),(2.34)-(2.38).

We mention here that the constraint (3.22) will be automatically satisfied for every solution we are going to describe.

[3] Nonzero entries of the upper $n_V + 1$ entries of the vector $\mathcal{G}$ correspond to magnetic gaugings, like anticipated in Chapter 1, eq. (1.86).

[4] This constraint is equivalent to the Hamiltonian constraint coming from the reduced action [82]. In this way solutions of the system of equations (3.15),(3.17)-(3.19),(3.21) coming from the squaring of the one-dimensional reduced action are solutions to the full system of Einstein-Maxwell-scalar equations of motion.

3.3 Rotation of charges: towards non-BPS solutions

As we already said, the equations of motion obtained from the squaring were shown to be equivalent to those obtained by the supersymmetry variations of the fermionic fields on the black hole solution [45]. However, the procedure of squaring the action (3.6) is not unique.

Indeed we can act with a linear transformation on the black hole charges and obtain a different set of first order equations with respect to those of [45]. The new solutions still satisfy the second order equations of motion of N=2 U(1)-gauged supergravity, but now correspond to non supersymmetric configurations. In the following we are going to apply to the $U(1)$-gauged theory the procedure presented in [93], to obtain a non-BPS flow.

Consider a symplectic rotation acting on the black hole charges, given by a constant matrix $\mathcal{S}$,

$$Q = (p^\Lambda, q_\Lambda) \to \widetilde{Q} \equiv \mathcal{S} Q \ , \tag{3.23}$$

such that

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \ , \qquad \mathcal{S}^T \mathcal{M} \mathcal{S} = \mathcal{M} \ . \tag{3.24}$$

$\mathcal{S}$ does not act on the symplectic sections, thus it leaves the scalars unchanged. It has the same role and the properties (3.24) are the same as those of the matrix $\mathcal{S}$ introduced by Ceresole and Dall'Agata in [93]. The black hole charges Q enter in the one-dimensional reduced action (3.6) only through the black hole potential

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q \ , \tag{3.25}$$

which is left invariant by a transformation whose matrix $\mathcal{S}$ obeys (3.23)[5]. Also the gauging potential V_g, that only depends on the scalars and Fayet-Iliopoulos terms, is left invariant, and so is the 1d action (3.6).

We re-do the computation of Sec. 3.1, introducing a *fake* central charge $\widetilde{\mathcal{Z}}$

$$\widetilde{\mathcal{Z}} = \langle \widetilde{Q}, \mathcal{V} \rangle \ , \tag{3.26}$$

[5]While it is immediate to see that the second of equations (3.24) is also required for the invariance of V_{BH}, one could wonder why the matrix $\mathcal{S}$ is required to be symplectic (the first equation (3.24)). One can easily verify that this equation is needed in order to be able to rewrite the black hole potential V_{BH} in terms of the fake central charge $\widetilde{\mathcal{Z}}$ in this way:

$$V_{BH} = -\frac{1}{2} \widetilde{Q}^T \mathcal{M} \widetilde{Q} = \widetilde{\mathcal{Z}} \overline{\widetilde{\mathcal{Z}}} + g^{l\overline{m}} D_l \widetilde{\mathcal{Z}} D_{\overline{m}} \overline{\widetilde{\mathcal{Z}}}$$

This ensures that the squaring of the action proceeds in the exact same way as in the BPS case, provided that $Q \to \widetilde{Q}$ and $\mathcal{Z} \to \widetilde{\mathcal{Z}}$.

so that the 1d effective action (3.6) can be squared as

$$\begin{aligned} S_{1d} &= \int dr \Bigg\{ -\frac{1}{2h^2}\widetilde{\mathcal{E}}^T\mathcal{M}\widetilde{\mathcal{E}} - h^2U^2\left[(\alpha'+\mathcal{A}_r)+\frac{2}{U}\mathrm{Re}(e^{-i\alpha}\mathcal{L})\right]^2 \\ &\quad -h^2U^2\left[\left(\frac{h'}{h}+\frac{U'}{U}\right)-\frac{2}{U}\mathrm{Im}(e^{-i\alpha}\mathcal{L})\right]^2 - \left(1+\langle\mathcal{G},\widetilde{Q}\rangle\right) \\ &\quad -2\frac{d}{dr}\left[h^2U\,\mathrm{Im}(e^{-i\alpha}\mathcal{L})+U\,\mathrm{Re}(e^{-i\alpha}\widetilde{\mathcal{Z}})\right]\Bigg\}\,, \end{aligned} \tag{3.27}$$

where now

$$\widetilde{\mathcal{E}}^T \equiv 2h^2U^2\left(\frac{1}{U}\mathrm{Im}(e^{-i\alpha}\mathcal{V})\right)'^{\,T} - h^2\mathcal{G}^T\Omega\mathcal{M}^{-1}+\frac{4}{U}(\alpha'+\mathcal{A}_r)\mathrm{Re}(e^{-i\alpha}\mathcal{V})^T+\widetilde{Q}^T \tag{3.28}$$

The first order equations are then

$$\widetilde{\mathcal{E}} = 0\,, \tag{3.29}$$

$$\frac{h'}{h}+\frac{U'}{U} = \frac{2}{U}\mathrm{Im}(e^{-i\alpha}\mathcal{L})\,, \tag{3.30}$$

$$(\alpha'+\mathcal{A}_r) = -\frac{2}{U}\mathrm{Re}(e^{-i\alpha}\mathcal{L})\,, \tag{3.31}$$

supplemented by the constraint

$$1+\langle\mathcal{G},\widetilde{Q}\rangle = 0\,, \tag{3.32}$$

and describe possibly extremal non-BPS black hole solutions.

In other words, the configuration obtained by performing the transformation (3.23) on the BPS solution is a solution to the second order equations of motion of the same 1d Lagrangian (3.6).

However, it will not preserve supersymmetry, since the SUSY equations depend on the black hole charges Q (defined, as usual, as the fluxes of the abelian gauge fields at infinity), while the flow equations above depend on the vector $\widetilde{Q}$.

Eventually, the black hole solution will differ from the BPS one because of the dependence of first order equations on $\widetilde{Q}$ instead of Q.

The symplectic rotation is possible whenever a non-trivial matrix satisfying (3.23) exists. Let us notice that, analogously to the ungauged case, the choice $S = -\mathbf{1}$ gives the second branch of the BPS equations, obtained by a different choice of the phases for Killing spinor projectors.

It is easy to find a matrix $\mathcal{S}$ for instance for the stu model with zero axions and all the moduli identified. This is the model treated in [45]. Indeed, in the case of the cubic stu prepotential

$$F = \frac{X^1X^2X^3}{X^0}\,, \tag{3.33}$$

if we identify all the moduli and look for the zero axions solutions

$$s = t = u = -i\lambda \ , \tag{3.34}$$

the most general matrix satisfying (3.23) is

$$\left(\begin{array}{cc|cc} a & 0 & 0 & 0 \\ 0 & A_{3\times 3} & 0 & 0 \\ \hline 0 & 0 & a & 0 \\ 0 & 0 & 0 & A_{3\times 3} \end{array}\right) \ , \qquad a = \pm 1 \ , \qquad A^2_{3\times 3} = \mathbf{1}_{3\times 3} \ , \tag{3.35}$$

with $A_{3\times 3}$ a 3x3 matrix.

The model $F = -2i\sqrt{X^0X^1X^2X^3}$ is obtained from the stu one by means of a symplectic rotation $\mathcal{R}$ of the form

$$\mathcal{R} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1}_{3\times 3} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \mathbf{1}_{3\times 3} & 0 & 0 \end{array}\right) \ . \tag{3.36}$$

Since $[\mathcal{R}, \mathcal{S}] = 0$ for $\mathcal{S}$ satisfying (3.35), matrices $\mathcal{S}$ of the form (3.35) will also work in generating non-BPS solutions in this $F = -2i\sqrt{X^0X^1X^2X^3}$ symplectic frame.

We focus now on the specific example of the model $F = -2i\sqrt{X^0(X^1)^3}$, describing in detail a simple example of non-BPS solution, making comparison with the BPS solution of the same model in Section 2.4.3.

3.4 A non-BPS solution for the $F = -2i\sqrt{X^0(X^1)^3}$ model

In this section we construct regular non-BPS solutions of the model with prepotential $F = -2i\sqrt{X^0(X^1)^3}$, in presence of purely electric gaugings: $\mathcal{G} = g(0, 0, \xi_0, \xi_1) = (0, 0, g_0, g_1)$.

These solutions are purely magnetic:

$$q_0 = q_1 = 0 \, ,$$

and they are obtained from the 1/4 BPS solution found by [30] and [70] by means of the rotation trick described in the previous section.

An easy way to obtain a extremal non-BPS configuration is flipping the sign of one charge with respect to the BPS case. This corresponds to performing the trick of using a 4×4 matrix

$$\mathcal{S} = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{3.37}$$

We start from ansatz for the metric (2.19) and vector fields (2.20) with the same warp factors as in the BPS solution:

$$U^2(r) = e^{\mathcal{K}} \frac{g^2(r^2 - r_h^2)^2}{r^2}, \qquad h(r) = r\, e^{-\mathcal{K}/2}. \tag{3.38}$$

In complete analogy with the BPS case, also the sections are still given by harmonic functions

$$X_0 = a_0 + \frac{b_0}{r}, \qquad X_1 = a_1 + \frac{b_1}{r} \qquad z = \frac{X_1}{X_0}, \tag{3.39}$$

and $e^{\mathcal{K}} = i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)$. We decide to take the + sign in (3.37), such that the sign of p^1 is flipped with respect to the BPS configuration. This corresponds to new charges $(\widetilde{p}^0\ , \widetilde{p}^1)$

$$\widetilde{Q} = (\widetilde{p}^0\ , \widetilde{p}^1, 0, 0) = (p^0\ , -p^1, 0, 0). \tag{3.40}$$

We checked that the equations (3.29)-(3.32) are satisfied for this choice of parameters:

$$a_0 = \frac{1}{4\xi_0}, \qquad b_0 = -\frac{\xi_1 b_1}{\xi_0}, \qquad a_1 = \frac{3}{4\xi_1}, \qquad r_h^2 = \frac{16}{3}(g_1 b_1)^2 - \frac{1}{2}. \tag{3.41}$$

and the new condition on the charges becomes

$$g_0 p^0 - g_1 p^1 = -1. \tag{3.42}$$

Furthermore, the charges are written in terms of the other parameters as

$$p^0 = -\frac{2}{g_0}\left(\frac{1}{8} + \frac{8(g_1 b_1)^2}{3}\right), \qquad p^1 = +\frac{2}{g_1}\left(\frac{3}{8} - \frac{8(g_1 b_1)^2}{3}\right). \tag{3.43}$$

Alternatively, we can write the other parameters in terms of p^1

$$p^0 = \frac{g_1 p^1 - 1}{g_0}, \qquad b_1 = -\frac{3\sqrt{1-4(p^1 g_1)/3}}{8 g_1}, \qquad r_h = \frac{\sqrt{1-4 g_1 p^1}}{2}. \tag{3.44}$$

All this corresponds exactly to change the sign of the p^1 charge on the BPS solution.

At the end of the day the warp factor in this case turns out to be

$$U^2 = \frac{2\sqrt{g_0 (g_1)^3}(r^2 - r_h^2)^2}{\left(r + \frac{3}{2}\sqrt{1-4(p^1 g_1)/3}\right)^{1/2}\left(3r - \frac{3}{2}\sqrt{1-4(p^1 g_1)/3}\right)^{3/2}}, \tag{3.45}$$

$$\tag{3.46}$$

while the scalar profile is

$$z = z_\infty \sqrt{\frac{2r - \sqrt{1-4p^1 g_1/3}}{2r + 3\sqrt{1-4p^1 g_1/3}}}, \qquad \text{with} \qquad z_\infty = \frac{3\xi_0}{\xi_1}. \tag{3.47}$$

The non-BPS solution looks qualitatively similar to the BPS one for what concerns the general behaviour of the warp factor and the location of the singularities. The configuration depends on the parameters g_0, g_1 and p^1 (or alternatively, b_1), like in the BPS case. The singularities are located at the poles of $e^{\mathcal{K}}$, namely $r_{1,s} = -4/3 b_1 \xi_1$ and $r_{s,2} = 4 b_1 \xi_1$ (these expressions can be written in terms of the charge p^1 too). Like in the BPS case, requiring that the horizon shields the singularity sets the constraint $g \xi_1 b_1 < -3/8$.

3.4.1 Dirac quantization condition

We see now that the non-BPS black holes we obtained by construction satisfy relation (3.32). In the case of the magnetic non-BPS solutions of the previous subsection, (3.32) reads

$$g_0 p^0 - g_1 p^1 = -1 \,. \tag{3.48}$$

Therefore, the Dirac quantization condition (3.1) is not automatically satisfied. We need to impose it by hand:

$$g_0 p^0 + g_1 p^1 = n \,, \tag{3.49}$$

where we left a generic value for $n \in \mathbb{Z}$.

This constraint, together with (3.43), yields the relation

$$g_\Lambda p^\Lambda = \frac{1}{2} - \frac{32}{3}(g_1 b_1)^2 = -\frac{1}{2} - 2r_h^2 \,. \tag{3.50}$$

It turns out that the value of r_h, that determines the radial position of the horizon, enters in the quantization condition (3.49), and it is constrained to satisfy

$$-2r_h^2 \in \mathbb{Z} + \frac{1}{2} \,. \tag{3.51}$$

Whenever r_h fulfills this condition, we are able to build a tower of states of extremal black holes with more generic n integer. We recall that, in order to have proper black holes (finite nonzero area of the event horizon) the parameter r_h has to be positive and this restricts the possible values of n to be negative. In particular, defining $m = -n$, we have then that only the values $m \in \mathbb{N} \backslash \{0,1\}$ correspond to proper black holes[6]. The constraints of the magnetic charges are as follows:

$$g_0 p^0 - g_1 p^1 = -1 \,, \tag{3.52}$$

$$g_0 p^0 + g_1 p^1 = n = -m \,. \tag{3.53}$$

This gives

$$2g_0 p^0 = -1 - m \qquad 2g_1 p^1 = 1 - m \,. \tag{3.54}$$

[6]The other branch, that correspond to flipping the sign of the Fayet-Iliopoulos terns, or alternatively of both the magnetic charges, gives configurations with $-m \in \mathbb{N} \backslash \{0,1\}$. This gives the full spectrum of values of $m \in \mathbb{Z} \backslash 0$.

Note that the charges p^0 and p^1 are always negative; furthermore, notice that the configuration with $p^1 = 0$ is a naked singularity. The scalar field solution is of the form:

$$z = z_\infty \sqrt{\frac{2r - \sqrt{3(2m+1)}}{2r + 3\sqrt{3(2m+1)}}}\,, \tag{3.55}$$

and the warp factor reduces to:

$$U^2 = \frac{2\sqrt{g_0(g_1)^3}(r^2 - r_h^2)^2}{\left(r + \frac{3}{2}\sqrt{(2m+1)/3}\right)^{1/2}\left(3r - \frac{3}{2}\sqrt{(2m+1)/3}\right)^{3/2}}\,. \tag{3.56}$$

From (3.50) $r_h = \frac{\sqrt{2m-1}}{2}$, and consequently the entropy takes the form

$$h^2(r_h) = 2\sqrt{p^0(p^1)^3}\,\frac{\left(r_h + \frac{3}{2}\sqrt{(2m+1)/3}\right)^{1/2}\left(3r_h - \frac{3}{2}\sqrt{(2m+1)/3}\right)^{3/2}}{\sqrt{(m^2-1)(1-m)^2}} =$$

$$= \frac{3\sqrt{3}}{2}\sqrt{p^0(p^1)^3}\,\frac{\left(\sqrt{2m-1} + \sqrt{3(2m+1)}\right)^{1/2}\left(\sqrt{2m-1} - \sqrt{(2m+1)/3}\right)^{3/2}}{|m-1|\sqrt{(m^2-1)}}\,. \tag{3.57}$$

Notice that the entropy is given by an expression which is nothing but the entropy of the black hole in the corresponding ungauged supergravity configuration, corrected by a factor that only depends on the quantization parameter.

The quantity $p^0(p^1)^3$ corresponds to the quartic invariant of the duality group of the theory. It would be interesting to analyze further the duality properties for the gauged supergravity solutions, in analogy with the ungauged case.

Discussion

In this chapter we constructed extremal non-supersymmetric black hole solutions, thanks to the generating technique pioneered in [93]. This allowed us to get new solutions simply by rotating the charges of the BPS configuration. The conditions on the parameters of these new configurations are incompatible with BPS conditions (as one can see by comparing equations (3.42) and (2.38)) therefore no supersymmetry is preserved.

By no means have we proven that these solutions are the unique extremal configurations of the theory: in principle other ways of squaring the action can exist, and these could lead to other extremal configurations. In particular, in Chapter 6, looking at the plot 6.3 we will see that there exist other kind of extremal but non-supersymmetric configurations different than the ones treated here.

The lack of supersymmetry of these black holes could in principle lead to instabilities. It would be interesting to study the (thermodynamical and mechanical) stability properties of these non-BPS black holes, in order to compare them with their BPS cousins.

Chapter 4

Nonextremal black holes

This chapter is devoted to the construction and analysis of thermal (also called nonextremal) Anti-de Sitter black holes. These solutions are characterized by the presence of noncoincident inner and outer horizons, therefore they radiate at a nonzero temperature. They provide thermal gravitational backgrounds interesting for applicative sides of the AdS/CFT correspondence.

In the nonextremal case, in general, one cannot derive the solution from a set of first order equations (the possibility of a first order flow will be discussed Chapter 8). In general we need to solve the system of Einstein's, Maxwell's and scalar equations of motion in its entirety[1].

Solving the Einstein's equations provides a difficult task, given that the equations are no longer first order, but second order differential equations. However, knowledge of the details of the BPS extremal configurations turns out to be extremely useful and simplify the procedure of finding a solution. It turns out that we can construct nonextremal black holes by suitably modifying the warp factors in the metric, keeping the form of the scalar fields unchanged.

This procedure is used also in ungauged supergravity, see for instance [97]. In the realm of gauged supergravity, this was used to construct thermal AdS black holes in five dimensions [98] and also electric nonextremal AdS black holes [64]. The latter reduce in the BPS limit to the singular electric supersymmetric solutions of Section 2.3.

We present here a detailed review of the derivation of these electric solutions and we also describe nonextremal deformation of the magnetic BPS solution of Cacciatori and Klemm [30]. This provides the first genuine AdS thermal black hole that is nonsingular in the BPS limit [96, 99].

The explicit black hole solutions we present are given for the particular case of prepotential $F = -2i\sqrt{X^0(X^1)^3}$, for which a string/M-theory embedding exists,

[1] Another approach to this problem is the one based on dimensional reduction and the real formulation of special geometry, as developed in [95]. Within this formalism, the problem of constructing stationary solutions of $4D, \mathcal{N} = 2$ Fayet-Iliopoulos gauged supergravity reduces to solving a particular three-dimensional Euclidean non-linear sigma model (with potential). We do not treat it here, but further details can be found in [94, 96].

but we believe that the same qualitative features should appear also in presence of other prepotentials. We compute the mass with the Ashtekhar-Magnon-Das (AMD) formalism [100, 101] that provide finite conserved quantities without need of renormalization. The mass and other conserved quantities for these configurations will also be computed with the formalism developed in [102, 103] in Chapter 7 and by means of the holographic renormalization techniques [104–106] in Chapter 8. Furthermore, for all black hole solutions it is explicitly verified that the product of the areas of the horizons depends just on the gauging and black hole charges and not on (parameters depending on) the mass.

4.1 Supergravity lagrangian and equations of motion

We report here once again the bosonic part of the action for abelian gauged $D=4$ $\mathcal{N}=2$ supergravity with n_V vector multiplets and in absence of hypermultiplets is:

$$S=\int \mathrm{d}^4x\, e\left[\frac{1}{2}R+g_{i\bar{l}}\partial^\mu z^i\partial_\mu\bar{z}^{\bar{l}}+I_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\mu\nu|\Sigma}+\frac{1}{2}R_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}-V_g(z,\bar{z})\right], \tag{4.1}$$

with $\Lambda,\Sigma=0,1...,n_V$, $i,j=1,...,n_V$. The complex scalars z^i are written in terms of the holomorphic symplectic sections (X^Λ, F_Λ). Furthermore, the scalar potential can be written as:

$$V=g^2(g^{i\bar{l}}f_i^\Lambda\bar{f}_{\bar{l}}^\Sigma-3\bar{L}^\Lambda L^\Sigma)\xi_\Lambda\xi_\Sigma\,, \tag{4.2}$$

or alternatively as (1.80).

Einstein's equations then are (conventions are in Appendix A)

$$\begin{aligned}-(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R)&=g_{\mu\nu}V_g(z,\bar{z})-g_{\mu\nu}\partial^\sigma z^i\partial_\sigma\bar{z}^{\bar{l}}g_{i\bar{l}}+2g_{i\bar{l}}\partial_{(\mu}z^i\partial_{\nu)}\bar{z}^{\bar{l}}+\\&\quad-I_{\Lambda\Sigma}g_{\mu\nu}F^\Lambda_{\rho\sigma}F^{\rho\sigma|\Sigma}+4I_{\Lambda\Sigma}F^\Lambda_{\mu\alpha}F_\nu{}^{\alpha|\Sigma}\,.\end{aligned} \tag{4.3}$$

The equations of motion for the scalar fields z^i read:

$$g_{i\bar{l}}\partial_\mu(e\partial^\mu\bar{z}^{\bar{l}})+e\frac{\partial g_{i\bar{k}}}{\partial\bar{z}^{\bar{l}}}\partial^\mu\bar{z}^{\bar{l}}\partial_\mu\bar{z}^{\bar{k}}+e\frac{\partial I_{\Lambda\Sigma}}{\partial z^i}F^\Lambda_{\rho\sigma}F^{\rho\sigma|\Sigma}+\frac{e}{2}\frac{\partial R_{\Lambda\Sigma}}{\partial z^i}\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}-e\frac{\partial V_g}{\partial z^i}=0\,, \tag{4.4}$$

and the Maxwell's equations for the vector fields A^Λ_ν are:

$$\partial_\mu(eF^{\mu\nu|\Sigma}I_{\Sigma\Lambda}-\frac{e}{2}\epsilon^{\mu\nu\rho\sigma}F^\Sigma_{\rho\sigma}R_{\Lambda\Sigma})=0\,. \tag{4.5}$$

4.1.1 Plugging in the ansatz

At this point we restrict ourselves to the static and spherically symmetric ansatz (2.19), as usual:

$$\mathrm{d}s^2=U^2(r)\mathrm{d}t^2-\frac{1}{U^2(r)}\mathrm{d}r^2-h^2(r)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\,, \tag{4.6}$$

and agan, due to spherical symmetry and the Bianchi identity, the most general form of the field strengths solving the Maxwell's equations (4.5) is

$$F^{\Lambda}_{tr} = -\frac{1}{2h^2(r)} I^{\Lambda\Sigma}(R_{\Sigma\Gamma}p^{\Gamma} - q_{\Sigma}), \qquad F^{\Lambda}_{\theta\varphi} = \frac{1}{2}p^{\Lambda}\sin\theta, \tag{4.7}$$

while all other components vanish. Here $I^{\Lambda\Sigma}$ is the inverse of $I_{\Lambda\Sigma}$, and the magnetic and electric charges satisfy

$$p^{\Lambda} = -\frac{1}{4\pi}\int_{S^2_{\infty}} F^{\Lambda}, \qquad q_{\Lambda} = -\frac{1}{4\pi}\int_{S^2_{\infty}} G_{\Lambda}, \tag{4.8}$$

with G_{Λ} as in (1.32). The matrices $I_{\Lambda\Sigma}$ and $R_{\Lambda\Sigma}$ depend on the specific model taken into consideration and for the moment we keep q_{Λ} and p^{Λ} unconstrained.

The scalar field equation reduces to:

$$\frac{1}{h^2(r)} g_{i\bar{l}}\partial_r\left(h^2(r)U^2(r)\partial_r\bar{z}^{\bar{l}}\right) + \frac{\partial g_{i\bar{l}}}{\partial\bar{z}^{\bar{k}}}\partial_r\bar{z}^{\bar{l}}\partial^r\bar{z}^{\bar{k}} + \frac{\partial I_{\Lambda\Sigma}}{\partial z^i}F^{\Lambda}_{\mu\nu}F^{\mu\nu|\Sigma} +$$
$$+\frac{1}{2}\frac{\partial R_{\Lambda\Sigma}}{\partial z^i}\epsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} - \frac{\partial V_g}{\partial z^i} = 0\,. \tag{4.9}$$

The relevant components of the Einstein's equation yield:
tt component:

$$-\frac{\left(-1+2h\,h'\,U'U + U^2(h'^2+2h\,h'')\right)}{h^2} = V_g(z,\bar{z}) - \partial^r z^i\partial_r\bar{z}^{\bar{l}}g_{i\bar{l}} - \frac{V_{BH}}{h^4}\,, \tag{4.10}$$

rr component:

$$-\frac{-1+U^2h'^2+2h\,h'\,U'U}{h^2} = V_g(z,\bar{z}) + \partial^r z^i\partial_r\bar{z}^{\bar{l}}g_{i\bar{l}} - \frac{V_{BH}}{h^4}\,, \tag{4.11}$$

$\theta\theta$ component:

$$-\frac{\left(hU'^2+U^2h''+U(2h'U'+hU'')\right)}{h} = V_g(z,\bar{z}) - \partial^r z^i\partial_r\bar{z}^{\bar{l}}g_{i\bar{l}} + \frac{V_{BH}}{h^4}\,, \tag{4.12}$$

where V_{BH} was already defined in (3.8) and here we rewrite for convenience:

$$V_{BH}(z,\bar{z},q_{\Lambda},p^{\Lambda}) = -\frac{1}{2}\begin{pmatrix} p^{\Lambda} & q_{\Lambda}\end{pmatrix}\begin{pmatrix} I_{\Lambda\Sigma}+R_{\Lambda\Gamma}I^{\Gamma\Delta}R_{\Delta\Sigma} & -R_{\Lambda\Gamma}I^{\Gamma\Sigma} \\ -I^{\Lambda\Gamma}R_{\Gamma\Sigma} & I^{\Lambda\Sigma}\end{pmatrix}\begin{pmatrix} p^{\Sigma} \\ q_{\Sigma}\end{pmatrix}. \tag{4.13}$$

The $\varphi\varphi$ component gives the $\theta\theta$ one multiplied by $\sin^2\theta$. All other nondiagonal components are trivial.

We now manipulate the equations in order to get simpler ones, along the same lines as [90, 91]. Adding tt and rr we get

$$-2\frac{h''}{h} = 2g_{i\bar{j}}\partial_r z^i \partial_r \bar{z}^{\bar{j}}\,, \tag{4.14}$$

while adding rr and $\theta\theta$ we obtain

$$\frac{1-(U^2h^2)''/2}{h^2} = 2V_g(z,\bar{z})\,. \tag{4.15}$$

Finally, $\theta\theta - tt$ gives

$$-1+U^2h'^2+hh''U^2-h^2U'^2-h^2UU'' = \frac{2V_{BH}}{h^2}\,. \tag{4.16}$$

Furthermore we have n_V equations for the scalar fields. It is easy to show that for nonconstant scalars if the Einstein's rr, tt component and the scalar equation of motion are satisfied, the $\theta\theta$ component is satisfied too. So there are actually n_V+1 independent equations to be satisfied. In what follows we will focus on the case of one single scalar field. The above considerations therefore tell us that if we find a solution to the Einstein's equations, the scalar equation is automatically satisfied and does not impose any other constraint. We verified this explicitly for all solutions with nonconstant scalars.

For constant scalars, however, this is not true and one needs to solve an extra equation. We treat that case separately in the next section.

4.2 Constant scalar solutions

We start now with the simplest case in which the scalar profile does not depend on the radial variable, namely the scalars are constant throughout the flow. We would like to construct nonextremal solutions with frozen scalar fields in presence of an arbitrary prepotential. For the moment keep constant complex scalar fields.

First of all we analyze the equation (4.14). The right hand side is identically zero, so we integrate twice and find $h(r)$:

$$h(r) = a\,r+b\,, \qquad a,b \text{ constant.} \tag{4.17}$$

We keep for the moment also the integration constant b, in order to deal with the solution in its full generality[2].

Analyzing the scalar equation of motion (4.9), then, due to the different radial dependence, the two remaining terms have to vanish separately

$$\frac{\partial I_{\Lambda\Sigma}}{\partial z^i}F^{\Lambda}_{\mu\nu}F^{\mu\nu|\Sigma} + \frac{1}{2}\frac{\partial R_{\Lambda\Sigma}}{\partial z^i}\epsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} = 0\,, \tag{4.18}$$

[2] Solutions of the form (4.17) are found for instance when we impose constant scalars and sections X^Λ proportional to each other. When we impose also constant sections, we find solutions with $b=0$.

$$\frac{\partial V_g}{\partial z^i} = 0\,. \tag{4.19}$$

This sets the scalar at their constant value that extremizes that scalar potential

$$z^i = z^i_* \qquad \text{such that} \qquad \left.\frac{\partial V_g}{\partial z^i}\right|_{z^i = z^i_*} = 0 \tag{4.20}$$

and imposes the constraint (4.18), that is quadratic in the electric and magnetic charges.

We now integrate the equation (4.15), that gives the form of the warp factor

$$U^2(r) = \frac{-V_{g*}\left(\frac{a^2 r^4}{3} + \frac{4a\,b\,r^3}{3} + 2b^2 r^2\right) + r^2 + \mu r + Q}{(ar+b)^2}\,. \tag{4.21}$$

When $V_{g*} = V_g(z^i_*, \bar{z}^{\bar{\imath}}_*)$ is negative the solution asymptotes to AdS$_4$. The parameters μ, Q are further (real) integration constants. Finally equation (4.16) gives another constraint:

$$2(b^4 V_{g*} - b^2 + a\,b\,\mu - a^2 Q) = V_{BH}\,, \tag{4.22}$$

where V_{BH} is given in (4.13). The most general solution with constant scalars is then of the form (4.17) (4.21) and has to satisfy the constraints (4.18) (4.22).

We now specialize to electric and magnetic black holes with constant scalars and prepotential $F = -2i\sqrt{X^0 (X^1)^3}$. The general solution is characterized by two magnetic charges p^0, p^1 and two electric ones q_0, q_1, and one scalar $z = \frac{X^1}{X^0}$ that we assume to be real for simplicity of the calculations presented below. The special Kähler quantities for this prepotential are given in Appendix A. Furthermore, the positivity of the Kähler metric requires the scalar field to be positive.

We notice that we can perform the shift in the radial variable (a is nonzero for a regular solution)

$$r \to r - b/a\,. \tag{4.23}$$

This amount to take $b = 0$ in (4.21) without loss of generality. Therefore in the explicit examples, for similarity with respect to Reissner-Nordström case, we set $b = 0$. Eq. (4.19) is satisfied if impose the constant value of the scalar field

$$z = z_* = \frac{3\xi_0}{\xi_1}\,, \tag{4.24}$$

that gives the value of the potential $V_{g*} = -\frac{2g^2}{\sqrt{3}}\sqrt{\xi_0 \xi_1^3}$, such that we have a AdS$_4$ extremum (in particular it is a local maximum). For the field z_* to be positive the FI terms then must have the same sign. We choose positive FI terms without loss of generality, since the Lagrangian remains invariant (provided the gravitinos flip charge) under the transformation for which both ξ_0 and ξ_1 change sign.

In the following subsection we will analyze separately purely electric and purely magnetic black holes.

We will compare the BPS bound with the extremality bound for these new solutions. To find the extremality bound first of all we rescale the radial and time component of the metric (4.21) with $b = 0$ such that we retrieve the familiar form

$$U^2(r) = 1 - \frac{2m}{r} + \frac{Z^2}{r^2} + g'^2 r^2 \,. \tag{4.25}$$

The parameters m, Z, g' are related to the ones of (4.21) through

$$m = \frac{1}{2} a\mu \,, \qquad Z^2 = a^2 Q \,, \qquad g'^2 = |V_{g*}| \,. \tag{4.26}$$

To compute the value of μ appearing in (4.21) at extremality, $\mu_{extr}(a, Q, g, V_*)$, we require that the function $U(r)$ and also its derivative with respect to r vanish at the horizon:

$$U = 0 \quad \text{and} \quad \frac{dU}{dr} = 0 \quad \text{at the horizon} \,. \tag{4.27}$$

Adding these two equations we get a condition on the radius of the event horizon:

$$r^2_{hor} = \frac{-1 + \sqrt{1 + 12g^2|V_*|a^2 Q}}{6|V_*|} \,, \tag{4.28}$$

and plugging in back in the first of eqq. (4.27) we finally get

$$\mu_{extr} = \frac{2}{3\sqrt{6}a\sqrt{|V_*|}} (\sqrt{1 + 12|V_*|a^2 Q} + 2)(\sqrt{1 + 12|V_*|a^2 Q} - 1)^{1/2} \,. \tag{4.29}$$

the same expression (in terms of g, Z) was also obtained in [68]. For $\mu \geq \mu_{extr}$ the solution represents a black hole, otherwise a naked singularity. We remind that for the solutions obtained by [61] in minimal gauged supergravity the BPS bound lies below the extremality bound; this means that supersymmetric solutions are naked singularities. In the examples with constant scalars that follow we will see that the situation remains the same.

Finally, let us mention that the Hawking temperature of these nonextremal black holes can be computed along the same lines as [61].

4.2.1 Explicit examples

Electric solution

For this configuration the magnetic charges are set to zero, $p^0 = p^1 = 0$, meanwhile the two electric charges are unconstrained. We remind the reader that the real scalar is fixed at the value $z = \frac{3\xi_0}{\xi_1}$, so that (4.19) is satisfied. We construct a solution of the form (4.21) with $b = 0$ and $a = \frac{\sqrt{2}(\xi_0\xi_1^3)^{1/4}}{3^{3/4}}$. This cumbersome factor is somewhat convenient for the comparison of these solutions with the ones with running scalars of Section 4.3.

$$h(r) = \frac{\sqrt{2}(\xi_0\xi_1^3)^{1/4} r}{3^{3/4}}, \qquad U^2(r) = \frac{3\sqrt{3}\left(\frac{4}{27}\xi_0\xi_1^3 g^2 r^2 + 1 - \frac{\mu}{r} + \frac{Q}{r^2}\right)}{2\sqrt{\xi_0\xi_1^3}}. \tag{4.30}$$

Equation (4.22) then imposes a constraint on Q

$$Q = \frac{1}{4}\left(\frac{q_0^2}{\xi_0^2} + 3\frac{q_1^2}{\xi_1^2}\right), \tag{4.31}$$

while (4.18) gives this condition:

$$q_0 = \pm\xi_0 \frac{q_1}{\xi_1}. \tag{4.32}$$

The warp factor then reduces to:

$$U^2(r) = \frac{3\sqrt{3}\left(\frac{4}{27}\xi_0\xi_1^3 g^2 r^2 + 1 - \frac{\mu}{r} + \frac{q_1^2}{\xi_1^2 r^2}\right)}{2\sqrt{\xi_0\xi_1^3}}, \tag{4.33}$$

Solutions of this kind admit two noncoincident horizons for a suitable range of parameters, giving rise to nonextremal black holes. When we set $\mu = 0$ and we choose the positive sign in (4.32) we retrieve the supersymmetric solution with constant scalars found by [63] [3]. The solutions (4.33) asymptote to ordinary AdS_4. The mass M can be computed with the AMD formalism [100, 101], whose formulas can be found in Appendix C:

$$M = \frac{\mu}{2} \tag{4.34}$$

and results proportional to the $1/r$ coefficient μ as in the case of AdS-RN black hole. The BPS configuration is retrieved when $\mu = 2\frac{q_1}{\xi_1}$ [84], so that

$$\mu_{BPS} = 2\frac{q_1}{\xi_1}. \tag{4.35}$$

We can then compute the quantity ΔM, namely the mass gap between extremality (4.29) and BPSness (4.35):

$$\Delta\mu = \mu_{extr} - \mu_{BPS} =$$

[3] In particular the solution (4.33) with $\mu = 0$ is related through a redefinition of the coordinate r to the BPS one in [63]. This last has constant scalars but nonconstant sections, so the parameter b is nonzero. From here the need to redefine r to get exactly (4.33) with zero μ.

$$= \frac{2}{9\sqrt{2}\,a^2 g}\left(\sqrt{1+36g^2a^4\frac{q_1^2}{\xi_1^2}}+2\right)\left(\sqrt{1+36g^2a^4\frac{q_1^2}{\xi_1^2}}-1\right)^{1/2} - 2\frac{q_1}{\xi_1} =$$

$$= \frac{1}{a^2 g}\left[\frac{2}{9\sqrt{2}}\left(\sqrt{1+36Y^2}+2\right)\left(\sqrt{1+36Y^2}-1\right)^{1/2} - 2Y\right] \tag{4.36}$$

where we used $-V_* = 3a^2$ and we defined $Y = \frac{ga^2q_1}{\xi_1}$.

The mass gap $\Delta\mu$ is indeed never negative. We compute the expansion of this quantity in two interesting limits. The first case is the limit in which both $g << 1$, small coupling (large radius of AdS), and $gq_1 << 1$ (the radius of AdS much larger than the scale of the black hole). The expansion gives:

$$\Delta\mu = \frac{3g^2a^4q_1^3}{\xi_1^3} + O\left(\left(\frac{ga^2q_1}{\xi_1}\right)^5\right) \tag{4.37}$$

The other interesting limit is the one of large charges, $q_1 >> 1$:

$$\Delta\mu = \frac{4a\sqrt{g}}{\sqrt{3}}\left(\frac{q_1}{\xi_1}\right)^{3/2} + O(q_1)\,. \tag{4.38}$$

More details about the superalgebra underlying these configurations and the BPS bound arising from it will be given in Chapter 7.

Magnetic solution

In this section we will deal with purely magnetic black holes. These solutions have a constant magnetic flux at infinity; the warp factors are given by

$$U^2 = \frac{2\sqrt{\xi_0\xi_1^3}\left(1+\frac{Q}{r^2}+\frac{\mu}{r}+g^2r^2\right)}{3\sqrt{3}} \qquad h^2 = \frac{3\sqrt{3}}{2}\frac{r^2}{\sqrt{\xi_0\xi_1^3}} \tag{4.39}$$

Then (4.22) imposes this constraint

$$Q = \frac{1}{3}\left(3\xi_0^2(p^0)^2 + (\xi_1)^2(p^1)^2\right)\,, \tag{4.40}$$

while (4.18) gives the following:

$$-9\xi_0^2(p^0)^2 + \xi_1^2(p^1)^2 = 0\,, \tag{4.41}$$

We recall that the magnetic charges should satisfy the quantization condition (1.87). If we restrict ourselves to

$$g\xi_\Lambda p^\Lambda = -1\,, \tag{4.42}$$

eq. (4.41) has these two solutions:

$$\textit{solution } 1 \qquad p_*^1 = -\frac{3}{4g\xi_1}\,, \qquad p_*^0 = -\frac{1}{4g\xi_0}\,, \tag{4.43}$$

$$\textit{solution } 2 \qquad p_{**}^1 = -\frac{3}{2g\xi_1}\,, \qquad p_{**}^0 = \frac{1}{2g\xi_0}\,. \tag{4.44}$$

For both these solutions the mass is: $M = \mu/2$.

Let us examine more closely the two cases. For the case denoted by p_*^1 the warp factor is:

$$U^2{}_{p=p_*^1}(r) = \frac{2\sqrt{\xi_0\xi_1^3}}{3\sqrt{3}}\left(g^2r^2 + 1 - \frac{\mu}{r} + \frac{1}{4g^2r^2}\right)\,. \tag{4.45}$$

For $\mu = 0$ the warp factor is a perfect square, and supersymmetric solutions of this type include the 1/4 BPS magnetic solution found in [84]. These BPS solutions are singular.

In the second case, p_{**}^1, instead, the warp factor is

$$U^2{}_{p=p_{**}^1}(r) = \frac{2\sqrt{\xi_0\xi_1^3}}{3\sqrt{3}}\left(g^2r^2 + 1 - \frac{\mu}{r} + \frac{1}{g^2r^2}\right) \tag{4.46}$$

and when we set $\mu = 0$ it also gives a naked singularity. Furthermore, the warp factor in this case it is not a perfect square. By studying the supersymmetry variations one can see that this solution is not supersymmetric.

At this point we can repeat here the procedure used in section to compute the mass gap $\Delta\mu$ between extremality and BPSness. Notice that in this case $\mu_{BPS} = 0$ [84]. We take into consideration just the solution (4.43), for which $Q = \frac{1}{4g^2}$ and $-V_* = 3/a^2$. Then, from (4.29):

$$\begin{aligned}\Delta\mu = \mu_{extr} - \mu_{BPS} &= \frac{\sqrt{2}}{9g}(\sqrt{1+36g^2Q}+2)(\sqrt{1+36g^2Q}-1)^{1/2} = \\ &= \frac{\sqrt{2}}{9g}(\sqrt{10}+2)(\sqrt{10}-1)^{1/2}\,,\end{aligned} \tag{4.47}$$

that is always non-negative. Also in this case the extremality bound lies above the BPS bound, and the mass gap in inversely proportional to the gauge coupling constant.

4.3 Electric black holes with nonconstant scalars

This section gives a detailed description of the application of the minimal modification ansatz of [97] to the purely electric solutions described in Chapter 2, Section

2.3. Genuine nonextremal black holes were already found in [64], and represent a subset of the solutions found here.

In order to find nonextremal solutions, we make the modification to the ansatz for the warp factor:

$$U^2(r) = e^{\mathcal{K}}(1 - \frac{\mu}{r} + g^2 r^2 e^{-2\mathcal{K}})\,, \tag{4.48}$$

keeping

$$h(r) = e^{-\mathcal{K}/2} r\,. \tag{4.49}$$

where $\mathcal{K}$ is the Kähler potential computed as (2.24). With sections as harmonic functions:

$$\mathrm{Im} X^\Lambda = 0\,, \qquad 2\mathrm{Im} F_\Lambda = H_\Lambda = \xi_\Lambda + \frac{b_\Lambda}{r}\,. \tag{4.50}$$

At this moment a_Λ and b_Λ are unconstrained. Notice that we retrieve the BPS solution of 2.3 by choosing $a_\Lambda = \xi_\Lambda$ and $b_\Lambda = q_\Lambda$.

We illustrate the explicit example of prepotential $F = -2i\sqrt{X^0(X^1)^3}$; the holomorphic sections then are

$$X_0 = \frac{1}{6\sqrt{3}}\sqrt{\frac{H_1^3}{H_0}}\,, \qquad X_1 = \frac{1}{2\sqrt{3}}\sqrt{H_0 H_1}\,, \tag{4.51}$$

so that

$$e^{-\mathcal{K}} = \frac{2}{3\sqrt{3}}\sqrt{H_0 H_1^3}\,. \tag{4.52}$$

The solution is purely electric, with field strengths of the form (the sections are real, so by construction $R_{\Lambda\Sigma} = 0$)

$$F^\Lambda_{tr} = \frac{I^{\Lambda\Sigma} q_\Sigma}{h^2(r)}\,, \tag{4.53}$$

while the magnetic charges are set to zero $p^0 = p^1 = 0$. The relevant components of Einstein's equations to be satisfied are (4.14), (4.15), and (4.16) computed on the Special Kähler quantities given in appendix for the prepotential $F = -2i\sqrt{X^0(X^1)^3}$. We choose to solve first the three Einstein's equations: for the configurations with one running scalar, this is enough since the scalar equation of motion does not give further constraints.

The first equation (4.14) is automatically satisfied without imposing any other constraints on the sections, while the second one (4.15) imposes $a_\Lambda = \xi_\Lambda$. The remaining equation (4.16), gives the two independent constraints:

$$(b_1^2\xi_0^2 - b_0^2\xi_1^2)(b_0^2 + b_0\xi_0\mu - q_0^2) = 0\,, \tag{4.54}$$

$$3(b_1^2\xi_0^2 - b_0^2\xi_1^2)(b_1^2 + b_1\xi_1\mu - q_1^2) = 0\,. \tag{4.55}$$

At this point we have two choices: if impose that $(b_0^2 + b_0\xi_0\mu - q_0^2) = 0$ and $(b_1^2 + b_1\xi_1\mu - q_1^2) = 0$, parameterizing q_0 and q_1 as

$$q_0 = \xi_0\mu \cosh Q_0 \sinh Q_0 \,, \tag{4.56}$$

$$q_1 = \xi_1\mu \cosh Q_1 \sinh Q_1 \,, \tag{4.57}$$

with Q_1 and Q_0 real, we obtain two sets of solutions for b_0, b_1. The first set is

$$b_1 = \xi_1\mu \sinh^2 Q_1 \qquad b_0 = \xi_0\mu \sinh^2 Q_0 \,. \tag{4.58}$$

This family of solutions was found in the literature by [64], as already mentioned. It corresponds to genuine nonextremal black hole for a choice of parameters. The singularities are located at the points in which $e^{\mathcal{K}}$ blows up: $r_{s,1} = -\mu \sinh^2 Q_1$ and $r_{s,2} = -\mu \sinh^2 Q_0$, the horizons instead are at the real zeroes of (4.48). We can retrieve the BPS solution of [64] by taking $\mu \to 0$ and $Q_\Lambda \to \infty$, in such a way that the parameters q_Λ, b_Λ are finite and respectively equal to each other.

