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Jordanian deformations of the $AdS_5 \times S^5$ superstring

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Abstract

We consider Jordanian deformations of the $AdS_5 \times S^5$ superstring action. The deformations correspond to non-standard q-deformation. In particular, it is possible to perform partial deformations, for example, only for the S^5 part. Then the classical action and the Lax pair are constructed with a linear, twisted and extended R operator. It is shown that the action preserves the κ-symmetry.

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1 Introduction

One of the fascinating topics in string theory is the AdS/CFT correspondence [1–3]. The most well-studied example is the duality between type IIB superstring on the $\mathrm{AdS}_5\times\mathrm{S}^5$ background [4] (often called the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring) and the $\mathcal{N}=4$ $SU(N)$ super Yang-Mills (SYM) theory in four dimensions (in the large N limit). It has been revealed that the integrable structure exists behind the duality and it plays a fundamental role in testing the correspondence of physical quantities (For a comprehensive review, see [5]).

Our interest here is the integrability in the string-theory side. The classical integrable structure of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring is closely related to the Z_4-grading property of the supercoset [6][1],

$$PSU(2,2|4)/[SO(1,4)\times SO(5)]\,.$$

The supercosets with the grading property are classified, including the stringy conditions [9].

The next is to consider integrable deformations. There are two approaches, the one is based on 1) deformed S-matrices and the other is based on 2) deformed target spaces. For the first approach, the deformed S-matrices are constructed in a mathematically well-defined way [10–14], but the corresponding geometry of target space is unclear. In the second direction, the classical integrable structure has been well studied for three-dimensional examples such as squashed S^3 (For the classic works and the recent progress, see [15–17] and [18–23], respectively) and warped AdS_3 [24–26]. The deformed geometries are represented by non-symmetric cosets [27] and there is no general prescription to argue the integrability. For generalizations to higher dimensions, see [28, 29]. In particular, the method utilized in [29] is based on Yang-Baxter sigma models [30]. The standard q-deformation of $\mathfrak{su}(2)$ [31–33] and the affine extension are also presented [20, 29] and [21], respectively.

Recently, a q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action was constructed [34] by generalizing the result in [29]. Then the bosonic part of the action was determined and, by using the action, the world-sheet S-matrix of bosonic excitations was computed in [35]. The resulting S-matrix exactly agrees with the q-deformed S-matrix in the large tension limit. Thus the two approaches will now be related each other and there are many directions to study q-deformations of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring.

[1] For the classical integrability based on the Roiban-Siegel formulation [7], see [8].

In this paper, we consider how to twist the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action. This twisting is regarded as a non-standard q-deformation. Indeed, it would also be seen as a higher-dimensional generalization of 3D Schrödinger sigma models in which q-deformed Poincare algebra [36, 37] and its infinite-dimensional extension are realized as shown in a series of works [24, 25]. In particular, it is possible to perform partial deformations, for example, only for the S^5 part. It would make the resulting geometry much simpler. Some extensions of the twisted R operators are also discussed. Then the classical action and the Lax pair are constructed with a linear, twisted and extended R operator. It is shown that the action preserves the κ-symmetry.

The paper is organized as follows. Section 2 is a short review of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ action. Section 3 describes how to twist the q-deformed action. Then we construct the Jordanian deformed action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring preserving the κ-symmetry. The Lax pair is also presented. Section 4 is devoted to conclusion and discussion. Appendix A describes the notation of the superconformal generators. In Appendix B, the notation of the classical R-matrix is explained. A general prescription to twist the classical r-matrix for the standard q-deformation of Drinfeld-Jimbo type is also provided.

2 A review of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring

In this section, we will give a short review of the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action constructed in [34], using the notation therein.

2.1 The linear R operator

A key ingredient in the construction is the classical R-matrix, which is a linear map $R:\mathfrak{g}\to\mathfrak{g}$ over a Lie algebra $\mathfrak{g}$ satisfying the modified classical Yang-Baxter equation (mCYBE);

$$[R(M),R(N)]-R\left([R(M),N]+[M,R(N)]\right)=-c^2\,[M,N]\ , \tag{2.1}$$

where $M,N\in\mathfrak{g}$ and c is a complex parameter. When $c\neq 0$, the parameter is regarded as a scaling of the R-matrix and could be normalized as $c=1$. When $c=0$, the mCYBE is nothing but the classical Yang-Baxter equation (CYBE).

The standard q-deformation of the superstring action presented in [34] is described by the following R-matrix,

$$R(E_{ij}) = \begin{cases} +cE_{ij} & \text{for} \quad i < j \\ -cE_{ij} & \text{for} \quad i > j \end{cases} \qquad \text{and} \qquad R(E_{ii}) = 0\,, \tag{2.2}$$

where E_{ij} $(i,j = 1,\cdots,8)$ are the $\mathfrak{gl}(4|4)$ generators. For the standard notation of the superconformal generators, see Appendix A. The parity of the indices is given by $\bar{i} = 0$ for $i = 1,\cdots,4$ and $\bar{i} = 1$ for $i = 5,\cdots,8$. The associated tensorial r-matrix is

$$r_{\rm DJ} = c \sum_{1\leq i<j\leq 8} E_{ij} \wedge E_{ji}\, (-1)^{\bar{i}\,\bar{j}}\,, \tag{2.3}$$

where the super skew-symmetric symbol is introduced as

$$E_{ij} \wedge E_{kl} \equiv E_{ij} \otimes E_{kl} - E_{kl} \otimes E_{ij} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\,. \tag{2.4}$$

The relations between the linear R operator and the tensorial notation r are summarized in Appendix B. The classical r-matrix given in (2.3) describes the standard q-deformation of Drinfeld-Jimbo (DJ) type [31–33].

2.2 The classical action and the Lax pair

With the help of the linear R operator defined in (2.2), the q-deformed classical action S is given by[2]

$$S = -\frac{(1+\eta^2)^2}{2(1-\eta^2)} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, P_-^{\alpha\beta}\, {\rm Str}\left(A_\alpha\, d \circ \frac{1}{1-\eta R_g \circ d}(A_\beta)\right)\,. \tag{2.5}$$

Here τ and σ are time and spatial coordinates of the string world-sheet and the periodic boundary condition is imposed for the σ direction. The real constant $\eta \in [0,1)$ measures the deformation[3]. The super Maurer-Cartan one-form A_α is defined as

$$A_\alpha \equiv g^{-1}\partial_\alpha g\,, \qquad g \in SU(2,2|4)\,,$$

and A_α takes the value in the Lie superalgebra $\mathfrak{su}(2,2|4)$. The action of the R-matrix (2.2) on A_α is induced from $\mathfrak{gl}(4|4)$ by imposing a suitable reality condition. Note that A_α

[2]Here we have normalized the parameter as $c = 1$ in (2.2).

