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One-sided power sum and cosine inequalities

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Abstract

In this note we prove results of the following types. Let be given distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and for every z_j there exists an i such that $z_i = \overline{z_j}$. Then

$$\inf_k \sum_{j=1}^{n} z_j^k \leq -1.$$

If, moreover, none of the ratios z_i/z_j with $i \neq j$ is a root of unity, then

$$\inf_k \sum_{j=1}^{n} z_j^k \leq -\frac{1}{\pi^4} \log n.$$

The constant -1 in the former result is the best possible. The above results are special cases of upper bounds for $\inf_k \sum_{j=1}^{n} b_j z_j^k$ obtained in this paper.

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1. Introduction

Our colleague Marc N. Spijker asked the following question in view of an application in numerical analysis [6]:

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Problem 1. Is it true that for given real numbers $b_j \geq 1$ and distinct complex numbers z_j satisfying the conditions $|z_j| = 1, z_j \neq 1$ for $j = 1, \ldots, n$ and

for every z_j there exists an i such that $z_i = \overline{z_j}, b_i = b_j$

we have $\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k \leq -1$?

Note that by the conjugacy conditions on b_i, z_i the sum $\sum_{j=1}^{n} b_j z_j^k$ is real for all k.

In Section 2 we answer Spijker's question in a slightly generalized and sharpened form (see Theorem 1 and Corollary 1). The solution of Problem 1 has an application to numerical analysis, more particularly Linear multistep methods (LMMs). They form a well-known class of numerical step-by-step methods for solving initial-value problems for certain systems of ordinary differential equations. In many applications of such methods it is essential that the LMM has specific stability properties. An important property of this kind is named *boundedness* and has recently been studied by Hundsdorfer, Mozartova and Spijker [3]. In that paper the *stepsize-coefficient* γ is a crucial parameter in the study of boundedness. In [6] Spijker attempts to single out all LMMs with a positive stepsize-coefficient γ for boundedness. By using Corollary 1 below he is able to nicely narrow the class of such LMMs.

As a fine point we can remark that the bound -1 in Spijker's problem is the optimal one. Namely, take $z_j = \zeta^j$ where $\zeta = e^{2\pi i/(n+1)}$ and $b_j = 1$ for all j. Then the exponential sum equals n if k is divisible by $n+1$ and -1 if not.

If, moreover, none of z_j/z_i with $i \neq j$ is a root of unity, then the upper bound in Problem 1 can be improved to $-\log n/\pi^4$. We deal with this question in Theorem 3 and more particularly Corollary 2. The obtained results can easily be transformed into estimates for $\inf_{k\in\mathbb{Z}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k)$ where b_j, α_j are real numbers and $\alpha_1, \ldots, \alpha_n$ are strictly between 0 and 1/2. Theorem 4 states that this infimum is equal to $\inf_{t\in\mathbb{R}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j t)$, provided that the $\mathbb{Q}$-span of $\alpha_1, \ldots, \alpha_n$ does not contain 1.

2. The general case

We provide an answer to Problem 1.

Theorem 1. *Let n be a positive integer. Let $b_1, \ldots, b_n$ be nonzero complex numbers such that $b_{n+1-i} = \overline{b_i}$ for all $i = 1, 2, \ldots, n$. Let $z_1, \ldots, z_n$ be distinct complex numbers with absolute value 1, not equal to 1, such that $z_{n+1-i} = \overline{z_i}$ for all $i = 1, 2, \ldots, n$. Then*

$$\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k \leq -\frac{\sum_{j=1}^{n} |b_j|^2}{\sum_{j=1}^{n} |b_j|}.$$

Note that $\sum_{j=1}^{n} b_j z_j^k$ is real because of the conjugacy conditions.

By applying the Cauchy–Schwarz inequality we immediately obtain the following consequence.

