CONNECTIVITY OF COMPLEXES OF SEPARATING CURVES

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In memory of Fritz Grünewald (1949-2010)

ABSTRACT. We prove that the separating curve complex of a closed orientable surface of genus g is $(g-3)$-connected. We also obtain a connectivity property for a separating curve complex of the open surface that is obtained by removing a finite set from a closed one, where it is assumed that the removed set is endowed with a partition and that the separating curves respect that partition. These connectivity statements have implications for the algebraic topology of the moduli space of curves.

1. STATEMENTS OF THE RESULTS

Let S be a connected oriented surface of genus g with finite first Betti number $2g+n$ (i.e., a closed surface with n punctures) and make the customary assumption that S has negative Euler characteristic: if $g=0$, then $n\geq 3$ and if $g=1$, then $n\geq 1$. We recall that the *curve complex* $\mathcal{C}(S)$ of S is the simplicial complex whose vertex set consists of the isotopy classes of embedded (unoriented) circles in S which do not bound in S a disk or a cylinder. A finite set of vertices spans a simplex precisely when its elements can be represented by embedded circles that are pairwise disjoint. Thus, a closed 1-dimensional submanifold A of S with $k+1$ connected components such that every connected component of its complement has negative Euler characteristic defines a k-simplex σ_A of $\mathcal{C}(S)$ and every simplex of $\mathcal{C}(S)$ is thus obtained.

This complex has proven to be quite useful in the study of the mapping class group of S. For the purposes of studying the Torelli group of S a subcomplex $\mathcal{C}_{\mathrm{sep}}(S)$ of $\mathcal{C}(S)$ can render a similar service. It is defined as the full subcomplex of $\mathcal{C}(S)$ spanned by the separating vertices of $\mathcal{C}(S)$, where a vertex is called *separating* if a representative embedded circle separates S into two components. Our main result for the case when S is closed is contained in the following theorem.

Theorem 1.1 (A_g)**.** *If $n\leq 1$, then the simplicial complex $\mathcal{C}_{\mathrm{sep}}(S)$ is $(g-3)$-connected.*

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Previous work on this topic that we are aware of concerns the case $n = 0$. Farb and Ivanov announced in 2005 [1, Thm. 4] that $\mathcal{C}_{\mathrm{sep}}(S)$ is connected for $g \geq 3$. Putman gave in [5, Thm. 1.4] another proof of this and showed that $\mathcal{C}_{\mathrm{sep}}(S)$ is simply connected for $g \geq 4$ (*op. cit.*, Thm. 1.11). In that paper he also mentions that Hatcher and Vogtmann have proved that $\mathcal{C}_{\mathrm{sep}}(S)$ is $\lfloor \frac{1}{2}(g-3) \rfloor$-connected for all g (unpublished).

Remark 1.2. Possibly the connectivity bound in Theorem 1.1 is the best possible for every positive genus. In a paper with Van der Kallen [3] we showed that the quotient of $\mathcal{C}_{\mathrm{sep}}(S)$ by the action of the Torelli group of S has the homotopy type of a bouquet of $(g-2)$-spheres. In the situation of the theorem, $\mathcal{C}_{\mathrm{sep}}(S)$ has dimension $2g-4+n$ so that its connectivity is half the dimension (as $n \leq 1$). In particular, we cannot conclude that $\mathcal{C}_{\mathrm{sep}}(S)$ is spherical.