[3]Since the deformation is measured by η, it is often called "η-deformation". On the other hand, η is related to the q parameter of the standard q-deformation by Drinfeld-Jimbo [31–33] as shown in [29]. Hence we will refer this deformation as to q-deformation, following [34].

automatically satisfies the flatness condition,

$$\mathcal{Z} \equiv \frac{1}{2}\epsilon^{\alpha\beta}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]\right) = 0\,. \tag{2.6}$$

The projection operators $P^{\alpha\beta}_{\pm}$ are defined as

$$P^{\alpha\beta}_{\pm} \equiv \frac{1}{2}\left(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta}\right)\,.$$

Then operators d and $\tilde{d}$ are linear combinations of the projection operators P_i $(i = 1, 2, 3)$,

$$d \equiv P_1 + \frac{2}{1-\eta^2}\,P_2 - P_3\,, \qquad \tilde{d} \equiv -P_1 + \frac{2}{1-\eta^2}\,P_2 + P_3\,. \tag{2.7}$$

The symbol R_g indicates a chain of the adjoint operation and the linear R operation,

$$R_g(M) \equiv Ad_g^{-1}\circ R \circ Ad_g(M) = g^{-1}R(gMg^{-1})g\,. \tag{2.8}$$

Note that the usual $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring action is reproduced from (2.5) when $\eta = 0$. For a pedagogical review of the undeformed $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring, see [42].

It is convenient to introduce the following notations,

$$J_\alpha \equiv \frac{1}{1-\eta R_g\circ d}\,(A_\alpha)\,, \quad \widetilde{J}_\alpha \equiv \frac{1}{1+\eta R_g\circ \tilde{d}}\,(A_\alpha)\,, \quad J^\alpha_- \equiv P^{\alpha\beta}_- J_\beta\,, \quad \widetilde{J}^\alpha_+ \equiv P^{\alpha\beta}_+ \widetilde{J}_\beta\,. \tag{2.9}$$

Then the equations of motion are written in a simpler form,

$$\mathcal{E} = d(\partial_\alpha J^\alpha_-) + \tilde{d}(\partial_\alpha \widetilde{J}^\alpha_+) + [\widetilde{J}_{+\alpha}, d(J^\alpha_-)] + [J_{-\alpha}, \tilde{d}(\widetilde{J}^\alpha_+)] = 0\,. \tag{2.10}$$

The Lax pair is given by

$$\begin{aligned} L^\alpha_+ &= \widetilde{J}^{\alpha(0)}_+ + \lambda\sqrt{1+\eta^2}\,\widetilde{J}^{\alpha(1)}_+ + \lambda^{-2}\left(\frac{1+\eta^2}{1-\eta^2}\right)\widetilde{J}^{\alpha(2)}_+ + \lambda^{-1}\sqrt{1+\eta^2}\,\widetilde{J}^{\alpha(3)}_+\,, \\ M^\alpha_- &= J^{\alpha(0)}_- + \lambda\sqrt{1+\eta^2}\,J^{\alpha(1)}_- + \lambda^{2}\left(\frac{1+\eta^2}{1-\eta^2}\right)J^{\alpha(2)}_- + \lambda^{-1}\sqrt{1+\eta^2}\,J^{\alpha(3)}_-\,, \end{aligned} \tag{2.11}$$

where λ is the spectral parameter that takes a complex value. The flatness condition (2.6) can be rewritten in terms of J^α_- and $\widetilde{J}^\alpha_+$ like

$$\mathcal{Z} = \partial_\alpha \widetilde{J}^\alpha_+ - \partial_\alpha J^\alpha_- + [J_{-\alpha}, \widetilde{J}^\alpha_+] + \eta^2[d(J_{-\alpha}), \tilde{d}(\widetilde{J}^\alpha_+)] + \eta R_g(\mathcal{E}) = 0\,. \tag{2.12}$$

With the definition $\mathcal{L}_\alpha \equiv L_{+\alpha} + M_{-\alpha}$, the zero-curvature condition

$$\partial_\alpha\mathcal{L}_\beta - \partial_\beta\mathcal{L}_\alpha + [\mathcal{L}_\alpha, \mathcal{L}_\beta] = 0 \tag{2.13}$$

is equivalent to the equations of motion given in (2.10) and the flatness condition (2.12). For the κ-symmetry argument, see [34].

3 Jordanian deformations of the $AdS_5 \times S^5$ superstring

In this section we shall consider Jordanian deformations of the $AdS_5 \times S^5$ superstring action. The deformations correspond to a non-standard q-deformation and contain twists of the linear R operator. The twist procedure is realized as an adjoint operation for the linear R operator with an arbitrary bosonic root. Also, see Appendix B.

We first explain how to construct Jordanian R operators by twisting the linear R operator used in the q-deformed $AdS_5 \times S^5$ superstring action (2.5) . There are two remarkable features of Jordanian R operators. The first is that they satisfy CYBE rather than mCYBE (2.1). The second is the nilpotency of them. That is

$$[R(M), R(N)] - R\left([R(M), N] + [M, R(N)]\right) = 0\,, \tag{3.1}$$

$$R^n(M) = 0 \qquad \text{for} \quad n \geq 3\,, \tag{3.2}$$

for $M, N \in \mathfrak{g}$.

Then, by using the Jordanian R operators, the Jordanian deformed action with the κ-symmetry and the Lax pair are presented.

3.1 Jordanian R operators from twists and their extension

We shall give a description to twist the linear R operator for basic examples of Jordanian R operators here. Then some extensions of twisted R operators are discussed.

First of all, note that the classical r-matrix of Drinfeld-Jimbo type (2.3) has the vanishing Cartan charges,

$$[\Delta(E_{ii}), r_{\rm DJ}] = 0 \qquad \text{for} \qquad i = 1, \cdots, 8\,, \tag{3.3}$$

where the coproduct is given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad \text{for} \quad X \in \mathfrak{g}\,.$$

On the other hand, one may introduce a classical r-matrix which has non-zero Cartan charges for the deformation of $AdS_5 \times$ S^5 superstring. In this sense, we refer those as to Jordanian r-matrices. In general, such an r-matrix can be constructed by a twist of $r_{\rm DJ}$ with an arbitrary bosonic root E_{ij} with $i < j$,

$$r_{\rm tw}^{(i,j)} \equiv [\Delta(E_{ij}), r_{\rm DJ}]\,. \tag{3.4}$$

One may also consider twists by negative bosonic roots E_{ij} $(i > j)$, but the corresponding r-matrix has the same property because $\mathfrak{gl}(4|4)$ algebra enjoys the automorphism

$$E_{ij} \mapsto E_{9-j,9-i}\,. \tag{3.5}$$

Thus positive roots E_{ij} $(i < j)$ are enough for our later argument. The twisted, linear R operator is defined as

$$\begin{aligned} R^{(i,j)}_{\rm tw}(X) &\equiv \langle r^{(i,j)}_{\rm tw}, 1\otimes X\rangle \\ &= [E_{ij}, R(X)] - R([E_{ij}, X]) \qquad \text{for} \quad X \in \mathfrak{g}\,. \end{aligned} \tag{3.6}$$

It is straight forward to read off the R operator from the tensorial r-matrix via (3.6) and inner product (A.2).

