Phase transitions of magnetic AdS$_4$ black holes with scalar hair

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We determine the thermodynamic properties of a class of spherically symmetric and static black holes in AdS$_4$ with magnetic charges and scalar hair. These black holes are solutions in four-dimensional $N = 2$ gauged supergravity that can arise from 11-dimensional supergravity compactified on S^7. At zero temperature, they preserve supersymmetry and hence are stable. At nonzero temperatures, we explore in detail the canonical ensemble and stability of solutions and find a first-order phase transition between small and large hairy black holes. The transition emerges as a liquid-gas phase transition in the dual three-dimensional field theory on $\mathbb{R} \times$ S^2 with magnetic flux through S^2.

I. INTRODUCTION AND SUMMARY

Black holes behave much like thermodynamical systems. They have an internal energy (the mass), a temperature (due to Hawking radiation) and other quantum numbers like charges and angular momentum. The entropy is determined by the area of the event horizon [1,2], and from these data one can compute the free energy. Black holes can undergo phase transitions, which reveal a particularly interesting structure in anti–de Sitter (AdS) spacetimes. Most famous is perhaps the Hawking-Page phase transition between a thermal gas of gravitons and Schwarzschild black holes in AdS [3]. Using the AdS/CFT correspondence, this phase transition was interpreted as the transition from a confining to a deconfining phase in the dual gauge theory [4]. Similarly, phase transitions of charged black branes (black holes with planar symmetry) with scalar hair have been used in modeling holographic superconductors on the boundary, where a charged condensate forms via a second-order phase transition and a global U(1) symmetry is spontaneously broken [5].

In this work we present a new phase transition for spherically symmetric black holes, based on a class of magnetic solutions with nontrivial scalar profiles. These magnetic black holes were recently discovered in Refs. [6,7] and can be thought of as thermal excitations over supersymmetric Bogomol'nyi-Prasad-Sommerfeld (BPS) static black holes with fixed graviphoton magnetic charge [8–10]. Our analysis follows closely the study of charged AdS$_4$ black holes presented in Refs. [11–13], but it is applied to a different class of black hole solutions. Nevertheless, the phase transition we find shows some similarities to the one found in Refs. [11,13] for electrically charged black holes and in the absence of scalar fields [14].

Anticipating our final results, we already present the phase diagram in the canonical ensemble in Fig. 1, the main result of our study. We denote by $P = p^1$ one

of the two magnetic charges carried by the black hole. In region I we find a small/large black hole first-order phase transition for $|P| < P_c$ and a smooth crossover between the two phases for $|P| > P_c$. Apart from a very small region close to criticality, the phase diagram consists of solutions with positive specific heat that are stable against thermal fluctuations. In regions II and III, as well as the two lines with thermal black holes for constant scalars, BPS solutions are absent, and one finds a smooth crossover between small and large black holes that are thermally stable everywhere in phase space [16].

Figure 1 needs to be supplemented with the phase diagram for the scalar hair, which we present in the last section of the paper. There, we also discuss the implications of this result for the AdS/CFT correspondence.

II. BLACK HOLE SOLUTIONS

We will discuss properties of black hole solutions of gauged four-dimensional $N = 2$ supergravity [17,18]. We consider the particular case of the $N = 2$ gravitational multiplet coupled to a single vector multiplet with gauge coupling constant g and prepotential $F = -2i\sqrt{X^0(X^1)^3}$. The gauge group is $U(1) \times U(1)$, and the gauging introduces charges for the two gravitini and a scalar potential V that contains Fayet-Iliopoulos (FI) parameters. The Lagrangian is written in e.g. Ref. [6], whose conventions and notation we follow here. This is the simplest four-dimensional theory that allows for static supersymmetric black holes in AdS$_4$ with spherical symmetry. Moreover, for certain values of the FI parameters, the model can be directly embedded in 11-dimensional supergravity [10,15], and hence this could lead to an underlying microscopic string/M-theory description.