The second set of solutions, keeping the parameterization (4.56) (4.57), is this:

$$b_1 = -\xi_1\mu \cosh^2 Q_1 \,, \qquad b_0 = -\xi_0\mu \cosh^2 Q_0 \,, \tag{4.59}$$

and seems to present always naked singularities. The BPS solution is retrieved if the take the limit $\mu \to 0$ and $Q_\Lambda \to -\infty$.

We have a second choice when we insist in the vanishing of $(b_1^2\xi_0^2 - b_0^2\xi_1^2)$, namely the first term in the product of (4.54), (4.55):

$$b_1 = \pm \frac{b_0\xi_1}{\xi_0} \,. \tag{4.60}$$

Let us just mention briefly what happens to the solutions: when we choose the $+$ sign we have a solution with constant scalars. The equations of motions are satisfied if we impose these other two constraints:

$$q_0 = \pm \frac{\xi_0 q_1}{\xi_1} \,, \qquad \mu = \frac{\xi_0^2 q_1^2 - b_0^2 \xi_1^2}{b_0\xi_0\xi_1^2} \,. \tag{4.61}$$

The solution found represents genuine black hole for a suitable choice of parameters. Setting $\mu = 0$ in the case of the $+$ sign we retrieve the BPS solution of [63] with constant scalars. This solution, in the notation of (4.17) has $b \neq 0$, nevertheless the scalar is constant due to the fact that sections are proportional to each other.

If we instead choose the minus sign in (4.60) we get new constraints but all the solutions seem to have naked singularities and so we omit them.

Mass of the electric configurations

We are ready now to compute the mass of the electric nonextremal black hole configurations by means of the Ashtekar-Magnon-Das (AMD) techniques. By making use of formulas (C.1) and (C.3) of Appendix C, we have

$$M = \frac{1}{4}\left(2\mu + \frac{b_0}{\xi_0} + 3\frac{b_1}{\xi_1}\right) . \tag{4.62}$$

In the particular case of the solution (4.58), it gives

$$M = \frac{\mu(2 + 3\sinh^2 Q_1 + \sinh^2 Q_0)}{4} . \tag{4.63}$$

4.4 Magnetic black holes with nonconstant scalars

We are interested here in the nonextremal generalization of the supersymmetric solution of Cacciatori and Klemm [30] that we described in Chapter 2, Section 2.4.3. We work in the model with prepotential $F = -2i\sqrt{X^0(X^1)^3}$. We set the electric charges to zero,

$$q_0 = q_1 = 0 , \tag{4.64}$$

since we are interested in a purely magnetic configuration, in analogy to [30]. Like in the BPS case, the scalar is truncated to be purely real (no axions). $R_{\Lambda\Sigma}$ vanishes in this case (see formulas in Appendix A), so then

$$F^0_{rt} = F^1_{rt} = 0 . \tag{4.65}$$

The magnetic field strengths are of the form

$$F^\Lambda_{\theta\phi} = \frac{p^\Lambda}{2}\sin\theta . \tag{4.66}$$

For the moment we keep the charges unconstrained, knowing that at the end of the day they need to satisfy the Dirac quantization condition (1.87).

The static and spherically symmetric metric is of the type (4.6). We choose an ansatz of the form

$$U^2(r) = e^{\mathcal{K}}\left(g^2 r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2}\right) , \tag{4.67}$$

such that the warp factor could in principle have two real noncoincident roots. These correspond to the inner and outer horizon of the configuration. We keep the other warp factor $h(r)$ unchanged with respect to the BPS case:

$$h(r) = r e^{-\mathcal{K}/2} . \tag{4.68}$$

$\mathcal{K}$ is given as usual by (2.24). The scalar z is real and positive and we keep the same sections as in the BPS case,

$$z = \frac{X_1}{X_0} \qquad X_0 = a_0 + \frac{b_0}{r}, \qquad X_1 = a_1 + \frac{b_1}{r}. \tag{4.69}$$

This guess for the form of the nonextremal solution is then followed by brute-force solving the Einstein's equations of motion. Indeed the first equation (4.14) is automatically satisfied given the form of the sections. For (4.15) to be satisfied, instead, we have to impose the same relations (4.70) as for the BPS case,

$$a_0 = \frac{1}{4\xi_0}, \qquad b_0 = -\frac{\xi_1 b_1}{\xi_0}, \qquad a_1 = \frac{3}{4\xi_1}, \qquad c = 1 - \frac{32}{3}(g\xi_1 b_1)^2, \tag{4.70}$$

meanwhile solving the last equation (4.16) requires [96, 99]:

$$Q = \xi_0^2 (p^0)^2 + \frac{1}{3}\xi_1^2 (p^1)^2 - \frac{16}{3} b_1^2 \xi_1^2 - \frac{256}{27} b_1^4 g^2 \xi_1^4, \tag{4.71}$$

and

$$b_1 \mu = \frac{8}{3} b_1^2 \xi_1 + \frac{512}{27} b_1^4 g^2 \xi_1^3 - \frac{3\xi_0^2 (p^0)^2}{4\xi_1} + \frac{\xi_1 (p^1)^2}{12}, \tag{4.72}$$

The scalar equation of motion does not impose further constraints. We recall furthermore that the charges p^Λ need to satisfy the condition (1.87).

We can now make connection with the constant scalar solutions of section 4.2.1. Indeed if we impose $b_1 = 0$ the sections become constant. Then the last equation leaves μ unconstrained, as we found in Section 4.2.1, provided that we impose the values (4.43) (4.44) for the magnetic charges, once we impose the quantization condition $g\xi_\Lambda p^\Lambda = -1$. This is remnant of the further constraint that one has to solve in the case of constant scalars.

Furthermore, to obtain the extremal BPS solution described in Section 2.4.3, we need to impose the following constraints (2.48) between the charges and the scalar parameter b_1:

$$p^0 = \pm \frac{2}{g\xi_0}\left(\frac{1}{8} + \frac{8(g\xi_1 b_1)^2}{3}\right), \qquad p^1 = \pm \frac{2}{g\xi_1}\left(\frac{3}{8} - \frac{8(g\xi_1 b_1)^2}{3}\right), \tag{4.73}$$

which gives $g\xi_\Lambda p^\Lambda = \pm 1$.

To sum up, our solution depends on the FI parameters ξ_1, ξ_0 and on other three unconstrained parameters: b_1, p^1 and p^0 (alternatively, one of the two charges is fixed in terms of the other one and the integer n appearing in (1.87)). In contrast, the BPS solution of [70] depends just on the FI parameters and on p^1: the nonextremal solution with the same quantization condition $g\xi_\Lambda p^\Lambda = \pm 1$ has one parameter more, b_1, that is considered for our purposes the non-extremality parameter.

We can say then that we have found a nonextremal generalization of the magnetic solution found in [70], that by construction at extremality remains regular (no naked singularities). Expressing the non-extremality in terms of μ seems more natural and would in principle be possible inverting the relation (4.72) in order to obtain b_1 in function of μ.

For completeness we give the value of the prefactor $e^{\mathcal{K}}$

$$e^{\mathcal{K}} = \frac{1}{8\sqrt{\left(\frac{3}{4\xi_1}+\frac{b_1}{r}\right)^3\left(\frac{1}{4\xi_0}-\frac{b_1\xi_1}{\xi_0 r}\right)}} \tag{4.74}$$

so, together with (4.70), (4.71) and (4.72) one has all the necessary values in order to retrieve the full warp factor $U^2(r)$. We omit the full lengthy writing of it, nevertheless it is clear that the singularities lie at the zeros of formula (4.74)

$$r_{sing,1} = 4\xi_1 b_1\,, \qquad r_{sing,2} = -\frac{4}{3}\xi_1 b_1\,. \tag{4.75}$$

These nonextremal black holes share with the BPS solution described in the previous section the feature of having a singularity at finite nonzero r. Still, to have a proper black hole the singularity must be shielded by a horizon, namely $r_{sing} < r_{hor}$. However, reading off the coordinate of the horizon is quite cumbersome, since we should solve explicitly a quartic order equation with the nasty coefficients (4.71), (4.72). Nevertheless, we have explicitly checked that for some values of parameters the black hole has a singularity surrounded by two noncoincident horizons. In Chapter 6 we will present a plot of the full range in which the solution is a genuine black hole, in the particular case in which $g_\Lambda p^\Lambda = -1$.

The same minimal modification ansatz works also in the case of more general horizons. In particular, we can generate black brane configurations. One example of nonextremal magnetic black brane obtained by means of this procedure can be found in [99].

Mass of the magnetic configurations

The mass of the configuration can again be computed by means of the AMD techniques. In this case the formulas (C.1) (C.3) of Appendix C, computed on the magnetic solution (4.71), (4.72) found in the previous subsection, give

$$M_{AMD} = \frac{4b_1\xi_1}{3} - \frac{3\xi_0^2(p^0)^2}{8b_1\xi_1} + \frac{\xi_1(p^1)^2}{24b_1}\,, \tag{4.76}$$

which in the BPS case reduces to

$$M_{ADM,BPS} = \pm\frac{256}{27}\xi_1^3 b_1^3 g^2\,, \tag{4.77}$$

respectively for the two branches $g\xi_\Lambda p^\Lambda = \pm 1$.

This expression for the mass satisfies the first law of thermodynamics, as we will explain in Chapter 6. Nevertheless, some subtleties arise when we try to compute the BPS bound satisfied by these configurations. We postpone this discussion to Chapter 7.

Comment

The ansatz (4.67) of the metric was chosen in order to make completely manifest the relation with the BPS solution. Indeed, the BPS warp factor (2.41) contains a perfect square, hence the configuration possess two coincident horizons and it is extremal. The expression (4.67), instead, contains, other than the term with the exponential of the Kähler potential, a polynomial of fourth degree in the variable r. This was required to possess at least two noncoincident real roots, corresponding to the internal and external horizons.

We stress once more that this corresponds to a minimal deformation of the BPS solution that allows for a tunable parameter, namely the temperature of the black hole. By no means it is proven that it is most general solution to the Einstein's equations (4.14), (4.15), (4.16). Indeed, while the Maxwell's equations can be directly integrated and solutions (4.7) are the most general for this kind of model with uncharged scalars, we have chosen a very specific form for sections and warp factors. The same holds for the ansatz chosen by in [64].

The fact that this minimal modification procedure works is remarkable. It is compatible with the form of the sections (4.69), and it could be that it needs to be modified when the sections are not just given by harmonic functions. It does not seem to us particularly straightforward to give a reason why the ansatz works just by analysing the Einstein's equations (in particular, (4.16) looks daunting and the analysis of the full system of equations is not particularly illuminating). Of course, solutions of more general form that reduce to ours in certain limits might exist, but we do not treat them here.

Another comment is in order here: it turns out that at least a subset of the magnetic solutions we have found in Section 4.4 could also be obtained by duality rotations from the electric solutions found by Duff and Liu that we described in section 4.3.

Indeed the potential appearing in the Lagrangian (4.1), once we have imposed the reality of the scalar field, is invariant under the exchange

$$z \to \frac{9\xi_0^2}{\xi_1^2}\frac{1}{z}\,. \tag{4.78}$$

The transformation (4.78), supplemented by the corresponding symplectic rotation $\mathcal{T}$

on the charges (the kinetic matrix transforming as (1.36)):

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & \frac{\xi_1^2 \upsilon}{3\xi_0^2} & 0 \\ 0 & 0 & 0 & \upsilon \\ -\frac{3\xi_0^2}{\xi_1^2 \upsilon} & 0 & 0 & 0 \\ 0 & -\frac{1}{\upsilon} & 0 & 0 \end{pmatrix}, \qquad \upsilon = \frac{2\xi_1}{9\xi_0}, \tag{4.79}$$

is a symmetry of the bosonic equations of motion, and turns the solution into a magnetic one, *up to a coordinate shift*. The details about the rotation, applied to the specific example of Section 4.4 can be found in Appendix D.

4.5 Product of the areas

With the many nonextremal solutions at hand we can check a conjecture about the area product formula of black hole horizons [107]. Indeed it seems true in a lot of examples [108–110] that the product of the areas of the inner and outer horizons of a black hole is just function of the quantized charges (and, if present, of the cosmological constant), in particular it is independent of the mass. Such a product area law might be a calling for an underlying microscopic interpretation in string theory or conformal field theory.

As discussed in [108], for AdS_4 the previous statement holds if we take into consideration the product over the four areas A_α, $\alpha = 1, .., 4$, correspondent to the four roots of the warp factor. Indeed there may be zeroes for complex values of r, but they come in complex conjugate pairs: the product of all the areas then remains real. A physical meaning of this is still not clear; nevertheless we check this for our solutions, at least for comparison with the other existing black holes.

We deal with constant scalar solutions first. For these solutions (section 4.2) the warp factor (4.21) ($b = 0$) can be decomposed in this way:

$$r^2 U^2(r) = \frac{|V_{g*}|}{3r^2} \prod_{\alpha=1}^{4} (r - r_\alpha) \,. \tag{4.80}$$

The product of the four roots $r_1 r_2 r_3 r_4$, then, is the coefficient of lower degree in r, namely is proportional to the quantity denoted with Q in (4.21).

For the electric black hole, then, using (4.31) together with (4.32), we have:

$$\prod_{\alpha=1}^{4} A_\alpha = (4\pi)^4 \prod_{\alpha=1}^{4} r_\alpha^2 = (4\pi)^4 \frac{q_1^4}{(g\xi_1)^4} = (4\pi)^4 \frac{q_1^4}{g_1^4} \,. \tag{4.81}$$

We recall that the quantities $g_\Lambda = g\xi_\Lambda$ are the electric charges of the gravitinos.

Fot the magnetic black hole, instead, from (4.40) and (4.43), the product reads

$$\prod_{\alpha=1}^{4} A_\alpha = (4\pi)^4 \prod_{\alpha=1}^{4} r_\alpha^2 = (4\pi)^4 \frac{3^6}{2^8 (g_0 g_1^3)^2} = (4\pi)^4 \frac{27 p^0 (p^1)^3}{g_0 g_1^3}\,, \tag{4.82}$$

For both constant scalar cases then the product does not depend on the mass parameter μ.

Things get more involved when one takes into account running scalars[4]. In both cases (for the electric black holes we will refer in particular to formulas (4.56), (4.57) (4.58)) the product of the areas is:

$$\prod_{\alpha=1}^{4} A_\alpha = (4\pi)^4 \prod_{\alpha=1}^{4} e^{-\mathcal{K}(r_\alpha)} r_\alpha^2\,, \tag{4.83}$$

where the function $h^2(r)$ is of the form

$$h^2(r) = \text{const} \times \sqrt{(r - r_{s,1})(r - r_{s,2})^3}\,. \tag{4.84}$$

In particular $r_{s,1/2}$ are the two values are where the singularities are located. We now rewrite the warp factor as:

$$U^2(r) = \text{const2} \times \frac{e^{\mathcal{K}}}{r^2} \prod_{\alpha=1}^{4} (r - r_\alpha)\,. \tag{4.85}$$

for instance, in the magnetic case we have:

$$U^2(r) = e^{\mathcal{K}} \left(g^2 r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2} \right) = \frac{e^{\mathcal{K}} g^2}{r^2} \left(r^4 + \frac{c r^2}{g^2} - \frac{\mu r}{g^2} + \frac{Q}{g^2} \right) = \frac{e^{\mathcal{K}} g^2}{r^2} \prod_{\alpha=1}^{4} (r - r_\alpha)\,. \tag{4.86}$$

The coefficient of lowest degree in r, namely $\frac{Q}{g^2}$, gives the value the product of all the roots $r_1 r_2 r_3 r_4$. We now first make this redefinition:

$$r' = r - r_{s,1}\,, \tag{4.87}$$

and we express the warp factor in terms of r'. In a similar way the coefficient of lowest degree in r', from now on denoted by κ_1, represents the product of all the r' roots: $r'_1 r'_2 r'_3 r'_4$. For instance, in the magnetic case this coefficient turns of to be:

$$\kappa_1 = r_{s,1}^4 + \frac{c r_{s,1}^2}{g^2} + \frac{\mu r_{s,1}}{g^2} + \frac{Q}{g^2}\,. \tag{4.88}$$

[4]We first assume value of the parameters such that the singularity is shielded by two horizons.

where the values of Q and μ are given respectively in (4.72) (4.71). This factor κ_1 appears in (4.84) and consequently in (4.83). Repeating the procedure gives κ_2, so that we have what we need to compute the area product:

$$\prod_{\alpha=1}^{4} A_\alpha = (\text{const})^4 (4\pi)^4 \sqrt{\kappa_1 \kappa_2^3}\,. \tag{4.89}$$

Once we have this explicit formula, for the electric solution we get

$$\prod_{\alpha=1}^{4} A_\alpha = \prod_{\alpha=1}^{4} e^{-\mathcal{K}(r_\alpha)} r_\alpha^2 = (4\pi)^4 \frac{q_0 q_1^3}{g_0 g_1^3}\,. \tag{4.90}$$

We see then that the result depends just on the electric charges present in the solution and on the gauging charges g_Λ. This result agrees with (the static limit of) the one of [108].

For the magnetic solution:

$$\prod_{\alpha=1}^{4} A_\alpha = \frac{27(4\pi)^4}{g_0 g_1^3} (p^0)(p^1)^3\,. \tag{4.91}$$

Also this product is function solely of the magnetic charges p^Λ and on g_Λ.

A few comments are in order here: first of all, we can see that the formula obtained for constant scalars solutions can be derived as particular cases of the ones of nonconstant scalars; this does not come as a surprise. Secondly, the dependence on the charges resembles the form of the prepotential of the model, but we could not find any direct explanation of this. Finally we rewrite more suggestively the formulas in the electric and magnetic case in terms of the cosmological constant $\Lambda = -3V_{g*}$ and the charges:

$$\prod_{\alpha=1}^{4} A_{\alpha,electric} = (4\pi)^4 \frac{12 q_0 q_1^3}{\Lambda^2}\,, \qquad \prod_{\alpha=1}^{4} A_{\alpha,magnetic} = \frac{3^4 2^2 (4\pi)^4}{\Lambda^2} (p^0)(p^1)^3\,. \tag{4.92}$$

Discussion

We have discussed here examples of nonextremal solutions that reduce for a certain choice of parameters to the BPS ones described in Chapter 2. Just the magnetic ones are regular in the BPS limit. Chapter (6) is devoted to the analysis of the thermodynamics of these solutions and some aspects in relation to the AdS/CFT correspondence are treated there.

Regarding more applicative sides of AdS/CFT, let us mention that the solutions we described here have been used as thermal background in [111], where the existence of multiple center black holes in Anti-de Sitter in the probe approximation was investigated. Thermodynamics and phase transitions of such configurations are believed to model the glassy phase transition in the dual field theory [111].

Moreover, thermal solutions in presence of charged scalars are useful and interesting for modelling holographic superconductors. In this case one could try and generalize the BPS solutions obtained by [60] in presence of nontrivial hypermultiplet fields with the procedure we described in this chapter. Models of superconductors were obtained from maximal gauged supergravity already by Bobev, Pilch and Warner [112]. Their black holes with charged scalars were found numerically.

Coming back to the solution generating technique, we have seen that it is possible to solve the system of Einstein's-Maxwell-scalar equations of motion analytically. This involves second order differential equations. The possibility of achieving a squaring of the action, and consequently a first order flow for nonextremal AdS configurations is treated in Chapter 8. The squaring can be achieved in the case of five-dimensional AdS nonextremal black holes [113]. We refer to Chapter 8 for details about the squaring of the action for four-dimensional configurations.

Let us mention that dyonic solutions also exist. An example of dyonic solution is derived in Appendix B, where we perform a symplectic rotation of the magnetic solutions. The dyonic black hole is a solutions to the theory with magnetic gaugings. One can also rotate the solutions via duality, performing the same procedure applied in the BPS case by [71]. The goal in this case would be to obtain dyonic solutions in theories with purely electric gaugings. Existence of these this kind of solutions is under investigation. Finally, let us mention that a branch of dyonic solutions of the theory with electric gaugings were recently found in [114], but these solutions present some peculiarities, in particular they do not satisfy the standard first law of thermodynamics, but a modified one[114].

Hairy uncharged nonextremal solutions of the models treated in this chapter were found by Hertog and Maeda [115]. We will spend a few words on them in the Chapter devoted to the phase transitions. Other uncharged nonextremal solutions with scalar hair were found in [116], and recently also in [117] where a theory of maximal gauged supergravity with a ω-deformation [118] was considered.

Chapter 5

Rotating AdS solutions

In this chapter we describe the results found in [28] and also in [29] towards the construction of more general AdS black holes solution of FI abelian gauged supergravity in absence of hypermultiplets. In particular, such configurations are characterized by nonvanishing angular momentum, in addition to electric and magnetic charges, mass and NUT charge.

The analysis of rotating supersymmetric solutions in minimal $\mathcal{N} = 2$ gauged supergravity was performed in [68, 69, 119]. It was noticed that rotating configurations, opposite to static ones, can be supersymmetric and extremal (nonzero area of the event horizon) even in absence of scalars. Rotating electric solutions with scalar fields were subsequently found in [120–122], and include both extremal and thermal solutions with angular momentum, and in the BPS limit the solutions are necessarily rotating.

For magnetic configurations the situation is different. So far no magnetic BPS black hole with angular momentum was discovered. Namely, there is no rotating generalization (extremal or nonextremal) of the spherical static Cacciatori-Klemm [30] solution. Only one magnetic BPS rotating solution with hyperbolic event horizon is known [123].

The aim of this chapter is to fill this gap at least in part. We will manage to construct a rotating magnetic configuration with nonconstant scalar field with spherical horizon topology in the simple model with prepotential $F = -iX^0X^1$.

This chapter is structured as follows. Since we are interested in a systematic approach to construct black hole solutions with every prepotential, we first investigate on the ansatz for the metric of the rotating configurations. We find that all known Anti-de Sitter rotating solutions found so far [68, 69, 120–125] can be cast in a universal form, that we will subsequently take as the starting point for the construction of the solution. We anticipate here that the rotating configuration will be supported by complex scalar fields with a nontrivial scalar profile in the radial and angular (θ) components.

The general ansatz accommodates also for the presence of nonvanishing electric and NUT charge. This allows us to find a large parameter space of black hole solutions in one of the possible models (with prepotential $F = -iX^0X^1$) and gives strong hints on how to tackle the same problem with more complicated scalar manifolds.

We compute the thermodynamic quantities of these new configurations, in particular we calculate the mass by means of the ADM procedure. The study of the thermodynamics and the phase transitions of this new class of black holes is work in progress and some details will be shown in Chapter 6, Section 6.6.

Lastly, at the end of the chapter we also show how the new solutions can be written in terms of harmonic functions and special geometry quantities and how in the limit of vanishing gauging a class of known solutions to ungauged supergravity [126] is recovered. Finally, a proposal for the general ansatz suitable for different and more complicated prepotentials is given.

5.1 Universal structure of rotating black holes

This section is devoted to showing that all known rotating black holes in matter-coupled $\mathcal{N} = 2$ gauged supergravity in four dimensions have a universal metric structure. It turns out that in all cases the metric can be cast in the form

$$ds^2 = f(dt + \omega_y dy)^2 - f^{-1}\left[v\left(\frac{dq^2}{Q} + \frac{dp^2}{P}\right) + PQdy^2\right], \tag{5.1}$$

where $Q(q)$ and $P(p)$ are polynomials of fourth degree respectively in the variables q (radial variable) and p (function of the angular variable θ). The warp factors f, v, ω_y are more general functions of q and p. We start now with the examples and finish the section by commenting on some novel general features of this metric, such as the difference between over- and under-rotating solutions and the relation between the function $P(p)$ and the choice of horizon topology.

5.1.1 Carter-Plebański solution

The metric and U(1) field strength of the Carter-Plebański solution [124] [125] of minimal gauged supergravity are respectively given by

$$ds^2 = \frac{Q(q)}{p^2+q^2}(d\tau - p^2 d\sigma)^2 - \frac{p^2+q^2}{Q(q)}dq^2 - \frac{p^2+q^2}{P(p)}dp^2 - \frac{P(p)}{p^2+q^2}(d\tau + q^2 d\sigma)^2, \tag{5.2}$$

$$F = \frac{\mathrm{Q}(p^2-q^2)+2\mathrm{P}pq}{(p^2+q^2)^2}\mathrm{d}q\wedge(\mathrm{d}\tau - p^2\mathrm{d}\sigma) + \frac{\mathrm{P}(p^2-q^2)-2\mathrm{Q}pq}{(p^2+q^2)^2}\mathrm{d}p\wedge(\mathrm{d}\tau + q^2\mathrm{d}\sigma), \tag{5.3}$$

where the quartic structure functions read

$$\begin{aligned} P(p) &= \alpha - \mathrm{P}^2 + 2np - \varepsilon p^2 + (-\Lambda/3)p^4\,, \\ Q(q) &= \alpha + \mathrm{Q}^2 - 2mq + \varepsilon q^2 + (-\Lambda/3)q^4\,. \end{aligned} \tag{5.4}$$

Here, Q, P and n denote the electric, magnetic and NUT-charge respectively, m is the mass parameter, while α and ε are additional non-dynamical constants.

By making the coordinate transformation

$$\tau = At + By\,, \qquad \sigma = Ct + Dy\,, \qquad AD - BC = 1\,, \tag{5.5}$$

(5.2) can be cast into the form (5.1), where

$$v = Q(A - p^2C)^2 - P(A + q^2C)^2\,, \qquad f = \frac{v}{p^2 + q^2}\,, \tag{5.6}$$

and

$$\omega_y = \frac{1}{v}\left[Q(A - p^2C)(B - p^2D) - P(A + q^2C)(B + q^2D)\right]. \tag{5.7}$$

We see that there is actually more than one way to write (5.2) as a fibration (5.1) over a three-dimensional base space. A simple choice would be for instance $A = D = 1$, $B = C = 0$, such that

$$v = Q - P\,, \qquad \omega_y = \frac{Qp^2 + Pq^2}{P - Q}\,.$$

5.1.2 Rotating magnetic BPS black holes, prepotential $F = -iX^0X^1$

Our second example is the family of BPS magnetic rotating black holes in the model with prepotential $F = -iX^0X^1$, constructed in [123].

This model has just one complex scalar τ. The symplectic sections in special coordinates are $v^T = (1, \tau, -i\tau, -i)$. The Kähler potential, metric and vector kinetic matrix are respectively of this form:

$$e^{-\mathcal{K}} = 2(\tau + \bar\tau)\ , \qquad g_{\tau\bar\tau} = \partial_\tau\partial_{\bar\tau}\mathcal{K} = (\tau + \bar\tau)^{-2}\ , \tag{5.8}$$

$$\mathcal{N} = \begin{pmatrix} -i\tau & 0 \\ 0 & -\frac{i}{\tau} \end{pmatrix}\ , \tag{5.9}$$

thus requiring $\mathrm{Re}\tau > 0$. For our choice of electric gauging, the scalar potential is

$$V = -\frac{4}{\tau + \bar\tau}(g_0^2 + 2g_0g_1\tau + 2g_0g_1\bar\tau + g_1^2\tau\bar\tau)\ , \tag{5.10}$$

which has an extremum at $\tau = \bar\tau = |g_0/g_1|$.

The metric of the BPS solution of [123] reads

$$\begin{aligned} ds^2 = \quad & -\frac{p^2 + q^2 - \Delta^2}{P}dp^2 - \frac{P}{p^2 + q^2 - \Delta^2}\left(dt + (q^2 - \Delta^2)dy\right)^2 \\ & -\frac{p^2 + q^2 - \Delta^2}{Q}dq^2 + \frac{Q}{p^2 + q^2 - \Delta^2}\left(dt - p^2dy\right)^2\ , \end{aligned} \tag{5.11}$$

with the structure functions

$$P = (1+A)\frac{\mathrm{E}^2 l^2}{4} - \mathrm{E}p^2 + \frac{p^4}{l^2}\,, \qquad Q = \frac{1}{l^2}\left(q^2 + \frac{\mathrm{E}l^2}{2} - \Delta^2\right)^2\,. \tag{5.12}$$

The upper parts of the (nonholomorphic) symplectic section (L^Λ, M_Λ) and the U(1) gauge potentials are given by

$$L^0 = \frac{1}{2}\left(\frac{g_1}{g_0}\right)^{\frac{1}{2}}\left(\frac{p^2+(q-\Delta)^2}{p^2+q^2-\Delta^2}\right)^{\frac{1}{2}}\,, \quad L^1 = \frac{1}{2}\left(\frac{g_0}{g_1}\right)^{\frac{1}{2}}\frac{p^2+q^2-\Delta^2+2ip\Delta}{\left[(p^2+q^2-\Delta^2)(p^2+(q-\Delta)^2)\right]^{\frac{1}{2}}}\,,$$

$$A^\Lambda = \frac{\mathrm{E}p\,\sqrt{-A}}{4g_\Lambda(p^2+q^2-\Delta^2)}(dt+(q^2-\Delta^2)dy)\,, \qquad \Lambda = 0,1\,.$$

The solution is thus specified by three free parameters A, E, Δ. (The asymptotic AdS curvature radius l is related to the gauge coupling constants by $l^{-2} = 4g_0g_1$). The new rotating solution that we are going to describe in section 5.2 is a nonextremal deformation of this solution. For $\Delta = 0$, the moduli are constant, and the solution reduces to a subclass of (5.2), (5.3).

The metric (5.11) can again be written in the form (5.1), where now

$$v = Q - P\,, \qquad f = \frac{v}{p^2+q^2-\Delta^2}\,, \tag{5.13}$$

$$\omega_y = \frac{P(q^2-\Delta^2)+Qp^2}{P-Q}\,. \tag{5.14}$$

5.1.3 Rotating black holes of Chong, Cvetič, Lu, Pope and Chow

Finally, there are three other examples of rotating black hole solutions, described in [120–122]. They all fit in the form of the metric (5.1). We report the details of the rotating black holes with two pair-wise equal charges in SO(4)-gauged $\mathcal{N} = 4$ supergravity constructed in [120], since they are the most relevant for the new configurations described in section 5.2. The metric, dilaton, axion and gauge fields read respectively

$$
\begin{aligned}
ds^2 &= \frac{\Delta_r}{W}(dt - a\sin^2\theta d\phi)^2 - W\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) \qquad (5.15)\\
&- \frac{\Delta_\theta \sin^2\theta}{W}\Big[a dt - (r_1 r_2 + a^2)d\phi\Big]^2\,, \\
e^{\varphi_1} &= \frac{r_1^2 + a^2\cos^2\theta}{W} = 1 + \frac{r_1(r_1 - r_2)}{W}\,, \qquad \chi_1 = \frac{a(r_2 - r_1)\cos\theta}{r_1^2 + a^2\cos^2\theta}\,, \\
A^1 &= -\frac{2\sqrt{2}ms_1c_1\left[a dt - (r_1r_2 + a^2)d\phi\right]\cos\theta}{W}\,, \\
A^2 &= -\frac{2\sqrt{2}ms_2c_2r_1(dt - a\sin^2\theta d\phi)}{W}\,, \qquad (5.16)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_r &= r^2 + a^2 - 2mr + g^2 r_1 r_2(r_1 r_2 + a^2)\,, \qquad \Delta_\theta = 1 - a^2 g^2\cos^2\theta\,, \qquad (5.17)\\
W &= r_1 r_2 + a^2\cos^2\theta\,, \qquad r_I = r + 2ms_I^2\,, \qquad s_I = \sinh\delta_I\,, \qquad c_I = \cosh\delta_I\,.
\end{aligned}
$$

Notice that the other scalar fields $\varphi_2, \varphi_3, \chi_2, \chi_3$ are set to zero in the truncation of [120]. Also, the two electromagnetic charges of the solution are carried by fields in U(1) subgroups of the two SU(2) factors in SO(4) $\sim$ SU(2) $\times$ SU(2). In the case $\delta_1 = \delta_2$, the dilaton φ_1 and the axion χ_1 vanish. Then, the solution boils down to the Kerr-Newman-AdS geometry with purely electric charge if A^1 is dualized, or to purely magnetic KNAdS if we dualize A^2. Note also that, after dualizing A^1, the model considered in [120] (their Lagrangian (52)) can be embedded into $\mathcal{N} = 2$ gauged supergravity as well, by choosing the prepotential $F = -iX^0X^1$ [127].

After the rescaling $\phi \to ay$ and the redefinition $q = r$, $p = a\cos\theta$, the metric in (5.16) can again be cast into the form (5.1), with (quartic) structure functions[1]

$$
P = (1 - g^2p^2)(a^2 - p^2)\,, \qquad Q = \Delta_r\,, \qquad (5.18)
$$

and

$$
v = Q - P\,, \qquad f = \frac{v}{W}\,, \qquad (5.19)
$$

$$
\omega_y = \frac{Q(p^2 - a^2) + P(r_1 r_2 + a^2)}{v}\,. \qquad (5.20)
$$

As mentioned before, the solutions of [121, 122] can be recast in the form (5.1) too. The reader can find the complete form of the solution in the original papers,

[1] It should be emphasized that, like in the case of the Carter-Plebański solution, also here and in the previous example there is an SL(2,**R**) gauge freedom that consists in sending $t \mapsto \alpha t + \beta y$, $y \mapsto \gamma t + \delta y$, $\alpha\delta - \beta\gamma = \pm 1$, which preserves the form (5.1) of the metric while transforming the functions f, v and ω_y. This freedom can prove useful for the explicit construction of new solutions.

here we skip the details, since the procedure is straightforward and along the same lines as for the previous ones.

We first rescale $t \to \Xi t$, $\phi \to a y$, and redefine $q = r$, $p = a\cos\theta$. Then, for the single-charged black hole solution of [121] the structure functions are:

$$P = (1-g^2p^2)(a^2-p^2)\,, \qquad Q = \Delta_r\,, \tag{5.21}$$

$$v = \Big((1-g^2p^2)Q - V_r^2(a^2-p^2)\Big)\,(1-g^2p^2)\,, \tag{5.22}$$

$$f = \frac{v}{(H)^{\frac{1}{2}}(p^2+q^2)}\,, \qquad \omega_y = \frac{2mqcP\sqrt{1+a^2g^2s^2}}{v\Xi}\,, \tag{5.23}$$

while for the the two-charge rotating black holes of [122] the functions P, Q and v have the same form as (5.21) (5.22), whereas

$$f = \frac{v}{(H_1H_2)^{\frac{1}{2}}(p^2+q^2)}\,, \qquad \omega_y = \frac{2mrc_1c_2\tilde{c}_1\tilde{c}_2P}{v\Xi}\,. \tag{5.24}$$

The simplicity of the geometry (5.1), and the fact that it is particularly suited for a formalism based on timelike dimensional reduction like the one used in [95], should help constructing new nonextremal rotating black holes in matter-coupled gauged supergravity with an arbitrary number of vector multiplets and general prepotentials. Unfortunately even if the universal structure should remain the same, the equations of motion of gauged supergravity depend crucially on the given model and cannot be solved in complete generality, therefore we first restrict ourselves to considering the simplest interesting prepotentials with a single vector multiplet.

5.1.4 Over- vs. under-rotating solutions

An interesting possibility arises in the extremal limit of rotating black holes (see e.g. [128, 129]). One can sometimes find several extremal limits that correspond to either of two physically different solutions, called over-rotating and under-rotating solutions. The over-rotating solutions (a typical example here is the extremal Kerr black hole) have an ergoregion, while the under-rotating (that resemble more the extremal Reissner-Nordström spacetime) do not have ergoregions. Due to the AdS asymptotics, allowing for a wide range of coordinate choices, it might not be easy to see immediately whether one can have both types of extremal limits. The key to determining this is the following - one first needs to write the black hole metric in asymptotically AdS coordinates, from which the asymptotic time direction can be extracted. Once we know the correct Killing vector $k = \partial_t$, we can follow its behavior on the horizon. For an under-rotating solution, k is null, $|k|^2 = 0$, while an overrotating solution has $|k|^2 > 0$, indicating the existence of an ergosphere.

To make the discussion more explicit, let us take the example of our metric (5.1),

$$ds^2 = f(dt + \omega_y dy)^2 - f^{-1}\left[v\left(\frac{dq^2}{Q} + \frac{dp^2}{P}\right) + PQdy^2\right], \tag{5.25}$$

and assume for the sake of argument that this metric was already written in asymptotically AdS coordinates[2] (this means that in the limit $q \to \infty$, one has $\omega_y = 0, f \sim q^2, v \sim q^4, Q \sim q^4$). In the extremal limit, with horizon at q_h, $Q(q_h) = Q'(q_h) = 0$, the norm of the timelike Killing vector is f and $f(q_h)$ will be either vanishing or positive. Typically both these possibilities will exist for some choice of parameters that determine the solution. This leads to three distinct physical possibilities for the complete geometry:

- $f(q_h) > 0$, only possible if $\omega_y(q_h) \neq 0$: this corresponds to the over-rotating solution; typically there is a lower bound for the angular momentum, $|J| > J_{\min}$ (sometimes $J_{\min} = 0$).
- $f(q_h) = 0$ and $\omega_y(q_h) = 0$: static attractor, leading to a static black hole, typically resulting from the limit $J = 0$.
- $f(q_h) = 0$, $\omega_y(q_h) \neq 0$: under-rotating solution; typically there is an upper bound for the angular momentum, $0 < |J| < J_{\max}$.

The first two cases exist as extremal limits for all known rotating solutions with electromagnetic charges, while the third case is quite special and exists only in the presence of nontrivial scalar fields. Under-rotating solutions are known to exist in ungauged supergravity with cubic prepotentials, [128, 129], but not for quadratic ones of the type $F = -iX^0X^1$. We were not able to find explicit examples of under-rotating solutions in AdS among the general solutions of the $F = -iX^0X^1$ discussed in the present paper, but their existence in other models is an interesting possibility.

Note that it is important to identify the asymptotic time to be able to properly distinguish between the two rotating cases, thus only an analysis of the near-horizon geometry is in principle not enough, even if it gives good hints of the nature of the solution. In particular, observe that the near-horizon geometries of the asymptotically flat under- and over-rotating solutions are exactly the same (cf. (5.38) and (5.60) of [129]), but the former are defined in the parameter space $J^2 < \mathrm{P}^2\mathrm{Q}^2$, while the latter only for $J^2 > \mathrm{P}^2\mathrm{Q}^2$, where P and Q denote the magnetic and electric charge respectively.

5.1.5 Relation between $P(p)$ and horizon topology

The horizon topology of the black hole with metric (5.1) can be studied through the analysis of the function $P(p)$. It is a quartic polynomial in p and here we choose to

[2]Note that this in general will not be true for the explicit solutions we find and one needs to first perform a coordinate change from the Plebański form of the metric to the asymptotically AdS form.

write it in the form

$$P(p) = (p - p_a)(p - p_b)(p - p_c)(p - p_d) \ , \tag{5.26}$$

where the roots $p_{a,b,c,d}$ depend on the explicit values of the physical parameters of the metric (mass, NUT charge, electric and magnetic charges).

If we first look at the simple case without NUT charge, we have pairs of roots such that $p_c = -p_b, p_d = -p_b$. In order for the induced metric on the horizon $q = q_\mathrm{h}$ (where $Q(q_\mathrm{h}) = 0$) to have the right signature, we need $P \geq 0$. Let us first assume that the polynomial P has four real roots with $0 < p_a < p_b$. Then P is non-negative for $|p| \leq p_a$ or $|p| \geq p_b$. Choosing $-p_a \leq p \leq p_a$ leads to a function $P(p)$ that is bounded and vanishes at two points, which are coordinate singularities. Such a function can be defined for horizons with spherical topology, where the two singularities correspond to the north and the south pole of the (possibly squashed) sphere. Choosing the other possibility, $p_b \leq p$, leads to hyperbolic topology since the function P is not bounded anymore. The coordinate singularity at $p = p_b$ is at the origin of the hyperbolic space in the standard hyperbolic coordinates. One can see that in this case we have two disjoint types of black holes within the same solutions, depending on whether we choose the compact or non-compact range. The third main type of topology arises in the case when p_a and p_b are both complex, thus P is everywhere positive and non-vanishing for real p - this corresponds to the flat topology of black branes, where no coordinate singularities are encountered.

To summarize the three basic types of topology and their relation with $P(p)$, the possibilities are

- spherical topology: $P(p)$ bounded and vanishing at two points, north and south pole.
- hyperbolic topology: $P(p)$ unbounded and vanishing at a single point.
- flat topology: $P(p)$ unbounded and never vanishing.

On top of those topologies and their quotients, we can have some new exotic situations in some special cases. In section 5.2.3 we will show the situation where the two positive roots of P coincide, $p_a = p_b$. It turns out choosing $-p_a \leq p \leq p_a$ in this case leads to a sphere with two punctures on the place of the two poles, i.e. the horizon has a cylindrical topology but finite area. Thus we are lead to think that whenever the function $P(p)$ has a double root the horizon is punctured at that point, which is no longer just a coordinate singularity.

The situation with NUT charge is even more complex, since then all four roots can be a priori unrelated to each other. One can therefore have situations with $p_d < p_c < p_b = p_a$ for example, where the choice of bounded region for $P(p)$ will lead to one pole and one puncture and therefore to a bottle-shaped horizon topology. Even more exotic possibilities would be three coinciding roots, a case which is yet to be analyzed carefully. In any case, the three main types of horizon topologies continue to exist whether one allows for NUT charge or not.

5.2 Thermal rotating solutions with magnetic charges

We shall now construct a nonextremal deformation of the BPS solution to the model with prepotential $F = -iX^0X^1$, constructed in [123], and described in the previous section. Inspired by the form (58) of [120], we can make the ansatz

$$ds^2 = \frac{Q}{W}\left[dt - p^2 dy\right]^2 - \frac{P}{W}[dt + q_1 q_2 dy]^2 - W\left(\frac{dq^2}{Q} + \frac{dp^2}{P}\right), \tag{5.27}$$

with

$$Q = a_0 + a_1 q + a_2 q^2 + a_4 q^4 , \qquad P = b_0 + b_1 p + b_2 p^2 + b_4 p^4 , \tag{5.28}$$

and

$$W = q_1 q_2 + p^2 , \qquad q_i = q - \Delta_i , \tag{5.29}$$

where a_i, b_i, and Δ_i are constants. (5.27) fits into the general form of the metric (5.1) with

$$v = Q - P , \qquad f = \frac{v}{W} , \qquad \omega_y = -\frac{1}{v}\left[P q_1 q_2 + Q p^2\right] . \tag{5.30}$$

It boils down to the BPS solution (5.11) when Q, P reduce to the functions (5.12), and $\Delta_1 = -\Delta_2 \equiv \Delta$. The ansatz for the gauge potentials and the scalars is

$$A^\Lambda = -\frac{\mathrm{P}^\Lambda (dt + q_1 q_2 dy)}{W} p , \qquad \tau = e^{-\varphi} + i\chi = \frac{X^1}{X^0} = \frac{\mu W + i\nu p}{q_1^2 + p^2} , \tag{5.31}$$

where the constants P^Λ are proportional to the magnetic charges, and μ, ν are real constants.

In order to reduce the number of free parameters, we will first restrict to the case $\Delta_1 = -\Delta_2 \equiv \Delta$, and take μ, ν in the scalar to be the same as in the BPS case. We have then checked that the system of Einstein-Maxwell-scalar equations of motion of the FI-gauged Lagrangian (4.1), namely (4.3)-(4.4)-(4.5), is satisfied if

$$a_0 = b_0 - a_2\Delta^2 - \frac{\Delta^4}{l^2} + 2l^2\left(g_0^2 \mathrm{P}^{0\,2} + g_1^2 \mathrm{P}^{1\,2}\right) , \qquad a_1 = \frac{2l^2(g_0^2 \mathrm{P}^{0\,2} - g_1^2 \mathrm{P}^{1\,2})}{\Delta} , \tag{5.32}$$

$$b_1 = 0 , \qquad b_2 = -a_2 - 2\frac{\Delta^2}{l^2} , \qquad b_4 = a_4 = 1/l^2 \equiv 4 g_0 g_1 , \tag{5.33}$$

$$\mu = \frac{g_0}{g_1} , \qquad \nu = 2\frac{\Delta g_0}{g_1} , \tag{5.34}$$

where we assumed that g_0, g_1 are positive. We can check that the scalar field τ has the correct behaviour at infinity, since in this model the AdS_4 asymptotic geometry is obtained for $\tau_\infty = g_0/g_1$. If we fix the Fayet-Iliopuolos constants g_0 and g_1, the solution depends on the five parameters b_0, a_2, Δ, P^0, P^1, thus two more parameters with respect to the BPS solution. From the second equation of (5.32) we see that, in the case $g_0\mathrm{P}^0 = g_1\mathrm{P}^1$, one has either $a_1 = 0$ or $\Delta = 0$. If a_1 vanishes, one can thus have equal charges and yet a nontrivial scalar profile (i.e., $\Delta \neq 0$). This behaviour is qualitatively different from that of the solutions constructed for instance in [120].

Notice that the form of the scalar field and of the vector field strengths is the same as in the BPS case. The latter is recovered for

$$a_2 = \mathrm{E} - 2\frac{\Delta^2}{l^2}\,, \qquad b_0 = (1+A)\frac{\mathrm{E}^2 l^2}{4}\,, \qquad \mathrm{P}^\Lambda = \frac{\sqrt{-A}\mathrm{E}}{4g_\Lambda}\,. \tag{5.35}$$

A further generalization to a black hole with electric and NUT-charges (that would include also the solutions of [123] and [120]) is straightforward, however, we postpone this discussion to the next section and first elaborate on the physical properties and novelties of the magnetic solutions. In this case, in fact, the absence of closed timelike curves makes them interesting thermodynamical and gravitational systems.

