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Influence of Dzyaloshinskii-Moriya interactions on the properties of domain walls in a magnetic nanowire

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We study the influence of the Dzyaloshinkii-Moriya interaction on the structure of domain walls in magnetic nanowires. Using a variational approach, we show how the type of domain wall changes under the influence of the Dzyaloshinkii-Moriya interaction. In paticular, we show how the domain wall gradually transforms from a Bloch to Neel wall for increasing strength of the Dzyaloshinkii-Moriya interactios. These results are relevant for the development of modern technologies that make use of domain walls, such as the magnetic race-track memory.

Contents

I. INTRODUCTION

We consider a conducting magnetic wire. The magnetization is treated classically, so that it can be explicitly written as a steradian vector of unit length $\vec{\Omega}(x)$. The orientation is dependent on the position in the wire; however, this dependence on x shall be implied throughout this thesis, and the brackets will be omitted. The steradian $\vec{\Omega}$, can be expanded in polar coordinate θ, and azimuthal angle φ, via the usual identity $\vec{\Omega} = (\cos\varphi \sin\theta\,, \sin\varphi \sin\theta\,, \cos\theta)$, where dependence of θ and φ on x is also implicit. The nature of this dependence is the main subject of this thesis.

The approach towards analytically describing the orientation of the electron spins, is mainly mathematical. However, a few phenomenological remarks on the physics involved shall be made in this section.

A. Energy functional

Like in all physical systems, the energy of a magnetic wire 'prefers' to be minimized. This means that the magnetization is such, that making an arbitrary change would increase the energy. There are numerous physical effects that have an impact on the energy of a configuration. The primary effects are described below.

1. *Exchange interactions*

In ferromagnetic materials, spins of neighbouring particles, prefer to have the same orientation. For two electrons the energy due to this effect looks like $E \sim -J_s(\vec{\Omega}_A \cdot \vec{\Omega}_B)$, where J_s is called the exchange constant. It has units [Energy × distance^2]. The continuum limit of the energy due to spin-orbit coupling yields

$$\int_V \frac{J_s}{2}(\vec{\Omega} \cdot \nabla^2 \vec{\Omega}) \frac{d\vec{x}}{a^3},$$

where a is the distance between neighbouring sites, ie. the lattice constant. It has units [distance]. Rather than writing $(\vec{\Omega} \cdot \nabla^2 \vec{\Omega})$, it is sometimes chosen to write $(\vec{\nabla}\vec{\Omega})^2$, in which case a minus sign has to be factored in.

In antiferromagnetic materials the sign is inverted, and neighbouring spins prefer opposite configurations.

2. *Anisotropy*

Anisotropy, or directional asymmetry, causes certain directions of spin to be preferred over others. Anisotropic effects could be induced by applying an external field, or they could be due to the specific crystalline structure of the material in which the electrons are located, thus giving the spins a preferred orientation. Anisotropy affects the energy like

$$\int_V \frac{K_{\hat{e}}}{2}(\vec{\Omega} \cdot \hat{e})^2 \frac{d\vec{x}}{a^3}.$$

Here $\hat{e}$ is a unit vector, in the direction of the anisotropy (usually simply $\hat{x}$,$\hat{y}$, or $\hat{z}$). $K_{\hat{e}}$ is called the anisotropy constant in the respective direction. It has units [Energy]. Note that in order to make a certain direction energetically advantageous, rather than energetically disadvantageous, a minus sign could be added.

3. *Breaking of inversion symmetry*

The last type of interaction with the spins discussed here is called Dzyaloshinskii-Moriya interaction. (DM-interaction). It occurs in materials, in which inversion symmetry is broken, and where spint-orbit coupling is present. (In practice this is always the case). Usually, these materials are composed from two, or more, different kinds of metal. The spins couple with the material. The details of DM-interaction are beyond this thesis; however a crude description is given.

For two particles, the DM-interaction has the form: $E \sim A_{DM}[\vec{r} \times (\vec{x}_A - \vec{x}_B)] \cdot (\Omega_A \times \Omega_B)$, where $\vec{r}$ denotes the displacement of some ligand from the line connecting the two particles. Note that the way $\vec{r}$ influences the energy, depends completely on the properties of the material, and that there are several distinctive types of $\vec{r}$ one could imagine. Throughout this thesis, it has units [Energy × distance].

We make the distinction between two types of DM-interaction. Bulk DM-interaction, and interface-induced DM-interaction. The former has continuum limit

$$\int_V A_{DM}\vec{\Omega} \cdot (\vec{\nabla} \times \vec{\Omega}) \frac{d\vec{x}}{a^3},$$

and the latter has

$$\int_V A_{DM}\hat{y} \cdot \vec{\Omega} \times (\nabla\vec{\Omega}) \frac{d\vec{x}}{a^3},$$

for a one-dimensional wire along the x-axis, and with the $\hat{z}$ the direction of the interface normal.

B. Motivation

In magnetic wires, some or all of the above interactions play a part in the magnetization, depending on the material. Now, if we take a ferromagnetic wire, and create boundary conditions so that the magnetization has opposite direction, ie. $+\hat{z}$, and $-\hat{z}$, at either end of the wire; at some point the magnetization will have to switch. It has been found that this switch actually takes place at a small interval. This interval is called the domain wall.

One of practical applications of these domain walls is currently under development; an attempt is made to create circumstances on a wire, such that at regular intervals, there is either a domain wall, or not. The idea is to use this as a data point, ie. whether or not there is a change in magnetic orientation at a certain point in the wire, will signifiy a '1', or a '0'. This envisioned system is called 'racetrack memory'. A great benefit of this kind of data storage, is that is very small; an even better one still, is that the data can be moved through the wire by applying an external field; hence no more vulnerable moving parts!

A system with many adjacent domain walls, is only realizable if domain walls can remain stable in each other's proximity. Since across two domain walls the magnetization is back to where it came from, they have to be far enough apart so as not to affect each other. Knowing how various properties of materials influence the magnitude of the domain wall length is therefore crucial to creating the desire configuration.

One of the systems subject to experimental research uses a cobalt wire, with platinum layers on the bottom and top of the wire. It has properties such that the domain walls are relatively small; the nature and behaviour of domain walls where the main components that make up the energy are exchange interactions, and anisotropy, are well understood. However, since the material has broken inversion symmetry, Dzyaloshinkii-Moriya interaction occurs within the material. In this thesis the influence of DM-interaction on the properties of domain walls shall be investigated.

II. A MAGNETIC NANOWIRE WITHOUT DM-INTERACTION

In this section we consider a magnetic nanowire without DM-interactions and determine the domain-wall configuration. The preferred direction of magnetization is chosen to be the $\hat{z}$ direction, modeled by writing a positive anisotropy term in the y-direction, and a negative one in the z-direction. DM-interaction is considered negligible for now. The wire is assumed to be along the x-axis. Hence, the energy of this system is modeled as

$$E[\vec{\Omega}] = \int \frac{d\vec{x}}{a^3}\left[\frac{J_s}{2}\left(\vec{\Omega}\cdot\nabla^2\vec{\Omega}\right) + \frac{K_\perp}{2}\Omega_y^2 - \frac{K_z}{2}\Omega_z^2\right]. \tag{1}$$

The wire in considered to have a small enough y-, and z-width, typically in the region of nanometers, so that the gradient $\vec{\nabla}$ can be reduced to ∂_x, ie. $\partial_y\vec{\Omega} = 0$, $\partial_z\vec{\Omega} = 0$. Also, by this assumption, the y, and z component can be explicitly integrated; yielding section $\ell_y\ell_z$, where ℓ_y and ℓ_z are the respective widths of the wire in the y, and z-direction. The both have units [distance]. The energy can now be rewritten in terms of the polar angle θ and the azimuthal angle φ_0, as

$$E[\theta,\varphi] = \frac{\ell_y\ell_z}{a^3}\int dx\left\{-\frac{J_s}{2}[(\partial_x\theta)^2 + (\partial_x\varphi)^2\sin^2\theta] + \frac{K_\perp}{2}\sin^2\theta\sin^2\varphi - \frac{K_z}{2}\cos^2\theta\right\}. \tag{2}$$