Before we state a version for the case $n \geq 2$, we point out a consequence that pertains to the moduli space of curves. Consider the Teichmüller space $\mathcal{T}(S)$ of S, a contractible manifold on which acts the mapping class group $\Gamma(S)$. The action is proper and the orbit space may be identified with the moduli space $\mathcal{M}_g$ of curves of genus g. The *Harvey bordification* of $\mathcal{T}(S)$, here denoted by $\mathcal{T}(S)^+ \supset \mathcal{T}(S)$, is a (noncompact) manifold with boundary with corners to which the action of $\Gamma(S)$ naturally extends. This action is also proper and according to [4] the orbit space $\mathcal{M}_g^+ := \Gamma(S)\backslash\mathcal{T}(S)^+$ is a compactification of $\mathcal{M}_g$ that can also be obtained from the Deligne-Mumford compactification $\overline{\mathcal{M}}_g \supset \mathcal{M}_g$ as a 'real oriented blowup' of its boundary $\Delta_g := \overline{\mathcal{M}}_g - \mathcal{M}_g$. The walls of $\mathcal{T}(S)^+$ define a closed covering of the boundary $\partial\mathcal{T}(S)^+$ and any nonempty corner closure is an intersection of walls. As is well-known, the curve complex $\mathcal{C}(S)$ can be identified with the nerve of this covering of $\partial\mathcal{T}(S)^+$. Since the corner closures are contractible, Weil's nerve theorem implies that $\partial\mathcal{T}(S)^+$ has the same homotopy type as $\mathcal{C}(S)$.

Let $\Delta_{g,0} \subset \Delta_g$ denote the irreducible component of the Deligne-Mumford boundary whose generic point parameterizes irreducible curves with one singular point. We may understand $\mathcal{M}_g^c := \overline{\mathcal{M}}_g - \Delta_{g,0}$ as the moduli space of stable genus g curves all of whose nodes are separating (which is equivalent to the irreducible components of the curve being smooth and with their genera summing up to g) and $\Delta_g^c := \Delta_g - \Delta_{g,0}$ as the locus in $\mathcal{M}_g^c$ that parameterizes the singular ones among them. If Γ is a subgroup of $\Gamma(S)$ with the property that every Dehn twist along a separating curve in S has a positive power lying in Γ, then this defines a (not necessarily finite) cover $\tilde{\mathcal{M}}_g^c \to \mathcal{M}_g^c$.

Corollary 1.3. *Suppose $\Gamma \subset \Gamma(S)$ is as above and is in addition torsion free. If we denote by $\tilde{\Delta}_g^c \subset \tilde{\mathcal{M}}_g^c$ the preimage of Δ_g^c, then the pair $(\tilde{\mathcal{M}}_g^c, \tilde{\Delta}_g^c)$ is $(g-2)$-connected. Moreover, $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ for $k \leq g-2$.*

Proof. Let $\mathcal{T}(S)^+_{\mathrm{sep}}$ be obtained from $\mathcal{T}(S)^+$ by removing the walls that correspond to the nonseparating vertices of $\mathcal{C}(S)$. Then $\mathcal{T}(S)^+_{\mathrm{sep}}$ is the preimage of $\mathcal{M}_g^c$ in $\mathcal{T}(S)^+$. The same reasoning as above shows that $\partial\mathcal{T}(S)^+_{\mathrm{sep}}$

is homotopy equivalent to $\mathcal{C}(S)_{sep}$ and so $\partial\mathcal{T}(S)^+_{sep}$ is $(g-3)$-connected. It follows that we can construct a relative CW complex $(Z, \partial\mathcal{T}(S)^+_{sep})$ obtained from $\partial\mathcal{T}(S)^+_{sep}$ by attaching cells of dimension $\geq g-1$ in a $\Gamma(S)$-equivariant manner as to ensure that Z is contractible and no nontrivial element of $\Gamma(S)$ fixes a cell. Then Γ acts freely on Z (as it does on the contractible space $\mathcal{T}(S)^+_{sep}$) and so there is a Γ-equivariant homotopy equivalence $Z \to \mathcal{T}(S)^+_{sep}$ relative to $\partial\mathcal{T}(S)^+_{sep}$. It follows that we also have a homotopy equivalence $\Gamma\backslash Z \to \tilde{\mathcal{M}}^c_g$ relative to $\tilde{\Delta}^c_g$ and we conclude that $(\tilde{\mathcal{M}}^c_g, \tilde{\Delta}^c_g)$ is $(g-2)$-connected.