5.2.1 Physical discussion

Following section 5.1.5, we assume P has four distinct roots, $\pm p_a$, $\pm p_b$, where $0 < p_a < p_b$. Then P is non-negative for $|p| \leq p_a$ or $|p| \geq p_b$. Since we are interested in black holes with compact horizon[3], we consider the range $|p| \leq p_a$, and set $p = p_a \cos\theta$, where $0 \leq \theta \leq \pi$. By using the scaling symmetry

$$p \to \lambda p\,, \qquad q \to \lambda q\,, \qquad t \to t/\lambda\,, \qquad y \to y/\lambda^3\,, \qquad \Delta \to \lambda\Delta\,, \tag{5.36}$$
$$a_0 \to \lambda^4 a_0\,, \qquad a_1 \to \lambda^3 a_1\,, \qquad a_2 \to \lambda^2 a_2\,, \qquad b_0 \to \lambda^4 b_0\,, \qquad b_2 \to \lambda^2 b_2\,,$$

one can set $p_b = l$ without loss of generality. If we define the rotation parameter j by $p_a^2 = j^2$, this amounts to the choice

$$b_0 = j^2\,, \qquad b_2 = -1 - \frac{j^2}{l^2}\,, \tag{5.37}$$

which implies

$$a_0 = (j^2 - \Delta^2)\left(1 - \frac{\Delta^2}{l^2}\right) + 2l^2\left(g_0^2 {\mathrm{P}^0}^2 + g_1^2 {\mathrm{P}^1}^2\right)\,, \qquad a_2 = 1 - \frac{\Delta^2}{l^2} + \frac{j^2 - \Delta^2}{l^2}\,. \tag{5.38}$$

Taking also

$$t \to t + \frac{j\phi}{\Xi}\,, \qquad y \to \frac{\phi}{j\Xi}\,, \qquad \Xi \equiv 1 - \frac{j^2}{l^2}\,, \tag{5.39}$$

[3] As already discussed, noncompact hyperbolic horizons can be obtained by restricting to the region $p \geq p_b$ and setting $p = p_b \cosh\theta$, where $0 \leq \theta < \infty$. In this case, the rotation parameter is defined by $p_b^2 = j^2$. The resulting black holes represent generalizations of the solutions of minimal gauged supergravity constructed in [130].

the metric (5.27) becomes

$$
\begin{aligned}
ds^2 = \quad & \frac{Q}{(q^2-\Delta^2+j^2\cos^2\theta)}\left[dt+\frac{j\sin^2\theta}{\Xi}d\phi\right]^2 \\
& -(q^2-\Delta^2+j^2\cos^2\theta)\left(\frac{dq^2}{Q}+\frac{d\theta^2}{\Delta_\theta}\right) \\
& -\frac{\Delta_\theta\sin^2\theta}{(q^2-\Delta^2+j^2\cos^2\theta)}\left[j dt+\frac{q^2+j^2-\Delta^2}{\Xi}d\phi\right]^2 ,
\end{aligned}
\tag{5.40}
$$

where we defined

$$\Delta_\theta = 1-\frac{j^2}{l^2}\cos^2\theta .$$

From (5.32) it is clear that for $g_0\mathrm{P}^0 = g_1\mathrm{P}^1$ and $\Delta = 0$, the mass parameter a_1 can be arbitrary; this leads to the Kerr-Newman-AdS solution with magnetic charge and constant scalar. On the other hand, for zero rotation parameter, $j = 0$, (5.40) boils down to the static nonextremal black holes with running scalar constructed in [96].

The Bekenstein-Hawking entropy of the black holes described by (5.40) is given by

$$S = \frac{\pi}{\Xi G}(q_\mathrm{h}^2+j^2-\Delta^2) , \tag{5.41}$$

where G denotes Newton's constant and q_h is the location of the horizon, i.e., $Q(q_\mathrm{h}) = 0$. In order to compute the temperature and angular velocity, we write the metric in the canonical (ADM) form

$$ds^2 = N^2dt^2-\sigma(d\phi-\omega dt)^2-(q^2-\Delta^2+j^2\cos^2\theta)\left(\frac{dq^2}{Q}+\frac{d\theta^2}{\Delta_\theta}\right) , \tag{5.42}$$

with

$$\sigma = \frac{\Sigma^2\sin^2\theta}{(q^2-\Delta^2+j^2\cos^2\theta)\Xi^2} , \qquad \omega = \frac{j\Xi}{\Sigma^2}(Q-\Delta_\theta(q^2+j^2-\Delta^2)) , \tag{5.43}$$

and the lapse function

$$N^2 = \frac{Q\Delta_\theta(q^2-\Delta^2+j^2\cos^2\theta)}{\Sigma^2} , \tag{5.44}$$

where

$$\Sigma^2 \equiv \Delta_\theta(q^2+j^2-\Delta^2)^2-Qj^2\sin^2\theta .$$

The angular velocity of the horizon is thus

$$\omega_\mathrm{h} = \omega|_{q=q_\mathrm{h}} = -\frac{j\Xi}{q_\mathrm{h}^2+j^2-\Delta^2} , \tag{5.45}$$

whereas at infinity one has

$$\omega_\infty = \frac{j}{l^2}\,. \tag{5.46}$$

The angular momentum computed by means of the Komar integral reads

$$J = \frac{1}{16\pi G}\oint_{S^2_\infty} dS^{\mu\nu}\nabla_\mu m_\nu\,, \tag{5.47}$$

with $m = \partial_\phi$ and the oriented measure

$$dS^{\mu\nu} = (v^\mu u^\nu - v^\nu u^\mu)\sqrt{\hat\sigma}d\theta d\phi\,.$$

Here, $u = N^{-1}(\partial_t + \omega\partial_\phi)$ is the normal vector of a constant t hypersurface, $v = (Q/(q^2 - \Delta^2 + j^2\cos^2\theta))^{1/2}\partial_q$, and

$$\sqrt{\hat\sigma} = \frac{\Sigma\sin\theta}{\Xi\Delta_\theta^{1/2}}\,,$$

where $\hat\sigma$ denotes the induced metric on a two-sphere of constant q and t. Evaluation of (5.47) yields

$$J = \frac{a_1 j}{2\Xi^2 G}\,. \tag{5.48}$$

The Komar mass

$$M = -\frac{1}{8\pi G}\oint_{S^2_\infty} dS^{\mu\nu}\nabla_\mu k_\nu \tag{5.49}$$

has to be computed with respect to the Killing vector $k = \Xi^{-1}\partial_t$ [131], leading to

$$M = -\frac{1}{8\pi G}\lim_{q\to\infty}\int d\theta d\phi\frac{\sin\theta}{(j^2 - l^2)^2}\left[-2l^2q^3 - 2(j^2-\Delta^2)l^2q + l^4a_1 + \mathcal{O}(q^{-1})\right]\,,$$

which is of course divergent. If we subtract the background with $a_1 = 0$ and the same j and Δ, we get the finite result

$$M = -\frac{a_1}{2\Xi^2 G}\,. \tag{5.50}$$

Notice that the 'ground state' with $a_1 = 0$ is a naked singularity (contrary to the case of hyperbolic horizons addressed in footnote 3): The curvature singularity $W = 0$[4] is shielded by a horizon if $q_\text{h}^2 - \Delta^2 + j^2\cos^2\theta > 0$, which is equivalent to

$$\frac{2}{l^2}q_\text{h}^2 + a_2 > 1 + \frac{j^2}{l^2}\,.$$

[4]Note also that for $W < 0$, the real part of the scalar field becomes negative, so that ghost modes appear.

Now, using $Q(q_{\mathrm{h}})=0$, this can be rewritten as

$$\sqrt{a_2^2-\frac{4a_0}{l^2}}>1+\frac{j^2}{l^2}\,,$$

which can be easily shown to lead to a contradiction by using (5.38).

An alternative mass definition, that does not require any background subtraction, is based on the Ashtekar-Magnon-Das (AMD) formalism [100, 101]. (Cf. also [121] for an application to rotating AdS black holes and for more details). First of all we compute the Weyl tensor of a conformally rescaled metric (in this case the conformal rescaling factor is $\Omega=l/q$), to leading order in q. This reads

$$\overline{C}^{t}_{qtq}=\frac{-g_0^2{\mathrm{P}^0}^2+g_1^2{\mathrm{P}^1}^2}{8\Delta g_0^2g_1^2q^5}+O(1/q^6)\,. \tag{5.51}$$

Once we have this quantity, we can compute the mass associated to the Killing vector $K=\Xi^{-1}\partial_t$, given by

$$M=\frac{1}{8\pi G(4g_0g_1)^{3/2}}\int_\Sigma d\overline{\Sigma}_a\Omega^{-1}\bar{n}^c\bar{n}^d\overline{C}^a_{cbd}K^b=-\frac{(g_0^2{\mathrm{P}^0}^2-g_1^2{\mathrm{P}^1}^2)}{4\Delta g_0g_1\,G\,\Xi^2}=-\frac{a_1}{2\,G\,\Xi^2}\,, \tag{5.52}$$

so that the AMD procedure gives the same result as the regularized Komar integral.

The magnetic charges π^Λ are given by

$$\pi^\Lambda=\frac{1}{4\pi}\oint_{S^2_\infty}F^\Lambda=-\frac{\mathrm{P}^\Lambda}{\Xi}\,. \tag{5.53}$$

Now that we have computed the physical quantities of our solution, a comment on the number of free parameters is in order. We already mentioned that the metric (5.27) and the gauge potentials and scalar (5.31) depend on five parameters. However, due to the scaling symmetry (5.36), one of them is actually redundant and can be scaled away. There remain thus four free parameters, for instance P^Λ,Δ,j, or alternatively π^Λ,M,J. Our black holes are therefore labelled by two independent magnetic charges, mass and angular momentum. Note also that the parameter Δ related to the running of the scalar is not independent of the mass; for this reason our solution does not carry primary hair.

The product of the horizon areas, given formula (5.41), is

$$\prod_{\alpha=1}^{4}A_\alpha=\frac{(4\pi)^4}{\Xi^4}\prod_{\alpha=1}^{4}(q_{\mathrm{h}_\alpha}^2+j^2-\Delta^2)=\frac{(4\pi)^4}{\Xi^4}\prod_{\alpha=1}^{4}(q_{\mathrm{h}_\alpha}-q_+)(q_{\mathrm{h}_\alpha}-q_-)\,, \tag{5.54}$$

with $q_\pm=\pm\sqrt{\Delta^2-j^2}$. At this point the formulas resemble the ones given in the static case, and we can use the procedure explained in [99]. We define

$$\kappa_+=q_+^4+\frac{a_2}{a_4}q_+^2+\frac{a_1}{a_4}q_++\frac{a_0}{a_4}\qquad \kappa_-=q_-^4+\frac{a_2}{a_4}q_-^2+\frac{a_1}{a_4}q_-+\frac{a_0}{a_4}\,, \tag{5.55}$$

so that the area product will be given by $\prod_{\alpha=1}^{4} A_\alpha = (4\pi)^4 \kappa_+\kappa_-/\Xi^4$. Plugging in the values of the coefficients and using the expression (5.48) for J we have

$$\prod_{\alpha=1}^{4} A_\alpha = (4\pi)^4 l^2 \left((\pi^0\pi^1)^2 + J^2\right) . \tag{5.56}$$

The charge-dependent term on the rhs of (5.56) is directly related to the prepotential; a fact that was first noticed in [99] for static black holes.

5.2.2 Thermodynamics and extremality

A quasi-Euclidean section of the metric can be obtained by analytically continuing $t \to -it_{\mathrm{E}}$. It turns out that this is regular at $q = q_{\mathrm{h}}$ provided t_{E} is identified modulo $4\pi\Xi(q_{\mathrm{h}}^2 + j^2 - \Delta^2)/Q'_{\mathrm{h}}$, where Q'_{h} denotes the derivative of Q w.r.t. q, evaluated at the horizon. This yields the Hawking temperature

$$T = \frac{Q'_{\mathrm{h}}}{4\pi(q_{\mathrm{h}}^2 + j^2 - \Delta^2)} . \tag{5.57}$$

Using the expressions (5.41), (5.48) and (5.50) for the entropy, angular momentum and mass respectively, as well as the fact that Q vanishes for $q = q_{\mathrm{h}}$, one obtains by simple algebraic manipulations the Christodoulou-Ruffini-type mass formula

$$\begin{aligned} M^2 \quad = \quad & \frac{S}{4\pi G} + \frac{\pi J^2}{SG} + \frac{\pi}{4SG^3}(\pi^0\pi^1)^2 + \left(\frac{l^2}{G^2} + \frac{S}{\pi G}\right)\left((g_0\pi^0)^2 + (g_1\pi^1)^2\right) \\ & + \frac{J^2}{l^2} + \frac{S^2}{2\pi^2 l^2} + \frac{S^3 G}{4\pi^3 l^4} . \end{aligned} \tag{5.58}$$

Note that this reduces correctly to equ. (43) of [131] in the KNAdS case $(g_0\pi^0)^2 = (g_1\pi^1)^2$, $\Delta = 0$, a_1 arbitrary.

Since S, J, π^Λ form a complete set of extensive parameters, (5.58) represents also the black hole thermodynamic fundamental relation $M = M(S, J, \pi^\Lambda)$. The quantities conjugate to S, J, π^Λ are the temperature

$$\begin{aligned} T = \left(\frac{\partial M}{\partial S}\right)_{J,\pi^\Lambda} = \frac{1}{8\pi GM} \qquad & \left[1 - \frac{4\pi^2 J^2}{S^2} - \frac{\pi^2}{S^2G^2}(\pi^0\pi^1)^2 + 4\left((g_0\pi^0)^2 + (g_1\pi^1)^2\right)\right. \\ & \left. + \frac{4SG}{\pi l^2} + \frac{3S^2G^2}{\pi^2 l^4}\right] , \end{aligned} \tag{5.59}$$

the angular velocity

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_{S,\pi^\Lambda} = \frac{\pi J}{MGS}\left[1 + \frac{SG}{\pi l^2}\right] , \tag{5.60}$$

and the magnetic potentials

$$\Phi_\Lambda = \left(\frac{\partial M}{\partial \pi^\Lambda}\right)_{S,J,\pi^{\Sigma\neq\Lambda}} = \frac{1}{MG}\left[\frac{\pi}{4SG^2}\pi^0\pi^1\eta_{\Lambda\Sigma}\pi^\Sigma + \left(\frac{l^2}{G}+\frac{S}{\pi}\right)g_\Lambda^2\pi^\Lambda\right], \qquad (5.61)$$

where

$$\eta_{\Lambda\Sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and there is no summation over Λ in the last term. The obtained quantities satisfy the first law of thermodynamics

$$dM = TdS + \Omega dJ + \Phi_\Lambda d\pi^\Lambda . \qquad (5.62)$$

Furthermore, by eliminating M from (5.59)-(5.61) using (5.58), it is possible to obtain four equations of state for the black holes (5.40). It is straightforward to verify that the relation (5.59) for the temperature coincides with equ. (5.57), whereas (5.60) yields

$$\Omega = \omega_{\mathrm{h}} - \omega_\infty , \qquad (5.63)$$

with ω_{h} and ω_∞ given respectively by (5.45) and (5.46). It is thus the difference between the angular velocities at the horizon and at infinity which enters the first law; a fact that was stressed in [131] for the case of the KNAdS black hole.

The Hawking temperature (5.57) vanishes in the extremal case, when q_{h} is at least a double root of Q. The structure function Q can then be written as

$$Q = (q-q_{\mathrm{h}})^2\left(\frac{q^2}{l^2}+\frac{2q_{\mathrm{h}}q}{l^2}+a_2+\frac{3q_{\mathrm{h}}^2}{l^2}\right),$$

and we must have

$$a_0 = a_2 q_{\mathrm{h}}^2 + \frac{3q_{\mathrm{h}}^4}{l^2}, \qquad a_1 = -2q_{\mathrm{h}}a_2 - \frac{4q_{\mathrm{h}}^3}{l^2}. \qquad (5.64)$$

These equations restrict of course the number of free parameters compared to the nonextremal case. To obtain the near-horizon geometry of the extremal black holes, we define new (dimensionless) coordinates $z, \hat{t}, \hat{\phi}$ by

$$q = q_{\mathrm{h}} + \epsilon q_0 z, \qquad t = \frac{\hat{t}q_0}{\Xi\epsilon}, \qquad \phi = \hat{\phi} + \omega_{\mathrm{h}}\frac{\hat{t}q_0}{\epsilon}, \qquad q_0^2 \equiv \frac{\Xi l^2(q_{\mathrm{h}}^2+j^2-\Delta^2)}{6q_{\mathrm{h}}^2+a_2l^2}, \qquad (5.65)$$

and take $\epsilon \to 0$ keeping $z, \hat{t}, \hat{\phi}$ fixed. This leads to

$$\begin{aligned} ds^2 = \quad & -\frac{q_{\mathrm{h}}^2-\Delta^2+j^2\cos^2\theta}{C}\left(-z^2d\hat{t}^2+\frac{dz^2}{z^2}+C\frac{d\theta^2}{\Delta_\theta}\right) \\ & -\frac{\Delta_\theta\sin^2\theta(q_{\mathrm{h}}^2+j^2-\Delta^2)^2}{(q_{\mathrm{h}}^2-\Delta^2+j^2\cos^2\theta)}\left(\frac{d\hat{\phi}}{\Xi}+\frac{2q_{\mathrm{h}}\omega_{\mathrm{h}}}{C}zd\hat{t}\right)^2, \end{aligned} \qquad (5.66)$$

where the constant C is given by

$$
\begin{aligned}
C &= \frac{6q_{\mathrm{h}}^2}{l^2} + a_2 \\
&= \left\{ \left(1 - \frac{\Delta^2}{l^2}\right)^2 + \frac{(j^2-\Delta^2)^2}{l^4} + \frac{14}{l^2}\left(1 - \frac{\Delta^2}{l^2}\right)(j^2-\Delta^2) + 24(g_0^2 \mathrm{P}^{0^2} + g_1^2 \mathrm{P}^{1^2}) \right\}^{1/2} .
\end{aligned}
$$

If q_{h} is at least a triple root of Q, C vanishes, and one has an ultracold black hole. In this case, the zooming procedure (5.65) does not conform to Geroch's criteria of limiting spaces [132], and thus the resulting geometry would not even solve the equations of motion. This problem was first pointed out by Romans [61], and discussed also in [133]. There exists an alternative limiting procedure [134, 135] which basically consists in going first to the situation where Q has a double root, and then taking the near-horizon limit simultaneously with the ultracold limit in a particular way. We postpone a discussion of the ultracold case to a future publication.

Note that in the extremal limit, when $T = 0$, it is easy to see that the entropy is only a function of the discrete charges J and π^I by inverting (5.59) in terms of S.

5.2.3 Noncompact horizon with finite area

We shall now discuss the special case where the polynomial $P(p)$ has two double roots, i.e., $p_a = p_b$ in the notation adopted at the beginning of section 5.2.1. This corresponds to $j^2 = l^2$, which means that the conformal boundary rotates at the speed of light. For the Kerr-AdS solution, this limit (in which the metric (5.40) is of course singular) was explored in [136][5], where it was argued that it represents an interesting example in which to study AdS/CFT.

Using again the scaling symmetry (5.36), we can set $p_a = l$ without loss of generality, so that

$$P(p) = \frac{1}{l^2}(p^2 - l^2)^2$$

in this case. The induced metric on the horizon $q = q_{\mathrm{h}}$ (where Q vanishes) is given by

$$ds_{\mathrm{h}}^2 = -\frac{P}{q_{\mathrm{h}}^2 - \Delta^2 + p^2}(q_{\mathrm{h}}^2 - \Delta^2 + \alpha)^2 dy^2 - \frac{q_{\mathrm{h}}^2 - \Delta^2 + p^2}{P} dp^2 , \qquad (5.67)$$

where the constant α takes into account a possible shift $t \to t + \alpha y$, similar to (5.39). If we want y to be a compact coordinate, the absence of closed timelike curves requires setting $\alpha = l^2$, since otherwise g_{yy} will be negative close to $p^2 = l^2$. Note that we consider the coordinate range $-l \leq p \leq l$, and that (5.67) becomes singular for $p^2 = l^2$. To understand more in detail what happens at these singularities, take

[5] See also [137, 138].

for instance the limit $p \to l$, in which (5.67) simplifies to

$$ds_{\rm h}^2 = -(q_{\rm h}^2 - \Delta^2 + l^2)\left[\frac{d\rho^2}{4\rho^2} + 4\rho^2 dy^2\right] . \tag{5.68}$$

Here, the new coordinate ρ is defined by $\rho = l - p$. (5.68) is clearly a metric of constant negative curvature on the hyperbolic space H^2 (or on a quotient thereof, if we want y to be a compact coordinate). Since (5.67) is symmetric under $p \to -p$, an identical result holds for $p \to -l$. Thus, for $p \to \pm l$, the horizon approaches a space of constant negative curvature, and there is no true singularity there. In particular, this implies that the horizon is noncompact, which comes as a surprise, since one might have expected the limit of coincident roots $p_a = p_b$ to be smooth, and for $p_a \neq p_b$ the horizon was topologically a sphere. Moreover, the horizon area reads

$$A_{\rm h} = \int (q_{\rm h}^2 - \Delta^2 + l^2) dy dp = 2Ll(q_{\rm h}^2 - \Delta^2 + l^2) , \tag{5.69}$$

where we assumed y to be identified modulo L. We see that, in spite of being noncompact, the event horizon has finite area, and the entropy of the corresponding black hole is thus also finite. To the best of our knowledge, this represents the first instance of a black hole with noncompact horizon, but still finite entropy.

In order to visualize the geometry (5.67), we can embed it in $\mathbf{R}^3$ as a surface of revolution[6]. To this end write the flat metric in cylindrical coordinates,

$$ds_3^2 = -dz^2 - dr^2 - r^2 d\phi^2 ,$$

and consider $z = z(p)$, $r = r(p)$. Setting $\phi = 2\pi y/L$, and identifying the resulting line element with (5.67), one gets

$$r = \frac{L}{2\pi l}(q_{\rm h}^2 - \Delta^2 + p^2)^{-1/2}(l^2 - p^2)(q_{\rm h}^2 - \Delta^2 + l^2) , \tag{5.70}$$

as well as

$$\left(\frac{dr}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2 = \frac{q_{\rm h}^2 - \Delta^2 + p^2}{P} , \tag{5.71}$$

which is a differential equation for dz/dp. By expanding near $p = \pm l$, one easily sees that z diverges logarithmically for $|p| \to l$, and that r goes to zero in this limit. We integrated (5.71) numerically for the values $l = 1$, $L = 2\pi$ and $q_{\rm h}^2 - \Delta^2 = 5$. The resulting surface of revolution is shown in figure 5.1, where the z-axis is vertical. Note that the two cusps extend up to infinity, with $z \to \pm\infty$ for $p \to \pm l$ respectively, while the 'equator' $z = 0$, where r becomes maximal, is reached for $p = 0$.

The metric on the conformal boundary $q \to \infty$ of the black hole solution reads

$$ds_{\rm bdry}^2 = dt^2 - 2dtdy(p^2 - l^2) - l^2\frac{dp^2}{P} , \tag{5.72}$$

and hence y becomes a lightlike coordinate there.

[6] This is possible if $L(q_{\rm h}^2 - \Delta + l^2)$ is not too large, since otherwise $(dz/dp)^2$ in (5.71) will become negative in some region.

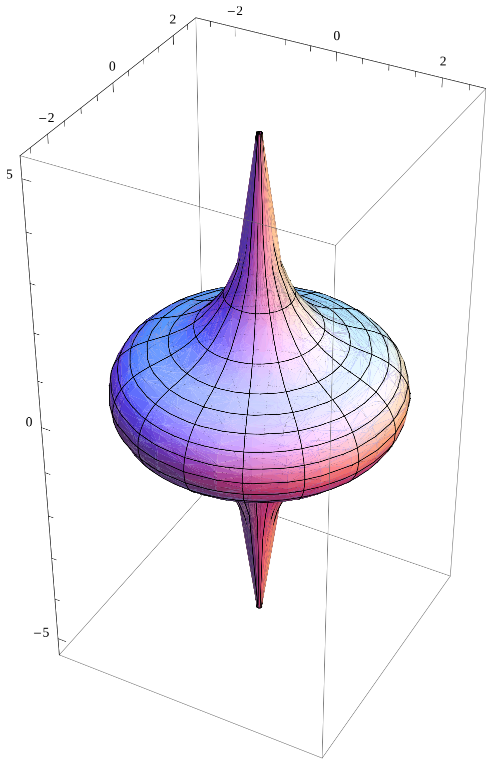

Figure 5.1: The event horizon of a black hole in the case where $P(p)$ has two double roots, embedded in $\mathbf{R}^3$ as a surface of revolution.

5.3 Inclusion of NUT- and electric charges

Inspired by the solution in section 5 of [120], we make the following ansatz to include also NUT- and electric charges:

$$ds^2 = \frac{Q}{W}[dt - p_1 p_2 dy]^2 - \frac{P}{W}[dt + q_1 q_2 dy]^2 - W\left(\frac{dq^2}{Q} + \frac{dp^2}{P}\right), \tag{5.73}$$

where Q, P are again given by (5.28) (with $a_4 = b_4 = 1/l^2$), and

$$W = q_1 q_2 + p_1 p_2\,, \qquad q_1 = q - \Delta\,, \qquad q_2 = q + \Delta\,, \qquad p_1 = p - \delta\,, \qquad p_2 = p + \delta\,.$$

The ansatz for the gauge potentials and the scalars is

$$A^\Lambda = -\frac{\mathrm{P}^\Lambda(dt + q_1 q_2 dy)}{W} p_1 - \frac{\mathrm{Q}^\Lambda(dt - p_1 p_2 dy)}{W} q_1\,, \tag{5.74}$$

$$\tau = e^{-\varphi} + i\chi = \frac{X^1}{X^0} = \frac{\mu W + i(\nu p + \lambda q)}{q_1^2 + p_1^2}\,, \tag{5.75}$$

where the constants Q^Λ are proportional to the electric charges, and μ, ν, λ are constants to be determined.

We have checked that the equations of motion of the $F = -iX^0X^1$ model are satisfied if the parameters assume the following form:

$$a_4 = b_4 = 1/l^2\,, \tag{5.76}$$

$$\begin{aligned} a_0 = b_0 - a_2\Delta^2 - \frac{\Delta^4}{l^2} + \frac{g_0 {P^0}^2}{2g_1} + \frac{g_1 {P^1}^2}{2g_0} + \frac{g_0 {Q^0}^2}{2g_1} + \frac{g_1 {Q^1}^2}{2g_0} + \\ - \frac{a_2\Delta^2 {Q^1}^2}{{P^1}^2} - \frac{2\Delta^4 {Q^1}^2}{l^2\, {P^1}^2} - \frac{\Delta^4 {Q^1}^4}{l^2\, {P^1}^4}\,, \end{aligned} \tag{5.77}$$

$$b_1 = -\frac{2l^2 P^1 (g_0^2(-2P^0P^1Q^0 + {P^0}^2 Q^1 - {Q^0}^2 Q^1) + g_1^2 Q^1 ({P^1}^2 + {Q^1}^2))}{\Delta({P^1}^2 + {Q^1}^2)}\,, \tag{5.78}$$

$$a_1 = -\frac{2l^2 P^1 (g_0^2(-{P^0}^2 P^1 + P^1 {Q^0}^2 - 2P^0Q^0Q^1) + g_1^2 P^1 ({P^1}^2 + {Q^1}^2))}{\Delta({P^1}^2 + {Q^1}^2)}\,, \tag{5.79}$$

$$b_2 = -a_2 - \frac{2\Delta^2({P^1}^2 + {Q^1}^2)}{l^2\, {P^1}^2}\,, \tag{5.80}$$

$$\mu = \frac{g_0}{g_1}\,, \qquad \nu = \frac{2\Delta g_0}{g_1}\,, \qquad \lambda = \frac{2\Delta g_0 Q^1}{g_1 P^1}\,, \qquad \delta = -\frac{\Delta Q^1}{P^1}\,. \tag{5.81}$$

The solution has free parameters Δ, b_0 a_2, P^Λ and Q^Λ. It reduces to the one we have previously found for $Q^1 = 0 = Q^0$.

Let us now keep the ansatz (5.73)-(5.74) for the metric and the gauge fields and look for a solution with a scalar field of the form

$$\tau = e^{-\varphi} + i\chi = \frac{X^1}{X^0} = \frac{\mu W + i(\nu p + \lambda q)}{q_2^2 + p_2^2}\,. \tag{5.82}$$

In this case, the equations of motions are satisfied for the following parameters

$$a_4 = b_4 = 1/l^2\,, \tag{5.83}$$

$$\begin{aligned} a_0 = b_0 - a_2\Delta^2 - \frac{\Delta^4}{l^2} + \frac{g_0 {P^0}^2}{2g_1} + \frac{g_1 {P^1}^2}{2g_0} + \frac{g_0 {Q^0}^2}{2g_1} + \frac{g_1 {Q^1}^2}{2g_0} + \\ - \frac{a_2\Delta^2 {Q^0}^2}{{P^0}^2} - \frac{2\Delta^4 {Q^0}^2}{l^2\, {P^0}^2} - \frac{\Delta^4 {Q^0}^4}{l^2\, {P^0}^4}\,, \end{aligned} \tag{5.84}$$

$$b_1 = -\frac{2l^2 \mathrm{P}^0 (g_1^2(-2\mathrm{P}^1\mathrm{P}^0\mathrm{Q}^1 + \mathrm{P}^{1^2}\mathrm{Q}^0 - \mathrm{Q}^{1^2}\mathrm{Q}^0) + g_0^2 \mathrm{Q}^0(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2}))}{\Delta(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2})}, \tag{5.85}$$

$$a_1 = -\frac{2l^2 \mathrm{P}^0 (g_1^2(-\mathrm{P}^{1^2}\mathrm{P}^0 + \mathrm{P}^0\mathrm{Q}^{1^2} - 2\mathrm{P}^1\mathrm{Q}^0\mathrm{Q}^1) + g_0^2 \mathrm{P}^0(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2}))}{\Delta(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2})}, \tag{5.86}$$

$$b_2 = -a_2 - \frac{2\Delta^2(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2})}{l^2\,\mathrm{P}^{0^2}}, \tag{5.87}$$

$$\mu = \frac{g_0}{g_1}, \qquad \nu = -\frac{2\Delta g_0}{g_1}, \qquad \lambda = -\frac{2\Delta g_0 \mathrm{Q}^0}{g_1 \mathrm{P}^0}, \qquad \delta = -\frac{\Delta \mathrm{Q}^0}{\mathrm{P}^0}. \tag{5.88}$$

In order to discuss more in detail the solution (5.76)-(5.81), we assume that the polynomial P has four distinct roots $p_a < p_b < p_c < p_d$. Since we are interested in black holes with compact horizon[7], we consider the region $p_b \le p \le p_c$ (where P is positive), and set $p = N + j\cos\theta$, where

$$N \equiv \frac{p_b + p_c}{2}, \qquad j \equiv \frac{p_c - p_b}{2}, \tag{5.89}$$

and $0 \le \theta \le \pi$. Using the scaling symmetry (5.36), supplemented by

$$\delta \to \lambda\delta, \qquad b_1 \to \lambda^3 b_1,$$

one can set $p_d = -N + \sqrt{l^2 + 4N^2}$ without loss of generality[8]. This implies

$$b_0 = (j^2 - N^2)\left(1 + \frac{3N^2}{l^2}\right), \quad b_1 = 2N\left(1 - \frac{j^2}{l^2} + \frac{4N^2}{l^2}\right), \quad b_2 = -1 - \frac{j^2}{l^2} - \frac{6N^2}{l^2},$$

$$a_0 = b_0 + b_2(\Delta^2 + \delta^2) + \frac{(\Delta^2 + \delta^2)^2}{l^2} + 2l^2\left[g_0^2(\mathrm{P}^{0^2} + \mathrm{Q}^{0^2}) + g_1^2(\mathrm{P}^{1^2} + \mathrm{Q}^{1^2})\right],$$

$$a_2 = 1 + \frac{j^2}{l^2} + \frac{6N^2}{l^2} - \frac{2}{l^2}(\Delta^2 + \delta^2). \tag{5.90}$$

Taking also

$$t \to t + \left(\frac{j}{\Xi} + \frac{N^2 - \delta^2}{j\Xi}\right)\phi, \qquad y \to \frac{\phi}{j\Xi},$$

[7] Black holes with hyperbolic horizons can be obtained by taking the region $p \ge p_d$.

[8] This choice is made in order to correctly reproduce the KNTN-AdS solution of minimal gauged supergravity as a special subcase, see e.g. [69].

the metric (5.73) becomes

$$
\begin{aligned}
ds^2 = \quad & \frac{Q}{q^2-\Delta^2+(N+j\cos\theta)^2-\delta^2}\left[dt+\frac{j\sin^2\theta}{\Xi}d\phi-\frac{2N}{\Xi}\cos\theta d\phi\right]^2 \\
& -\left[q^2-\Delta^2+(N+j\cos\theta)^2-\delta^2\right]\left(\frac{dq^2}{Q}+\frac{d\theta^2}{\Delta_\theta}\right) \qquad (5.91)\\
& -\frac{\Delta_\theta\sin^2\theta}{q^2-\Delta^2+(N+j\cos\theta)^2-\delta^2}\left[jdt+\frac{q^2+j^2+N^2-\Delta^2-\delta^2}{\Xi}d\phi\right]^2,
\end{aligned}
$$

where now

$$
\Delta_\theta = 1-\frac{j^2}{l^2}\cos^2\theta-\frac{4Nj}{l^2}\cos\theta\,,
$$

while the fluxes and the scalar field are given respectively by

$$
\begin{aligned}
F^\Lambda \quad = \quad & \frac{\mathrm{P}^\Lambda(q^2-\Delta^2-(p-\delta)^2)-2(q-\Delta)p\mathrm{Q}^\Lambda}{\left[q^2-\Delta^2+(N+j\cos\theta)^2-\delta^2\right]^2}\sin\theta\left[jdt+\frac{q^2+j^2+N^2-\Delta^2-\delta^2}{\Xi}d\phi\right]\wedge d\theta \\
+ \quad & \frac{\mathrm{Q}^\Lambda(p^2-\delta^2-(q-\Delta)^2)-2(p-\delta)q\mathrm{P}^\Lambda}{\left[q^2-\Delta^2+(N+j\cos\theta)^2-\delta^2\right]^2}dq\wedge\left[dt+\frac{j\sin^2\theta}{\Xi}d\phi-\frac{2N}{\Xi}\cos\theta d\phi\right],
\end{aligned}
$$

$$
\tau = \frac{g_0}{g_1}\frac{q+\Delta-i(p+\delta)}{q-\Delta-i(p-\delta)}\,,
$$

with $p = N+j\cos\theta$.

If one turns off the rotation ($j=0$), and fixes the charges in terms of N,Δ,δ according to

$$
g_1\mathrm{P}^1 = \frac{N^2}{l^2}-\frac{N\delta}{l^2}+\frac{1}{4}\,, \qquad g_1\mathrm{Q}^1 = -\frac{\delta}{\Delta}g_1\mathrm{P}^1\,, \qquad (5.92)
$$

$$
g_0\mathrm{P}^0 = g_1\mathrm{P}^1+\frac{2N\delta}{l^2}\,, \qquad g_0\mathrm{Q}^0 = g_1\mathrm{Q}^1+\frac{2N\Delta}{l^2}\,, \qquad (5.93)
$$

one recovers the spherical NUT-charged BPS solution constructed in [127][9]. With the charges fixed as above, and

$$
a_1 = -\frac{4N}{\Delta}\left[\frac{2N\delta^2}{l^2}+\frac{2N\Delta^2}{l^2}-\frac{2N^2\delta}{l^2}-\frac{\delta}{2}\right],
$$

all the constraints (5.77)-(5.80) are satisfied.

From (5.91), we can also get a dyonic solution without NUT charge. Setting $N=0$ one has

$$
b_0 = j^2\,, \qquad b_1 = 0\,, \qquad b_2 = -1-\frac{j^2}{l^2}\,, \qquad a_2 = 1+\frac{j^2}{l^2}-\frac{2}{l^2}(\Delta^2+\delta^2)\,,
$$

[9]The flat or hyperbolic BPS solutions of [127] can be obtained in a similar way. Notice that only the latter represent genuine black holes, while in the spherical or flat case one has naked singularities [127].

$$a_0 = j^2 - \left(1 + \frac{j^2}{l^2}\right)(\Delta^2 + \delta^2) + \frac{(\Delta^2 + \delta^2)^2}{l^2} + 2l^2 \left[g_0^2({\rm P}^{0^2} + {\rm Q}^{0^2}) + g_1^2({\rm P}^{1^2} + {\rm Q}^{1^2}) \right] ,$$

and a_1 is given by (5.79). Since b_1 vanishes, (5.78) implies $\mathrm{P}^1 = 0$ or

$$g_0^2 \left(\mathrm{Q}^0\mathrm{Q}^1 + \mathrm{P}^0\mathrm{P}^1\right)^2 = \left(\mathrm{Q}^{1^2} + \mathrm{P}^{1^2}\right)\left(g_1^2 \mathrm{Q}^{1^2} + g_0^2 \mathrm{P}^{0^2}\right) , \tag{5.94}$$

which allows to express e.g. Q^0 in terms of the other charges. The solution is thus specified by the five parameters $\mathrm{P}^0, \mathrm{P}^1, \mathrm{Q}^1, j, \Delta$, or alternatively by three charges, angular momentum and mass.

Note that, also in the case with nonvanishing N, (5.78) together with the second eqn. in (5.90) fix one of the electromagnetic charges in terms of the other parameters, and therefore the solution is labelled by three independent U(1) charges, NUT charge, angular momentum and mass. It is thus not the most general solution, which should have four independent U(1) charges.

5.4 Solution with harmonic functions and flat limit

We can partially rewrite the ansatz in terms of complex harmonic functions in order to make the dependence on the prepotential more suggestive. If we define the variable $\rho = q - ip$, we can use the harmonic functions in ρ,

$$X^0 \equiv H_0 = h_0 (1 - \frac{\Delta - i\delta}{\rho}) , \qquad X^1 \equiv H_1 = h_1 (1 + \frac{\Delta - i\delta}{\rho}) , \tag{5.95}$$

with $h_0 = g_0^{-1}, h_1 = g_1^{-1}$. We can then rewrite the scalar ansatz as

$$\tau = \frac{H_1}{H_0} , \tag{5.96}$$

while the function W appearing in the metric and gauge field ansätze (5.73), (5.74) can be cast into the form

$$W = l^2 (q^2 + p^2) e^{-K(X^\Lambda)} , \tag{5.97}$$

where K is the Kähler potential of special geometry that depends on the prepotential[10]. In the case of $F = -iX^0X^1$, we have

$$e^{-K}_{F=-iX^0X^1} = \frac{1}{l^2(q^2+p^2)} (q_1 q_2 + p_1 p_2) , \tag{5.98}$$

as needed.

[10]Note that here we do not mean the physical Kähler potential $K(\tau, \bar\tau)$, but the one that is obtained directly from the sections (5.95).

Rewriting the solution in this form makes it easy to take the limit of vanishing gauging. We take $g_0, g_1 \to 0$, keeping the ratio an arbitrary finite constant (which is the value of the scalar field at infinity). This leads to a simplification in the explicit parameters a_i, b_i that parametrize the functions $P(p)$ and $Q(q)$. We can again write the metric in the form (5.91), but now with $\Delta_\theta = \Xi = 1$. A further redefinition of the radial coordinate $q = r + m$ for $a_1 = -2m$ leads to

$$Q = r^2 + (j^2 + m^2 - N^2 - \Delta^2 - \delta^2) \ .$$

Written this way, the solution can be seen to sit inside the general class of solutions of [126] with arbitrary mass, angular momentum, electric and magnetic and NUT charges. Just like in the case with cosmological constant, we cannot recover the most general class due to the restriction that the NUT charge is fixed in terms of the electric and magnetic charges, c.f. (5.78) and (5.90).

5.4.1 General case: possible ansatz

Given the various examples and solutions we presented in the preceding sections, we can make an ansatz that is likely to yield solutions for more general prepotentials with arbitrary number of vector multiplets. The metric ansatz would remain

$$ds^2 = f(dt + \omega_y dy)^2 - f^{-1}\left[v\left(\frac{dq^2}{Q} + \frac{dp^2}{P}\right) - PQdy^2\right], \tag{5.99}$$

with

$$Q = a_0 + a_1 q + a_2 q^2 + g^2 q^4\,, \qquad P = b_0 + b_1 p + b_2 p^2 + g^2 p^4\,,$$

$$v = Q - P\ , \qquad f = ve^{2U}\ , \qquad e^{-2U} = i(\overline{X}^\Lambda F_\Lambda - X^\Lambda \overline{F}_\Lambda)\,,$$

$$\omega_y = -\frac{1}{v}\left(Q(c_0 + c_1 p + p^2) + P(d_0 + d_1 q + q^2)\right)\ , \tag{5.100}$$

scalar fields given by the symplectic sections

$$X^\Lambda = h^\Lambda(q - ip + \Delta^\Lambda - i\delta^\Lambda)\ , \tag{5.101}$$

and gauge fields

$$\begin{aligned} A^\Lambda \quad = \quad & \frac{1}{W}\Big(\mathrm{P}^\Lambda(p + k_0)(\mathrm{d}t + (d_0 + d_1 q + q^2)\mathrm{d}y) \\ & + \mathrm{Q}^\Lambda(q + l_0)(\mathrm{d}t - (c_0 + c_1 p + p^2)\mathrm{d}y)\Big)\ . \end{aligned} \tag{5.102}$$

The real constant parameters $a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, d_0.d_1, k_0, l_0, h^\Lambda, \Delta^\Lambda, \delta^\Lambda$, P^Λ, Q^Λ will eventually have to be expressed in terms of the physical parameters of a given solution (mass, angular momentum, NUT charge, electric and magnetic charges) upon solving the equations of motion in a chosen model. The question whether the above ansatz leads to solutions in models of the STU type is left for future research.

Discussion

In this chapter we have explicitly constructed a general rotating black hole solution of the FI-gauged model with prepotential $F=-iX^0X^1$. The solution is characterized by mass, 2 electric charges, 2 magnetic charges, angular momentum, NUT charge and cosmological constant (gauge coupling constant). However, for this configuration the NUT charge is determined in function of the other parameters of the solution, so in total we have $8-1=7$ free parameters.

During the write-up of our work [28] another paper [29] on the subject appeared. The authors of [29] managed to find a solution to the same model with all 8 parameters unconstrained. We verified explicitly that the solutions we presented here corresponds to a subset of the solutions presented in [29]. In particular, in their case the two parameters Δ_1, Δ_2 appearing in (5.29) are unconstrained, while for our configuration $\Delta_1 = -\Delta_2$.

It would be interesting to understand if the solution with 8 independent parameters exhaust the space of possible solutions to that model, namely if the solution is the most general solution to the Einstein's equations for this kind of prepotentials. This is a highly nontrivial task that requires techniques that are beyond the scope of this thesis.

We mention finally that for the simple prepotential $F=-iX^0X^1$ the BPS limit in both the static and the rotating case contains naked singularities. One could try to construct the rotating configurations considering the more interesting model $F=-2i\sqrt{X^0(X^1)^3}$, where the static BPS limit is regular. The obstruction in this case is that, given the ansatz in section 5.4.1, annoying square root factors in the function e^{-2U} (5.100) appear, that make the analytic computation less tractable with respect to the case of the prepotential $F=-iX^0X^1$.

Chapter 6

Thermodynamics and phase transitions

In this chapter we analyze the thermodynamics of the magnetic thermal solution found in [96, 99] described in Chapter 4. We first discuss the thermodynamic quantities associated to these configurations. The temperature and the entropy of the solutions are unambiguously defined and straightforwardly computed. We need to take additional care for what concerns the definition of the mass. We make use of the Ashtekar-Magnon-Das (AMD) [100, 101] procedure for the computation of conserved quantities for asymptotically Anti-de Sitter configurations; we verified that for the magnetic solutions the AMD mass indeed satisfies the first law of thermodynamics.

We subsequently elaborate on the phase transitions in the canonical ensemble. We find that a first order phase transition arises between small and large black hole solutions [139]. The picture on the gravity side presents the same qualitative features found by [140] in the case of AdS-RN black holes, namely in absence of scalars. However, in our case the scalar field also undergoes some change and the phase transition corresponds to the process in which a small black hole with scalar hair turns into a large, less-hairy one. The holographic dictionary allows us to interpret this as a liquid-gas-like phase transition in the dual field theory.

First, let us briefly recap and highlight the properties of black hole thermodynamics in Anti-de Sitter spacetime.

6.1 Thermodynamics of Anti-de Sitter black holes

It is a well-known fact that the Anti-de Sitter arena stabilizes the thermodynamics of black holes. Indeed, in Anti-de Sitter multiple branches of stable black hole exist, characterized by positive specific heat. Phase transitions between them can arise, resulting in a very rich thermodynamics.

Let us consider first, for comparison, the Schwarzschild black hole solution in absence of cosmological constant ($\Lambda = 0$). The temperature of the black hole is $T = 1/(8\pi M)$ where M is the mass of the black hole. The black hole has negative specific heat

$$C_s = T\left(\frac{\partial S}{\partial T}\right) = -8\pi M^2\,. \tag{6.1}$$

Such configuration can never be in equilibrium with a reservoir at a fixed temperature: a tiny increase in the black hole energy will result in a decrease of its temperature. The black hole will keep on absorbing heat from the reservoir, cooling down in the meantime. Vice versa, a decrease in energy will heat up the black hole more and more. In this sense a Schwarzschild black hole is thermodynamically instable in an ensemble of fixed temperature.