In order to achieve a physical solution for the spin configuration, the energy must be minimized, under the appropriate boundary conditions. Note that the expression for the energy (2) becomes trivial when a solution of the form $\{\theta = m\pi, \varphi = k\pi\}$, is chosen, where m and k are arbitrary integers; for all terms are minimized by these solutions. The solution we are interested in, however, has an upward pointing orientation on the one side, and a downward pointing orientation on the other, ie

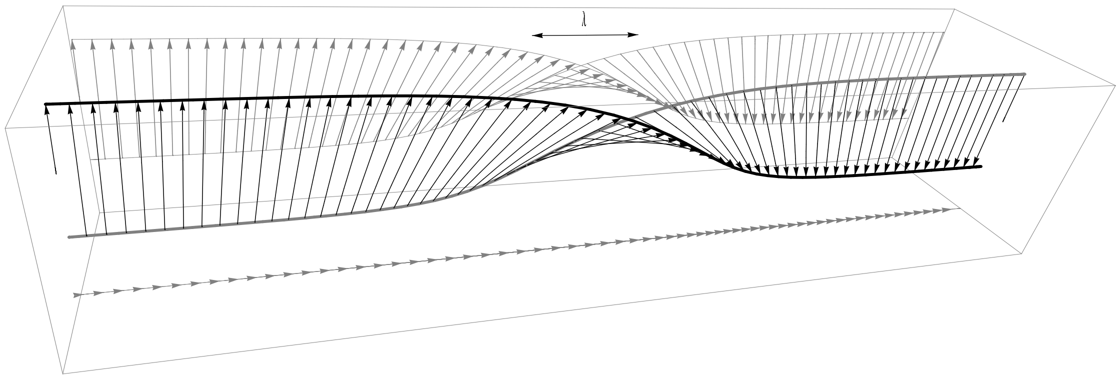
$$\lim_{x\to\pm\infty}\vec{\Omega} = \begin{pmatrix}\pm1\\0\\0\end{pmatrix}. \tag{3}$$

These shall be the boundary conditions used in this section. The lowest-energy configuration obeying these boundary condition corresponds to a domain wall.

Since the system described in this section favours orientations that have an x-component to those that have a y-component, the spins are expected to gradually change their polar angle θ from 0 to π, within the xz-plane, as a function of the position x in the wire. As shall be seen later in this section, the major part of the change in orientation takes place within a small region. This region is called the domain wall, and has length λ, which shall be calculated analytically in this section. A domain wall that has $\varphi = 0$ is called a Néel type domain wall, after the French physicist Louis Néel. Such a domain wall is depicted in

FIG. 1.

FIG. 1. Néel type domain wall. ($\varphi = 0$)



An attempt is made to find a solution for this type of system, satisfying the above boundary conditions. The integrand in the expression for the energy (2) denotes the energy density $\mathcal{E}$ of the system. After filling in $\varphi = k\pi$, the energy density reduces to

$$\mathcal{E} = \frac{\ell_y \ell_z}{a^3}\left[\frac{J_s}{2}(\partial_x \theta)^2 - \frac{K_z}{2}\cos^2\theta\right].$$

In order to minimize the energy we make use of the Euler-Lagrange equation $\partial_x\left(\frac{\partial \mathcal{E}}{\partial(\partial_x\theta)}\right) = \partial_\theta \mathcal{E}$, from which differential equation

$$\partial_{xx}\theta = \partial_\theta\left(-\frac{K_z}{2J_s}\cos^2\theta\right), \tag{4}$$

is obtained. It is solved using a few tricks. First of all, by filling in the boundary conditions, the system is reduced from second order to first order. The chain rule for differentiation prescribes that for any continuously differentiable function f, $[\partial_x f(\theta)] = (\partial_x\theta)[\partial_\theta f(\theta)]$, as long as f does not explicitly depend on $\partial_x\theta$, or x. Realizing this, and multiplying previous expression by $2(\partial_x\theta)$ yields

$$2(\partial_x\theta)(\partial_{xx}\theta) = \partial_x\left(-\frac{K_z}{J_s}\cos^2\theta\right).$$

The preceding step allows for integration along the whole x-axis, producing

$$(\partial_x\theta)^2 = C - \frac{K_z}{J_s}\cos^2\theta, \tag{5}$$

where C is an integration constant, to be determined by applying boundary conditions, (3).

$$\lim_{x\to-\infty}\theta = 0, \quad \lim_{x\to+\infty}\theta = \pi, \quad \lim_{x\to\pm\infty}(\partial_x\theta) = 0.$$

Plugging these conditions into differential equation (5) gives the solution $C = \frac{K_z}{J_s}$. The identity $\cos^2 u + \sin^2 u = 1$, gives rise to differential equation

$$(\partial_x\theta)^2 = \frac{K_z}{J_s}\sin^2\theta. \tag{6}$$

To solve (6), note that from the mean value theorem it follows that θ will, somewhere along the x-axis, assume the value $\pi/2$. Define the domain wall position r_{dw} to be the associated x, ie: $\theta(r_{dw}) = \pi/2$. Integrating the square root of (6) from the position of the domain wall r_{dw} leads to

$$\int_{\frac{\pi}{2}}^{\theta(x)} \frac{d\theta}{\sin\theta} = \ln\tan\frac{\theta}{2} = \pm\sqrt{\frac{K_z}{J_s}}(x - r_{dw}).$$

This expression can be further simplified to obtain the solution

$$\theta = 2\arctan\left(e^{\frac{Q(x-r_{dw})}{\lambda}}\right). \tag{7}$$

Here λ denotes the domain wall width, a measure for the length over which the magnetization changes direction. It has the value

$$\lambda \equiv \sqrt{\frac{J_S}{K_z}}. \tag{8}$$

Parameter $Q \equiv \pm 1$ is called the charge, and determines whether the magnetization changes from $+\hat{z}$ to $-\hat{z}$ or oppositely, when moving in the positive x-direction. The exponent in the argument of the arctangent of the solution for θ, is a dimensionless term that will occur many times throughout this thesis. Hence we introduce the definition

$$\xi \equiv \frac{Q(x - r_{dw})}{\lambda}.$$

III. INFLUENCE OF BULK DM-INTERACTION ON DOMAIN WALL CONFIGURATION

For now, the above result concludes our investigation on inversion-symmetric materials, such that there is no DM-interaction. We have seen that if the boundary conditions are such that the magnetization changes from $+\hat{z}$ to $-\hat{z}$, the magnetization along the x-axis will show a domain wall. In this section it is investigated what happens when the bulk inversion symmetry of the system is broken, and therefore there is a form of DM-interaction, bulk in this case. Mathematically, this leads to adding the term

$$A_{DM}\vec{\Omega}\cdot(\vec{\nabla}\times\vec{\Omega}),$$

to the energy density. This term favours a gradient in the configuration. Since we assume the y- and z-direction of the wire to be very small, we can take $\partial_y\vec{\Omega}$ and $\partial_z\vec{\Omega}$ to be zero. In terms of the polar angle θ, and the azimuthal angle φ, the contribution of this type of Dzyaloshinskii-Moriya interaction then becomes

$$A_{DM}\big((\partial_x\theta)\sin\varphi + (\partial_x\varphi)\cos\theta\sin\theta\cos\phi\big).$$

In the presence of the above DM interactions the energy is given by

$$\begin{aligned} E[\theta,\varphi] = \frac{\ell_y\ell_z}{a^3}\int dx \bigg\{&\frac{J_S}{2}[(\partial_x\theta)^2 + (\partial_x\varphi)^2\sin^2\theta] + \frac{K_\perp}{2}\sin^2\theta\sin^2\varphi - \frac{K_z}{2}\cos^2\theta \\ &+ A_{DM}\left[(\partial_x\theta)\sin\varphi + \frac{1}{2}(\partial_x\varphi)\sin\theta\cos\theta\cos\varphi\right]\bigg\}. \end{aligned} \tag{9}$$