So far, we have constructed the Jordanian R operators via twists of $r_{\rm DJ}$. One may also consider the extension of the twisted R operators by adding bilinear terms of fermionic root generators. It should be noted that the latter cannot be obtained with the twists. Thus there are the two classes: 1) Jordanian R operators stemming from twists and 2) extended Jordanian R operators. We will introduce some examples below.

1) Jordanian R operators from twists

The first example is twists by simple roots. Then the corresponding subsectors of the superstring action are deformed. For instance, let us consider twists by positive simple root generators $E_{k,k+1}$ $(k = 1,\ldots,\check{4},\ldots,7)$[4]. Then the associated classical r-matrix is given by

$$r^{(k,k+1)}_{\rm tw} = [\Delta(E_{k,k+1}), r_{\rm DJ}] = cE_{k,k+1} \wedge \left(E_{kk}(-1)^{\bar{k}} - E_{k+1,k+1}(-1)^{\overline{k+1}}\right)\,. \tag{3.7}$$

The twists give rise to deformations of the AdS_3 or S^3 subspace. For each of the values $k = 1,2,3$, the resulting geometry is given by a deformed AdS_3 spacetime. It would contain a three-dimensional Schrödinger spacetime and may be regarded as a generalization of the previous works [24, 25]. The explicit relation will be presented in [43].

More interesting examples are deformations of either AdS_5 or S^5. These partial deformations are realized by twists with the maximal bosonic generators $E_{14} = P_{14}$ in $\mathfrak{su}(2,2)$ and

[4] For $k = 4$, the simple root $E_{45} = \bar{S}_{45}$ is fermionic and it is regarded as a fermionic twist.

$E_{58} = R_{58}$ in $\mathfrak{su}(4)$, respectively [5];

$$\mathrm{AdS}_5 : \quad r_{\mathrm{tw}}^{(1,4)} = [\Delta(E_{14}), r_{\mathrm{DJ}}] = c\Big(E_{14} \wedge (E_{11} - E_{44}) - 2 \sum_{\kappa=2,3} E_{1\kappa} \wedge E_{\kappa 4}\Big)\,, \tag{3.8}$$

$$\mathrm{S}^5 : \quad r_{\mathrm{tw}}^{(5,8)} = [\Delta(E_{58}), r_{\mathrm{DJ}}] = c\Big(E_{58} \wedge (-E_{55} + E_{88}) + 2 \sum_{k=6,7} E_{5k} \wedge E_{k8}\Big)\,. \tag{3.9}$$

The deformation of S^5 should be interesting because it would provide a simpler geometry without deforming AdS_5 . The associated linear operator acts on the generators as follows:

$$\begin{aligned}
&R_{\mathrm{tw}}^{(5,8)}(E_{55}) = +cE_{58}\,, && R_{\mathrm{tw}}^{(5,8)}(E_{k5}) = +2cE_{k8}\,, \\
&R_{\mathrm{tw}}^{(5,8)}(E_{88}) = -cE_{58}\,, && R_{\mathrm{tw}}^{(5,8)}(E_{8k}) = -2cE_{5k}\,, \\
&R_{\mathrm{tw}}^{(5,8)}(E_{85}) = c(-E_{55} + E_{88})\,, && R_{\mathrm{tw}}^{(5,8)}(\mathrm{others}) = 0\,,
\end{aligned}$$

where $k = 6, 7$.

Remarks More generally, the Reshetikhin twist [44] or the Jordanian twist [45, 46] is closely related to the present prescription. The Jordanian twists for Lie superalgebras are considered in [38–41]. The relation will be elaborated somewhere else.

As a side remark, we have worked with a particular choice of the simple roots associated with the Dynkin diagram O-O-O-X-O-O-O of the superconformal algebra. It would be also interesting to see the twisting based on the different choice of simple roots such as O-X-O-O-O-X-O.

2) Extended Jordanian R operators

Let us now consider some extensions of the twisted classical r-matrices given in (3.7), (3.8) and (3.9). Recall that these are obtained by twisting r_{DJ} . Here we are concerned with some extensions of the twisted r-matrices, which are not described as twists.

It is easy to see that a linear combination of (3.8) and (3.9)

$$r_{\mathrm{tw}}^{(1,4),(5,8)} \equiv c_1 r_{\mathrm{tw}}^{(1,4)} + c_2 r_{\mathrm{tw}}^{(5,8)} \tag{3.10}$$

with $c = 1$ is also a solution of CYBE, due to the relation

$$[r_{\mathrm{tw}}^{(1,4)}, r_{\mathrm{tw}}^{(5,8)}] = 0\,. \tag{3.11}$$

[5]For the map between the E_{ij} generators and the superconformal generators, see Appendix A.

The r-matrix $r_{\rm tw}^{(1,4),(5,8)}$ implies independent deformations of AdS^5 and S^5 with different parameters c_1 and c_2 respectively.

Furthermore, these r-matrices may be extended to contain supercharges in their tails, including two parameters, like

$$\widetilde{r}_{\rm tw}^{(1,4)} = E_{14} \wedge (\alpha E_{11} - \beta E_{44}) - (\alpha + \beta) \sum_{j \neq 1,4} E_{1j} \wedge E_{j4}\,, \tag{3.12}$$

$$\widetilde{r}_{\rm tw}^{(5,8)} = E_{58} \wedge (\alpha' E_{55} - \beta' E_{88}) - (\alpha' + \beta') \sum_{j \neq 5,8} E_{5j} \wedge E_{j8}\,. \tag{3.13}$$

Here $\alpha, \beta, \alpha', \beta'$ are arbitrary parameters. The extended r-matrices satisfy CYBE (3.1).

As a remark, it would not be obvious that the multi-parameter deformations may lead to consistent string theories. For the vanishing β-function on the world-sheet, there may be additional constraints on the deformation parameters.

Comments on fermionic twists

One may think of twists by fermionic generators. However, it seems that the resulting r-matrices do not satisfy CYBE (3.1), in general.

As an example, let us consider $E_{45} = \bar{S}_{45}$. This is a simple root generator but it gives rise to the maximal twist. That is, the corresponding geometry is also maximally deformed. The associated classical r-matrix is given by

$$r_{\rm tw}^{(4,5)} = [\Delta(E_{45}), r_{\rm DJ}] = c\Big[(E_{44} + E_{55}) \wedge E_{45} + 2\sum_{\kappa=1}^{3} E_{4\kappa} \wedge E_{\kappa 5} - 2\sum_{k=6}^{8} E_{4k} \wedge E_{k5}\Big]\,. \tag{3.14}$$

Note that c is a Grassmann odd element [38], so that the r-matrix is Grassmann even. However it does not seem to be a solution of CYBE.