Like black holes in asymptotically flat space, Anti-de Sitter black hole solutions have thermodynamic properties including a characteristic temperature and entropy equal to one quarter of the area of the event horizon. However, in contrast with its asymptotically flat counterpart, the Schwarzschild AdS solution can have positive specific heat C_s. There exist a minimal temperature above which the black hole has positive specific heat, hence it can be in equilibrium with thermal radiation at a fixed temperature.

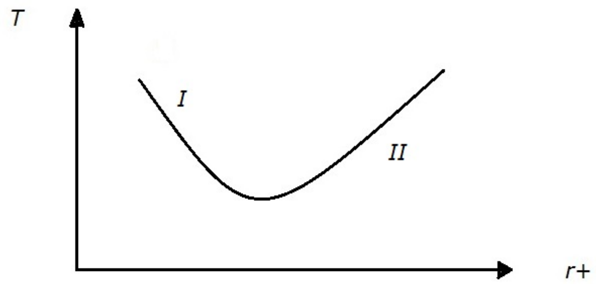

Figure 6.1: Temperature VS radius of the event horizon for Schwarzschild-AdS black holes. The minimum in the curve is reached for $r_+ = R_{AdS} = \sqrt{-\frac{3}{\Lambda}}$.

It is instructive to look at the plot of the temperature in function of the radius of the horizon, Fig. 6.1. For a given temperature, there are two distinct solutions, with different radius of outer event horizon r_+. We call them small (branch I, negative slope) and large (branch II, positive slope) black holes, with reference to the area of the event horizon. The small ones are unstable, namely they have negative specific heat, while for the large ones $C_S > 0$. It was found by Hawking and Page [19] that four-dimensional Anti-de Sitter spacetime filled with thermal radiation undergoes a phase transition to a large black hole when the temperature is raised above a specific value T_{HP},

$$T_{HP} \propto \sqrt{|\Lambda|}\,. \tag{6.2}$$

Using the AdS/CFT dictionary, this phase transition in case of AdS_5 uncharged black holes was interpreted as a confinement-deconfinement phase transition in the

$\mathcal{N} = 4$ SYM dual field theory [141]. In this case, AdS plus thermal radiation is the confined phase, while the black hole phase represents the deconfined phase.

The presence of electric and magnetic charge alters the phase diagram. Indeed, in the canonical ensemble (fixed charge), if the charge is small enough another stable black hole branch appears in the $T - r_+$ plane:

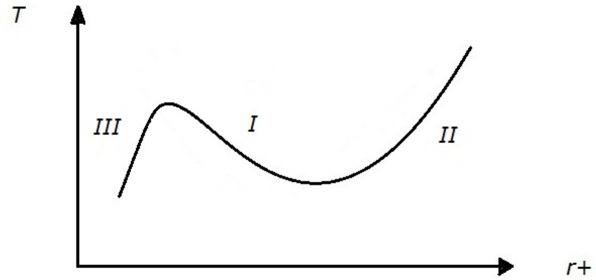

Figure 6.2: Temperature VS radius of the event horizon for a Reissner-Nordström (RN) AdS black hole, for small values of electromagnetic charge. Notice that the presence of a nonvanishing negative cosmological constant Λ is responsible for branch II. Switching off the cosmological constant branch II disappears, and the diagram corresponds to that of an asymptotically flat RN black hole. The flat Schwarzschild solution corresponds to the absence of charge, namely just branch I. In both cases $\Lambda < 0$ and $\Lambda = 0$ the presence of charge makes the appearance of branch III possible.

For a given temperature we have now at most three AdS-RN black holes branches: small (branch III), medium-sized (branch I) and large (branch II) [140, 142]. Branch III was absent in the uncharged case. These black holes have a radius of event horizon r_+ respectively smaller, comparable and bigger with respect to the AdS radius.

In [140, 142] it was discovered that phase transitions can arise between small (III) and large (II) AdS-RN black holes, for a specific interval of electric charge. Branch I is unstable (negative specific heat), in analogy with the case of Schwarzschild-AdS, therefore it never enters in the physics of the system[1].

In the following we analyze the thermodynamics and the phase transitions of the class of magnetic solutions with non-trivial scalar profiles described in Chapter 4. This small-large black hole phase transition described a few lines above represents the prototype for the processes we study in this chapter. In our case, however, the phase diagram of the black hole configuration should be supplemented by the analysis of the scalar field behaviour. Indeed the presence of scalar fields will allow the interpretation of processes involving black holes in terms of phase transitions in the dual field theory, via the AdS/CFT correspondence.

[1]We highlight here a difference with respect to the Hawking-Page [19] scenario. In the charged case, one does not consider transitions from a black hole to "pure AdS space plus charge Q", since the latter is not a solution of the Einstein-Maxwell equations with fixed electric charge Q as boundary condition. In other words, the transitions considered here will concern only different branches of black holes.

Before proceeding, let us mention that further study of thermodynamics and stability of AdS electric configurations with scalars in dimensions $d = 4,5,7$ was undertaken in [83].

6.2 Magnetic black hole solution: a recap

The magnetically charged black hole solutions of [99], also described in Chapter 4, have all fermions set to zero and are given by the spacetime metric

$$\mathrm{d}s^2 = U^2(r)\mathrm{d}t^2 - U^{-2}(r)\mathrm{d}r^2 - h^2(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\,, \tag{6.3}$$

with

$$U^2(r) = e^{\mathcal{K}}\left(g^2r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2}\right)\,, \qquad h^2(r) = e^{-\mathcal{K}}r^2\,. \tag{6.4}$$

In these coordinates, the field strengths are given by $F^{\Lambda}_{\theta\phi} = p^{\Lambda}\sin\theta/2$, purely magnetic. We do not allow for electric charges. The vector multiplet contains a complex scalar field z that parametrizes a special Kähler manifold with Kähler potential $\mathcal{K}$ given by

$$z = \frac{X^1}{X^0}\,, \qquad \mathcal{K} = -log[X^0\overline{X}^0(\sqrt{z} + \sqrt{\bar{z}})^3]. \tag{6.5}$$

The general thermal solutions with running scalars have

$$X^0 = \frac{1}{4\xi_0} - \frac{\xi_1 b_1}{\xi_0 r}\,, \quad X^1 = \frac{3}{4\xi_1} + \frac{b_1}{r}\,, \quad c = 1 - \frac{32(g\xi_1 b_1)^2}{3}\,, \tag{6.6}$$

In this case we focus our attentions to configurations with Dirac quantization condition (1.87)

$$g\xi_\Lambda p^\Lambda = -1\,, \tag{6.7}$$

in order to have an explicit one-parameter deformation of the BPS solution. The parameters appearing in the warp factors assume this form:

$$Q = -\frac{16}{3}b_1^2\xi_1^2 + \frac{1}{g^2} - \frac{256}{27}b_1^4\xi_1^4 g^2 + \frac{2\xi_1 p^1}{g} + \frac{4}{3}\xi_1^2(p^1)^2\,, \tag{6.8}$$

and

$$\mu = \frac{8}{3}\xi_1 b_1 - \frac{3}{4g^2\xi_1 b_1} + \frac{512}{27}g^2\xi_1^3 b_1^3 - \frac{3p^1}{2gb_1} - \frac{2\xi_1(p^1)^2}{3b_1}\,. \tag{6.9}$$

The configuration (6.3-6.9) is a solution to the full non-linear set of coupled equations of motion for the metric, gauge fields and complex scalar. For fixed gravitino charges g_Λ, the solution contains two parameters, which we can choose to be b_1 and p^1 (with p^0 fixed by the quantization condition (6.7)).

The solution for the scalar field is real. Its asymptotic expansion is $z = 3\frac{\xi_0}{\xi_1} + 16\xi_0\frac{b_1}{r} + \mathcal{O}(r^{-2})$ where the constant term is dictated by the value of the scalar in the (magnetic) AdS$_4$ vacuum at infinity. The extremum of the potential V_* sets the value of the cosmological constant which in our conventions is

$$\Lambda = 3V_* = -2\sqrt{3}g^2\sqrt{\xi_0\xi_1^3}\,.$$

After truncating the imaginary part of z, we can write down the Lagrangian for the real part, canonically normalized as $\phi = \sqrt{\frac{3}{8}}\ln z$. We then get, using the special Kähler quantities in App. A:

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}R(g) - e^{\sqrt{6}\phi}F^0_{\mu\nu}F^{0\mu\nu} - 3e^{-\sqrt{\frac{2}{3}}\phi}F^1_{\mu\nu}F^{1\mu\nu} \\ &+ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g^2(\xi_0\xi_1 e^{-\sqrt{\frac{2}{3}}\phi} + \frac{\xi_1^2}{3}e^{\sqrt{\frac{2}{3}}\phi})\ .\end{aligned} \tag{6.10}$$

For values for which $3\xi_0 = \xi_1$, one gets the cosh-potential of [64], obtained from a truncation of eleven-dimensional supergravity compactified on S^7. The extremum of the potential is at $\phi_* = \sqrt{\frac{3}{8}}\ln(3\xi_0/\xi_1)$, which is the maximum. Expanding around this value, one finds the mass of the scalar to be $m^2_\phi = -2|V_*|/3$. One can then verify that the mass satisfies the Breitenlohner-Freedman bound [23], and in the conventions of e.g. [143] in which $V_* = -3$, we have $m^2_{BF} = -9/4 < m^2_\phi = -2 < m^2_{BF}+1$. Asymptotically, the scalar field behaves as

$$\phi - \phi_* = \frac{\alpha}{r} + \frac{\beta}{r^2} + \mathcal{O}(r^{-3})\ , \tag{6.11}$$

with

$$\beta = \alpha^2/\sqrt{6} \qquad \text{and} \qquad \alpha = 4\sqrt{2/3}\,b_1\xi_1\,. \tag{6.12}$$

We have, again in the conventions of [143], $\beta = k\alpha^2$ with a fixed value of $k = 1/\sqrt{6}$. More general solutions with scalar hair that allow for arbitrary values of k might exist and we will comment on this towards the end of the chapter.

The magnetic charge $P \equiv p^1$ can be varied freely and defines three regions of black hole solutions with running scalars: $P > P_I$, $P_I > P > P_{III}$ and $P < P_{III}$. The values $P_I = -\frac{3}{4g\xi_1}$ and $P_{III} = -\frac{3}{2g\xi_1}$ are the only ones allowed for constant scalar ($b_1 = 0$) solutions, see Chapter 4, eq. (4.43)- (4.44). For other values of the magnetic charge, the limit to constant scalars $b_1 = 0$ is not smooth. This is different compared with the situation in [140, 142], where there are no scalars and the magnetic charge is free.

A black hole solution can therefore be plotted in the (p^1, b_1) plane, as shown in Fig. 6.3. Not all points correspond to solutions with horizons. Singular (denoted by white background in the figure) and regular (shaded areas) solutions are separated by two curves. The first one (green curve) corresponds to the 1/4-BPS solutions of [30, 45, 70] of Chapter 2, and is defined by the relation $p^1 = -\frac{1}{4g_1}(3-2\alpha^2 g^2)$, such that $U^2(r) = e^K(gr + \frac{c}{2gr})^2$ with horizon located at $r_+^2 = -c/(2g^2)$ with $c < 0$.

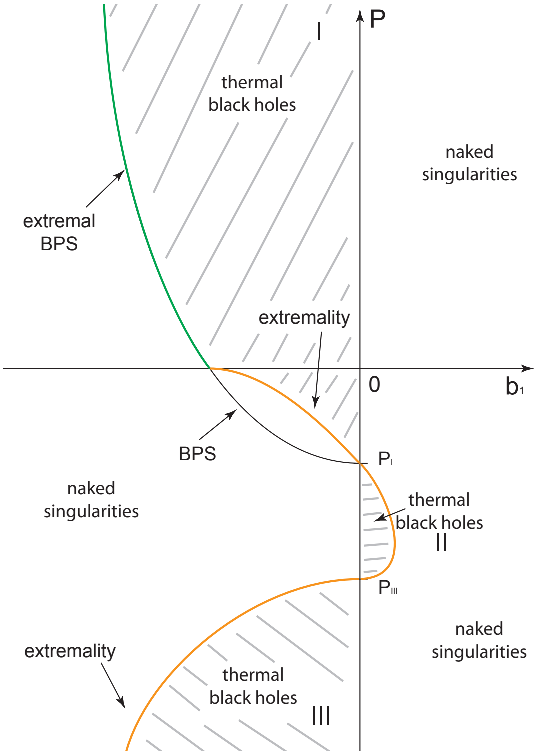

Figure 6.3: Parameter space for genuine black hole solutions (shaded) in regions I, II and III.

The second curve (in orange) defines the extremal, but non-BPS black hole solutions. It can be obtained from setting $T = U^2 = 0$, excluding the BPS black holes and solutions with naked singularities. It yields conditions that can be analyzed numerically and the result is in Fig. 6.3.

From now on in the explicit computations we will set the Fayet-Iliopoulos parameters to a constant value, namely $\xi_1 = 3/\sqrt{2}$, $\xi_0 = 1/\sqrt{2}$, such that $V_* = -3g^2$.

6.3 First law of thermodynamics

Four dimensional AdS black holes satisfy the first law of thermodynamics that reads[2]:

$$\mathrm{d}M = T\mathrm{d}S - \phi^{\Lambda}\mathrm{d}q_{\Lambda} + \chi_{\Lambda}\mathrm{d}p^{\Lambda} + \omega_r \mathrm{d}J\,. \tag{6.13}$$

where T is the temperature of the black hole, S is the entropy, M is the mass, q_Λ and p^Λ are as usual respectively the electric and magnetic charges, ϕ^Λ and χ_Λ are the electrostatic and magnetostatic potentials, ω_r is the difference between the angular velocities at the horizon and at infinity [131] and J is the angular momentum.

For our static magnetic black hole case the first law reduces to

$$\mathrm{d}M = T\mathrm{d}S + \chi_{\Lambda}\mathrm{d}p^{\Lambda}\,. \tag{6.14}$$

We give now definitions for the quantities appearing in this equation.

6.3.1 Temperature

The metric is of the form (6.3), and we denote the position of the outer and inner horizon respectively with r_+, r_-. Then, introducing a new radial coordinate $\rho = \sqrt{2\kappa^{-1}|r-r_+|}$ we can write the near horizon $r \to r_+$ metric as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2(i\kappa \mathrm{d}t)^2 + h^2(r_+)\mathrm{d}\Omega^2\,, \tag{6.15}$$

where the surface gravity κ is defined as

$$\kappa = \frac{1}{2}|2U(r)U(r)'|_{r=r_+}\,. \tag{6.16}$$

The metric (6.15) has a regular point for $\rho = 0$ if $it\kappa$ has period 2π, namely if it is an angular variable with period $2\pi/\kappa$. The Hawking temperature at the horizon is given by the inverse of this period:

$$T = \frac{\kappa}{2\pi}\,. \tag{6.17}$$

For the nonextremal solution (6.6)-(6.9) we have:

$$T = \frac{1}{4\pi}\left|e^{\mathcal{K}(r_+)}\left(4g^2 r_+ + \frac{\mu}{r_+^2} + 2\frac{c}{r_+}\right)\right|\,, \tag{6.18}$$

[2] We will not consider ensembles in which the cosmological constant Λ varies. In those cases a further term appears in the first law [144, 145]

$$\mathrm{d}M = T\mathrm{d}S - \phi^{\Lambda}\mathrm{d}q_{\Lambda} + \chi_{\Lambda}\mathrm{d}p^{\Lambda} + \omega_r\mathrm{d}J + \theta\mathrm{d}\Lambda\,,$$

where the cosmological constant provides a "pressure" term, and Θ is a so-called "effective volume" inside the event horizon [144]. Θ appears also in the Smarr relation [144, 146].

where we used the fact that at the horizon the function $U(r)$ vanishes. The radius of the outer event horizon r_+ is given by the largest root of the quartic polynomial obtained by setting $U^2 = 0$. A closed formula for computing the roots of a quartic polynomial exists, nevertheless it is quite cumbersome and not particularly illuminating for our purposes. For this reason so we do not write down the explicit expression for r_+, but we leave it implicit.

We can actually rewrite the expression for the temperature in a slightly more convenient way, namely in function of the difference $(r_+ - r_-)$. Indeed, the metric can be expressed as:

$$U^2(r) = e^{\mathcal{K}}(r-r_+)(r-r_-)(x+\frac{y}{r}+\frac{t}{r^2})\,, \tag{6.19}$$

with parameters:

$$g^2 = x\,, \quad c = xr_+r_- + t - y(r_+ + r_-)\,, \quad \mu = r_+r_-y - (r_+ + r_-)t\,, \quad c_2 = r_+r_-t\,,$$

with the constraint $y = x(r_+ + r_-)$. Then the temperature is given by

$$T = \frac{1}{4\pi}|e^{\mathcal{K}}(r_+ - r_-)(x+\frac{y}{r_+}+\frac{t}{r_+^2})|\,. \tag{6.20}$$

6.3.2 Entropy

The Bekenstein-Hawking entropy is related to the area of the event horizon of the magnetic black hole. We can straightforwardly compute it as

$$S = \frac{A}{4} = \pi h^2(r_+) = \pi r_+^2 e^{-\mathcal{K}(r_+)}\,. \tag{6.21}$$

where r_+ is the radius of the outer event horizon.

6.3.3 Magnetostatic potentials

The electrostatic potential ϕ^Λ for a black hole is defined as

$$\phi^\Lambda = -\int_{r_+}^{\infty} F^\Lambda_{tr}\mathrm{d}r\,, \tag{6.22}$$

where $F_{\mu\nu}$ is the eletric field strength. In complete analogy, for the magnetostatic potential we consider the dual field strength in eq. (1.32):

$$G_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{\delta\mathscr{L}}{\delta F^\Lambda_{\rho\sigma}}\,, \tag{6.23}$$

where $\mathscr{L}$ is (6.10), and we compute the corresponding potential as

$$\chi_\Lambda = -\int_{r_+}^{\infty} G_{\Lambda,tr}\mathrm{d}r\,. \tag{6.24}$$

For the case at hand, the magnetic potentials χ_Λ can be straightforwardly computed to be

$$\chi_0 = -\frac{1}{\sqrt{2}}\frac{\sqrt{2}+3p^1}{(\sqrt{2}r_+ - 12b_1)}\,, \qquad \chi_1 = \frac{1}{\sqrt{2}}\frac{3p^1}{(\sqrt{2}r_+ + 4b_1)}\,. \tag{6.25}$$

6.3.4 Mass

Computation of the mass in an Anti-de Sitter background requires care. The Komar integral associated to the Killing vector K^μ reads

$$Q[K] = -\frac{1}{8\pi}\oint_{S^2} d\Sigma_{\mu\nu}D^\mu K^\nu\,, \tag{6.26}$$

where S^2 is a 2-sphere at spatial infinity. $Q[K]$ is a conserved quantity, and when K^μ is the time translation vector field (in this case $K^t = (1,0,0,0)$) formula (6.26) gives the mass of the black hole configuration. For asymptotically AdS configurations the Komar definition suffers from the problem that the integral diverges for $r \to \infty$. One therefore has to perform a subtraction of the background contribution in order to obtain a finite result. Finding an unambiguous procedure to do this can be problematic.

In the Abbott and Deser [22] approach, one takes pure AdS space as the reference background, so a pure AdS term is subtracted by the Komar integral (other approaches to Komar integrals in AdS spacetime were developed in [147]). This procedure is not unambiguously defined, especially for configurations supported by scalar fields with weaker metric fall off conditions at infinity. Moreover, in some cases the result obtained with the background subtraction failed to satisfy [148] the standard first law of thermodynamics (6.13).

One could alternatively define the mass integrating the first law. One advantage to do that is that the quantities on the right-hand side of (6.13) are unambiguously defined, and their computation present no complications associated with divergent behaviour at radial infinity. One could compute the right-hand side of (6.13) and then integrate the result and get a finite expression for M.

The Ashtekar-Magnon-Das (AMD) [100, 101] definition expresses the mass in terms of an integral of certain components of the Weyl tensor over the spatial conformal boundary at infinity. Since the metric approaches AdS asymptotically, the integrand falls off and the integral is inherently well-defined. It was shown in [148, 149] that the mass obtained by integration of the first law and the AMD calculation are in agreement for the uncharged and charged 4d rotating black holes. This was found to be valid also for various black hole configurations in five [150] and higher dimensions [149].

This pattern seems to be confirmed by our computations: indeed for all solutions presented in this thesis we verified that the mass obtained from the AMD [100, 101] procedure satisfies the first law as in (6.13)[3].

[3]With the exception of the rotating solutions with NUT charge of Section 5.3, for which the computation

This holds in particular also for the magnetic configuration (6.3)-(6.9). Formulas for the computation of the mass through the AMD procedure are shown in Appendix C. The value of the mass obtained for the magnetic black hole was already found in Chapter 4, formula (4.76) and in this case it specializes to[4]:

$$M = 2\sqrt{2}b_1 - \frac{1}{4\sqrt{2}g^2 b_1} - \frac{3p^1}{4gb_1} - \frac{(p^1)^2}{\sqrt{2}b_1}\,. \tag{6.29}$$

6.4 Phase transition

In the model taken into consideration here, the only charged particles are gravitini and gaugini. Given that we have electric gaugings, they possess electric charge $\pm g\xi_\Lambda$. There are no magnetically charged particles in the game, so the black hole cannot lose its magnetic charge[5].

In this section we discuss therefore the thermodynamical properties of the black holes in the canonical ensemble. This corresponds to describing a closed system at fixed temperature and charge, $\mathrm{d}T = \mathrm{d}p^\Lambda = 0$. We do not treat here the case of the grand-canonical ensemble, where the charge is allowed to vary but the chemical potentials are kept fixed.

In the canonical ensemble the free energy

$$F = M - TS\,, \tag{6.30}$$

is extremised at equilibrium, $\mathrm{d}F = 0$, due to the first law (6.14). One can therefore compare the free energy $F(T,P)$ of different solutions at fixed T and $P \equiv p^1$. The state of lowest free energy is thermodynamically preferred.

It is instructive to compare how the temperature behaves as a function of the outer horizon radius r_+ for constant magnetic charge in the region $-P_c \leq P \leq P_c$ and outside it (Fig. 6.4).

was not performed.

[4]Let us mention lastly that the first law implies the Smarr [146] relation, that in AdS spacetime reads [131, 144]

$$M = 2TS + \phi_\Lambda p^\Lambda - 2\Theta\Lambda\,, \tag{6.27}$$

where Θ has this expression:

$$\Theta = \frac{1}{\Lambda}\int_{r_{sing}}^{r_+} V_g(r)h^2(r)dr = \frac{1}{\Lambda}\int_{-\frac{4}{3}b_1}^{r_+} V_g(r)e^{-\mathcal{K}}r^2 dr\,. \tag{6.28}$$

The physical meaning of this quantity dubbed "effective volume" in [144] is somewhat mysterious. However, in the simple case of constant scalars we find that Θ is indeed proportional to the volume inside the horizon, hence it is seen as the thermodynamical variable conjugate to the pressure Λ. More details can be found in [144].

[5]We do not take into consideration the possibility for a black hole to split into smaller ones (hence forming a multicenter configuration). Possible presence of multicenter solutions in Anti-de Sitter has been recently investigated in [111, 151], but exact solutions of this kind have not been discovered so far. We also do not consider the possible presence of other magnetic solitonic objects.

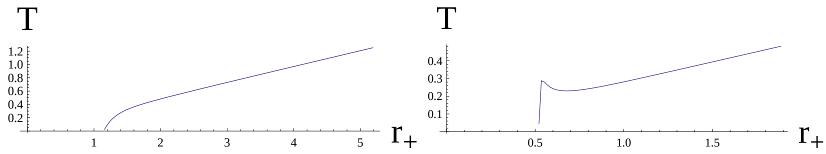

Figure 6.4: Temperature vs. outer horizon radius r_+ for fixed values of $|P| > P_c$ (left, $P = 0.5$) and $|P| < P_c$ (right, $P = 0.01$). Numerical analysis shows that $P_c \approx 0.025$ for the values $\xi_1 = 3/\sqrt{2}$, $\xi_0 = 1/\sqrt{2}$ and $g = 1$.

For $|P| > P_c$, the temperature is a monotonic function of the radius, and so there exists only one black hole for fixed T. In the region $|P| < P_c$, for fixed T there exist up to three different black holes - a small, a medium-sized, and a large black hole. To see which black hole is preferred in the canonical ensemble, we plot the free energy for these solutions:

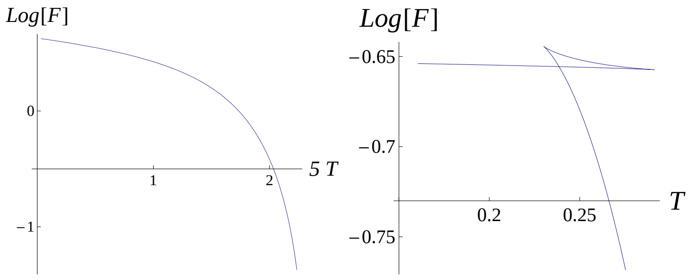

Figure 6.5: Free energy as a function of temperature for the same fixed values of P as in Fig. 6.4.

From Fig. 6.5 we see that for $|P| < P_c$, increasing T leads to the appearance of medium-sized and large black hole branches, whose free energy is initially higher than for small black holes [6]. Upon further increase of T, the free energy of the large black holes rapidly decreases, leading to a crossing point. The free energy is not smooth here, and so a first order phase transition between small and large black holes occurs, similar to [140, 142][7].

To check the intrinsic stability of the different phases against thermal fluctuations, one computes the specific heat C_P at fixed magnetic charge,

$$C_P \equiv T \left(\frac{\partial S}{\partial T} \right)_P \geq 0 \ , \tag{6.31}$$

[6] We have an exception of this only at $P = 0$, where the branch with larger black holes dominates over the smaller black holes everywhere. Therefore the critical lines do not cross the $P = 0$ axis, just as in [140, 142].

[7] It would be interesting to verify if at criticality the transition becomes of second-order. One should determine the exact location of the critical point, and this is difficult to perform just by looking at the plots. A more careful numerical analysis of the free energy and its derivatives is needed.

which must be positive for stable solutions. It turns out that the medium-sized black holes have negative specific heat and are therefore unstable under temperature fluctuations. They are never part of the physical picture. The other regions on Fig. 6.4, where the temperature grows with size, have a positive specific heat and are stable. This means that both the small and the large black hole phases are stable. There is a tiny region where this is not exactly true, as shown on Fig. 6.6 - close to the critical temperature $T_c \approx 0.226$, the phase with larger size black holes contains some negative specific heat solutions according to our numerical analysis. It would be interesting to perform further checks on these solutions, for example check the mechanical stability, in order to shed light on their unstable nature.

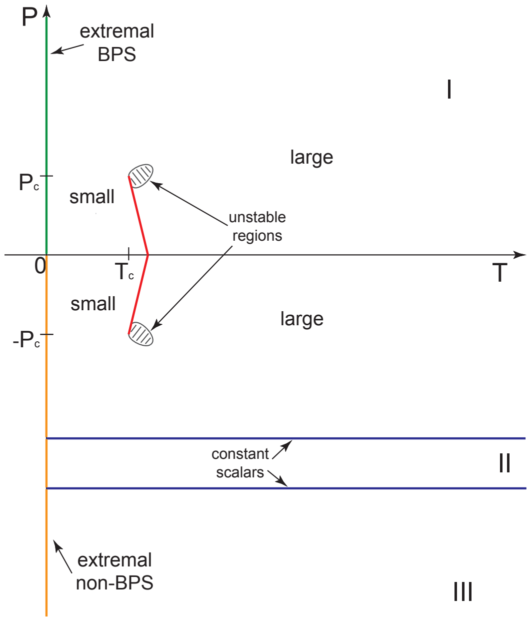

Figure 6.6: The phase diagram in the canonical ensemble. Regions I, II and III are separated by the two blue lines of thermal black holes with constant scalar profiles.

The exact values and sizes of each feature on the phase diagram depend on the values of the free positive constants g, ξ_0, ξ_1. One finds, for instance, that the value of P_c scales inversely with the coupling constant g, and we expect $P_c \sim (-\Lambda)^{-1/2}$, and $T_c \sim (-\Lambda)^{1/2}$, with cosmological constant Λ.

The phase diagram in the canonical ensemble is shown in Fig. 6.6, and it summarizes the result of [139]. We denote by $P = p^1$ one of the two magnetic charges carried by the black hole. In region I we find a small/large black hole first order phase transition for $|P| < P_c$ and a smooth crossover between the two phases for $|P| > P_c$.

As said before, apart from a very small region close to criticality, the phase diagram consists of solutions with positive specific heat that are stable against thermal fluctuations. In regions II and III, as well as the two lines with thermal black holes for constant scalars, BPS solutions are absent, and one finds a smooth crossover between small and large black holes that are thermally stable everywhere in phase space. For constant scalars, the magnetic charge is fixed by the Dirac quantization condition and falls outside the critical region. Hence there is no phase transition in our case for constant scalars.

6.5 Dual description via AdS/CFT

The supergravity models we have discussed can be embedded in M-theory compactified on $AdS_4\times S^7$ [70, 83], and have field theory duals in the class of ABJM models [27] defined on $\mathbb{R}\times S^2$. The bulk gauge group $U(1)\times U(1)$ corresponds to part of the global R-symmetry group on the boundary, which can be gauged to produce background magnetic fields on the boundary with flux through the S^2.

The bulk model contains a scalar field whose mass $m_\phi^2 = -2$ lies within the interval $m_{BF}^2 < m_\phi^2 < m_{BF}^2 + 1$, therefore both modes in (6.11),(6.12) are normalizable[8]. The boundary values provide sources for operators of dimensions equal to one or two, depending on the choice of quantization ($\alpha = 0$ being the standard choice of boundary conditions). Examples of such operators in three dimensions are bilinears of boundary scalars φ or fermions ψ transforming under the global R-symmetry group, namely

$$\mathcal{O}_1 = \mathrm{Tr}(\varphi^I a_{IJ} \varphi^J)\ , \qquad \mathcal{O}_2 = \mathrm{Tr}(\psi^I b_{IJ} \psi^J)\ , \tag{6.32}$$

for some constant matrices a and b.

So-called *mixed boundary conditions* [115], for which $\beta = \beta(\alpha)$, are also allowed. In this case, the ABJM action S_0 is deformed by multitrace operators, along the lines of [143, 152–154]. In particular, boundary conditions of the form

$$\beta = k\alpha^2 \tag{6.33}$$

are responsible for triple trace operator deformations [115, 152] of ABJM:

$$S = S_0 + k\int \mathcal{O}_1^3\,. \tag{6.34}$$

[8] We recall here that in AdS_d, solutions to $\Box\phi - m^2\phi^2 = 0$ fall off asymptotically ($r\to\infty$) like

$$\phi - \phi* = \frac{\alpha}{r^{\Delta_-}} + \frac{\beta}{r^{\Delta_+}}\,,$$

where

$$\Delta_\pm = \frac{d-1\pm\sqrt{(d-1)^2+4m^2}}{2}\,.$$

In the case of our solutions (6.3)-(6.9), k is fixed to the value $k = 1/\sqrt{6}$, see eq. (6.12).

The holographic dictionary in presence of mixed boundary conditions was worked out by Papadimitriou in [154]. It turns out that the values of source and VEV get modified due to the mixed boundary conditions. In particular, the source for the operator $\mathcal{O}_1$ is function of β (further details are in the original paper, [154], Table 3), while α is the expectation value for such an operator.

Since $\alpha \sim b_1$, we can use b_1 as an order parameter and express it as a function of the temperature. Doing so, we get the following plots,

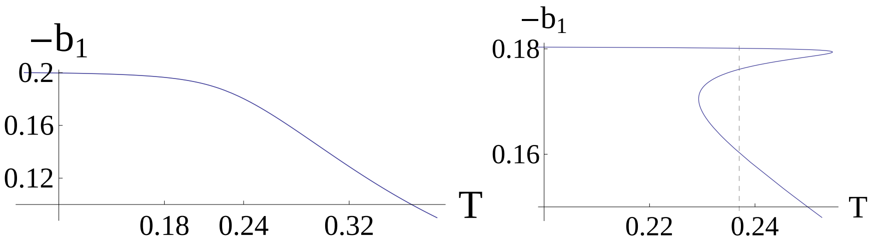

Figure 6.7: Behaviour of the order parameter b_1 in function of the temperature T for values $P > P_c$ (left, $P = 0.1$) and $P < P_c$ (right, $P = 0.015$).

The behavior of the order parameter mimics that of a first order liquid-gas phase transition, and one can determine the critical temperature from Maxwell's 'equal area' construction.

Furthermore, the parameter b_1 is responsible for the nontrivial radial profile ($b_1 = 0$ sets the scalar at a constant value throughout all the solution). The transition can then be seen as a (discontinuous) change in the scalar hair. If we combine this with the results of the previous section, the transition can be seen as a process turning a small hairy black hole into a large, less hairy one.

From Fig. 6.7 we see that the condensate is actually never vanishing for any finite temperature, and it goes to zero in the limit $T \to \infty$, so the global R-symmetry is always broken. This is very different with respect to what happens in the case of the holographic superconductor [17]. The latter process is indeed a spontaneous symmetry breaking: for temperatures $T < T_c$ below a critical value an operator gets a nonzero expectation value, with formation of a condensate. The R-symmetry is then spontaneously broken if $T < T_c$ and the transition is of second order. In the case [139], instead, the R-symmetry is always broken, since the expectation value of the operator is always nonvanishing. For this reason, the transition is liquid-gas-like, in a theory deformed by triple trace deformation. There is no spontaneous symmetry breaking.

Remark

Let us mention now that in [115, 155] a class of hairy uncharged black holes was discovered and the thermodynamics was subsequently studied. These are numerical solutions of maximal gauged supergravity in four dimensions satisfying mixed boundary conditions in the form (6.33), in which the parameter k is not fixed, but is tunable.

As we have seen, for our nonextremal magnetic solutions $k = 1/\sqrt{6}$ is fixed. Therefore, a natural question to ask would be: could we find a broader class of magnetic solutions that allows for more generic values of k? The answer requires the search for a further generalization of the solutions found in Chapter 4, and this comes back to the problem of finding exact solutions to the Einstein's equations, for sure a nontrivial task. Nevertheless, in Chapter 8 we will develop a first order formalism for nonextremal black holes that provide a concrete strategy in finding such solutions. We will give a set of first order equations, whose solutions correspond to configurations with tunable k. Finding solutions to these equations is work in progress. The interested reader will find more details in Chapter 8, section 8.1.4.

As a last remark, we verified that for the electric solutions of [64] the parameter k is fixed the value $k = -1/\sqrt{6}$. Since k is related to the CFT deformation and BPS solutions of [30] and [64] should correspond to supersymmetric states in the dual field theory, one might think that these two values of k, $k = \pm 1/\sqrt{6}$, are the only compatible with supersymmetry in the dual field theory. It would be interesting to see if this is indeed true.

6.6 Rotating solutions

Our analysis [139] so far dealt just with static configurations. With the inclusion of additional parameters in the solution, for instance angular momentum, new phase transitions could arise.

The thermodynamics of rotating Anti-de Sitter black holes was already studied in [136], for the case of Kerr-AdS black holes. In this case the dual CFT lives in a rotating Einstein universe, and in [136, 156] interesting correspondences have been found between the divergences in the partition function of the conformal field theory and the action of the black hole at the critical angular velocity at which the boundary rotates at the speed of light.

In the paper of Caldarelli, Cognola and Klemm [131] the thermodynamics of four-dimensional Kerr-Newman-AdS black holes was analyzed, both in the canonical and grand-canonical ensembles. Thermodynamic stability of these black holes was also investigated. It was shown there that in the canonical ensemble a first order transition exists between small and large black holes, and such transition disappears for sufficiently large electric charge or angular momentum.

It would be interesting to see if similar features appear in the rotating solution described in Chapter 5 and find their interpretation via AdS/CFT. The metric (5.37)-

(5.38) is more complicated and it is supported by complex scalar fields. For what concerns the gravity side, we can already gain some information by looking at Plot 6.8, which gives the temperature in function of the radius of the horizon for specific values of magnetic charge and angular momentum for the configuration (5.37)-(5.38):

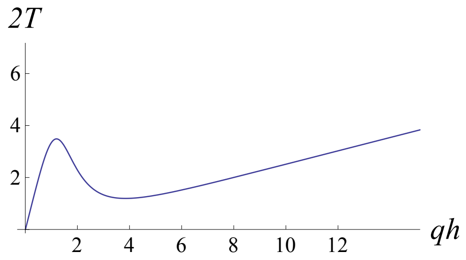

Figure 6.8: Plot of the temperature T in function of the radius of the event horizon q_h for the rotating configuration with $p^1 = 2$, $p^0 = 1$, $g_0 = g_1 = 1$, $J = 0.1$, $\theta = 0$

We can already see from Plot 6.8 that there are multiple configurations with the same charge, angular momentum and temperature. There is therefore the possibility of a phase transition arising between them. Notice that in the rotating solution (5.37)-(5.38) if the charges p^0, p^1 are set to zero, also the angular momentum vanishes (5.48). Therefore we have to analyze the thermodynamics with all these quantity at a finite nonzero value. Study of the thermodynamics of this configuration is work in progress.

Discussion

In this chapter we have analyzed the thermodynamics of the nonextremal magnetic black hole solutions of Section 4.4. In the canonical ensemble we have found a first order phase transition between small and large black hole for a sufficiently low value of the magnetic charge. This phase transition was interpreted via AdS/CFT in the dual field theory as a liquid-gas-like phase transition.

The thermodynamical behavior of the black holes discussed here should be reproduced by the boundary field theory thermal partition function on $S^1 \times S^2$. At zero temperature, degeneracies of the ground state of this system should explain the finite entropy of the BPS-black hole, presumably through the presence of Landau levels that arise because of the magnetic field. At finite temperature, we predict that the system undergoes a phase transition for magnetic charges below a critical value. It would be interesting to understand this phase transition directly in the boundary field theory, at the level of the thermal partition function, perhaps using the techniques recently developed by e.g. [157].

The phase transition illustrated in this chapter confirmed the observation that there is a remarkable resemblance between the thermodynamics of Reissner-Nordström AdS black holes and that of the Van der Waals-Maxwell liquid gas system. This resemblance persists also in dimensions greater than four. The analysis of the thermodynamics of AdS black holes in higher dimensions was performed in [158–160], where the inclusion of angular momentum was also treated. In higher dimensions some other curious features arise: for instance, the canonical ensemble of $d = 6$ rotating black holes [158] exhibit a large-small-large black hole transition, with features that are reminiscent of the solid/liquid/gas phase transition, along with the presence of a triple point.

Let us mention here that the thermodynamics of black holes with different topology of the event horizon also attracted some attention. In particular, phase transitions of black brane solutions in AdS are useful in AdS/CFT applications to condensed matter systems [161, 162], since in these cases the dual field theory lives in flat space. Lastly, hyperbolic black holes admit a dual description as thermal Rindler states of the CFT in flat space. The thermodynamics of black holes with hyperbolic horizon in Anti-de Sitter was studied in [163–165].

Chapter 7

BPS bounds in gauged supergravity

In this chapter we analyze the BPS bounds arising from minimal and non-minimally coupled $\mathcal{N}=2$ gauged supergravity, and the superalgebras underlying these theories.

The main motivation for our analysis comes from the following paradox. Similarly to BPS states in asymptotically flat spacetime, the authors of [119] provided a BPS bound in asymptotically anti-de Sitter spacetime that in the static case reduces to:

$$M \geq \frac{1}{2}\sqrt{Q_e^2+Q_m^2}\,. \tag{7.1}$$

Supersymmetric configurations would have to saturate the bound with $M^2=(Q_e^2+Q_m^2)/4$, for a given mass M, electric charge Q_e, and magnetic charge Q_m in appropriate units. However, in $\mathcal{N}=2$ minimal gauged supergravity, Romans [61] found two classes of supersymmetric solutions, that we already described in Chapter 2, Section 2.1. The first one is the 1/2 BPS electric AdS Reissner-Nordström solution with

$$Q_m=0\,,\qquad M=|Q_e|/2\,. \tag{7.2}$$

The other one is the 1/4 BPS cosmic dyon, characterized by

$$M=0\,,\qquad Q_m=\pm\frac{1}{g}\,,\qquad Q_e\ \text{arbitrary}\,. \tag{7.3}$$

While the first configuration (7.2) saturates the BPS bound (7.1), the cosmic dyon (7.3) solution does not. It actually violates the bound (7.1), therefore an apparent paradox arises[1]. The main aim of this chapter is to resolve this conflict. The resolution of the paradox lie in understanding the BPS ground states of gauged supergravity and the associated superalgebras.

We find that the cosmic monopole solution, due to the presence of magnetic charge, defines another vacuum of the theory (since $M=0$). This vacuum is topologically distinct from pure AdS_4 in which $Q_m=0$. We call this vacuum magnetic Anti-de Sitter, or $mAdS_4$.

[1]The Killing spinors for the cosmic dyon were explicitly constructed in [61] (see also Appendix E). States that admit a Killing spinor should saturate the BPS bound, hence the contradiction.

We show that the cosmic dyon in fact satisfies a different BPS bound that follows from a superalgebra different from the usual AdS_4 superalgebra. We will determine the new BPS bound starting from the explicit calculation of the supercharges and computing the anticommutator[2]. In summary, to state the main results, for stationary configurations the new BPS bounds are:

- For asymptotically AdS_4 solutions with vanishing magnetic charge, $Q_m = 0$, the BPS bound is
$$M \geq \frac{1}{2}|Q_e| + g|\vec{J}| \ , \tag{7.4}$$
where $\vec{J}$ is the angular momentum.

- For asymptotically magnetic AdS_4 solutions with[3] $Q_m = \pm 1/(g)$, the BPS bound is simply
$$M \geq 0 \ , \tag{7.5}$$
with unconstrained electric charge Q_e and angular momentum $\vec{J}$.

We will show the explicit computation for the case of minimal gauged supergravity (only gravity multiplet). In [103] this procedure was extended to the case in which matter is present (vector and hypermultiplets). We report in Section 7.4 the main results found in [103], and we discuss their applications to the case of the extremal and nonextremal (magnetic and electric) solutions with scalars described in Chapters 2 and 4.

7.1 Supercurrents and charges from the Noether theorem

In this section, we define and determine the Noether currents in minimally gauged supergravity. The currents define conserved supercharges, and the Poisson (or Dirac) brackets between these charges produce a superalgebra. In the next section, we derive BPS bounds from the superalgebra. We start this section by reviewing some (well-known) facts about Noether currents for local symmetries.

7.1.1 Generalities

Given a Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$, depending on fields collectively denoted by ϕ, we have that under general field variations
$$\delta\mathcal{L} = \sum_\phi \mathcal{E}_\phi \delta\phi + \partial_\mu N^\mu \ , \tag{7.6}$$

[2]An alternative approach based on the Witten-Nester energy was proposed in [166], leading to similar conclusions.

[3]Other values for the magnetic charges are not considered: it is not known if any other supersymmetric vacua can exist with (1.87) and $n \neq 0, 1$.

where $\mathcal{E}_\phi$ vanishes upon using the equation of motion of ϕ and

$$N^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta\phi \ . \tag{7.7}$$

Under a symmetry variation, parametrized by ϵ, the Lagrangian must transform into a total derivative, such that the action is invariant for appropriate boundary conditions,

$$\delta_\epsilon \mathcal{L} = \partial_\mu K^\mu_\epsilon \ . \tag{7.8}$$

Combining this with (7.6) for symmetry variations, we obtain

$$\sum_\phi \mathcal{E}_\phi \delta_\epsilon \phi = \partial_\mu (K^\mu_\epsilon - N^\mu_\epsilon) \, . \tag{7.9}$$

From the previous expression we see that the quantity

$$J^\mu_\epsilon \equiv K^\mu_\epsilon - N^\mu_\epsilon \ , \tag{7.10}$$

is the (on-shell) conserved current associated with symmetry transformations. For the case of supersymmetry, we call J^μ_ϵ the supercurrent. It depends on the (arbitrary) parameter ϵ and is defined up to improvement terms of the form $\partial_\nu I^{\mu\nu}$ where I is an antisymmetric tensor, as usual for conserved currents. The associated conserved supercharge is then

$$\mathcal{Q} \equiv \int \mathrm{d}^3 x \, J^0_\epsilon(x) \, . \tag{7.11}$$

This supercharge should also generate the supersymmetry transformations of the fields,

$$\delta_\epsilon \phi = \{\mathcal{Q}, \phi\} \, , \tag{7.12}$$

via the classical Poisson (or Dirac in case of constraints) brackets. Since the supercurrent and correspondingly the supercharge are defined up to improvement terms and surface terms respectively, it is not directly obvious that the Noether procedure will lead to the correct supersymmetry variations using (7.12). In practice one always has the information of the supersymmetry variations together with the supergravity Lagrangian, so it is possible to cross check the answers and thus derive uniquely the correct expression of the supercharge. We now illustrate this in detail for the case of minimally gauged supergravity.