A. Configurations for the modified energy density

For this modified energy, it is again desirable to find solutions for angles θ and φ. Like in the previous section, we start with the Euler-Lagrange equations; which, after a few rearrangements look like so:

$$\begin{cases} \partial_{xx}\theta = \dfrac{1}{2}(\partial_x\varphi)^2\sin 2\theta + \dfrac{K_\perp}{2J_s}\sin 2\theta\sin^2\varphi + \dfrac{K_z}{J_s}\sin 2\theta - \dfrac{2A_{DM}}{J_s}(\partial_x\varphi)\sin^2\theta\cos\varphi\,; \\ (\partial_{xx}\varphi)\sin^2\theta = -(\partial_x\theta)(\partial_x\varphi)\sin 2\theta + \dfrac{K_\perp}{2J_s}\sin 2\varphi\sin^2\theta + \dfrac{2A_{DM}}{J_s}(\partial_x\theta)\sin^2\theta\cos\varphi. \end{cases} \tag{10}$$

We collect terms; and in the equation for φ, dividing by $\sin^2\theta$; thus excluding solutions $\theta = k\pi$, which we can safely do since by our boundary conditions this only happens at infinity.

$$\begin{cases} \partial_{xx}\theta = \frac{1}{2}(\partial_x\varphi)^2 \sin 2\theta + \frac{K_\perp}{2J_s}\sin 2\theta \sin^2\varphi + \frac{K_z}{J_s}\sin 2\theta - \frac{2A_{DM}}{J_s}(\partial_x\varphi)\sin^2\theta\cos\varphi\,; \\ \partial_{xx}\varphi = -2(\partial_x\theta)(\partial_x\varphi)\cot\theta + \frac{K_\perp}{2J_s}\sin 2\varphi + \frac{2A_{DM}}{J_s}(\partial_x\theta)\cos\varphi\,. \end{cases} \tag{11}$$

Since this set of equations is quite difficult to solve; it seems sensible, to treat some special cases, so as to gain insight in the behaviour of the system.

1. *Configurations of a system with zero anisotropy.*

When the anisotropy terms are removed, ie. $K_\perp = 0$, and $K_z = 0$, the system of equations (11) reduces to

$$\begin{cases} \partial_{xx}\theta = \frac{1}{2}(\partial_x\varphi)^2 \sin 2\theta - \frac{2A_{DM}}{J_s}(\partial_x\varphi)\sin^2\theta\cos\varphi \\ \partial_{xx}\varphi = -2(\partial_x\theta)(\partial_x\varphi)\cot\theta + \frac{2A_{DM}}{J_s}(\partial_x\theta)\cos\varphi \end{cases}. \tag{12}$$

With no anisotropy, physically we expect the spins to avoid the x-direction, since, due to the DM-interaction, it is no longer unfavourable to have such an orientation, thus having $\varphi = \frac{\pi}{2} + k\pi$, where k is some integer. Mathematically, one can observe that, by assuming this solution for φ, the second equation of (12) is indeed solved, and that the first is reduced to: $\partial_{xx}\theta = 0$. This implies the spins have a constantly changing polar angle, (in other words, θ has a constant derivative), and therefore configure like a helicoid. The solution can be written as $\theta = q'(x + c)$, where q' and c are constants to be determined.

Plugging the set of solutions into the energy equation (9), yiels the expression

$$E[\theta,\varphi]_{K_\perp=0,K_z=0} = \frac{\ell_y\ell_z}{a^3}\int\left[\frac{J_s}{2}q'^2 + A_{DM}(-1)^{k+1}q'\right]dx. \tag{13}$$

Since the integrand does not depend on x, it is minimized by solving $\frac{J_s}{2}q'^2 + (-1)^{k+1}q'A_{DM} = 0$, and so $q' = (-1)^{k+1}\left(\frac{2A_{DM}}{J_s}\right)$.

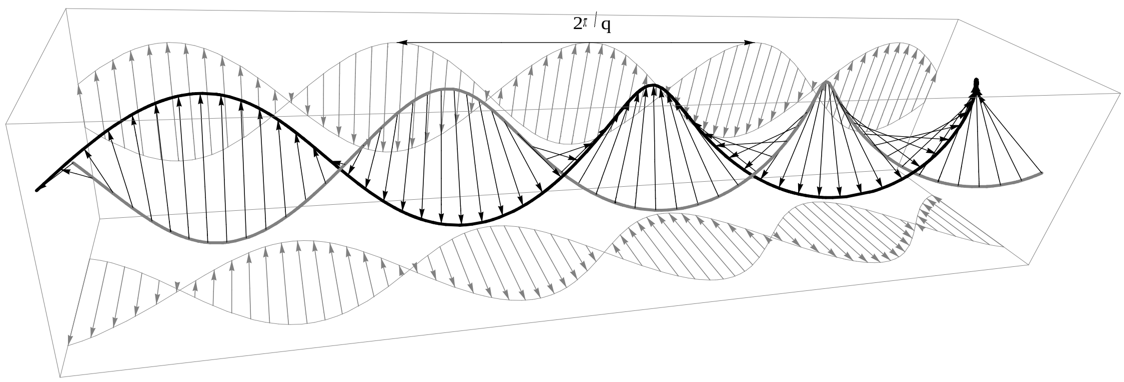
Summarizing the above; for $K_\perp = 0, K_z = 0$, we have solutions

$$\begin{cases} \theta = q(-1)^{k+1}(x + c) \\ \varphi = \frac{\pi}{2} + k\pi \end{cases}, \tag{14}$$

where $q = \left(\frac{2A_{DM}}{J_s}\right)$.

Since there is no anisotropy, and since the DM-interaction induces a gradient, the spins form a helix, as seen in FIG. 2. Hence q is a measure for the wavelength. As an example, in MnSi $q \sim \frac{2\pi}{20}\,\mathrm{nm}^{-1}$.

FIG. 2. Helicoid magnetization.



2. Configurations of a system with different anisotropy.

So as to expand the results found in the previous paragraph, in this section it will be shown how the separate anisotropy terms, $K_\perp$, and K_z, influence the configurations. Recall that the physical interpretation of the anisotropy terms, is the 'energy cost' for the spins to be oriented in the respective direction.

From this, one expects K_z to have no influence on the solution of φ. Assuming this solution to be true for nonzero K_z, will lead to Euler-Lagrange equation:

$$\partial_{xx}\theta = \frac{K_z}{2J_s}\sin 2\theta.$$

Note that this could be rewritten to equation (4), which is solved by (7). So when $J_s, K_z, A_{DM} \neq 0$, but $K_\perp = 0$, the system is solved by:

$$\begin{cases} \theta = 2\arctan\left(e^{\xi}\right) \\ \varphi = \dfrac{\pi}{2} + k\pi \end{cases}.$$

This form of configuration, perpendicular to the xz-plane, is called Bloch-type. It is described in detail in section IV, and it is depicted in FIG. 3.

The above result can be interpreted as follows: For nonzero K_z, the current type of DM-interaction pushes the magnetization out of the xz-plane. Since the term with $K_\perp$ actually favours magnetization within the xz-plane, these two quantities have opposing interest. It is expected that the magnitude determines which of these two 'wins'. An exact analytical solution is difficult to obtain, however, by making some justified guesses, an estimation can be made.

