Duality, non-standard elements, and dynamic properties of r.e. sets

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Abstract. We investigate the Priestley dual $(\mathcal{E}^*)^\star$ of the lattice $\mathcal{E}^*$ of r.e. sets modulo finite sets. Connections with non-standard elements of r.e. sets in models of 1st order true arithmetic as well as with dynamic properties of r.e. sets are pointed out. Illustrations include the Harrington–Soare dynamic characterization of small subsets, a model-theoretic characterization of promptly simple sets, and relations between the inclusion ordering of prime filters on $\mathcal{E}^*$ (a.k.a. points of $(\mathcal{E}^*)^\star$) and the complexity of their index sets.

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0. Introduction

The lattice $\mathcal{E}^* = \mathcal{E}/\mathrm{fin}$ of r.e. sets modulo finite differences has seen much effort invested in its study over the last half-century or so. A variety of methods is employed in its investigation. Duality for distributive lattices has to my best knowledge not been among these methods even though adjacent branches of recursion theory have brushed shoulders with duality — see Nerode [30] and the more recent Selivanov [35]. It's one thing when an approach path is abandoned, not having been found particularly promising, but to have never tried what has for quite some time been a fairly standard instrument in analysis of individual distributive lattices, that is something I find surprising.

There is a certain kinship between $\mathcal{E}^*$ and the lattice Σ_1/T of Σ_1 sentences modulo provability in a consistent formal theory T such as Peano Arithmetic PA. This kinship manifests itself in the isomorphism between $\mathcal{E}$ and $\Sigma_1(\boldsymbol{x})/\mathrm{TA}$, the lattice of Σ_1 formulas with a single free variable modulo equivalence in $\mathbb{N}$, or, equivalently, modulo full True Arithmetic TA. In other words, r.e. sets are like $\Sigma_1(\boldsymbol{x})$ formulas. When you quotient $\mathcal{E}$ by finite differences, you are in fact strengthening TA by the requirement that $\boldsymbol{x}$ be non-standard, for non-standard numbers are precisely those whose membership in an r.e. set is not affected by finite variations of the latter. Thus $\mathcal{E}^* \cong \Sigma_1(\boldsymbol{x})/(\mathrm{TA} + \boldsymbol{x} > \mathbb{N})$.

Our interest in the (Priestley) dual space $(\mathcal{E}^*)^\star$ of $\mathcal{E}^*$ was motivated by progress in Σ_1/T where the dual (or at least the underlying ordering of prime filters of Σ_1/T) is known as the *E-tree* and has been around for a long time — see Jensen & Ehrenfeucht [14, § 3] or Simmons [38]. Throughout the paper we point out similarities and differences between $(\mathcal{E}^*)^\star$ and E-trees of formal theories.

The typical prime filter of Σ_1/T is the collection of Σ_1 sentences holding in a given model of T. Similarly, a prime filter of $\mathcal{E}^*$ is the collection of $\Sigma_1(\boldsymbol{x})$ formulas that hold in a model of TA with a distinguished non-standard element $\boldsymbol{x}$. Thus models of $\mathrm{TA} + \boldsymbol{x} > \mathbb{N}$ play a role in the study of $(\mathcal{E}^*)^\star$ similar to the role of models of T for the E-tree.

While the subject matter of the present paper was inspired by developments in provability theory, none of the results herein have any immediate connection to provability. The choice of TA as the umbrella theory is to some extent arbitrary and reflects personal preferences. Another obvious candidate would be $TA_2 = Th_{\forall\exists}\,\mathbb{N}$ where certain model-theoretic issues are simpler than with full TA (see e.g. Hirschfeld [11] or Hirschfeld & Wheeler [13]), while $\Sigma_1(\boldsymbol{x})$ formulas behave exactly the same — at least as long as $\boldsymbol{x}$ does not vary. Schmerl & Shavrukov [34] show, among other things, some positive influence on investigations into models of TA_2 from connections we pursue in the present paper.

With the dual space on the one hand and non-standard elements on the other, many of our arguments exhibit a geometric/visual flavour — we include many illustrations to emphasize this point. Oftentimes this turns out to relate to what Harrington & Soare [8] call *dynamic properties* of r.e. sets (or arrangements thereof). These properties are hallmarked by being formulated in terms of speed at which elements enter an r.e. set, in other words, a key role is given to enumeration stages. Through relations such as 'at most total recursively later than', a model of TA allows for coarser measurement of when a generic element enters a given r.e. set than do the natural numbers. This allows some dynamic properties to find their model-theoretic equivalents, as well as gain alternative proofs of known equivalences to lattice-theoretic properties.

Our ambition in this enterprise is to suggest that bringing together r.e. sets, duality, and models of arithmetic can lead to some useful synergy. Keeping accessibility in mind, we spend somewhat more ink on the basics of these three ingredients than would be appropriate for readers well versed in respective fields.

We start section 1 with a brief introduction to Priestley duality for bounded (that is, possessing 0 and 1) distributive lattices. Next come some first basic properties of $(\mathcal{E}^*)^\star$. Most of the section is taken up by examples of dual-space characterizations of prominent classes of r.e. sets (or relations between those) such as r-maximal sets and major subsets.

Section 2 introduces models of TA into the fray. After recalling preliminary facts and definitions and articulating the connection between points of $(\mathcal{E}^*)^\star$ and models of $TA + \boldsymbol{x} > \mathbb{N}$, we present a characterization of simple sets via end-extensions of models. This exploits earlier ideas of Hirschfeld, Wilkie, and Schmerl — unlike $(\mathcal{E}^*)^\star$, non-standard elements of r.e. sets do have a history.

Section 3 turns to dynamic properties of r.e. sets using models of arithmetic as an instrument. With the help of an old lemma of Wilkie, we treat major and small subsets, re-obtaining the Harrington–Soare dynamic charaterization of the latter. We also produce a model-theoretic equivalent to prompt simplicity. It turns out that the behaviour of promptly simple r.e. sets in models of TA is not unlike that of inconsistency statements in models of formal theories. Along the way we isolate the class of *hinged* prime filters which is also going to play a part in the succeeding section.

In section 4 we look at index sets of prime filters of $\mathcal{E}^*$ (= points of $(\mathcal{E}^*)^\star$). This section is directly inspired by the earlier ascent of the E-tree in Shavrukov & Solovay [36]. We establish the key Jump-the-Gap Lemma which relates the Turing complexity of hinged prime filters in an inclusion chain to the ordering of that chain, and draw some consequences for order types of branches through $(\mathcal{E}^*)^\star$.

Our line of approach to $\mathcal{E}^*$ should, I believe, at least be a plentiful source of new questions. Some evidence to that effect is presented in section 5.

I would like to thank Peter Cholak, Othman Echi, Ali Enayat, James Schmerl, Andrea Sorbi, Michael Stob, Marcus Tressl, and Alex Wilkie for helpful and stimulating input, and Vladimir Nikolaevich Krupskiĭ for a very timely comment.

1. Priestley duality for $\mathscr{E}^*$

1.1. Convention. Throughout this paper, L will always denote some non-trivial ($0 \neq 1$) bounded distributive lattice.

1.A. General Priestley duality

The *Priestley dual* $L^\star$ of L is a (partailly) ordered topological space $(\mathcal{P}_L, \leq, \pi)$, where

— $\mathcal{P}_L$ is the set of proper (i.e. not containing 0) prime filters of L;

— $\leq$ is inclusion: $p \leq q \;\Leftrightarrow\; p \subseteq q$;

— $\{\, x^\star - y^\star \mid x, y \in L \,\}$ is a base for π, where

— $x^\star = \{\, p \in \mathcal{P}_L \mid x \in p \,\}$ is the *picture of* x (in $L^\star$)

(see e.g. Davey & Priestley [3, Chapter 11]). One has $x \in p \;\Leftrightarrow\; p \in x^\star$.

Caution: The ordering $\leq$ on $L^\star$ above coincides with those in Priestley [32] or Cornish [2], but is *opposite* to the one in Davey & Priestley [3].

The topology π is the *Priestley* (a.k.a. *patch* or *constructible*) topology. It should not be confused with the *spectral* (a.k.a. *hull-kernel* or *Zariski*) topology σ which has $\{\, x^\star \mid x \in L \,\}$ for a base. The topological space $\operatorname{Spec} L = (\mathcal{P}_L, \sigma)$, the *spectrum* of L, provides another format for duality that predates Priestley duals. Cornish [2] explains the relation between Priestley and spectral (a.k.a. Stone) duality for bounded distributive lattices. The difference between the two versions is almost linguistic, but the choice between them does influence the selection, or at least the relative priority of questions that come to the fore. Our preference for the Priestley format may be a matter of familiarity perceived as convenience.

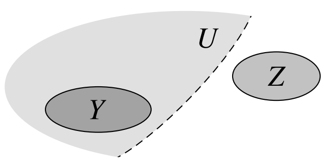
An $\uparrow$-set is any subset of P that is upwards closed w.r.t. $\leq$. $\downarrow$-sets are defined symmetrically. The key property of Priestley duals is that L is isomorphic to the lattice of π-clopen $\uparrow$-subsets of $L^\star$ via $x \mapsto x^\star$. In particular, pictures of elements of L exhaust π-clopen $\uparrow$-subsets of $L^\star$.

A *Priestley* (a.k.a. *ordered Boolean* or *CTOD*) space is an ordered topological space $(P, \leq, \tau)$ which is compact and *totally order-disconnected*:

— for each $p, q \in P$ such that $p \nleq q$ there is a τ-clopen $\uparrow$-set U such that $p \in U \not\ni q$.

All Priestley duals are Priestley spaces, and each Priestley space is order-homeomorphic to some Priestley dual. *Order-homeomorphisms* between ordered topological spaces are mappings that are at the same time homeomorphisms and poset isomorphisms. (The reader may find it amusing to construct two Priestley spaces which are both homeomorphic as spaces and isomorphic as posets, but not order-homeomorphic.) Priestley spaces are Hausdorff and satisfy a beefed-up version of total order-disconnectedness that we are going to find useful:

— for each pair Y, Z of τ-closed subsets such that $p \leq q$ holds for no $p \in Y$ and $q \in Z$ there is a τ-clopen $\uparrow$-set $U \supseteq Y$ which is disjoint from Z. (See Lemma 11.21(ii)(b) in Davey & Priestley [3].)



For $Y \subseteq P$, the least $\uparrow$-set containing Y is denoted $\uparrow Y$. $\downarrow Y$ has the symmetric definition. Typical examples of closed sets in Priestley spaces are singletons, $\uparrow\{p\}$, $\downarrow\{p\}$, and, more generally, $\uparrow C$ and $\downarrow C$ where C is any closed set.

1.2. Exercise. *Any maximal chain in a Priestley space is closed.*

Hint. Given a point p outside the maximal chain μ, there is a point $q \in \mu$ incomparable with p. There is a clopen set $C \ni p$ disjoint from the closed set $\uparrow\{q\} \cup \downarrow\{q\} \supseteq \mu$. ■

Another useful property of Priestley spaces/duals is that any non-empty closed set C contains points that are $\leq$-minimal in C — as well as points that are maximal in C (Exercise 11.15 in [3]). The sets of those are denoted by $\min C$ and $\max C$. Since $x^\star$ is an $\uparrow$-set for each $x \in L$, we have $\max x^\star = x^\star \cap \max L^\star$.

This subsection has (almost) been limited to bare necessities so that we can address our central example without having to go through lengthy preliminaries. Further instalments of duality generalities will be released as needed.

1.B. R.e. sets

$\mathcal{E}^* = \mathcal{E}/\text{fin}$ is the lattice of r.e. sets modulo the ideal of finite sets (see Soare [41, Chapter X]). To our best knowledge, the (Priestley) dual $(\mathcal{E}^*)^\star$ of $\mathcal{E}^*$ has not previously been considered in the literature. Points of $(\mathcal{E}^*)^\star$ will henceforth be called *primes*.

We freely confuse elements of $\mathcal{E}^*$ with individual r.e. sets representing the mod-finite equivalence classes. For r.e. X and Y one has $X^\star = Y^\star \Leftrightarrow X =^* Y$ where $=^*$ stands for finite difference. $X \subset_\infty Y$ is short for $X \subseteq Y \;\&\; X \neq^* Y$. For $X \subseteq \omega$, $\overline{X}$ denotes the complement $\omega - X$.

1.C. Reduction Principle

L satisfies the *Reduction Principle* if

$$\forall x, y \in L \; \exists x', y' \in L \, (x' \leq x \;\&\; y' \leq y \;\&\; x' \vee y' = x \vee y \;\&\; x' \wedge y' = 0).$$

(Equivalently, L is *separated* — see Lachlan [18]. Monteiro [29, p.26] considers *Ore's Axiom*, the antipode of the Reduction Principle.) The Reduction Principle is clearly preserved by quotients.

1.3. Fact (see Soare [41, Corollary II.1.10]). *$\mathcal{E}$ and hence $\mathcal{E}^*$ satisfy the Reduction Principle.* ■

In bounded distributive lattices, the Reduction Principle implies that the dual of the lattice is an $\uparrow$-growing forest: the set of predecessors of any point is a chain. This is well known, but we still spell out the proof for didactical reasons:

1.4. Corollary. *$(\mathcal{E}^*)^\star$ is an $\uparrow$-growing forest.*

Proof. Suppose q and r were incomparable primes. Using total order-diconnectedness of $(\mathcal{E}^*)^\star$, fix r.e. sets X and Y such that $q \ni X \notin r$ and $q \not\ni Y \in r$. The Reduction Principle supplies r.e. $X' \subseteq X$ and $Y' \subseteq Y$ such that $X' \cap Y' = \varnothing$ and $X' \cup Y' = X \cup Y$. Now

$$X' \cup Y \supseteq X' \cup Y' = X \cup Y \in q \not\ni Y,$$

so $X' \in q$ by primeness of q. Similarly, $Y' \in r$. But as $X' \cap Y' = \varnothing$, there can be no proper prime filter including both q and r. Thus the set of predecessors of any prime is a chain. ■

The Reduction Principle also holds in Σ_1/T, so essentially the same proof works for the E-tree (Jensen & Ehrenfeucht [14, Theorem 6]), partially justifying that name.

The distributive lattice equivalent of the forest-likeness of the dual space is *relative normality*:

$$\forall x, y \in L\ \exists x', y' \in L\, (x' \leq x\ \ \&\ \ y' \leq y\ \ \&\ \ x' \vee y = x \vee y' = x \vee y\ \ \&\ \ x' \wedge y' = 0).$$

Monteiro [29, Définition I.5.1 and Remarques on pp.25–27] introduced the antipode of this property characterizing lattices of closed subsets of hereditarily normal topological spaces among all topological spaces.

1.5. Fact (Monteiro [29, Théorèmes V.3.1–2]). *L is relatively normal if and only if $L^\star$ is $\uparrow$-forest-like.* ■

The Reduction Principle implies relative normality. The converse does not generally hold (Monteiro [29, p.27]), although the two properties are equivalent for finite distributive lattices (see Lindström & Shavrukov [24, Lemma 1.2]).

Unlike relative normality, the Reduction Principle cannot be expressed as a property of the ordering of the dual without involving the topology.

1.D. Recursive sets

We would like to be able to talk about pictures in $(\mathcal{E}^*)^\star$ of complements and differences of r.e. sets. This is possible because the Priestley dual of L subsumes the Stone representation of BL, the *Boolean envelope* of L (a.k.a. *free* or *minimal Boolean extension*). The extension $L \hookrightarrow BL$ is universal for homomorphisms of L to Boolean lattices. Each element of BL is equal to the value of some Boolean term applied to a tuple of elements of L.

1.6. Fact. (a) *If $L^\star = (\mathcal{P}_L, \leq, \pi)$ then $(BL)^\star = (\mathcal{P}_L, =, \pi)$.*

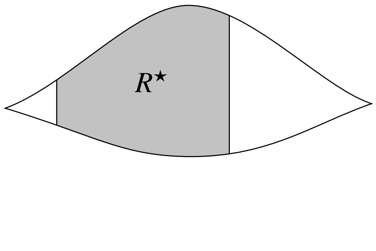
(b) *L is Boolean iff $L = BL$ iff $L^\star$ is an antichain.*

Comment. (a) See Davey & Priestley [3, 11.17]. Throwing away the ordering on $L^\star$, one obtains the Stone dual space of BL.

(b) follows from (a). ■

It follows from clause (a) that for a Boolean term $t(\cdot)$ and $\vec{x} \in L$ we have $(t(\vec{x}))^\star = t(\vec{x}^\star)$ where Boolean operations on the r.h.s. are understood as set operations.

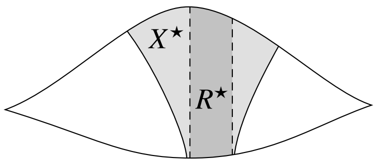
1.7. Lemma. *An r.e. set R is recursive if and only if $R^\star$ is a $\downarrow$-set.*



Proof. (if) If $R^\star$ is a $\downarrow$-set then $\overline{R^\star} = \overline{R}^\star$ must be a clopen $\uparrow$-set. Hence $\overline{R}$ is r.e., so R is recursive.

(only if) For recursive R, $\overline{R^\star}$ is an $\uparrow$-set because $\overline{R}$ is r.e. Thus $R^\star$ is a $\downarrow$-set. ∎

Every infinite r.e. set X has an infinite recursive subset R (see Soare [41, Exercise II.1.21]). This means that $X^\star \supseteq R^\star$. With $R^\star$ being, by Lemma 1.7, a non-empty $\updownarrow$-set, we conclude



1.8. Corollary. *$X^\star$ intersects $\min(\mathcal{E}^*)^\star$ for every infinite r.e. X.* ∎

The analogue of this corollary fails in the E-tree.

The complemented elements of the lattice Σ_1/T are sentences that are Δ_1 in T. Non-trivial instances of these are present if and only if T is Σ_1-ill (see Lindström [22, Exercise 2.25]).

1.E. Minimal and minimax primes

The Boolean lattice $\mathcal{R}^*$ of recursive sets mod finite is a sublattice of $\mathcal{E}^*$. Accordingly, the recursive elements of each prime form an ultrafilter of $\mathcal{R}^*$. Each minimal prime is generated by its recursive elements:

1.9. Lemma. *$p \in \min(\mathcal{E}^*)^\star$ if and only if*

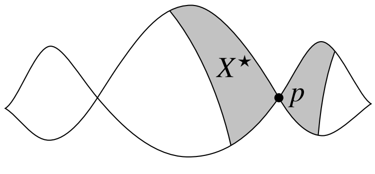
$$p = \{\, \text{r.e. } X \mid X \supseteq R \text{ for some recursive } R \in p \,\}.$$

Proof. (if) Any prime $q \le p$ will have to contain all recursive elements $R \in p$ because $R^\star$ are $\downarrow$-sets. Hence $p \le q$.

(only if) Given any ultrafilter u of $\mathcal{R}^*$, we verify that $p = \{\, \text{r.e. } X \mid X \supseteq R \in u \,\}$ is a prime: If $p \ni U \cup V \supseteq R \in u$, then let $U' \subseteq U$ and $V' \subseteq V$ be disjoint r.e. sets corresponding to U and V by the Reduction Principle. Then $U' \cup V' = U \cup V \supseteq R$, and $R \cap U'$ and $R \cap V'$ are recursive sets partitioning R. Hence one of them must be an element of u. Therefore one of U and V belongs to p. ∎

A prime is *minimax* if it is both minimal and maximal in $(\mathcal{E}^*)^\star$. Such primes exist — the construction below is due to Hirschfeld [11, 4.5] (or see Hirschfeld & Wheeler [13, 9.6(ii)]).

1.10. Proposition. *For any infinite r.e. X there is a minimax prime $p \in X^\star$.*



Proof. We construct a sequence $(R_i)_{i\in\omega}$ of infinite recursive sets with $R_0 \subseteq X$ and $R_{i+1} \subseteq R_i$ and such that $(R_i)_{i\in\omega}$ decides every r.e. set: $R_{i+1} \subseteq W_i$ or $R_i \cap W_i$ is finite:

At each step, see if $R_i \cap W_i$ is infinite and if so, let R_{i+1} be an infinite recursive subset of $R_i \cap W_i$. Otherwise, put $R_{i+1} = R_i$.

Clearly, $\{R_i\}_{i\in\omega}$ generates an ultrafilter of $\mathcal{R}^*$. Let p be the minimal prime including $\{R_i\}_{i\in\omega}$. Then $p \in R_0^\star \subseteq X^\star$ and $p \in \min(\mathcal{E}^*)^\star$, for p is generated by recursive sets (Lemma 1.9). Finally, $p \in \max(\mathcal{E}^*)^\star$ because for each W_i, p either contains W_i or a recursive set disjoint from W_i. ∎

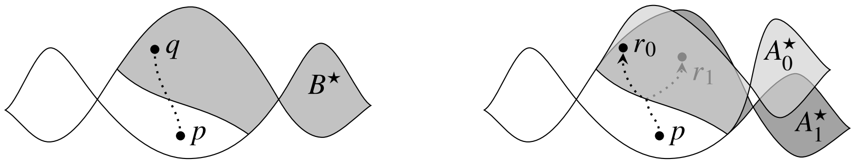
The construction above readily adapts to show that $(\mathcal{E}^*)^\star$ sports $2^{\aleph_0}$ many minimax primes. See also Hirschfeld [12, 5.5], where $2^{\aleph_0}$ many primes are detected within an even narrower class.