The FI parameters of the theory, denoted by ξ_0 and ξ_1, determine the electric charges for the two gravitini [19] [with respect to the $U(1)$ gauge fields A^0_μ and A^1_μ]. They are $e_\Lambda \equiv g\xi_\Lambda$ ($\Lambda = 0, 1$) for one gravitino and opposite for
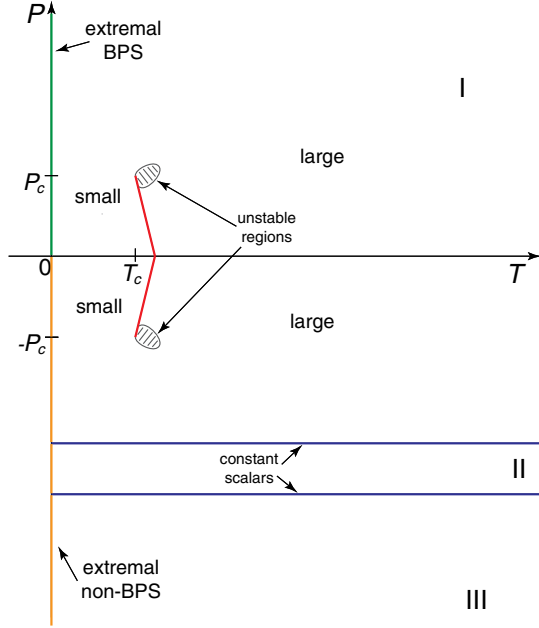
FIG. 1 (color online). The phase diagram in the canonical ensemble. Regions I, II and III are separated by the two blue lines of thermal black holes with constant scalar profiles.

the other. The magnetic charges p^0 and p^1 of the black hole therefore obey a Dirac quantization condition,

$$g\xi_\Lambda p^\Lambda = n, \qquad n \in \mathbb{Z}. \tag{1}$$

In this work we choose $n = -1$ (the case of $n = +1$ is then also covered by a sign flip on the gravitino charges), which ensures that the black holes asymptote to the stable magnetic AdS ground state [21]. For values of $n \neq \pm 1$, spherical black holes do not preserve supersymmetry at infinity, so we expect instabilities to occur.

The magnetically charged black hole background solutions of Ref. [6] have all fermions set to zero and are given by the spacetime metric

$$\mathrm{d}s^2 = U^2(r)\mathrm{d}t^2 - U^{-2}(r)\mathrm{d}r^2 - h^2(r)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{2}$$

with

$$U^2(r) = e^{\mathcal{K}}\left(g^2r^2 + c - \frac{\mu}{r} + \frac{Q}{r^2}\right), \quad h^2(r) = e^{-\mathcal{K}}r^2. \tag{3}$$

In these coordinates, the field strengths are given by $F^\Lambda = p^\Lambda \sin\theta\mathrm{d}\theta \wedge \mathrm{d}\varphi$ and we do not allow for electric charges. The vector multiplet contains a complex scalar field z that parametrizes a special Kähler manifold with Kähler potential $\mathcal{K}$ given by

$$z = \frac{X^1}{X^0}, \qquad \mathcal{K} = -\log[X^0\bar{X}^0(\sqrt{z} + \sqrt{\bar{z}})^3]. \tag{4}$$

The general thermal solutions with running scalars have

$$X^0 = \frac{1}{4\xi_0} - \frac{\xi_1 b_1}{\xi_0 r}, \qquad X^1 = \frac{3}{4\xi_1} + \frac{b_1}{r}, \tag{5}$$

$$c = 1 - \frac{32(g\xi_1 b_1)^2}{3},$$

$$Q = -\frac{16}{3}b_1^2\xi_1^2 + \frac{1}{g^2} - \frac{256}{27}b_1^4\xi_1^4 g^2 + \frac{2\xi_1 p^1}{g} + \frac{4}{3}\xi_1^2(p^1)^2, \tag{6}$$

and

$$\mu = \frac{8}{3}\xi_1 b_1 - \frac{3}{4g^2\xi_1 b_1} + \frac{512}{27}g^2\xi_1^3 b_1^3 - \frac{3p^1}{2gb_1} - \frac{2\xi_1(p^1)^2}{3b_1}. \tag{7}$$

The configuration (2)–(7) is a solution to the full nonlinear set of coupled equations of motion for the metric, gauge fields and complex scalar. For fixed gravitino charges e_Λ, the solution contains two parameters, which we can choose to be b_1 and p^1 [with p^0 fixed by the quantization condition (1)].

The solution for the scalar field is real. Its asymptotic expansion is $z = 3\frac{\xi_0}{\xi_1} + 16\xi_0\frac{b_1}{r} + \mathcal{O}(r^{-2})$, where the constant term is dictated by the value of the scalar in the (magnetic) AdS$_4$ vacuum at infinity. The extremum of the potential V_* sets the value of the cosmological constant which in our conventions is $\Lambda = 3g^2V_* = -2\sqrt{3}g^2\sqrt{\xi_0\xi_1^3}$ [6]. After truncating the imaginary part of z, we can write down the Lagrangian for the real part, canonically normalized as $\phi = \sqrt{\frac{3}{8}}\ln z$. We then get, using the results of Ref. [6],

$$L = \frac{1}{2}R(g) - e^{\sqrt{6}\phi}F^0_{\mu\nu}F^{0\mu\nu} - 3e^{-\sqrt{\frac{2}{3}}\phi}F^1_{\mu\nu}F^{1\mu\nu}$$
$$+ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + g^2\left(\xi_0\xi_1 e^{-\sqrt{\frac{2}{3}}\phi} + \frac{\xi_1^2}{3}e^{\sqrt{\frac{2}{3}}\phi}\right). \tag{8}$$