B. Variational approach to approximate the energy

Now that a few limiting solutions have been explored, the following procedure can be used so as to improve a found solution. First, the most suitable (guessed) solution is plugged back into the energy functional. Since the dependence of the energy on x is now explicit, the integration can be carried out. After integration, the energy can be minimized over some parameter left free, by setting the derivative of the energy with respect to that variable equal to zero, and solving. Based on the previous part, the most suitable ansätze seem:

$$\begin{cases} \theta = 2\arctan\left(e^{\xi}\right) \\ \varphi = \varphi_0 \end{cases}, \tag{15}$$

where φ_0 is some constant, expected to be $\frac{\pi}{2}$ for comparatively small $K_\perp$, and 0 for large $K_\perp$. When filling in the ansatz $\varphi = \varphi_0$, all derivatives of φ become zero. Hence the energy equation (9) becomes:

$$E[\theta,\varphi] = \frac{\ell_y \ell_z}{a^3}\int dx \left\{\frac{J_s}{2}(\partial_x\theta)^2 + \frac{K_\perp}{2}\sin^2\theta\sin^2\varphi_0 - \frac{K_z}{2}\cos^2\theta + A_{DM}(\partial_x\theta)\sin\varphi_0\right\}. \tag{16}$$

In order to fill in $\theta = 2\arctan\left(e^{\xi}\right)$, the following relations from the appendix are used: $\partial_x\theta = \frac{Q}{\lambda}\operatorname{sech}\xi$, and $\sin\theta = \operatorname{sech}\xi$. This leads to

$$E[\varphi_0] = \frac{\ell_y \ell_z}{a^3}\int dx \left(-\frac{J_s}{2}\operatorname{sech}^2\xi + \frac{K_\perp}{2}\operatorname{sech}^2\xi\sin^2\varphi_0 - \frac{K_z}{2}\tanh^2\xi + A_{DM}\frac{Q}{\lambda}\operatorname{sech}\theta\sin\varphi_0\right). \tag{17}$$

Rather than going into the details of the mathematical steps, we simply present the most important results of the approach described above. The same approach is taken, for a slightly different system; in equations (26), through (35). There, the mathematics are explained in detail. The resulting energy of the system described above, is

$$E[\varphi_0] = \frac{\ell_y \ell_z}{a^3}\left[-\frac{J_s}{\lambda} + \lambda\left(K_y + K_z\sin^2\varphi_0\right) + A\pi Q\sin\varphi_0\right]. \tag{18}$$

When explicitly filling in λ it becomes

$$E[\varphi_0] = \frac{\ell_y \ell_z}{a^3}\left(A\pi Q\sin\varphi_0 + \frac{\sqrt{J_s}K_y}{\sqrt{K_z}}\sin^2\varphi_0\right). \tag{19}$$

It is minimized by

$$\varphi_0 = 0, \quad \varphi_0 = \pi, \quad \varphi_0(A_{DM}) = -\frac{\pi}{2} \pm \arccos\left(\frac{A\pi Q\sqrt{K_z}}{2K_y\sqrt{J_s}}\right)$$

Define $A_c \equiv \frac{2K_y\sqrt{J_s}}{\pi\sqrt{K_z}}$ so that the energy is minimized by:

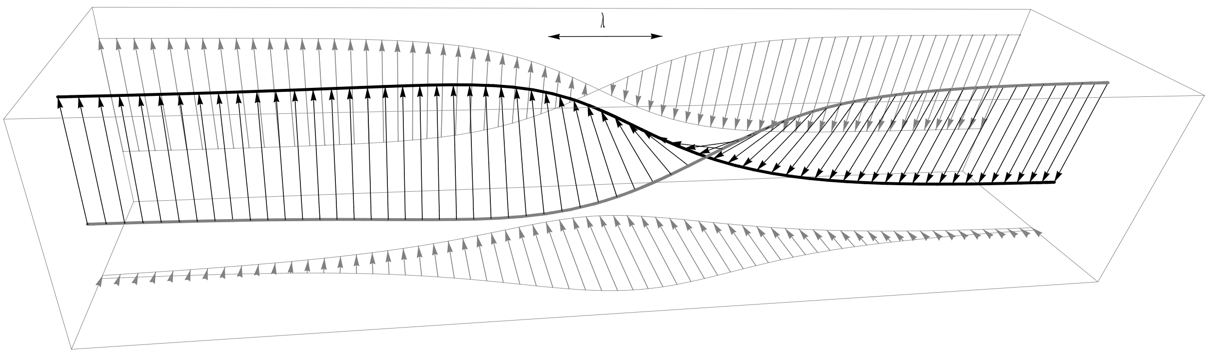
$$\varphi_0 = \begin{cases} -\dfrac{\pi}{2} & A_{DM} < -A_c \\ -\dfrac{\pi}{2} \pm \arccos\left(-\dfrac{QA_{DM}}{A_c}\right) & |A_{DM}| \leq A_c. \\ \dfrac{\pi}{2} & A_{DM} > A_c \end{cases} \tag{20}$$

The above result signifies a smooth phase transition between Néel-type ($\varphi_0 = 0$) domain walls, for zero A_{DM}, to a Bloch-type ($\varphi_0 = \pi/2$) domain wall. This result is in correspondence to the limits that have been found in the preceding paragraphs. We will refrain from going into many details about this solution, but we will refer back to it and compare the result to (35) in the next section.

IV. INTERFACE INDUCED DM-INTERACTION

In this section we consider layered magnetic systems, where the interface can be shown to lead to DM-interactions. These materials have perpendicular magnetic anisotropy. This causes the magnetization of the domain wall to be perpendicular to the x-axis (in the absence of DM-interactions) It is called a Bloch domain wall, after the Swiss physicist Felix Bloch. A domain wall of this type is depicted in FIG. 3.

FIG. 3. Bloch type domain wall. ($\varphi = \pi/2$)



The energy associated with this kind of materials, looks like

$$E_{MM}[\vec{\Omega}] = \int \frac{d\vec{x}}{a^3} \left\{\frac{J_s}{2}\vec{\Omega} \cdot (\partial_x^2\vec{\Omega}) + \frac{K_x}{2}\Omega_x^2 + \frac{K_z}{2}(1 - \Omega_z^2) + A_{DM}\hat{y} \cdot \vec{\Omega} \times (\partial_x\vec{\Omega})\right\}. \tag{21}$$

Note that the term $-\frac{K_z}{2}\Omega_z^2$, has been adjusted to $+\frac{K_z}{2}(1 - \Omega_z^2)$, so that the minimal contribution of this term is zero, rather than -1. Since the wire is set to equal the x-axis; and assumed to be thin enough to neglect contribution of the width, the y, and z components can be integrated out. Recall that $\vec{\Omega} \equiv (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$, so that

$$\begin{aligned} E_{MM}[\theta, \varphi] = \ell_y \ell_z \int \frac{dx}{a^3} \Big\{ -\frac{J}{2}[(\partial_x\theta)^2 + \sin^2\theta\,(\partial_x\varphi)^2] + \frac{K_x}{2}\cos^2\varphi \sin^2\theta + \frac{K_z}{2}\sin^2\theta \\ + A_{DM}[(\partial_x\theta)\cos^2\varphi - (\partial_x\varphi)\cos\theta \sin\theta \sin\varphi]\Big\}. \end{aligned} \tag{22}$$

This leads to Euler-Lagrange equations:

$$\begin{cases} \partial_{xx}\theta = \dfrac{1}{2}(\partial_x\varphi)^2 \sin 2\theta + \dfrac{K_x}{2J}\sin 2\theta \cos^2\varphi + \dfrac{K_z}{2J}\sin 2\theta + 2\dfrac{A_{DM}}{J}(\partial_x\varphi)\sin^2\theta \sin\varphi\,; \\ \partial_{xx}\varphi = 2(\partial_x\theta)(\partial_x\varphi)\cot\theta - \dfrac{K_x}{2J}\sin 2\varphi + \dfrac{A_{DM}}{J}2(\partial_x\theta)\sin\varphi\,. \end{cases} \tag{23}$$

Inspired by the calculations of the previous paragraph, assume the solution to be of the form:

$$\theta = 2\arctan(\xi)\,, \qquad \varphi = \frac{\pi}{2}, \qquad \xi = \frac{Q(x - r_{dw})}{\lambda}. \tag{24}$$