1.F. Friedberg splittings

A partitioning $\{A_0, A_1\}$ of an r.e. set B into two r.e. halves is called an *r.e. splitting* of B. Such a splitting is a *Friedberg splitting* if for each r.e. X one has that $X - B$ is r.e. if at least one of $X - A_i$ is. Any non-recursive r.e. set admits a Friedberg splitting (see Soare [41, Theorem X.2.1]).

The following example is prompted by Hirschfeld & Wheeler [13, 8.21].

1.11. Proposition. *An r.e. splitting $\{A_0, A_1\}$ of B is a Friedberg splitting if and only if for all primes $p \notin B^\star \ni q$ with $q \geq p$ there are $r_i \in A_i^\star$ such that $r_i \geq p$ for both i.*



Proof. (if) Suppose $X - B$ failed to be r.e., so $X^\star - B^\star$ is not an $\uparrow$-set. Then there are $p \leq q$ with $p \in X^\star - B^\star \not\ni q$. This means that $p \in X^\star$ while $p \notin B^\star \ni q$. Our assumption supplies $r_i \in A_i^\star$ with $r_i \geq p$. So $r_i \notin X^\star - A_i^\star$ whereas $p \in X^\star - A_i^\star$. This shows that both $X - A_i$ are not r.e. Thus $\{A_0, A_1\}$ is a Friedberg splitting.

(only if) Suppose $p \notin B^\star$ and there is no $r_0 \geq p$ with $r_0 \in A_0^\star$. By total order-disconnectedness, there must then exist an r.e. set X disjoint from A_0 such that $p \in X^\star$. Then $X - A_0$ is r.e. Hence by the assumption so is $X - B$. Therefore, as $p \in X^\star - B^\star$ and $X^\star - B^\star$ is an $\uparrow$-set, there can be no $q \geq p$ with $q \in B^\star$. This establishes the property on the r.h.s. ∎

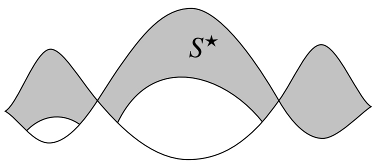
The example above, together with the existence of Friedberg splittings, implies that the $\uparrow$-growing forest that is $(\mathcal{E}^*)^\star$ is properly branching. In other words, $(\mathcal{E}^*)^\star$ is *not* $\downarrow$-forest-like.

The same holds for the E-tree, for Friedberg splittability holds for Σ_1 sentences as well — this follows from Theorems 0 and 3 in Hájek [7].

1.G. Simple sets

Recall that an r.e. set is *simple* if it intersects every infinite r.e. set.

1.12. Proposition. *An r.e. set S is simple if and only if $S^\star \supseteq \max(\mathcal{E}^*)^\star$.*

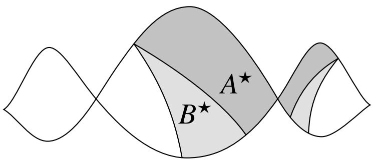


Proof. (if) The picture $D^\star$ of any infinite r.e. set D intersects $\max(\mathcal{E}^*)^\star$. Hence $S^\star \cap D^\star \neq \varnothing$. Thus $S \cap D$ is infinite and in particular non-empty.

(only if) If $S^\star$ fails to cover $\max(\mathcal{E}^*)^\star$ then there is a prime $p \in \max(\mathcal{E}^*)^\star - S^\star$ so that $p \nleq r$ for any $r \in S^\star$. Both $S^\star$ and $\{p\}$ being closed, by total order-disconnectedness there is a clopen $\uparrow$-set $D^\star$ — where D is r.e. — such that $p \in D^\star$ and $D^\star \cap S^\star = \varnothing$. Hence $D \cap S$ is finite. Thus $D - S$ is an infinite r.e. set disjoint from S, so S is not simple. ∎

An r.e. subset $A \subseteq B$ is *simple in B* if A intersects every infinite r.e. subset of B.

1.13. Proposition. *For r.e. $A \subseteq B$, A is simple in B if and only if* $\max A^\star = \max B^\star$.



Comment. The argument is a straightforward adaptation of the one for Proposition 1.12. ∎

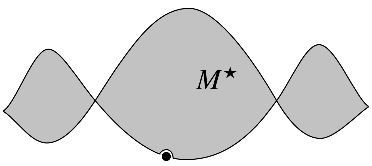
In lattices Σ_1/T, analogues of simple sets are the Σ_1 sentences which are Π_1-*conservative* over T: for each Π_1 sentence π, if $T \vdash \sigma \to \pi$ then $T \vdash \pi$ (see Lindström [22, Chapter 5]). Similarly, a Σ_1 sentence σ is the counterpart of a 'simple subset' of another Σ_1 sentence τ such that $T \vdash \sigma \to \tau$ if σ is Π_1-conservative over $T + \tau$.

In Σ_1/T there always are *doubly conservative* Σ_1 sentences, that is, σ is Π_1-conservative and $\neg\sigma$ is Σ_1-conservative over T (see Lindström [22, Theorem 5.3(a)]). The picture of σ in the E-tree covers all maximal but none of the minimal points. Therefore the E-tree knows no minimax points.

1.H. Maximal sets

An r.e. M is *maximal* if for each r.e. superset $S \supseteq M$ one either has $S =^* \omega$ or $S =^* M$. (Maximality of r.e. sets is not to be confused with maximality of primes.)

1.14. Proposition. *An r.e. set M is maximal if and only if $\overline{M^\star}$ consists of a single prime. That prime, called the* heel *of M, has to be minimal and cannot be maximal.*



Proof. (if) For any r.e. $S \supseteq M$ we must have $S^\star \supseteq M^\star$. So when $\overline{M^\star}$ is a singleton we either have $S^\star = (\mathcal{E}^*)^\star$ in which case $S =^* \omega$, or $S^\star = M^\star$ meaning $S =^* M$.

(only if) If there are two distinct primes p and q in $\overline{M^\star}$ then w.l.o.g. $p \nleq q$. Hence there is an r.e. S with $p \in S^\star \not\ni q$. Then $M^\star \subsetneq S^\star \cup M^\star \subsetneq (\mathcal{E}^*)^\star$. Thus $S \cup M$ is a superset of M with $S \cup M \neq^* M$ and $S \cup M \neq^* \omega$. So M cannot be maximal.

Let $\overline{M^\star} = \{p\}$. p is minimal because $\overline{M^\star}$ is a $\downarrow$-set. Since each maximal set is simple, p cannot be maximal by Proposition 1.12. ∎

Principal ideals $(M]$ of $\mathcal{E}^*$ with maximal M are maximal and, in particular, prime. Proposition 1.14 also tells us that heels of maximal sets are isolated in $(\mathcal{E}^*)^\star$. We shall see in Corollary 1.34 that all principal prime ideals and isolated points of $\mathcal{E}^*$ stem from maximal sets.

1.I. Quotients, ideals, filters

At this point, we would like to review the duality theory of congruences, ideals, filters and quotients in general bounded distributive lattices.

1.15. Fact (see Davey & Priestley [3, 11.32]). *There is a one-one correspondence between non-empty closed subsets of $L^\star$ and proper congruences on L given by*

$$C \mapsto \theta_C = \{\, (x, y) \in L^2 \mid x^\star \cap C = y^\star \cap C \,\}.$$

Under this correspondence, $(L/\theta_C)^\star$ is order-homeomorphic to C with inherited order and topology. ■

Suppose I is an ideal of L. Put $I^\star = \bigcup_{x\in I} x^\star$. Symetrically, put $F^\star = \bigcap_{y\in F} y^\star$ for any filter F of L. Note that $I^\star$ is open, $F^\star$ is closed, and both of them are $\uparrow$-sets.

Let $a \in L$. Then $(a]$ and $[a)$ denote the principal ideal and the principal filter corresponding to a respectively. Observe that $(a]^\star = [a)^\star = a^\star$.

If p is a prime filter then $p^\perp$ denotes the complementary prime ideal.

1.16. Fact (Davey & Priestley [3, Exercise 11.17]). *Let I be an ideal in L.*

(a) *I is principal if and only if $I^\star$ is clopen.*

(b) *I is prime if and only if $\overline{I^\star}$ is $\uparrow$-directed if and only if $I^\star = \overline{\downarrow\{p\}}$ for some $p \in L^\star$.*

(c) *I is maximal if and only if $I^\star = \overline{\{p\}}$ for some $p \in \min L^\star$.*

In clauses (b) *and* (c) *we have $I = p^\perp$.*

Hint. In clause (b), to obtain $\uparrow$-directedness from primeness, assume there are $p, q \notin I^\star$ with no $s \geq p, q$ outside $I^\star$. Then there is an $a \in L$ such that $a^\star \ni p$ and $a^\star \cap (\uparrow\{q\} - I^\star) = \varnothing$. There is a $b \in L$ such that $b^\star \ni q$ and $b^\star \cap (a^\star - I^\star) = \varnothing$. Thus a and b witness the failure of primeness of I. ■

1.17. Fact. (a) *Let I be an ideal of L. Then $(L/I)^\star$ is order-homeomorphic to $\overline{I^\star}$.*

(b) *Let F be a filter of L. Then $(L/F)^\star$ is order-homeomorphic to $F^\star$.*

(c) *Let $a < b$ be elements of L. Then the dual of the interval $[a, b]$ is order-homeomorphic to $b^\star - a^\star$.*

Comment. (a) is implicit in Proposition 13 of Priestley [32]. The prime filters of L/I are those prime filters p of L that do not intersect I. Equivalently, $p \notin I^\star$.

(b) is symmetric to (a).

(c) follows from (a) and (b) because the interval $[a, b]$ of L is isomorphic to the double quotient $(L/(a])/[b)$. ■

1.J. r-maximal sets

L is *local* if $x \vee y = 1$ always implies $x = 1$ or $y = 1$.

1.18. Lemma. *L is local if and only if $L^\star$ has a least point.*

Proof. (if) If $x^\star \cup y^\star = L^\star$ then the least point of $L^\star$ is w.l.o.g. an element of $x^\star$. Hence $x^\star = L^\star$ because $x^\star$ is an $\uparrow$-set. So $x = 1$.

(only if) Suppose $p \neq q$ were two distinct minimal points of $L^\star$. By total order-disconnectedness there is a clopen $\uparrow$-set $x^\star$ such that $p \in x^\star \not\ni q$. Since $\overline{x^\star}$ is closed, there is also a clopen $\uparrow$-set $y^\star$ such that $\overline{x^\star} \subseteq y^\star \not\ni p$. Thus $x \neq 1 \neq y$ whereas $x \vee y = 1$. So L cannot be local. ∎

$(\mathscr{E}^*)^\star$ is not local because there are many minimal primes. The lattice Σ_1/T is local if and only if T is Σ_1-sound. Thus $(\mathscr{E}^*)^\star$ sooner resembles the E-tree of a Σ_1-ill theory than that of a Σ_1-sound one.

Maximality of r.e. sets can be generalized in two orthogonal directions. The first one of these is embodied by r-maximal sets: An r.e. set Q is *r-maximal* if no recursive set splits $\overline{Q}$ into two infinite subsets. All r-maximal sets are simple.

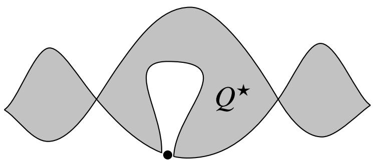
$\mathscr{L}^*(A)$ denotes the interval $[A, \omega]$ of $\mathscr{E}^*$.

1.19. Proposition. *Let Q be an r.e. set. The following are equivalent:*

(i) *Q is r-maximal;*

(ii) *$\mathscr{L}^*(Q)$ is local: $X \cup Y \cup Q = \omega \ \Rightarrow\ X \cup Q =^* \omega$ or $Y \cup Q =^* \omega$;*

(iii) *$\overline{Q^\star} \cap \min(\mathscr{E}^*)^\star$ is a singleton.*



Proof. (ii) ⇔ (iii) is Lemma 1.18, for the subspace $\overline{Q^\star}$ of $(\mathscr{E}^*)^\star$ is the dual of $\mathscr{L}^*(Q)$ by Fact 1.17(a).

(iii) ⇒ (i) Let $\overline{Q^\star} \cap \min(\mathscr{E}^*)^\star = \{p\}$. Given a recursive R, we may assume $p \in R^\star$ — otherwise replace R by $\overline{R}$. But then $\overline{Q^\star} \subseteq \uparrow\{p\} \subseteq R^\star$, thus $\overline{Q} \subseteq^* R$, so R fails to split $\overline{Q}$.

(i) ⇒ (ii) Consider r.e. X and Y with $X \cup Y \cup Q =^* \omega$. By the Reduction Principle there is a recursive R such that $R \subseteq X \cup Q$ and $\overline{R} \subseteq Y \cup Q$. By the r-maximality of Q we have $\overline{Q} \subseteq^* R$ or $\overline{Q} \subseteq^* \overline{R}$. In the former case $\overline{Q} \subseteq^* R \subseteq X \cup Q$ entails $\overline{Q} \subseteq^* X$, or $X \cup Q =^* \omega$. The other case is symmetric. ∎

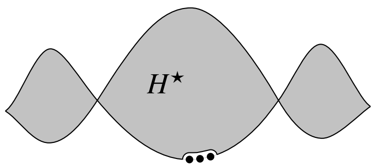
Since r-maximal sets Q are simple, $\overline{Q^\star}$ does not intersect $\max(\mathscr{E}^*)^\star$. Thus the existence of non-maximal r-maximal sets tells us that in $(\mathscr{E}^*)^\star$, there are primes that are neither minimal nor maximal. Also note that by relative normality of $(\mathscr{E}^*)^\star$ it follows that $\overline{Q^\star}$ is a rooted tree. In this respect it resembles the E-tree of a Σ_1-sound formal theory.

1.20. Definition. Let Q be r-maximal. The unique minimal prime $p \notin Q^\star$ is called the *heel* of Q.

1.K. Hyperhypersimple sets

The second generalization of maximal sets are the hhsimple sets. An r.e. set H is *hhsimple* if $\mathscr{L}^*(H)$ is a non-trivial Boolean lattice. For our purposes, this is going to serve as the definition. See Lachlan [19, Theorem 3] or Soare [41, Corollary X.2.7] for its equivalence to the original one. All hhsimple sets are simple.

1.21. Proposition. *An r.e. set H is hhsimple if and only if $\varnothing \neq \overline{H^\star} \subseteq \min(\mathscr{E}^*)^\star$.*
In this case, $\max(\mathscr{E}^*)^\star \subseteq H^\star$.



Proof. Seeing as $\overline{H^\star}$ is the dual of $\mathscr{L}^*(H)$, it follows from Fact 1.6(b) that hhsimplicity of H is equivalent to $\overline{H^\star}$ being a non-empty antichain. Since $\overline{H^\star}$ is a $\downarrow$-set, this antichain can only consist of minimal primes.

None of the primes in $\overline{H^\star}$ are maximal because H is simple. ∎

1.22. Definition. A *branch through* $(\mathscr{E}^*)^\star$ is a maximal chain in $(\mathscr{E}^*)^\star$.

1.23. Corollary. *Let H be hhsimple and let* **b** *be any branch through $(\mathscr{E}^*)^\star$ whose least prime p lies outside $H^\star$. Then p has an immediate successor on* **b**.

Proof. The immediate successor is $\mathbf{b} \cap \min H^\star$. ∎

1.L. D-hyperhypersimple sets

Given an r.e. set A, Herrmann & Kummer [10, Definition 2.3] define

$$\mathscr{D}^*(A) = \{\, B \in \mathscr{L}^*(A) \mid B - A \text{ is r.e.} \,\}.$$

Then $\mathscr{D}^*(A)$ is an ideal of $\mathscr{L}^*(A)$. They further put $\mathscr{C}(A) = \mathscr{L}^*(A)/\mathscr{D}^*(A)$.

A is *D-hhsimple* if $\mathscr{C}(A)$ is non-trivial and Boolean. All hhsimple sets are D-hhsimple. Other examples of D-hhsimple sets are provided, but not exhausted by non-recursive halves of splittings of hhsimple sets (Lerman & Soare [21, Theorem 2.15] and Herrmann & Kummer [10, p. 71]).

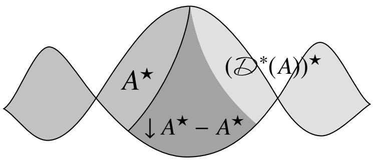
1.24. Definition. Let A be r.e. The *shadow* of A is the closed $\downarrow$-subset $\downarrow A^\star - A^\star$ of $(\mathscr{E}^*)^\star$. In other words, the shadow is the collection of all those primes that lie outside $A^\star$ but see a prime in $A^\star$ above.

1.25. Lemma. $(\mathscr{C}(A))^\star$ *is order-homeomorphic to the shadow of A for any r.e. A.*

Proof. We first note that the ideal $\mathscr{D}^*(A)$ of $\mathscr{L}^*(A)$ consists of those r.e. $B \supseteq^* A$ whose picture $B^\star$ is disjoint from the shadow of A. Indeed, suppose $B - A$ is r.e. Then any prime p in the shadow of A cannot be an element of $B^\star$, for otherwise $B^\star - A^\star$ would not be an $\uparrow$-set. Conversely, if $B^\star \supseteq A^\star$ is disjoint from the shadow of A then $B^\star - A^\star$ must be an $\uparrow$-set.

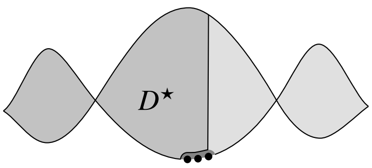
Given a prime $r \notin A^\star$ outside the shadow of A, there is a clopen $\uparrow$-set $B^\star \ni r$ such that $B^\star$ is disjoint from $A^\star$ because $r \not\leq q$ for all $q \in A^\star$. $B^\star$ is then also disjoint from the shadow of A. Since $B - A =^* B$ is r.e., $A \cup B \in \mathscr{D}^*(A)$.

Therefore $(\mathscr{D}^*(A))^\star$, as a subspace of $\overline{A^\star}$, the dual of $\mathscr{L}^*(A)$, is order-homeomorphic to $\overline{\downarrow A^\star}$. By Fact 1.17(a), the dual of $\mathscr{C}(A) = \mathscr{L}^*(A)/\mathscr{D}^*(A)$ is then order-homeomorphic to $\downarrow A^\star - A^\star$, the shadow of A.



■

1.26. Proposition. *An r.e. D is D-hhsimple if and only if the shadow of D is non-empty and consists of minimal primes only.*



Proof. $\mathscr{C}(D)$ is non-trivial and Boolean if and only if $(\mathscr{C}(D))^\star$ is a non-empty antichain (Fact 1.6(b)). The shadow of D is a $\downarrow$-set order-homeomorphic to $(\mathscr{C}(D))^\star$. It is an antichain if and only if it consists of minimal primes only. ■

1.27. Corollary. *Let D be D-hhsimple and let $\mathbf{b}$ be a branch through $(\mathscr{E}^*)^\star$ that intersects $D^\star$ but whose least prime p lies outside $D^\star$. Then p has an immediate successor on $\mathbf{b}$.* ■

There are no analogues of maximal, hhsimple nor D-hhsimple sets in Σ_1/T, for no non-trivial interval of Σ_1/T, being itself isomorphic to Σ_1/S for some consistent theory S, can admit primes that are minimax in that interval.

1.M. Major subsets

When $A \subset_\infty B$ are r.e., A is called a *major* subset of B if $B \cup X = \omega \;\Rightarrow\; A \cup X =^* \omega$ for each r.e. X. The equivalence (i) ⇔ (ii) of the following proposition is well known (see Maass & Stob [28, Lemma 2.1]), as is the fact that major subsets are simple subsets.

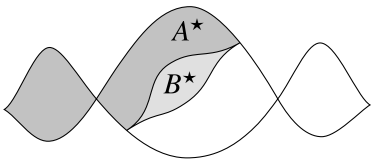
1.28. Proposition. *For r.e. $A \subset_\infty B$, the following are equivalent:*

(i) *A is a major subset of B;*

(ii) *$R \subseteq B \;\Rightarrow\; R \subseteq^* A$ for each recursive R;*

(iii) *$A^\star \cap \min(\mathscr{E}^*)^\star = B^\star \cap \min(\mathscr{E}^*)^\star$;*

(iv) $A^\star \cap \min(\mathcal{E}^*)^\star = B^\star \cap \min(\mathcal{E}^*)^\star$ *and* $A^\star \cap \max(\mathcal{E}^*)^\star = B^\star \cap \max(\mathcal{E}^*)^\star$. *In particular, A is simple in B (Proposition 1.13).*



Proof. (i) ⇒ (ii) $R \subseteq B \Rightarrow \overline{R} \cup B = \omega \Rightarrow \overline{R} \cup A =^* \omega \Rightarrow R \subseteq^* A$.