Corollary 1. *Let n, b_j, z_j be as in Theorem 1. Then*

$$\liminf_{k\to\infty} \sum_{j=1}^{n} b_j z_j^k \leq -\frac{1}{n} \sum_{j=1}^{n} |b_j|.$$

Obviously this answers Problem 1, since in that case $b_j \in \mathbb{R}_{\geq 1}$ for all j.

In the special case when the b_j are positive real numbers we can even drop the distinctness condition on the z_j.

Theorem 2. *Let n, b_j, z_j be as in Theorem 1 with the additional condition $b_j \in \mathbb{R}_{>0}$ for all j and let the distinctness condition on the z_j be dropped. Then the conclusions of Theorem 1 and Corollary 1 still hold.*

Furthermore, since $\sum_{j=1}^{n} b_j z_j^k$ is almost periodic we can apply Dirichlet's Theorem on simultaneous Diophantine approximation, and find that the liminf coincides with the infimum in the above theorems and corollary.

Proof of Theorem 1. Put $s_k = b_1 z_1^k + \cdots + b_n z_n^k$. Put $c = -\liminf_{k\to\infty} s_k$. Note that the z_j^k equal 0 on average for each j. Hence the same holds for the values s_k and thus we see that $c \geq 0$. Let $\varepsilon > 0$. Choose N so large that $s_k > -(c+\varepsilon)$ for all $k \geq N$. For any positive integer K consider the sum

$$\Sigma_1 := \sum_{k=N}^{N+K-1} s_k.$$

Since none of the z_i is 1, we have

$$\Sigma_1 = \sum_{j=1}^{n} \sum_{k=N}^{N+K-1} b_j z_j^k = \sum_{j=1}^{n} b_j \frac{z_j^N - z_j^{N+K}}{1 - z_j}.$$

Thus there exists C_1, independent of N and K, such that

$$|\Sigma_1| \leq \sum_{j=1}^{n} \frac{2|b_j|}{|1-z_j|} = C_1.$$

Define Σ_1^+ to be the subsum of Σ_1 of all nonnegative s_k and Σ_1^- to be minus the subsum of Σ_1 of all negative s_k. Let P be the number of nonnegative s_k for $k = N, \ldots, N+K-1$. Then

$$\Sigma_1^+ \leq \Sigma_1^- + C_1 \leq (K-P)(c+\varepsilon) + C_1$$

and

$$\Sigma_1^- \leq \Sigma_1^+ + C_1 \leq P \sum_{i=1}^{n} |b_i| + C_1.$$

Consider the sum $\Sigma_2 := \sum_{k=N}^{N+K-1} s_k^2$. Then, using $\overline{s_k} = s_k$,

$$\Sigma_2 = \sum_{k=N}^{N+K-1} |s_k|^2 = \sum_{i,j} \sum_{k=N}^{N+K-1} b_i \overline{b_j} z_i^k \overline{z_j}^k$$

$$= K \sum_{i=1}^{n} |b_i|^2 + \sum_{i\neq j} b_i \overline{b_j} \frac{(z_i/z_j)^N - (z_i/z_j)^{N+K}}{1 - z_i/z_j}.$$

Hence there exists C_2, independent of N and K, such that

$$\left| \Sigma_2 - K \sum_{i=1}^{n} |b_i|^2 \right| \leq \sum_{i\neq j} |b_i b_j| \frac{2}{|1 - z_i/z_j|} = C_2.$$

The terms s_k^2 in Σ_2 can be estimated above by $(\sum_{i=1}^n |b_i|)\Sigma_1^+$ when $s_k \geq 0$ and by $(c+\varepsilon)\Sigma_1^-$ when $s_k < 0$. So we get the upper bound

$$\Sigma_2 \leq \left(\sum_{i=1}^n |b_i|\right) \Sigma_1^+ + (c+\varepsilon)\Sigma_1^-.$$