For reasons that will be clear later on, λ is left undefined for now. Ignoring the contribution of the Dzyaloshinskii-Moriya interaction, ie. $A_{DM} = 0$, for now, it will be shown that solutions (24) indeed satisfy (23). First, take the derivative of θ, ie,

$$LHS = \partial_{xx} 2\arctan\left(e^{\xi}\right)$$

$$= 2\frac{Q}{\lambda}\partial_x\left(\frac{e^{\xi}}{e^{2\xi}+1}\right),$$

Which, by the quotient rule results in

$$= -\frac{2}{\lambda^2}\left(\frac{e^{\xi}\left(e^{2\xi}-1\right)}{(e^{2\xi}+1)^2}\right)$$

$$= -\frac{1}{\lambda^2}\operatorname{sech}(\xi)\tanh(\xi).$$

Realizing that derivatives of φ, and $\sin\varphi$, become zero, the right hand side, using ansätze (24) becomes

$$RHS = \frac{1}{2}(\partial_x\varphi)^2\sin 2\theta + \frac{K_z}{2J_s}\sin 2\theta + \frac{K_x}{2J_s}\sin 2\theta\cos^2\varphi + 2\frac{A_{DM}}{J_s}(\partial_x\varphi)\sin^2\theta\sin\varphi$$

$$= \frac{1}{2}\left(\frac{K_x + K_z}{J_s}\right)\sin\left[4\arctan\left(e^{\xi}\right)\right].$$

using (48) from the appendix

$$= -2\left(\frac{K_x + K_z}{J_s}\right)\frac{e^{\xi}\left(e^{2\xi}-1\right)}{(e^{2\xi}+1)^2}$$

$$= -\left(\frac{K_x + K_z}{J_s}\right)\operatorname{sech}(\xi)\tanh(\xi).$$

Hence the equality is satisfied, up to domain wall width λ^2, which, for this type of material, is redefined as

$$\lambda^{-1} \equiv \sqrt{\frac{K_x + K_z}{J_s}}. \tag{25}$$

In the second equation of (23), each term has either a derivative of φ, or a term that contains a factor $\sin\varphi$. Since the ansatz for $\varphi = 0$, all terms of the equality become zero, therefore it is also satisfied.

A. Inclusion of Dzyaloshinskii-Moriya interaction

We are now interested in solving the system of equations, where we do not neglect the DM-interaction. Since the extra terms make the system of differential equations analytically troublesome to solve, it seems sensible to make variations on the solutions found so far. The assumption that the azimuthal angle φ is some constant φ_0, independent of x, gets rid of all of the terms with derivatives $\partial_x\varphi$. This is physically justifiable, by noting that a constant azimuthal angle satisfies the boundary solutions, and it is a solution to the Euler Lagrange equations (23), and therefore minimizes the energy.

After filling in $\varphi \to \varphi_0$, the energy (22) becomes

$$E_{MM}[\theta,\varphi_0] = \ell_y \ell_z \int \frac{dx}{a^3} \left\{-\frac{J_s}{2}(\partial_x\theta)^2 + \frac{K_x}{2}\cos^2\varphi_0 \sin^2\theta + \frac{K_z}{2}\sin^2\theta + A_{DM}\cos\varphi_0\,(\partial_x\theta)\right\}. \tag{26}$$

As for the polar angle of the magnetization θ, it seems appropriate to start with the solution from the ansätze (24), ie. $\theta = 2\arctan e^{\xi}$. Using identities $\sin\theta = \operatorname{sech}\xi$, and $\partial_x\theta = \frac{Q}{\lambda}\operatorname{sech}\xi$; from the appendix, we can write

$$E_{MM}[\varphi_0] = \ell_y \ell_z \int \frac{dx}{a^3} \left\{\left(-\frac{J_s}{2}\frac{Q^2}{\lambda^2} + \frac{K_z + K_x\cos^2\varphi_0}{2}\right)\operatorname{sech}^2\xi + \frac{Q}{\lambda}A_{DM}\cos\varphi_0\operatorname{sech}\xi\right\}. \tag{27}$$

Since all terms explicitly depend on x, the integration can be performed, to obtain

$$E_{MM}\,[\varphi_0] = \frac{\ell_y\ell_z}{a^3}\left[\left(-\frac{J_s}{2}\frac{Q}{\lambda} + \frac{\lambda}{Q}\frac{K_z + K_x\cos^2\varphi_0}{2}\right)\tanh(\xi) + 2A_{DM}\cos\varphi_0\arctan\left(e^{\xi}\right)\right]_{x=-\infty}^{\infty}. \tag{28}$$

Here we have used standard integrals, to be found in the appendix. Note that $\lim_{x\to\pm\infty}\tanh(\xi) = \pm Q$, and that $\lim_{x\to\pm\infty}\arctan\left(e^{\xi}\right) = Q\pi\left(\frac{1}{4}\pm\frac{1}{4}\right)$. This allows for evaluation of the integration limits, so as to acquire energy

$$E_{MM}\,[\varphi_0] = \frac{\ell_y\ell_z}{a^3}\left[\left(-\frac{J_s}{2}\frac{Q}{\lambda} + \frac{\lambda}{Q}\frac{K_z + K_x\cos^2\varphi_0}{2}\right)(2Q) + 2A_{DM}\cos\varphi_0\left(\frac{\pi}{2}Q\right)\right]. \tag{29}$$

Finally, using $Q^2 = 1$, the expression for the energy becomes

$$E_{MM}\,[\varphi_0] = \frac{\ell_y\ell_z}{a^3}\left[-\frac{J_s}{\lambda} + \lambda(K_z + K_x\cos^2\varphi_0) + Q\pi A_{DM}\cos\varphi_0\right]. \tag{30}$$

At this point, it seems suitable to fill in λ explicitly, from (25), so as to obtain the final expression for the energy as a function of azimuthal angle φ_0. By using the identity $1-\cos^2 u = \sin^2 u$, the expression becomes

$$E_{MM}\,[\varphi_0] = \frac{\ell_y\ell_z}{a^3}\left(Q\pi A_{DM}\cos\varphi_0 - \frac{\sqrt{J_s}K_x\sin^2\varphi_0}{\sqrt{K_x + K_z}}\right). \tag{31}$$

1. *Minimzation of the obtained energy with respect to the azimuthal angle*

Now that an expression for the energy has been found, an expression for the azimuthal angle φ_0 can be found, such that the energy is minimized. To find the minimal energy, first take the derivative of (31) with respect to φ_0:

$$\partial_{\varphi_0}E_{MM}[\varphi_0] = -\frac{\ell_y\ell_z}{a^3}\sin\varphi_0\left(Q\pi A_{DM} + \frac{2\sqrt{J_s}K_x\cos\varphi_0}{\sqrt{K_x + K_z}}\right). \tag{32}$$

Setting this derivative to zero gives rise to three classes of solutions. Note that, since it concerns an angle, each of the solutions could have an arbitrary integer multiple of 2π added to them. The solutions are

$$\varphi_0 = 0, \quad \varphi_0 = \pi, \quad \varphi_0(A_{DM}) \equiv \pm\arccos\left(-\frac{\pi\sqrt{K_x + K_z}QA_{DM}}{2\sqrt{J_s}K_x}\right). \tag{33}$$

The last solutions has brackets, to signify the dependence on A_{DM}. In order to find out which of these three solutions will manifest itself, the energy of each of the solutions will be written. The lowest of the three will be the global minimum. Plugging both solutions back into the expression for the energy (30) yields:

$$
\begin{aligned}
E_{MM}\,[\varphi_0 \to 0] &= \frac{\ell_y \ell_z Q \pi A_{DM}}{a^3} \\
E_{MM}\,[\varphi_0(A_{DM})] &= \frac{\ell_y \ell_z}{a^3}\sqrt{J_s(K_z+K_x)}\left(\frac{K_x}{K_x+K_z} - \frac{\pi^2 A_{DM}^2}{4 J_s K_x}\right), \qquad (34)\\
E_{MM}\,[\varphi_0 \to \pi] &= -\frac{\ell_y \ell_z Q \pi A_{DM}}{a^3}
\end{aligned}
$$