For values for which $3\xi_0 = \xi_1$, one gets the cosh-potential of Ref. [15], obtained from a truncation of 11-dimensional supergravity compactified on S^7. The extremum of the potential is at $\phi_* = \sqrt{\frac{3}{8}}\ln(3\xi_0/\xi_1)$, which is the maximum. Expanding around this value, one finds the mass of the scalar to be $m_\phi^2 = -2g^2|V_*|/3$. One can then verify that the mass satisfies the Breitenlohner-Freedman (BF) bound, and in the conventions of e.g. Ref. [22] in which $V_* = -3$, we have $m_{BF}^2 = -9/4 < m_\phi^2 = -2 < m_{BF}^2 + 1$. Asymptotically, the scalar field behaves as

$$\phi - \phi_* = \frac{\alpha}{r} + \frac{\beta}{r^2} + \mathcal{O}(r^{-3}), \tag{9}$$

with $\beta = \alpha^2/\sqrt{6}$ and $\alpha = 4\sqrt{2/3}b_1\xi_1$. We have, again in the conventions of Ref. [22], $\beta = k\alpha^2$ with a fixed value of $k = 1/\sqrt{6}$ [23].

The magnetic charge $P \equiv p^1$ can be varied freely and defines three regions of black hole solutions with running scalars: $P > P_{\text{I}}$, $P_{\text{I}} > P > P_{\text{III}}$ and $P < P_{\text{III}}$. The values $P_{\text{I}} = -\frac{3}{4g\xi_1}$ and $P_{\text{III}} = -\frac{3}{2g\xi_1}$ correspond to the only smooth solutions with constant scalars ($b_1 = 0$) (see Ref. [6]). For other values of the magnetic charge, the limit to constant scalars $b_1 = 0$ is not smooth. This is an important difference compared with the situation in Refs. [11,13], where there are no scalars and the magnetic charge is free.

The temperature of the black hole can be calculated from the surface gravity, and is given by $T = \frac{1}{2\pi}|UU'|_{r=r_+}$, or

$$T = \frac{1}{4\pi}\left| e^{\mathcal{K}(r_+)}\left(4g^2 r_+ - \frac{\mu}{r_+^2} + 2\frac{c}{r_+}\right)\right|, \quad (10)$$

with the value of the radius at the outer horizon r_+ found as the largest root of the equation $U^2(r) = 0$. The resulting Bekenstein-Hawking entropy is therefore

$$S = \frac{A}{4} = \pi h^2(r_+) = \pi r_+^2 e^{-\mathcal{K}(r_+)}. \quad (11)$$

A black hole solution can therefore be plotted in the (p^1, b_1) plane, as shown in Fig. 2. Not all points correspond to solutions with horizons. Singular and regular solutions are separated by two curves. The first one (green curve) corresponds to the 1/4-BPS solutions of Refs. [8–10], and is defined by the relation $p^1 = -\frac{1}{4e_1}(3 - 2\alpha^2 g^2)$, such that $U^2(r) = e^K(gr + \frac{c}{2gr})^2$ with the horizon located at $r_+^2 = -c/(2g^2)$ with $c < 0$.

The second curve (in orange) defines the extremal, but non-BPS black hole solutions. It can be obtained from setting $T = U^2 = 0$, excluding the BPS black holes and
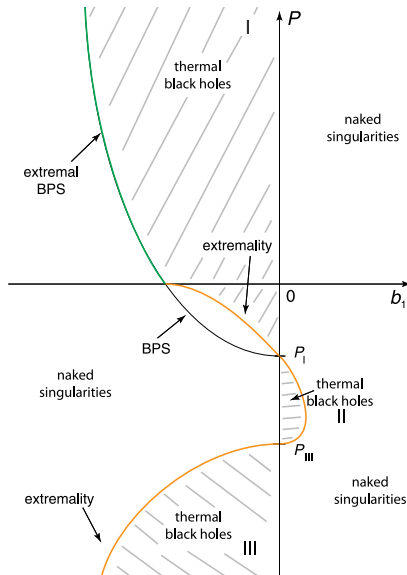

FIG. 2 (color online). Parameter space for genuine black hole solutions (shaded) in regions I, II and III.

solutions with naked singularities. It yields conditions that can be analyzed numerically and the result is in Fig. 2.