7.1.2 Supercharges of minimal gauged supergravity

First we compute the supercurrent from the Lagrangian of minimal $D = 4$ $N = 2$ gauged supergravity following the conventions of [69] (which is written in 1.5-formalism):

$$\begin{aligned} S = \int \mathrm{d}^4 x \, e \frac{1}{2} \big[& R(e,\omega) + 6g^2 + \frac{2}{e} \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_\mu \gamma_5 \gamma_\nu \big(\hat{\mathcal{D}}_\rho + i \frac{g}{2} A_\rho \sigma^2 \big) \psi_\sigma - \mathcal{F}^2 \\ & - \frac{1}{2e} \epsilon^{\mu\nu\rho\sigma} \overline{\psi}_\rho \gamma_5 \sigma^2 \psi_\sigma (i \overline{\psi}_\mu \sigma^2 \psi_\nu - \frac{1}{2e} \epsilon_{\mu\nu}{}^{\tau\lambda} \overline{\psi}_\tau \gamma_5 \sigma^2 \psi_\lambda) \big] \, , \end{aligned} \tag{7.13}$$

where $\overline{\psi} = i\psi^\dagger\gamma_0$ and

$$\hat{\mathcal{D}}_\rho = \partial_\rho - \frac{1}{4}\omega_\rho^{ab}\gamma_{ab} - \frac{i}{2}g\gamma_\rho \ , \tag{7.14}$$

and

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) + i\overline{\psi}_\mu\sigma^2\psi_\nu - \frac{1}{2e}\epsilon_{\mu\nu}{}^{\rho\sigma}\overline{\psi}_\rho\gamma_5\sigma^2\psi_\sigma = \\ &= F_{\mu\nu} + i\overline{\psi}_\mu\sigma^2\psi_\nu - \frac{1}{2e}\epsilon_{\mu\nu}{}^{\rho\sigma}\overline{\psi}_\rho\gamma_5\sigma^2\psi_\sigma \, .\end{aligned} \tag{7.15}$$

The spin connection satisfies

$$\mathrm{d}e^a - \omega^a{}_b \wedge e^b = 0 \tag{7.16}$$

for a given vielbein $e^a = e_\mu^a \mathrm{d}x^\mu$. The g^2-term in the Lagrangian is related to the presence of a negative cosmological constant $\Lambda = -3g^2$.

In most of our calculations, such as in the supercurrents and supercharges, we only work to lowest order in fermions since higher order terms vanish in the expression of the (on shell) supersymmetry algebra, where we set all fermion fields to zero. The supersymmetry variations are:

$$\delta_\varepsilon\psi_\mu = \widetilde{\mathcal{D}}_\mu\varepsilon = (\partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - \frac{i}{2}g\gamma_\mu + i\frac{g}{2}A_\mu\sigma^2 + \frac{1}{4}F_{\lambda\tau}\gamma^{\lambda\tau}\gamma_\mu\sigma^2)\varepsilon \ , \tag{7.17}$$

$$\delta_\varepsilon e_\mu^a = -i\overline{\varepsilon}\gamma^a\psi_\mu \ , \tag{7.18}$$

$$\delta_\varepsilon A_\mu = -i\overline{\varepsilon}\sigma^2\psi_\mu \ . \tag{7.19}$$

$U(1)$ gauge transformations act on the gauge potential and on the spinors in this way:

$$A'_\mu = A_\mu + \partial_\mu\alpha \ , \tag{7.20}$$

$$\psi'_\mu = e^{-ig\alpha\sigma^2}\psi_\mu \ . \tag{7.21}$$

We use the conventions in which all the spinors are real Majorana ones[4], and the gamma matrix conventions and identities of Appendix A.

The quantities N^μ and K^μ for this theory are:

$$N^\mu = \frac{\partial\mathcal{L}}{\partial_\mu\omega}\delta\omega + 2\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu\gamma_5\gamma_\rho\widetilde{\mathcal{D}}_\sigma\varepsilon + 4ieF^{\mu\nu}\overline{\varepsilon}\sigma^2\psi_\nu \ . \tag{7.23}$$

[4] In this chapter, the two real gravitini in the gravity multiplet are packaged together in the notation:

$$\psi_\mu = \begin{pmatrix} \psi_\mu{}^1 \\ \psi_\mu{}^2 \end{pmatrix} , \tag{7.22}$$

where each gravitino is itself a 4-component Majorana spinor. Similar conventions are used for the supersymmetry parameters. In other words, the $SU(2)_R$ indices are completely suppressed in our notation of this chapter.

$$K^\mu = \frac{\partial \mathcal{L}}{\partial_\mu \omega}\delta\omega - 2\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu \gamma_5 \gamma_\rho \widetilde{\mathcal{D}}_\sigma \varepsilon + 4ieF^{\mu\nu}\overline{\varepsilon}\sigma^2\psi_\nu \ . \tag{7.24}$$

Hence the supercurrent has the form:

$$J^\mu = -4\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu \gamma_5 \gamma_\rho \widetilde{\mathcal{D}}_\sigma \varepsilon \ . \tag{7.25}$$

This expression is gauge invariant due to the cancelation between the variation of the gravitino, the vector field and the supersymmetry parameter. Furthermore we can show that the supercurrent is conserved on shell:

$$\partial_\mu J^\mu = \partial_\mu(-4\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\nu \gamma_5 \gamma_\rho \widetilde{\mathcal{D}}_\sigma \varepsilon) \ , \tag{7.26}$$

which vanishes if we enforce the equation of motion for $\overline{\psi}_\nu$ and use the antisymmetry of the Levi-Civita symbol.

The Dirac brackets defined for the given theory read (we only need those containing gravitinos):

$$\{\psi_\mu(x), \psi_\sigma(x')\}_{t=t'} = 0 \ , \tag{7.27}$$

$$\{\overline{\psi}_\mu(x), \overline{\psi}_\sigma(x')\}_{t=t'} = 0 \ , \tag{7.28}$$

$$\{\psi_\mu(x), 2\epsilon^{0\nu\rho\sigma}\overline{\psi}_\rho(x')\gamma_5\gamma_\sigma\}_{t=t'} = \delta_\mu{}^\nu \delta^3(\vec{x}-\vec{x}') \ . \tag{7.29}$$

We can now check if (7.12) holds with the above form of the supercurrent. It turns out that, up to overall normalization, we indeed have the right expression without any ambiguity of improvement terms. We only need to rescale, since the factor of 4 in (7.25) does not appear in the supersymmetry variations (7.17)-(7.19). The supercharge is then defined as the volume integral[5]

$$\mathcal{Q} \equiv 2\int_V \mathrm{d}\Sigma_\mu \epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\sigma \gamma_5 \gamma_\rho \widetilde{\mathcal{D}}_\nu \varepsilon \overset{e.o.m.}{=} 2\oint_{\partial V} \mathrm{d}\Sigma_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\sigma\gamma_5\gamma_\rho\varepsilon, \tag{7.31}$$

where the second equality follows from the Gauss theorem via the equations of motion (in what follows we will always deal with classical solutions of the theory). The Dirac bracket of two supersymmetry charges is then straightforwardly derived as the supersymmetry variation of (7.31):

$$\{\mathcal{Q},\mathcal{Q}\} = 2\oint_{\partial V} \mathrm{d}\Sigma_{\mu\nu}(\epsilon^{\mu\nu\rho\sigma}\overline{\varepsilon}\gamma_5\gamma_\rho\widetilde{\mathcal{D}}_\sigma\varepsilon) \ , \tag{7.32}$$

which is again a boundary integral. The above formula is reminiscent of the expression for the Witten-Nester energy [25, 167], which has already been implicitly assumed to generalize for supergravity applications [166, 168, 169]. Thus the correspondence between BPS bounds and positivity of Witten-Nester energy is confirmed also in the case of minimal gauged $\mathcal{N}=2$ supergravity by our explicit calculation of the supercharge anticommutator.

[5] For volume and surface integrals, we use the notation that

$$\mathrm{d}\Sigma_\mu = \epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ , \qquad \mathrm{d}\Sigma_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ . \tag{7.30}$$

7.2 Two different BPS bounds

In this section, we derive two BPS bounds based on the two BPS sectors that we consider. What is relevant for the BPS bound are the properties of the asymptotic geometries and corresponding Killing spinors. The Killing spinors of AdS_4 and magnetic AdS_4 ("cosmic monopole") are given in App. E, see also [61, 166]. Since only the asymptotics are important, we can set $Q_e = 0$ in the cosmic dyon solution. The AdS_4 solution is characterized by $M = Q_e = Q_m = 0$, while $mAdS_4$ by $M = 0$, $Q_e = 0$, $Q_m = \pm 1/g$. The corresponding Killing spinors take a very different form:

$$\varepsilon_{AdS} = e^{\frac{i}{2} arcsinh(gr)\gamma_1} e^{\frac{i}{2} gt\gamma_0} e^{-\frac{1}{2}\theta\gamma_{12}} e^{-\frac{1}{2}\varphi\gamma_{23}} \varepsilon_0 \ , \tag{7.33}$$

$$\varepsilon_{mAdS} = \frac{1}{4}\sqrt{gr + \frac{1}{2gr}}(1 + i\gamma_1)(1 \mp i\gamma_{23}\sigma^2)\varepsilon_0 \ , \tag{7.34}$$

where ϵ_0 is a doublet of constant Majorana spinors, carrying 8 arbitrary parameters. From here we can see that AdS_4 is fully supersymmetric and its Killing spinors show dependence on all the four coordinates. $mAdS_4$ on the other hand is only 1/4 BPS: its Killing spinors satisfy a double projection that reduces the independent components to 1/4 and there is no angular or time dependence. We will come back to this remarkable fact in Section 7.3.

The form of the Killing spinors is important because the bracket of two supercharges is a surface integral at infinity (7.32). Writing out the covariant derivative in (7.32), one obtains

$$\{\mathcal{Q},\mathcal{Q}\} = 2\oint_{\partial V} \mathrm{d}\Sigma_{\mu\nu}\left[\epsilon^{\mu\nu\rho\sigma}\overline{\varepsilon}\gamma_5\gamma_\rho(\partial_\sigma - \frac{1}{4}\omega_\sigma^{ab}\gamma_{ab} - \frac{i}{2}g\gamma_\sigma + i\frac{g}{2}A_\sigma\sigma^2 + \frac{1}{4}F_{\lambda\tau}\gamma^{\lambda\tau}\gamma_\sigma\sigma^2)\varepsilon\right] \ , \tag{7.35}$$

and it depends on the asymptotic value of the Killing spinors of the solution taken into consideration. Therefore the superalgebra will be different in the two cases and there will be two different BPS bounds.

The procedure to compute the BPS bound is the following. From (7.31) we have a definition of the supercharges $\mathcal{Q}_{AdS}(\varepsilon_{AdS})$ and $\mathcal{Q}_{mAdS}(\varepsilon_{mAdS})$. We will then make use of the following definition for the fermionic supercharges Q_{AdS}, Q_{mAdS}:

$$\mathcal{Q}_{AdS} \equiv Q_{AdS}^T\varepsilon_0 = \varepsilon_0^T Q_{AdS} \ , \qquad \mathcal{Q}_{mAdS} \equiv Q_{mAdS}^T\varepsilon_0 = \varepsilon_0^T Q_{mAdS} \ , \tag{7.36}$$

i.e. any spacetime and gamma matrix dependence of the bosonic supercharges $\mathcal{Q}$ is left into the corresponding fermionic Q. We are thus able to strip off the arbitrary constant ε_0 in any explicit calculations and convert the Dirac brackets for $\mathcal{Q}$ into an anticommutator for the spinorial supercharges Q that is standardly used to define the superalgebra. Therefore now we compute the surface integrals (7.32) for the Killing spinors of AdS_4 and $mAdS_4$ respectively. After stripping off the ε_0's, we find the anticommutator of fermionic supercharges given explicitly in terms of the other conserved charges in the respective vacua. The BPS bound is then derived in the standard way by requiring the supersymmetry anticommutator to be positive definite, see e.g. [170] for details.

7.2.1 Asymptotically AdS$_4$ states

We now derive the resulting supersymmetry algebra from the asymptotic spinors of AdS$_4$. For this we use the general expression (7.35) for the Dirac brackets of the supersymmetry charges, together with the asymptotic form of the Killing spinors, (7.33). Inserting the Killing spinors ε_{AdS} of (7.33) in (7.35), we recover something that can be written in the following form:

$$\{\mathcal{Q},\mathcal{Q}\} = -i\,\overline{\varepsilon_0}(A + B_a\gamma^a + C\gamma^5 + D_{ij}\gamma^{ij} + E_i\gamma^{0i} + F_a\gamma^{a5})\varepsilon_0 \ , \tag{7.37}$$

where the charges $A, B, \ldots$ can be written down explicitly from the surface integral (7.35). They will define the electric charge (A), momentum (B), angular momentum (D, with $i, j = 1,2,3$ spatial indices), and boost charges (E). The charge C would correspond to a magnetic charge, which we assumed to vanish by construction. Without the charge F_a, the above bracket will fit in the $Osp(2|4)$ superalgebra (see more below). We will therefore take as definition of asymptotically AdS solutions the ones for which F_a vanishes. This choice of fall-off conditions is similar to the case of $N = 1$ supergravity, where the asymptotic charges are required to generate the $Osp(1|4)$ superalgebra [171]. Extensions of the $\mathcal{N} = 2$ superalgebra where the charges C and F_a are non-zero have been discussed in [172].

From the previous expression we see that conserved charges like Q_e, M et cetera will arise as surface integrals of the five terms (or their combinations) appearing in the supercovariant derivative. We are going to see how this works analyzing each term in the supercovariant derivative, explicitly in terms of the ansatz of the metric (2.4) and vector fields (2.2). This will provide us with a definition of the asymptotic charges in AdS$_4$ with no need to use background subtraction or holographic renormalization techniques. As an explicit example one can directly read off the definition of mass $M \equiv B_0/(8\pi)$ from the explicit form of the asymptotic Killing spinors. In the stationary case,

$$\boxed{\begin{aligned} M = \frac{1}{8\pi}\lim_{r\to\infty}\oint \mathrm{d}\Sigma_{tr}\{ & e^t_{[0}e^r_1 e^\theta_{2]} + \sin\theta e^t_{[0}e^r_1 e^\varphi_{3]} \\ & + g^2 r e^t_{[0}e^r_{1]} - \sqrt{g^2r^2+1}(\omega_\theta^{ab} e^t_{[0}e^r_a e^\theta_{b]} + \omega_\varphi^{ab} e^t_{[0}e^r_a e^\varphi_{b]})\} \ . \end{aligned}} \tag{7.38}$$

We are going to take into consideration both static and rotating solutions, but we will carry out our procedure and explain the calculation in full detail only the case of the electric RN-AdS black hole and comment more briefly on the rotating generalizations.

- Electric RN-AdS

Here we take into consideration solutions of the form (2.4)-(2.2) with zero magnetic charge, $Q_m = 0$. We now evaluate the various terms in (7.35).

To begin, it is easy to determine the piece concerning the field strength, namely

$$2\oint \mathrm{d}\Sigma_{\mu\nu}\left[\epsilon^{\mu\nu\rho\sigma}\overline{\varepsilon}(t,r,\theta,\varphi)\gamma_5\gamma_\rho\frac{1}{4}F_{\lambda\tau}\gamma^{\lambda\tau}\gamma_\sigma\sigma^2\varepsilon(t,r,\theta,\varphi)\right] \tag{7.39}$$

Inserting the Killing spinors for AdS_4 described in (7.33), and exploiting the Clifford algebra relations we get

$$\begin{aligned}
4\oint \mathrm{d}\Sigma_{tr}\overline{\varepsilon}_{AdS}(t,r,\theta,\varphi)F^{tr}\sigma^2\varepsilon_{AdS}(t,r,\theta,\varphi) \; = \\
= 4\oint \mathrm{d}\Sigma_{tr}{\varepsilon_0}^T e^{\frac{1}{2}\varphi\gamma_{23}}e^{\frac{1}{2}\theta\gamma_{12}}e^{-igt\gamma_0}e^{\frac{i}{2}arcsinh(gr)\gamma_1}\gamma_0F^{tr}\sigma^2e^{\frac{i}{2}arcsinh(gr)\gamma_1}\cdot \\
\cdot\, e^{igt\gamma_0}e^{-\frac{1}{2}\theta\gamma_{12}}e^{-\frac{1}{2}\varphi\gamma_{23}}\varepsilon_0 = \\
= 4i\pi\overline{\varepsilon_0}Q_e\sigma^2\varepsilon_0 \;,
\end{aligned} \tag{7.40}$$

with the normalization of electric charge

$$Q_e = \frac{1}{4\pi}\oint_{S^2}\mathrm{d}\Sigma_{tr}F^{tr} \;. \tag{7.41}$$

From here we can identify the term $A = -4\pi Q_e\sigma^2$ appearing in (7.37).

Next we consider the term containing the "bare" gauge field A_μ. This gives a possible contribution to the charge F_a in (7.37). As we mentioned above, we assumed this contribution to vanish for asymptotically AdS solutions. One can explicitly check this for the class of electric RN-AdS solutions given in (2.4), since the only nonzero component of the vector field is A_t (see (2.2)), hence

$$\oint \mathrm{d}\Sigma_{tr}\left[\epsilon^{tr\rho\sigma}\overline{\varepsilon}_{AdS}(t,r,\theta,\varphi)\gamma_5\gamma_\rho i\frac{g}{2}A_\sigma\sigma^2\varepsilon_{AdS}(t,r,\theta,\varphi)\right] = 0\,. \tag{7.42}$$

The term with the partial derivative ∂_σ in (7.35) in the supercovariant derivative gives nonvanishing contributions for $\sigma = \theta, \varphi$ and it amounts to the integral:

$$2\oint \mathrm{d}\Sigma_{tr}\left[\epsilon^{tr\rho\sigma}\overline{\varepsilon}_{AdS}(t,r,\theta,\varphi)\gamma_5\gamma_\rho\partial_\sigma\varepsilon_{AdS}(t,r,\theta,\varphi)\right] = -2i\oint r\overline{\epsilon_0}\gamma_0\epsilon_0\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\,. \tag{7.43}$$

Clearly, this term will contribute, together with other terms, to the mass.

The integral containing the spin connection is:

$$-2i\oint \mathrm{d}\Sigma_{tr}\overline{\varepsilon}_{AdS}(t,r,\theta,\varphi)\gamma^0 e_0^t \frac{2}{4}(\omega_\theta^{12}e_1^r e_2^\theta + \omega_\varphi^{13}e_1^r e_3^\varphi)\varepsilon_{AdS}(t,r,\theta,\varphi) =$$

$$= 2i\oint \overline{\varepsilon_0}\gamma^0 r\sqrt{1+g^2r^2}\sqrt{1+g^2r^2-\frac{2M}{r}+\frac{Q_e^2}{4r^2}}\,\varepsilon_0 \sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\,, \tag{7.44}$$

where we have used

$$e^{i\,arcsinh(gr)\gamma_1} = \sqrt{1+g^2r^2}+ig\gamma_1 r\,, \tag{7.45}$$

and the value of the spin connection:

$$\omega_t^{01} = U\partial_r U, \quad \omega_\theta^{12} = -U, \quad \omega_\varphi^{13} = -U\sin\theta, \quad \omega_\varphi^{23} = -\cos\theta\ . \tag{7.46}$$

Also (7.44) will contribute to B^0, and therefore to the mass.

The last contribution of the supercovariant derivative, the term proportional to $g\gamma_\sigma$, yields

$$-2\oint \mathrm{d}\Sigma_{tr}\overline{\varepsilon}_{AdS}(t,r,\theta,\varphi)\frac{ig}{2}\gamma^{01}e_0^t e_1^r \varepsilon_{AdS}(t,r,\theta,\varphi) =$$

$$= -2i\oint \overline{\varepsilon_0}\gamma^0 r^3 g^2 \varepsilon_0 \sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi. \tag{7.47}$$

In deriving this we have used the formula $\gamma^{tr\mu}\gamma_\mu = 2\gamma^{tr}$. Again, this term contributes to the mass formula.

Collecting all the terms that contribute to the mass (the derivative term (7.43), the sum of the spin connection term (7.44), and the gamma term (7.47)) gives rise to:

$$2i\oint \overline{\varepsilon_0}\gamma^0\left[r\sqrt{1+g^2r^2}\sqrt{1+g^2r^2-\frac{2M}{r}+\frac{Q_e^2}{4r^2}}-r^3g^2-r\right]\varepsilon_0 \sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi\,. \tag{7.48}$$

The integral has to be performed on a sphere with $r\to\infty$. Taking this limit one can see that in this expression all the positive powers of r are cancelled. Hence all possible divergences cancel out, and we are left with a finite contribution. In this sense, our method provides a holographic renormalization of the mass. In the cases we can compare, our method agrees with previously known results.

Performing the integral on the remaining finite part we find:

$$-8i\pi\overline{\varepsilon_0}M\gamma^0\varepsilon_0 = -i\overline{\varepsilon_0}\gamma^0 B_0\,\varepsilon_0\,. \tag{7.49}$$

To sum up, for the electric RN-AdS solution, the brackets between supercharges read:

$$\begin{aligned}\{\mathcal{Q},\mathcal{Q}\} &= -8\pi i\overline{\varepsilon_0}(M\gamma^0 - \frac{1}{2}Q_e\sigma^2)\varepsilon_0 \\ \Rightarrow \{\varepsilon_0^T Q, Q^T\varepsilon_0\} &= 8\pi\varepsilon_0^T(M - \frac{1}{2}Q_e\gamma^0\sigma^2)\ \varepsilon_0 .\end{aligned} \tag{7.50}$$

Now we can strip off the constant linearly independent doublet of spinors ϵ_0 on both sides of the above formula to restore the original $SO(2)$ and spinor indices:

$$\{Q^{A\alpha}, Q^{B\beta}\} = 8\pi\left(M\delta^{AB}\delta^{\alpha\beta} - \frac{i}{2}Q_e\epsilon^{AB}(\gamma^0)^{\alpha\beta}\right) . \tag{7.51}$$

This expression coincides with the one expected from the algebra $Osp(2|4)$ (see (7.68) in the next section) if we identify $M_{-10} = 8\pi M$ and $T^{12} = 4\pi Q_e$.

The BPS bound for the electric RN-AdS solution is then[6]:

$$M \geq |Q_e|/2 . \tag{7.52}$$

The state that saturates this bound, for which $M = |Q_e|/2$, preserves half of the supersymmetries, i.e. it is half-BPS. It is the ground state allowed by (7.52) and represents a naked singularity. All the excited states have higher mass and are either naked singularities or genuine black holes.

It is interesting to look at the case of extremal black holes, in which inner and outer horizon coincide. This yields a relation between the mass and charge, which can be derived from the solution given in (2.4) and (2.5). Explicit calculation gives the following result [68]:

$$M_{extr} = \frac{1}{3\sqrt{6}g}(\sqrt{1+3g^2Q_e^2}+2)(\sqrt{1+3g^2Q_e^2}-1)^{1/2} . \tag{7.53}$$

This lies above the BPS bound unless $Q_e = 0$, in which case we recover the fully supersymmetric AdS_4 space. Thus,

$$M_{extr} > M_{BPS} . \tag{7.54}$$

- Kerr-AdS
  The Kerr-AdS black hole is an example of a stationary spacetime without charges but with non-vanishing angular momentum. It is most standardly written in Boyer-Lindquist-type coordinates and we refer to [68] for more details. More details on how to calculate the angular momenta from the

[6] See e.g. [170] for details on the general procedure of deriving of BPS bounds from the superalgebra.

anticommutator of the supercharges can be found in Appendix F. The BPS bound is straightforward to find also in this case, leading to

$$M \geq g|\vec{J}| \ , \tag{7.55}$$

where the BPS state satisfies $M = g|\vec{J}|$ and in fact corresponds to a singular limit of the Kerr-AdS black hole because the AdS boundary needs to rotate as fast as the speed of light [136]. Note that in general for the Kerr black hole we have $|\vec{J}| = aM$, where a is the rotation parameter appearing in the Kerr solution in standard notation. Thus $M = g|\vec{J}|$ implies $a = 1/g$, which is exactly the singular case. All the excited states given by $a < 1/g$ are however proper physical states, corresponding to all the regular Kerr-AdS black holes, including the extremal one. Thus the BPS bound is always satisfied but never saturated by any physical solution of the Kerr-AdS type,

$$M_{extr} > M_{BPS} \ , \tag{7.56}$$

as is well-known.

- KN-AdS
The BPS bound for Kerr-Newman-AdS (KN-AdS) black holes[7] is a bit more involved due to the presence of both electric charge and angular momentum. We will not elaborate on the details of the calculation which is straightforward. The resulting BPS bound is

$$M \geq \frac{1}{2}|Q_e| + g|\vec{J}| = \frac{1}{2}|Q_e| + agM \ , \tag{7.57}$$

and the ground (BPS) state is in fact quarter-supersymmetric. This BPS bound is also well-known and is described in [170]. The BPS bound does in general not coincide with the extremality bound, which in the case of the KN-AdS black holes is a rather complicated expression that can be found in [68, 136]. Interestingly, the BPS bound and the extremality bound coincide at a finite non-zero value for the mass and charge (with $ag < 1$),

$$|Q_{e,crit}| \equiv \sqrt{\frac{a}{g}\frac{1}{1-ag}} \ . \tag{7.58}$$

Now we have two distinct possibilities for the relation between the BPS state and the extremal KN-AdS black hole depending on the actual value for the electric charge (there is exactly one BPS state and exactly one extremal black hole for any value of charge Q_e):

$$\begin{aligned} M_{extr} &> M_{BPS} \ , \qquad |Q_e| \neq |Q_{e,crit}| \ , \\ M_{extr} &= M_{BPS} \ , \qquad |Q_e| = |Q_{e,crit}| \ . \end{aligned} \tag{7.59}$$

[7] See again [68] for more detailed description of the KN-AdS black holes.

So for small or large enough electric charge the BPS solution will be a naked singularity and the extremal black hole will satisfy but not saturate the BPS bound, while for the critical value of the charge the extremal black hole is supersymmetric and all non-extremal solutions with regular horizon will satisfy the BPS bound.

7.2.2 Magnetic AdS$_4$

Unlike the standard AdS$_4$ case above, the Killing spinors of magnetic AdS$_4$ already break 3/4 of the supersymmetry, c.f. (E.9). The projection that they obey is,

$$\varepsilon_{mAdS} = P\varepsilon_{mAdS} \ , \qquad\qquad P \equiv \frac{1}{4}(1+i\gamma_1)(1\mp i\gamma_{23}\sigma^2) \ , \tag{7.60}$$

for either the upper or lower sign, depending on the sign of the magnetic charge. Furthermore, one has the following properties of the projection operators,

$$P^\dagger P = P^\dagger i\gamma_1 P = \pm P^\dagger i\gamma_{23}\sigma^2 P = \pm P^\dagger(-i\gamma_0\gamma_5\sigma^2)P = P \ , \tag{7.61}$$

and all remaining quantities of the form $P^\dagger \Gamma P$ vanish, where Γ stands for any of the other twelve basis matrices generated by the Clifford algebra.

These identities allow us to derive, from (7.35), the bracket

$$\{\mathcal{Q},\mathcal{Q}\} = \overline{P\varepsilon}_0\gamma_0(-i\,8\pi)\mathcal{M}^* P\varepsilon_0 \qquad \Rightarrow \qquad \{\varepsilon_0^T PQ,(PQ)^T\varepsilon_0\} = \varepsilon_0^T(8\pi\mathcal{M}^*)P\varepsilon_0 \ . \tag{7.62}$$

provided that the quantity $\mathcal{M}^*$ is given by

$$\boxed{\begin{aligned}\mathcal{M}^* = \frac{1}{8\pi}\lim_{r\to\infty}\oint \mathrm{d}\Sigma_{tr}\left(gr+\frac{1}{2gr}\right)\Big(& g(A_\theta e^t_{[0}e^r_2 e^\theta_{3]} + A_\varphi e^t_{[0}e^r_2 e^\varphi_{3]}) \\ &+\frac{1}{g}e^t_{[0}e^r_1 e^\theta_2 e^\varphi_{3]} + g e^t_{[0}e^r_{1]} - (\omega_\theta^{ab}e^t_{[0}e^r_a e^\theta_{b]} + \omega_\varphi^{ab}e^t_{[0}e^r_a e^\varphi_{b]})\Big) \ .\end{aligned}} \tag{7.63}$$

This expression simplifies further if we choose to put the vielbein matrix in an upper triangular form, such that we have nonvanishing $e_t^{0,1,2,3}, e_r^{1,2,3}, e_\theta^{2,3}, e_\varphi^3$, and the inverse vielbein has only components $e_0^{t,r,\theta,\varphi}, e_1^{r,\theta,\varphi}, e_2^{\theta,\varphi}, e_3^\varphi$. The mass is then

$$\mathcal{M}^* = \frac{1}{8\pi}\lim_{r\to\infty}\oint \mathrm{d}\Sigma_{tr}\left(gr+\frac{1}{2gr}\right)\left(\frac{1}{g}e^t_0 e^r_1 e^\theta_2 e^\varphi_3 + g e^t_0 e^r_1 - (\omega_\theta^{12}e^t_0 e^r_1 e^\theta_2 + \omega_\varphi^{13}e^t_0 e^r_1 e^\varphi_2)\right) . \tag{7.64}$$

Notice that this mass formula is different from the one for asymptotically AdS$_4$ spacetime.

Stripping off the parameters ϵ_0 in (7.62), leaves us with a matrix equation in spinor space. Due to the projection operators, one is effectively reducing the number of supercharges to two instead of eight. These two supercharges are scalars, since the Killing spinors are invariant under rotation as they don't depend on the angular coordinates (see also in the next section). Denoting them by Q^1 and Q^2, the anticommutator then becomes

$$\{Q^I, Q^J\} = 8\pi\mathcal{M}^*\delta^{IJ}\,, \qquad I,J = 1,2\,. \tag{7.65}$$

Hence the BPS bound is just

$$\mathcal{M}^* \geq 0\,. \tag{7.66}$$

Saturating the bound leads to a quarter-BPS solution. Given the form of (7.65) and the previous computations, we see that many terms, that in the AdS_4 case composed different conserved charges, contribute now to a single one, namely $\mathcal{M}^*$. This quantity, as we will see below, coincides with the mass M of the configuration in absence of scalars and when scalars are constant.

In particular, for the case of of the Reissner-Nordström solution (2.4) with fixed magnetic charge $Q_m = \pm 1/g$ and Q_e arbitrary, the mass integral (7.63) yields

$$\begin{aligned}
\mathcal{M}^* &= \frac{1}{8\pi}\times \\
&\times \lim_{r\to\infty}\oint\left(gr+\frac{1}{2gr}\right)\left(r\sqrt{1+g^2r^2-\frac{2M}{r}+\frac{Q_e^2+1/(g^2)}{4r^2}}-g\,r^2-\frac{1}{2g}\right)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi \\
&= \frac{1}{8\pi}\oint\left(-g^2r^3-\frac{r}{2}+M-\frac{r}{2}+g^2r^3+r\right)\sin\theta\,\mathrm{d}\theta\mathrm{d}\varphi = M\,.
\end{aligned} \tag{7.67}$$

This is exactly the mass parameter M appearing in (2.5). The supersymmetric solutions found by Romans (the so-called cosmic monopole/dyons) have vanishing mass parameter hence indeed saturate the BPS bound (7.66).

Of course in the context of a rotating black hole vanishing mass results in vanishing angular momentum due to the proportionality between the two, i.e. an asymptotically mAdS Kerr-Newman with non-zero angular momentum spacetime can never saturate the BPS bound (7.66). Nevertheless, excitations over the magnetic AdS$_4$ include all Reissner-Nordström and Kerr-Newman AdS black holes that have fixed magnetic charge $gQ_m = \pm 1$ and arbitrary (positive) mass, angular momentum and electric charge. All these solutions satisfy the magnetic AdS$_4$ BPS bound.

7.3 Superalgebras

7.3.1 AdS$_4$

The procedure we used to find the BPS bound determines also the superalgebras of AdS$_4$ and mAdS$_4$, which are found to be different. In fact, given the Killing spinors and Killing vectors, there is a general algorithm to determine the superalgebra, see [173] and chapter 13 in [174].

For what concerns the pure AdS_4, in $\mathcal{N}=2$ gauged supergravity the superalgebra is $Osp(2|4)$, which contains as bosonic subgroup $SO(2,3)\times SO(2)$: the first group is the isometry group of AdS_4 and the second one corresponds to the gauged R-symmetry group that acts by rotating the two gravitinos. The algebra contains the generators of the $SO(2,3)$ group M_{MN} ($M,N=-1,0,1,2,3$), and $T^{AB}=-T^{BA}=T\epsilon^{AB}, A,B=1,2$, the generator of $SO(2)$. Furthermore we have supercharges $Q^{A\alpha}$ with $A=1,2$ that are Majorana spinors. The non-vanishing (anti-)commutators of the $Osp(2|4)$ superalgebra are:

$$
\begin{aligned}
[Q^{A\alpha},T] &= \epsilon^{AB}Q^{B\alpha}\\
[M_{MN},M_{PQ}] &= -\eta_{MP}M_{NQ}-\eta_{NQ}M_{MP}+\eta_{MQ}M_{NP}+\eta_{NP}M_{MQ}\\
[Q^{A\alpha},M_{MN}] &= \frac{1}{2}(\hat{\gamma}_{MN})^{\alpha}{}_{\beta}Q^{A\beta}\\
\{Q^{A\alpha},Q^{B\beta}\} &= \delta^{AB}(\hat{\gamma}^{MN}C^{-1})^{\alpha\beta}M_{MN}-(C^{-1})^{\alpha\beta}T\epsilon^{AB}\ ,
\end{aligned}
\tag{7.68}
$$

where $\eta_{MN}=\mathrm{diag}(1,1,-1,-1,-1)$, the gamma matrices are $\hat{\gamma}_M\equiv\{\gamma_5,i\gamma_\mu\gamma_5\}$, and $\hat{\gamma}_{MN}=\frac{1}{2}[\hat{\gamma}_M,\hat{\gamma}_N]$. Further details can be found in [174]. T does not have the role of a central charge, as it doesn't commute with the supercharges. Nevertheless it is associated to the electric charge[8]. The isometry group of AdS_4 is $SO(2,3)$, isomorphic to the conformal group in three dimensions, whose generators are 3 translations, 3 rotations, 3 special conformal transformations (conformal boosts) and the dilatation.

7.3.2 mAdS_4

In the case of mAdS_4, the symmetry group is reduced due to the presence of the magnetic charge. Spatial translations and boosts are broken, because of the presence of the magnetic monopole. There are 4 Killing vectors related to the invariance under time translations and rotations. The isometry group of this spacetime is then $\mathbb{R}\times SO(3)$. Furthermore, we have also gauge invariance. The projector (7.60) reduces the independent components of the Killing spinors to 1/4, consequently the number of fermionic symmetries of the theory is reduced, as we explained in section 7.2.2. We have denoted the remaining two real supercharges with Q^I ($I=1,2$). To sum up, the symmetry generators of mAdS_4 include[9]: two supercharges Q^I where $I=1,2$, the charge associated with time translations, that we denote by H, the gauge transformation generator T, the angular momentum J_i, $i=1,2,3$.

[8] If we perform a Wigner–Inönü contraction of the algebra, T^{AB} gives rise to a central charge in the Poincaré superalgebra. See [174] for further details.

[9] We have not determined all solutions of the Killing equation. However, we expect that the generators above are all continuous symmetries, and there is no remnant of a dilatation symmetry.

From (7.65) the anticommutator between two supercharges is

$$\{Q^I, Q^J\} = H\delta^{IJ} \ . \tag{7.69}$$

Since mAdS$_4$ is static and spherically symmetric, we have the commutation relations

$$[H, J_i] = 0 \ , \qquad [J_i, J_j] = \epsilon_{ijk} J_k \ . \tag{7.70}$$

The following commutators are then determined by imposing the Jacobi identities:

$$[Q^I, J_i] = [Q^I, H] = 0 \ . \tag{7.71}$$

Next, we add the gauge generator T to the algebra. Because of gauge invariance, we have the commutators

$$[T, J_i] = [T, H] = 0 \ . \tag{7.72}$$

From the Jacobi identities one now derives that

$$[Q^I, T] = \epsilon^{IJ} Q^J \ , \tag{7.73}$$

with a fixed normalization of T. This commutator also follows from the observation that gauge transformations act on the supersymmetry parameters in gauged supergravity, together with the fact that T commutes with the projection operator P defined in the previous section. The commutation relations derived in this section define the algebra U(1,1) [175].

The first commutator in (7.71) implies that the supercharges Q^I are singlet under rotations. This is a consequence of the fact that the mAdS$_4$ Killing spinors have no angular dependence [61]. Group theoretically, this follows from the fact that the group of rotations entangles with the $SU(2)_R$ symmetry, as explained in [55].

7.4 Matter couplings

The results we have just obtained can be generalized to $\mathcal{N} = 2$ gauged supergravity in presence of additional matter, with both hypermultiplets and vector multiplets. This was done in [103], where the following expression was found for the Poisson bracket:

$$\{\mathcal{Q}, \mathcal{Q}\} = 2 \oint_{\partial V} \mathrm{d}\Sigma_{\mu\nu} \left[\epsilon^{\mu\nu\rho\sigma} \overline{\varepsilon}^A \gamma_\rho \widetilde{\mathcal{D}}_\sigma \varepsilon_A - \epsilon^{\mu\nu\rho\sigma} \overline{\varepsilon}_A \gamma_\rho \widetilde{\mathcal{D}}_\sigma \varepsilon^A \right] , \tag{7.74}$$

where $\widetilde{\mathcal{D}}_\sigma$ is defined as in (1.66). The form of the asymptotic Killing spinors for electric and magnetic configurations is the same as in the minimal case, and this reflects on the fact that again the two underlying superalgebras are of the AdS-form 7.3.1 for electric (Duff-Liu) configurations and $mAdS$-form 7.3.2 for magnetic (Cacciatori-Klemm) ones.

The only difference in the matter coupled case is the definition of the definition of the conserved charges. Indeed the quantities appearing in the superalgebra get dressed with scalar fields, and we will recap the definitions of [103] in the next subsections.

7.4.1 Ordinary AdS_4

The solutions (4.33) asymptote to ordinary AdS_4, with underlying $osp(2|4)$ superalgebra[10]; for this reason the BPS bound, in analogy with the case of minimal gauged supergravity, is

$$M \geq |Z| \,, \tag{7.75}$$

where M is defined as [103]

$$M = \frac{1}{8\pi} \lim_{r\to\infty} \oint \mathrm{d}\Sigma_{tr} \Bigg(e^t_{[0} e^r_1 e^\theta_{2]} + \sin\theta e^t_{[0} e^r_1 e^\varphi_{3]} + 2gg' r |P^a_\Lambda L^\Lambda| e^t_{[0} e^r_{1]} + \\ - \sqrt{g'^2 r^2 + 1} (\omega^{12}_\theta e^t_{[0} e^r_1 e^\theta_{2]} + \omega^{13}_\varphi e^t_{[0} e^r_1 e^\varphi_{3]}) \Bigg) \,. \tag{7.76}$$

where $g' = \sqrt{-\Lambda} = \sqrt{-V_g(r=\infty)/3}$. For the concrete examples at our disposal, the quantity M coincides with the AMD mass of the solution multiplied by an overall factor that depends on the FI terms. Z is the charge related to the $O(2)$ generator in the $Osp(2|4)$ superalgebra and reads:

$$Z = \frac{1}{4\pi} \lim_{r\to\infty} \oint_{S^2} Re(\mathcal{Z}) = \lim_{r\to\infty} Re(L^\Lambda q_\Lambda - M_\Lambda p^\Lambda) \,. \tag{7.77}$$

where $\mathcal{Z}$ is defined as (2.33). Further details can be found in [103].

Duff-Liu solutions

Let us analyze now the BPS bound for the electric nonextremal configurations found in [64] and described in Chapter 4, Section 4.3. The formulas given in the previous subsection can be used to find the conserved charges of the solution. In particular, Z reads

$$Z = \frac{(\xi_0 \xi_1^3)^{1/4}}{2^{3/2} 3^{3/4}} \left(\frac{q_0}{\xi_0} + 3 \frac{q_1}{\xi_1} \right) \,. \tag{7.78}$$

The quantity M is computed from (7.76). After an appropriate rescaling in order to have the correct AdS asymptotics we get:

$$M = \frac{3b_1 \left(\frac{\xi_0}{\xi_1}\right)^{1/4} + b_0 \left(\frac{\xi_1}{\xi_0}\right)^{3/4} + 2(\xi_0 \xi_1^3)^{1/4} \mu}{2\sqrt{2} 3^{3/4}} \,. \tag{7.79}$$

In the particular case of the solution (4.58), it gives [99]

$$M = \frac{\mu (\xi_0 \xi_1^3)^{1/4} (2 + 3\sinh^2 Q_1 + \sinh^2 Q_0)}{2\sqrt{2} 3^{3/4}} \,. \tag{7.80}$$

[10] The full conditions for a configuration to be endowed with the AdS_4 superalgebra of Section 7.3.1 are in [103], formulas (3.1).

This expression differs with respect to the one found via the AMD procedure, formulas (4.62),(4.63) just due to an overall multiplicative factor $\sqrt{2}(\xi_0\xi_1^3)^{1/4}/3^{3/4}$. This reflects the fact that the Killing spinor is determined up to a multiplicative constant factor. In other words, I can multiply the Killing spinor with an arbitrary constant c and this will still solve the BPS equations. The formulas for the conserved charges will therefore acquire an overall c^2 factor. I can of course choose the factor in order to find agreement with the AMD mass, in our case this would correspond to setting $c^2 = (\sqrt{2}(\xi_0\xi_1^3)^{1/4}/3^{3/4})^{-1}$.

For BPS solutions the mass and the charge was computed in [103] by means of the formulas (7.76) (7.77), and give

$$M = \frac{(\xi_0\xi_1^3)^{1/4}}{2^{3/2}3^{3/4}}\left(\frac{q_0}{\xi_0}+3\frac{q_1}{\xi_1}\right) = Z\,. \tag{7.81}$$

The BPS bound $M \geq |Z|$ is saturated by the BPS solutions, if we restrict to positive charges. In particular, the condition for the saturation of the BPS bound $M = |Z|$ is:

$$2\mu = \frac{q_0-b_0}{\xi_0}+3\frac{q_1-b_1}{\xi_1}\,. \tag{7.82}$$

We recover the supersymmetric solution by imposing $\mu = 0$, and $q_\Lambda = b_\Lambda$; but there are also in this case other solutions that saturate the bound, namely those for which (7.82) in satisfied with $\mu \neq 0$ [99].

7.4.2 Magnetic AdS$_4$

The concept of magnetic AdS superalgebra was introduced in Section 7.3.1. This is the superalgebra underlying the magnetic black hole configurations of [30] and [99], described in Chapters 2 and 4. The conditions for a solution with scalars to asymptote to mAdS are listed in [103], in particular there is a constraint on the magnetic charges

$$g_\Lambda p^\Lambda = \mp 1\,. \tag{7.83}$$

The mAdS is characterized by the BPS bound

$$\mathcal{M}^* \geq 0\,. \tag{7.84}$$

where the formula for $\mathcal{M}^*$ was given in [103] and reads:

$$\mathcal{M}^* = \frac{1}{8\pi}\lim_{r\to\infty}\oint d\Sigma_{tr}\left(g'r+\frac{g'}{2g^2r}\right)\Bigg(\pm 2\mathrm{Im}(L^\Lambda q_\Lambda - M_\Lambda p^\Lambda)\sin\theta e_0^t e_1^r e_2^\theta e_3^\varphi +$$
$$+2g|P^a_\Lambda L^\Lambda| e_0^t e_1^r - (\omega_\theta^{12} e_0^t e_1^r e_2^\theta + \omega_\varphi^{13} e_0^t e_1^r e_3^\varphi)\Bigg)\,. \tag{7.85}$$

We now compute this quantity on the nonextremal magnetic configurations of Section 4.4, and elaborate on its physical meaning.

Cacciatori-Klemm solutions and their nonextremal counterparts

Once we rescale the radial and the time coordinate in order to have the right AdS asymptotics, formula (7.85) computed on the solution (4.71)-(4.72) in Section 4.4, provided that we impose that $g_\Lambda p^\Lambda = -1$, gives

$$\mathcal{M}^* = -\frac{\big(-9+2\xi_1 g(8b_1^2\xi_1 g-3p^1)\big)\big(-9+4\xi_1 g(16b_1^2\xi_1 g-3p^1)\big)}{72\sqrt{2}\,3^{1/4}b_1\xi_1(\xi_0\xi_1^3)^{1/4}g^2}\,. \tag{7.86}$$

Some comments are in order here: first of all, we verified that for all values of parameters that correspond to genuine regular black holes, $\mathcal{M}^*$ it is positive. Therefore for generic nonextremal solutions the bound (7.84) is satisfied.

The mass formula (7.85) yields a finite value if and only if the quantization condition $g_\Lambda p^\Lambda = \mp 1$ is imposed. It diverges for other values n on the right-hand side of the Dirac quantization condition (1.87).

Furthermore, the mass formula in (7.86) admits two zeros: one is in correspondence with the value assumed by p^1 for the solution of [30]

$$p^1 = -\frac{2}{\xi_1 g}\left(\frac{3}{8}-\frac{8}{3}(b_1\xi_1 g)^2\right)\,. \tag{7.87}$$

We retrieved then the expected result, namely that the supersymmetric solution saturates the BPS bound given by (7.84). The other zero of the mass formula is obtained for

$$p^1 = -\frac{2}{\xi_1 g}\left(\frac{3}{4}-\frac{4}{3}(b_1\xi_1 g)^2\right)\,. \tag{7.88}$$

To our knowledge, this value does not correspond to any BPS configuration; all solutions of this family correspond to naked singularities. The existence of non-BPS solutions saturating the bound happens also in the electric case (see end of previous section).