Now, since all solutions for the energy depend on the DM-interaction, the global minimum can be found in terms of A_{DM}. Setting the above energies equal, quite simply solved by noting that $\arccos(1) = 0$ and $\arccos(-1) = \pi$, results in the critical interaction parameter:

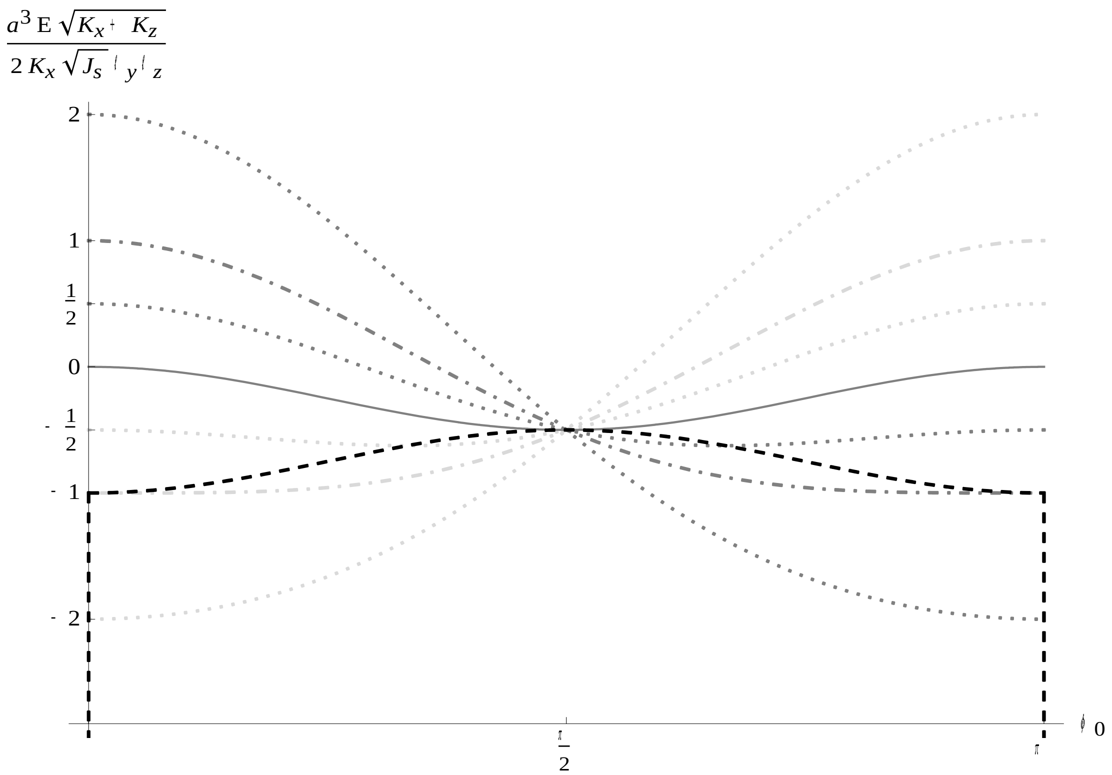
$$
A_c \equiv \frac{2\sqrt{J_s}K_x}{\sqrt{K_x+K_z}\pi}, \qquad (35)
$$

so that the azimuthal angle for which the energy is minimized as a function of $A_{DM}$is obtained:

$$
\varphi_0 = \begin{cases} 0 & A_{DM} < -A_c \\ \pm\arccos\left(-\frac{QA_{DM}}{A_c}\right) & |A_{DM}| \le A_c. \\ \pi & A_{DM} > A_c \end{cases}
$$

The above result can be interpreted as a phase transition, from a mixed-type system, for low values of $|A_{DM}|$, to a Néel type-system, for $QA_{DM} > A_c$. It is a second order phase transition, because the global minimum energy changes continuously, ie. there are no jumps. In the limit $A_{DM} \to 0$ the system is a pure Bloch-type system. Plots of the dimensionless energy as a function of φ_0 are presented in FIG. 4. The energy has been made dimensionless through dividing it by $\ell_y \ell_z \pi A_c / a^3$.

FIG. 4. Plots of the dimensionless energy, for various values of (QA_{DM}/A_c). Plotted are 0 (solid), ± 1 (dash-dotted), $\pm 1/2$, and ± 2, (both dotted). Dark grey lines have positive value, light grey lines have negative value. The dashed black line marks position of the global minimum.



As a final remark on this, not the similarity between this solution and (20). The plot for that system will look the same, with the only differences being a different expression for the respective critical DM-interaction parameters, and a shift of the origin, from $\pi/2$ to 0.

B. Refinement of the expression for the domain wall width

In the previous section, a local solution has been found for the azimuthal angle $\varphi_0$under which the energy of a Bloch-type material with subcritical DM-interaction is minimized. The obtained solution, ie. the third solution from (33), can be used to as a new ansatz. This allows for an attempt of solving the Euler-Lagrange equations for this system, with a stronger ansatz. The solution for the polar angle θ, is expected to be the same; however, with the newly achieved information about the local behaviour of $\varphi_0(A_{DM})$, a refinement could be made on the expression for the domain wall width. Starting with the energy equation (26), along with the solution from (33), $\varphi_0(A_{DM}) = \arccos(-QA_{DM}/A_c)$, the energy equation can be rewritten as

$$E_{MM}[\theta, \varphi_0(A_{DM})] = \ell_y \ell_z \int \frac{dx}{a^3} \left[-\frac{J_s}{2}(\partial_x \theta)^2 + \frac{K_x A_{DM}^2}{4A_c^2} \sin^2\theta + \frac{K_z}{2} \sin^2\theta - \frac{A_{DM}^2}{2A_c}(\partial_x \theta) \right]. \tag{36}$$

It is once again only dependent on θ. Recall that the Euler-Lagrange equation on θ reads $\partial_x \left(\frac{\partial \mathcal{E}}{\partial(\partial_x \theta)} \right) = \partial_\theta \mathcal{E}$, where $\mathcal{E}$ in this case in the integrand from (36). After dividing by $-J_s$, it reads

$$\partial_{xx}\theta = -\partial_\theta \left[\left(\frac{K_x A_{DM}^2}{4J_s A_c^2} + \frac{K_z}{2J_s} \right) \sin^2\theta \right].$$

The procedure followed here, to decrease the order of the differential equation from second, to first order, is just as in part II, eqs. (4) through (6). First, the above equation is multiplied on both sides by $(\partial_x \theta)$. Then, by integrating over x, the following relation of obtained:

$$(\partial_x \theta)^2 + \left(\frac{K_x A_{DM}^2}{2J_s A_c^2} + \frac{K_z}{J_s} \right) \sin^2\theta = C,$$

where C is the integration constant, which should satisfy boundary conditions (3). Since at infinity both $\partial_x \theta$ and $\sin\theta$ equal zero, $C = 0$. As will be seen later, choosing C, so as to invert the sign of the sine, makes sure the solution for θ becomes 0 at minus infinity, and π at plus infinity, rather than vice versa. Taking the square root yields first order differential equation:

$$\partial_x \theta = \sqrt{\frac{K_x A_{DM}^2}{2J_s A_c^2} + \frac{K_z}{J_s}} \sin\theta.$$

It is solved by $\theta(x) = \arctan\left[\exp\left(\frac{Q(x - r_{dw})}{\mu} \right) \right]$. Here $\mu^{-1} \equiv \sqrt{\frac{K_x A_{DM}^2}{2J_s A_c^2} + \frac{K_z}{J_s}}$, is the refined expression for the domain wall width, for small values of A_{DW} in a Bloch-type system. After explicitly filling in A_c from equation (35) it reads

$$\mu^{-1} = \sqrt{\frac{K_z}{J_s} + \frac{A_{DM}^2 \pi^2 (K_x + K_z)}{J_s^2 K_x}}. \tag{37}$$

Observe that for small A_{DM}, it is just the same as the previously achieved λ, as written in (8).