Now use the upper bounds for $\Sigma_1^\pm$ we found above to get

$$\Sigma_2 \leq (c+\varepsilon)(K-P)\left(\sum_{i=1}^n |b_i|\right) + (c+\varepsilon)P\left(\sum_{i=1}^n |b_i|\right) + C_3,$$

where $C_3 = C_1(c+\varepsilon+\sum_{i=1}^n |b_i|)$. Combine this with the lower bound $\Sigma_2 \geq -C_2 + K\sum_{i=1}^n |b_i|^2$ to get

$$-C_2 + K\sum_{i=1}^n |b_i|^2 \leq (c+\varepsilon)K\sum_{i=1}^n |b_i| + C_3.$$

Dividing on both sides by K and letting $K \to \infty$ yields

$$\sum_{i=1}^n |b_i|^2 \leq (c+\varepsilon)\sum_{i=1}^n |b_i|.$$

Since ε can be chosen arbitrarily small, the assertion follows. □

Proof of Theorem 2. Take the distinct elements from $\{z_1, \ldots, z_n\}$ and write them as $w_1, \ldots, w_m$. Denote for any r the sum of the b_j over all j such that $z_j = w_r$ by B_r. Then $s_k = \sum_{r=1}^m B_r w_r^k$ for every k. We can apply Theorem 1 to this sum to obtain

$$\liminf_{k\to\infty} s_k \leq -\frac{\sum_{r=1}^m B_r^2}{\sum_{r=1}^m B_r}.$$

Since the b_j are positive reals, we get that $\sum_{r=1}^m B_r = \sum_{j=1}^n b_j$ and $\sum_{r=1}^m B_r^2 \geq \sum_{j=1}^n b_j^2$. Our theorem now follows. □

3. The non-degenerate case

In the next theorem we make an additional assumption on the z_i, which allows us to improve on the upper bound in Theorem 1 considerably.

Theorem 3. *Let $b_j \in \mathbb{C}$ and let $z_j \in \mathbb{C}$ be as in Theorem 1. Assume in addition that $z_i \neq -1$ for all i and that none of the ratios z_j/z_i with $i \neq j$ is a root of unity. Then*

$$\inf_k \sum_{j=1}^n b_j z_j^k < -\frac{1}{\pi^4}(\min_j |b_j|)\log n.$$

If the z_i satisfy the conditions of Theorem 3 we say that we are in the *non-degenerate case*. Notice in particular that $z_i/z_{n+1-i} = z_i^2$ and hence none of the z_i are roots of unity in the non-degenerate case. For the proof of Theorem 3 we use the following result.

Lemma 1. *Let $b_1, \ldots, b_n \in \mathbb{C}$. Let $q_1, \ldots, q_n$ be distinct integers. Set $f(t) = \sum_{j=1}^{n} b_j e^{iq_j t}$. Then*

$$\int_{-\pi}^{\pi} |f(t)| dt \geq \frac{4}{\pi^3} (\min_j |b_j|) \log n.$$

Proof. See Stegeman, [7]. □

Lemma 1 is a refinement of a result independently obtained by McGehee, Pigno, Smith [5] and Konyagin [4] who thereby established a conjecture of Littlewood [1]. Already Littlewood noticed that the constant in Lemma 1 cannot be better than $4/\pi^2$, cf. [7] p. 51. Stegeman expects that the optimal constant in Lemma 1 is $4/\pi^2$ indeed. As a fine point we mention that the choice of $4/\pi^3$ by Stegeman is for esthetical reasons only, the best possible value with his method happens to lie close to this value. See also [8].

The following lemma connects Littlewood's conjecture with minima of sums of exponentials.