What is more puzzling about the result (7.86), is the fact that $\mathcal{M}^*$ *computed on the nonextremal solution* (4.71)-(4.72) *does not coincide with its AMD mass* M_{AMD} (4.76), not even up to multiplicative factors.

$$\mathcal{M}^* \neq M_{AMD} \qquad \text{for magnetic with nonconstant scalars}\,.$$

The difference Δ between the two values $\mathcal{M}^*$ and M_{AMD} is finite and gives (we have chosen here the explicit values $\xi_1 = 1$, $\xi_0 = 27/4$, $g = 1$ for simplicity):

$$\Delta = \mathcal{M}^* - M_{AMD} = 2b_1 - \frac{128b_1^3}{27} + \frac{8}{3}b_1 p^1\,. \tag{7.89}$$

In particular, Δ remains nonzero also in the BPS limit: $\Delta_{BPS} = \frac{256b_1^3}{27}$.

We remind the reader that the two quantities M_{AMD} and $\mathcal{M}^*$ instead match in the minimal case, see (7.67). In the constant scalars case, the mass computed with (7.85) coincides with the AMD mass apart from a multiplicative factor.

The quantity $\mathcal{M}^*$ provides the correct BPS bound for these configurations, in the sense that $\mathcal{M}^* = 0$ holds for the BPS configuration of [30]. This is in complete analogy with the analysis in minimal gauged supergravity [102]. Yet, this expression, as defined in (7.85), does not satisfy the first law of thermodynamics for nonconstant scalars (6.14) (that we have seen is satisfied by M_{AMD}). Therefore, $\mathcal{M}^*$ provides another quantity, different than the mass M_{ADM}, that is conserved for solutions (4.71)-(4.72), and defines the BPS bound as in (7.84).

Discussion

The procedure outlined in this chapter to compute the BPS bound is general and can be extended to other horizon topologies (i.e. [176]). One disadvantage of our approach is that our renormalized mass formulas (7.38) and (7.63) are written in specific coordinates and therefore not manifestly diffeomorphism invariant. A coordinate independent formulation might be useful also in comparison with the other methods for computing the mass in Anti-de Sitter.

The mismatch of the expression for $\mathcal{M}^*$ appearing in the superalgebra and the AMD mass for magnetic configurations with nonconstant scalars remains unsettling.

Indeed both quantities $\mathcal{M}^*$ and M_{AMD} are conserved charges for the magnetic solutions. They both arise as results of well-defined procedures and they are both automatically finite, without any need for holographic renormalization. The logical expectation that they would agree is not satisfied in this case. The difference between the two values $\Delta = \mathcal{M}^* - M_{AMD}$ corresponds to a conserved charge, too. It would be interesting to elucidate the physical meaning of this quantity.

This unexpected disagreement fueled the computations we present in the next chapter: there the mass of the magnetic configuration is obtained via another method, different than the ones used so far (integration of the first law, AMD, BPS bound). We will use there holographic renormalization techniques [104–106]. As anticipation, also the result obtained with holographic techniques will agree with the AMD computation.

We should be careful then in understanding the physical meaning of $\mathcal{M}^*$. Despite this, all the considerations we made concerning the superalgebra are still valid. There are suggestions in the literature [177] that excitations of the dual theory are relevant for condensed matter physics in the presence of external magnetic field, e.g. quantum Hall effect and Landau level splitting at strong coupling. A better understanding of the mAdS superalgebra could then provide us with more insights about the dual field theory. We leave this for future research.

Chapter 8

First order flow and mass from holographic renormalization

We compute here the mass for Anti-de Sitter black holes making use of the Hamilton-Jacobi holographic renormalization techniques [104, 178–180], that remove the divergencies of the boundary stress-energy tensor by adding additional surface terms to the action. These counterterms are built up with curvature invariants of the boundary and thus they do not alter the bulk equations of motion. This yields a well-defined boundary stress tensor and a finite action and mass of the system.

We focus on solutions of $\mathcal{N}=2$ four-dimensional Fayet-Iliopoulos gauged supergravity, where the effect of the gauging is to introduce a potential V_g for the scalars z^i in the bosonic action. In deriving the suitable counterterms we need to identify a function called superpotential $\mathcal{W}(z^i)$, related to the gauging potential by ($\mathcal{G}^{ij}$ is the metric on the manifold of the scalars):

$$V_g(z) = g^2 \left(\mathcal{G}^{ij} \frac{\partial \mathcal{W}}{\partial z^i} \frac{\partial \mathcal{W}}{\partial z^j} - 3\mathcal{W}^2 \right) . \tag{8.1}$$

Such superpotential $\mathcal{W}$ drives the flow of the metric field and the scalars at radial infinity [104]. The flow equations at infinity, in appropriate coordinates, are of the form [104]

$$\frac{dz^i}{dr} = g\mathcal{G}^{ij}\partial_j \mathcal{W} , \qquad \frac{dg_{\mu\nu}}{dr} = g\mathcal{W} g_{\mu\nu} . \tag{8.2}$$

We find here first that order equations involving the superpotential $\mathcal{W}$ are satisfied for nonextremal AdS black holes throughout all the flow. At radial infinity these equations are indeed of the form (8.2). We find this by performing a specific squaring the one-dimensional reduced action, tailored on the static black holes of Chapter 4, in the same spirit of what was done in [113] for five-dimensional solutions.

We begin the Chapter by illustrating the procedure of the squaring of the action. We then deal with the holographic renormalization techniques for the mass computation. Finally, we show that the mass that we obtain through holographic renormalization coincides with the one obtained with the Ashtekar-Magnon-Das procedure, upon choosing an appropriate prescription for the counterterms. This chapter contains new material, result of some work in progress in collaboration with A. Gnecchi.

8.1 First order flow for non-Extremal AdS solutions

8.1.1 Setup and conventions

We consider a generic bosonic action for gravity coupled to a set of n_s scalar and n_f vector fields given in the form

$$S_{4d} = \int d^4x \sqrt{-g} \left(\frac{R}{2} + g_{ij}(z)\partial_\mu z^i \partial^\mu z^j + \mathcal{I}_{\Lambda\Sigma}(z) F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} - V_g \right) , \tag{8.3}$$

where z^i, $i = 0, 1, \ldots n_s - 1$ are real scalars, $F^\Lambda_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ are the field strengths for the vector fields ($\Lambda, \Sigma = 0, 1, ..n_f - 1$), and V_g is the scalar potential. We assume that the potential can be written as $V_g = g^2 \left(-3\mathcal{W}^2 + g^{ij}\partial_i \mathcal{W} \partial_j \mathcal{W} \right)$ and we find this to be true for the examples we treat here.

For instance, all solutions of the model $F = \sqrt{X^0 (X^1)^3}$ in Chapter 4 have vanishing axions, the bosonic action therefore can be cast in this form (8.3), with just one real scalar field z and $\Lambda = 0, 1$ (further details in Appendix A):

$$V_g = -g^2 \left(\frac{\xi_0 \xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3} \sqrt{z} \right) , \quad g_{ij} = g_{zz} = \frac{3}{16z^2} , \quad \mathcal{I}_{\Lambda\Sigma} = I_{\Lambda\Sigma} = diag[-\sqrt{z^3}, -\frac{3}{\sqrt{z}}] . \tag{8.4}$$

We do not specify a superpotential $\mathcal{W}$ yet.

Our procedure of the squaring of the action is however more general, namely we do not need to assume the form of the prepotential. In addition to the usual assumption of staticity and spherical symmetry, we furthermore assume that the sections X^Λ, and therefore the scalars, are real (no axions) and that $Re(\mathcal{N}) = 0$, necessary if we want the supersymmetric Lagrangian (1.78) to fit in (8.3).

Static and spherically symmetric black hole configurations can be cast in this form:

$$ds^2 = U^2(r)dt^2 - \frac{dr^2}{U^2(r)} - h^2(r)(d\theta^2 + \sin^2\theta d\phi^2) , \tag{8.5}$$

with

$$U^2 = e^K f(r) , \qquad h^2 = e^{-K} r^2 , \tag{8.6}$$

where for the moment we leave the functions $K(r)$ and $f(r)$ unspecified. Furthermore, the real scalar fields z^i depend just on the radial coordinate $z^i = z^i(r)$, and the Maxwell's and Bianchi equations are solved by

$$F^\Lambda_{tr} = \frac{1}{2h^2(r)} I^{\Lambda\Sigma} q_\Sigma , \qquad F^\Lambda_{\theta\varphi} = \frac{1}{2} p^\Lambda \sin\theta . \tag{8.7}$$

8.1.2 Squaring for the electric case

At this point we consider electrically charged solutions $p^\Lambda = 0$ with line element (8.5)-(8.6), where (highlighting $g = \mathbf{g}$ for future computations)

$$f = \left(1 - \frac{\mu}{r} + \mathbf{g}^2 r^2 e^{-2K}\right) , \tag{8.8}$$

It turns out that we are able to identify first order equations for the warp factor $K(r)$ and the scalar fields $z^i(r)$ in function of the superpotential $\mathcal{W}$ thanks to a suitable squaring of the action. To do this we plug the ansatz (8.6)-(8.8) in the action (8.3), and we rewrite the action as a sum of squares, as performed in [113] for nonextremal five-dimensional gauged supergravity black hole solutions.

We find it useful to divide the action in terms of S_2, containing factors of $\mathbf{g}^2$, and S_0, with zero powers of $\mathbf{g}$, the gauge coupling constant. Terms in $\mathbf{g}^1$ are absent.

$$S = S_2 + S_0 \tag{8.9}$$

In the following, η is the determinant of the 2-sphere element, it depends just on the angular variable θ. The explicit form for the part in $\mathbf{g}^2$ is ($'$ denotes differentiation with respect to the radial variable r):

$$S_2 = \int d^4x\,\sqrt{\eta}\,\mathbf{g}^2 \Bigg[-r^4 e^{-2K}\left(z^{i\prime} + \frac{e^{K/2}}{r} g^{ik}\partial_k \mathcal{W}\right) g_{ij} \left(z^{j\prime} + \frac{e^{K/2}}{r} g^{jl}\partial_l \mathcal{W}\right) +$$
$$+ 3r^2 e^{-K}\left(\mathcal{W} + \frac{e^{-K/2} r}{2}(K' - \frac{2}{r})\right)^2 \Bigg] + S_{td}^{(2)} , \tag{8.10}$$

where

$$S_{td}^{(2)} = \int \sqrt{\eta}\,\mathbf{g}^2 \frac{d}{dr}\left[\frac{3}{2} e^{-2K} K' r^4 - 3e^{-2K} r^3 + 2\,e^{-3K/2} r^3 \mathcal{W}\right] . \tag{8.11}$$

It turns out that then also S_0 can be written as a sum of squares[1] and a total derivative. We introduce harmonic functions H_Λ of the form

$$H_\Lambda = a_\Lambda + \frac{b_\Lambda}{r} , \tag{8.12}$$

and S_0 can be squared as:

$$S_0 = \int d^4x\,\sqrt{\eta} \Bigg[-4\mu \frac{e^{K/2}}{r^2} b_\Lambda I^{\Lambda\Sigma}\left(iM_\Lambda - e^{K/2} H_\Lambda \frac{1}{2}\right) +$$
$$+ e^{-K} r^2 \left(F_{tr}^\Lambda - \frac{I^{\Lambda\Gamma} q_\Gamma}{h^2}\right) I_{\Lambda\Sigma} \left(F_{tr}^\Sigma - \frac{I^{\Sigma\Delta} q_\Delta}{h^2}\right) +$$

[1] Plus a term that vanishes once one enforces the equations of motion, see the discussion before (8.15).

$$+2\left(1-\frac{\mu}{r}\right)r^2\left(iM'_\Lambda - i\frac{K'}{2}M_\Lambda+\frac{e^{K/2}b_\Lambda}{r^2}\right)I^{\Lambda\Sigma}\left(iM'_\Sigma - i\frac{K'}{2}M_\Sigma+\frac{e^{K/2}b_\Sigma}{r^2}\right), \quad (8.13)$$

where M_Λ is defined in (1.14) as the lower part of the covariantly holomorphic vector $\mathcal{V}$. For the electric solutions at hand the quantities M_Λ are purely imaginary, hence the appearance of imaginary factors i in the action. The total derivative part is

$$\begin{aligned} S^{(0)}_{td} &= \int d^4x\,\sqrt{\eta}\;\frac{d}{dr}\left[(r^2-\mu r)\left(\frac{K'}{2}+4b^\Lambda I^{\Lambda\Sigma} iM_\Sigma\frac{e^{K/2}}{r}\right)\right]+ \\ &+ \int d^4x\,\sqrt{\eta}\;\left(-\mu\frac{K'}{2}+2q_\Lambda F^\Lambda_{tr}\right). \end{aligned}$$

In performing the squaring we have made use of the special Kähler identities valid for purely real sections in Appendix A, and of the following constraints between the charges q_Λ and the parameters appearing in the harmonic functions (8.12):

$$q_\Lambda I^{\Lambda\Sigma} q_\Sigma = 4\left(b_\Lambda I^{\Lambda\Sigma} b_\Sigma+\mu\, a_\Lambda I^{\Lambda\Sigma} b_\Sigma\right), \quad (8.14)$$

and $a_\Lambda=\xi_\Lambda$. The first factor in the S_0 term is not a perfect square but it vanishes under variations with respect to K and also under variations of M_Λ, provided that

$$M_\Lambda = -ie^{K/2}H_\Lambda \quad (8.15)$$

holds.

At the end of the day, through the squaring of the action we found that a nonextremal electric solution of the kind (8.8), satisfies the first order equations obtained by setting to zero each squared term in S_0 and S_2 :

$$z^{i\prime} = -\frac{e^{K/2}}{r}g^{ij}\partial_j\mathcal{W}, \qquad (r\,e^{-K/2})' = \mathcal{W}, \quad (8.16)$$

plus (8.15) and (8.14), for a superpotential $\mathcal{W}$ that satisfies (8.1) with $V_g(z)$ given by (8.4).

Finally, we explicitly verified that the Einstein's equations and the scalars equations do not give further constraints. In other words, we verified that eq. (8.15) (8.16), plus the form of the field strengths (8.7) are sufficient to solve all equations of motion.

In particular, the Duff-Liu [64] electric solutions described in Chapter 4 exactly fall into the class described by (8.6)-(8.8), and satisfy the above equations with

$$\mathcal{W}_{el}(z) = \frac{\xi_0+\xi_1 z}{2z^{3/4}}. \quad (8.17)$$

One can verify that for the electric solutions in Chapter 4 the superpotential (8.17) can be written as $\mathcal{W}_{el}=\frac{1}{2}\xi_\Lambda L^\Lambda$.

8.1.3 Squaring for the magnetic case

For the magnetic solution we start from this ansatz for the warp factors:

$$U^2(r) = e^K \left(1 - \frac{\mu}{r} + \widetilde{\mathbf{g}}^2 r^2 e^{-2K}\right), \qquad h^2(r) = e^{-K} r^2, \tag{8.18}$$

with $\widetilde{\mathbf{g}}^2 \equiv \tilde{\xi}^2 \mathbf{g}^2 \equiv \frac{4\xi_1^3\xi_0}{27}\mathbf{g}^2$. The field strengths are magnetic

$$F^\Lambda_{tr} = 0, \qquad F^\Lambda_{\theta\phi} = \frac{p^\Lambda}{2}\sin\theta. \tag{8.19}$$

Magnetic solutions found in [99] and described in Section (4.4) can be cast in this form, as explained in Appendix (D). Plugging this ansatz in the action (8.3), we see that again we collect terms in $\widetilde{\mathbf{g}}^2$ and $\widetilde{\mathbf{g}}^0$:

$$S = S_2 + S_0 \tag{8.20}$$

and in this case we obtain:

$$S_2 = \widetilde{\mathbf{g}}^2 \int d^4x\,\sqrt{\eta}\Bigg[-r^4 e^{-2K}\left(z^{i\prime} + \frac{e^{K/2}}{r} g^{ik}\frac{\partial_k \mathcal{W}}{\tilde{\xi}}\right) g_{ij}\left(z^{i\prime} + \frac{e^{K/2}}{r} g^{il}\frac{\partial_l \mathcal{W}}{\tilde{\xi}}\right) + \\ + 3r^2 e^{-K}\left(\frac{\mathcal{W}}{\tilde{\xi}} - (re^{-K/2})\right)^2\Bigg] + S^{(2)}_{td}, \tag{8.21}$$

with

$$S^{(2)}_{td} = \widetilde{\mathbf{g}}^2 \int d^4x\,\sqrt{\eta}\left(-\frac{3}{4}\frac{d^2}{dr^2}\left[r^4 e^{-2K}\right] + \frac{d}{dr}\left[2(r^3 e^{-3K/2})\frac{\mathcal{W}}{\tilde{\xi}}\right]\right). \tag{8.22}$$

Also in this case we introduce harmonic functions $H^\Lambda = a^\Lambda + \frac{b^\Lambda}{r}$, so that the part S_0 can be squared as

$$S_0 = \int d^4x\,\sqrt{\eta}\Bigg[2r^2\left(1 - \frac{\mu}{r}\right)\left(L^{\Lambda\prime} - \frac{K'}{2}L^\Lambda + e^{K/2}\frac{b^\Lambda}{r^2}\right) I_{\Lambda\Sigma}\left(L^{\Sigma\prime} - \frac{K'}{2}L^\Sigma + e^{K/2}\frac{b^\Sigma}{r^2}\right) \\ -\mu\frac{e^{K/2}}{r^2} I_{\Lambda\Sigma} 4b^\Sigma\left(L^\Lambda - e^{K/2}H^\Lambda\frac{1}{2}\right)\Bigg] + S^{(0)}_{td},$$

with L^Λ defined in (1.14) and total derivative term

$$S^{(0)}_{td} = \int d^4x\,\sqrt{\eta}\ \frac{d}{dr}\left[(r^2 - \mu r)\left(\frac{K'}{2} + 4b^\Lambda I_{\Lambda\Sigma} L^\Sigma \frac{e^{K/2}}{r}\right)\right] + \\ - \int d^4x\,\sqrt{\eta}\ \left(\mu\frac{K'}{2}\right). \tag{8.23}$$

We have used the identities of special geometry derived in Appendix A for real sections L^Λ. Furthermore, the charges need to satisfy the following constraint

$$p^\Lambda I_{\Lambda\Sigma} p^\Sigma = 4\left(b^\Lambda I_{\Lambda\Sigma} b^\Sigma + \mu\, a^\Lambda I_{\Lambda\Sigma} b^\Sigma\right)\,, \tag{8.24}$$

with $a_0 = \frac{1}{4\xi_0}$, $a_1 = \frac{3}{4\xi_1}$. As in the electric case, there is a factor in the action that is not a perfect square, nonetheless it vanishes under variations with respect to K and also L^Λ provided that this holds:

$$L^\Lambda = e^{K/2}\left(a^\Lambda + \frac{b^\Lambda}{r}\right) \equiv e^{K/2} H^\Lambda\,. \tag{8.25}$$

The first order equations coming from this squaring are given in the magnetic case by

$$z^{i\prime} = \frac{e^{K/2}}{\tilde{\xi} r} g^{ij}\partial_j \mathcal{W}\,, \qquad (r\, e^{-K/2})' = \frac{\mathcal{W}}{\tilde{\xi}}\,, \tag{8.26}$$

with $\tilde{\xi} = 2\sqrt{\xi_0\xi_1^3}/3\sqrt{3}$ and $\mathcal{W}(z)$ satisfying (8.1) with (8.4). Also in this case the Einstein's and scalar equations of motion do not give further constraints.

The magnetic solutions of [99] satisfy the first order flow equations (8.26), plus (8.25) and the constraint (8.24), with superpotential W given by:

$$\mathcal{W}_{magn}(z) = \frac{\sqrt{3}\sqrt{\xi_0\xi_1}}{2z^{1/4}} + \frac{\sqrt{\xi_1^3}\, z^{3/4}}{6\sqrt{3}\sqrt{\xi_0}}\,. \tag{8.27}$$

We see now that the superpotential driving the flow for electric and magnetic solutions is not the same. Indeed both $\mathcal{W}_{el}$ and $\mathcal{W}_{magn}$ generate through equation (8.1) the same gauging potential (8.4). Furthermore, these two quantities $\mathcal{W}_{el}$ and $\mathcal{W}_{magn}$ are related via the duality transformations described in Appendix D, acting on the scalar as

$$z \to \frac{3\xi_0^2}{\xi_1^2 z}\,. \tag{8.28}$$

Comments on the first order equations

We here show that asymptotically the equations (8.16) and (8.26) reduce to the ones (8.2) found by de Boer, Verlinde and Verlinde [104].

To see this, we first need to cast the four dimensional metric (8.5) in the form used by [104] that reads

$$ds^2 = dr^2 - g_{IJ}(x,r)dx^I dx^J\,, \qquad I,J = 1,2,3\,. \tag{8.29}$$

To do this, we perform a change of coordinates $d\tilde{r} = \frac{dr}{U}$, so that the equations (8.16) - (8.26) become

$$\frac{dz}{d\tilde{r}} = \frac{dz}{dr}\frac{dr}{d\tilde{r}} = \frac{U}{h} g^{zz}\partial_z W\,, \qquad \frac{dg_{IJ}}{d\tilde{r}} = \frac{dg_{IJ}}{dr}\frac{dr}{d\tilde{r}} = W\,U\,. \tag{8.30}$$

Given that the warp factors $U(r)$ (8.8) and (8.18) at radial infinity behave as

$$U^2 = e^K\left(\tilde{g}^2 r^2 e^{-2K} + \dots\right) \qquad \to \qquad U = \tilde{g} r e^{-K/2} + \dots \tag{8.31}$$

at leading order in r, the equations (8.30) take this form

$$\frac{dz}{d\tilde{r}} = g g^{zz}\partial_j W\,, \qquad \frac{dg_{IJ}}{d\tilde{r}} = g W g_{IJ}\,, \tag{8.32}$$

exactly as in (8.2), for both electric and magnetic solutions.

8.1.4 Generating new solutions with the first order flow

Let us focus for the moment on the first order flow for the magnetic and electric configurations. It turns out that solving the equations (8.26) plus (8.25)-(4.14) for a given $\mathcal{W}$ satisfying (8.1) gives a solution to the equation of motion.

We have found two concrete examples of $\mathcal{W}$, namely (8.17) and (8.27). They both satisfy (8.1) with (8.4), nevertheless, they are not the only solutions to this equation.

Let us now define

$$\varphi = \phi - \phi*\,, \tag{8.33}$$

so that the potential in (8.4) becomes

$$V_g = g^2 \frac{2\sqrt{\xi_0 \xi_1^3}}{\sqrt{3}} \cosh\left(\sqrt{\frac{2}{3}}\varphi\right)\,. \tag{8.34}$$

It was found in [154, 181] that all solutions to the equation (8.1) with potential (8.34) assume the following form:

$$\mathcal{W}_\nu(\varphi) = \frac{2}{3}\frac{\sqrt{2}(\xi_0\xi_1^3)^{1/4}}{3^{1/4}} \frac{1}{(1-\rho^2)^{3/4}} \frac{1-\rho^2+\sqrt{1+2\nu\rho+\rho^2}}{\sqrt{2(1+\nu\rho+\sqrt{1+2\nu\rho+\rho^2})}}\,, \tag{8.35}$$

namely they are parameterized by a continuous real parameter ν. In the formula above

$$\rho = \tanh\left(\sqrt{\frac{3}{2}}\varphi\right)\,, \tag{8.36}$$

and $\nu \geq -1$. Expression (8.35) reduces to the superpotentials that we have already found $\mathcal{W}_{el}$ and $\mathcal{W}_{magn}$ respectively for $\nu = 1$ and $\nu = -1$.

At this point the strategy to find new solutions is clear: one tries to integrate equations (8.26) with generic $\mathcal{W}$ in (8.35), in order to find a one-parameter family of solutions. These solutions will have different boundary conditions, in particular the coefficient k in $\beta = k\alpha^2$ assumes a continuous value [154]. This technique was already exploited in [181] for the construction of domain wall solutions with different asymptotic boundary conditions.

8.1.5 Comment: asymptotic expansion of the superpotentials

We would like to emphasize the fact that both superpotentials $\mathcal{W}_{el}$ and $\mathcal{W}_{magn}$ generate through (8.1) the same potential V_g. Furthermore, their asymptotic expansion is the same up to order $O(1/r^2)$:

$$\mathcal{W}_{el} = A + \frac{B}{r^2} + \frac{C_{el}}{r^3} + O\left(\frac{1}{r^4}\right) , \tag{8.37}$$

$$\mathcal{W}_{magn} = A + \frac{B}{r^2} + \frac{C_{magn}}{r^3} + O\left(\frac{1}{r^4}\right) . \tag{8.38}$$

The two asymptotic expasions differ at order $O(1/r^3)$ since $C_{magn} = 2C_{el}$. Therefore at leading order they both satisfy the asymptotic equations (8.2) for both solutions. As predicted in [104], the terms that are not determined by the HJ constraint have dimension 3.

We can also expand the superpotential in terms of the field $\phi = \sqrt{\frac{3}{8}} Log(z)$, and in that case we have ($l' = \frac{3^{3/4}}{\sqrt{2}(\xi_1^3 \xi_0)^{1/4}}$):

$$\mathcal{W}_{el} = \frac{1}{l'} + \frac{1}{4l'}\phi^2 + \frac{\phi^3}{l' \times 6\sqrt{3}} + O\left(\phi^4\right) \tag{8.39}$$

$$\mathcal{W}_{magn} = \frac{1}{l'} + \frac{1}{4l'}\phi^2 - \frac{\phi^3}{l' \times 6\sqrt{3}} + O\left(\phi^4\right) \tag{8.40}$$

This difference reflects the two kind of boundary conditions $\beta = k\alpha^2$, where $k = -1/\sqrt{6}$ for the electric configuration and $k = 1/\sqrt{6}$ for the magnetic one. Furthermore, in particular, we have the expansion (6.8) of [181] holds in our conventions with $\Delta_- = 1$, therefore we could in principle use either one or the other as counterterm in the holographic renormalization procedure. More details about this on the next subsection.

8.2 Mass from holographic renormalization

We here compute the mass of the black holes in Anti-de Sitter by means of holographic renormalization techniques.

There exist nowadays well established procedures for computing the boundary counterterms and removing the divergencies[2], see for instance [187, 188]. Here we will make use of the Hamilton-Jacobi (HJ) method, first used in the context of AdS/CFT by [104]. A comparison between the HJ approach and other holographic renormalization techniques [187, 188] can be found for instance in [189]. The analysis suitable for black hole solutions such as ours is the one of Papadimitriou [154], where the presence of mixed boundary conditions was taken into account in the renormalization of the stress-energy tensor. We follow closely this procedure. Let us finally mention that the analysis of the mass obtained from the HJ renormalization technique was performed in [106], in which the authors compute the mass for electric black holes solutions in the truncation of $\mathcal{N}=8$ $SO(8)$-gauged theory to the $\mathcal{N}=2$ $U(1)$ gauged subsector.

8.2.1 Regularized action

To properly compute the conserved quantities in a 4-dimensional spacetime with boundary $\partial\mathcal{M}$ we have to consider the bulk action together with the contribution coming from the Gibbons-Hawking boundary term

$$\begin{aligned} I &= I_{bulk} + I_{GH} = \\ &= \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\left(\frac{R}{2} - g_{ij}\partial_\mu z^i \partial^\mu z^j + I_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\mu\nu\Sigma} - V_g\right) \\ &\quad - \int_{\partial\mathcal{M}} d^3x\,\sqrt{-h}\Theta\,. \end{aligned} \tag{8.41}$$

In the Gibbons-Hawking term Θ is the trace of the extrinsic curvature

$$\Theta_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu) \equiv -\nabla_{(\mu} n_{\nu)} \tag{8.42}$$

where we choose $n^\mu = (0, \sqrt{-g^{rr}}, 0, 0)$ as an outward-pointing normal vector to $\partial\mathcal{M}$, and $h = \det(h_{\mu\nu})$ is the determinant of the induced metric $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ on $\partial\mathcal{M}$ [184].

For any Killing vector field K^a associated with an isometry of the boundary induced metric $h_{\mu\nu}$, we can define the conserved quantity

$$Q_K = \frac{1}{8\pi}\int_\Sigma d^2x\,\sqrt{\sigma} u_a \tau^{ab} K_b\,, \tag{8.43}$$

where Σ is the spacelike section of the boundary surface $\partial\mathcal{M}$, $u^a = \sqrt{h^{tt}}(1,0,0)$ is the unit normal vector to Σ in $\partial\mathcal{M}$, σ_{ab} is the induced metric on Σ and finally the

[2] Let us mention that the removal of divergencies of the on-shell action by adding counterterms was first performed in [182, 183], while the notion of energy and black hole mass in terms of the renormalized Brown-York [184] boundary stress-energy tensor was analyzed first in [185, 186].

local surface energy momentum tensor is defined as the variation of the boundary action with respect to the induced metric

$$\tau^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ab}} \ . \tag{8.44}$$

The mass of the black hole is the conserved quantity associated with the killing vector $K^a = (1,0,0,0)$ of the metric $h_{\mu\nu}$ at the boundary.

Notice that, since the boundary stress energy tensor computed for the action (8.41) is divergent, we need to regulate it and then add an appropriate counterterm action I_{ct} , so the resulting contribution to the black hole mass will be given by

$$I = I_{reg} + I_{ct} \ , \tag{8.45}$$

or equivalently

$$\tau^{ab} = \tau^{ab}_{reg} + \tau^{ab}_{ct} \ . \tag{8.46}$$

We choose to regularize (8.41) by introducing a cutoff radius r_0 in the parametrization of the spacetime, thus leaving a truncated spacetime $\mathcal{M}_0$ with boundary $\partial\mathcal{M}_0$ located at $r = r_0$. Removing the cutoff corresponds to taking the limit $r_0 \to \infty$. The regulated boundary stress tensor receives contribution from the Gibbons-Hawking term and has the form

$$\tau^{ab}_{reg} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h_{ab}}\Big|_{r_0} = \left(\Theta^{ab} - \Theta h^{ab}\right)\Big|_{r_0} \ . \tag{8.47}$$

The mass of the black hole solution is the finite on-shell quantity remaining after removing the cutoff in the expression

$$M_{ren} = Q_K(\tau^{ab}_{reg}) + Q_K(\tau^{ab}_{ct}) \equiv E_{reg} + E_{ct} \ . \tag{8.48}$$

We will discuss in the rest of the section how to compute the contribution from the counterterms and how to derive a finite formula for the black hole mass.

8.2.2 Counterterms

The analysis of the counterterms needed for the action (8.41) has been performed in [106, 154, 181], by solving the Hamilton-Jacobi equations. We refer to those papers for the details of the computation. The allowed possible counterterms assume this form:

- The counterterm involving the Ricci scalar of the boundary metric $\mathcal{R}$

$$I_{ct,1} = \int_{\partial\mathcal{M}_0} d^3x \sqrt{-h} \left(\frac{1}{\tilde{g}}\mathcal{R}\right) \ . \tag{8.49}$$

- The counterterm involving the superpotential

$$I_{ct,2} = \int_{\partial\mathcal{M}_0} d^3x\,\sqrt{-h}\,(2g\mathcal{W}(z))\,, \tag{8.50}$$

where $\mathcal{W}(z)$ satisfies the relation

$$\frac{V_g}{g^2} = -3\mathcal{W}^2 + g^{ij}\partial_i\mathcal{W}\partial_j\mathcal{W}\,, \tag{8.51}$$

namely it is a superpotential for the gauging potential V_g. One is allowed to use as counterterm for the renormalization any of the superpotentials which are solutions of (8.51), each of them corresponding to a different renormalization scheme. For our electric and magnetic solutions, in particular, such counterterm is of the form (see subsection 8.1.5)

$$\mathcal{W} = \frac{1}{l'} + \frac{1}{4l'}\phi^2 + \mathcal{C}\frac{\phi^3}{l'\times 6\sqrt{3}} + O\left(\phi^4\right)\,, \tag{8.52}$$

with $\mathcal{C}$ a real number, choice that determines our renormalization scheme. Quantities relative to solutions with the same boundary conditions should be computing with the same renormalization scheme.

- The term involving the gauge field:

$$I_{ct,3} = \int_{\partial\mathcal{M}_0} d^4x\,\sqrt{-h}\;C_{\Lambda\Sigma}(z)F^{\Lambda}_{ab}F^{ab,\Sigma}\,, \tag{8.53}$$

where C is a scalar-dependent matrix to be determined from the Hamilton-Jacobi equations. Given the falloff conditions for the fields

$$z(r) = z_* + O\left(\frac{1}{r}\right)\,,\quad f(r) = g^2r^2 + O(r)\,,\quad e^{K(r)} = const + O\left(\frac{1}{r}\right)\,, \tag{8.54}$$

and field strengths (8.7) this counterterm gives a vanishing contribution once the cutoff is removed ($r_0 \to \infty$).

- Terms including higher powers of $\mathcal{R}$, namely

$$I_{ct,4} = \int_{\partial\mathcal{M}_0} d^4x\,\sqrt{-h}\,(E(z)\mathcal{R}^2 + D(z)\mathcal{R}_{ab}\mathcal{R}^{ab})\,, \tag{8.55}$$

also vanish once we remove the cutoff for solutions with falloff conditions (8.54).

- A term that accounts for the presence of mixed boundary conditions for the scalar field[3]

$$I_{ct,mixed} = \int_{\partial\mathcal{M}_0} d^3x\,\sqrt{-h}\,\mathbf{f}(\phi_-)\,, \tag{8.56}$$

[3]Details about mixed boundary conditions for scalar fields were already introduced in Chapter 6.5.

where ϕ_- is the expectation value of the operator of dimension one, namely the parameter α defined as in (6.11). The function $\mathbf{f}(\phi_-)$ is such that $\beta = \mathbf{f}'(\alpha)$, and in our specific case this yields

$$\mathbf{f}(\alpha) = \frac{1}{\sqrt{6}}\frac{1}{3}\alpha^3 . \tag{8.57}$$

Further details about the holographic dictionary in presence of mixed boundary conditions are given in [154], table 3.

To sum up, the counterterms that give a nonvanishing contribution in the renormalization of the stress-energy tensor, for our falloff conditions, are $I_{ct,1}$, $I_{ct,2}$, $I_{ct,mixed}$. In what follows, we show the details of the computation of the mass for the electric and magnetic solutions.

8.2.3 Renormalized black hole mass

We have now all the ingredients to compute the mass from the formula (8.48).

From the definition (8.43) we can compute the regulated energy, corresponding to τ^{ab}_{reg}

$$E_{reg} = \frac{1}{8\pi}\int_\Sigma \sqrt{\sigma} u_t \tau^{tt}_{reg}\xi_t = -\frac{r_0^2}{4} f(r_0)K'(r_0) + \frac{1}{2} r_0 f(r_0) , \tag{8.58}$$

where we have used the extrinsic curvature expressions

$$\Theta_{tt} = h_{tt}\, e^{K/2}\sqrt{f(r)}\left(\frac{K'}{2} + \frac{f'}{2f}\right), \qquad \Theta = e^{K/2}\sqrt{f(r)}\left(-\frac{K'}{2} + \frac{f'}{2f} + \frac{2}{r}\right). \tag{8.59}$$

When the counterterms described in the previous section 8.2.2 are added to action, the boundary stress-energy tensor gets the contributions

$$E_{ct,i} = \frac{1}{8\pi}\int_\Sigma \sqrt{\sigma} u_t \tau^{tt}_{ct,i}\xi_t . \tag{8.60}$$

where

$$\tau^{ab}_{ct,i} = \frac{2}{\sqrt{-h}}\frac{\delta I_{ct,i}}{\delta h_{ab}}\Big|_{r_0} . \tag{8.61}$$

Adding up all these terms to the regularized energy, and expanding for $r_0 \to \infty$ one finally obtains the mass for the electric and magnetic black hole solutions we described.

For the electric solutions parameterized as in 4.3, we obtain

$$\begin{aligned} M \;=\;& \frac{b_0}{4\xi_0} + \frac{3b_1}{4\xi_1} + \frac{\mu}{2} - \frac{b_0^3}{32\xi_0^3 l^2} + \frac{b_0^3\xi_1^3}{432\xi_0^2 l^2} + \frac{3b_0^2 b_1}{32\xi_0^2\xi_1 l^2} - \frac{b_0^2 b_1\xi_1^2}{144\xi_0 l^2} + \\ &+\; \mathcal{C}\left(-\frac{b_0^3\xi_1^3}{432\xi_0^2 l^2} + \frac{b_0^2 b_1\xi_1^2}{144\xi_0 l^2} - \frac{b_0 b_1^2\xi_1}{144 l^2} + \frac{b_1^3\xi_0}{432 l^2}\right) - \frac{3b_0 b_1^2}{32\xi_0\xi_1^2 l^2} + \frac{b_0 b_1^2\xi_1}{144 l^2} - \frac{b_1^3\xi_0}{432 l^2} + \frac{b_1^3}{32\xi_1^3 l^2} \end{aligned} \tag{8.62}$$

where the value of $\mathcal{C}$ is related to the choice of prescription of the renormalization. Notice that the choice $\mathcal{C} = 1 - \frac{27}{2\xi_0\xi_1^3}$ gives the mass formula

$$M_{ren} = \frac{1}{4}\left(2\mu + \frac{b_0}{\xi_0} + 3\frac{b_1}{\xi_1}\right), \tag{8.63}$$

that coincides with the value in (4.62), where the mass was computed by means of the AMD technique. This was already proven in [106] for electric solutions in maximal FI gauged supergravity.

For the magnetic configuration in the parameterization (4.67), gives

$$M = \frac{64\mathcal{C}b_1^3\xi_1^3}{27l^2} + \frac{64b_1^3\xi_1^3}{27l^2} - \frac{3\xi_0^2(p^0)^2}{8b_1\xi_1} + \frac{\xi_1(p^1)^2}{24b_1} + \frac{4b_1\xi_1}{3}. \tag{8.64}$$

If we now choose the remormalization scheme[4] such that $\mathcal{C} = -1$, we obtain:

$$M_{ren} = \frac{4b_1\xi_1}{3} - \frac{3\xi_0^2(p^0)^2}{8b_1\xi_1} + \frac{\xi_1(p^1)^2}{24b_1}. \tag{8.65}$$

This value coincides with the AMD mass of the solution (4.76), but disagrees with formula (7.86), which gives the expression for the quantity related to the BPS bound. Whatever choice of renormalization scheme we choose, we cannot achieve the expression $\mathcal{M}^*$ in formula (7.86).

Discussion

In this chapter we have computed the mass with the Hamilton-Jacobi holographic renormalization procedure, and we have shown that this computation agrees with the result obtained with the AMD technique for a specific choice of renormalization scheme.

Other choices lead to different expressions for the mass: in particular different finite terms contribute to its expression. This ambiguity concerning finite terms in the HJ procedure was also pointed out in [104] and [181], where it was noticed that eq. (8.1) fixes the divergent contribution of E_{ct} coming from the superpotential $\mathcal{W}$, but not its finite contribution.

[4]We are allowed to choose different renormalization schemes for electric and magnetic solutions because they obey different boundary conditions (see remark at the end of 6.5).

Conclusions and outlook

This thesis dealt with the development of the supergravity flow equations and solution generating techniques for Anti-de Sitter black hole solutions in $\mathcal{N} = 2$ gauged supergravity, and the analysis of their aspects such as thermodynamics, phase transitions and superalgebras.

After a recap of the main features of gauged $\mathcal{N} = 2$ supergravity in Chapter 1, Chapters 2, 3, 4, 5 dealt with the construction and description of BPS and non-BPS static and stationary AdS black holes. We have seen that the construction of BPS black holes arises from solving first order differential equations coming from the Killing spinor equations or alternatively from the BPS squaring of the action. We have examined, among the others, the first example of static supersymmetric AdS black hole found in [30] and [45], [70]. Once we know the form of the BPS solution, we can deform it in order to obtain zero temperature non-supersymmetric ones, or thermal configurations. Moreover, we have studied a systematic approach to obtain more general Anti-de Sitter stationary solutions described by angular momentum, electromagnetic charges, mass and NUT charge.

Despite the fact that we work in a specific supergravity model, we expect that the techniques we analyzed work also in more general settings.

The ansatz for the AdS solutions and some of the generating techniques were inspired from the much better known case of ungauged supergravity, which yields asymptotically flat solutions. Indeed at the level of supergravity equations, and BPS squaring, there are some analogies between gauged and ungauged supergravity that we can exploit in order to construct solutions.

Chapters 6, 7, 8 instead dealt with the thermodynamics of a specific class of static configurations, and with the definition of conserved charges and thermodynamic quantities. As we said before, the thermodynamics of Anti-de Sitter black holes exhibit present a lot of interesting features, in particular phase transitions between different black hole branches can arise. Transitions are important for their interpretation in terms of the AdS/CFT correspondence. This analysis comes with some subtleties in computing mass in AdS, and in particular with BPS bound. It remains somehow unsettling that the expression for the quantity appearing in the supersymmetry anticommutator (7.85) does not coincide, in the case of configurations running scalars, with the expression coming from the AMD techniques. Nevertheless, we would like here to stress the point that these two quantities are per se consistent. The ADM mass satisfies the first law and it coincides also with the quantity obtained by the holographic renormalization techniques. The quantity that appears in the superalgebra provides the correct BPS bound, retrieved also in the case of minimal gaugings.

These last few years in particular have witnessed a good amount of developments regarding Anti-de Sitter black hole solutions, in particular the ones arising from gauged supergravity. Being less explored than their asymptotically flat counterparts, and appealing for applicative sides of AdS/CFT, AdS black holes provide a fertile field of investigation, rapidly developing. We hope that this thesis fulfilled its goal to give an overview at least in part of the state of the art in the realm of gauged supergravity. Even if progress has been made there are still plenty of questions left for future exploration. If we compare the results explained in this thesis with the outlook of the thesis [190], we see that we can put a check-mark on a good amount of points. For this reason I am optimistic that at least some points of the to-do list we are giving in this last part (and the other observations regarding future directions discussed at the end of each chapter) will be fulfilled soon enough.

- **New AdS solutions from gauged supergravity**

  One obvious long-term ambition is to further enlarge the set of known AdS black hole solutions (and black objects with different horizon topology) in gauged supergravity theories, and classify the spectrum of solutions in the case of general gaugings. It would be interesting to see if the only allowed kind of supersymmetry compatible with the presence of a horizon in the static case falls into the class of [30] also in presence of arbitrary gaugings.

  Furthermore, a long-standing challenge is proving the existence of multicenter AdS solutions. Indeed it was believed that such solutions could not exist, since the negative cosmological constant effectively acts as a negative force between the centers. However, recently the existence of stable probe states in a thermal AdS black hole background with scalar fields was proven [111] (also, the analysis of multicenter configurations in a dynamical AdS background was undertaken in [151]), and this is a hint that black hole bound states are possible even in presence of a negative cosmological constant. It would be interesting to see if these solutions remain stable even when the backreaction is taken into account, and/or to find exact solutions of this kind.

  Furthermore, another recent development in the field was the discovery of nonequivalent models of maximal gauged supergravity [118]. First of all, it would be interesting to understand the meaning of the omega deformation on the level of the dual field theory. Secondly, one could study the possibility for the existence of new black hole solutions in these omega-deformed theories, along the lines of [117, 191], in appropriate truncations to $\mathcal{N}=2$ models.

  Last but not least, it is important to learn how the supergravity solutions we have dealt with in this thesis with could change due to the addition of higher derivative terms typically coming from string theory and how/if singular solutions can resolved. In particular, one might wonder what the fate of the singular solutions is once higher order corrections are taken into account.

- **Phase transitions and AdS/CFT**

  New solutions most likely alter the picture of the thermodynamic ensemble. Novel phase transitions can arise and their analysis in terms of AdS/CFT correspondence might reveal interesting connections with condensed matter systems. A reason to search for phenomenological models for AdS/CMT in the realm of gauged supergravity (rather than in a so called "bottom-up" approach) is the possibility of having much better control over the dual field theory because of the well-established dualities.

  For instance, the authors of [112] dealt with a holographic superconductor setup for which a string theory embedding exists: they considered solutions of maximal gauged supergravity with charged scalars. Along the same lines, for instance, in our case one could try to find the thermal generalization of the solutions with hypermultiplets found in [60], with the techniques developed in Chapter 4. Such thermal solutions would contain charged scalar fields, and could be useful in modeling the holographic superconductor phase transition.

  Furthermore, study of bound states of AdS black holes, namely multicenter solutions, are relevant in applications of the AdS/CFT correspondence to the study the glass phase transition, the physics of supercooled systems and their relaxation dynamics [111].

  As we have seen, regarding the thermodynamics of these black holes and the applications of AdS/CFT it is crucial to know how to consistently define the thermodynamic quantities and conserved charges like the mass, for AdS configurations.

  Let us mention lastly two of the obvious directions to expand the analysis developed in this thesis. One immediate follow-up is the development of the thermodynamic properties and phase transitions of the rotating solutions presented in Chapter 5. Some preliminary results have been already presented in Chapter 6. Another interesting development would be to retrieve the small-large phase transition of Chapter 6 through the computation of the ABJM partition function on $S^1 \times S^2$, in presence of background magnetic field.