C. Further refinement by using domain wall width depending on the azimuthal angle

Thus far, the domain wall width λ was assumed to be independent of φ_0. In this paragraph, a more general expression for the domain wall width is suggested, dependent on φ_0, denoted $\lambda_{DW}(\varphi_0)$. In order to find an appropriate formula, we once again assume a constant φ_0. Thus obtaining (26),

$$E_{MM}[\theta, \varphi_0] = \ell_y \ell_z \int \frac{dx}{a^3} \left\{ -\frac{J_s}{2}(\partial_x \theta)^2 + \frac{K_x}{2} \cos^2\varphi_0 \sin^2\theta + \frac{K_z}{2} \sin^2\theta + A_{DM} \cos\varphi_0 (\partial_x \theta) \right\}.$$

The corresponding Euler-Lagrange equation for θ is

$$\partial_{xx}\theta = -\left(\frac{K_x \cos^2 \varphi_0 + K_z}{2J_s}\right) \sin 2\theta.$$

Note that this can be rewritten to resemble (4) up to terms that are constant with respect to x. It is solved using the procedure introduced in II, leading to $\theta = 2 \arctan\left[\exp \frac{Q(x-r_{dw})}{\lambda_{DW}(\varphi_0)}\right]$, where

$$\lambda_{DW}^{-1}(\varphi_0) \equiv \sqrt{\frac{K_x \cos^2 \varphi_0 + K_z}{J_s}}. \tag{38}$$

Note that it indeed depends on the azimuthal angle φ_0. Also note that when φ_0 is taken to be zero, the domain wall width reduces to (25), which is quite desirable. The energy is now expressed in terms of φ_0; since $\lambda_{DW}(\varphi_0)$ is independent of x, the exact same steps from (26) through (30) can be taken, so as to obtain

$$E_{MM}\,[\varphi_0] = \frac{\ell_y \ell_z}{a^3}\left[-\frac{J_s}{\lambda_{DW}(\varphi_0)} + \lambda_{DW}(\varphi_0)(K_z + K_x \cos^2 \varphi_0) + Q\pi A_{DM} \cos \varphi_0\right].$$

When taking the derivative with respect to φ_0 the dependence of $\lambda_{DW}(\varphi_0)$ has to be taken into account. Notice that all the terms cancel (!), so that the above expression reduces to

$$E_{MM}\,[\varphi_0] = \frac{\ell_y \ell_z}{a^3} Q\pi A_{DM} \cos \varphi_0. \tag{39}$$

Setting its derivative with respect to φ_0 equal to zero in order to minimize gives a sine, solved by the set of solutions $\varphi_0 = 2k\pi$ or $\varphi_0 = \pi(1 + 2k)$. Their respective energy is

$$\begin{cases} E_{MM}\,[\varphi_0 \to\ 2k\pi\,] = \dfrac{\ell_y \ell_z}{a^3} Q\pi A_{DM} \\ E_{MM}\,[\varphi_0 \to\ \pi(1 + 2k)] = -\dfrac{\ell_y \ell_z}{a^3} Q\pi A_{DM} \end{cases}, \tag{40}$$

and so the azimuthal angle φ_0 for which the energy is minimized depends solely on the sign of (QA_{DM}). The solutions are all Néel-type domain walls, except when $A_{DM} = 0$, when all values for φ_0 are permitted. This is at the very least an odd solution. We previously found that the type of system treated in this section has Bloch-type domain walls, however even for infinitesimal A_{DM}, the system exhibits Néel-type domain walls.

A way to get an even better expression for the energy, is keeping the same ansätze, whilst improving the equation for the energy. An attempt is made in the next section.

V. INCLUSION OF HIGHER ORDER TERMS

It seems suitable to reapply the above procedure on the energy with higher other terms. Starting from (21), along with the extra anisotropy term $\frac{K_4}{4!}\Omega_x^4$ one obtains expression for the energy

$$E[\vec{\Omega}] = \frac{\ell_y \ell_z}{a^3}\int \frac{J_s}{2}(\vec{\Omega} \cdot \partial_{xx}\vec{\Omega}) + \frac{K_x}{2}\Omega_x^2 + \frac{K_4}{4!}\Omega_x^4 + \frac{K_z}{2}(1 - \Omega_z^2) + A[\hat{y} \cdot (\vec{\Omega} \times \partial_x \vec{\Omega})]\, dx. \tag{41}$$

In terms of the polar angle θ, and azimuthal angle φ, the term $+\frac{K_4}{4!}\Omega_x^4$ becomes $+\frac{K_4}{4!}\cos^4 \varphi \sin^4 \theta$.

The same procedure as before is being followed; by taking the azimuthal angle to be constant, ie. $\varphi = \varphi_0$, and the polar angle $\theta = 2\arccos(e^{\xi})$, where $\xi = Q(x - r_{dw})/\lambda_{dw}(\varphi_0)$, so that the domain wall width explicitly depends on φ_0 as in equation (38):

$$\lambda_{DW}^{-1}(\varphi_0) = \sqrt{\frac{K_x}{J_s}\cos^2 \varphi_0 + \frac{K_z}{J_s}}.$$

We can once again, plug our ansätze into the equation for the energy, and integrate. However, since this has already been done in the previous section for all terms but $\frac{K_4}{4!}\Omega_x^4$, in this section the integration will be limited to that term. Using from the appendix $\sin\theta = \operatorname{sech}\xi$ we obtain

$$\int dx \frac{K_4}{4!}\cos^4\varphi_0 \operatorname{sech}^4\xi \qquad (42)$$

Note that φ_0, and therefore $\lambda_{dw}(\varphi_0)$, are independent of x, so that we can integrate (42), finding

$$\left[\frac{QK_4\lambda}{72}\cos^4\varphi_0 \tanh\xi\,(2+\operatorname{sech}^2\xi)\right]_{\xi=-\infty}^{\infty}. \qquad (43)$$

Note that $\tanh(u)$ is an odd function, and that $\operatorname{sech}(u)$ is an even function, so that we can take out Q of the argument. Equations (43) becomes

$$\left[\frac{K_4\lambda}{72}\cos^4\varphi_0 \tanh\frac{\xi}{Q}\left(2+\operatorname{sech}^2\frac{\xi}{Q}\right)\right]_{\xi=-\infty}^{\infty}, \qquad (44)$$

where it is used that $Q^2 = 1$. This gives rise to the integration limit $\frac{\sqrt{J_s}K_4\cos^4\varphi_0}{18\sqrt{K_z+K_x\cos^2\varphi_0}}$, leading to the final expression for the energy:

$$E = \frac{\ell_y\ell_z}{a^3}\left(QA_{DM}\pi\cos\varphi_0 + \frac{\sqrt{J_s}K_4\cos^4\varphi_0}{18\sqrt{K_z+K_x\cos^2\varphi_0}}\right). \qquad (45)$$

A. Minimization of the energy

An attempt is made to minimize the energy found in the above paragraph. Taking the derivative of (45) leads to

$$\frac{dE}{d\varphi_0} = \frac{\ell_y\ell_z}{a^3}\left(QA_{DM}\pi\sin\varphi_0 + \frac{\sqrt{J_s}K_4\sin\varphi_0\,(4K_z\cos^3\varphi_0 + 3K_x\cos^5\varphi_0)}{18(K_z+K_x\cos^2\varphi_0)^{3/2}}\right). \qquad (46)$$

Setting it to zero, it can immediately be seen that $\sin\varphi_0 = 0$ is a solution, which implies $\varphi_0 = k\pi$, where k is an arbitrary integer. To find any further extremes, the equality

$$\sqrt{J_s}K_4(4K_z\cos^3\varphi_0 + 3K_x\cos^5\varphi_0) + 18QA\pi(K_z+K_x\cos^2\varphi_0)^{3/2} = 0 \qquad (47)$$

needs be solved. Substituting $u = \cos^2\varphi_0$, leads to polynomial

$$-324A^2K_z^3\pi^2 - 972A^2K_xK_z^2\pi^2u - 972A^2K_x^2K_z\pi^2u^2 + (16\,J_sK_4^2K_z^2 - 324A^2K_x^3\pi^2)u^3 + 24J_sK_xK_4^2K_zu^4 + 9J_sK_x^2K_4^2u^5.$$

Using $\varphi_0 = \arccos\sqrt{u}$, numerical procedures can be used to find the roots of the above polynomial, and therefore the minimum energy.