Lemma 2. *Let $b_1, \ldots, b_n \in \mathbb{C}$. Let $q_1, \ldots, q_n$ be distinct nonzero integers. Suppose that $f(t) = \sum_{j=1}^{n} b_j e^{iq_j t}$ is real-valued for all real t. Then*

$$\min_{t \in \mathbb{R}} f(t) < -\frac{1}{\pi^4} (\min_j |b_j|) \log n.$$

Proof. Denote the minimum of $f(t)$ by $-c$. Define $f^+(t) = \max(f(t), 0)$ and $f^-(t) = -\min(f(t), 0)$. Then $f = f^+ - f^-$. Since the exponents q_j are nonzero, we have that $\int_{-\pi}^{\pi} f(t) dt = 0$, hence that $\int_{-\pi}^{\pi} f^+(t) dt = \int_{-\pi}^{\pi} f^-(t) dt$ and

$$\int_{-\pi}^{\pi} |f(t)| dt = \int_{-\pi}^{\pi} f^+(t) dt + \int_{-\pi}^{\pi} f^-(t) dt = 2 \int_{-\pi}^{\pi} f^-(t) dt < 4\pi c.$$

Now combine this upper bound with the lower bound from Lemma 1 to find the assertion of our lemma. □

Proof of Theorem 3. Consider the subgroup G of $\mathbb{C} \setminus \{0\}$ generated by $z_1, \ldots, z_n$. By the fundamental theorem of finitely generated abelian groups, G is isomorphic to $T \times \mathbb{Z}^d$ for some d and some finite group T consisting of roots of unity. More concretely this means that there exist $w_1, \ldots, w_d \in G$ and $\mu \in T$ such that $w_1, \ldots, w_d$ are multiplicatively independent and every z_j can be written in the form

$$z_j = \mu^{r_j} w_1^{a_{j1}} \cdots w_d^{a_{jd}}, \quad r_j, a_{ji} \in \mathbb{Z},\ 0 \leq r_j < |T|.$$

It follows from the condition in Theorem 1 that $a_{n+j-1,h} = -a_{j,h}$ for all $j = 1, \ldots, n$ and $h = 1, \ldots, d$. Our exponential sum can be rewritten as

$$s_k := \sum_{j=1}^{n} b_j \mu^{kr_j} w_1^{ka_{j1}} \cdots w_d^{ka_{jd}}.$$

By Kronecker's approximation theorem, the closure of the set of points $(w_1^k, \ldots, w_d^k)$ for $k \in \mathbb{Z}_{\geq 0}$ equals the set $(S^1)^d$ consisting of points $(\omega_1, \ldots, \omega_d)$ with $|\omega_j| = 1$ for $j = 1, \ldots, d$. The same holds true if we restrict ourselves to values of k that are divisible by $|T|$. Hence

$$\inf s_k \leq \min_{|\omega_1| = \cdots = |\omega_d| = 1} \sum_{j=1}^{n} b_j \omega_1^{a_{j1}} \cdots \omega_d^{a_{jd}}.$$

Because there are no roots of unity among the z_j, for every j at least one coefficient a_{ji} is nonzero. Since the ratios z_i/z_j are not a root of unity for every $i \neq j$, the vectors $(a_{j1}, \ldots, a_{jd})$ are pairwise distinct. Hence we can choose $p_1, \ldots, p_d \in \mathbb{Z}$ such that the numbers $q_j = a_{j1}p_1 + \cdots + a_{jd}p_d$, $j = 1, \ldots, n$ are distinct and nonzero. Let us now restrict to the points with $\omega_l = e^{ip_l t}$, $t \in \mathbb{R}$, $l = 1, \ldots, d$. Then we get

$$\inf s_k \leq \min_{t\in\mathbb{R}} \sum_{j=1}^{n} b_j e^{iq_j t},$$

where the sum on the right-hand side is real for all t in view of $b_{n+1-j} = \overline{b_j}$, $q_{n+1-j} = -q_j$ for $j = 1, \ldots, n$. By Lemma 2 the right-hand side is bounded above by $-\frac{1}{\pi^4}(\min_j |b_j|) \log n$. □

In the special case $b_j = 1$ for all j we have the following corollary.