- **Microstates counting for AdS$_4$ black holes**

  As we mentioned before, the AdS black holes treated in this thesis have uplift in M-theory and represent the gravitational backreaction of bound states of M2 and M5-branes wrapped on curved internal dimensions.

  However, the microscopic entropy counting cannot be performed as it is done in the case of BPS Minkovski black holes, since for AdS ones the scalar fields get stabilized at a value that extremizes the potential.

One can try instead to analyze them as BPS states in the dual field theory. The entropy should be retrieved by computing appropriate degeneracy indices, counting supersymmetric states in the CFT. This research program has been undertaken in the case of rotating BPS AdS_5 black holes [192–195].

In our case of four-dimensional AdS black holes, the BPS states to count are those of the dual field theory that fall in the class of ABJM models with magnetic background field. One possible approach explained in [175] for the counting consists in applying holography to the black hole near-horizon geometry, which contains an AdS_2 factor: the aim in this case is to exploit the the AdS_2/CFT_1 correspondence to find a one-dimensional superconformal quantum mechanics capturing the microscopic degrees of freedom (see for example [196, 197]). Usually there is no systematic way to obtain the correct CFT, whose states reproduce the black hole entropy. For black holes in AdS, however, one can also use holography at the boundary. The magnetic BPS black holes described in Chapter 2 have AdS_4 asymptotics and $AdS_2 \times S^2$ in the near-horizon limit, and can be uplifted to M-theory. The M-brane geometry and holography at the boundary help in identifying the correct dual CFT_1, where the microscopic counting is be performed. Some preliminary steps in this direction, namely checks on the superconformal algebras, were already performed for instance in [175], but the full program of microstate counting for the magnetic BPS black holes is to be considered still work-in-progress.

Appendix A

Conventions and special Kähler quantities

General conventions

In our conventions the signature is $[+,-,-,-]$ and Riemann-Christoffel tensor and the Ricci tensor are defined as

$$R^{\rho}_{\sigma\mu\nu} = -(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\mu \Gamma^{\rho}_{\nu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma})\,, \qquad R^{\rho}{}_{\sigma\rho\nu} = R_{\sigma\nu}\,. \tag{A.1}$$

The Einstein's equation then read:

$$-(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = \kappa^2 T_{\mu\nu}\,, \tag{A.2}$$

where T_{00} is positive. We furthermore take $\kappa = 1$.

The Levi-Civita tensor is defined in this way:

$$\epsilon^{0123} = 1 = -\epsilon_{0123}\,, \tag{A.3}$$

with

$$\epsilon^{\mu\nu\rho\sigma} = \sqrt{-det\,g}\; e^{\mu}{}_{a} e^{\nu}{}_{b} e^{\rho}{}_{c} e^{\sigma}{}_{d} \epsilon^{abcd}\,. \tag{A.4}$$

Our convention for the field strengths is:

$$F_{\mu\nu} = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\,. \tag{A.5}$$

The electric and magnetic charges are denoted respectively as q_Λ and p^Λ and are defined as

$$-\frac{1}{4\pi}\int_{S^2} F^{\Lambda} \equiv p^{\Lambda}\,, \qquad -\frac{1}{4\pi}\int_{S^2} G_{\Lambda} \equiv q_{\Lambda}\,. \tag{A.6}$$

Gamma matrix conventions

The Dirac gamma-matrices in four dimensions satisfy

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} \ , \tag{A.7}$$

and we define

$$[\gamma_a, \gamma_b] \equiv 2\gamma_{ab} \ , \qquad \gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{A.8}$$

In addition, they can be chosen such that

$$\gamma_0^\dagger = \gamma_0 , \quad \gamma_0\gamma_i^\dagger\gamma_0 = \gamma_i , \quad \gamma_5^\dagger = \gamma_5 , \quad \gamma_a^* = -\gamma_a \ . \tag{A.9}$$

An explicit realization of such gamma matrices is the Majorana basis, given by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} , \qquad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} ,$$

$$\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} , \qquad \gamma_5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \ , \tag{A.10}$$

where the σ^i $(i = 1, 2, 3)$ are the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{A.11}$$

For the charge conjugation matrix, we choose

$$C = i\gamma^0 \ , \tag{A.12}$$

hence Majorana spinors have real components.

We also make use of the following identities, with curved indices:

$$\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\rho = ie\gamma^{\mu\nu\sigma} \, , \tag{A.13}$$

$$\gamma_\mu\gamma_{\rho\sigma} = -\gamma_\rho g_{\mu\sigma} + \gamma_\sigma g_{\mu\rho} + \frac{i}{e}\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\nu \, , \tag{A.14}$$

$$\gamma_\mu\gamma_{\nu\rho}\gamma_\sigma - \gamma_\sigma\gamma_{\nu\rho}\gamma_\mu = 2g_{\mu\nu}g_{\rho\sigma} - 2g_{\mu\rho}g_{\nu\sigma} + 2\frac{i}{e}\epsilon_{\mu\nu\rho\sigma}\gamma_5 \, . \tag{A.15}$$

Antisymmetrizations are taken with weight one half, and the totally antisymmetric Levi Civita symbol is defined by

$$\epsilon^{0123} = 1 = -\epsilon_{0123} \, . \tag{A.16}$$

With curved indices,

$$\epsilon^{\mu\nu\rho\sigma} \equiv e\, e^{\mu}_{a} e^{\nu}_{b} e^{\rho}_{c} e^{\sigma}_{d}\, \epsilon^{abcd} \ , \tag{A.17}$$

is a tensor-density.

Another important property that ensures the super-Jacobi identities of $Osp(2|4)$ hold is

$$(\hat{\gamma}^{MN} C^{-1})^{\alpha\beta} (\hat{\gamma}_{MN} C^{-1})^{\gamma\delta} = (C^{-1})^{\alpha\gamma} (C^{-1})^{\beta\delta} + (C^{-1})^{\alpha\delta} (C^{-1})^{\beta\gamma} \ , \tag{A.18}$$

where $\hat{\gamma}_{MN}$ are defined in section 7.3.1.

Special Kähler quantities for $F = -2i\sqrt{X^0(X^1)^3}$

Our conventions on special Kähler quantities are as in. We restrict ourselves to a model with prepotential $F(X) = -2i\sqrt{X^0(X^1)^3}$, just one scalar field. The relevant special Kähler quantities read:

$$\mathcal{K} = -log[i(\overline{X}^{\Lambda} F_{\Lambda} - \overline{F}_{\Lambda} X^{\Lambda})] = -log[X^0 \overline{X}^0 (\sqrt{z} + \sqrt{\bar{z}})^3], \tag{A.19}$$

such that the Kähler metric is

$$g_{z\bar{z}} = \partial_z \partial_{\bar{z}} \mathcal{K} = \frac{3}{4} \frac{1}{(\sqrt{z} + \sqrt{\bar{z}})^2 \sqrt{z\bar{z}}} \, . \tag{A.20}$$

This requires $Re(z) > 0$. The scalar potential turns out to be

$$V(z,\bar{z}) = (g^{ij} f_i^{\Lambda} \bar{f}_j^{\Sigma} - 3\bar{L}^{\Lambda} L^{\Sigma}) \xi_{\Lambda} \xi_{\Sigma} =$$

$$= \frac{1}{(\sqrt{z} + \sqrt{\bar{z}})^3} \left(\left(-2z - 2\bar{z} - 4\sqrt{z\bar{z}} \right) \xi_0 \xi_1 - \left(\frac{4}{3} z\bar{z} + \frac{2}{3}(z+\bar{z})\sqrt{z\bar{z}} \right) \xi_1^2 \right) , \tag{A.21}$$

and the period matrix is

$$\mathcal{N}_{00} = i \frac{2\sqrt{z^3}\sqrt{\bar{z}}}{\sqrt{\bar{z}} - 3\sqrt{z}} \, , \quad \mathcal{N}_{01} = i \frac{3(z - \sqrt{z\bar{z}})}{\sqrt{\bar{z}} - 3\sqrt{z}} \, , \quad \mathcal{N}_{11} = i \frac{6}{\sqrt{\bar{z}} - 3\sqrt{z}} \, . \tag{A.22}$$

In the explicit examples the scalar is taken to be real. The Kähler metric reduces to

$$g_{z\bar{z}} = \frac{3}{16 z^2} \, . \tag{A.23}$$

Given the real sections the period matrix is purely imaginary and diagonal:

$$\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} -i\sqrt{z^3} & 0 \\ 0 & -3i\sqrt{\frac{1}{z}} \end{pmatrix} , \tag{A.24}$$

and the scalar potential reduces to

$$V(z,\bar{z}) = -g^2 \left(\frac{\xi_0 \xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z} \right) . \tag{A.25}$$

Special geometry in case of zero axions

In the following some relations of special geometry valid in the case of no axionic fields are derived. These relations are used in the squaring of the action in Chapter 8.

In case of zero axions $\mathcal{N}_{\Lambda\Sigma} = iI_{\Lambda\Sigma}$. The sections satisfy the relation

$$< \mathcal{V}, \mathcal{V}' >= 0 = M_\Lambda (L^\Lambda)' - L^\Lambda (M_\Lambda)' . \tag{A.26}$$

However, the sections are normalized as[1]

$$L^\Lambda M_\Lambda = i(L^\Lambda I_{\Lambda\Sigma} L^\Sigma) = -\frac{i}{2} , \qquad \rightarrow \qquad (L^\Lambda I_{\Lambda\Sigma} L^\Sigma)' = 0 \tag{A.27}$$

but that means

$$M_\Lambda (L^\Lambda)' + L^\Lambda (M_\Lambda)' = 0 . \tag{A.28}$$

This, together with equation (A.26) implies

$$L^\Lambda (M_\Lambda)' = 0 . \tag{A.29}$$

By definition[1], in absence of axions i.e. when $\mathrm{Re}\mathcal{N}_{\Lambda\Sigma} = 0$ we have

$$M_\Lambda = iI_{\Lambda\Sigma} L^\Sigma , \qquad D_i M_\Lambda = -iI_{\Lambda\Sigma} D_i L^\Sigma , \tag{A.30}$$

and, again, if we restrict to the real submanifold we have

$$\mathcal{Q} = \frac{1}{2i}(dz^i \partial_i \mathcal{K} - d\bar{z}^{\bar{\imath}} \partial_{\bar{\imath}} \mathcal{K}) = 0 . \tag{A.31}$$

Then,

$$M'_\Lambda = z^{i\prime} D_i M_\Lambda + i\mathcal{Q}_r = (\text{zero axions}) = z^{i\prime} D_i M_\Lambda = -i\, z^{i\prime} I_{\Lambda\Sigma} D_i L^\Sigma = -iI_{\Lambda\Sigma}(L_\Sigma)'$$

thus

$$(I_{\Lambda\Sigma} L^\Sigma)' = -I_{\Lambda\Sigma}(L_\Sigma)' . \tag{A.32}$$

This, together with (A.29), imply also that

$$L^\Lambda I_{\Lambda\Sigma}(L_\Sigma)' = 0 . \tag{A.33}$$

[1]See eq. 4.35 of [34].

Special Kähler quantities for $F = -iX^0X^1$

This model has just one complex scalar τ. The symplectic sections in special coordinates are $v^T = (1, \tau, -i\tau, -i)$. The Kähler potential and metric are:

$$e^{-\mathcal{K}} = 2(\tau + \overline{\tau}) \ , \qquad g_{\tau\overline{\tau}} = \partial_\tau \partial_{\overline{\tau}} \mathcal{K} = (\tau + \overline{\tau})^{-2} \ , \tag{A.34}$$

and the vector kinetic matrix is of this form:

$$\mathcal{N} = \begin{pmatrix} -i\tau & 0 \\ 0 & -\frac{i}{\tau} \end{pmatrix} \ , \tag{A.35}$$

thus requiring $\mathrm{Re}\tau > 0$. For our choice of electric gauging in Chapter 5, the scalar potential is

$$V = -\frac{4}{\tau + \overline{\tau}} (g_0^2 + 2g_0 g_1 \tau + 2g_0 g_1 \overline{\tau} + g_1^2 \tau \overline{\tau}) \ , \tag{A.36}$$

which has an extremum at $\tau = \overline{\tau} = |g_0/g_1|$.

Appendix B

Symplectic rotation to the t^3 prepotential

The gauging procedure in general breaks the electric-magnetic duality that Einstein-Maxwell theories possess. We consider here an approach that restores symplectic covariance in $\mathcal{N}=2$ FI gauged supergravity, allowing for both electric and magnetic gaugings. We briefly elucidate here some aspects of the gauging procedure and we describe dyonic black hole solutions (BPS, non BPS) of a particular theory with magnetic gaugings.

Magnetic gaugings are those for which the image in the symplectic group allows for nonzero entries of the quantity B in formula (1.34). There have been various proposals in the literature for introducing them in a consistent way [57, 58], among which we quote making use the embedding tensor formalism [54, 198, 199]. Such formalism requires the introduction of additional tensor fields in the Lagrangian and, in case the charges are mutually local, the symplectic covariance is restored[1].

The most general gauged supergravity theory with electric and magnetic gaugings is not known. Nevertheless, we know how the bosonic Lagrangian and supersymmetry rules look in the case of FI mixed (electric and magnetic) gaugings. We follow here the embedding tensor approach of [54, 55]. We follow section 5.1 of [54], where it is explained that it is possible to integrate out the tensor fields, along with gauge fixing half of the gauge fields, that we assumed to be both electric A_μ^Λ and magnetic $A_{\mu,\Lambda}$, $\Lambda=0,..n_V$. In the starting Lagrangian there are $2n_V+2$ vector fields (electric and magnetic). At the end of the day the procedure of [54] yields a Lagrangian with n_V+1 physical vectors, that we denote $(A_\mu^{\Lambda'}, A_{\mu,\Lambda''})$, where we split the index Λ in two parts $\{\Lambda\}=\{\Lambda',\Lambda''\}$. The corresponding vector fields $(A_\mu^{\Lambda''}, A_{\mu,\Lambda'})$ have been integrated out.

Thus the linear combination of vector fields

$$\xi_{\Lambda'}A_\mu^{\Lambda'}-\xi^{\Lambda''}A_{\mu,\Lambda''} \tag{B.1}$$

[1]Even if the quantities appearing in the equations are symplectic invariant, we avoid the term "symplectic invariance of the theory", since the prepotential changes (hence the theory per se is not invariant). Symplectic covariance has to be rather understood as equivalence between theories.

is used for performing the $U(1)$ FI gauging. The coefficients $g\xi^{\Lambda''} = g^{\Lambda''}$ are the magnetic charges of the gravitini, and the Dirac quantization condition for the electric and magnetic charges of the solution $(p^\Lambda, q_\Lambda) = Q$ then reads:

$$g_{\Lambda'}p^{\Lambda'} - g^{\Lambda''}q_{\Lambda''} = \langle Q,\mathcal{G}\rangle = n \qquad n\in\mathbb{Z}\,, \tag{B.2}$$

where we defined $\mathcal{G} = (g^\Lambda, g_\Lambda)$, as anticipated at the end of Chapter 2. The resulting Lagrangian and susy variations for FI gauged supergravity with magnetic gaugings (and local charges) are obtained simply as a symplectic completion of the theory described in section 1.4. In particular, the scalar potential assumes this form:

$$V_g(z,\overline{z}) = (g^{l\overline{k}} f_l^{\Lambda'} f_{\overline{k}}^{\Sigma'} - 3\overline{L}^{\Lambda'}L^{\Sigma'})g_{\Lambda'}g_{\Sigma'} - (g^{l\overline{k}} h_{l,\Lambda''}h_{\overline{k},\Sigma''} - 3\overline{M}_{\Lambda''}M_{\Sigma''})g^{\Lambda''}g^{\Sigma''}\,, \tag{B.3}$$

and can be rewritten as (1.80)

$$V_g = g^{i\overline{\jmath}}\, D_i\mathcal{L}\overline{D}_{\overline{\jmath}}\overline{\mathcal{L}} - 3|\mathcal{L}|^2\,, \tag{B.4}$$

where $D_i\mathcal{L} \equiv \partial_i\mathcal{L} + \frac{1}{2}\partial_i K\,\mathcal{L}$ and in this case

$$\mathcal{L} = \langle\mathcal{V},\mathcal{L}\rangle = e^{\mathcal{K}/2}(X^\Lambda g_\Lambda - F_\Lambda g^\Lambda)\,. \tag{B.5}$$

Once we have this symplectic covariant formulation, we can easily rotate solutions to a different symplectic frame. For our black hole case, the equations one gets from the supersymmetry variations (or the one-dimensional squaring, see Chapter 3) are constructed with symplectic invariant objects. Therefore, a solution of the model where $\mathcal{L} = \langle\mathcal{G},\mathcal{V}\rangle$, will be mapped into a solution of a different model, where $\mathcal{L} = \langle\mathcal{G}',\mathcal{V}'\rangle$ where $\mathcal{V}' = \mathcal{S}\mathcal{V}$ and FI parameters $\mathcal{G}' = \mathcal{S}\mathcal{G}$. The symplectic rotation acts on all the symplectic vectors, namely the electromagnetic charges Q, the sections $\mathcal{V}$ and the Fayet-Iliopoulos $\mathcal{G}$.

In our specific case, the magnetic BPS solution constructed in 2.4.3 in the symplectic frame $F = -2i\sqrt{X^0(X^1)^3}$ with purely electric gaugings g_0, g_1 is equivalent to a dyonic one, in a theory with mixed gaugings g_0, g^1, with prepotential $F = \frac{X^1X^2X^3}{X^0}$ (also known as t^3 model).

The symplectic matrix $\mathcal{S}$ performing such rotation is (see for example [45, 70])

$$\mathcal{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}\,, \tag{B.6}$$

We report here the BPS solution in the rotated t^3 frame, presented for the first time in [45]. The advantage of having this solution at hand is that we can directly apply the formulas of [78], that are valid for stu models. These formulas express the entropy of the BPS black hole configuration as function of the quartic invariant of specific vectors $\mathbf{p}$ and $\mathbf{q}$ containing the black hole and gauging charges.

Rotated BPS solution

We discuss here the details of the dyonic solution of the model *stu* with the three scalar fields identified (*ttt*-model). The solution can be obtained by performing the symplectic rotation (B.6) on the configuration obtained in the model $F = -2i\sqrt{X^0X^1X^2X^3}$ where the scalars are identified at the end. The explicit procedure can be found in [73], here we limit ourselves to describing the configuration, since the main aim is rewriting the entropy in terms of the formulas of [78].

In our conventions the metric is written as (2.19) with:

$$U^2(r) = e^{\mathcal{K}}\frac{\left(r^2 - r_h^2\right)^2}{r^2}\,, \qquad h^2(r) = e^{-\mathcal{K}}r^2\,, \tag{B.7}$$

and

$$X^0 = \alpha_0 + \frac{\beta^0}{r}\,, \qquad X^1 = X^2 = X^3 = \alpha_1 + \frac{\beta^1}{r}\,. \tag{B.8}$$

The three scalars are identified and the real part can be truncated away, leaving with a purely imaginary scalar t:

$$\lambda = -Im(t) = \sqrt{\frac{X^1}{X^0}}\,. \tag{B.9}$$

The parameters appearing in the harmonic functions are:

$$\alpha_0 = \frac{1}{4\xi_0 g}\,, \qquad \alpha_1 = \frac{1}{4\xi^1 g}\,, \qquad \beta^0 = -3\beta^1\frac{\xi^1}{\xi_0}\,. \tag{B.10}$$

The solution is dyonic with charges p^0, $q_1 = q_2 = q_3$, while the others are set to zero $p^1 = p^2 = p^3 = 0 = q_0$. Supersymmetry requires the Dirac quantization condition $g\xi_0 p^0 - 3\xi^1 q_1 = \pm 1$ (we choose the minus sign in the subsequent computations). β_1 and r_h are then expressed in terms of q_1:

$$\beta^1 = -\frac{\sqrt{1-4g\xi^1 q_1}}{8g\xi^1}\,, \qquad r_h = \frac{\sqrt{1-12g\xi^1 q_1}}{2}\,. \tag{B.11}$$

The warp factor assumes this form:

$$U^2 = \frac{\sqrt{g_0(g^1)^3}\left(r^2 - \frac{1-12g^1q_1}{4}\right)^2}{2\left(2r - \sqrt{1-4g^1q_1}\right)^{3/2}\left(2r + 3\sqrt{1-4g^1q_1}\right)^{1/2}}\,. \tag{B.12}$$

and the scalars are:

$$s = t = u = -i\lambda = -i\sqrt{\frac{g_0}{g^1}}\sqrt{\frac{2r - \sqrt{1-4g^1q_1}}{2r + 3\sqrt{1-4g^1q_1}}}\,. \tag{B.13}$$

Area and quartic invariant

We are able now to directly compute the area of the event horizon for the BPS solution. It reads

$$A = h^2(r_h) = \sqrt{\frac{\left(-\sqrt{1+4\xi^1 g q1}+2r_h\right)^3 \left(3\sqrt{1+4\xi^1 g q1}+2r_h\right)}{64\xi_0(\xi^1)^3 g^4}} =$$

$$= \frac{1}{4}\sqrt{\frac{1+2(1-4gq_1)\sqrt{1-16gq_1+48g^2q_1^2}-3(1-4gq_1)^2}{g^4\xi_0(\xi^1)^3}}\,. \tag{B.14}$$

In [78] it is shown that, when the scalar manifold is homogeneous, the expression of the area of the event horizon is related to the quartic invariant of the model. We are now going to recap their results and to show that (B) can be retrieved also from the much more compact formulation of [78].

We define as in [78] complex vectors of the charges q_Λ and p_Λ and the gauging charges ξ_Λ (and ξ^Λ, in case of magnetic or mixed gaugings) called $\mathbf{p}^\Lambda$ and $\mathbf{q}_\Lambda$:

$$\mathbf{p}^\Lambda = p^\Lambda + iR_2^2 g\xi^\Lambda\,, \tag{B.15}$$

$$\mathbf{q}_\Lambda = q_\Lambda + iR_2^2 g\xi_\Lambda\,, \tag{B.16}$$

where R_2 is defined in (2.55).

It was noticed in [78] that regularity of the solution requires the vanishing of a quantity called quartic invariant $\mathcal{I}_4$ of the model taken into consideration[2]:

$$\mathcal{I}_4(\mathbf{p},\mathbf{q}) = 0\,. \tag{B.17}$$

For the stu model the expression for the quartic invariant is [74]:

$$\mathcal{I}_{4,STU} = -(p^\Lambda q_\Lambda)^2 + 4\left((p^1q_1)(p^2q_2)+(p^3q_3)(p^1q_1)+(p^3q_3)(p^2q_2)\right) +$$

$$+4q_0p^1p^2p^3 - 4p^0q_1q_2q_3\,. \tag{B.18}$$

Indeed in our case (B.17) holds. Separating the equation (B.17) in real and imaginary part, we obtain two polynomial in R_2 that need to vanish

$$0 = a_0 + a_4R_2^4 + a_8R_2^8\,, \tag{B.19}$$

$$0 = a_2R_2^2 + a_6R_2^6\,. \tag{B.20}$$

[2] The expression of the quartic invariant depends on the model taken into consideration. See [78] for the form of $\mathcal{I}_4$ in presence of more general prepotentials.

In our case, it turns out that

$$a_2 = 0\,, \qquad a_6 = 0\,, \tag{B.21}$$

so eq. (B.20) is automatically satisfied. The values for a_4, a_0 and a_8 computed on our solutions are:

$$\begin{aligned} a_0 &= -4p^0 q_1^3\,, && \text{(B.22)}\\ a_4 &= -12(\xi^1)^2 q_1^2 + (\xi_0 p^0 + 3\xi_1 q_1)^2\,, && \text{(B.23)}\\ a_8 &= 4\xi_0(\xi^1)^3\,. && \text{(B.24)} \end{aligned}$$

R_2, from formula (B.19) is

$$R_2^4 = \frac{-a_4 \pm \sqrt{a_4^2 \pm 4a_0 a_8}}{2a_8}\,, \tag{B.25}$$

and computed on our solution, choosing the $+$ sign in (B.25) this gives

$$R_2^2 = A = \frac{1}{4}\sqrt{\frac{1 - 3(1-g\xi^1 q_1)^2 + 2(1+4\xi^1 g q_1)\sqrt{(1+4\xi^1 g q_1)(1+12\xi^1 g q_1)}}{\xi_0(\xi^1)^3 g^4}}$$

$$= \quad \frac{1}{4}\sqrt{\frac{1+2(1-4gq_1)\sqrt{1-16gq_1+48g^2q_1^2} - 3(1-4gq_1)^2}{g^4\xi_0(\xi^1)^3}}\,. \tag{B.26}$$

The area of the event horizon is given by R_2^2 and it coincides indeed with (B).

Nonextremal generalization

To find the nonextremal generalization we use this ansatz for the warp factor:

$$U^2(r) = e^{\mathcal{K}}\left(r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2}\right)\,, \tag{B.27}$$

keeping the same function $h^2(r)$ as in the BPS case, and also the same sections X^Λ.

This guess for the form of the nonextremal solution is then followed by brute-force solving the Einstein's equations of motion (4.14) (4.15) (4.16). The potential for the ttt model is:

$$V = -\frac{(3\xi^1(\xi_0 + \xi^1\lambda^2)}{\lambda}\,, \tag{B.28}$$

where we already plugged in the ansatz for the scalar field ($s = t = u = -i\lambda$). It turns out that the equations of motion (Einstein, Scalar, Maxwell) are satisfied if the parameters assume the form (B.8) (B.10) and furthermore:

$$c = 1 - 96 b_1^2 (\xi^1)^2 g^2 \,, \tag{B.29}$$

$$\mu = \frac{1}{4 b_1 \xi^1 g} - 8 b_1 \xi_1 g - 512 b_1^3 (\xi^1)^3 g^3 + \frac{3 q_1}{2 b_1} + \frac{2 \xi^1 g q_1^2}{b_1} \,, \tag{B.30}$$

$$Q = -48 b_1^2 (\xi^1)^2 g^2 - 768 b_1^4 (\xi^1)^4 g^4 + \xi_0^2 g^2 (p^0)^2 + 3 (\xi^1)^2 g^2 q_1^2 \,. \tag{B.31}$$

Here we have left the charges p^0, q_1 unconstrained. For a certain range of parameters the solution represents a nonextremal black hole.

We can also compute the product of the area of the four horizon, as done in Section 4.5 for the nonextremal purely magnetic solution. With reference to the formulas (4.88), we have that

$$\kappa_1 \;=\; r_{s,1}^4 + \frac{c r_{s,1}^2}{g^2} - \frac{\mu r_{s,1}}{g^2} + \frac{Q}{g^2} = (2 \xi^1 g q_1)^2 \,, \tag{B.32}$$

$$\kappa_2 \;=\; r_{s,2}^4 + \frac{c r_{s,2}^2}{g^2} - \frac{\mu r_{s,2}}{g^2} + \frac{Q}{g^2} = (2 \xi_0 g p^0)^2 \,, \tag{B.33}$$

where the values of μ and Q are given respectively in (B.30) (B.31). This gives (4.89)

$$\prod_{\alpha=1}^{4} A_\alpha = (\mathrm{const})^4 (4\pi)^4 \sqrt{\kappa_1 \kappa_2^3} = (4\pi)^4 \frac{p^0 (q_1)^3}{e_0 (e^1)^3} \,. \tag{B.34}$$

We see then that the result depends only on the black hole and gravitino electric charges. In this way, the electric charges of the gravitini are traded for the cosmological constant, and only the black hole charges appear explicitly.

Appendix C

Ashtekar-Magnon-Das (AMD) formalism

This is a recap of the main formulas of the mass computation for Anti-de Sitter black holes by means of the AMD procedure [100, 101]. The AMD techniques are valid for d-dimensional asymptotically AdS spacetime, but we restrict here our attention to four spacetime dimensions.

The AMD procedure expresses the mass in terms of the integral of suitable contractions of the Weyl tensor over the conformal boundary at infinity. Since the black hole metric approaches asymptotically AdS, the integral is not divergent and well defined.

It is not our intention to report the complete details of the computation, as they are somewhat lengthy and they can be found in the original papers [100, 101], and for instance [148]. Instead we give here a (very brief) summary of the formulas used and an explicit example for the computation of the mass.

In the work of AMD a proposal was given for the expression of charges in Anti-de Sitter spacetime; these quantities are shown to be conserved. We recall here that we have explicitly verified that the mass obtained with the AMD procedure satisfies the first law of thermodynamics as in (6.13) for all the solutions without NUT charge presented in this thesis.

Given an asymptotically Anti-de Sitter configuration X with metric $g_{\mu\nu}$ with negative cosmological constant $\Lambda = -l^2$, with a conformal boundary ∂X, one introduces a conformally rescaled metric $\overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ such that on the conformal boundary ∂X both $\Omega = 0$ and $d\Omega \neq 0$ (Ω is defined up to a function f that is nonzero on the boundary). As future reference, we will choose for our solutions $\Omega = \frac{l}{r}$.

If we denote as $\overline{C}^{\mu}_{\nu\rho\sigma}$ the Weyl tensor of the metric $g_{\mu\nu}$, with indices raised and lowered by the conformally rescaled metric $\overline{g}_{\mu\nu}$, and a vector $n_\mu = \partial_\mu \Omega$, one defines the quantity

$$\overline{E}^{\mu}{}_{\nu} = l^2 \, \Omega \, \overline{n}^{\rho} \overline{n}^{\sigma} \overline{C}^{\mu}_{\nu\rho\sigma} \, . \tag{C.1}$$

The contraction of this quantity with an asymptotic Killing vector K^μ will give a conserved quantity, in this way:

$$Q[K] = \frac{l}{8\pi}\oint_\Sigma \overline{E}^{\mu}{}_{\nu} K^\nu d\overline{\Sigma}_\mu\,, \tag{C.2}$$

Here $d\overline{\Sigma}_\mu$ is the area element of the spherical section of the conformal boundary. The authors of [100, 101] shown that $Q[K]$ is indeed a conserved charge, and this quantity does not depend on the conformal rescaling factor Ω defined before.

We are interested in the mass M of the configuration, therefore we choose the time Killing vector $K = \partial/\partial_t$, therefore, from (C.2) we have

$$M = \frac{l}{8\pi}\oint_\Sigma \overline{E}^t_t K^t d\overline{\Sigma}_t\,. \tag{C.3}$$

We show now how to compute the mass for the magnetic solution described in section (4.4). In that case $l^2 = \frac{3\sqrt{3}}{2\sqrt{\xi_0\xi_1^3}}$ and we take $\Omega = l/r$. The electric part of the Weyl tensor reads:

$$C^t_{rtr} = \frac{1}{g^2 r^5}\left(\frac{8}{3}b_1\xi_1 - \frac{3}{4}\frac{\xi_0^2(p^0)^2}{b_1\xi_1} + \frac{\xi_1(p^1)^2}{12b_1}\right) + O\left(\frac{1}{r^6}\right)\,. \tag{C.4}$$

Furthermore

$$\overline{E}^t_t = \frac{l^2}{\Omega}\overline{g}^{\alpha r}\overline{g}^{\beta r}\overline{n}_r\overline{n}_r C^t_{\alpha t\beta} = \frac{l^4}{r^4\Omega^5}(g^{rr})^2 C^t_{rtr}\,, \tag{C.5}$$

so that the mass turns out to be:

$$M = \frac{l}{8\pi}\oint_\Sigma \overline{E}^t_t K^t d\overline{\Sigma}_t = \frac{4}{3}b_1\xi_1 - \frac{3}{8}\frac{\xi_0^2(p^0)^2}{b_1\xi_1} + \frac{\xi_1(p^1)^2}{24b_1}\,. \tag{C.6}$$

Appendix D

Duality rotation of the magnetic solution

The goal of this appendix is twofold:

- Showing that the same nonextremal magnetic solutions of [99] described in Section 4.4, obtained starting from

$$U^2(r) = e^{\mathcal{K}}\left(g^2 r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2}\right) \tag{D.1}$$

  can be obtained also using the ansatz for the warp factor U^2 used by Duff and Liu for the electric solutions 4.3, namely

$$U^2(r) = e^{\mathcal{K}}(1 - \frac{\mu'}{r} + \mathcal{X} g^2 r^2 e^{-2\mathcal{K}})\,. \tag{D.2}$$

  The solutions obtained with these two different ansatzes are related by a coordinate shift; hence they are the same physical solution

- Showing that the electric and magnetic configurations of Sections 4.3 and 4.4 are related by a particular duality transformation (that acts on the electromagnetic charges, sections but leaves the FI terms invariant) and a shift in the radial variable.

To prove the first point, we solve the Einstein's equations of motion with the ansatz (D.2), where we have chosen $\mathcal{X} = \frac{4}{27}\xi_0\xi_1^3$ in order for the solutions to have the same asymptotic behaviour (same cosmological constant). The warp factor h has the usual form:

$$h(r) = r e^{-\mathcal{K}/2}\,, \tag{D.3}$$

where $\mathcal{K}$ is given by (2.24). The scalar z is real again given by sections

$$z = \frac{X_1}{X_0}\,, \qquad X_0 = a_0 + \frac{b_0}{r}\,, \qquad X_1 = a_1 + \frac{b_1}{r}\,. \tag{D.4}$$

In other words, the only change in our ansatz is replacing (D.1) with (D.2).

The constraints coming from the Einstein's equations are $a_1 = \frac{3}{4\xi_1}$ and $a_0 = \frac{1}{4\xi_0}$, so that the scalar is

$$z = \frac{\xi_0(4b_1\xi_1 + 3r)}{\xi_1(4b_0\xi_0 + r)} . \tag{D.5}$$

Furthermore, the parameters appearing in the solution need to satisfy:

$$4b_0^2\xi_0 - \xi_0(p^0)^2 + b_0\mu' = 0 , \tag{D.6}$$

$$4b_1^2\xi_1 - \xi_1(p^1)^2 + 3b_1\mu' = 0 . \tag{D.7}$$

The solution now depends on the parameters b_1, b_0, μ', p^1, p^0, with two constraints (D.6) -(D.7). Therefore just three parameters are independent. Performing the change of coordinates

$$r = r' + \mathcal{Y} , \tag{D.8}$$

with $\mathcal{Y}$ satisfying the system of equations

$$4b_1\xi_1 + 3\mathcal{Y} = 4b_1'\xi_1 , \tag{D.9}$$

$$4b_0\xi_0 + \mathcal{Y} = -4b_1'\xi_1 , \tag{D.10}$$

we can recast the solution found before in another one. Indeed through equations (D.10)-(D.10) we have traded b_0 and b_1 for b_1' and $\mathcal{Y}$, so the solution now depends on $b_1', p^0, p^1, \mu', \mathcal{Y}$ (again subject to the two constraints (D.6)-(D.7)). It turns out that the metric in the new coordinate r' assumes exactly the form

$$U^2(r') = e^{\mathcal{K}(r')} \left(g^2 r'^2 + c - \frac{\mu}{r'} + \frac{Q}{r'^2} \right) , \tag{D.11}$$

with

$$c = 1 - \frac{32}{3} {b_1'}^2 \xi_1^2 g^2 , \tag{D.12}$$

$$\mu = \frac{8}{3} b_1' \xi_1^2 + \frac{512}{27} (b_1')^3 g^2 \frac{\xi_1^3}{b_1'} - \frac{\xi_0^2 (p^0)^2}{4\xi_1 b_1'} + \frac{\xi_1 (p^1)^2}{12 b_1'} , \tag{D.13}$$

$$Q = \frac{16{b_1'}^2 {\xi_1}^2}{3} - {\xi_0}^2 (p^0)^2 - \frac{{\xi_1}^2 (p^1)^2}{3} - \frac{256}{27} {b_1'}^4 \xi_1^4 g^2 , \tag{D.14}$$

and the scalar field has this form:

$$z = \frac{\xi_0(3r' + 4b_1'\xi_1)}{\xi_1(r' - 4b_1'\xi_1)} , \tag{D.15}$$

which is exactly of the form we wanted. To sum up, the solution found by means of the "Duff-Liu" type of ansatz (D.2) is the same one found by the ansatz (D.1), because they are related by a change of coordinates.

We are ready now to transform the solution (D.2)-(D.5)-(D.6)-(D.7) by means of a duality transformation. Let's first recap the features of this transformation. The scalar field transforms as:

$$z \to \frac{9\xi_0^2}{\xi_1^2 z}\,. \tag{D.16}$$

The symplectic matrix $\mathcal{T}$ acting on the charges and on the sections is

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & \frac{\xi_1^2 \upsilon}{3\xi_0^2} & 0 \\ 0 & 0 & 0 & \upsilon \\ -\frac{3\xi_0^2}{\xi_1^2 \upsilon} & 0 & 0 & 0 \\ 0 & -\frac{1}{\upsilon} & 0 & 0 \end{pmatrix}, \qquad \upsilon = \frac{2\xi_1}{9\xi_0}\,. \tag{D.17}$$

The FI parameters do not change, as already mentioned. The period matrix $\mathcal{N}$ transforms according to (1.36)

The rotation described turns the magnetic solution of the form (D.2)-(D.5)-(D.6)-(D.7) into the electric one characterized by

$$U^2(r) = e^{\mathcal{K}}(1 - \frac{\mu''}{r} + \frac{16\xi_1^6}{729\xi_0^2} g^2 r^2 e^{-2\mathcal{K}})\,, \qquad h(r) = e^{-\mathcal{K}/2} r\,, \tag{D.18}$$

and holomorphic sections

$$X_0 = \frac{1}{6\sqrt{3}} \sqrt{\frac{H_1^3}{H_0}}\,, \qquad X_1 = \frac{9\xi_0^2}{\xi_1^2} \frac{1}{2\sqrt{3}} \sqrt{H_0 H_1}\,, \tag{D.19}$$

so that

$$e^{-\mathcal{K}} = \frac{6\sqrt{3}\xi_0^3}{\xi_1^3} \sqrt{H_0 H_1^3}\,, \tag{D.20}$$

and $a_0 = \frac{1}{4\xi_0}$ and $a_1 = \frac{9}{4\xi_1}$. The constraints get turned exactly[1] into the correct "electric" ones, needed to satisfy the Einstein's equations

$$\begin{aligned} 4b_0^2\xi_0^6 + b_0\xi_0^5\mu'' - \frac{4\xi_1^6 {q_0}^2}{729} &= 0\,, & \text{(D.21)} \\ 4b_1^2\xi_0^2\xi_1 + 9b_1\xi_0^2\mu'' - \frac{4\xi_1^3 q_1^2}{9} &= 0\,, & \text{(D.22)} \end{aligned}$$

[1] We rescaled $b_1 \to 3b_1$.

The scalar field reads:

$$z = \frac{9\xi_0(4b_0\xi_0 + r)}{\xi_1(4b_1\xi_1/3 + 3r)}\,. \tag{D.23}$$

This configuration solves the equations of motion with electric field strengths. We have used sections with different multiplicative numerical factors with respect to the solution in 4.3, in order to achieve the correct matching in the rotation. The Duff-Liu solutions correspond to the subcase in which we solve equations (D.21) by setting

$$q_0 = \frac{27\xi_0^2}{4\xi_1^3}\mu'' \cosh(Q_0)\sinh(Q_0)\,, \tag{D.24}$$

$$q_1 = \frac{27\xi_0}{4\xi_1^2}\mu'' \cosh(Q_1)\sinh(Q_1)\,, \tag{D.25}$$

$$b_0 = \frac{\mu''}{4\xi_0}\sinh^2(Q_0)\,, \tag{D.26}$$

$$b_1 = \frac{9\mu''}{4\xi_1}\sinh^2(Q_1)\,. \tag{D.27}$$

It is related to the original solution in 4.3, equations (4.56)-(4.58) just via a redefinition of parameters b_Λ, Q_Λ. Therefore we have shown that the Duff-Liu solutions represent a subset of all the solutions obtained if we rotate the magnetic solution described in 4.4.

One last remark: this duality transformation is a symmetry of the bosonic equations of motion, but not of the supersymmetry variations. Indeed, for instance, if we rotate the 1/2 BPS electric solution of Section 2.3 by means of the transformation (D.16) -(D.17) (namely we rotate sections, charges but we maintain the FI parameters intact) we do not obtain a supersymmetric 1/2 BPS magnetic one. This was already noticed in [64].

Appendix E

Asymptotic Killing spinors

AdS_4

Here we give details about the Killing spinors for AdS_4. We consider the metric in spherical coordinates

$$\mathrm{d}s^2 = (1+g^2r^2)\,\mathrm{d}t^2 - (1+g^2r^2)^{-1}\,\mathrm{d}r^2 - r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)\ , \tag{E.1}$$

and corresponding vielbein

$$e^a_\mu = \mathrm{diag}\Big(\sqrt{1+g^2r^2}, (\sqrt{1+g^2r^2})^{-1}, r, r\sin\theta\Big)\ . \tag{E.2}$$

The non-vanishing components of the spin connection turn out to be:

$$\omega_t^{01} = g^2 r,\quad \omega_\theta^{12} = -\sqrt{1+g^2r^2},\quad \omega_\varphi^{13} = -\sqrt{1+g^2r^2}\sin\theta,\quad \omega_\varphi^{23} = -\cos\theta\,. \tag{E.3}$$

For the AdS_4 solution, the field strength vanishes,

$$F_{\mu\nu} = 0\ . \tag{E.4}$$

To find the Killing spinors corresponding to this spacetime we need to solve $\widetilde{\mathcal{D}}_\mu \epsilon = 0$. This equation has already been solved in [166] and we have explicitly checked that the resulting Killing spinors are given by

$$\epsilon_{AdS} = e^{\frac{i}{2} arcsinh(gr)\gamma_1}\, e^{\frac{i}{2} gt\gamma_0}\, e^{-\frac{1}{2}\theta\gamma_{12}}\, e^{-\frac{1}{2}\varphi\gamma_{23}}\,\epsilon_0\ , \tag{E.5}$$

where ϵ_0 is a doublet of arbitrary constant Majorana spinors, representing the eight preserved supersymmetries of the configuration.

It is important to note that the asymptotic solution of the Killing spinor equations as $r \to \infty$ (given the same asymptotic metric) cannot change unless $A_\varphi \neq 0$. This is easy to see from the form of the supercovariant derivative (7.14) since any other term would necessarily vanish in the asymptotic limit. More precisely, any gauge field carrying an electric charge that appears in the derivative vanishes asymptotically, the only constant contribution can come when a magnetic charge is present. In other words, any spacetime with vanishing magnetic charge and asymptotic metric (E.1) has asymptotic Killing spinors given by (E.5).

Magnetic AdS$_4$

Now we will show that the asymptotic Killing spinors take a very different form when magnetic charge is present. In this case the metric is

$$\mathrm{d}s^2 = (1+g^2r^2+\frac{Q_m^2}{4r^2})\,\mathrm{d}t^2-(1+g^2r^2+\frac{Q_m^2}{4r^2})^{-1}\,\mathrm{d}r^2-r^2\,(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)\ , \quad \text{(E.6)}$$

with corresponding vielbein:

$$e_\mu^a = \mathrm{diag}\Big(\sqrt{1+g^2r^2+\frac{Q_m^2}{4r^2}},\frac{1}{\sqrt{1+g^2r^2+\frac{Q_m^2}{4r^2}}},r,r\sin\theta\Big)\ . \quad \text{(E.7)}$$

The non-vanishing components of the spin connection turn out to be:

$$\omega_t^{01} = g^2r-\frac{Q_m^2}{4r^3}\ , \qquad \omega_\theta^{12} = -\sqrt{1+g^2r^2+\frac{Q_m^2}{4r^2}}\ ,$$

$$\omega_\varphi^{13} = -\sqrt{1+g^2r^2+\frac{Q_m^2}{4r^2}}\sin\theta\ , \qquad \omega_\varphi^{23} = -\cos\theta\ . \quad \text{(E.8)}$$

As opposed to the previous section, now we have a non-vanishing gauge field component $A_\varphi = -Q_m\cos\theta$, resulting in $F_{\theta\varphi} = Q_m\sin\theta/2$. If we require $\widetilde{\mathcal{D}}_\mu\epsilon = 0$ and insist that $Q_m \neq 0$, we get a solution described by Romans in [61] as a "cosmic monopole" (which we call magnetic AdS$_4$). The magnetic charge satisfies $gQ_m = \pm 1$, such that the metric function is an exact square $(gr+\frac{1}{2gr})^2$. The Killing spinors corresponding to solutions with $Q_m = \pm 1/g$ in our conventions are given by

$$\epsilon_{mAdS} = \frac{1}{4}\sqrt{gr+\frac{1}{2gr}}(1+i\gamma_1)(1\mp i\gamma_{23}\sigma^2)\,\epsilon_0\ , \quad \text{(E.9)}$$

preserving two of the original eight supersymmetries. Note that in the limit $r\to\infty$ the Killing spinor projections continue to hold. Furthermore, the functional dependence is manifestly different in the expressions (E.5) and (E.9) for the Killing spinors of ordinary AdS$_4$ and its magnetic version. This leads to the conclusion that these two vacua and their corresponding excited states belong to two separate classes, i.e. they lead to two independent superalgebras and BPS bounds. Note that one can also add an arbitrary electric charge Q_e to the above solution, preserving the same amount of supersymmetry (the "cosmic dyon" of [61]). The corresponding Killing spinors [61] have the asymptotic form of (E.9), i.e. the cosmic dyons are asymptotically magnetic AdS$_4$.