The last assertion follows from the existence of a normal subgroup $\Gamma \subset \Gamma(S)$ of finite index that is torsion free. For if Γ is such a group, then $H_k(\mathcal{M}^c_g, \Delta^c_g; \mathbb{Q}) \cong H_k(\tilde{\mathcal{M}}^c_g, \tilde{\Delta}^c_g; \mathbb{Q})^{\Gamma(S)/\Gamma} = 0$ for $k \leq g-2$. □

A similar statement holds for the universal curve $\mathcal{M}_{g,1}$.

When $n > 1$, we need to come to terms with the fact that the separability notion has no good heriditary properties: if T is a closed surface, $A \subset T$ a compact 1-dimensional submanifold representing a simplex of $\mathcal{C}(T)$ and S a connected component of $T - A$, then a vertex of $\mathcal{C}(S)$ may split S, but not T. This happens precisely when the vertex in question separates two boundary components of ∂S that lie on the same connected component of $T-S$. So the basic object should be, what Andy Putman calls in [6], a partitioned surface: a closed surface minus a finite set, for which the removed set comes with a partition. This leads to the following definition.

Definition 1.4. Let N be the set of points of S at infinity (the cusps) and let P be a partition of N. We call a vertex of $\mathcal{C}(S)$ *separating relative to* P if a representative embedded circle $\alpha \subset S$ has the property that $S - \alpha$ has two connected components each of which meets N in a union of parts of P. We denote by $\mathcal{C}(S, P)$ the full subcomplex of $\mathcal{C}(S)$ spanned by such vertices.

One might also understand $\mathcal{C}(S, P)$ as the full subcomplex of $\mathcal{C}(S)$ spanned by the isotopy classes of embedded cycles which are separating on the surface S^P that is obtained by capping off for each part of P the corresponding set of cusps by a sphere with that many holes. Notice that $\mathcal{C}(S, P) \subset \mathcal{C}_{sep}(S)$ and that we have equality when P is discrete or N is empty.

We shall prove Theorem 1.1 by induction and simultaneously with

Theorem 1.5 $(A_{g,n})$**.** *Suppose* $g > 0$ *and* $n = |N| > 1$. *Let* P *be a partition of* N. *Then* $\mathcal{C}(S, P)$ *is* $(g-2)$*-connected.*

To be precise, the induction starts with $g = 0$, where the statements (A_g) and $(A_{0,n})$ are trivially true and the induction strategy will be to show that

(i) $(A_{h,n})$ for $h < g$ implies (A_g) and
(ii) (A_g) and $(A_{h,k})$ for $(h, k) < (g, n)$ (for the lexicographic ordering) imply $(A_{g,n})$.

I am indebted to Allen Hatcher for pointing out that the stronger version of Theorem 1.5 that I stated in a previous version was incorrect. Yet it may be that some such statement might hold. For instance, if $r(P)$ denotes the number of nonempty parts of P and $s(P)$ the number of parts with at least two elements, is it true that $\mathcal{C}(S,P)$ is $(g+r(P)+s(P)-4)$-connected when $g>0$ (as I claimed in the earlier version)? In case $g=0$, $\mathcal{C}(S,P)$ is a complex of dimension $r(P)+s(P)-4$. Is this $(r(P)+s(P)-5)$-connected? In other words, is this complex spherical?

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2. Proofs

Before we start off, we mention the following elementary fact that we will frequently use.