III. THERMODYNAMICS AND PHASE TRANSITIONS

In this section we discuss the thermodynamical properties of the black holes in the canonical ensemble. For magnetically charged black holes, the grand-canonical ensemble seems less appealing, since there are no magnetically charged particles that the black hole can emit. One can also allow the cosmological constant Λ to vary, as was done in Refs. [24,25]. We however keep Λ fixed and we set $V_* = -3$. In particular, we set $\xi_1 = 3/\sqrt{2}$, $\xi_0 = 1/\sqrt{2}$, which allows for an embedding into 11-dimensional supergravity, as discussed before.

Black holes with magnetic charges must satisfy the first law of thermodynamics,

$$\mathrm{d}M = T\mathrm{d}S + \chi_\Lambda \mathrm{d}p^\Lambda, \quad (12)$$

where the magnetostatic potentials are defined as $\chi_\Lambda = -\int_{r_+}^\infty G_{\Lambda,\text{tr}}\mathrm{d}r$, and G is the dual field strength. For the case at hand, these potentials can be computed to be

$$\chi_0 = -\frac{1}{\sqrt{2}}\frac{\sqrt{2} + 3p^1}{(\sqrt{2}r_+ - 12b_1)},$$
$$\chi_1 = \frac{1}{\sqrt{2}}\frac{3p^1}{(\sqrt{2}r_+ + 4b_1)}. \quad (13)$$

All quantities on the right-hand side of Eq. (12) have now been determined, so we can integrate the expression to find the mass M on the left-hand side,

$$M = 2\sqrt{2}b_1 - \frac{1}{4\sqrt{2}g^2 b_1} - \frac{3p^1}{4gb_1} - \frac{(p^1)^2}{\sqrt{2}b_1}. \quad (14)$$

This expression for the mass can be reproduced by using the Ashtekar-Das-Magnon (AMD) formalism [26,27], which we explicitly verified. The AMD mass in asymptotically AdS space guarantees that M is a conserved quantity [28].

The canonical ensemble describes a closed system at fixed temperature and charge, $\mathrm{d}T = \mathrm{d}p^\Lambda = 0$. The free energy

$$F = M - TS \quad (15)$$

is extremized at equilibrium, $\mathrm{d}F = 0$, due to the first law (12). One can therefore compare the free energy $F(T, P)$ of different solutions at fixed T and $P \equiv p^1$—the state with the lowest F is the one that is thermodynamically preferred.

The phase diagram in the canonical ensemble was already presented in Fig. 1, but now we can discuss it in more detail. It is instructive to compare how the temperature behaves as a function of the outer horizon radius r_+ for constant magnetic charge in the region $-P_c \leq P \leq P_c$ and outside it (Fig. 3). For $|P| > P_c$, the temperature is a
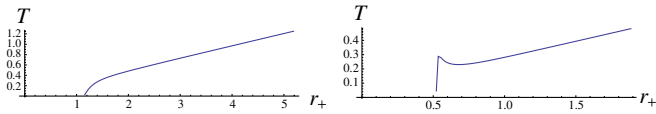
FIG. 3 (color online). Temperature vs outer horizon radius r_+ for fixed values of $|P| > P_c$ (left, $P = 0.5$) and $|P| < P_c$ (right, $P = 0.01$). Numerical analysis shows that $P_c \approx 0.025$ for the values $\xi_1 = 3/\sqrt{2}$, $\xi_0 = 1/\sqrt{2}$ and $g = 1$.

monotonic function of the radius, and so there exists only one black hole for fixed T. In the region $|P| < P_c$, for fixed T there exist up to three different black holes: a small, a medium-sized, and a large black hole. To see which black hole is preferred in the canonical ensemble, we plot the free energy for these solutions. From Fig. 4 we see that for $|P| < P_c$, increasing T leads to the appearance of medium-sized and large black hole branches, whose free energy is initially higher than for small black holes [30]. Upon further increase of T, the free energy of the large black holes rapidly decreases, leading to a crossing point. The free energy is not smooth here, and so a first-order phase transition between small and large black holes occurs, similar to Refs. [11,13].