Corollary 2. *Let $z_1, \ldots, z_n$ be as in Theorem 1. Suppose in addition that none of the ratios z_i/z_j with $i \neq j$ is a root of unity. Then*

$$\inf_{k\in\mathbb{Z}} \sum_{j=1}^{n} z_j^k < -\frac{1}{\pi^4} \log n.$$

Proof. Note that we have not excluded the possibility that $z_i = -1$ for some i. When $z_i \neq -1$ for all i we are in the non-degenerate case and can apply Theorem 3 immediately. When $z_i = -1$ for some i we can take $z_n = -1$. We now consider the subsequence of sums for odd k. Put $k = 2\kappa + 1$. Note that

$$\inf_{k\in\mathbb{Z}} \sum_{j=1}^{n} z_j^k \leq -1 + \inf_{\kappa} \sum_{j=1}^{n-1} z_j^{2\kappa+1}.$$

Apply Theorem 3 to the numbers $b_j = z_j$ for $j = 1, \ldots, n-1$ and z_j^2 instead of z_j for $j = 1, \ldots, n-1$. Then we find

$$\inf_{k\in\mathbb{Z}} \sum_{j=1}^{n} z_j^k \leq -1 - \frac{1}{\pi^4} \log(n-1) < -\frac{1}{\pi^4} \log n. \quad \square$$

4. The continuous case

The conditions on b_j, z_j in Theorem 1 can be seen as an invitation to write the power sum as a cosine sum. We consider the easier case when $b_j \in \mathbb{R}$ for all j. Then we have

$$\sum_{j=1}^{n} b_j z_j^k = \mathrm{Re}\left(\sum_{j=1}^{n} b_j z_j^k\right) = \sum_{j=1}^{n} b_j \cos(2\pi\alpha_j k),$$

where we have written $z_j = \exp(2\pi i\alpha_j)$ for all j. To make things simpler assume that $z_j \neq -1$ for all j. Then n is even and the arguments α_j come in pairs which are opposite modulo $\mathbb{Z}$. Letting $m = n/2$ we rewrite our sum as

$$2\sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k),$$

where we can assume that $0 < \alpha_j < 1/2$ for all j.

Corollary 1 immediately implies the following.

Corollary 3. *Let $b_1, \dots, b_m, \alpha_1, \dots, \alpha_m$ be real numbers such that the α_j are distinct and strictly between 0 and 1/2 for all j. Then*

$$\inf_{k\in\mathbb{Z}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k) < -\frac{1}{2m}\sum_{j=1}^{m} |b_j|.$$

Proof. Apply Corollary 1 with $n = 2m$ and $b_j = b_{j-m}$ when $j > m$ and $z_j = \exp(2\pi i\alpha_j)$ when $j \le m$ and $z_j = \exp(-2\pi i\alpha_{j-m})$ when $j > m$. $\square$

Similarly Theorem 3 implies the following.

Corollary 4. *Let $b_1, \dots, b_m, \alpha_1, \dots, \alpha_m$ be real numbers such that the α_j are distinct and strictly between 0 and 1/2 for all j. Suppose in addition that none of the differences $\alpha_i - \alpha_j$ with $i \neq j$ and none of the sums $\alpha_i + \alpha_j$ is rational. Then*

$$\inf_{k\in\mathbb{Z}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k) < -\frac{\log(2m)}{2\pi^4}\min_j |b_j|.$$

We introduce the notation

$$c_S = -\inf_{k\in\mathbb{Z}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k),$$

$$c_T = -\inf_{t\in\mathbb{R}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j t).$$

In the notation c_S, c_T we suppress the dependence on the α's and b's. Of course, $c_S \le c_T$ for all numbers α_j and b_j.

Problem 2. What are the corresponding upper bounds for c_T?

The following result shows that $c_T = c_S$ under a general condition.

Theorem 4. *Let $b_1, \dots, b_m$ be real numbers and let $\alpha_1, \dots, \alpha_m$ be real numbers such that their $\mathbb{Q}$-span does not contain 1. Then $c_S = c_T$.*

In the proof we use the following consequence of Kronecker's theorem on simultaneous Diophantine approximation.