Lemma 2.1. *Let X_i be a d_i-connected space ($d_i=-1$ means $X_i\neq\emptyset$), where $i=1,\dots,k$. Then the iterated join $X_1*\cdots*X_k$ is $(-2+\sum_{i=1}^k(d_i+2))$-connected.*

Proof that ($A_{h,n}$) for $h<g$, all n, implies (A_g). So here $n\le 1$. We must show that $\mathcal{C}_{sep}(S)$ is $(g-3)$-connected. For $g<2$, there is nothing to show and so we may assume that $g\geq 2$. A theorem of Harer [2, Thm. 1.2] asserts that $\mathcal{C}(S)$ is $(2g-3)$-connected. So it is certainly $(g-3)$-connected. Let $\mathcal{C}_k$ be the subcomplex of $\mathcal{C}(S)$ that is the union of $\mathcal{C}_{sep}(S)$ and the k-skeleton of $\mathcal{C}(S)$. So $\mathcal{C}_{-1}=\mathcal{C}_{sep}(S)$ and $\mathcal{C}_k=\mathcal{C}(S)$ for k large. Notice that a finite set of vertices of $\mathcal{C}(S)$ spans a simplex of $\mathcal{C}_k$ if and only if no more than $k+1$ of these are nonseparating. Hence a minimal simplex of $\mathcal{C}_k-\mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A\subset S$ with $k+1$ connected components, each of which is nonseparating. We prove that the boundary of the star of such a simplex in $\mathcal{C}_k$ is a $(g-3)$-connected subcomplex of $\mathcal{C}_{k-1}$. This property implies that the pair $(\mathcal{C}(S),\mathcal{C}_{sep}(S))$ is $(g-2)$-connected. Since $\mathcal{C}(S)$ is $(g-3)$-connected, it then follows that $\mathcal{C}_{sep}(S)$ is. Let $\{S_i\}_{i\in I}$ be the set of connected components of $S-A$. Notice that if g_i is the genus of S_i, then $g_i<g$. An Euler characteristic argument shows that

$$g-1=k+1+\sum_{i\in I}(g_i-1).$$

We denote by N_i the set of 'cusps' of S_i, i.e., the finitely many points needed to make S_i a closed surface. So an element of N_i is given by possibly a cusp of S (if it exists and if it is also a cusp of S_i) or by a connected component of A in the boundary of S_i *endowed with the orientation it receives as such*. The set N_i comes with an evident partition P_i: if S has a cusp and N_i contains it, then this cusp makes up a singleton part of P_i and any other two elements of N_i belong to the same part of P_i if and only if they come from connected components of A that lie on the same connected component of $S-S_i$. (NB:

beware that a connected component of $S - S_i$ could be simply a connected component A_o of A; then its two orientations define a 2-element part of P_i.) connected components of A that bound S_i. Note that since the connected components of A are nonseparating, we always have $|N_i| \geq 2$. By our induction hypothesis $\mathcal{C}(S_i, P_i)$ is then $(g_i - 2)$-connected. The boundary of the star of the k-simplex σ_A defined by A in $\mathcal{C}_k$ lies in $\mathcal{C}_{k-1}$ and can be identified with the $(|I| + 1)$-fold join

$$\partial\sigma_A * \big(*_{i\in I}\mathcal{C}(S_i, P_i)\big).$$

Since $\partial\sigma_A$ is a combinatorial $(k-1)$-sphere, this join has by Lemma 2.1 connectivity at least $-2 + k + \sum_{i\in I} g_i$. By the displayed formula above this is equal to $g - 4 + |I|$ and is therefore $\geq g - 3$. □

The proof of $A_{g,n}$ begins with a discussion. We now assume that $g > 0$ and $n \geq 2$. Let $x \in N$. Notice that $S' := S \cup \{x\}$ has still negative Euler characteristic. We put $N' := N - \{x\}$ and $P' := P|N'$. The goal is to compare $\mathcal{C}(S', P')$ with $\mathcal{C}(S, P)$. There is in general no forgetful map $\mathcal{C}(S, P) \to \mathcal{C}(S', P')$ because there will be vertices of $\mathcal{C}(S, P)$ that do not give vertices of $\mathcal{C}(S', P')$. Let us first identify this set of vertices.