(ii) ⇒ (iii) Let $p \in B^\star \cap \min(\mathcal{E}^*)^\star$. By total order-disconnectedness there is an r.e. X such that $\overline{B^\star} \subseteq X^\star \not\ni p$. Note that $B \cup X =^* \omega$. By the Reduction Principle there must be a recursive $R \subseteq B$ with $\overline{R} \subseteq X$. By (ii), $R \subseteq^* A$. Observe that $p \in R^\star$ because $p \notin X^\star$. Hence $p \in A^\star$.

(iii) ⇒ (i) Suppose $B \cup X = \omega$ with X r.e. Then $(B \cup X) \cap \min(\mathcal{E}^*)^\star = \min(\mathcal{E}^*)^\star$. By clause (iii), $(A \cap \min(\mathcal{E}^*)^\star) \cup (X \cap \min(\mathcal{E}^*)^\star) = \min(\mathcal{E}^*)^\star$. Both $A^\star$ and $X^\star$ being ↑-sets, $A^\star \cup X^\star = (\mathcal{E}^*)^\star$, so $A \cup X =^* \omega$. Thus A is major in B.

(iv) ⇒ (iii) is trivial.

(ii) & (iii) ⇒ (iv) Suppose there was a maximal prime in $B^\star - A^\star$. Then A is not simple in B by Proposition 1.13. There is then an infinite subset $R \subseteq B$ disjoint from A. We may assume that R is recursive. But this contradicts (ii). ■

1.29. Fact (Maass & Stob [28]). *If A is a major subset of B and A' is a major subset of B', then the intervals $[A, B]$ and $[A', B']$ of $(\mathcal{E}^*)^\star$ are isomorphic.* ■

We let $\mathcal{M}^*$ denote the common isomorphism type of major intervals. According to Fact 1.17(c), the dual $(\mathcal{M}^*)^\star$ is order-homeomorphic to any subspace of $(\mathcal{E}^*)^\star$ of the form $B^\star - A^\star$ where A is major in B.

In the E-tree, clause (iii) fails to imply (iv) of Proposition 1.28 because Corollary 1.8 does not hold in Σ_1/T. Stob [44] calculates the ∀∃ fragment of the 1st order theory of $\mathcal{M}^*$. It turns out to coincide with that of any lattice of the form Σ_1/T where T is a Σ_1-ill formal theory (Lindström & Shavrukov [24, Proposition 6.3]). No analogue of Fact 1.29 is known for any Σ_1/T.

1.N. Small subsets

1.30. Definition (see Soare [41, Definition X.4.10(i)]). An r.e. subset A of an r.e. set B is a *small* subset of B if $A \subset_\infty B$ and

$$\forall \text{r.e. } X, Y \big(X \cap (B - A) \subseteq Y \Rightarrow Y \cup (X - B) \text{ is r.e.}\big).$$

This definition makes sense in any bounded distributive lattice: call $a \in L$ *small in* $b \in L$ if $a < b$ and

$$\forall x, y \in L \big(x \wedge (b - a) \leq y \Rightarrow y \vee (x - b) \in L\big).$$

Relations involving − are interpreted in the Boolean envelope of L. Alternatively, using the equivalence $a \wedge \overline{b} \leq c \Leftrightarrow a \leq b \vee c$, where $\overline{b}$ is short for $1 - b$, each relation involving − rewrites equivalently as a conjunction of several lattice relations — for example, $z = y \vee (x - b)$ is equivalent in any bounded distributive lattice to $y \leq z \;\&\; z \leq y \vee x \;\&\; z \wedge b \leq y \;\&\; x \leq b \vee z$.

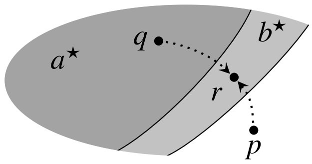
The equivalence (i) ⇔ (ii) of the following proposition was established by Harrington & Soare for the particular case $L = \mathcal{E}^*$ by a different method.

1.31. Proposition (after Harrington & Soare [8, Theorem 3.2]). *In any relatively normal L, the following conditions on elements $a < b$ of L are equivalent:*

(i) *a is small in b;*

(ii) $\forall y \in L\,(b - a \leq y \;\Rightarrow\; y \vee \overline{b} \in L)$;

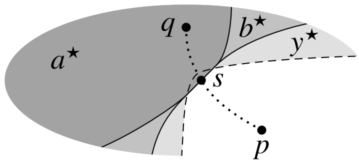
(iii) *For all $p \leq q$ in $L^\star$ with $p \notin b^\star \ni q$ there is $r \in (p, q]$ such that $r \in b^\star - a^\star$. In other words,*



(iv) $\min a^\star \cap \min b^\star \subseteq \min L^\star$.

Proof. (i) ⇒ (ii) In the definition of *a small in b*, put $x = 1$.

(ii) ⇒ (iii) By contraposition: Suppose we had $p \leq q$ in $L^\star$ with $p \notin b^\star \ni q$ and no points from $b^\star - a^\star$ in $(p, q]$. Let $s \in \min(b^\star \cap [p, q])$ — the minimum is non-empty because $[p, q]$ is closed. Then $p < s \in \min b^\star$ because $L^\star$ is an ↑-growing forest. Also, $s \in a^\star$ by our assumption. Therefore for all $r \in b^\star - a^\star$ one has $s \not\geq r$. Both $\{s\}$ and $b^\star - a^\star$ being closed, there is by total order-disconnectedness a clopen ↑-set $y^\star$ such that $y^\star \supseteq b^\star - a^\star$ so that $b - a \leq y$, and $s \notin y^\star$.



We have $p \in \bar{b}^\star \subseteq y^\star \cup \overline{b}^\star = (y \vee \overline{b})^\star$ but $s \notin y^\star \cup \overline{b}^\star$ while $s \geq p$. Thus $y \vee \overline{b} \notin L$ because $(y \vee \overline{b})^\star$ is not an ↑-set. This refutes (ii).

(iii) ⇒ (i) Assuming (iii) and $x \wedge (b - a) \leq y$, we are going to show that $y \vee (x - b) \in L$, or rather that $y^\star \cup (x^\star - b^\star)$ is an ↑-set.

Let $q \geq p \in y^\star \cup (x^\star - b^\star)$. If $p \in y^\star$ then $q \in y^\star$ because $y^\star$ is an ↑-set, so assume $p \notin y^\star$, hence $p \in x^\star - b^\star$. We have $q \in x^\star$, so the only interesting case is $q \in b^\star$. Let's assume as much. Then by (iii) there is $r \in (p, q]$ such that $r \in b^\star - a^\star$. Also $r \in x^\star$ because $r \geq p$. So $r \in (x \wedge (b - a))^\star \subseteq y^\star$. Hence $q \in y^\star$, for $r \leq q$.

Thus $q \in (y \vee (x - b))^\star$ in all cases which completes the proof of (i).

(iii) ⇔ (iv) is an easy exercise. ∎

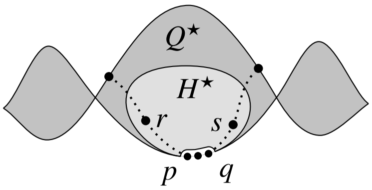
We return with a non-standard and a dynamic characterization — the latter one also due to Harrington & Soare — of small subsets in Theorem 3.25.

For relatives of small (as well as small major) subsets in Σ_1/T, see our comments to Question 5.6.

1.32. Fact (Lachlan [18, Theorem 3] or Soare [41, Exercise X.4.12]). *Each non-recursive r.e. set has a small major subset.* ∎

1.33. Exercise. *L is a chain if and only if $L^\star$ is a chain.* ∎

Let H be hhsimple and non-maximal. Let Q be a small major subset of H. There are distinct $p, q \notin H^\star$. The maximal primes above p as well as those above q have to lie in $H^\star$ because H is simple. By smallness of Q and (i) ⇒ (iii) of Proposition 1.31 there are $r, s \in H^\star - Q^\star$ with $r \geq p$ and $s \geq q$. Note that r and s are incomparable because $(\mathcal{E}^*)^\star$ is forest-like.



$H^\star - Q^\star$ being an instance of $(\mathcal{M}^*)^\star$, we conclude that $(\mathcal{M}^*)^\star$ is not a chain, hence neither is $\mathcal{M}^*$, which is of course well known.

There are no principal prime filters in $\mathcal{E}^*$, for every infinite r.e. set can be split into two infinite r.e. halves. Clause (b) of the following corollary addresses the situation with principal prime ideals:

1.34. Corollary. (a) *The only isolated points of $(\mathcal{E}^*)^\star$ are the heels of maximal r.e. sets.*

(b) *The only principal prime ideals of $\mathcal{E}^*$ are given by maximal r.e. sets.*

Proof. (a) Suppose p is isolated. If p is minimal then by Proposition 1.14 it is the heel of some maximal set.

So assume that p is not minimal. Since p is isolated, there are r.e. sets A and B such that $\{p\} = B^\star - A^\star$ because sets of this form constitute a base for the topology of $(\mathcal{E}^*)^\star$. Since $A^\star$ and $B^\star$ differ by a single non-minimal point, we conclude by (iii) ⇒ (i) of Proposition 1.28 that $B^\star - A^\star$ is an instance of $(\mathcal{M}^*)^\star$. But we know that $(\mathcal{M}^*)^\star$ is not a singleton. Contradiction.

(b) According to Fact 1.16(b–c), for a principal prime ideal $(P]$, the set $\overline{P^\star}$ is of the form $\downarrow\{p\}$ for some $p \in (\mathcal{E}^*)^\star$.

So suppose p is not minimal. Note that since $(\mathcal{E}^*)^\star$ is forest-like, $\downarrow\{p\}$ is a chain. In particular, $\downarrow\{p\} \cap \min(\mathcal{E}^*)^\star$ is a singleton, so P is r-maximal by Proposition 1.19 and hence simple. Take some $q < p$. There is an r.e. set A such that $q \notin A^\star \ni p$. Then $A \cap P$ is a major subset of A and $A^\star - P^\star$ is a chain. Which $(\mathcal{M}^*)^\star$ is not.

The contradiction shows that p is minimal, hence P is maximal. ∎

There are no principal prime ideals nor filters in Σ_1/T, just as there are no isolated points in the E-tree.

2. Models of arithmetic

We deal with 1st order arithmetic in the orthodox language $L = (0, 1, +, \times, \leq)$. When we wish to extend L by a finite tuple of constants, we write $L(\vec{c})$.

An L formula $\delta(\vec{x})$ is Δ_0 if no unbounded quantifiers occur in $\delta(\vec{x})$. A formula $\sigma(\vec{x})$ is Σ_1 if it has the form $\exists\vec{y}\,\delta(\vec{x}, \vec{y})$ where $\delta(\vec{x}, \vec{y})$ is Δ_0. Formulas of the form $\forall\vec{y}\,\delta(\vec{x}, \vec{y})$ are Π_1. The classes Σ_n and Π_n also carry their usual meaning. A formula $\gamma(\vec{x})$ is Δ_1 if it is TA-equivalent to both a Σ_1 and a Π_1 formula. The relations represented in $\mathbb{N}$ by Δ_1 formulas are exactly the recursive relations. The following is a direct consequence of Matiyasevich's Theorem:

2.1. Fact (Gaifman [4, Theorem 1]). *Suppose* $M \subseteq K$ *are models of* TA *and* $\vec{c} \in M$.

(Σ_1 persistence) *If* $\sigma(\vec{x})$ *is* Σ_1 *then* $M \models \sigma(\vec{c}) \;\Rightarrow\; K \models \sigma(\vec{c})$.

(Δ_1 absoluteness) *If* $\gamma(\vec{x})$ *is* Δ_1 *then* $M \models \gamma(\vec{c}) \;\Leftrightarrow\; K \models \gamma(\vec{c})$.

By Δ_1 *absoluteness, total recursive functions are absolute for extensions.* ∎

2.2. Convention. As we are particularly interested in non-standard models $M \models \mathrm{TA}$ with a distinguished non-standard element $\boldsymbol{x}$, notation $M_{\boldsymbol{x}}$, $K_{\boldsymbol{x}}$ etc. will always presuppose $M_{\boldsymbol{x}} \models \mathrm{TA} + \boldsymbol{x} > \mathbb{N}$.

2.A. Primes and models

Since Σ_1 formulas represent r.e. sets, we will henceforth identify $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}}$ with $\{\, \text{r.e. } X \mid M_{\boldsymbol{x}} \models \boldsymbol{x} \in X \,\}$.

2.3. Proposition. *A subset* p *of* $\mathscr{E}^*$ *is a prime filter if and only if there is a model* $M_{\boldsymbol{x}}$ *such that* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$.

Proof. (if) If $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A$ then A is infinite because $M_{\boldsymbol{x}} \models \boldsymbol{x} > n$ for each $n \in \omega$. To see primeness, note that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A \cup B$ entails $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A$ or $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B$. Closure under supersets and the intersection property are equally straightforward.

(only if) Let p be a prime. Consider the theory

$$T = \mathrm{TA} + \{\, \boldsymbol{x} \in P \,\}_{P \in p} + \{\, \boldsymbol{x} \notin X \,\}_{X \in p^{\perp}}.$$

In any model of T we have $\boldsymbol{x} > n$ because $\omega - n \in p$ for each $n \in \omega$. So all we have to show is that T is consistent. If it were not, then $\mathrm{TA} \vdash \boldsymbol{x} \in P \to \boldsymbol{x} \in X$ for some $P \in p$ and $X \notin p^{\perp}$, for p is closed under intersection and $p^{\perp}$, under union. Hence $\forall x\, (x \in P \to x \in X)$, or, equivalently, $P \subseteq X$ which contradicts $P \in p \not\ni X$. ∎

2.B. Scott sets and standard systems

Scott sets and standard systems are instrumental for construction of (sub)models of arithmetic. In this subsection we visit the hardware store to assemble a minimalist toolkit.

2.4. Definition. A non-empty collection $\mathfrak{X}$ of subsets of ω is a *Scott set* (a.k.a. *completion-closed algebra*) if $\mathfrak{X}$ is closed under Turing reducibility (from several members of $\mathfrak{X}$), and whenever $T \in \mathfrak{X}$ is an axiom set for a consistent theory in a finite language, some completion of T must be an element of $\mathfrak{X}$. (See section 13.1 and Exercise 13.2 in Kaye [15]).

2.5. Exercise. *Let* $\mathfrak{X}$ *be a Scott set and* $\mathcal{S} \subseteq \mathfrak{X}$ *be countable. Then there is a countable Scott set* $\mathfrak{Y}$ *such that* $\mathcal{S} \subseteq \mathfrak{Y} \subseteq \mathfrak{X}$.

Hint. Iterate closure of $\mathcal{S}$ under Turing reducibility and $\mathfrak{X}$-completion. ∎

2.6. Definition. Let $M \models \mathrm{PA}$ be non-standard. SSy M, the *standard system* of M, is the collection of subsets of ω of the form

$$X = \{\, i \in \omega \mid M \models (c)_i = 0 \,\},$$

where $c \in M$ and $(x)_y = z$ is any of the conventional coding devices for "the yth element of (the sequence coded by) c is z". The element c is then a *code* for X, and X is *coded* in M. (See Kaye [15, section 11.1].)

2.7. Fact. *Let M be a non-standard model of* PA.

(a) SSy M *is a Scott set.*

(b) $\mathrm{Th}_{\Sigma_n}(M, \vec{c})$ *is coded in M for each $\vec{c} \in M$ and $n \in \omega$.*

References. (a) Kaye [15, Theorem 13.2].

(b) Kaye [15, Lemma 12.1] — this uses truth definitions for Σ_n formulas. ∎

Where T is a theory and Γ is a class of formulas, we write $T|_\Gamma$ for the set of Γ consequences of T.

2.8. Fact. *Let $T \vdash$ PA be a complete theory in $L(\vec{c})$, and let $\mathfrak{X}$ be a Scott set.*

(a) *The following are equivalent:*

(i) $T|_{\Sigma_n(\vec{c})} \in \mathfrak{X}$ *for all $n \in \omega$.*

(ii) *There exists a (countable) model $M \models T$ with* SSy $M \subseteq \mathfrak{X}$*;*

(b) *If $\mathfrak{X}$ is countable then the clauses above are also equivalent to*

(iii) *There exists a (countable) model $M \models T$ with* SSy $M = \mathfrak{X}$.

Comment. (i) ⇔ (iii) for countable $\mathfrak{X}$ is Theorem 3.4 in Smoryński [39] who generously ascribes this to Jensen & Ehrenfeucht [14] and Guaspari [5]. Where Smoryński speaks of subsets of ω *representable* in T, it is clear that sets of the form $T|_{\Sigma_n(\vec{c})}$ are cofinal among the representable ones modulo Turing (or even m-) reducibility.

(ii) ⇒ (i) is just as straightforward as (iii) ⇒ (i).

(i) ⇒ (ii) Exercise 2.5 allows one to select a countable $\mathfrak{Y} \subseteq \mathfrak{X}$ with $T|_{\Sigma_n(\vec{c})} \in \mathfrak{Y}$ for all n, whereafter we invoke (i) ⇒ (iii). ∎

The following lemma has much similarity with Theorem 1 in Hájek [6].

2.9. Lemma. *Let $\mathfrak{X}$ be a Scott set and $T \vdash$ PA a consistent theory in $L(\vec{c})$ of the form $T = S + X$ where $\mathfrak{X} \ni X \subseteq \Sigma_k(\vec{c})$ and $S|_{\Sigma_n(\vec{c})} \in \mathfrak{X}$ for each $n \in \omega$.*

Then there exists a completion C of T such that $C|_{\Sigma_n(\vec{c})} \in \mathfrak{X}$ for all $n \in \omega$.

Proof. We construct C as the union of an increasing sequence $(C_i)_{i\in\omega}$ of sets of $L(\vec{c})$ sentences. Let $T_0 = S|_{\Pi_k(\vec{c})} + X$. By our assumptions, $T_0 \in \mathfrak{X}$, and T_0 is consistent because $T_0 \subseteq T$. Since $\mathfrak{X}$ is Scott, there is a completion $\tilde{T}_0 \in \mathfrak{X}$ of T_0. Put $C_0 = \tilde{T}_0|_{\Sigma_k(\vec{c})}$. Note that $X \subseteq C_0 \subseteq \Sigma_k(\vec{c})$. C_0 is consistent with S for otherwise there'd be a $\Sigma_k(\vec{c})$ sentence $\sigma(\vec{c})$ such that $\tilde{T}_0 \vdash \sigma(\vec{c})$ and $S \vdash \neg\sigma(\vec{c})$ contradicting the consistency of $\tilde{T}_0$.

Generally, given C_i consistent with S and $\mathfrak{X} \ni C_i \subseteq \Sigma_{k+i}(\vec{c})$, define

$$C_{i+1} = \Big(\text{an } \mathfrak{X}\text{-completion of } \big(S|_{\Pi_{k+i+1}(\vec{c})} + C_i\big)\Big)\Big|_{\Sigma_{k+i+1}(\vec{c})}.$$

Then $C_i \subseteq C_{i+1} \subseteq \Sigma_{k+i+1}(\vec{c})$, $C_{i+1} \in \mathfrak{X}$, and C_{i+1} is consistent. Hence C_{i+1} is also consistent with S.

The theory $C = \bigcup_{i\in\omega} C_i$ is a completion of T, for $X \subseteq C_0 \subseteq C$ and $S|_{\Pi_{k+i}(\vec{c})} \subseteq C_{i+1} \subseteq C$. Furthermore, $C|_{\Sigma_{k+i}(\vec{c})} = C_{i+1}|_{\Sigma_{k+i}(\vec{c})} \in \mathfrak{X}$. ∎

2.C. Primes and extensions

The following fact is a one-way version of Friedman's Embedding Theorem:

2.10. Fact (see Kaye [15, Exercise 12.7]). *Let $M, K \models$ PA with M countable, $a \in M$ and $b \in K$. Then there is an embedding $h : M \to K$ such that $h(a) = b$ if and only if* $\mathrm{SSy}\, M \subseteq \mathrm{SSy}\, K$ *and* $\mathrm{Th}_{\Sigma_1}(M, a) \subseteq \mathrm{Th}_{\Sigma_1}(K, b)$. ∎

2.11. Gaifman's Cofinal Theorem (Gaifman [4, Theorem 4] or Kaye [15, Corollary 7.10]). *Let $M \subseteq K$ be models of* PA. *Then M is an elementary submodel of*

$$\bar{M} = \{\, a \in K \mid \exists c \in M\ \ K \models a \leq c \,\},$$

the initial closure *of M in K.* ∎

2.12. Definition. The *index set* $\underset{\sim}{p}$ of a prime p is defined by

$$\underset{\sim}{p} = \{\, i \in \omega \mid W_i \in p \,\},$$

where $(W_i)_{i\in\omega}$ is the traditional numbering of r.e. sets.

2.13. Proposition. *Let p and q be primes. The following are equivalent:*

(i) *$p \leq q$;*

(ii) *Every model $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$ extends to a model $K_{\boldsymbol{x}}$ such that $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} = q$;*

(iii) *Every model $N_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} N_{\boldsymbol{x}} = q$ has an elementary extension $K_{\boldsymbol{x}}$ which has an initial segment $M_{\boldsymbol{x}}$ such that $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$.*

Proof. That either of (ii) and (iii) implies (i) is clear by Σ_1 persistence.