The only exception is obtain by the maximal root, (Also see Appendix B.2)

$$r_{\rm tw}^{(1,8)} = [\Delta(E_{18}), r_{\rm DJ}] = -cE_{18} \wedge (E_{11} + E_{88})\,. \tag{3.15}$$

This is a solution of CYBE. For this fermionic twist, we have no clear understanding for the physical interpretation because the deformation is measured by a Grassmann odd parameter. It would be interesting to interpret the fermionic twist in type IIB supergravity.

3.2 Jordanian deformed action

The next is to consider Jordanian deformations of the classical action of the $\mathrm{AdS}_5\times\mathrm{S}^5$ superstring. Although the construction is almost parallel to the one in [34], it is necessary to take account of small modifications coming from the fact that the Jordanian linear operator R_{Jor} satisfies CYBE rather than mCYBE.

In the following, R_{Jor} is used as a representative of arbitrary (extended) Jordanian R operators[6]. The detail expression of R_{Jor} is not relevant to the subsequent analysis.

The Jordanian deformed classical action is given by

$$S=-\frac{1}{2}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,P_-^{\alpha\beta}\,\mathrm{Str}\left(A_\alpha\,d\circ\frac{1}{1-\eta\,[R_{\mathrm{Jor}}]_g\circ d}(A_\beta)\right)\,. \tag{3.16}$$

Here, by using Jordanian R-matrix R_{Jor} , a chain of the operations $[R_{\mathrm{Jor}}]_g$ is defined as

$$[R_{\mathrm{Jor}}]_g\,(M)\equiv Ad_g^{-1}\circ R_{\mathrm{Jor}}\circ Ad_g(M)=g^{-1}R_{\mathrm{Jor}}(gMg^{-1})g\,. \tag{3.17}$$

In the present case, d and $\tilde{d}$ are not deformed and do not contain η like

$$d\equiv P_1+2P_2-P_3\,,\qquad \tilde{d}\equiv -P_1+2P_2+P_3\,, \tag{3.18}$$

and the overall factor of the action (2.5) is not needed to be multiplied. As in the case of [34], the equations of motion can be written simply with the following quantities:

$$\begin{aligned} J_\alpha&\equiv\frac{1}{1-\eta\,[R_{\mathrm{Jor}}]_g\circ d}\,(A_\alpha)\,, \qquad & J_-^\alpha&\equiv P_-^{\alpha\beta}J_\beta\,,\\ \widetilde{J}_\alpha&\equiv\frac{1}{1+\eta\,[R_{\mathrm{Jor}}]_g\circ \tilde{d}}\,(A_\alpha)\,, \qquad & \widetilde{J}_+^\alpha&\equiv P_+^{\alpha\beta}\widetilde{J}_\beta\,. \end{aligned} \tag{3.19}$$

There are two ways to rewrite the action given in (3.16). The first is based on J_α and the action is written as

$$\begin{aligned} S=&-\frac{1}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\left(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{Str}\left(J_\alpha d\left(J_\beta\right)\right)\\ &+\frac{\eta}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\left(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{Str}\left([R_{\mathrm{Jor}}]_g\circ d(J_\alpha)d\left(J_\beta\right)\right)\\ =&-\frac{1}{2}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\gamma^{\alpha\beta}\mathrm{Str}\left(J_\alpha^{(2)}J_\beta^{(2)}\right)-\frac{1}{2}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\epsilon^{\alpha\beta}\mathrm{Str}\left(J_\alpha^{(1)}J_\beta^{(3)}\right) \end{aligned}$$

[6] The (extended) Jordanian operators are easily derived from the tensorial r-matrix presented in Sec. 3.1 by using the relations (B.4) and the inner product (A.2).

$$
+\frac{\eta}{4}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\epsilon^{\alpha\beta}\mathrm{Str}\left(d(J_\alpha)\,[R_{\mathrm{Jor}}]_g\circ d\,(J_\beta)\right)\,. \tag{3.20}
$$

The second is based on $\widetilde{J}_\alpha$ and the action becomes

$$
\begin{aligned}
S=&-\frac{1}{4}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\left(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{Str}\left(\widetilde{d}(\widetilde{J}_\alpha)\widetilde{J}_\beta\right)\\
&-\frac{\eta}{4}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\left(\gamma^{\alpha\beta}-\epsilon^{\alpha\beta}\right)\mathrm{Str}\left(\widetilde{d}(\widetilde{J}_\alpha)\,[R_{\mathrm{Jor}}]_g\circ\widetilde{d}(\widetilde{J}_\beta)\right)\\
=&-\frac{1}{2}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\gamma^{\alpha\beta}\mathrm{Str}\left(\widetilde{J}^{(2)}_\alpha\widetilde{J}^{(2)}_\beta\right)-\frac{1}{2}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\epsilon^{\alpha\beta}\mathrm{Str}\left(\widetilde{J}^{(1)}_\alpha\widetilde{J}^{(3)}_\beta\right)\\
&+\frac{\eta}{4}\int_{-\infty}^{\infty}d\tau\int_{0}^{2\pi}d\sigma\,\epsilon^{\alpha\beta}\mathrm{Str}\left(\widetilde{d}(\widetilde{J}_\alpha)\,[R_{\mathrm{Jor}}]_g\circ\widetilde{d}(\widetilde{J}_\beta)\right)\,.
\end{aligned} \tag{3.21}
$$

The two expressions are useful to discuss the Virasoro conditions and the κ-invariance.

Then equations of motion are given by

$$
\mathcal{E}=d(\partial_\alpha J^\alpha_-)+\tilde{d}(\partial_\alpha\widetilde{J}^\alpha_+)+[\widetilde{J}_{+\alpha},d(J^\alpha_-)]+[J_{-\alpha},\tilde{d}(\widetilde{J}^\alpha_+)]=0\,, \tag{3.22}
$$

and the flatness condition is represented by

$$
\begin{aligned}
\mathcal{Z}&=\frac{1}{2}\epsilon^{\alpha\beta}\left(\partial_\alpha A_\beta-\partial_\beta A_\alpha+[A_\alpha,A_\beta]\right)\\
&=\partial_\alpha\widetilde{J}^\alpha_+-\partial_\alpha J^\alpha_-+[J_{-\alpha},\widetilde{J}^\alpha_+]+\eta\,[R_{\mathrm{Jor}}]_g\,(\mathcal{E})=0\,.
\end{aligned} \tag{3.23}
$$

Note that the flatness condition does not contain the η^2 terms, in comparison to the one given in (2.12). This modification comes from the fact that the Jordanian operator R_{Jor} satisfies CYBE, rather than mCYBE.