Appendix F

BPS bound for nonvanishing angular momentum

Here we focus on stationary spacetimes with rotations. From the supersymmetry Dirac brackets in asymptotic AdS$_4$ spaces,

$$\{\mathcal{Q}, \mathcal{Q}\} = -8\pi i \bar{\epsilon}_0 \left(\ldots + g J_{ij} \gamma^{ij} + \ldots \right) \epsilon_0 \ , \tag{F.1}$$

we can derive a definition of the conserved angular momenta. The explicit expressions are somewhat lengthy and assume a much simpler form once we choose the vielbein matrix e_μ^a in an upper triangular form, such that its inverse e_a^μ is also upper triangular. More explicitly, in spherical coordinates we choose nonvanishing $e_t^{0,1,2,3}, e_r^{1,2,3}, e_\theta^{2,3}, e_\varphi^3$, such that the inverse vielbein has only non-vanishing components $e_0^{t,r,\theta,\varphi}, e_1^{r,\theta,\varphi}, e_2^{\theta,\varphi}, e_3^\varphi$. The resulting expressions for the angular momenta in this case become:

$$J_{12} = \frac{1}{8\pi} \lim_{r\to\infty} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \Bigg((e_0^t e_1^r e_\varphi^3 \omega_\theta^{01} + e_0^t e_1^r e_\theta^2 e_\varphi^3 e_2^\varphi \omega_\varphi^{01}) r \cos\varphi + \\ + (e_0^t e_1^r e_\theta^2 \omega_\varphi^{01}) r \cos\theta \sin\varphi \Bigg) \ , \tag{F.2}$$

$$J_{13} = \frac{1}{8\pi} \lim_{r\to\infty} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \Bigg((e_0^t e_1^r e_\varphi^3 \omega_\theta^{01} + e_0^t e_1^r e_\theta^2 e_\varphi^3 e_2^\varphi \omega_\varphi^{01}) r \sin\varphi + \\ + (e_0^t e_1^r e_\theta^2 \omega_\varphi^{01}) r \cos\theta \cos\varphi \Bigg) \ , \tag{F.3}$$

$$J_{23} = \frac{1}{8\pi} \lim_{r\to\infty} \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\theta \left(e_0^t e_1^r e_\theta^2 \omega_\varphi^{01} r \sin\theta \right) . \tag{F.4}$$

It is easy to see that in case of axisymmetric solutions around φ, such as the Kerr and Kerr-Newman metrics in AdS, the angular momenta J_{12} and J_{13} automatically vanish due to $\int_0^{2\pi} \mathrm{d}\varphi \sin\varphi = \int_0^{2\pi} \mathrm{d}\varphi \cos\varphi = 0$.

One can then use the formula for J_{23} to derive the value of the angular momentum for the Kerr black hole. This is still somewhat non-trivial because one needs to change the coordinates from Boyer-Lindquist-type to spherical. The leading terms at large r were found in appendix B of [171] and are enough for the calculation of the angular momentum since subleading terms vanish when the limit is taken in (F.4). The calculation of the relevant component of the spin connection leads to

$$\omega_\varphi^{01} = -\frac{3am\sin^2\theta(1-g^2a^2\sin^2\theta)^{-5/2}}{r^2} + \mathcal{O}(r^{-3}) \tag{F.5}$$

and gives the exact same result as in (B.8) of [171],

$$J_{23} = \frac{am}{(1-g^2a^2)^2} \ . \tag{F.6}$$

This expression has also been derived from different considerations in [131], thus confirming the consistency of our results.

One can also verify the result for the asymptotic mass of the Kerr and Kerr-Newman spacetimes using (7.38) and the metric in appendix B of [171]. After a somewhat lengthy but straightforward calculation one finds

$$M = \frac{m}{(1-g^2a^2)^2} \ , \tag{F.7}$$

as expected from previous studies (see, e.g., [68, 131]).

References

[1] R. Schodel, T. Ott, R. Genzel, A. Eckart, N. Mouawad, *et al.*, "Stellar dynamics in the central arcsecond of our Galaxy", *Astrophys.J.* **596** (2003) 1015–1034, arXiv:astro-ph/0306214.

[2] F. Eisenhauer, R. Genzel, T. Alexander, R. Abuter, T. Paumard, *et al.*, "SINFONI in the Galactic Center: Young stars and IR flares in the central light month", *Astrophys.J.* **628** (2005) 246–259, arXiv:astro-ph/0502129.

[3] http://www.astro.ucla.edu/ ghezgroup/gc/pictures/orbitsMovie.shtml.

[4] K. Schwarzschild, "Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie", *Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften zu Berlin*, 1916.

[5] C. Sivaram, "Entropy of Stars, Black Holes and Dark Energy", arXiv:0710.1377.

[6] J. M. Bardeen, B. Carter, and S. Hawking, "The Four laws of black hole mechanics", *Commun.Math.Phys.* **31** (1973) 161–170.

[7] J. D. Bekenstein, "Black holes and entropy", *Phys.Rev.* **D7** (1973) 2333–2346.

[8] S. Hawking, "Particle Creation by Black Holes", *Commun.Math.Phys.* **43** (1975) 199–220.

[9] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, "Black Holes: Complementarity or Firewalls?", *JHEP* **1302** (2013) 062, arXiv:1207.3123.

[10] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy", *Phys.Lett.* **B379** (1996) 99–104, arXiv:hep-th/9601029.

[11] G. T. Horowitz and A. Strominger, "Counting states of near extremal black holes", *Phys.Rev.Lett.* **77** (1996) 2368–2371, arXiv:hep-th/9602051.

[12] J. M. Maldacena, A. Strominger, and E. Witten, "Black hole entropy in M theory", *JHEP* **9712** (1997) 002, arXiv:hep-th/9711053.

[13] G. 't Hooft, "Dimensional reduction in quantum gravity", arXiv:gr-qc/9310026.

[14] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions", *Nucl.Phys.* **B72** (1974) 461.

[15] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200.

[16] A. Strominger, "Black hole entropy from near horizon microstates", *JHEP* **9802** (1998) 009, arXiv:hep-th/9712251.

[17] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, "Building a Holographic Superconductor", *Phys.Rev.Lett.* **101** (2008) 031601, arXiv:0803.3295.

[18] S. S. Gubser, "Drag force in AdS/CFT", *Phys.Rev.* **D74** (2006) 126005, arXiv:hep-th/0605182.

[19] S. Hawking and D. N. Page, "Thermodynamics of Black Holes in anti-De Sitter Space", *Commun.Math.Phys.* **87** (1983) 577.

[20] S. Carlip and S. Vaidya, "Phase transitions and critical behavior for charged black holes", *Class.Quant.Grav.* **20** (2003) 3827–3838,

arXiv:gr-qc/0306054.
[21] M. Grana, "Flux compactifications in string theory: A Comprehensive review", *Phys.Rept.* **423** (2006) 91–158, arXiv:hep-th/0509003.
[22] L. Abbott and S. Deser, "Stability of Gravity with a Cosmological Constant", *Nucl.Phys.* **B195** (1982) 76.
[23] P. Breitenlohner and D. Z. Freedman, "Stability in Gauged Extended Supergravity", *Annals Phys.* **144** (1982) 249.
[24] G. Gibbons, C. Hull, and N. Warner, "The Stability of Gauged Supergravity", *Nucl.Phys.* **B218** (1983) 173.
[25] E. Witten, "A Simple Proof of the Positive Energy Theorem", *Commun.Math.Phys.* **80** (1981) 381.
[26] G. Gibbons, S. Hawking, G. T. Horowitz, and M. J. Perry, "Positive Mass Theorems for Black Holes", *Commun.Math.Phys.* **88** (1983) 295.
[27] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals", *JHEP* **0810** (2008) 091, arXiv:0806.1218.
[28] A. Gnecchi, K. Hristov, D. Klemm, C. Toldo, and O. Vaughan, "Rotating black holes in 4d gauged supergravity", arXiv:1311.1795.
[29] D. D. K. Chow and G. Compère, "Dyonic AdS black holes in maximal gauged supergravity", arXiv:1311.1204.
[30] S. L. Cacciatori and D. Klemm, "Supersymmetric AdS(4) black holes and attractors", *JHEP* **1001** (2010) 085, arXiv:0911.4926.
[31] B. de Wit, P. Lauwers, and A. Van Proeyen, "Lagrangians of N=2 Supergravity - Matter Systems", *Nucl.Phys.* **B255** (1985) 569.
[32] B. de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models", *Nucl.Phys.* **B245** (1984) 89.
[33] R. D'Auria, S. Ferrara, and P. Fre, "Special and quaternionic isometries: General couplings in N=2 supergravity and the scalar potential", *Nucl.Phys.* **B359** (1991) 705–740.
[34] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, *et al.*, "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map", *J.Geom.Phys.* **23** (1997) 111–189, arXiv:hep-th/9605032.
[35] A. Bilal, "Introduction to supersymmetry", arXiv:hep-th/0101055.
[36] S. P. Martin, "A Supersymmetry primer", arXiv:hep-ph/9709356.
[37] J. Wess and J. Bagger, "Supersymmetry and supergravity", 1992.
[38] D. Z. Freedman and A. Van Proeyen, "Supergravity", 2012.
[39] v. H. J. de Wit, B. and T. van Proeyen, "Transformation rules of $\mathcal{N} = 2$ Supergravity Multiplets", *Nucl.Phys.* **B167** (1980) 186.
[40] v. H. J. de Wit, B. and T. van Proeyen, "Structure of $\mathcal{N} = 2$ supergravity", *Nucl.Phys.* **B184** (1981) 77.
[41] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, "What is special Kahler geometry?", *Nucl.Phys.* **B503** (1997) 565–613, arXiv:hep-th/9703082.
[42] A. Strominger, "Special geometry", *Communications in Mathematical Physics*

133 (1990), no. 1, 163–180.
[43] A. Ceresole, R. D'Auria, and S. Ferrara, "The Symplectic structure of N=2 supergravity and its central extension", *Nucl.Phys.Proc.Suppl.* **46** (1996) 67–74, `arXiv:hep-th/9509160`.
[44] M. K. Gaillard and B. Zumino, "Duality Rotations for Interacting Fields", *Nucl.Phys.* **B193** (1981) 221.
[45] G. Dall'Agata and A. Gnecchi, "Flow equations and attractors for black holes in N = 2 U(1) gauged supergravity", *JHEP* **1103** (2011) 037, `arXiv:1012.3756`.
[46] J. Bagger and E. Witten, "Matter Couplings in N=2 Supergravity", *Nucl.Phys.* **B222** (1983) 1.
[47] R. D'Auria and S. Ferrara, "On fermion masses, gradient flows and potential in supersymmetric theories", *JHEP* **0105** (2001) 034, `arXiv:hep-th/0103153`.
[48] G. Dall'Agata and M. Zagermann, "Supergravity (lecture notes)", *Series of lecture notes*, 2010.
[49] H. Looijestijn, "Aspects of Conpactifications and Black holes in four dimenaional gauged supergravity", *PhD Thesis* **Utrecht University** (2010).
[50] P. Meessen and T. Ortin, "Supersymmetric solutions to gauged N=2 d=4 sugra: the full timelike shebang", *Nucl.Phys.* **B863** (2012) 65–89, `arXiv:1204.0493`.
[51] K. Galicki, "A Generalization of the Momentum Mapping Construction for Quaternionic Kahler Manifolds", *Commun.Math.Phys.* **108** (1987) 117.
[52] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren, *et al.*, "N = 2 supergravity in five-dimensions revisited", *Class.Quant.Grav.* **21** (2004) 3015–3042, `arXiv:hep-th/0403045`.
[53] E. Bergshoeff, S. Cucu, T. De Wit, J. Gheerardyn, R. Halbersma, *et al.*, "Superconformal N=2, D = 5 matter with and without actions", *JHEP* **0210** (2002) 045, `arXiv:hep-th/0205230`.
[54] B. de Wit, H. Samtleben, and M. Trigiante, "Magnetic charges in local field theory", *JHEP* **0509** (2005) 016, `arXiv:hep-th/0507289`.
[55] B. de Wit and M. van Zalk, "Electric and magnetic charges in N=2 conformal supergravity theories", *JHEP* **1110** (2011) 050, `arXiv:1107.3305`.
[56] L. Andrianopoli, R. D'Auria, and L. Sommovigo, "D=4, N=2 supergravity in the presence of vector-tensor multiplets and the role of higher p-forms in the framework of free differential algebras", *Adv.Stud.Theor.Phys.* **1** (2008) 561–596, `arXiv:0710.3107`.
[57] G. Dall'Agata, R. D'Auria, L. Sommovigo, and S. Vaula, "D = 4, N=2 gauged supergravity in the presence of tensor multiplets", *Nucl.Phys.* **B682** (2004) 243–264, `arXiv:hep-th/0312210`.
[58] R. D'Auria, L. Sommovigo, and S. Vaula, "N = 2 supergravity Lagrangian coupled to tensor multiplets with electric and magnetic fluxes", *JHEP* **0411** (2004) 028, `arXiv:hep-th/0409097`.
[59] U. Theis and S. Vandoren, "N=2 supersymmetric scalar tensor couplings",

JHEP **0304** (2003) 042, **arXiv:hep-th/0303048**.

[60] N. Halmagyi, M. Petrini, and A. Zaffaroni, "BPS black holes in AdS_4 from M-theory", *JHEP* **1308** (2013) 124, **arXiv:1305.0730**.

[61] L. Romans, "Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory", *Nucl.Phys.* **B383** (1992) 395–415, **arXiv:hep-th/9203018**.

[62] W. Sabra, "Anti-de Sitter BPS black holes in N=2 gauged supergravity", *Phys.Lett.* **B458** (1999) 36–42, **arXiv:hep-th/9903143**.

[63] A. H. Chamseddine and W. Sabra, "Magnetic and dyonic black holes in D = 4 gauged supergravity", *Phys.Lett.* **B485** (2000) 301–307, **arXiv:hep-th/0003213**.

[64] M. Duff and J. T. Liu, "Anti-de Sitter black holes in gauged N = 8 supergravity", *Nucl.Phys.* **B554** (1999) 237–253, **arXiv:hep-th/9901149**.

[65] L. Vanzo, "Black holes with unusual topology", *Phys.Rev.* **D56** (1997) 6475–6483, **arXiv:gr-qc/9705004**.

[66] R. Kallosh, A. D. Linde, T. Ortin, A. W. Peet, and A. Van Proeyen, "Supersymmetry as a cosmic censor", *Phys.Rev.* **D46** (1992) 5278–5302, **arXiv:hep-th/9205027**.

[67] T. Hertog, G. T. Horowitz, and K. Maeda, "Update on cosmic censorship violation in AdS", **arXiv:gr-qc/0405050**.

[68] M. M. Caldarelli and D. Klemm, "Supersymmetry of Anti-de Sitter black holes", *Nucl.Phys.* **B545** (1999) 434–460, **arXiv:hep-th/9808097**.

[69] N. Alonso-Alberca, P. Meessen, and T. Ortin, "Supersymmetry of topological Kerr-Newman-Taub-NUT-AdS space-times", *Class.Quant.Grav.* **17** (2000) 2783–2798, **arXiv:hep-th/0003071**.

[70] K. Hristov and S. Vandoren, "Static supersymmetric black holes in AdS_4 with spherical symmetry", *JHEP* **1104** (2011) 047, **arXiv:1012.4314**.

[71] N. Halmagyi and T. Vanel, "AdS Black Holes from Duality in Gauged Supergravity", **arXiv:1312.5430**.

[72] S. L. Cacciatori, D. Klemm, D. S. Mansi, and E. Zorzan, "All timelike supersymmetric solutions of N=2, D=4 gauged supergravity coupled to abelian vector multiplets", *JHEP* **0805** (2008) 097, **arXiv:0804.0009**.

[73] A. Gnecchi and C. Toldo, "On the non-BPS first order flow in N=2 U(1)-gauged Supergravity", *JHEP* **1303** (2013) 088, **arXiv:1211.1966**.

[74] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong, "STU black holes and string triality", *Phys.Rev.* **D54** (1996) 6293–6301, **arXiv:hep-th/9608059**.

[75] S. Ferrara and R. Kallosh, "Universality of supersymmetric attractors", *Phys.Rev.* **D54** (1996) 1525–1534, **arXiv:hep-th/9603090**.

[76] S. Ferrara and M. Gunaydin, "Orbits of exceptional groups, duality and BPS states in string theory", *Int.J.Mod.Phys.* **A13** (1998) 2075–2088, **arXiv:hep-th/9708025**.

[77] B. Pioline, "Lectures on black holes, topological strings and quantum attractors", *Class.Quant.Grav.* **23** (2006) S981, **arXiv:hep-th/0607227**.

[78] N. Halmagyi, "BPS Black Hole Horizons in N=2 Gauged Supergravity", `arXiv:1308.1439`.
[79] A. Gnecchi and N. Halmagyi, "Supersymmetric Black Holes in AdS4 from Very Special Geometry", `arXiv:1312.2766`.
[80] K. Hristov, H. Looyestijn, and S. Vandoren, "Maximally supersymmetric solutions of D=4 N=2 gauged supergravity", *JHEP* **0911** (2009) 115, `arXiv:0909.1743`.
[81] S. Cucu, H. Lu, and J. F. Vazquez-Poritz, "Interpolating from AdS(D-2) x S**2 to AdS(D)", *Nucl.Phys.* **B677** (2004) 181–222, `arXiv:hep-th/0304022`.
[82] K. Hristov, S. Katmadas, and V. Pozzoli, "Ungauging black holes and hidden supercharges", *JHEP* **1301** (2013) 110, `arXiv:1211.0035`.
[83] M. Cvetic, M. Duff, P. Hoxha, J. T. Liu, H. Lu, *et al.*, "Embedding AdS black holes in ten-dimensions and eleven-dimensions", *Nucl.Phys.* **B558** (1999) 96–126, `arXiv:hep-th/9903214`.
[84] A. H. Chamseddine and W. Sabra, "Magnetic strings in five-dimensional gauged supergravity theories", *Phys.Lett.* **B477** (2000) 329–334, `arXiv:hep-th/9911195`.
[85] D. Klemm and W. Sabra, "Supersymmetry of black strings in D = 5 gauged supergravities", *Phys.Rev.* **D62** (2000) 024003, `arXiv:hep-th/0001131`.
[86] J. B. Gutowski and H. S. Reall, "Supersymmetric AdS(5) black holes", *JHEP* **0402** (2004) 006, `arXiv:hep-th/0401042`.
[87] J. B. Gutowski and H. S. Reall, "General supersymmetric AdS(5) black holes", *JHEP* **0404** (2004) 048, `arXiv:hep-th/0401129`.
[88] H. K. Kunduri, J. Lucietti, and H. S. Reall, "Supersymmetric multi-charge AdS(5) black holes", *JHEP* **0604** (2006) 036, `arXiv:hep-th/0601156`.
[89] Z. Chong, M. Cvetic, H. Lu, and C. Pope, "Five-dimensional gauged supergravity black holes with independent rotation parameters", *Phys.Rev.* **D72** (2005) 041901, `arXiv:hep-th/0505112`.
[90] S. Bellucci, S. Ferrara, A. Marrani, and A. Yeranyan, "d=4 Black Hole Attractors in N=2 Supergravity with Fayet-Iliopoulos Terms", *Phys.Rev.* **D77** (2008) 085027, `arXiv:0802.0141`.
[91] T. Kimura, "Non-supersymmetric Extremal RN-AdS Black Holes in N=2 Gauged Supergravity", *JHEP* **1009** (2010) 061, `arXiv:1005.4607`.
[92] J. F. Morales and H. Samtleben, "Entropy function and attractors for AdS black holes", *JHEP* **0610** (2006) 074, `arXiv:hep-th/0608044`.
[93] A. Ceresole and G. Dall'Agata, "Flow Equations for Non-BPS Extremal Black Holes", *JHEP* **0703** (2007) 110, `arXiv:hep-th/0702088`.
[94] D. Klemm and O. Vaughan, "Nonextremal black holes in gauged supergravity and the real formulation of special geometry II", *Class.Quant.Grav.* **30** (2013) 065003, `arXiv:1211.1618`.
[95] T. Mohaupt and O. Vaughan, "The Hesse potential, the c-map and black hole solutions", *JHEP* **1207** (2012) 163, `arXiv:1112.2876`.
[96] D. Klemm and O. Vaughan, "Nonextremal black holes in gauged supergravity and the real formulation of special geometry", *JHEP* **1301** (2013) 053,

arXiv:1207.2679.
[97] M. Duff, H. Lu, and C. Pope, "The Black branes of M theory", *Phys.Lett.* **B382** (1996) 73–80, arXiv:hep-th/9604052.
[98] K. Behrndt, M. Cvetic, and W. Sabra, "Nonextreme black holes of five-dimensional N=2 AdS supergravity", *Nucl.Phys.* **B553** (1999) 317–332, arXiv:hep-th/9810227.
[99] C. Toldo and S. Vandoren, "Static nonextremal AdS4 black hole solutions", *JHEP* **1209** (2012) 048, arXiv:1207.3014.
[100] A. Ashtekar and A. Magnon, "Asymptotically anti-de Sit ter space-times", *Class. Quant. Grav.* **1L39** (1984).
[101] A. Ashtekar and S. Das, "Asymptotically Anti-de Sitter space-times: Conserved quantities", *Class.Quant.Grav.* **17** (2000) L17–L30, arXiv:hep-th/9911230.
[102] K. Hristov, C. Toldo, and S. Vandoren, "On BPS bounds in D=4 N=2 gauged supergravity", *JHEP* **1112** (2011) 014, arXiv:1110.2688.
[103] K. Hristov, "On BPS Bounds in D=4 N=2 Gauged Supergravity II: General Matter couplings and Black Hole Masses", *JHEP* **1203** (2012) 095, arXiv:1112.4289.
[104] J. de Boer, E. P. Verlinde, and H. L. Verlinde, "On the holographic renormalization group", *JHEP* **0008** (2000) 003, arXiv:hep-th/9912012.
[105] I. Papadimitriou and K. Skenderis, "Thermodynamics of asymptotically locally AdS spacetimes", *JHEP* **0508** (2005) 004, arXiv:hep-th/0505190.
[106] A. Batrachenko, J. T. Liu, R. McNees, W. Sabra, and W. Wen, "Black hole mass and Hamilton-Jacobi counterterms", *JHEP* **0505** (2005) 034, arXiv:hep-th/0408205.
[107] F. Larsen, "A String model of black hole microstates", *Phys.Rev.* **D56** (1997) 1005–1008, arXiv:hep-th/9702153.
[108] M. Cvetic, G. Gibbons, and C. Pope, "Universal Area Product Formulae for Rotating and Charged Black Holes in Four and Higher Dimensions", *Phys.Rev.Lett.* **106** (2011) 121301, arXiv:1011.0008.
[109] A. Castro and M. J. Rodriguez, "Universal properties and the first law of black hole inner mechanics", *Phys.Rev.* **D86** (2012) 024008, arXiv:1204.1284.
[110] P. Galli, T. Ortin, J. Perz, and C. S. Shahbazi, "Non-extremal black holes of N=2, d=4 supergravity", *JHEP* **1107** (2011) 041, arXiv:1105.3311.
[111] D. Anninos, T. Anous, F. Denef, and L. Peeters, "Holographic Vitrification", arXiv:1309.0146.
[112] N. Bobev, A. Kundu, K. Pilch, and N. P. Warner, "Minimal Holographic Superconductors from Maximal Supergravity", *JHEP* **1203** (2012) 064, arXiv:1110.3454.
[113] G. Cardoso and V. Grass, "On five-dimensional non-extremal charged black holes and FRW cosmology", *Nucl.Phys.* **B803** (2008) 209–233, arXiv:0803.2819.
[114] H. Lu, Y. Pang, and C. Pope, "AdS Dyonic Black Hole and its Thermodynamics", *JHEP* **1311** (2013) 033, arXiv:1307.6243.

[115] T. Hertog and K. Maeda, "Black holes with scalar hair and asymptotics in N = 8 supergravity", *JHEP* **0407** (2004) 051, `arXiv:hep-th/0404261`.
[116] A. Anabalon, "Exact Black Holes and Universality in the Backreaction of non-linear Sigma Models with a potential in (A)dS4", *JHEP* **1206** (2012) 127, `arXiv:1204.2720`.
[117] A. Anabalon and D. Astefanesei, "Black holes in ω-defomed gauged $N = 8$ supergravity", `arXiv:1311.7459`.
[118] G. Dall'Agata, G. Inverso, and M. Trigiante, "Evidence for a family of SO(8) gauged supergravity theories", *Phys.Rev.Lett.* **109** (2012) 201301, `arXiv:1209.0760`.
[119] V. A. Kostelecky and M. J. Perry, "Solitonic black holes in gauged N=2 supergravity", *Phys.Lett.* **B371** (1996) 191–198, `arXiv:hep-th/9512222`.
[120] Z.-W. Chong, M. Cvetic, H. Lu, and C. Pope, "Charged rotating black holes in four-dimensional gauged and ungauged supergravities", *Nucl.Phys.* **B717** (2005) 246–271, `arXiv:hep-th/0411045`.
[121] D. D. Chow, "Single-charge rotating black holes in four-dimensional gauged supergravity", *Class.Quant.Grav.* **28** (2011) 032001, `arXiv:1011.2202`.
[122] D. D. Chow, "Two-charge rotating black holes in four-dimensional gauged supergravity", *Class.Quant.Grav.* **28** (2011) 175004, `arXiv:1012.1851`.
[123] D. Klemm, "Rotating BPS black holes in matter-coupled AdS_4 supergravity", *JHEP* **1107** (2011) 019, `arXiv:1103.4699`.
[124] B. Carter, "Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations", *Commun.Math.Phys.* **10** (1968) 280.
[125] J. F. Plebański, "A class of solutions of Einstein-Maxwell equations", *Annals Phys.* **90** (1975) 196.
[126] E. Lozano-Tellechea and T. Ortin, "The General, duality invariant family of nonBPS black hole solutions of N=4, D = 4 supergravity", *Nucl.Phys.* **B569** (2000) 435–450, `arXiv:hep-th/9910020`.
[127] M. Colleoni and D. Klemm, "Nut-charged black holes in matter-coupled N=2, D=4 gauged supergravity", *Phys.Rev.* **D85** (2012) 126003, `arXiv:1203.6179`.
[128] D. Rasheed, "The Rotating dyonic black holes of Kaluza-Klein theory", *Nucl.Phys.* **B454** (1995) 379–401, `arXiv:hep-th/9505038`.
[129] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, "Rotating attractors", *JHEP* **0610** (2006) 058, `arXiv:hep-th/0606244`.
[130] D. Klemm, V. Moretti, and L. Vanzo, "Rotating topological black holes", *Phys.Rev.* **D57** (1998) 6127–6137, `arXiv:gr-qc/9710123`.
[131] M. M. Caldarelli, G. Cognola, and D. Klemm, "Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories", *Class.Quant.Grav.* **17** (2000) 399–420, `arXiv:hep-th/9908022`.
[132] R. P. Geroch, "Limits of spacetimes", *Commun.Math.Phys.* **13** (1969) 180–193.
[133] P. Meessen and T. Ortin, "Ultracold spherical horizons in gauged N=1, d=4 supergravity", *Phys.Lett.* **B693** (2010) 358–361, `arXiv:1007.3917`.
[134] P. H. Ginsparg and M. J. Perry, "Semiclassical Perdurance of de Sitter

Space", *Nucl.Phys.* **B222** (1983) 245.
[135] V. Cardoso, O. J. Dias, and J. P. Lemos, "Nariai, Bertotti-Robinson and anti-Nariai solutions in higher dimensions", *Phys.Rev.* **D70** (2004) 024002, `arXiv:hep-th/0401192`.
[136] S. Hawking, C. Hunter, and M. Taylor, "Rotation and the AdS / CFT correspondence", *Phys.Rev.* **D59** (1999) 064005, `arXiv:hep-th/9811056`.
[137] M. M. Caldarelli, R. Emparan, and M. J. Rodriguez, "Black Rings in (Anti)-deSitter space", *JHEP* **0811** (2008) 011, `arXiv:0806.1954`.
[138] M. M. Caldarelli, R. G. Leigh, A. C. Petkou, P. M. Petropoulos, V. Pozzoli, *et al.*, "Vorticity in holographic fluids", *PoS* **CORFU2011** (2011) 076, `arXiv:1206.4351`.
[139] K. Hristov, C. Toldo, and S. Vandoren, "Phase transitions of magnetic AdS4 black holes with scalar hair", *Phys.Rev.* **D88** (2013) 026019, `arXiv:1304.5187`.
[140] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Charged AdS black holes and catastrophic holography", *Phys.Rev.* **D60** (1999) 064018, `arXiv:hep-th/9902170`.
[141] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories", *Adv.Theor.Math.Phys.* **2** (1998) 505–532, `arXiv:hep-th/9803131`.
[142] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, "Holography, thermodynamics and fluctuations of charged AdS black holes", *Phys.Rev.* **D60** (1999) 104026, `arXiv:hep-th/9904197`.
[143] T. Hertog and G. T. Horowitz, "Designer gravity and field theory effective potentials", *Phys.Rev.Lett.* **94** (2005) 221301, `arXiv:hep-th/0412169`.
[144] M. Cvetic, G. Gibbons, D. Kubiznak, and C. Pope, "Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume", *Phys.Rev.* **D84** (2011) 024037, `arXiv:1012.2888`.
[145] D. Kubiznak and R. B. Mann, "P-V criticality of charged AdS black holes", *JHEP* **1207** (2012) 033, `arXiv:1205.0559`.
[146] L. Smarr, "Mass fomula for Kerr black holes", *Phys.Rev.Lett.* **30** (1973) 71–73.
[147] A. Magnon, "On Komar integrals in asymptotically anti-de Sitter space-times", *J.Math.Phys.* **26** (1985) 3112–3117.
[148] W. Chen, H. Lu, and C. Pope, "Mass of rotating black holes in gauged supergravities", *Phys.Rev.* **D73** (2006) 104036, `arXiv:hep-th/0510081`.
[149] G. Gibbons, M. Perry, and C. Pope, "The First law of thermodynamics for Kerr-anti-de Sitter black holes", *Class.Quant.Grav.* **22** (2005) 1503–1526, `arXiv:hep-th/0408217`.
[150] H. K. Kunduri and J. Lucietti, "Notes on non-extremal, charged, rotating black holes in minimal D=5 gauged supergravity", *Nucl.Phys.* **B724** (2005) 343–356, `arXiv:hep-th/0504158`.
[151] S. Chimento and D. Klemm, "Multi-centered black holes with a negative cosmological constant", `arXiv:1311.6937`.
[152] E. Witten, "Multitrace operators, boundary conditions, and AdS / CFT

correspondence", arXiv:hep-th/0112258.
[153] M. Berkooz, A. Sever, and A. Shomer, "'Double trace' deformations, boundary conditions and space-time singularities", *JHEP* **0205** (2002) 034, arXiv:hep-th/0112264.
[154] I. Papadimitriou, "Multi-Trace Deformations in AdS/CFT: Exploring the Vacuum Structure of the Deformed CFT", *JHEP* **0705** (2007) 075, arXiv:hep-th/0703152.
[155] T. Hertog and K. Maeda, "Stability and thermodynamics of AdS black holes with scalar hair", *Phys.Rev.* **D71** (2005) 024001, arXiv:hep-th/0409314.
[156] S. Hawking and H. Reall, "Charged and rotating AdS black holes and their CFT duals", *Phys.Rev.* **D61** (2000) 024014, arXiv:hep-th/9908109.
[157] S. Jain, S. Minwalla, T. Sharma, T. Takimi, S. R. Wadia, *et al.*, "Phases of large N vector Chern-Simons theories on $S^2 \times S^1$", *JHEP* **1309** (2013) 009, arXiv:1301.6169.
[158] N. Altamirano, D. Kubiznak, R. B. Mann, and Z. Sherkatghanad, "Kerr-AdS analogue of triple point and solid/liquid/gas phase transition", arXiv:1308.2672.
[159] N. Altamirano, D. Kubiznak, R. B. Mann, and Z. Sherkatghanad, "Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume", arXiv:1401.2586.
[160] N. Altamirano, D. Kubiznak, and R. B. Mann, "Reentrant Phase Transitions in Rotating AdS Black Holes", *Phys.Rev.* **D88** (2013) 101502, arXiv:1306.5756.
[161] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics", *Class.Quant.Grav.* **26** (2009) 224002, arXiv:0903.3246.
[162] C. P. Herzog, "Lectures on Holographic Superfluidity and Superconductivity", *J.Phys.* **A42** (2009) 343001, arXiv:0904.1975.
[163] R. Emparan, "AdS / CFT duals of topological black holes and the entropy of zero energy states", *JHEP* **9906** (1999) 036, arXiv:hep-th/9906040.
[164] D. Birmingham, "Topological black holes in Anti-de Sitter space", *Class.Quant.Grav.* **16** (1999) 1197–1205, arXiv:hep-th/9808032.
[165] M. M. Caldarelli and D. Klemm, "M theory and stringy corrections to Anti-de Sitter black holes and conformal field theories", *Nucl.Phys.* **B555** (1999) 157–182, arXiv:hep-th/9903078.
[166] M. P. Izquierdo, J.M and T. Ortin, "Bogomol'nyi Bounds in AdS", *Unpublished*, 1999.
[167] J. A. Nester, "A New gravitational energy expression with a simple positivity proof", *Phys.Lett.* **A83** (1981) 241.
[168] R. Argurio, F. Dehouck, and L. Houart, "Supersymmetry and Gravitational Duality", *Phys.Rev.* **D79** (2009) 125001, arXiv:0810.4999.
[169] E. Sezgin and Y. Tanii, "Witten-Nester Energy in Topologically Massive Gravity", *Class.Quant.Grav.* **26** (2009) 235005, arXiv:0903.3779.
[170] J. P. Gauntlett, G. W. Gibbons, C. M. Hull, and P. K. Townsend, "BPS states of D = 4 N=1 supersymmetry", *Commun.Math.Phys.* **216** (2001) 431–459,

arXiv:hep-th/0001024.
[171] M. Henneaux and C. Teitelboim, "Asymptotically anti-de sitter spaces", *Communications in Mathematical Physics* **98** (1985), no. 3, 391–424.
[172] G. Dibitetto and D. Klemm, "Magnetic charges in the AdS(4) superalgebra osp(4|2)", *JHEP* **1012** (2010) 005, arXiv:1005.4334.
[173] J. M. Figueroa-O'Farrill, "On the supersymmetries of Anti-de Sitter vacua", *Class.Quant.Grav.* **16** (1999) 2043–2055, arXiv:hep-th/9902066.
[174] T. Ortin, "Gravity and strings", 2004.
[175] K. Hristov, A. Tomasiello, and A. Zaffaroni, "Supersymmetry on Three-dimensional Lorentzian Curved Spaces and Black Hole Holography", *JHEP* **1305** (2013) 057, arXiv:1302.5228.
[176] K. Hristov, C. Toldo, and S. Vandoren, "Black branes in AdS: BPS bounds and asymptotic charges", *Fortsch.Phys.* **60** (2012) 1057–1063, arXiv:1201.6592.
[177] S. A. Hartnoll and P. Kovtun, "Hall conductivity from dyonic black holes", *Phys.Rev.* **D76** (2007) 066001, arXiv:0704.1160.
[178] E. P. Verlinde and H. L. Verlinde, "RG flow, gravity and the cosmological constant", *JHEP* **0005** (2000) 034, arXiv:hep-th/9912018.
[179] J. de Boer, "The Holographic renormalization group", *Fortsch.Phys.* **49** (2001) 339–358, arXiv:hep-th/0101026.
[180] J. Kalkkinen, D. Martelli, and W. Mueck, "Holographic renormalization and anomalies", *JHEP* **0104** (2001) 036, arXiv:hep-th/0103111.
[181] I. Papadimitriou, "Non-Supersymmetric Membrane Flows from Fake Supergravity and Multi-Trace Deformations", *JHEP* **0702** (2007) 008, arXiv:hep-th/0606038.
[182] H. Liu and A. A. Tseytlin, "D = 4 superYang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity", *Nucl.Phys.* **B533** (1998) 88–108, arXiv:hep-th/9804083.
[183] M. Henningson and K. Skenderis, "The Holographic Weyl anomaly", *JHEP* **9807** (1998) 023, arXiv:hep-th/9806087.
[184] J. D. Brown and J. York, James W., "Quasilocal energy and conserved charges derived from the gravitational action", *Phys.Rev.* **D47** (1993) 1407–1419, arXiv:gr-qc/9209012.
[185] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity", *Commun.Math.Phys.* **208** (1999) 413–428, arXiv:hep-th/9902121.
[186] R. Emparan, C. V. Johnson, and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence", *Phys.Rev.* **D60** (1999) 104001, arXiv:hep-th/9903238.
[187] M. Bianchi, D. Z. Freedman, and K. Skenderis, "Holographic renormalization", *Nucl.Phys.* **B631** (2002) 159–194, arXiv:hep-th/0112119.
[188] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence", *Commun.Math.Phys.* **217** (2001) 595–622, arXiv:hep-th/0002230.
[189] D. Martelli and W. Mueck, "Holographic renormalization and Ward identities with the Hamilton-Jacobi method", *Nucl.Phys.* **B654** (2003) 248–276,

arXiv:hep-th/0205061.
[190] K. Hristov, "Lessons from the Vacuum Structure of 4d N=2 Supergravity", arXiv:1207.3830.
[191] H. Lu, Y. Pang, and C. Pope, "An ω Deformation of Gauged STU Supergravity", arXiv:1402.1994.
[192] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An Index for 4 dimensional super conformal theories", *Commun.Math.Phys.* **275** (2007) 209–254, arXiv:hep-th/0510251.
[193] M. Berkooz, D. Reichmann, and J. Simon, "A Fermi Surface Model for Large Supersymmetric AdS(5) Black Holes", *JHEP* **0701** (2007) 048, arXiv:hep-th/0604023.
[194] M. Berkooz and D. Reichmann, "Weakly Renormalized Near 1/16 SUSY Fermi Liquid Operators in N=4 SYM", *JHEP* **0810** (2008) 084, arXiv:0807.0559.
[195] C.-M. Chang and X. Yin, "1/16 BPS States in N=4 SYM", *Phys.Rev.* **D88** (2013) 106005, arXiv:1305.6314.
[196] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend, *et al.*, "Black holes and superconformal mechanics", *Phys.Rev.Lett.* **81** (1998) 4553–4556, arXiv:hep-th/9804177.
[197] D. Gaiotto, A. Strominger, and X. Yin, "Superconformal black hole quantum mechanics", *JHEP* **0511** (2005) 017, arXiv:hep-th/0412322.
[198] B. de Wit, H. Samtleben, and M. Trigiante, "The Maximal D=5 supergravities", *Nucl.Phys.* **B716** (2005) 215–247, arXiv:hep-th/0412173.
[199] B. de Wit and H. Samtleben, "Gauged maximal supergravities and hierarchies of nonAbelian vector-tensor systems", *Fortsch.Phys.* **53** (2005) 442–449, arXiv:hep-th/0501243.

Nederlandse samenvatting

Egli e' scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi e' impossibile a intenderne umanamente parola; senza questi e' un aggirarsi vanamente per un oscuro laberinto.[1]

In deze uitspraak van Il Saggiatore stelt Galileo Galilei dat het heelal is geschreven in wiskundige taal. Er is een discussie gaande over of wiskunde een menselijke creatie is of de ontdekking van *a priori* bestaande werkelijkheden (Galileo had waarschijnlijk een voorkeur voor het laatste). Desondanks geloof ik dat logica, abstractie en wiskunde de manieren zijn waarop het verstand het universum om ons heen te doorgronden en modelleren.

De natuurkunde beschrijft de wereld om ons heen door middel van wiskundige modellen. In het bijzonder beoogt de hoge-energie theoretische natuurkunde een zogenaamde 'theorie van alles' te formuleren waarin alle bekende interacties in een uniek wiskundig raamwerk worden beschreven.

Op dit moment weten we dat er vier fundamentele wisselwerkingen zijn. Het standaardmodel van elementaire deeltjes beschrijft drie van deze: de sterke, zwakke en elektromagnetische wisselwerkingen. Wat hier nog ontbreekt is de zwaartekracht. Er wordt nog gezocht naar een kwantummechanische theorie die de zwaartekracht wel, de theorie van de kwantumzwaartekracht. Een dergelijke theorie wordt gebruikt om bijvoorbeeld te beschrijven wat er direct na de oerknal is gebeurd of wat er binnenin een zwart gat is.

Zwarte gaten zijn hemellichamen met een zeer hoge dichtheid, die bij elkaar worden gehouden door de zwaartekracht. De zwaartekracht van een zwart gat zorgt ervoor dat niks eruit kan ontsnappen, zelfs licht niet; daarom zijn ze zwart. Een waarnemer die een zwart gat in valt kan alle hoop om terug te komen opgeven zodra hij voorbij de zogeheten waarnemingshorizon is, een oppervlak rond het zwarte gat. Zwarte gaten ontstaan uit zware sterren die onder hun eigen zwaartekracht in elkaar storten. Recente observaties hebben sterk bewijs gevonden voor hun bestaan in ons heelal. De zwaartekracht in het binnenste van een zwart gat is extreem sterk,

[1] Het (universum) is geschreven in wiskundige taal, en de letters zijn driehoeken, cirkels en andere meetkundige figuren, zonder welke, voor een mens, het onmogelijk is om een woord the begrijpen; zonder dezen is het als dwalen door een obscuur doolhof.

daarom kan de studie van zwarte gaten belangrijke kwalitatieve informatie geven over hoe de zwaartekracht zich gedraagt bij hele hoge energieën.

Dit proefschrift is gewijd de studie van de theoretische aspecten van een specifieke klasse van zwarte gaten. De zwarte gaten waar we ons mee bezig houden zijn verre van realistisch. Ze corresponderen in zekere zin met vereenvoudigde versies van echte zwarte gaten, en kunnen worden gebruikt als een laboratorium voor het bestuderen van fundamentele aspecten van kwantumzwaartekracht. De zwarte gaten die we beschrijven zijn oplossingen van een specifieke theorie genaamd geijkte superzwaartekracht. Dit maakt een beschrijving in termen van snaartheorie mogelijk, op dit moment de hoofdkandidaat voor een consistente kwantumtheorie van zwaartekracht. In snaartheorie worden puntdeeltjes vervangen door eendimensionale objecten die we snaren noemen. Snaartheorie bevat op een natuurlijke manier zwaartekracht, daarom is het een kandidaat voor een theorie van alles, een op zichzelf staand wiskundig model dat alle fundamentele krachten en materie beschrijft.

Daarnaast bevinden de zwarte gaten die we bestuderen zich in een ruimte die wordt gekarakteriseerd door een negatieve kosmologische constante, de anti-De Sitter (AdS) ruimte. Deze objecten zijn interessant omdat er veel toepassingen zijn binnen de AdS/CFT correspondentie, dat theorieën van zwaartekracht in d-dimensionale anti-De Sitter ruimte relateert aan theorieën zonder zwaartekracht, meer specifiek aan conforme veldentheorieën in een ruimte met één dimensie lager. De kracht van de AdS/CFT correspondentie zelf ligt in het feit dat het een eenvoudigere duale beschrijving mogelijk maakt: eigenschappen van een zwart gat hebben misschien een duidelijkere interpretatie in termen van de duale veldentheorie. De holografische correspondentie kan ook de andere kant op gebruikt worden. Zwarte gaten gedragen zich als thermische ensembles en kunnen faseveranderingen ondergaan. Deze faseovergangen hebben voor AdS zwarte gaten een interpretatie in de duale theorie, en kunnen zo ingewikkelde fenomenen modelleren, bijvoorbeeld supergeleiding.

Dit proefschrift gaat over verschillende aspecten van oplossingen die AdS zwarte gaten beschrijven, en bestaat uit twee delen.

In het eerste gedeelte gebruiken we de theorie van superzwaartekracht om wiskundige oplossingen voor zwarte gaten te vinden. In het bijzonder kijken we naar stabiele configuraties van elektrisch geladen zwarte gaten bij het absolute nulpunt en naar zwarte gaten die straling uitzenden bij een eindige temperatuur. Ook beschrijven we AdS zwarte gaten met een impulsmoment: roterende anti-De Sitter zwarte gaten. De relevante details van de constructietechnieken die ook van toepassing zijn in andere gevallen worden benadrukt. De nieuwe oplossingen die wij hebben gevonden leveren interessante zwaartekracht achtergronden voor toepassingen van de AdS/CFT correspondentie.

In het tweede gedeelte van dit proefschrift bestuderen we de thermodynamica van statische configuraties van zwarte gaten. We leggen hierbij de nadruk op faseovergangen. Als de lading van het zwarte gat voldoende klein is, blijkt er een faseovergang plaats te vinden tussen kleine en grote zwarte gaten. Met behulp van de AdS/CFT correspondentie kunnen we dit fenomeen begrijpen in termen van een faseovergang tussen een gas en een vloeistof, die plaatsvindt in een conforme veldentheorie met één ruimtelijke dimensie lagen. We eindigen met een discussie over hoe men relevante grootheden voor AdS zwarte gaten kan definiëren, zoals hun massa.

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Chiara Toldo,

28 April 2014

Curriculum Vitae

I was born on 19th April 1986 in Schio (Italy). I attended high school at Liceo Scientifico Tron in Schio from 2000 to 2005 and just afterwards I enrolled in Physics. I followed the bachelor (2005-2008) and masters (2008-2010) programme in Padua University, where I graduated both cum laude in 2010. Shortly after, I moved to the Netherlands to start my PhD research under the supervision of Prof. S. Vandoren. The results of this research are the main subject of this thesis.