(i) ⇒ (ii) Consider the theory

$$T = \mathrm{TA} + \mathrm{Diag}_0\, M_{\boldsymbol{x}} + \{\, \boldsymbol{x} \in Q \,\}_{Q\in q} + \{\, \boldsymbol{x} \notin X \,\}_{X\in q^{\perp}}$$

where $\mathrm{Diag}_0\, M_{\boldsymbol{x}}$ is the open diagram of $M_{\boldsymbol{x}}$. If T failed to be consistent then by compactness the following would be a consequence of TA and hence true:

$$\forall x \left(\exists \vec{y}\ \delta(x, \vec{y})\ \ \&\ \ x \in Q \longrightarrow x \in X\right)$$

for some quantifier-free formula $\delta(x, \vec{y})$ such that $M_{\boldsymbol{x}} \models \exists \vec{y}\ \delta(\boldsymbol{x}, \vec{y})$, some $Q \in q$ and some r.e. X outside q, for q is closed under intersection and $q^{\perp}$ is closed under union. As $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$, the r.e. set $P = \{\, x \in \omega \mid \exists \vec{y}\ \delta(x, \vec{y}) \,\}$ is an element of $p \subseteq q$. Therefore $P \cap Q \subseteq X$ which contradicts $X \notin q \ni P \cap Q$.

Let $K_{\boldsymbol{x}} \models T$. Then $K_{\boldsymbol{x}} \models \boldsymbol{x} > \mathbb{N}$ for $\omega - n \in q$ for each $n \in \omega$, $K_{\boldsymbol{x}}$ extends $M_{\boldsymbol{x}}$, and $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} = q$ as required.

(i) ⇒ (iii) The elementary extension $K_{\boldsymbol{x}}$ of $N_{\boldsymbol{x}}$ obtains as a model of the theory

$$\mathrm{Diag}\, N_{\boldsymbol{x}} + \{\, (c)_i = 0 \,\}_{W_i\in p} + \{\, (c)_j = 1 \,\}_{W_j\in p^{\perp}}$$

where Diag $N_{\boldsymbol{x}}$ is the full 1st order diagram of $N_{\boldsymbol{x}}$ and c is a fresh constant. The consistency of all finite subtheories of that theory is witnessed in $N_{\boldsymbol{x}}$. We have $\underline{p} \in \mathrm{SSy}\, K$.

Let $\mathfrak{X}$ be a countable Scott subset of $\mathrm{SSy}\, K$ which contains $\underline{p}$ and all arithmetical sets (Exercise 2.5). By Lemma 2.9 and Fact 2.8(b) there is a countable model $I_{\boldsymbol{x}} \models \mathrm{TA}$ such that $\mathrm{Th}_{\Sigma_1} I_{\boldsymbol{x}} = p$ and $\mathrm{SSy}\, I = \mathfrak{X} \subseteq \mathrm{SSy}\, K$ — one considers the theory $\mathrm{TA} + \{\, \boldsymbol{x} \in P \,\}_{P \in p} + \{\, \boldsymbol{x} \notin X \,\}_{X \in p^{\perp}}$. By Fact 2.10, $I_{\boldsymbol{x}}$ is isomorphic to a submodel $M_{\boldsymbol{x}}$ of $K_{\boldsymbol{x}}$ because $\mathrm{Th}_{\Sigma_1} I_{\boldsymbol{x}} = p \subseteq q = \mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}}$.

By Gaifman's Cofinal Theorem 2.11, the initial closure $\bar{M}_{\boldsymbol{x}}$ of $M_{\boldsymbol{x}}$ in K is such that $\mathrm{Th}_{\Sigma_1} \bar{M}_{\boldsymbol{x}} = \mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$. ∎

The awkward form of clause (iii) could be an argument for using models of TA_2 rather than models of full TA: every TA_2-model $N_{\boldsymbol{x}}$ such that $\mathrm{Th}_{\Sigma_1} N_{\boldsymbol{x}} = p$ actually has an initial TA_2-segment $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$.

2.14. Exercise. *Let p be a prime and $M \models \mathrm{TA}$. There exists a non-standard $\boldsymbol{x} \in M$ such that $p = \mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}}$ if and only if $\underline{p} \in \mathrm{SSy}\, M$.*

Hint. Kaye [15, Lemmas 12.1 and 12.2]. ∎

2.D. Simple sets

The following theorem of Schmerl extends to the uncountable case the earlier result of Wilkie [46].

2.15. Fact (Schmerl [33, Theorem 1] or Kossak & Schmerl [17, Theorem 2.4.2]). *Let $M \models \mathrm{PA}$ be non-standard an let T be a complete theory in $L(\vec{c})$. Then M end-extends to a model of T if and only if $\mathrm{Th}_{\Sigma_1}(M, \vec{c}) \subseteq T$ and $T|_{\Sigma_n(\vec{c})} \in \mathrm{SSy}\, M$ for each n.* ∎

Precursors of the next proposition are Hirschfeld & Wheeler [13, Lemma 8.23] (see also Hirschfeld [11, 4.6]), Wilkie [45, Theorem 4.5], and Schmerl [33, Theorem 2] (see also Kossak & Schmerl [17, Theorem 2.4.3]).

2.16. Proposition. *Let S be r.e. The following are equivalent:*

(i) *S is simple;*

(ii) *Each $M_{\boldsymbol{x}}$ admits an extension $K_{\boldsymbol{x}} \models \boldsymbol{x} \in S$;*

(iii) *Each $M_{\boldsymbol{x}}$ end-extends to some $K_{\boldsymbol{x}} \models \boldsymbol{x} \in S$.*

Proof. (iii) ⇒ (ii) is trivial.

(ii) ⇒ (i) Suppose S failed to be simple, so there is an infinite r.e. D disjoint from S. Then in any non-standard model $M \models \mathrm{TA}$ there is a non-standard element $\boldsymbol{x}$ such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in D$. By (ii) there is an extension $K_{\boldsymbol{x}}$ of $M_{\boldsymbol{x}}$ such that $K_{\boldsymbol{x}} \models \boldsymbol{x} \in S$. We also have $K_{\boldsymbol{x}} \models \boldsymbol{x} \in D$ by Σ_1 persistence. Thus $K_{\boldsymbol{x}} \models S \cap D \neq \varnothing$ contradicting the disjointness of S and D. Which shows that S is simple.

(i) ⇒ (iii) Consider the theory $T = \mathrm{TA} + \mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} + \boldsymbol{x} \in S$. Is T consistent? If not, there is by compactness an r.e. set D such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in D$ (hence D is infinite) and $\mathrm{TA} \vdash \forall x\, (x \in D \to x \notin S)$, so D is disjoint from S, contradicting the simplicity of the latter.

Since T satisfies the conditions of Lemma 2.9 w.r.t. $\mathfrak{X} = \mathrm{SSy}\, M_{\boldsymbol{x}}$, there is a completion C of T such that $C|_{\Sigma_n(\boldsymbol{x})} \in \mathrm{SSy}\, M_{\boldsymbol{x}}$ for all n. Further, $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} \subseteq T \subseteq C$. Therefore by Fact 2.15 there is an end extension $K_{\boldsymbol{x}}$ of $M_{\boldsymbol{x}}$ such that $K_{\boldsymbol{x}} \models C$, so $K_{\boldsymbol{x}} \models \boldsymbol{x} \in S$. ∎

Proposition 2.16 parallels Theorem 6.5(i) of Guaspari [5] which characterizes, among other things, sentences Π_1-conservative over T as those which can be modelled in an appropriate end extension of every model of a formal theory T.

2.17. Exercise. *For r.e. $A \subseteq B$, A is simple in B if and only if every $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B$ end-extends to a* TA*-model of $\boldsymbol{x} \in A$.* ∎

3. Dynamic properties

We use $M \subseteq_e K$ to indicate that M is an initial segment of K, and $\equiv$ for 1st order equivalence.

3.A. Wilkie's Lemma

3.1. Notation. Suppose a and b are elements of a model $M \models$ TA. We write $M \models a \ll b$ if $M \models f(a) < b$ for each total recursive function f (with standard index).

The negation of $a \ll b$ is $b \lesssim a$. Note that both $\ll$ and $\lesssim$ are transitive.

The following lemma was established by Alex Wilkie a long long time ago but appears to have thus far miraculously escaped documentation.

3.2. Wilkie's Lemma. *Suppose $a, b \in M \models$ PA.*

(a) *The following are then equivalent:*

(i) *There is $I \subseteq_e M$ with $a < I < b$ and $I \equiv M$;*

(ii) *$M \models \forall x\, \exists y\, \delta(x, y)$ implies $M \models \exists y < b\ \delta(a, y)$ for each Δ_0 formula $\delta(x, y)$.*

(b) *If M is countable then the above clauses are also equivalent to*

(iii) *There is $I \subseteq_e M$ with $a < I < b$ and $I \cong M$.*

Note that when $M \models$ TA, clause (ii) is equivalent to $M \models a \ll b$.

We derive the Lemma as a consequence of

3.3. Friedman's Embedding Theorem (see Kaye [15, Theorem 12.3]). *Let a, b, c be elements of a countable model $M \models$ PA. Then there is an isomorphism i of M onto some initial segment $I \subseteq_e M$ with $i(c) = a \in I < b$ if and only if for all Δ_0 formulas $\delta(x, y)$ one has*

$$M \models \exists y\ \delta(c, y) \quad \Longrightarrow \quad M \models \exists y < b\ \delta(a, y).$$ ∎

3.4. Proof of Wilkie's Lemma. (i) ⇒ (ii) Suppose $M \models \forall x\, \exists y\, \delta(x, y)$. Since $I \equiv M$, it follows that $I \models \exists y\, \delta(a, y)$. Hence there is $d \in I$ such that $I \models \delta(a, d)$. As $I < b$ and Δ_0 formulas are absolute for initial segments modelling PA, we have $M \models \exists y < b\ \delta(a, y)$ as required.

(ii) ⇒ (iii) for countable M: From the assumption of clause (ii) it follows by 1st order logic that for each Δ_0 formula $\delta(x, y)$ there is a $d \in M$ such that

$$M \models \exists y\, \delta(d, y) \quad \Longrightarrow \quad M \models \exists y < b\ \delta(a, y).$$

Therefore the Π_1 type

$$T(x) = \{\, \forall y\, \neg\delta(x,y) \mid \delta(x,y) \text{ is } \Delta_0 \;\&\; M \models \forall y < b\ \neg\delta(a,y) \}$$

is realized point-wise in M, and since $T(x)$ is closed under conjunction (modulo 1st order logic), M also realizes $T(x)$ locally. Observe that $T(x)$ is coded in M (see Fact 2.7(b)). Therefore by Lemma 12.2 in Kaye [15] (bounded-complexity coded types are realized), $T(x)$ is also globally realized in M. Let $c \in M$ be a realization of $T(x)$. Then for each Δ_0 formula $\delta(x,y)$ there holds

$$M \models \exists y\ \delta(c,y) \quad \Longrightarrow \quad M \models \exists y < b\ \delta(a,y).$$

By Friedman's Embedding Theorem 3.3, there is an embedding i of M onto an initial segment I of M with $i(c) = a \in I < b$.

(iii) ⇒ (i) is clear.

To show (ii) ⇒ (i) for uncountable M, let $K \ni a, b$ be a countable elementary submodel of M. Then K also satisfies the condition of clause (ii). Hence by (ii) ⇒ (iii) there is an initial segment $I \cong K$ of K such that $a \in I < b$. Let $\bar{I}$ be the initial closure of I in M. By Gaifman's Cofinal Theorem, $\bar{I} \equiv I \cong K \equiv M$. Clearly, $a < \bar{I} < b$, which completes the proof. ∎

3.B. Hinges and hinged primes

3.5. Definition. An *enumeration* $(A_s)_{s\in\omega}$ of an r.e. set A is an effective sequence of finite sets such that $A = \bigcup_{s\in\omega} A_s$ and $A_s \subseteq A_{s+1}$. Unless we need to explicitly compare two enumerations, we will always silently assume that some enumeration of each r.e. set we consider is chosen and stays the same throughout an argument.

Define $A_{\text{at } s} = \begin{cases} A_s - A_{s-1} & \text{if } s > 0, \\ A_0 & \text{if } s = 0. \end{cases}$

Suppose $(U_i)_{i\in\omega}$ is a uniformly r.e. sequence of r.e. sets, that is, the binary relation $x \in U_s$ is r.e. An *enumeration* of $(U_i)_{i\in\omega}$ is an effective double sequence $(U_{i,s})_{i,s\in\omega}$ of finite sets such that $U_i = \bigcup_{s\in\omega} U_{i,s}$, $U_{i,s} \subseteq U_{i,s+1}$, the set $\{\,(x,i) \mid x \in U_{i,s}\,\}$ is finite for each s, and its code is recursive in s. (cf. Soare [41, Definition II.2.8].)

An enumeration is Δ_0 if the ternary relation $x \in U_{i,s}$ is. It follows from e.g. Matiyasevich's Theorem that any uniformly r.e. sequence can be supplied with a Δ_0 enumeration. Smullyan [40, Theorem IV.8] offers a Matiyasevich-free construction of a Δ_0 enumeration of all r.e. sets similar to Kleene's T-predicate.

Throughout the paper we assume that the traditional numbering $(W_i)_{i\in\omega}$ of all r.e. sets is accompanied by one of the usual enumerations.

3.6. Definition. Let A, B be r.e., and f be a recursive function. Define the r.e. set

$$A \searrow_f B = \Big\{\, x \;\Big|\; \exists s \big(x \in B_{\text{at } s} \;\&\; f(s)\!\downarrow \;\&\; x \in A_{f(s)} \big) \Big\}.$$

We shall mostly be interested in total recursive f. The requirement $f(s)\!\downarrow$ is only there for definiteness, so that an r.e. index for $A \searrow_f B$ be effective in those of A, B, and f. Note that the set $A \searrow_f B$ depends on the distinguished enumerations of A and B, but we always have $A \searrow_f B \subseteq A \cap B$.

Where f is a recursive function, $\bar{f}(t) = \max_{s\le t} f(s)$.

3.7. Lemma. *Let $(A_t)_{t\in\omega}$ and $(\tilde{A}_t)_{t\in\omega}$ be enumerations of the r.e. set A, let $(B_t)_{t\in\omega}$ and $(\tilde{B}_t)_{t\in\omega}$ be enumerations of the r.e. B, and f a total recursive function.*

There is a total recursive g such that $\tilde{A} \searrow_g \tilde{B} \supseteq A \searrow_f B$.

PROOF. Let h and j be total recursive functions such that $A_t \subseteq \tilde{A}_{h(t)}$ and $\tilde{B}_t \subseteq B_{j(t)}$. Put $g = h \circ \bar{f} \circ j$. ∎

The proof of the next lemma is straightforward.

3.8. LEMMA. *$\{A \searrow_f B, B - (A \searrow_f B)\}$ is an r.e. splitting of B whenever A and B are r.e. and f is total recursive.* ∎

The following important definition is borrowed from the E-tree where one uses provably recursive functions in place of total recursive ones (Shavrukov & Solovay [36]).

3.9. DEFINITION. Say that a prime p is *hinged on* an r.e. set P, or that $P \in p$ is a *hinge* for p, if for each $A \in p$ there is a total recursive f with $A \searrow_f P \in p$. (Observe that by Lemma 3.7 the choice of enumerations of A and P is not an issue.) Informally, p hinges on P if P is, up to total recursive functions, 'the latest' set in p.

p is *hinged* if it has a hinge.

3.10. LEMMA. *Let P be a hinge for a prime p. Then $Q \in p$ is also a hinge for p if and only if there is a total recursive f such that $P \searrow_f Q \in p$.*

PROOF. (only if) follows directly from the definition.

(if) For any $X \in p$, $X \searrow_g P \in p$ and $P \searrow_f Q \in p$ imply $X \searrow_{\bar{g} \circ f} Q \in p$, for $X \searrow_g P \cap P \searrow_f Q \subseteq X \searrow_{\bar{g} \circ f} Q$. ∎

3.11. PROPOSITION. *Let p be a prime and P be r.e. The following are equivalent:*

(i) *p hinges on P;*

(ii) *$P \in p$, and whenever* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} \supseteq p$ *and $M_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$, we have*

$$p = \{\text{r.e. } A \mid M_{\boldsymbol{x}} \models \exists t \leqslant s \ \boldsymbol{x} \in A_t\};$$

(iii) *$p \in \min P^{\star}$.*

PROOF. (i) ⇒ (ii) Suppose $M_{\boldsymbol{x}}$ and $s \in M$ are as specified in (ii).

If $A \in p$ then $A \searrow_f P \in p$ for some total recursive f because p hinges on P. Hence, in M one has $\boldsymbol{x} \in A \searrow_f P$ and therefore $\boldsymbol{x} \in A_{f(s)}$. Thus $M_{\boldsymbol{x}} \models \exists t \leqslant s \ \boldsymbol{x} \in A_t$.

Suppose $A \notin p$. Let f be an arbitrary total recursive function. Since $p \ni P$ is prime and $A \searrow_f P \subseteq A \notin p$, we have $P - (A \searrow_f P) \in p$ by Lemma 3.8. Hence $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin A_{f(s)}$. As f is arbitrary, $M_{\boldsymbol{x}} \not\models \exists t \leqslant s \ \boldsymbol{x} \in A_t$.

(ii) ⇒ (iii) Consider any prime q with $P \in q \le p$. By Proposition 2.13 there are models $K_{\boldsymbol{x}} \subseteq M_{\boldsymbol{x}}$ such that $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$ and $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} = q$. Since $P \in q$, we have $K_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$ for some $s \in K$. By Δ_1 absoluteness, $M_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$.

Given any $A \in p$, we must have $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A_{f(s)}$ for some total recursive f by the assumption. Hence $K_{\boldsymbol{x}} \models \boldsymbol{x} \in A_{f(s)}$ by Δ_1 absoluteness. Therefore $A \in q = \mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}}$, so we may conclude $p \subseteq q$ and hence $q = p$. Thus p is minimal among the primes that contain P.

(iii) ⇒ (i) Suppose p failed to hinge on P: there is an $A \in p$ with $A \searrow_f P \notin p$ for all total recursive f. Consider $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$ and $s \in M$ such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$. Since $A \in p$, there is an element t of M with $\boldsymbol{x} \in A_{\mathrm{at}\, t}$. By our assumption, $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin A_{f(s)}$ for all total recursive f. Thus $s \ll t$. By Wilkie's Lemma there is a TA-initial segment I of M such that $\boldsymbol{x} \lesssim s < I < t$. We have $I_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$ by Δ_1 absoluteness but $I_{\boldsymbol{x}} \models \boldsymbol{x} \notin A$. Therefore $p \notin \min P^\star$, for $P \in \mathrm{Th}_{\Sigma_1} I_{\boldsymbol{x}} \not\ni A \in p$. The contradiction concludes our proof. ■

3.12. Exercise. *Suppose $M_{\boldsymbol{x}} \models s \gtrsim \boldsymbol{x}$. Then* $\{$ r.e. $A \mid M_{\boldsymbol{x}} \models \exists t \lesssim s\ \boldsymbol{x} \in A_t \}$ *is a prime.* ■

3.13. Lemma. *Suppose P is a hinge for a prime p such that* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$, $t \in M$, $M_{\boldsymbol{x}} \models \boldsymbol{x} \in P_t$, *and $M_{\boldsymbol{x}} \models \exists y\, \sigma(\boldsymbol{x}, y)$ where $\sigma(x, y)$ is a Σ_1 formula.*

Then $M_{\boldsymbol{x}} \models \exists y \lesssim t\, \sigma(\boldsymbol{x}, y)$.

Proof. Consider the r.e. set $S = \{ x \mid \exists y\, \sigma(x, y) \}$. Suppose $\sigma(x, y)$ is $\exists \vec{z}\, \delta(x, y, \vec{z})$ where $\delta(x, y, \vec{z})$ is Δ_0. Define the enumeration $(S_t)_{t \in \omega}$ by

$$x \in S_t \iff t \geq x \ \&\ \exists y \leq t\, \exists \vec{z} \leq t\ \delta(x, y, \vec{z}).$$

We have $M_{\boldsymbol{x}} \models \boldsymbol{x} \in S$. Let $s \in M$ satisfy $M_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{\mathrm{at}\, s}$ so that $s \leq t$. By (i) ⇒ (ii) Proposition 3.11 there is $b \lesssim s \leq t$ such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in S_b$, hence $M_{\boldsymbol{x}} \models \exists y \lesssim t\ \sigma(\boldsymbol{x}, y)$. ■

3.C. Hinged points for bounded distributive lattices

A prime filter $p \in L^\star$ is a *Goldman point* if there is an $x \in L$ such that $p \in \min x^\star$. According to Proposition 3.11, in $(\mathscr{E}^*)^\star$, Goldman is a synonym for hinged. Bouacida & al. [1] study Goldman points in spectral spaces where they are exactly the locally closed points.