For later computations, it is convenient to decompose the equations of motion (3.22) and the flatness condition (3.23) as follows:

$$
\begin{aligned}
&\partial_\alpha\widetilde{J}^{\alpha(0)}_+-\partial_\alpha J^{\alpha(0)}_-+\left[J^{(0)}_{-\alpha},\widetilde{J}^{\alpha(0)}_+\right]+\left[J^{(1)}_{-\alpha},\widetilde{J}^{\alpha(3)}_+\right]+\left[J^{(2)}_{-\alpha},\widetilde{J}^{\alpha(2)}_+\right]+\left[J^{(3)}_{-\alpha},\widetilde{J}^{\alpha(1)}_+\right]=0\,, \quad (3.24)\\
&\left[J^{(3)}_{-\alpha},\widetilde{J}^{\alpha(2)}_+\right]=0\,,\\
&\partial_\alpha\widetilde{J}^{\alpha(1)}_+-\partial_\alpha J^{\alpha(1)}_-+\left[J^{(0)}_{-\alpha},\widetilde{J}^{\alpha(1)}_+\right]+\left[J^{(1)}_{-\alpha},\widetilde{J}^{\alpha(0)}_+\right]+\left[J^{(2)}_{-\alpha},\widetilde{J}^{\alpha(3)}_+\right]=0\,,\\
&\partial_\alpha\widetilde{J}^{\alpha(2)}_++\left[J^{(0)}_{-\alpha},\widetilde{J}^{\alpha(2)}_+\right]+\left[J^{(3)}_{-\alpha},\widetilde{J}^{\alpha(3)}_+\right]=0\,,\\
&\partial_\alpha J^{\alpha(2)}_--\left[J^{(1)}_{-\alpha},\widetilde{J}^{\alpha(1)}_+\right]-\left[J^{(2)}_{-\alpha},\widetilde{J}^{\alpha(0)}_+\right]=0\,,\\
&\left[J^{(2)}_{-\alpha},\widetilde{J}^{\alpha(1)}_+\right]=0\,,
\end{aligned}
$$

$$\partial_\alpha \widetilde{J}_+^{\alpha(3)} - \partial_\alpha J_-^{\alpha(3)} + \left[J_{-\alpha}^{(0)}, \widetilde{J}_+^{\alpha(3)}\right] + \left[J_{-\alpha}^{(1)}, \widetilde{J}_+^{\alpha(2)}\right] + \left[J_{-\alpha}^{(3)}, \widetilde{J}_+^{\alpha(0)}\right] = 0\,.$$

Then the Lax pair is given by

$$M_-^\alpha = J_-^{\alpha(0)} + \lambda J_-^{\alpha(1)} + \lambda^2 J_-^{\alpha(2)} + \lambda^{-1} J_-^{\alpha(3)}\,, \tag{3.25}$$

$$L_+^\alpha = \widetilde{J}_+^{\alpha(0)} + \lambda \widetilde{J}_+^{\alpha(1)} + \lambda^{-2} \widetilde{J}_+^{\alpha(2)} + \lambda^{-1} \widetilde{J}_+^{\alpha(3)}\,. \tag{3.26}$$

Note that the η^2 terms are not contained again, in comparison to the Lax pair given in (2.11), while the parameter η is still contained in $J_-^{\alpha(n)}$ and $\widetilde{J}_+^{\alpha(n)}$ $(n = 0, \ldots, 3)$. With $\mathcal{L}_\alpha \equiv L_{+\alpha} + M_{-\alpha}$, it is an easy task to show that the zero curvature condition (2.13) is equivalent to the equation of motion (3.22) and the flatness condition (3.23).

The next is to consider the Virasoro conditions. The expression given in (3.20) leads to the Virasoro conditions,

$$\mathrm{Str}\left(J_\alpha^{(2)} J_\beta^{(2)}\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\sigma}\mathrm{Str}\left(J_\rho^{(2)} J_\sigma^{(2)}\right) = 0\,. \tag{3.27}$$

On the other hand, the expression in (3.21) gives rise to

$$\mathrm{Str}\left(\widetilde{J}_\alpha^{(2)} \widetilde{J}_\beta^{(2)}\right) - \frac{1}{2}\gamma_{\alpha\beta}\gamma^{\rho\sigma}\mathrm{Str}\left(\widetilde{J}_\rho^{(2)} \widetilde{J}_\sigma^{(2)}\right) = 0\,. \tag{3.28}$$

The above two representations of the Virasoro conditions given in (3.27) and (3.28) should be equivalent.

3.3 κ-symmetry

Let us consider the κ-symmetry of the action (3.16).

We consider a fermionic local transformation (called the κ-transformation) of g given by

$$\delta g = g\epsilon\,, \qquad \epsilon \equiv \left(1 - \eta\,[R_{\mathrm{Jor}}]_g\right)\rho^{(1)} + \left(1 + \eta\,[R_{\mathrm{Jor}}]_g\right)\rho^{(3)}\,, \tag{3.29}$$

where $\rho^{(1)}$ and $\rho^{(3)}$ are arbitrary functions on the string world-sheet to be determined later, and hence ϵ also depends on the world-sheet coordinates. Then the variation of the action given in (3.16) is described as

$$\begin{aligned} \delta_g S &= \frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, \mathrm{Str}\left(\epsilon\,\mathcal{E}\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\, \mathrm{Str}\left(\rho^{(1)} P_3 \circ \left(1 + \eta\,[R_{\mathrm{Jor}}]_g\right)(\mathcal{E})\right. \end{aligned} \tag{3.30}$$

$$+\rho^{(3)} P_1 \circ \left(1 - \eta \left[R_{\rm Jor}\right]_g\right) (\mathcal{E})\Big)$$

$$= -2 \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \, {\rm Str} \left(\rho^{(1)} \left[J^{(2)}_{-\alpha}, \widetilde{J}^{\alpha(1)}_+ \right] + \rho^{(3)} \left[\widetilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_- \right] \right) .$$

Here the following relations have been used in the second equality,

$$P_1 \circ \left(1 - \eta \left[R_{\rm Jor}\right]_g\right) (\mathcal{E}) = -4 \left[\widetilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_- \right] - P_1(\mathcal{Z}) \,,$$
$$P_3 \circ \left(1 + \eta \left[R_{\rm Jor}\right]_g\right) (\mathcal{E}) = -4 \left[J^{(2)}_{-\alpha}, \widetilde{J}^{\alpha(1)}_+ \right] + P_3(\mathcal{Z}) \,. \tag{3.31}$$

Now let the forms of $\rho^{(1)}$ and $\rho^{(3)}$ be

$$\rho^{(1)} = i\kappa^{\alpha(1)}_+ J^{(2)}_{-\alpha} + J^{(2)}_{-\alpha} i\kappa^{\alpha(1)}_+ \,, \qquad \rho^{(3)} = i\kappa^{\alpha(3)}_- \widetilde{J}^{(2)}_{+\alpha} + \widetilde{J}^{(2)}_{+\alpha} i\kappa^{\alpha(3)}_- \,. \tag{3.32}$$