Denote by $\Sigma_x \subset N - \{x\}$ the set of $y \in N - \{x\}$ for which $\{x, y\}$ is a union of parts of P. In other words, if P_x denotes the part of P that contains x, then Σ_x is empty if P_x has more than 2 elements, equals $P_x - \{x\}$ if P_x is a 2-element set, and equals the set of $y \neq x$ for which P_y is a singleton in case $P_x = \{x\}$. Then the vertices of $\mathcal{C}(S, P)$ that have no image in $\mathcal{C}(S', P')$ are precisely the vertices α of $\mathcal{C}_{sep}(S)$ which for some $y \in \Sigma_x$ bound a disk neighborhood of $\{x, y\}$ in $S \cup \{x, y\}$. Such a disk neighborhood can be thought of as a regular neighborhood of an arc in $S \cup \{x, y\}$ connecting the two added cusps; this may help to explain why we have chosen to denote this set of vertices by $\mathrm{arc}_{(S,P)}(x)$. Denote by $\mathcal{C}(S, P)_x$ the full subcomplex of $\mathcal{C}(S, P)$ spanned by the vertices not in $\mathrm{arc}_{(S,P)}(x)$.

Observe that $\mathrm{arc}_{(S,P)}(x)$ is empty (so that $\mathcal{C}(S, P)_x = \mathcal{C}(S, P)$) if Σ_x is.

Lemma 2.2. *The link in* $\mathcal{C}(S, P)$ *of every vertex of* $\mathrm{arc}_{(S,P)}(x)$ *is a subcomplex of* $\mathcal{C}(S, P)_x$ *that projects isomorphically onto* $\mathcal{C}(S', P')$.

Proof. A vertex of $\mathrm{arc}_{(S,P)}(x)$ defines a $y \in \Sigma_x$ and (up to isotopy) a closed disk D in $S \cup \{x, y\}$ that is a neighborhood of $\{x, y\}$. The inclusion $S - D \subset S'$ identifies the link in question with $\mathcal{C}(S', P')$. □

Denote by $\tilde{P}$ the refinement of P which coincides with P on $N - P_x$ and partitions P_x further into $\{x\}$ and $P_x - \{x\}$. So $\tilde{P}' = P'$. It is clear that $\mathcal{C}(S, P)$ is a subcomplex of $\mathcal{C}(S, \tilde{P})$. Notice that $\mathrm{arc}_{(S,P)}(x) = \mathcal{C}(S, P) \cap \mathrm{arc}_{(S,\tilde{P})}(x)$ (we have $\mathrm{arc}_{(S,P)}(x) = \mathrm{arc}_{(S,\tilde{P})}(x)$ unless $|P_x| = 2$) and $\mathcal{C}(S, P)_x = \mathcal{C}(S, P) \cap \mathcal{C}(S, \tilde{P})_x$. We denote by f the forgetful simplicial map $\mathcal{C}(S, \tilde{P})_x \to \mathcal{C}(S', P')$ so that we have the diagram

$$\begin{array}{ccccc} C(S,P)_x & \subset & C(S,\tilde{P})_x & \xrightarrow{f} & C(S',P'). \\ \cap & & \cap & & \\ C(S,P) & \subset & C(S,\tilde{P}) & & \end{array}$$

Lemma 2.3. *The map* $f : \mathcal{C}(S,\tilde{P})_x \to \mathcal{C}(S',P')$ *is a homotopy equivalence.*

Proof. Choose an arc γ which connects x with another point of N and defines a vertex of $\mathrm{arc}_{(S,P)}(x)$ and observe that the full subcomplex $K \subset \mathcal{C}(S,\tilde{P})_x$ spanned by vertices that avoid γ defines a section of f (the inclusion $S-\gamma \subset S'$ is isotopic to a homeomorphism). We shall prove that $\mathcal{C}(S,\tilde{P})_x$ admits K as a deformation retract. (The proof will in fact show that each fiber of $|f|$ is a tree and essentially produces for every element of $|\mathcal{C}(S,\tilde{P})_x|$ the unique path in its $|f|$-fibre that connects it to the point of $|K|$.)