FIG. 5. Various plots of the domain wall angle φ_0, as a function of the strength of the DM-interaction A_{DM} for different values of the anisotropy constant K_4.

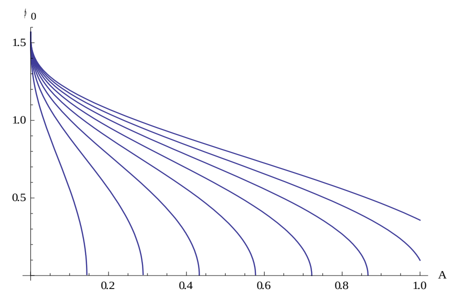
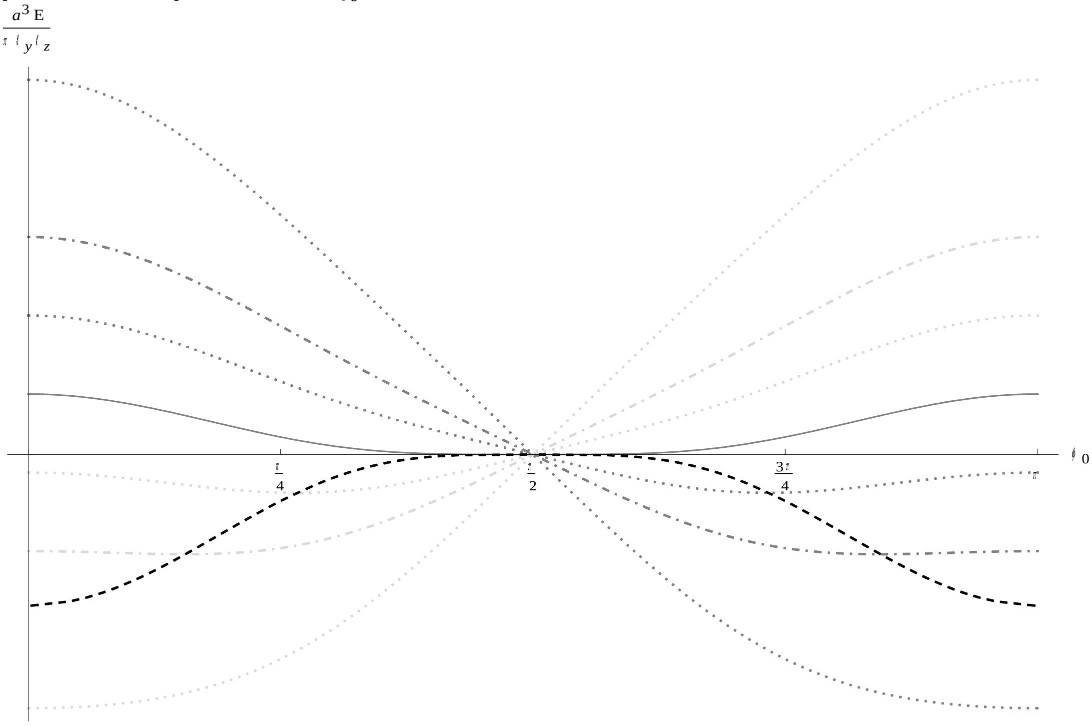


FIG. 6 A numerical sketch of the energy as a function of the azimuthal angle φ_0, for different values of A_{DM}, in multiples 0, (solid), 1/2 and 2, (dotted) , of $A_0 \sim 0.013\, J_s$ (dash-dotted). The fourth-order anisotropy constant is taken as $K_4 \sim 0.4\, J_s$. All other parameters in this plot are taken as $\sim J_s$.



VI. CONCLUSIONS AND OUTLOOK

In the preceding parts, two types of systems, namely, with bulk and interface induced Dzyaloshinkii-Moria interactions, have been investigated, satisfying the desired boundary conditions, so that they exhibit a domain wall. Inclusion of different types of Dzyaloshinkii-Moriya interaction in the energy equation for the respective systems, has led to a transition between Néel-, and Bloch-type domain walls. In most cases, an analytic expression for the critical strength of the DM-interaction has been found.

The results achieved throughout this thesis can be further improved, by refining the expression for the energy, for example by including higher order terms, or by iteratively improving the expressions for variable parameters λ and φ_0 within the assumed set of solutions, by minimizing their associated energy.

In addition to improving the stationary solution, future analysis can provide a dynamical solution for the magnetization. Taking into account time-dependence, allows for an even better grip on domain wall properties, so that the behaviour of domain walls in materials with non-negligible Dzyaloshinkii-Moriya interaction is better understood. This allows for the application of the found results on systems with changing magnetic fields, as is useful in the development of race-track memory.

VII. APPENDICES

A. Appendix 1: Identities

1. *Hyperbolic functions*

This section contains relations between hyperbolic functions and the exponential function.

$$\sinh u = \frac{e^u - e^{-u}}{2} = \frac{1}{\operatorname{csch} u}$$

$$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{1}{\operatorname{sech} u}$$

$$\tanh u = \frac{\sinh u}{\cosh u}$$

An extra note is made: $\operatorname{sech} u$ is often written as $\frac{2e^u}{e^{2u}+1}$.

2. *Sine of the arctangent*

This section contains expressions for the sine of various multiples of the inverse tangent function. The sine of the arctangent is given by

$$\sin(\arctan u) = \frac{u}{\sqrt{1+u^2}}.$$

From the half-tangent relation $\tan\frac{u}{2} = \frac{\sin u}{1+\cos u}$, one obtains

$$\arcsin u = 2\arctan\frac{u}{1+\sqrt{1-u^2}}.$$

Combining these two relations yields the desired relation

$$\sin(2\arctan u) = \frac{2u}{u^2+1}.$$

Repeating above procedure grants

$$\sin(4\arctan u) = -4\frac{(u^3-u)}{(u^2+1)^2}. \qquad (48)$$

3. *Derivatives*

This section contains some derivatives.

$$\partial_x \arctan u(x) = \frac{u'(x)}{u(x)^2+1}$$

$$\int dx \frac{2u'(x)}{u(x)^2+1} = \arctan u(x),$$

and

$$\int u'(x)\operatorname{sech}^2[u(x)]\,\mathrm{d}x = \int u'(x)\left(\frac{2e^{u(x)}}{e^{2u(x)}+1}\right)^2 \mathrm{d}x$$

$$= \frac{e^{u(x)} - e^{-u(x)}}{e^{u(x)} + e^{-u(x)}}$$

$$= \tanh u(x),$$

4. *Some explicit relations*

In this section some relations are explicitly shown, where it doesn't seem appriopriate to do so in the main part of this thesis.

The sine of θ:

$$\sin\theta = \sin\left[2\arctan\left(e^{\xi}\right)\right]$$

$$= \frac{2e^{\xi}}{e^{2\xi}+1}$$

$$= \operatorname{sech}\xi.$$

The cosine of θ:

$$\cos\theta = \cos\left[2\arctan\left(e^{\xi}\right)\right]$$

$$= \frac{1}{\sqrt{1+e^{2\xi}}}$$

$$= -\tanh\xi.$$

The derivative of θ

$$\partial_x\theta = \partial_x\left[2\arctan\left(e^{\xi}\right)\right]$$

$$= \frac{2(\partial_x\xi)e^{\xi}}{(e^{\xi})^2+1}$$

$$= \frac{\mathrm{Q}}{\lambda}\left(\frac{2e^{\xi}}{e^{2\xi}+1}\right)$$

$$= \frac{Q}{\lambda}\operatorname{sech}\xi.$$