Note that these forms are compatible to the grading assignment. Then one can show the relation

$${\rm Str} \left(\rho^{(1)} \left[J^{(2)}_{-\alpha}, \widetilde{J}^{\alpha(1)}_+ \right] \right) = {\rm Str} \left(J^{(2)}_{-\alpha} J^{(2)}_{-\beta} \left[\widetilde{J}^{\alpha(1)}_+, i\kappa^{\beta(1)}_+ \right] \right) . \tag{3.33}$$

The derivation is the following,

$$\begin{aligned}
{\rm Str} \left(\rho^{(1)} \left[J^{(2)}_{-\alpha}, \widetilde{J}^{\alpha(1)}_+ \right] \right) &= {\rm Str} \Big[\Big(i\kappa^{\tau(1)}_+ J^{(2)}_{-\tau} + J^{(2)}_{-\tau} i\kappa^{\tau(1)}_+ + i\kappa^{\sigma(1)}_+ J^{(2)}_{-\sigma} + J^{(2)}_{-\sigma} i\kappa^{\sigma(1)}_+ \Big) \\
&\qquad \times \Big(J^{(2)}_{-\tau} \widetilde{J}^{\tau(1)}_+ - \widetilde{J}^{\tau(1)}_+ J^{(2)}_{-\tau} + J^{(2)}_{-\sigma} \widetilde{J}^{\sigma(1)}_+ - \widetilde{J}^{\sigma(1)}_+ J^{(2)}_{-\sigma} \Big) \Big] \\
&= {\rm Str} \Big[J^{(2)}_{-\tau} J^{(2)}_{-\tau} \Big(\widetilde{J}^{\tau(1)}_+ i\kappa^{\tau(1)}_+ - i\kappa^{\tau(1)}_+ \widetilde{J}^{\tau(1)}_+ \Big) \\
&\qquad + J^{(2)}_{-\tau} J^{(2)}_{-\sigma} \Big(\widetilde{J}^{\tau(1)}_+ i\kappa^{\sigma(1)}_+ - i\kappa^{\sigma(1)}_+ \widetilde{J}^{\tau(1)}_+ \Big) \\
&\qquad + J^{(2)}_{-\sigma} J^{(2)}_{-\tau} \Big(\widetilde{J}^{\sigma(1)}_+ i\kappa^{\tau(1)}_+ - i\kappa^{\tau(1)}_+ \widetilde{J}^{\sigma(1)}_+ \Big) \\
&\qquad + J^{(2)}_{-\sigma} J^{(2)}_{-\sigma} \Big(\widetilde{J}^{\sigma(1)}_+ i\kappa^{\sigma(1)}_+ - i\kappa^{\sigma(1)}_+ \widetilde{J}^{\sigma(1)}_+ \Big) \Big] \\
&= {\rm Str} \left(J^{(2)}_{-\alpha} J^{(2)}_{-\beta} \left[\widetilde{J}^{\alpha(1)}_+, i\kappa^{\beta(1)}_+ \right] \right) .
\end{aligned}$$

The second equality comes from the fact that $J^{(2)}_{-\tau}$ is proportional to $J^{(2)}_{-\sigma}$. Similarly, one can show the relation,

$${\rm Str} \left(\rho^{(3)} \left[\widetilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_- \right] \right) = {\rm Str} \left(\widetilde{J}^{(2)}_{+\alpha} \widetilde{J}^{(2)}_{+\beta} \left[J^{\alpha(3)}_-, i\kappa^{\beta(3)}_- \right] \right) . \tag{3.34}$$

Furthermore, for any grade 2 traceless matrix $A^{(2)}_{\pm\alpha}$, the following relation is satisfied [42],

$$A^{(2)}_{\pm\alpha} A^{(2)}_{\pm\beta} = \frac{1}{8} {\rm Str} \left(A^{(2)}_{\pm\alpha} A^{(2)}_{\pm\beta} \right) \Upsilon + c_{\alpha\beta} \mathbf{1}_8 \,, \tag{3.35}$$

by using the matrix representation, where Υ is the following 8×8 matrix:

$$\Upsilon = \mathrm{diag}(\mathbf{1}_4, -\mathbf{1}_4)\,. \tag{3.36}$$

Thus the following relations are obtained,

$$\mathrm{Str}\left(\rho^{(1)}\left[J^{(2)}_{-\alpha}, \widetilde{J}^{\alpha(1)}_{+}\right]\right) = \frac{1}{8}\mathrm{Str}\left(J^{(2)}_{-\alpha}J^{(2)}_{-\beta}\right)\mathrm{Str}\left(\Upsilon\left[\widetilde{J}^{\alpha(1)}_{+}, i\kappa^{\beta(1)}_{+}\right]\right)\,, \tag{3.37}$$

$$\mathrm{Str}\left(\rho^{(3)}\left[\widetilde{J}^{(2)}_{+\alpha}, J^{\alpha(3)}_{-}\right]\right) = \frac{1}{8}\mathrm{Str}\left(\widetilde{J}^{(2)}_{+\alpha}\widetilde{J}^{(2)}_{+\beta}\right)\mathrm{Str}\left(\Upsilon\left[J^{\alpha(3)}_{-}, i\kappa^{\beta(3)}_{-}\right]\right)\,. \tag{3.38}$$

With the relations (3.37) and (3.38), the variation of the classical action (3.16) under the transformation (3.29) is evaluated as

$$\begin{aligned}\delta_g S = -\frac{1}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\mathrm{Str}\Big(&\mathrm{Str}\left(J^{(2)}_{-\alpha}J^{(2)}_{-\beta}\right)\Upsilon\left[\widetilde{J}^{\alpha(1)}_{+}, i\kappa^{\beta(1)}_{+}\right]\\ &+\mathrm{Str}\left(\widetilde{J}^{(2)}_{+\alpha}\widetilde{J}^{(2)}_{+\beta}\right)\Upsilon\left[J^{\alpha(3)}_{-}, i\kappa^{\beta(3)}_{-}\right]\Big)\,.\end{aligned} \tag{3.39}$$

Then we will show that this variation is canceled out with the variation of the action with respect to the world-sheet metric $\gamma^{\alpha\beta}$. Let the variation of $\gamma^{\alpha\beta}$ be

$$\begin{aligned}\delta\gamma^{\alpha\beta} = -\frac{1}{4}\mathrm{Str}\Big(&\Upsilon\left[\widetilde{J}^{\alpha(1)}_{+}, i\kappa^{\beta(1)}_{+}\right] + \Upsilon\left[\widetilde{J}^{\beta(1)}_{+}, i\kappa^{\alpha(1)}_{+}\right]\\ &+\Upsilon\left[J^{\alpha(3)}_{-}, i\kappa^{\beta(3)}_{-}\right] + \Upsilon\left[J^{\beta(3)}_{-}, i\kappa^{\alpha(3)}_{-}\right]\Big)\,.\end{aligned} \tag{3.40}$$