Denote by $K_r \subset \mathcal{C}(S,\tilde{P})_x$ the subcomplex whose simplices can be represented by a closed submanifold $A \subset S$ which meets γ transversally in at most r points. This defines a filtration $K = K_0 \subset K_1 \subset K_2 \subset \cdots$ whose union is $\mathcal{C}(S,\tilde{P})_x$. Although this filtration is infinite, it is enough to construct for every $r \geq 0$ a deformation retraction of $|K_{r+1}|$ onto $|K_r|$, for in the simplicial setting an infinite sequence of deformation retractions still gives a deformation retraction.

We do this per simplex: if σ is a simplex of K_{r+1} that is not in K_r and is minimal for this property, then its link in K_{r+1} lies in K_r and so it suffices to define for such a σ a deformation retraction h_σ of $|\operatorname{Star}_{K_{r+1}}(\sigma)|$ onto $|\operatorname{Link}_{K_{r+1}}(\sigma)|$.

The simplex σ is represented by a closed submanifold $A \subset S$ of which every connected component meets γ transversally and is such that $A\cap\gamma$ has cardinality $r+1$ (a number that cannot be made smaller in its isotopy class). Let x_0 be the point of $A\cap\gamma$ closest to x. Denote by α_0 the connected component of A which contains x_0 and choose in S' a thin regular neighborhood of the union of α_0 and the subarc of γ which connects x_0 with x. The boundary of that neighborhood has two connected components. Both lie in S and only one of them is isotopic to α_0. Denote by α_0' the other boundary component. If τ is a simplex of K_{r+1} which contains σ, then adding α_0' to τ gives also a simplex τ' of K_{r+1} and the codimension one face τ'' of τ' obtained by removing α_0 is contained in K_r. So if we regard $|\operatorname{Star}_{K_{r+1}}(\sigma)|$ as the cone over $|\operatorname{Link}_{K_{r+1}}(\sigma)|$ with the barycenter of σ as its vertex, then there is a simplicial map from this cone to its base which sends the barycenter to α_0' and is the identity on the base. Its geometric realization yields the desired h_σ. □

Corollary 2.4. *The complex* $\mathcal{C}(S,P)\cup\mathcal{C}(S,\tilde{P})_x$ *is canonically homotopy equivalent to the join* $\mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S',P')$ *(where* $\mathrm{arc}_{(S,P)}(x)$ *is discrete).*

Proof. The set of vertices of $\mathcal{C}(S,P)\cup\mathcal{C}(S,\tilde{P})_x$ not in $\mathcal{C}(S,\tilde{P})_x$ is $\mathrm{arc}_{(S,P)}(x)$. The link of any such vertex is contained in $\mathcal{C}(S,\tilde{P})_x$ and by Lemma 2.2 that link projects isomorphically onto $\mathcal{C}(S',P')$. In view of Lemma 2.3 this implies that the inclusion of this link in $\mathcal{C}(S,\tilde{P})_x$ is also a homotopy equivalence. Hence the natural inclusion $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x \subset \mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S,\tilde{P})_x$ is a homotopy equivalence. The corollary follows. □

From now on we assume that A_g *holds and that* $A_{h,k}$ *holds for all* (h,k) *smaller than* (g,n) *for the lexicographic ordering. Our goal is to prove* $A_{g,n}$.