3.14. Lemma. *Let $x \in p \in L^\star$. Then there is a $q \leq p$ such that $q \in \min x^\star$.*

Proof. Let q be any point minimal in the closed set $x^\star \cap \downarrow\{p\}$. ■

3.15. Lemma. (a) *If $p \in L^\star$ is a Goldman point, then $p \in \min L^\star$ or p has an immediate predecessor in each maximal chain $\pi \ni p$ within $L^\star$.*

(b) *Suppose L is relatively normal. A prime filter p is a Goldman point if and only if $p \in \min L^\star$ or p has an immediate predecessor in $L^\star$.*

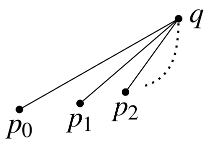
Proof. (a) Suppose $p \in \min x^\star$ and $p \notin \min L^\star$. Let $\pi \ni p$ be a maximal chain within $L^\star$. Then $q = \bigcup\{ r \in \pi \mid r < p \}$ is a prime filter because q is the union of a chain of primes. Also $q \subseteq p$ and $q \in \pi$ because π is maximal. Since $q \not\ni x$, we must have $q < p$, so q is an immediate predecessor of p.

(b) The (only if) direction is clause (a). We handle (if). Any minimal prime p is clearly Goldman, for $p \in \min L^\star = \min 1_L^\star$. Since $L^\star$ is an $\uparrow$-forest for relatively normal L, the set $\downarrow\{p\} - \{p\}$ of predecessors of p is a chain. Let q be the (unique) immediate predecessor of p. There is $x \in L$ such that $p \ni x \notin q$. Then $p \in \min x^\star$ because for each $r < p$ one has $q \geq r \not\ni x$, so p is a Goldman point. ■

We now point out that the implication in clause (a) of Lemma 3.15 cannot be reversed without further assumptions.

3.16. Example. *There is a bounded distributive lattice L and a non-Goldman prime filter p of L such that p has an immediate predecessor in each maximal chain $\pi \ni p$ within $L^\star$.*

DETAILS. Let C be the lattice of empty or cofinite subsets of ω with set-theoretical operations. Then $C^\star$ is the Priestley space



consisting of a countable collection $\{p_i\}_{i\in\omega}$ together with the prime q such that $p_i < q$ for all i. A subset $\pi \subseteq C^\star$ is open if and only if $q \in \pi$ implies that almost all p_i are also elements of π. Thus q is the (unique) limit point of $\{p_i\}_{i\in\omega}$. This Priestley space appears in Exercise 11.10(ii) of Davey & Priestley [3], although there it corresponds to the antipode of the lattice C because of opposite conventions concerning vertical orientation.

The only non-Goldman point of $C^\star$ is q, the prime filter formed by all cofinite subsets of ω. Observe that q has an immediate predecessor in each maximal chain. ∎

How much information about L is carried by the collection of Goldman primes? The answer turns out to depend on the structure one endows that collection with.

The spectral dual Spec L of L obtains as the sobrification of Gold L, its subspace consisting of Goldman points (see Proposition 3.6(2)(i), and Theorem 2.2 applied to the inclusion Gold $L \hookrightarrow$ Spec L in Bouacida & al. [1]). Thus Gold L completely determines L.

We shall see below that the situation with L^{G}, the subspace of $L^\star$ of Goldman points with Priestley subspace topology and restricted order, is less satisfactory. The lattice L is not generally uniquely determined by L^{G}. This shows that while $L^\star$ and Spec L are "essentially" the same thing (Cornish [2]), their respective subspaces L^{G} and Gold L are not, even though they are made of exactly the same points.

We first point out a particular case that admits successful reconstruction of L from (just the order structure of) L^{G}.

3.17. PROPOSITION. *If the lattice L is a chain, then $L - \{0\}$ is order-anti-isomorphic to L^{G}.*

HINT. The Goldman prime filters of L are exactly the proper principal filters. ∎

In relatively normal lattices, the order structure of L^{G} suffices to determine that of $L^\star$:

3.18. PROPOSITION. *If L is relatively normal, then $(\mathcal{P}_L, \leq)$, the order structure of $L^\star$, is order-isomorphic to the inclusion-ordered collection of non-empty linearly ordered subsets of L^{G} that are $\downarrow$-closed within L^{G}.*

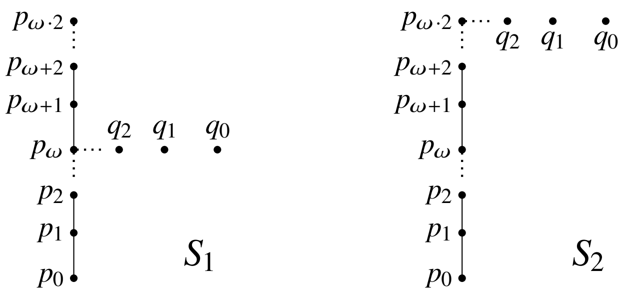
HINT. Since $L^\star$ is an $\uparrow$-forest, a non-empty collection $\pi \subseteq L^{\mathrm{G}}$ is that of all Goldman points contained in some prime if and only if π is linearly ordered and downward closed under inclusion. Any prime filter p is uniquely determined by the collection of Goldman primes contained in p because p is the union of that collection. ∎

The topology of $L^\star$ however cannot be decrypted from L^{G}:

3.19. EXAMPLE. *There exist two non-isomorphic bounded distributive lattices satisfying the Reduction Principle and such that the Goldman subspaces of their Priestley duals are order-homeomorphic.*

Details. It suffices to produce two non-order-homeomorphic Priestley spaces S_1 and S_2 whose Goldman subspaces are order-homeomorphic.

The points of S_j are two sequences $\{p_\alpha\}_{\alpha\in\omega\cdot 2+1}$ and $\{q_i\}_{i\in\omega}$ with the only non-reflexive order pairs being determined by $p_\alpha < p_\beta \Leftrightarrow \alpha < \beta$. The topologies do differ:



A subset π of S_1 is open in S_1 if $\pi \ni p_\omega$ implies that almost all of $\{p_i\}_{i\in\omega} \cup \{q_i\}_{i\in\omega}$ are elements of π, and $\pi \ni p_{\omega\cdot 2}$ implies that almost all of $\{p_\alpha\}_{\omega<\alpha\in\omega\cdot 2}$ are also in π. π is open in S_2 if $\pi \ni p_\omega$ implies that almost all elements of $\{p_i\}_{i\in\omega}$ are those of π, and $\pi \ni p_{\omega\cdot 2}$ implies that almost all of $\{p_\alpha\}_{\omega<\alpha\in\omega\cdot 2} \cup \{q_i\}_{i\in\omega}$ are in π. In particular, the unique limit point of the sequence $\{q_i\}_{i\in\omega}$ in S_j is $p_{\omega\cdot j}$. It is clear that S_1 and S_2 are not order-homeomorphic.

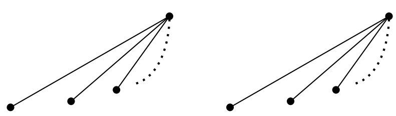
In both cases, p_ω and $p_{\omega\cdot 2}$ are the only non-Goldman points of S_i. Removing these, we obtain Goldman subspaces of S_j with discrete topology, for each remaining point is clopen. The ordering in both cases is the unordered union of a countably infinite antichain and a chain of ordertype $\omega \cdot 2$. Therefore the two Goldman subspaces are order-homeomorphic.

Given two clopen $\uparrow$-sets X and Y in S_j, we assume w.l.o.g. that $X \cap \{p_\alpha\}_{\alpha\in\omega\cdot 2+1} \supseteq Y \cap \{p_\alpha\}_{\alpha\in\omega\cdot 2+1}$, and put $X' = X$ and $Y' = Y - X$, so that Y' is a finite and hence clopen ($\uparrow$-)subset of $\{q_i\}_{i\in\omega}$. This shows that both S_j correspond to lattices satisfying the Reduction Principle. ∎

Without relative normality, the subspace L^{G} cannot even generally tell the number of non-Goldman points in $L^\star$:

3.20. Example. *There are two bounded distributive lattices, one with a single non-Goldman prime, the other with two non-Goldman primes, such that the Goldman subspaces of their Priestley duals are order-homeomorphic.*

Details. Recall the lattice C from Example 3.16. The Priestley dual of the lattice C^2 is the disjoint unordered union of two copies of the dual of C, for the product of bounded distributive lattices corresponds to the disjoint unordered union of their duals (Davey & Priestley [3, Exercise 11.8]).



The two maximal points are the non-Goldman primes of C^2. In C, there was a single non-Goldman prime. Both C^{G} and $(C^2)^{\mathrm{G}}$ are countably infinite discrete antichains, so they are order-homeomorphic. ∎

Contrary to what the examples above suggest, the topology of L^{G} need not always be discrete — consider the lattice of finite or cofinite subsets of ω, where all prime filters are Goldman.

It should be noted that, unlike * and Spec, neither $^{\mathrm{G}}$ nor Gold is functorial.

3.D. Minimal and maximal primes

3.21. Proposition. *For a prime p the following are equivalent:*

(i) $p \in \min(\mathcal{E}^*)^\star$;

(ii) *p hinges on each of its elements;*

(iii) *For each $X \in p$ there is a total recursive f such that for some/every $M_{\boldsymbol{x}}$ with* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$ *there holds $M_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{f(\boldsymbol{x})}$ (equivalently, $\{ x \mid x \in X_{f(x)} \} \in p$).*

Proof. (i) ⇒ (ii) If $X \in p$ then $p \in X^\star$ whence $p \in \min X^\star$ because $p \in \min(\mathcal{E}^*)^\star$. By Proposition 3.11, X is a hinge for p.

(ii) ⇒ (iii) Keeping in mind some fixed r.e. index for ω, there is a total recursive g such that $n \in \omega_{\mathrm{at}\ g(n)}$ for each n. Given any $X \in p$, there is a total recursive h such that $X \searrow_h \omega \in p$ because ω is a hinge for p. Hence in $M_{\boldsymbol{x}}$ we have $\boldsymbol{x} \in \omega_{\mathrm{at}\ g(\boldsymbol{x})}$ and $\boldsymbol{x} \in X \searrow_h \omega$ which entails $\boldsymbol{x} \in X_{h \circ g(\boldsymbol{x})}$.

(iii) ⇒ (i) Suppose $q \le p$. By Proposition 2.13 there are $M_{\boldsymbol{x}} \subseteq K_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = q$ and $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} = p$. For each $X \in p$ there is a total recursive f such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{f(\boldsymbol{x})}$. Therefore $K_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{f(\boldsymbol{x})}$ by Δ_1 absoluteness. Thus $X \in q$, so $q = p$. ∎

(iii) ⇒ (i) of the above proposition and Lemma 3.10 yield

3.22. Corollary. *Suppose q is non-minimal,* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = q$, *$Q$ is a hinge for q, and $M_{\boldsymbol{x}} \models \boldsymbol{x} \in Q_t$. Then $M_{\boldsymbol{x}} \models \boldsymbol{x} \ll t$.* ∎

Proposition 3.21 says that minimal primes are hinged (on ω). In particular, so are the minimax primes. The next proposition tells us that these are the only hinged maximal primes.

3.23. Proposition. *Suppose $p \in \max(\mathcal{E}^*)^\star$ is hinged. Then $p \in \min(\mathcal{E}^*)^\star$.*

Proof. Consider a maximal prime p hinged on some r.e. B, so $p \in \min B^\star$. Let A be a small simple subset of B (Fact 1.32). Then $p \in A^\star$, for $A^\star \cap \max(\mathcal{E}^*)^\star = B^\star \cap \max(\mathcal{E}^*)^\star$ in view of simplicity of A in B (Proposition 1.13). Therefore $p \in \min A^\star$ because $A \subseteq B$. By smallness of A in B and (i) ⇒ (iv) of Proposition 1.31 we conclude $p \in \min(\mathcal{E}^*)^\star$. ∎

3.E. Major subsets

The equivalence (i) ⇔ (ii) of the following proposition is based on the Marker Lemma (Maass & Stob [28, Lemma 1.3]).

3.24. Proposition. *Suppose $A \subset_\infty B$ are r.e. The following are equivalent:*

(i) *A is major in B;*

(ii) *For each total recursive f, $x \in B_{f(x)}$ holds for at most finitely many $x \in B - A$;*

(iii) *In any model $M_{\boldsymbol{x}}$, if $\boldsymbol{x} \in B - A$ and $\boldsymbol{x} \in B_t$, then $t \gg \boldsymbol{x}$.*

Proof. (i) ⇒ (ii) The recursive set $R = \{ x \mid x \in B_{f(x)} \}$ is a subset of B, hence by (i) ⇒ (ii) of Proposition 1.28, $R \subseteq^* A$. Therefore $R \cap (B - A) = \{ x \in B - A \mid x \in B_{f(x)} \}$ is finite.

(ii) ⇒ (iii) Consider any $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B - A$ and let f be any total recursive function. Since there are only finitely many $x \in B - A$ with $x \in B_{f(x)}$, we have $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin B_{f(\boldsymbol{x})}$. Thus $\boldsymbol{x}$ enters B at a stage that is more than total recursively larger than $\boldsymbol{x}$.

(iii) ⇒ (i) Let $p \in B^\star - A^\star$ and $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$. For the unique $t \in M_{\boldsymbol{x}}$ such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B_{\mathrm{at}\ t}$ we have $M_{\boldsymbol{x}} \models \boldsymbol{x} \ll t$ by our assumption. Hence by Wilkie's Lemma there is an initial TA-segment $I_{\boldsymbol{x}}$ of $M_{\boldsymbol{x}}$ with $I_{\boldsymbol{x}} < t$. So $\mathrm{Th}_{\Sigma_1} I_{\boldsymbol{x}} \subsetneq p$, for $I_{\boldsymbol{x}} \models \boldsymbol{x} \notin B$. Thus any $p \in B^\star - A^\star$ is non-minimal. By (iii) ⇒ (i) of Proposition 1.28, A is major in B. ∎

3.F. Small subsets

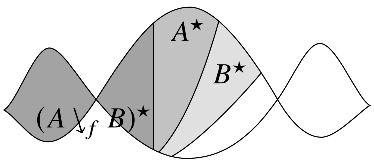
The equivalence (i) ⇔ (iv) of the following theorem is due to Harrington & Soare.

3.25. Theorem (after Harrington & Soare [8, Theorem 3.2]). *Suppose $A \subset_\infty B$ are r.e. The following are then equivalent:*

(i) *A is small in B;*

(ii) *For all $M_{\boldsymbol{x}} \subseteq_{\mathrm{e}} K_{\boldsymbol{x}}$ with $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin B$ and $K_{\boldsymbol{x}} \models \boldsymbol{x} \in B$ there is $I_{\boldsymbol{x}} \models \boldsymbol{x} \in B - A$ such that $M_{\boldsymbol{x}} \subseteq_{\mathrm{e}} I_{\boldsymbol{x}} \subseteq_{\mathrm{e}} K_{\boldsymbol{x}}$;*

(iii) *For each total recursive f, the r.e. set $A \searrow_f B$ is recursive.*



(iv) *For each total recursive f, there is a recursive $R \subseteq B$ such that $A \searrow_f B \subseteq R$.*

Proof. (i) ⇒ (ii) Let $M_{\boldsymbol{x}}$ and $K_{\boldsymbol{x}}$ be as specified in (ii). If $K_{\boldsymbol{x}} \models \boldsymbol{x} \notin A$, put $I_{\boldsymbol{x}} = K_{\boldsymbol{x}}$. Otherwise, since $K_{\boldsymbol{x}} \models \boldsymbol{x} \in A \subseteq B$, there are $s, t \in K_{\boldsymbol{x}}$ such that $K_{\boldsymbol{x}} \models \boldsymbol{x} \in B_{\mathrm{at}\ s}$ & $\boldsymbol{x} \in A_{\mathrm{at}\ t}$. One has $K_{\boldsymbol{x}} \models s, t > M_{\boldsymbol{x}}$, for $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin B \supseteq A$.

Suppose $K_{\boldsymbol{x}} \models t \lll s$, that is, there is a total recursive g with $g(s) \geq t$. Consider the r.e. set $Y = B - (A \searrow_g B)$ (Lemma 3.8). Clearly $B - A \subseteq Y$ and $K_{\boldsymbol{x}} \models \boldsymbol{x} \notin Y$. Now observe that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in \overline{B} \subseteq Y \cup \overline{B}$ but $K_{\boldsymbol{x}} \models \boldsymbol{x} \notin Y \cup \overline{B}$. By Σ_1 persistence, $Y \cup \overline{B}$ cannot be r.e. Thus A is not a small subset of B by (i) ⇒ (ii) of Proposition 1.31. Thus the assumption (i) outrules $K_{\boldsymbol{x}} \models t \lll s$.

Hence $K_{\boldsymbol{x}} \models s \ll t$. By Wilkie's Lemma 3.2(a) there is an $I_{\boldsymbol{x}} \subseteq_{\mathrm{e}} K_{\boldsymbol{x}}$ such that $K_{\boldsymbol{x}} \models M_{\boldsymbol{x}} < s < I_{\boldsymbol{x}} < t$, so by Δ_1 absoluteness, $I_{\boldsymbol{x}} \models \boldsymbol{x} \in B - A$.

(ii) ⇒ (iii) We will be done once we show that $(A \searrow_f B)^\star$ is a ↓-set, that is, for primes $r \geq p$ with $r \in (A \searrow_f B)^\star$, we show $p \in (A \searrow_f B)^\star$.

By Lemma 3.14, fix $q \leq r$ that hinges on B. Use Proposition 2.13 and Gaifman's Cofinal Theorem to find $K_{\boldsymbol{x}}$, $M_{\boldsymbol{x}}$ and $N_{\boldsymbol{x}}$ such that $M_{\boldsymbol{x}} \subseteq_{\mathrm{e}} K_{\boldsymbol{x}}$, $N_{\boldsymbol{x}} \subseteq_{\mathrm{e}} K_{\boldsymbol{x}}$, $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} = r$, $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = q$, and $\mathrm{Th}_{\Sigma_1} N_{\boldsymbol{x}} = p$. Let $s \in K_{\boldsymbol{x}}$ and $t \in M_{\boldsymbol{x}}$ satisfy $K_{\boldsymbol{x}} \models \boldsymbol{x} \in B_{\mathrm{at}\ s}$ & $\boldsymbol{x} \in A_{f(s)}$ and $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B_{\mathrm{at}\ t}$. By Δ_1 absoluteness we have $K_{\boldsymbol{x}} \models s = t$ and $M_{\boldsymbol{x}} \models \boldsymbol{x} \in B_{\mathrm{at}\ t}$ & $\boldsymbol{x} \in A_{f(t)}$. In particular, $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A \searrow_f B$. For the sake of interest, suppose $p \not\ni A \searrow_f B$, thus $p < q$ and $N_{\boldsymbol{x}} \subsetneq_{\mathrm{e}} M_{\boldsymbol{x}}$. As $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = q \in \min B^\star$, we have $N_{\boldsymbol{x}} \models \boldsymbol{x} \notin B$. By (ii) there must exist $I_{\boldsymbol{x}}$ with $N_{\boldsymbol{x}} \subseteq_{\mathrm{e}} I_{\boldsymbol{x}} \subseteq_{\mathrm{e}} M_{\boldsymbol{x}}$ and $I_{\boldsymbol{x}} \models \boldsymbol{x} \in B - A$. Thus by Δ_1 absoluteness $t \in I_{\boldsymbol{x}} \not\ni f(t)$ which contradicts $I_{\boldsymbol{x}} \models \mathrm{TA}$. The contradiction establishes $p \in (A \searrow_f B)^\star$.

(iii) ⇒ (iv) is trivial since $A \searrow_f B \subseteq B$.

(iv) ⇒ (i) By contraposition: Suppose A failed to be small in B. By (iv) ⇒ (i) of Proposition 1.31 there is a prime $q \in (\min A^\star \cap \min B^\star) - \min(\mathcal{E}^*)^\star$. Since $q \notin \min(\mathcal{E}^*)^\star$, there is a prime $p < q$. It follows that $p \notin B^\star$.

Since q hinges on B and $A \in q$, there is a total recursive f such that $A \searrow_f B \in q$. Consider any recursive $R \subseteq B$. We have $p \notin B^\star$, hence $p \notin R^\star$. From $q \geq p$ we get $q \notin R^\star$ by Lemma 1.7. Seeing as $A \searrow_f B \in q \not\ni R$, we cannot have $A \searrow_f B \subseteq R$. Thus (iv) fails to hold. ∎

In connection with the Harrington–Soare dynamic characterization of small subsets, even though their proof used neither dual spaces nor models of arithmetic, the authors developed the following intuition (Harrington & Soare [9, p.106]):

> ... *the intuition is that $A \subset_s C$ guarantees among other things that the A boundary is far below the C boundary.*

(Here the term 'boundary' is understood in an intuitive rather than a precisely defined sense.) This should be compared to clause (ii) of Theorem 3.25 and clauses (iii) and (iv) of Proposition 1.31, although we disagree with Harrington & Soare on vertical orientation.

3.G. The model theory of prompt simplicity

3.26. Definition. Let $\mathcal{C}$ be a class of r.e. sets. An (effective) *numbering* $(U_i)_{i\in\omega}$ of $\mathcal{C}$ is an onto mapping $i \mapsto U_i : \omega \to \mathcal{C}$ such that the binary predicate $x \in U_i$ is r.e. — thus $(U_i)_{i\in\omega}$ is a uniformly r.e. sequence of r.e. sets as in Definition 3.5. A numbering is *acceptable* if any other numbering $(V_i)_{i\in\omega}$ of $\mathcal{C}$ reduces to $(U_i)_{i\in\omega}$, that is, there is a total recursive f such that $V_i = U_{f(i)}$. (See Soare [41, Exercise I.5.9].)