To check the intrinsic stability of the different phases against thermal fluctuations, one computes the specific heat C_P at fixed magnetic charge,

$$C_P \equiv T\left(\frac{\partial S}{\partial T}\right)_P \geq 0, \quad (16)$$

which must be positive for stable solutions. It turns out that the medium-sized black holes have a negative specific heat and are therefore unstable under temperature fluctuations. The other regions in Fig. 3, where the temperature grows with size, have a positive specific heat and are stable. This means that both the small and the large black hole phases are stable. There is a tiny region where this is not exactly true, as shown on Fig. 1: close to the critical temperature $T_c \approx 0.226$, the phase with larger-size black holes contains some negative specific heat solutions according to our numerical analysis. The exact values and sizes of each feature on the phase diagram depend on the values of the free positive constants g, ξ_0, ξ_1. One finds, for instance, that the value of P_c scales inversely with the coupling

constant g, and we expect $P_c \sim (-\Lambda)^{-1/2}$ and $T_c \sim (-\Lambda)^{1/2}$, with cosmological constant Λ.

IV. DUAL DESCRIPTION VIA ADS/CFT

The supergravity models we have discussed can be embedded in M-theory compactified on $\text{AdS}_4 \times S^7$ [10,15], and will have field theory duals in the class of Aharony-Bergman-Jafferis-Maldacena (ABJM) models [31] defined on $\mathbb{R} \times S^2$. The bulk gauge group $U(1) \times U(1)$ corresponds to part of the global R-symmetry group on the boundary, which can be gauged to produce background magnetic fields on the boundary with flux through the S^2. The bulk model contains a scalar field with $m_\phi^2 = -2$, so the boundary values provide sources for operators of dimensions equal to one or two, depending on the choice of quantization. Examples of such operators in three dimensions are bilinears of boundary scalars φ or fermions ψ transforming under the global R-symmetry group,

$$\mathcal{O}_1 = \text{Tr}(\varphi^I a_{IJ} \varphi^J), \qquad \mathcal{O}_2 = \text{Tr}(\psi^I b_{IJ} \psi^J), \quad (17)$$

for some constant matrices a and b. According to the holographic dictionary, the expectation values of these operators are given by the boundary values of the bulk scalar field, $\langle \mathcal{O}_1 \rangle = \alpha$ and $\langle \mathcal{O}_2 \rangle = \beta$. Moreover, the ABJM action S_0 is deformed by multitrace operators, $S = S_0 + k \int \mathcal{O}_1^3$, along the lines of Refs. [22,32,33]. The dimensionless deformation parameter k is given by the boundary conditions of the scalar field, $\beta = k\alpha^2$, using the notation of Eq. (9). In our case, it is fixed to be $k = 1/\sqrt{6}$.

Since $\alpha \sim b_1$ and $\beta \sim b_1^2$, we can use b_1 as an order parameter and express it as a function of the temperature. Doing so, we get the plots shown in Fig. 5.

The condensate is actually never vanishing for any finite temperature, and it goes to zero in the limit $T \to \infty$, so the global R-symmetry is always broken. The behavior of the order parameter mimics more that of a liquid-gas transition, and one can determine the critical temperature from Maxwell's "equal area" construction.

The thermodynamical behavior of the black holes discussed in this paper should be reproduced by the boundary field theory thermal partition function on $S^1 \times S^2$.
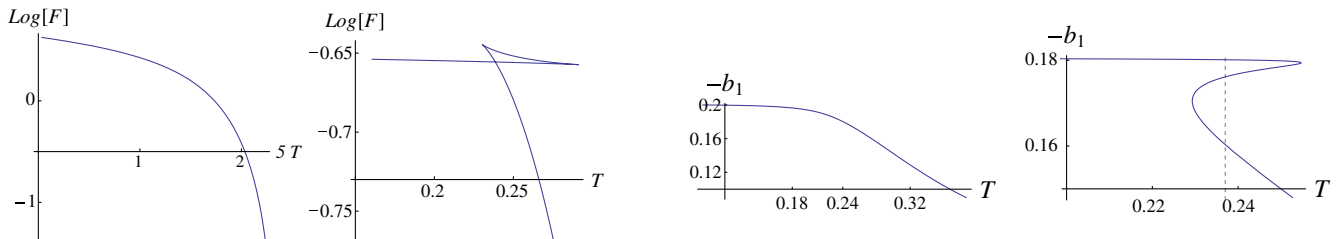

FIG. 4 (color online). Free energy as a function of temperature for the same fixed values of P as in Fig. 3.

FIG. 5 (color online). Behavior of the order parameter b_1 as a function of the temperature T for values $P > P_c$ (left, $P = 0.1$) and $P < P_c$ (right, $P = 0.015$).

At zero temperature, degeneracies of the ground state of this system should explain the finite entropy of the BPS black hole, presumably through the presence of Landau levels that arise because of the magnetic field. At finite temperature, we predict that the system undergoes a phase transition for magnetic charges below a critical value. It would be interesting to understand this phase transition directly in the boundary field theory, perhaps using the techniques recently developed by e.g. Ref. [34].

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