Lemma 2.5. *The pair* $(\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x, \mathcal{C}(S,P))$ *is* $(g-1)$*-connected.*

Proof. If $P_x = \{x\}$, then $\tilde{P} = P$ and there is nothing to show. We therefore assume that P_x has more than one element. Denote by $\mathcal{C}_k$ the subcomplex of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ that is the union of $\mathcal{C}(S,P)$ and the k-skeleton of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$: a finite set of vertices of $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ spans a simplex of $\mathcal{C}_k$ if and only if no more than $k+1$ of these separate x from $P_x - \{x\}$. Notice that $\mathcal{C}_{-1} = \mathcal{C}(S,P)$ and $\mathcal{C}_k = \mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ for k large. A minimal simplex of $\mathcal{C}_k - \mathcal{C}_{k-1}$ is represented by a compact 1-dimensional submanifold $A \subset S$ with $k+1$ connected components, each of which separates x from $P_x - \{x\}$ (the graph that is associated to A is then a string with $k+2$ nodes). We prove that the boundary of the star of such a simplex in $\mathcal{C}_{k-1}$ is $(g-2)$-connected if $g > 0$. We enumerate the connected components of A as $\alpha_0, \dots, \alpha_k$ and the connected components of $S - A$ as $S_0, \dots, S_{k+1}$ such that α_i is a boundary component of S_i and S_{i+1} and so that S_0 resp. S_{k+1} is punctured by x resp. $P_x - \{x\}$. The cusps of $S - A$ are naturally indexed by $\hat{N} := N \sqcup \{i_\pm\}_{i=0}^k$, where i_- resp. i_+ corresponds to the cusp defined by α_i on S_i resp. S_{i+1}. Let $\hat{N}_i \subset \hat{N}$ index the set of cusps on S_i. Denote by P_i the partition of $(N - P_x) \cap S_i$ that is simply the restriction of P and denote by $\hat{P}_i$ the partition of $\hat{N}_i$ that on $(N - P_x) \cap S_i$ is equal to P_i and has what remains of $\hat{N}_i$ as a single part. So this new part is $\{x, 0_+\}$ for $i = 0$, $\{(i-1)_-, i_+\}$ for $0 < i < k+1$ and $(P_x - \{x\}) \cup \{k_-\}$ for $i = k+1$.

The reason for introducing these partitions is that we can now observe that the boundary of the star of the k-simplex σ_A defined by A in $\mathcal{C}_k$ lies in $\mathcal{C}_{k-1}$ and can be identified with the iterated join

$$\partial\sigma_A * \mathcal{C}(S_0, \hat{P}_0) * \cdots * \mathcal{C}(S_{k+1}, \hat{P}_{k+1}).$$

It is then enough to show that this join is $(g-2)$-connected for $g > 0$. Since $\partial\sigma_A$ is a $(k-1)$-sphere, it is $(k-2)$-connected. The connectivity of a factor $\mathcal{C}(S_i, \hat{P}_i)$ with $g_i > 0$ is at least $g_i - 2$. So by Lemma 2.1 the connectivity of the above join is at least $-2 + k + \sum_{\{i:g_i>0\}} g_i = g + k - 2 \geq g - 2$. □

Proof of ($A_{g,n}$*).* We must show that $\mathcal{C}(S,P)$ is $(g-2)$-connected. In view of Lemma 2.5 it suffices to show that $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ has that property.

If $\mathrm{arc}_{(S,P)}(x) = \emptyset$, then $n > 2$ and so our induction hypothesis implies that $\mathcal{C}(S',P')$ is $(g-2)$-connected by $A_{g,n-1}$. It follows from Corollary 2.4 that $\mathcal{C}(S,P) \cup \mathcal{C}(S,\tilde{P})_x$ is homotopy equivalent to $\mathcal{C}(S',P')$ and hence is $(g-2)$-connected.

If $\mathrm{arc}_{(S,P)}(x) \neq \emptyset$, then we may have $n = 2$. At least we know that $\mathcal{C}(S', P')$ is $(g-3)$-connected (invoke A_g if $n = 2$). But since $\mathcal{C}(S, P) \cup \mathcal{C}(S, \tilde{P})_x$ is homotopy equivalent to $\mathrm{arc}_{(S,P)}(x) * \mathcal{C}(S', P')$ (by Corollary 2.4), it is $(g-2)$-connected. $\square$

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