The traditional numbering $(W_i)_{i\in\omega}$ is acceptable. When A is r.e., $(W_i \cap A)_{i\in\omega}$ is an acceptable numbering of all r.e. subsets of A.

3.27. Definition (Maass [26, Definition 3.1]). Suppose all of $B \subseteq P \subset_\infty A$ are r.e. P is *promptly simple in* the interval $[B, A]$ if there are an enumeration $(P_s)_{s\in\omega}$ of P, an acceptable numbering $(U_i)_{i\in\omega}$ of r.e. subsets of A with a matching enumeration, and a total recursive f such that

$$\forall e\,\big(U_e - B \text{ is infinite } \Rightarrow\ (P \searrow_f U_e) - B \neq \varnothing\big).$$

P is a *promptly simple set* if P is promptly simple in $[\varnothing, \omega]$ (Maass [25, Definition 10]).

Just as with promptly simple sets, the definition of sets promptly simple in a given interval is invariant w.r.t. the choice of acceptable numbering and accompanying enumerations, and there always is an infinite supply of prompt witnesses:

3.28. Lemma. *Let $B \subseteq P \subset_\infty A$ be r.e. Let $(V_i)_{i\in\omega}$ be an arbitrary acceptable numbering of r.e. subsets of A equipped with any matching enumeration. Fix any enumeration $(\tilde{P}_s)_{s\in\omega}$ of P.*

Then P is promptly simple in $[B, A]$ if and only if there is a total recursive h such that

$$\forall e\,\big(V_e - B \text{ is infinite } \Rightarrow\ (\tilde{P} \searrow_h V_e) - B \text{ is infinite}\big).$$

Comment. The proofs of Lemma 11 in Maass [25] and Proposition XIII.1.3 and Theorem XIII.1.4 in Soare [41] only need a few cosmetic touches after replacing acceptable numberings of all r.e. sets by those of r.e. subsets of A. The extra requirement that all witnesses lie outside B is harmless.

The argument proceeds by showing the equivalence of both properties to the existence of a total recursive g such that for each $e \in \omega$ one has

$$V_{g(e)} \subseteq V_e, \quad V_{g(e)} - P = V_e - P, \quad \text{and} \quad (V_e - B \text{ is infinite} \Rightarrow V_e - V_{g(e)} - B \text{ is infinite}). \quad \blacksquare$$

Hájek [7, Corollary 6] shows that to any non-standard element c of a countable model M of PA there is a model $K \models$ PA which coincides with M up to c and such that in K there is a PA-proof of $0 = 1$ with gödelnumber $\le 2^{2^{2^c}}$. Solovay [42, Theorem 1.1] refines this by arranging for an even shorter $\mathrm{I}\Sigma_a$-proof of inconsistency while keeping $\mathrm{I}\Sigma_{a-1}$ consistent in K, where $a < c$ is an appropriate non-standard element of M.

Corollary 3.31 below is inspired by Hájek's result (while borrowing heavily from his methods) and answers the question, how is a promptly simple set like an inconsistency statement: Any non-standard element can enter a promptly simple set at total recursively short notice. Theorem 3.30 also shows that one can at the same time keep that element from entering a suitable prescribed r.e. set, as Solovay does for inconsistency statements. Corollary 3.31 constitutes a non-standard characterization of promptly simple sets. Note that there can be no lattice characterization of prompt simplicity, as promptly simple sets are not invariant under automorphisms of $\mathscr{E}^*$ (Maass & al. [27, Corollary 1.13]).

We first quote an auxiliary fact. The language L' is a variant of L where $+$ and $\times$ are ternary predicates. This allows to view initial intervals $[0, c]_M$ of a model M as L' structures.

3.29. Fact. *Let $M, K \models$ PA be countable $L(\vec{c})$ models. Suppose $\vec{c} \le a \in M$ and $\vec{c} \le b \in K$ where both a and b are non-standard. Then the intervals $[0, a]_M$ and $[0, b]_K$ are isomorphic as $L'(\vec{c})$ structures if and only if $\mathrm{Th}_{\Delta_0}(M, \vec{c}, a) = \mathrm{Th}_{\Delta_0}(K, \vec{c}, b)$ and $\mathrm{SSy}\, M = \mathrm{SSy}\, K$.*

Reference. Theorem 15.23(a) plus Exercise 15.3 in Kaye [15]. In the exercise, the extra constants $\vec{c}$ enjoy a free ride. $\blacksquare$

3.30. Theorem. *Let $B \subseteq P \subset_\infty A$ be r.e. Then P is promptly simple in $[B, A]$ if and only if there is a total recursive f with the property that whenever $\boldsymbol{s} \ge \boldsymbol{x}$ is an element of a countable model $M_{\boldsymbol{x}} \models \mathrm{TA} + \boldsymbol{x} \in A_{\boldsymbol{s}}\ \&\ \boldsymbol{x} \notin B$, there exists a (countable) model*

$$(K_{\boldsymbol{x}}, \boldsymbol{s}) \models \mathrm{TA} + \boldsymbol{x} \in P_{f(\boldsymbol{s})}\ \&\ \boldsymbol{x} \in A_{\boldsymbol{s}}\ \&\ \boldsymbol{x} \notin B$$

such that $[0, \boldsymbol{s}]_M \cong [0, \boldsymbol{s}]_K$ as $L'(\boldsymbol{x})$ structures.

Proof. (if) For the distinguished enumeration $(A_s)_{s\in\omega}$ of A we have

$$x \in A_s \iff \exists t\ \gamma(x, s, t)$$

for some Δ_0 relation $\gamma(x, s, t)$. Take a Δ_0 enumeration $(W_{i,s})_{i,s\in\omega}$ of $(W_i)_{i\in\omega}$ (see 3.5). Put

$$x \in U_{i,s} \iff s \ge x\ \&\ x \in W_{i,s}\ \&\ \exists u \le s\, \exists t \le s\ \gamma(x, u, t).$$

Then $(U_{i,s})_{i,s\in\omega}$ is a Δ_0 enumeration for the acceptable numbering $(U_i)_{i\in\omega}$ of r.e. subsets of A, where $U_i = W_i \cap A$. This enumeration satisfies $x \in U_{i,s} \Rightarrow s \ge x$, and $U_{i,s} \subseteq A_s$.

Take an arbitrary countable $M \models$ TA and consider any e with $U_e - B$ infinite, so that there are nonstandard $\boldsymbol{x}, \boldsymbol{s} \in M$ such that $M \models \boldsymbol{x} \in U_{e,\mathrm{at}\,\boldsymbol{s}}\ \&\ \boldsymbol{x} \notin B$.

The assumption provides a model $K_{\boldsymbol{x}}$ with an element $\boldsymbol{s}$ such that $K_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{f(\boldsymbol{s})}$ & $\boldsymbol{x} \notin B$ and $[0,\boldsymbol{s}]_M \cong [0,\boldsymbol{s}]_K$ with the isomorphism preserving $\boldsymbol{x}$. But then $K_{\boldsymbol{x}} \models \boldsymbol{x} \in U_{e,\text{at}\,\boldsymbol{s}}$ because the enumeration $(U_{i,s})_{i,s\in\omega}$ is Δ_0 and $\mathrm{Th}_{\Delta_0}(K_{\boldsymbol{x}},\boldsymbol{s}) = \mathrm{Th}_{\Delta_0}(M_{\boldsymbol{x}},\boldsymbol{s})$ by Fact 3.29. Hence

$$K \models \exists x \notin B\, \exists s\, \big(x \in U_{e,\text{at}\, s} \;\&\; x \in P_{f(s)}\big).$$

K being a model of TA, we conclude that $(P \searrow_f U_e) - B \neq \varnothing$ and hence P is promptly simple in $[B,A]$.

Note that the (if) direction did not rely on $K_{\boldsymbol{x}} \models \boldsymbol{x} \in A_{\boldsymbol{s}}$.

(only if) We employ an individual numbering $(V_d)_{d\in\omega}$ of r.e. subsets of A:

$$x \in V_d \iff \exists t\,\big(t \geq x \;\&\; x \in A_t \;\&\; \delta(x,t)\big),$$

where d is the gödelnumber of the Δ_0 formula $\delta(\cdot,\cdot)$. $V_e = \varnothing$ if e is not a gödelnumber of an eligible formula. The numbering $(V_d)_{d\in\omega}$ is clearly acceptable, and

$$x \in V_{d,s} \iff s \geq d \;\&\; \exists t \leq s\,\big(t \geq x \;\&\; x \in A_t \;\&\; \delta(x,t)\big)$$

is a matching enumeration, for $x \in V_{d,s}$ entails $x, d \leq s$. Hence there is, by the prompt simplicity of P in $[B,A]$ and Lemma 3.28, a total recursive f such that $(P \searrow_f V_e) - B$ is infinite whenever $V_e - B$ is.

Consider any countable model $(M_{\boldsymbol{x}},\boldsymbol{s}) \models \mathrm{TA} + \boldsymbol{s} \geq \boldsymbol{x}$ & $\boldsymbol{x} \in A_{\boldsymbol{s}}$ & $\boldsymbol{x} \notin B$. Suppose the theory

$$T = \mathrm{TA} + \boldsymbol{s} \geq \boldsymbol{x} > \mathbb{N} + \mathrm{Th}_{\Delta_0}(M_{\boldsymbol{x}},\boldsymbol{s}) + \boldsymbol{x} \in P_{f(\boldsymbol{s})} \;\&\; \boldsymbol{x} \in A_{\boldsymbol{s}} \;\&\; \boldsymbol{x} \notin B,$$

$\boldsymbol{s}$ being viewed as a new constant, were inconsistent. By compactness there are then an $n \in \omega$ and a Δ_0 formula $\delta(\cdot,\cdot)$ such that $M_{\boldsymbol{x}} \models \delta(\boldsymbol{x},\boldsymbol{s})$ while in TA and hence in the real world we have

$$\forall x > n\, \forall s \geq x\, \big(\delta(x,s) \;\&\; x \in P_{f(s)} \;\&\; x \in A_s \to x \in B\big), \tag{*}$$

where n is w.l.o.g. larger than the gödelnumber d of $\delta(\cdot,\cdot)$. On the other hand, the set $V_d - B$ is infinite because in the model $(M_{\boldsymbol{x}},\boldsymbol{s}) \models \mathrm{TA}$ it has a non-standard element $\boldsymbol{x}$. Therefore

$$\exists x > n\, \exists s\, \big(x \in V_{d,\text{at}\, s} \;\&\; x \in P_{f(s)} \;\&\; x \notin B\big)$$

by the property we required of f. Observe that, in $\mathbb{N}$, $n < x \in V_{d,\text{at}\,s}$ implies $s \geq x$, $x \in A_s$, and also $\delta(x,s)$, for otherwise we would have $x \in V_{d,s-1}$ in view of $s \geq x > n > d$. Thus we have reached a contradiction with (*) which shows that T is consistent.

The $L(\boldsymbol{x},\boldsymbol{s})$ theory T has the form $\mathrm{TA} + X$ where $X \subseteq \Sigma_2(\boldsymbol{x},\boldsymbol{s})$ and $X \in \mathrm{SSy}\, M$ because $\mathrm{Th}_{\Delta_0}(M_{\boldsymbol{x}},\boldsymbol{s})$ is coded in M (Fact 2.7(b)). Also $\mathrm{TA}|_{\Sigma_n(\boldsymbol{x},\boldsymbol{s})} \in \mathrm{SSy}\, M$ for $M \models \mathrm{TA}$. Therefore by Fact 2.8(b) and Lemma 2.9 there is a countable model $(K_{\boldsymbol{x}},\boldsymbol{s}) \models T$ with $\mathrm{SSy}\, K = \mathrm{SSy}\, M$. In particular, $K_{\boldsymbol{x}} \models \mathrm{TA} + \boldsymbol{x} \in P_{f(\boldsymbol{s})}$ & $\boldsymbol{x} \in A_{\boldsymbol{s}}$ & $\boldsymbol{x} \notin B$ and $\mathrm{Th}_{\Delta_0}(M_{\boldsymbol{x}},\boldsymbol{s}) = \mathrm{Th}_{\Delta_0}(K_{\boldsymbol{x}},\boldsymbol{s})$. By Fact 3.29 the intervals $[0,\boldsymbol{s}]_M$ and $[0,\boldsymbol{s}]_K$ are $L'(\boldsymbol{x})$-isomorphic. ∎

3.31. Corollary. *An r.e. set P is promptly simple if and only if there is a total recursive f such that whenever $\boldsymbol{s} \geq \boldsymbol{x}$ is an element of a countable model $M_{\boldsymbol{x}}$, there exists a (countable) model $(K_{\boldsymbol{x}},\boldsymbol{s}) \models \mathrm{TA} + \boldsymbol{x} \in P_{f(\boldsymbol{s})}$ with $[0,\boldsymbol{s}]_M \cong [0,\boldsymbol{s}]_K$ as $L'(\boldsymbol{x})$ structures.* ∎

Next we draw some conclusions from Theorem 3.30 for the structure of $(\mathscr{E}^*)^\star$. First, a lemma.

3.32. Lemma. *Suppose* $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} \supseteq p$ *where* p *is a prime. If* $s \in M$ *satisfies* $M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{i,s}$ *for all* $W_i \in p$, *then* $M_{\boldsymbol{x}} \models s \gg t$ *whenever* $M_{\boldsymbol{x}} \models \boldsymbol{x} \in Q_{\mathrm{at}\,t}$ *and* $Q \in p$.

Proof. Given $Q \in p$ (together with a fixed enumeration) and any total recursive non-decreasing f, construct the set

$$W_e = \{\, x \mid \exists t\,(x \in Q_t - W_{e,f(t)}) \,\}$$

by the 2nd Recursion Theorem, slowing down the enumeration of Q by at least f — this is a variant of the Slowdown Lemma (see Soare [41, Lemma XIII.1.5]). Then $W_e = Q \in p$, so $M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{e,s}$. If $M_{\boldsymbol{x}} \models \boldsymbol{x} \in Q_{\mathrm{at}\,t}$ then $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin W_{e,f(t)}$, so $f(t) < s$. Thus $t \ll s$. ∎

3.33. Corollary. *Suppose* P *is promptly simple in* $[B, A]$ *and* $p \in A^\star - P^\star$. *Then*

(a) *p has an immediate successor* $q \in P^\star - B^\star$.

(b) *If p is hinged, then there is* $r \in P^\star - B^\star$ *which is a sibling of p, that is,* $\downarrow\{r\} - \{r\} = \downarrow\{p\} - \{p\}$.

Proof. Take a countable $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$.

(a) Spilling the formula $\exists y\, \forall i < n\,(\boldsymbol{x} \in W_i \to \boldsymbol{x} \in W_{i,y})$ over $\mathbb{N}$, we obtain an $s \geq \boldsymbol{x}$ such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_i \to \boldsymbol{x} \in W_{i,s}$ for all $i \in \omega$. Note that by Lemma 3.32, $M_{\boldsymbol{x}} \models s \gg t$ for any t with $M_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{\mathrm{at}\,t}$ for some $X \in p$. Next Theorem 3.30 applies to obtain a total recursive f and a model $K_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{f(s)}\ \&\ \boldsymbol{x} \notin B$ which coincides with $M_{\boldsymbol{x}}$ up to s. Therefore $p \subsetneq \mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}}$, for $p \not\ni P$. Let q be such that $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} \supseteq q \in \min P^\star$ (Lemma 3.14). Note that $p < q \not\ni B$.

Suppose $p < r \leq q$. Let $X \in r - p$ and let t be such that $K_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{\mathrm{at}\,t}$. Then $t \gtrsim s$ because $K_{\boldsymbol{x}}$ coincides with $M_{\boldsymbol{x}}$ up to s and $M_{\boldsymbol{x}} \models \boldsymbol{x} \notin X$. Therefore $K_{\boldsymbol{x}} \models \boldsymbol{x} \in P \searrow_g X$ for some total recursive g. Since $P \searrow_g X$ is half of an r.e. splitting of X (Lemma 3.8), $X \in q$, and $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} \supseteq q$, we must have $P \searrow_g X \in q$. Hence by Lemma 3.10, X is a hinge for q, so $q \in \min X^\star$ by Proposition 3.11. It follows that $q \leq r$ which shows that q is an immediate successor of p.

(b) With the help of Lemma 3.32 we can select a hinge X for p and a Δ_0 enumeration of X such that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{\mathrm{at}\,s}$ for some $s \geq \boldsymbol{x}$ with $M_{\boldsymbol{x}} \models \boldsymbol{x} \in A_s$. By Theorem 3.30 there is $K_{\boldsymbol{x}} \models \boldsymbol{x} \in P_{f(s)}\ \&\ \boldsymbol{x} \notin B$ coinciding with $M_{\boldsymbol{x}}$ up to s, so that $K_{\boldsymbol{x}} \models \boldsymbol{x} \in X_{\mathrm{at}\,s}$. Just as in the proof of (a), let $\mathrm{Th}_{\Sigma_1} K_{\boldsymbol{x}} \supseteq r \in \min P^\star$. We have $r \ni P \notin p$ while $p \ni X - (P \searrow_f X) \notin r$ with $X - (P \searrow_f X)$ being r.e., thus p and r are incomparable.

Suppose $Y \in q < p$. Then $M_{\boldsymbol{x}} \models \boldsymbol{x} \in Y_{\mathrm{at}\,t}$ for some $t \ll s$, for $t \gtrsim s$ would imply $p \in \min Y^\star$. Hence $K_{\boldsymbol{x}} \models \boldsymbol{x} \in Y_{\mathrm{at}\,t}\ \&\ t \ll s$ yielding $Y \in r$. Therefore $q \leq r$.

When $q < r$, one similarly shows $q \leq p$. We conclude that p and r are siblings. ∎

Maass [26, Theorem 3.3] shows that, given any r.e. $B \subseteq A$ with B major in A, there is an r.e. P between A and B which is promptly simple in $[B, A]$. (P has a further useful property.) Together with Corollary 3.33(a), this suggests a degree of ubiquity for $(\mathcal{E}^*)^\star$-primes with immediate successors.

In connection with clause (b) of Corollary 3.33, note that in any Priestly dual only hinged primes can have proper siblings, for any prime filter p is the union of hinged primes contained in p.

4. Index sets of prime filters

Recall that $\underline{p}$ denotes the index set $\{\, i \in \omega \mid W_i \in p \,\}$ of a prime p.

4.1. Lemma. *$\underline{p}$ is Π^0_2-hard w.r.t. m-reductions for any prime p. In particular, $\mathbf{0}'' \leq_{\mathrm{T}} \underline{p}$. In fact, a single m-reduction works for all primes p.*

Proof. We m-reduce the Π^0_2-complete set $\{\, i \mid W_i = \omega \,\}$ to $\underline{p}$. Let g be a total recursive function such that $W_{g(i)} = \{\, x \mid \forall y \leq x \;\; y \in W_i \,\}$. Note that $W_{g(i)}$ is finite unless $W_{g(i)} = W_i = \omega$. Thus

$$W_i = \omega \quad \Longleftrightarrow \quad g(i) \in \underline{p}. \qquad \blacksquare$$

Primes p with $\underline{p} \leq_{\mathrm{T}} \mathbf{0}''$ exist: take, for example, any maximal set M and let p be the heel of M. Then

$$\begin{aligned} W_i \in p \quad &\Longleftrightarrow \quad W_i \cup M =^* \omega \quad \text{and} \\ W_i \notin p \quad &\Longleftrightarrow \quad W_i \subseteq^* M, \end{aligned}$$

both r.h.s. conditions being Σ^0_3. Proposition 1.10 also constructs a minimax prime recursively in $\mathbf{0}''$. Schmerl & Shavrukov [34] show that only minimal primes can have Δ^0_3 index sets.

4.A. An ersatz indicator for TA

We employ a conventional numbering $(\{e\})_{e \in \omega}$ of partial recursive functions. We write $\{e\}(x)\!\downarrow$ or $\{e\}(x)\!\uparrow$ for convergence and divergence respectively. The Δ_1 formula $t : \{e\}(x)\!\downarrow$ expresses that $\{e\}(x)$ produces output in t or fewer steps.

4.2. Construction. Define the Δ_1 formula

$$T_{a,b}(e) \;\equiv\; \forall y \leq a \left(b : \{e\}(y)\!\downarrow\right)$$

together with total recursive functions

$$\begin{aligned} Y_{a,b}(c,d) &= \mu y < d\left[T_{a,b}(y) \;\&\; \neg(d : \{y\}(c)\!\downarrow)\right], \\ B_{a,b}(c,d) &= \mu y < d\left[Y_{a,b}(c,y) > Y_{a,b}(y,d)\right], \quad \text{and} \\ B^*(a,b) &= B_{a,b}(a,b) \end{aligned}$$

with the understanding that $\mu y < d\,[0 = 1] = d$.