Lemma 3. *Let $\alpha_1, \dots, \alpha_n$ be numbers such that their $\mathbb{Q}$-span does not contain 1. Let $t_0 \in \mathbb{R}$. Given $\delta > 0$ there exist integers $k, k_1, \dots, k_n$ such that $|\alpha_j t_0 - \alpha_j k - k_j| < \delta$ for $j = 1, \dots, n$.*

Proof. Let $\beta_1, \dots, \beta_d$ be a basis of the $\mathbb{Q}$-vector space spanned by the α_j. Choose $\lambda_{ij} \in \mathbb{Q}$ such that

$$\alpha_j = \sum_{i=1}^{d} \lambda_{ij}\beta_i.$$

By a convenient choice of the β_i we can see to it that $\lambda_{ij} \in \mathbb{Z}$ for all i, j. Put $\Lambda = \max_j(\sum_i |\lambda_{ij}|)$. By Kronecker's theorem ([2], Theorem 442) there exist integers $k, m_1, \ldots, m_d$ such that

$$|\beta_i t_0 - \beta_i k - m_i| < \delta/\Lambda$$

for $i = 1, \ldots, d$. Here we use the information that the $\mathbb{Q}$-span of the β_i's does not contain 1. Put $k_j = \sum_{i=1}^{d} \lambda_{ij} m_i$. Then we get, for $j = 1, \ldots, n$,

$$|\alpha_j t_0 - \alpha_j k - k_j| \leq \sum_{i=1}^{d} |\lambda_{ij}| \cdot |\beta_i t_0 - \beta_i k - m_i| < \sum_{i=1}^{d} |\lambda_{ij}| \frac{\delta}{\Lambda} \leq \delta. \quad \square$$

Proof of Theorem 4. It remains to prove that $c_T \leq c_S$. Let $\varepsilon > 0$. Choose t_0 such that

$$0 \leq c_T + \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j t_0) < \varepsilon.$$

We apply Lemma 3 with a δ which is so small that there exists an integer k_0 with

$$\left| \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j t_0) - \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k_0) \right| < \varepsilon.$$

Hence

$$0 \leq c_T + \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k_0) < 2\varepsilon.$$

We deduce that

$$\begin{aligned} c_T - c_S &= c_T + \inf_{k \in \mathbb{Z}} \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k) \\ &\leq c_T + \sum_{j=1}^{m} b_j \cos(2\pi\alpha_j k_0) < 2\varepsilon. \end{aligned}$$

Since ε can be chosen arbitrarily close to zero, we conclude $c_S = c_T$. $\square$

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References

[1] G.H. Hardy, J.E. Littlewood, A new proof of a theorem on rearrangements, J. Lond. Math. Soc. 23 (1948) 93–97.
[2] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fourth ed., Oxford at the Clarendon Press, 1962.
[3] W. Hundsdorfer, A. Mozartova, M.N. Spijker, Stepsize restrictions for boundedness and monotonicity of multistep methods, J. Sci. Comput. 50 (2012) 265–286.
[4] S.V. Konyagin, On a problem of Littlewood, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981) 241–265. 463 (in Russian); transl.: Math. USSR-Izv. 18 (1981) 205–225.

[5] O.C. McGehee, L. Pigno, B. Smith, Hardy's inequality and the L_1 norm of exponential sums, Ann. Math. 11 (1981) 613–618.
[6] M.N. Spijker, The existence of positive stepsize-coefficients for boundedness of linear multistep methods, Report no. MI-2011-16, Mathematical Institute, Leiden University, 2011, 16 pp.
[7] J.D. Stegeman, On the constant in the Littlewood problem, Math. Ann. 261 (1982) 51–54.
[8] R.M. Trigub, A lower bound for the L_1-norm of a Fourier series of power type, Mat. Zametki 73 (2003) 951–953 (in Russian); transl.: Math. Notes 73 (2003) 900–903.