Then, by using the expressions of the classical action given in (3.20) and (3.21), the variation of the action is evaluated as

$$\begin{aligned}\delta_\gamma S &= \frac{1}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\Big[\mathrm{Str}\left(\Upsilon\left[\widetilde{J}^{\alpha(1)}_{+}, i\kappa^{\beta(1)}_{+}\right]\right)\mathrm{Str}\left(J^{(2)}_{\alpha}J^{(2)}_{\beta}\right)\\ &\qquad+\mathrm{Str}\left(\Upsilon\left[J^{\alpha(3)}_{-}, i\kappa^{\beta(3)}_{-}\right]\right)\mathrm{Str}\left(\widetilde{J}^{(2)}_{\alpha}\widetilde{J}^{(2)}_{\beta}\right)\Big]\\ &= \frac{1}{4}\int_{-\infty}^{\infty}d\tau\int_0^{2\pi}d\sigma\,\Big[\mathrm{Str}\left(\Upsilon\left[\widetilde{J}^{\alpha(1)}_{+}, i\kappa^{\beta(1)}_{+}\right]\right)\mathrm{Str}\left(J^{(2)}_{-\alpha}J^{(2)}_{-\beta}\right)\\ &\qquad+\mathrm{Str}\left(\Upsilon\left[J^{\alpha(3)}_{-}, i\kappa^{\beta(3)}_{-}\right]\right)\mathrm{Str}\left(\widetilde{J}^{(2)}_{+\alpha}\widetilde{J}^{(2)}_{+\beta}\right)\Big]\,.\end{aligned} \tag{3.41}$$

In order to show the second equality, the following relations have been used,

$$A^{\alpha}_{\pm}B_{\alpha} = A^{\alpha}_{\pm}B_{\pm\alpha} + A^{\alpha}_{\pm}B_{\mp\alpha} = A^{\alpha}_{\pm}B_{\mp\alpha}\,. \tag{3.42}$$

Thus, the total variation of the classical action (3.16) becomes zero,

$$\delta_g S + \delta_\gamma S = 0\,. \tag{3.43}$$

That is, the action (3.16) is invariant under the κ-transformation.

4 Conclusion and Discussion

We have discussed Jordanian deformations of the $AdS_5 \times S^5$ superstring action. The description to construct Jordanian R operators via twists has been explained in detail. Notably the Jordanian R operators satisfy CYBE rather than mCYBE and they have non-vanishing Cartan charge. Then we have constructed the Jordanian deformed action that preserves the κ-symmetry. The Lax pair has also been presented.

It should be remarked that partial deformations are possible in our procedure. This fact implies that one may perform a deformation only for the S^5 part, for example. Then the background geometry would be much simpler because the AdS_5 part is not modified and the gauge-theory dual would be identified with a deformation of the scalar sector such as Leigh-Strassler deformations [47]. A promising way is to consider a twist of the q-deformation of the $SO(6)$ sector argued in [48, 49]. As a matter of course, even for the maximal twist, the metric of the twisted geometry can be determined, for example, by following [35]. For the background geometries, we will report on the result in the near future [43].

In principle, it should be possible to classify all of the skew-symmetric classical r-matrices of $\mathfrak{gl}(4|4)$ and its real form $\mathfrak{su}(2,2|4)$. The classification enables us to reveal all of the possible deformations of the $AdS_5 \times S^5$ superstring from the algebraic point of view.

We believe that the study of integrable deformations of the $AdS_5 \times S^5$ superstring would shed light on new aspects of the integrable structure behind the AdS/CFT correspondence.

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Appendix

A Notations of superconformal generators

In this paper, we work with the $\mathfrak{gl}(4|4)$ generators rather than $\mathfrak{u}(2,2|4)$ generators because the former generators are more convenient for the algebraic argument. The superconformal algebra $\mathfrak{u}(2,2|4)$ is obtained from $\mathfrak{gl}(4|4)$ by imposing a suitable condition. Thus, we will spell out the explicit relations among the generators. This is enough for our purpose.

The Lie superalgebra $\mathfrak{gl}(4|4)$ is a 32|32 dimensional algebra and generated by E_{ij} with $i,j=1,\cdots,8$ satisfying the relations[7],

$$[E_{ij},E_{kl}]=\delta_{kj}E_{il}-\delta_{il}E_{kj}(-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\,. \tag{A.1}$$

Here the parity of indices are defined as $\bar{i}=0$ for $i=1,\cdots,4$ and $\bar{i}=1$ for $i=5,\cdots,8$.

The invariant super-symmetric non-degenerate linear form is defined as

$$\langle E_{ij},E_{kl}\rangle=\delta_{kj}\delta_{il}(-1)^{\bar{j}}\,, \tag{A.2}$$

with $i,j,k,l=1,\cdots,8$, which satisfies the following properties

$$\begin{aligned}&\langle E_{ij},E_{kl}\rangle=\langle E_{kl},E_{ij}\rangle(-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\,,\\ &\langle E_{ij},E_{kl}\rangle=0 \qquad \text{for} \quad \bar{i}+\bar{j}\neq\bar{k}+\bar{l}\,.\end{aligned} \tag{A.3}$$

The bosonic part of the superconformal algebra is related to $\mathfrak{gl}(4|4)$ generators as

$$\begin{aligned}&L_{\alpha\beta}=E_{\alpha\beta}-\tfrac{1}{2}\delta_{\alpha\beta}E_{\lambda\lambda}\,, && D=\tfrac{1}{2}(E_{\lambda\lambda}-E_{\dot\lambda\dot\lambda})\,, && P_{\alpha\dot\beta}=E_{\alpha\dot\beta}\,,\\ &\bar{L}_{\dot\alpha\dot\beta}=E_{\dot\alpha\dot\beta}-\tfrac{1}{2}\delta_{\dot\alpha\dot\beta}E_{\dot\lambda\dot\lambda}\,, && C=\tfrac{1}{2}(E_{\lambda\lambda}+E_{\dot\lambda\dot\lambda}+E_{ll})\,, && K_{\dot\alpha\beta}=E_{\dot\alpha\beta}\,,\\ &R_{ab}=E_{ab}-\tfrac{1}{4}\delta_{ab}E_{ll}\,, && B=-\tfrac{1}{2}E_{ll}\,,\end{aligned} \tag{A.4}$$

where $\alpha,\beta,\lambda=1,2,\ \dot\alpha,\dot\beta,\dot\lambda=3,4$ and $a,b,l=5,\cdots,8$. The conformal algebra $\mathfrak{su}(2,2)$ contains two $\mathfrak{su}(2)$ subalgebras generated by $L_{\alpha\beta}$ and $\bar{L}_{\dot\alpha\dot\beta}$ as well as the translations $P_{\alpha\dot\beta}$

[7] The commutator is assumed to be supercommutator here and also in (2.1), (3.1), (A.1) and (B.9). The other commutators are not graded in constructing the action of the $AdS_5\times S^5$ superstring, because we consider a Grassmann envelope of the superalgebra by following [34].

and the conformal boosts $K_{\dot{\alpha}\beta}$. The R-symmetry $\mathfrak{su}(4)$ is generated by R_{ab}. The diagonal generators D, C, B are dilatation, central charge and hyper charge, respectively. The supertranslations $Q_{\alpha b}, \bar{Q}_{a\dot{\beta}}$ and superconformal boosts $S_{a\beta}, \bar{S}_{\dot{\alpha}b}$ are given by

$$Q_{\alpha b} = E_{\alpha b} \,, \qquad \bar{Q}_{a\dot{\beta}} = E_{a\dot{\beta}} \,, \qquad S_{a\beta} = E_{a\beta} \,, \qquad \bar{S}_{\dot{\alpha}b} = E_{\dot{\alpha}b} \,. \tag{A.5}$$

B Constant classical R-matrix

We summarize here the notation of the classical R-matrix, which is independent of the spectral parameter (For example, see [50]).