The function $Y_{a,b}(c,d)$ above is inspired by and constructed in the image and likeness of indicators (see Kaye [15, Chapter 14]). The next lemma shows that it has some of the nice properties that could be expected from an indicator for TA-submodels of TA-models. Corollary 5.5(a) in Kirby [16] tells us that there can be no Σ_1 indicator for TA-submodels of TA-models, so the use of parameters a and b appears unavoidable.

In contradistinction to glorious exploits of the original indicators, we just use our version as a kind of distance-like function.

4.3. Lemma. *Suppose a and b are elements of a model $M \models \mathrm{TA}$ such that $M \models \mathbb{N} < a \ll b$.*

(a) *If $e \in \omega$, then $M \models T_{a,b}(e)$ iff $\{e\}$ is total.*

(b) *If $n \in \omega$ then $M \models \forall c\, \exists d\;\, Y_{a,b}(c,d) \geq n$.*

(c) *If $c, d \in M$ then $M \models Y_{a,b}(c,d) > \mathbb{N} \leftrightarrow c \ll d$.*

(d) *If* $M \models c \ll d$ *then* $M \models c \ll B_{a,b}(c,d) \ll d$.

(e) $M \models a \ll B^*(a,b) \ll b$.

PROOF. (a) If $\{e\}$ fails to be total then $\{e\}(m)$ diverges for some $m \in \omega$. Hence $M \models \{e\}(m)\uparrow$. As $m \leq a$, $M \models \neg T_{a,b}(e)$ follows.

If $\{e\}$ is total then there is a total recursive g such that for all $m \in \omega$ one has $g(m) : \{e\}(m)\downarrow$. Hence $M \models \forall y\,(g(y) : \{e\}(y)\downarrow)$. Since $a \ll b$, this impies $M \models \forall y \leq a\; g(y) < b$, so $M \models \forall y \leq a\,(b : \{e\}(y)\downarrow)$. Thus $M \models T_{a,b}(e)$.

(b) Put $d = \max\{\, \mu y\,[y : \{e\}(c)\downarrow] \mid e \leq n \;\&\; \{e\} \text{ total}\,\}$.

(c) Suppose $Y_{a,b}(c,d) > \mathbb{N}$, $e \in \omega$, and $\{e\}$ is total. Then in M we have $T_{a,b}(e)$ by clause (a). Therefore $d : \{e\}(c)\downarrow$ because $Y_{a,b}(c,d) > e$. Thus $c \ll d$.

Conversely, let $c \ll d$ hold in M. If $e \in \omega$ is such that $T_{a,b}(e)$ then $\{e\}$ is total. Hence $d : \{e\}(c)\downarrow$. Conclude $Y_{a,b}(c,d) > \mathbb{N}$.

(d) Let $y = B_{a,b}(c,d)$ and $c \ll d$.

If $Y_{a,b}(c,y)$ were standard then so would be $Y_{a,b}(y,d)$, hence $d \lll y \lll c$ by clause (c), which contradicts $c \ll d$. Thus $Y_{a,b}(c,y) > \mathbb{N}$ so that $c \ll y$.

If $d \lll y$ held then this would imply $d \lll y - 1$ while from $c \ll y$ we have $c \ll y - 1$. By clause (c) it follows that $Y_{a,b}(c, y-1) > \mathbb{N} > Y_{a,b}(y-1, d)$ contradicting the minimality of y w.r.t. $Y_{a,b}(c,y) > Y_{a,b}(y,d)$. Therefore $y \ll d$.

(e) follows at once from (d). ∎

4.B. Hinged primes and e/m-cylinders

4.4. DEFINITION. Recall that Y is *e-reducible* to X ($Y \leq_{\mathrm{e}} X$) if there is an r.e. relation $S(n,\alpha)$ between integers and finite sets of integers such that for each $n \in \omega$

$$n \in Y \iff \exists\alpha\,\big(S(n,\alpha) \;\&\; \alpha \subseteq X\big).$$

An *index* of the e-reduction is an r.e. index of the relation $S(n,\alpha)$.

Taking cue from Polyakov & Rozinas [31, § 3], we call a set X an *(e/m-)cylinder* if for each $Y \subseteq \omega$ we have $Y \leq_{\mathrm{e}} X \Rightarrow Y \leq_{\mathrm{m}} X$, where $\leq_{\mathrm{m}}$ is m-reducibility. Equivalently, the m-degree of X is the largest m-degree within the e-degree of X.

When X is a cylinder, an index for the m-reduction can be obtained from the one for the e-reduction effectively, for let $U(y,\alpha)$ be an r.e. relation which is universal in the sense that there is a total recursive g satisfying

$$S_i(x,\alpha) \iff U\big(g(i,x),\alpha\big),$$

where $(S_i)_{i\in\omega}$ is an acceptable numbering of r.e. relations. Let

$$E(X) = \Big\{n \;\Big|\; \exists\alpha\,\big(U(n,\alpha) \;\&\; \alpha \subseteq X\big)\Big\}.$$

Since $E(X) \leq_{\mathrm{e}} X$, some total recursive f is an m-reduction of $E(X)$ to X. Hence $n \mapsto f(g(i,n))$ is an m-reduction of $\{\, n \mid \exists\alpha\,(S_i(n,\alpha) \;\&\; \alpha \subseteq X)\,\}$ to X.

A prime p is *cylindric* if $\underline{p}$ is a cylinder.

4.5. Definition. A minimal prime p is *multi-sky* if there is a total recursive f such that $M_{\boldsymbol{x}} \models \mathbb{N} < f(\boldsymbol{x}) \ll \boldsymbol{x}$ for any model $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$. Equivalently, for each $n \in \omega$ and each total recursive g, the recursive sets $\{ x \in \omega \mid f(x) > n \}$ and $\{ x \mid g(f(x)) < x \}$ belong to p.

A minimal prime p is *single-sky* if it is not multi-sky.

We shall encounter examples of single-sky minimal primes in Corollary 4.22. Multi-sky minimal primes exist as well, and Schmerl & Shavrukov [34] take a closer look at those ultrafilters on $\mathcal{R}^*$ that give rise to multi-sky minimal primes. These ultrafilters are characterized by the presence of infinitely many total recursive skies in corresponding recursive ultrapowers of $\mathbb{N}$ (see Hirschfeld [11, section 2] for the definition of recursive ultrapowers). Equivalently, these ultrapowers code $\mathbf{0}''$. Recursive ultrapowers corresponding to single-sky minimal primes only have a single non-standard total recursive sky.

4.6. Lemma. (a) *If p is a hinged non-minimal prime then p is cylindric.*

Moreover, an index for the m-reduction is obtained effectively from the one for e-reduction and an index for any hinge of p. In other words, there is a total recursive h such that whenever p is some non-minimal prime hinged on W_e and $S_i(\cdot,\cdot)$ e-reduces some set X to $\underline{p}$, one has that $h(i,e)$ is an index for an m-reduction of X to $\underline{p}$.

(b) *Any multi-sky minimal prime is cylindric.*

Proof. Suppose $X \leq_{\mathrm{e}} \underline{p}$, that is, for some r.e. $S(n,\alpha)$

$$n \in X \iff \exists \alpha \big(S(n,\alpha) \ \&\ \alpha \subseteq \underline{p} \big).$$

The expression $k : S(n,\alpha)$ says that (n,α) is enumerated by S in k or fewer steps. It is important that n and α be bounded by a total recursive function of k. We fix a model $M_{\boldsymbol{x}}$ with $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$.

(a) Let P be a hinge for p. Since p is non-minimal, we have $\boldsymbol{x} \ll t$ for any t such that $\boldsymbol{x} \in P_t$ by Corollary 3.22. We claim

$$n \in X \iff \big\{ x \mid \sigma_{P,S}(x,n) \big\} \in p, \quad \text{where}$$

$$\sigma_{P,S}(x,n) \equiv \exists t, y \Big(x \in P_t \ \&\ \exists \alpha, k \leq Y_{x,t}(t,y) \big(k : S(n,\alpha) \ \&\ \forall j \in \alpha \ x \in W_j \big) \Big).$$

(α is $\leq$-compared to a number in terms of α's code.) Note that $\sigma_{P,S}(x,n)$ is Σ_1.

Suppose $n \in X$. Fix some $k \in \omega$ such that $k : S(n,\alpha)$ with $\alpha \subseteq \underline{p}$. In $M_{\boldsymbol{x}}$, pick any t such that $\boldsymbol{x} \in P_t$. With the help of Lemma 4.3(b), select $y \in M$ so that $\alpha, k \leq Y_{\boldsymbol{x},t}(t,y)$. Since $\alpha \subseteq \underline{p}$, we have $\boldsymbol{x} \in W_j$ for each $j \in \alpha$. Thus $M_{\boldsymbol{x}} \models \sigma_{P,S}(\boldsymbol{x},n)$, therefore $\{ x \mid \sigma_{P,S}(x,n) \} \in p$.

Suppose $n \in \omega$ is such that $\{ x \mid \sigma_{P,S}(x,n) \} \in p$ and hence $M_{\boldsymbol{x}} \models \sigma_{P,S}(\boldsymbol{x},n)$. By Lemma 3.13 we may assume $y \lll t$. Hence $Y_{\boldsymbol{x},t}(t,y)$ is standard by Lemma 4.3(c) as are α and k. Therefore $S(n,\alpha)$ is in fact true. Since $M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_j$ for all $j \in \alpha$, we have $\alpha \subseteq \underline{p}$, for $p = \mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}}$. Thus $n \in X$.

(b) Since p is multi-sky, there is a total recursive f such that $M_{\boldsymbol{x}} \models \mathbb{N} < f(\boldsymbol{x}) \ll \boldsymbol{x}$. We show

$$n \in X \iff \big\{ x \mid \tau_{f,S}(x,n) \big\} \in p, \quad \text{where}$$

$$\tau_{f,S}(x,n) \equiv \exists y \, \exists \alpha, k \leq Y_{f(x),x}(x,y) \big(k : S(n,\alpha) \ \&\ \forall j \in \alpha \ x \in W_j \big).$$

Let $n \in X$. Choose $\alpha \subseteq \underline{p}$, $k \in \omega$ and $y \in M$ such that $k : S(n,\alpha)$ and $M_{\boldsymbol{x}} \models \alpha, k \leq Y_{f(\boldsymbol{x}),\boldsymbol{x}}(\boldsymbol{x},y)$. Since $\alpha \subseteq \underline{p}$, one has $M_{\boldsymbol{x}} \models \forall j \in \alpha \ \boldsymbol{x} \in W_j$. Thus $M_{\boldsymbol{x}} \models \tau_{f,S}(\boldsymbol{x},n)$ and $\{ x \mid \tau_{f,S}(x,n) \} \in p$.

Conversely, if there are $y, \alpha, k \in M$ as specified by $\tau_{f,S}(\boldsymbol{x}, n)$, then we may assume $y \lll \boldsymbol{x}$ by Lemma 3.13 and Proposition 3.21, so in view of Lemma 4.3(c) there are standard α and k for which $S(n, \alpha)$ and $\alpha \subseteq \underline{p}$. Hence $n \in X$.

In both (a) and (b), the displayed equivalences are m-reductions of X to $\underline{p}$. The m-reduction in (a) is clearly effective in the r.e. indices of $S(\cdot,\cdot)$ and P. (The m-reductions in (b) are effective in indices of f and $S(\cdot,\cdot)$, but we won't need that.) ∎

4.7. Corollary. *The index sets of minimal multi-sky primes and of non-minimal hinged primes are Σ^0_3-hard.*

Proof. Let $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = p$ with the prime p as in the statement. Let $\mathrm{Cof} = \{\, e \in \omega \mid W_e \text{ is cofinite} \,\}$. According to Corollary IV.3.5 in Soare [41], Cof is Σ^0_3-complete. Let $g(e, n)$ be the total recursive function defined by

$$\{g(e,n)\}(k) = \begin{cases} 0 & \text{if } k \leq n \\ \mu t\,[k \in W_{e,t}] & \text{if } k > n. \end{cases}$$

If p is minimal and multi-sky, let the total recursive f witness that fact. If p is non-minimal and hinged, let P be a hinge for p. In view of Lemma 4.6 it suffices to e-reduce Cof to $\underline{p}$:

$$e \in \mathrm{Cof} \iff \exists n\, \forall k > n\ k \in W_e \iff \exists n \in \omega\, \big(\{g(e,n)\} \text{ is total}\big)$$

$$\iff \begin{cases} \exists n \in \omega\ M_{\boldsymbol{x}} \models T_{f(\boldsymbol{x}),\boldsymbol{x}}(g(e,n)) & \text{if } p \text{ is minimal and multi-sky} \\ \exists n \in \omega\ M_{\boldsymbol{x}} \models \exists s\,(\boldsymbol{x} \in P_s\ \&\ T_{\boldsymbol{x},s}(g(e,n))) & \text{if } p \text{ is non-minimal and hinged} \end{cases}$$

$$\iff \begin{cases} \exists n \in \omega\ \{\, x \mid T_{f(x),x}(g(e,n)) \,\} \in p & \text{if } p \text{ is minimal and multi-sky} \\ \exists n \in \omega\ \{\, x \mid \exists s\,(x \in P_s\ \&\ T_{x,s}(g(e,n))) \,\} \in p & \text{if } p \text{ is non-minimal and hinged.} \end{cases}$$

Since the r.e. indices of the two last-mentioned sets are effective in e and n, we have an e-reduction of Cof to $\underline{p}$ in either case. ∎

Heels of r-maximal sets lie outside the scope of Corollary 4.7. Lempp & al. [20] discover an r-maximal set whose heel has a Σ^0_3-complete index set.

4.C. Jump-the-Gap Lemma

The following key lemma was motivated by sheer analogy with the case of the E-tree (Shavrukov & Solovay [36]). Its proof inherits the 'r.e. sets as Σ_1 formulas' approach already present in the proof of Lemma 4.6.

$\leq_{\mathrm{T}}$ is Turing reducibility and $'$ is the Turing jump. $\lessdot$ stands for the relation of immediate succession.

4.8. Jump-the-Gap Lemma. *Let p and q be hinged primes.*

(a) *If $p \leq q$, then $\underline{p} \leq_{\mathrm{m}} \underline{q}$, hence $\underline{p} \leq_{\mathrm{T}} \underline{q}$. When q is non-minimal, the index of the reduction is effective in the indices of any given hinges for p and q.*

(b) *If $p \lessdot q$, then $\underline{p}' \leq_{\mathrm{T}} \underline{q}$. The index of the reduction is effective in those of any hinges for p and q.*

Proof. Let p and q hinge on P and Q respectively, and let $\mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}} = q$.

(a) If q is minimal then $p = q$, so the identity reduction works. Assume that q is non-minimal. Suppose that $a, b \in M$ are such that $\boldsymbol{x} \in P_{\mathrm{at}\, a}$ and $\boldsymbol{x} \in Q_b$. Then $\boldsymbol{x} \ll b$ by Proposition 3.21 because q is non-minimal. To show:

$$W_i \in p \iff \exists n \in \omega \left\{ x \mid \rho_{P,Q}(x,i,n) \right\} \in q, \quad \text{where}$$

$$\rho_{P,Q}(x,i,n) \equiv \exists a,b,c \left(x \in P_{\mathrm{at}\, a} \;\&\; x \in Q_b \;\&\; x \in W_{i,c} \;\&\; Y_{x,b}(a,c) \leq n \right),$$

which is an e-reduction of $\underline{p}$ to $\underline{q}$ effective in the indices of P and Q.

When $W_i \in p$ and $\boldsymbol{x} \in P_{\mathrm{at}\, a}$, by Proposition 3.11 there is $c \lessapprox a$ in $M_{\boldsymbol{x}}$ with $\boldsymbol{x} \in W_{i,c}$. Let $b \in M$ be any number satisfying $x \in Q_b$. By Lemma 4.3(c), $Y_{\boldsymbol{x},b}(a,c)$ is standard. Thus for some standard n the model $M_{\boldsymbol{x}}$ satisfies the Σ_1 formula $\rho_{P,Q}(\boldsymbol{x},i,n)$ which puts $\{ x \mid \rho_{P,Q}(x,i,n) \}$ into q.

Suppose there is $n \in \omega$ such that $\{ x \mid \rho_{P,Q}(x,i,n) \} \in q$ and therefore $M_{\boldsymbol{x}} \models \rho_{P,Q}(\boldsymbol{x},i,n)$. Then $Y_{\boldsymbol{x},b}(a,c)$ is standard. Therefore $c \lessapprox a$ by Lemma 4.3(c). Since $\boldsymbol{x} \in W_{i,c}$, we have $W_i \in p$ by Proposition 3.11.

Finally, Lemma 4.6(a) turns the e-reduction into an m-reduction (effectively in the index of Q).

(b) In $M_{\boldsymbol{x}}$, let $a \ll b$ be such that $\boldsymbol{x} \in P_{\mathrm{at}\, a}$ and $\boldsymbol{x} \in Q_{\mathrm{at}\, b}$. Further, let $c = B^*(a,b)$. Lemma 4.3(e) ensures $a \ll c \ll b$.

We claim that, back in the real world, $p = \{ W_i \mid M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{i,c} \}$: Since $c \ll b$, there is by Wilkie's Lemma an initial TA-segment I of M such that $\boldsymbol{x} \leq c \in I < b$. In particular, $I_{\boldsymbol{x}} \models \boldsymbol{x} \notin Q$. We have

$$\begin{aligned} p &\subseteq \{ W_i \mid M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{i,c} \} && \text{(by Proposition 3.11 since } a \ll c) \\ &\subseteq \{ W_i \mid I_{\boldsymbol{x}} \models \boldsymbol{x} \in W_i \} && (\text{as } c \in I \subseteq_{\mathrm{e}} M) \\ &\subsetneq q && (\text{as } I_{\boldsymbol{x}} \models \boldsymbol{x} \notin Q). \end{aligned}$$

From $p \lessdot q$ it follows that $\mathrm{Th}_{\Sigma_1} I_{\boldsymbol{x}} = p$. Therefore $p = \{ W_i \mid M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{i,c} \}$.

The representation of the Turing jump that we use is $X' = \{ e \mid \{e\}^X\!\downarrow \}$ where $\{e\}^X$ is the eth input-free computational device with the oracle X plugged in — this helps reduce notation a little. We are now sufficiently prepared to establish

$$e \in \underline{p}' \iff \{e\}^{\underline{p}}\!\downarrow \iff \exists n \in \omega \left\{ x \mid \xi_{P,Q}(x,e,n) \right\} \in q, \quad \text{where}$$

$$\xi_{P,Q}(x,e,n) \equiv \exists a,b,c \left(x \in P_{\mathrm{at}\, a} \;\&\; x \in Q_{\mathrm{at}\, b} \;\&\; c = B^*(a,b) \;\&\; n : \{e\}^{\{ i \mid x \in W_{i,c} \}}\!\downarrow \right).$$

If for some standard n one has $\{ x \mid \xi_{P,Q}(x,e,n) \} \in q$ then $M_{\boldsymbol{x}} \models \xi_{P,Q}(\boldsymbol{x},e,n)$. The numbers a, b and c are uniquely determined. The computation $\{e\}^{\{ i \mid \boldsymbol{x} \in W_{i,c} \}}$ converges in standardly many steps, so the oracle is only queried on membership of standard numbers. For these queries, the answers of the oracle $\{ i \mid \boldsymbol{x} \in W_{i,c} \}$ in $M_{\boldsymbol{x}}$ are the same as those given by $\underline{p}$ in the real world. Thus $\{e\}^{\underline{p}}\!\downarrow$.

Conversely, if $n : \{e\}^{\underline{p}}\!\downarrow$ for some $n \in \omega$, then $M_{\boldsymbol{x}} \models n : \{e\}^{\{ i \mid \boldsymbol{x} \in W_{i,c} \}}\!\downarrow$ by the same argument. Thus $M_{\boldsymbol{x}} \models \xi_{P,Q}(\boldsymbol{x},e,n)$ and therefore $\{ x \mid \xi_{P,Q}(x,e,n) \} \in q$.

Since the formula $\xi_{P,Q}(x,e,n)$ is Σ_1, the displayed equivalence, now verified, is an e-reduction of $\underline{p}'$ to $\underline{q}$ effective in the indices of P and Q. To complete the proof, invoke Lemma 4.6(a). ∎

4.9. Exercise. *Suppose the prime p is hinged, r is arbitrary, and $p \leq r$. Then $\underline{p} \leq_{\mathrm{e}} \underline{r}$.* ∎

The next proposition should demonstrate the strength of Jump-the-Gap Lemma when the latter is combined with the fact of non-existence of certain uniformly defined sequences of Turing degrees.

4.10. Definition (Steel [43]). A sequence $(S_n)_{n\in\omega}$ of subsets of ω is a *Steel sequence* if for some arithmetical $A(X,Y)$ we have

$$\forall n \in \omega\, \forall Y \subseteq \omega\, \big(A(S_n, Y) \leftrightarrow Y = S_{n+1}\big) \quad \text{and} \quad \forall n \in \omega\ S'_{n+1} \leq_{\mathrm{T}} S_n.$$

4.11. Fact (Steel [43]). *No Steel sequences exist.* ∎

4.12. Proposition. *No interval in $(\mathcal{E}^*)^\star$ is order-isomorphic to ω^* (inverted ω).*

Proof. Suppose $(p_i)_{i\in\omega}$ were a sequence of primes with $p_{i+1} \lessdot p_i$, all i. Each p_i must be hinged because it has an immediate predecessor — see Lemma 3.15(b). We show that $(\underline{p}_i)_{i\in\omega}$ is a Steel sequence thus outruling the existence of $(p_i)_{i\in\omega}$ in view of Fact 4.11.