B.1 Classical Yang-Baxter equation

Let $\mathfrak{g}$ be a bosonic Lie algebra over $\mathbb{C}$. For $a_i, b_i \in \mathfrak{g}$, an element denoted by

$$r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g} \tag{B.1}$$

is called *classical r-matrix* if it satisfies the *classical Yang-Baxter equation (CYBE)*;

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0 \,, \tag{B.2}$$

where the action of r_{ij} is extended to three sites $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ such as

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1 \qquad r_{23} = \sum_i 1 \otimes a_i \otimes b_i \qquad r_{13} = \sum_i a_i \otimes 1 \otimes b_i \,. \tag{B.3}$$

Suppose that there exists the invariant non-degenerate symmetric bilinear form $\langle \ , \ \rangle$ on $\mathfrak{g}$. With the bilinear form, the linear operator $R : \mathfrak{g} \to \mathfrak{g}$ can be introduced though the following relation;

$$R(X) = \langle r, 1 \otimes X \rangle = \sum_i a_i \langle b_i, X \rangle \in \mathfrak{g} \tag{B.4}$$

for any $X \in \mathfrak{g}$. This operator R is also referred as to the classical R-matrix. With this notation, CYBE (B.2) is equivalent to

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0 \,, \tag{B.5}$$

if and only if the r-matrix is skew-symmetric;

$$r_{21} = \sum_i b_i \otimes a_i = -r\,. \tag{B.6}$$

Indeed, noting the following relations for any $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
[R(X), R(Y)] &= \langle [r_{12}, r_{13}], 1 \otimes X \otimes Y \rangle\,, \\
-R([R(X), Y]) &= \langle [r_{13}, -r_{32}], 1 \otimes X \otimes Y \rangle\,, \\
-R([X, R(Y)] &= \langle [r_{12}, r_{23}], 1 \otimes X \otimes Y \rangle\,,
\end{aligned} \tag{B.7}$$

one can see that the relation (B.5) is nothing but (B.2) if R is skew-symmetric.

Here it is worth mentioning the generalization of CYBE (B.5) such as

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = -c^2[X, Y]\,, \tag{B.8}$$

for any $X, Y \in \mathfrak{g}$ with a complex parameter $c \in \mathbb{C}$. The relation (B.8) is called *the modified classical Yang-Baxter equation* (mCYBE). The standard examples of the classical r-matrix (or R-matrix) satisfy CYBE (B.2) (or (B.5)), while (twice of) the skew-symmetric parts of them satisfy mCYBE (B.8).

B.2 Skew-symmetric r-matrix for $\mathfrak{gl}(M|N)$

Let us summarize typical constant r-matrices for the Lie superalgebra $\mathfrak{gl}(M|N)$. The Lie superalgebra $\mathfrak{gl}(M|N)$ is $(M+N)^2$-dimensional algebra over $\mathbb{C}$ and generated by E_{ij} with $i, j = 1, \cdots, M+N$ satisfying the relations;

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\,. \tag{B.9}$$

Here the parity of indices are defined as $\bar{i} = 0$ for $i = 1, \cdots, M$ and $\bar{i} = 1$ for $i = M + 1, \cdots, N+M$.

There are three typical solutions of (m)CYBE. The first one is the trivial solution $r = 0$ for CYBE. The second one is the classical r-matrix $r_{\rm DJ}$ of Drinfeld-Jimbo type [31–33]

$$r_{\rm DJ} = c \sum_{1 \le i < j \le M+N} E_{ij} \wedge E_{ji} (-1)^{\bar{i}\bar{j}}\,. \tag{B.10}$$

This is a solution of mCYBE.

The third solution is the non-standard classical r-matrix $r_{\rm tw}^{(i,i)}$, which are obtained by twisting $r_{\rm DJ}$ with a root generator E_{ij} .

The twists by the bosonic roots $E_{\alpha\beta}$ and E_{ab} with $\alpha < \beta$ and $a < b$ ($\alpha, \beta = 1, \cdots, M$ and $a, b = M+1, \cdots, N+M$) are given by

$$r_{\rm tw}^{(\alpha,\beta)} \equiv [\Delta(E_{\alpha\beta}), r_{\rm DJ}] = c\left[(-E_{\alpha\alpha} + E_{\beta\beta}) \wedge E_{\alpha\beta} - 2\sum_{\kappa=\alpha+1}^{\beta-1} E_{\alpha\kappa} \wedge E_{\kappa\beta}\right],$$

$$r_{\rm tw}^{(a,b)} \equiv [\Delta(E_{ab}), r_{\rm DJ}] = c\left[(E_{aa} - E_{bb}) \wedge E_{ab} + 2\sum_{k=a+1}^{b-1} E_{ak} \wedge E_{kb}\right],$$

where the coproduct is defined in (3.1). These are solutions of CYBE rather than mCYBE. We will call them the bosonic twists.

The fermionic root $E_{\alpha,b}$ gives rise to

$$r_{\rm tw}^{(\alpha,b)} \equiv [\Delta(E_{\alpha b}), r_{\rm DJ}] = c\left[(E_{\alpha\alpha} + E_{bb}) \wedge E_{\alpha b} + 2\sum_{\kappa=1}^{\alpha-1} E_{\alpha\kappa} \wedge E_{\kappa b} - 2\sum_{k=a+1}^{M+N} E_{\alpha k} \wedge E_{kb}\right],$$

where c is a Grassmann odd parameter rather than a complex number, so that the r-matrix should be Grassmann even [38]. We will refer the twists as to the fermionic twists. However it does not seem to be a solution of (m)CYBE.

An interesting example of the fermionic twist is given by $E_{1,M+N}$ as follows:

$$r_{\rm tw}^{(1,M+N)} = [\Delta(E_{1,M+N}), r_{\rm DJ}] = -cE_{1,M+N} \wedge (E_{11} + E_{M+N,M+N}) \,. \tag{B.11}$$

This is a solution of CYBE. When $M = N = 4$, it reduces to (3.15) .

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