Indeed, we have $\underline{p}'_{i+1} \leq_{\mathrm{T}} \underline{p}_i$ by Jump-the-Gap Lemma (b), and

$$Y = \underline{p}_{i+1} \iff Y \text{ is a prime filter of } \mathcal{E}^* \ \&\ Y \subsetneq \underline{p}_i$$
$$\&\ \forall j \in \underline{p}_i\, (j \in Y \text{ or } W_j \text{ is a hinge for } p_i), \quad \text{where}$$
$$W_j \text{ is a hinge for } p_i \iff j \in \underline{p}_i \ \&\ \forall k \in \underline{p}_i\, \exists e\, \big(\{e\} \text{ is total } \&\ W_k \setminus_{\{e\}} W_j \in p_i\big).$$ ∎

4.D. Jump-the-gap ladders and branches in $(\mathcal{E}^*)^\star$

A more systematic exploitation of Jump-the-Gap Lemma calls for the following

4.13. Definition (Shavrukov & Solovay [36]). A *jump-the-gap ladder* is a tuple $(J, \preceq, (C_i)_{i\in J})$, where $\varnothing \neq J \subseteq \omega$, $\preceq$ is a linear order on J, and $C_i \subseteq \omega$ are such that

(J1) for all $i \in J$, $\{\, j \in J \mid j \preceq i \,\} \leq_{\mathrm{T}} C'_i$ uniformly in i (i.e., there is a total recursive e such that $\{e(i)\}^{C'_i}$ decides $\{\, j \in J \mid j \preceq i \,\}$ for each $i \in J$);

(J2) for all $i, j \in J$, if $i \preceq j$ then $C_i \leq_{\mathrm{T}} C_j$ uniformly in (i, j);

(J3) for all $i, j \in J$, if $i \prec\!\cdot\, j$ then $C'_i \leq_{\mathrm{T}} C_j$.

It follows from (J1) and (J2) that $\preceq$ restricted to $\{\, j \in J \mid j \preceq i \,\}$ is uniformly recursive in C'_i.

A linear ordering $(X, \leq)$ *supports a jump-the-gap ladder* if there exists a jump-the-gap ladder $(J, \preceq, (C_i)_{i\in J})$ with $(J, \preceq) \cong (X, \leq)$.

Shavrukov & Solovay [36] investigate the order types of jump-the-gap ladders through the connection of the latter with jump pseudo-hierarchies.

4.14. Exercise. *Suppose $(J, \preceq, (C_i)_{i\in J})$ is a jump-the-gap ladder.*

(a) *If $D_i = C'_i$ for all $i \in J$ then $(J, \preceq, (D_i)_{i\in J})$ is also a jump-the-gap ladder satisfying*

(J1$^+$) for all $i \in J$, $\{\, j \in J \mid j \preceq i \,\} \leq_{\mathrm{T}} D_i$ uniformly in i.

(b) *If $E_i = C''_i$ for all $i \in J$ then $(J, \preceq, (E_i)_{i\in J})$ is a jump-the-gap ladder satisfying* (J1$^+$) *and*

(J3$^+$) for all $i, j \in J$, if $i \prec\!\cdot\, j$ then $E'_i \leq_{\mathrm{T}} E_j$ uniformly in (i, j). ∎

4.15. Definition. The *hinged skeleton* π^{G} of a closed subset $\pi \subseteq (\mathcal{E}^*)^\star$, in particular of a branch through $(\mathcal{E}^*)^\star$, is the collection of $(\mathcal{E}^*)^\star$-hinged primes in π. The hinged skeleta of linearly ordered subsets of $(\mathcal{E}^*)^\star$, in particular of branches, are countable because distinct hinged primes in a chain require distinct hinges.

A branch $\mathbf{b}$ through $(\mathscr{E}^*)^\star$, being its closed subset and a maximal chain, is, by Fact 1.15 and Exercise 1.33, the Priestley dual of a finest chain quotient Q of $\mathscr{E}^*$. According to Proposition 3.17, $\mathbf{b}^{\mathrm{G}}$ is order-anti-isomorphic to the lattice order on $Q - \{0\}$. Thus $\mathbf{b}$ together with its $(\mathscr{E}^*)^\star$-subspace topology can be restored from the ordering of its hinged skeleton $\mathbf{b}^{\mathrm{G}}$: Intervals of the form $[p, q)$ with $p, q \in \mathbf{b}^{\mathrm{G}}$ comprise a base for the Priestley subspace topology on $\mathbf{b}$.

Therefore to understand the structure of branches through $(\mathscr{E}^*)^\star$ (or of finest chain quotients of $\mathscr{E}^*$) it is in principle sufficient to understand the ordertypes of hinged skeleta of these branches.

4.16. Proposition. *The hinged skeleton of any branch* $\mathbf{b}$ *through* $(\mathscr{E}^*)^\star$ *supports a jump-the-gap ladder.*

Proof. Let $\mathbf{b}$ be given. For a hinged prime p on $\mathbf{b}$, let $i(p)$ be the minimal i such that W_i is a hinge for p. Note that i is injective, for no two distinct primes on a same branch can share the same hinge. Define

$$J = \{\, i(p) \mid p \in \mathbf{b}^{\mathrm{G}} \,\},$$

$$i(p) \preceq i(q) \iff p \le q,$$

$$C_{i(p)} = \underline{p}.$$

We are going to show that $(J, \preceq, (C_i)_{i \in J})$ is a jump-the-gap ladder by verifying (J1–3) of Definition 4.13.

(J1) $j = i(q)$ for some $q \le p$

$$\iff \quad j \in \underline{p} \;\&\; \forall k\, \forall e \big(\{e\} \text{ is total } \;\&\; W_k \setminus_{\{e\}} W_j \in p \;\&\; W_k \setminus_{\{e\}} W_j \in p \longrightarrow k \ge j\big),$$

which, thanks to Lemma 4.1, gives a uniform reduction of $\{\, j \in J \mid j \preceq i(p) \,\}$ to $\underline{p}' = C'_{i(p)}$.

(J2) To reduce $C_{i(p)} = \underline{p}$ to $C_{i(q)} = \underline{q}$, use the identity reduction if $i(p) = i(q)$. In the opposite case, use the reduction from Jump-the-Gap Lemma (a) with $W_{i(p)}$ and $W_{i(q)}$ in the roles of hinges.

(J3) See Jump-the-Gap Lemma (b). ∎

The hinged skeleta of all branches through the E-tree also support jump-the-gap ladders. This eventually leads Shavrukov & Solovay [36] to a characterization of the order types of E-branches: a countable linear order type $\boldsymbol{H}$ is that of the hinged skeleton of some branch through the E-tree if and only if $\boldsymbol{H}$ supports a jump-the-gap ladder, has a least element, and non-trivial densely ordered convex segments occur cofinally often in $\boldsymbol{H}$.

4.E. Germs at hinged primes

4.17. Definition. Let p be a non-maximal prime. Consider an equivalence relation on branches through $(\mathscr{E}^*)^\star$ containing p:

$$\mathbf{b}_0 \simeq_p \mathbf{b}_1 \iff \text{there is a prime } q \in \mathbf{b}_0 \cap \mathbf{b}_1 \text{ with } q > p.$$

Equivalence classes of $\simeq_p$ are called *germs at p*.

A germ $\mathscr{g}$ at p is a *successor* germ if p has an immediate successor on any or all branches from $\mathscr{g}$. $\mathscr{g}$ is *dense* if each branch from $\mathscr{g}$ contains a prime $q > p$ such that $[p, q]^{\mathrm{G}}$ is non-trivial and densely ordered (for hinged p, we may clearly omit non-triviality).

4.18. Proposition. *Let p be a non-maximal hinged prime. Then each germ at p is either a successor or a dense germ.*

Proof. In view of Proposition 4.16, it suffices to show that in any jump-the-gap ladder $(J, \preceq, (C_i)_{i\in J})$ any non-maximal point i either has an immediate successor or is the lower end-point of a non-trivial densely ordered convex segment. The only alternative to both scenarios could consist in $\preceq$-gaps descending onto i: for each $j \succ i$ there are k and ℓ with $i \prec k \prec\!\!\cdot\ \ell \preceq j$. We argue by reductio ad Steel sequence:

Let $j_0 \in J$ be an arbitrary point with $j_0 \succ i$. Put $S_0 = (j_0, C_{j_0})$.

Given $S_n = (j_n, C_{j_n})$ with $j_n \succ i$, let $j_{n+1} \in J$ be the least (in the conventional ordering of ω) such that

$$i \prec j_{n+1} \prec j_n \ \ \&\ \ j_{n+1} \text{ is the lower endpoint of a } \preceq\text{-gap.}$$

By our descending-gaps assumption and since $j_n \succ i$, such a j_{n+1} exists. Put $S_{n+1} = (j_{n+1}, C_{j_{n+1}})$.

We claim that $(S_n)_{n\in\omega}$ is a Steel sequence:

By property (J1) of $(J, \preceq, (C_i)_{i\in J})$, the point j_{n+1} is uniformly arithmetical in S_n. By (J2) so is $C_{j_{n+1}}$ and hence also S_{n+1}. In particular, S_{n+1} is a uniformly arithmetical in S_n singleton as required in Definition 4.10.

By (J3) and (J2) we have $C'_{j_{n+1}} \leq_T C_{j_n}$ because of the $\preceq$-gap that abuts on j_{n+1}. Hence

$$S'_{n+1} \equiv_T C'_{j_{n+1}} \leq_T C_{j_n} \equiv_T S_n,$$

which confirms that $(S_n)_{n\in\omega}$ is a Steel sequence. ∎

4.F. Dense germs

With Corollary 1.23 we have seen that any prime in the shadow of a hhsimple set only has successor germs. The next proposition shows that this behaviour is somewhat atypical for hinged primes.

4.19. Proposition. *Suppose Q is r.e. and p is a cylindric prime in the shadow of Q. Then there exists a prime $q > p$ hinged on Q such that $\underline{q} \leq_T \underline{p}$. If, on top of that, p is hinged, then $[p, q]^{G}$ is densely ordered.*

4.20. Construction. We are going to compile, recursively in $\underline{p}$, two non-decreasing sequences $(\alpha_n, \chi_n)_{n\in\omega}$ of finite sets of indices of r.e. sets together with a growing collection $(b_i)_{i\in\alpha_n}$ of (indices of) total recursive functions. Our intention is that $\underline{q} = \bigcup_{n\in\omega} \alpha_n$ be the index set of the required prime and $\bigcup_{n\in\omega} \chi_n$ complement $\underline{q}$ so that $\underline{q} \leq_T \underline{p}$. The functions $\{b_i\}$ will help to keep q hinged on Q.

We consruct α_n, χ_n, and b_n aiming to preserve the consistency of the $\mathcal{L}(\boldsymbol{x})$ theories

$$T_n = \mathrm{TA} + \{\, \boldsymbol{x} \in W_k \,\}_{k\in\underline{p}} + \boldsymbol{x} \in Q + \big\{\, \boldsymbol{x} \in W_i \searrow_{\{b_i\}} Q \,\big\}_{i\in\alpha_n} + \{\, \boldsymbol{x} \notin W_j \,\}_{j\in\chi_n}$$

which will be shown later.

Put $\alpha_0 = \chi_0 = \varnothing$.

Consider the total recursive function h which takes as arguments two natural numbers, two finite sets α and χ of (indices of) r.e. sets, and a recursive function index c_i for each $i \in \alpha$:

$$\begin{aligned}&\{h(k, \alpha, (c_i)_{i\in\alpha}, \chi, m)\}(t)\\&\quad = \mu y \Big[\forall z \Big(z \in W_{k,t} \ \&\ z \in Q_{\mathrm{at}\,t} \ \&\ \mathop{\&}_{i\in\alpha} z \in W_{i,\{c_i\}(t)} \longrightarrow z \in W_{m,y} \vee \bigvee_{j\in\chi} z \in W_{j,y}\Big)\Big].\end{aligned}$$

If any of $\{c_i\}(t)$ with $i \in \alpha$ fails to converge, then so does $\{h(k, \alpha, (c_i)_{i\in\alpha}, \chi, m)\}(t)$. Note that in view of Definition 3.5, the quantifier $\forall z$ is bounded by a total recursive function of t.

With α_n, χ_n and $(b_i)_{i\in\alpha_n}$ already in place, we decide whether W_n goes into α_{n+1} or into χ_{n+1} as follows: One can determine effectively in $\underline{p}$ whether

$$\exists k \in \underline{p} \left(\{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\} \text{ is total}\right),$$

for this question e-reduces via Lemma 4.1 to $\underline{p}$ and, since $\underline{p}$ is a cylinder, is also m-reducible to $\underline{p}$, recalling that for a fixed p, an index for the m-reduction is found effectively in the one for the e-reduction.

If such k exists, we find a witnessing instance effectively in $\underline{p}$ by exhaustive search — using Lemma 4.1 — and put

$$\alpha_{n+1} = \alpha_n \cup \{n\}, \quad b_n = h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n), \quad \text{and} \quad \chi_{n+1} = \chi_n.$$

If no $k \in \underline{p}$ such that $\{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\}$ is total exists, we let

$$\alpha_{n+1} = \alpha_n \quad \text{and} \quad \chi_{n+1} = \chi_n \cup \{n\}.$$

4.21. Proof of Proposition 4.19. We first note that in any model $M_{\boldsymbol{x}}$ of any T_n we have $\boldsymbol{x} > \mathbb{N}$ because $\boldsymbol{x} \in \omega - m \in p$ for each $m \in \omega$, so $\boldsymbol{x} > m$.

Let us show by induction on n that T_n is consistent. The consistency of T_0 follows from the assumptions of 4.19, for p extends to a larger prime that contains Q. Suppose T_n is consistent and consider T_{n+1}. There are two cases:

Case 1: $\exists k \in \underline{p}\,(\{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\}$ is total).

In this case $T_{n+1} = T_n + \boldsymbol{x} \in W_n \setminus_{\{b_n\}} Q$. Let $M_{\boldsymbol{x}} \models T_n$. In $M_{\boldsymbol{x}}$, let t be such that $\boldsymbol{x} \in Q_{\mathrm{at}\,t}$. Observe that $M_{\boldsymbol{x}} \models \boldsymbol{x} \in W_{i,\{b_i\}(t)}$ for all $i \in \alpha_n$ as $T_n \vdash \boldsymbol{x} \in W_i \setminus_{\{b_i\}} Q$. Since $Q \notin p$ and $W_k \in p \subseteq \mathrm{Th}_{\Sigma_1} M_{\boldsymbol{x}}$, it follows by (i)$\Rightarrow$(ii) of Proposition 3.11 that $\boldsymbol{x} \in W_{k,t}$. For $y = \{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\}(t) = \{b_n\}(t)$ we have $\boldsymbol{x} \in W_{n,y}$ by the construction of h because $\boldsymbol{x} \notin W_j$ for all $j \in \chi_n$. Hence $\boldsymbol{x} \in W_n \setminus_{\{b_n\}} Q$. Thus T_{n+1} is consistent, for $M_{\boldsymbol{x}} \models T_{n+1}$.

Case 2: $\forall k \in \underline{p}\,(\{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\}$ is not total).

Now $T_{n+1} = T_n + \boldsymbol{x} \notin W_n$. Suppose T_{n+1} were inconsistent. Then by compactness there exists $k \in \underline{p}$ — a single k suffices since p is closed under finite intersections — such that the following is a consequence of TA and hence true:

$$\forall x \Big(x \in W_k \ \&\ x \in Q \ \&\ \mathop{\&}_{i\in\alpha_n} x \in W_i \setminus_{\{b_i\}} Q \longrightarrow x \in W_n \vee \bigvee_{j\in\chi_n} x \in W_j\Big).$$

Since all $\{b_i\}$ with $i \in \alpha_n$ are total and x is bounded by a total recursive function of t for all $x \in W_{k,t}$, it follows that $\{h(k, \alpha_n, (b_i)_{i\in\alpha_n}, \chi_n, n)\}$ must be total as well which contradicts the assumption of Case 2. Hence T_{n+1} is consistent.

We conclude that $T = \bigcup_{n\in\omega} T_n$ is consistent. We have $T|_{\Sigma_1(\boldsymbol{x})} = q$ where $\underline{q} = \bigcup_{n\in\omega} \alpha_n$, and $q > p$ as $p \not\ni Q \in q$ and $k \in \alpha_{k+1} \subseteq \underline{q}$ for each $W_k \in p$, any function of the form $\{h(k,\dots,k)\}$ being total. The set Q is a hinge for q because $T \vdash \boldsymbol{x} \in W_i \setminus_{\{b_i\}} Q$ with $\{b_i\}$ total whenever $W_i \in q$, hence $W_i \setminus_{\{b_i\}} Q \in q$.

Finally, if p is hinged, then $[p,q]^{\mathrm{G}}$ is densely ordered because for hinged primes r and s, a gap of the form $p \le r \lessdot s \le q$ would by Jump-the-Gap Lemma imply $\underline{p}' \le_{\mathrm{T}} \underline{q}$, contradicting $\underline{q} \le_{\mathrm{T}} \underline{p}$. ■

A similar construction also makes sense in Σ_1/T (Shavrukov & Solovay [36]). There, instead of totality of recursive functions, one uses provability in T to distinguish between cases.

4.22. Corollary. *All (minimal) primes found in the shadow of any D-hhsimple set are single-sky.*

Proof. Let D be D-hhsimple. Any prime in the shadow of D is hinged because, by Proposition 1.26, it is minimal. If any such prime p was multi-sky then it would also be cylindric by Lemma 4.6(b). Hence by Proposition 4.19 there is a prime $q \in \min D^\star$ with $[p,q]^{\mathrm{G}}$ densely ordered. Therefore p fails to have successors on branches passing through (both p and) q. But this contradicts Corollary 1.27. (Alternatively, one argues that since $q \in \min D^\star$, the interval $[p,q)$ is an infinite chain within the shadow of D, which is impossible because that shadow is an antichain.) ■

In particular, all primes in the shadow of a hhsimple set are also single-sky. Schmerl & Shavrukov [34] find out that the class of single-sky minimal primes includes all minimax primes as well as all heels of r-maximal sets.

4.G. Complexity of immediate successors

Corollary 3.33(a) pointed out a large number of primes in $(\mathcal{E}^*)^\star$ with successor germs. The next proposition constructs immediate successors with Turing complexity of index sets that, for hinged primes, does not exceed that necessitated by Jump-the-Gap Lemma (b). This constitutes evidence that that lemma is reasonably optimal.

4.23. Proposition. *Let $B \subseteq Q \subset_\infty A$ be r.e. with Q promptly simple in $[B,A]$. Suppose p is a prime such that $Q \notin p \ni A$. Then there is a prime q hinged on Q with $p \lessdot q \not\ni B$ and $\underline{q} \le_{\mathrm{T}} \underline{p}'$.*

4.24. Construction. This follows the general pattern of 4.20. Thus we shall also build two non-decreasing sequences $(\alpha_n, \chi_n)_{n\in\omega}$ of finite sets of (indices of) r.e. sets and collections $(b_i)_{i\in\alpha_n}$ of total recursive indices — this time, recursively in $\underline{p}'$. As before, $\bigcup_{n\in\omega} \chi_n$ complements $\underline{q} = \bigcup_{n\in\omega} \alpha_n$. The theories we would like to keep consistent are

$$T_n = \mathrm{TA} + \{\, \boldsymbol{x} \in W_{e,\boldsymbol{s}} \,\}_{e\in\underline{p}} + \{\, \boldsymbol{x} \notin W_{d,\boldsymbol{s}} \,\}_{d\notin\underline{p}} + \boldsymbol{x} \in Q_{f(\boldsymbol{s})} + \left\{\, \boldsymbol{x} \in W_{i,\{b_i\}(\boldsymbol{s})} \,\right\}_{i\in\alpha_n} + \{\, \boldsymbol{x} \notin W_j \,\}_{j\in\chi_n} + \boldsymbol{x} \notin B,$$

where f is the total recursive function associated by Theorem 3.30 to Q's being promptly simple in $[B,A]$, and $\boldsymbol{s}$ is a new constant. Here we also assume that the distinguished enumeration $(W_{e,t})_{e,t\in\omega}$ is Δ_0 (see Definition 3.5).

Put $\alpha_0 = \chi_0 = \varnothing$.

Compared to 4.20, the function h has two finite-set arguments κ and λ instead of a single integer k:

$$\begin{aligned}&\{h(\kappa,\lambda,\alpha,(c_i)_{i\in\alpha},\chi,m)\}(s)\\ &\quad= \mu y\Big[\forall z\,\Big(\mathop{\&}_{k\in\kappa} z\in W_{k,s}\ \&\ \mathop{\&}_{\ell\in\lambda} z\notin W_{\ell,s}\ \&\ z\in Q_{f(s)}\ \&\ \mathop{\&}_{i\in\alpha} z\in W_{i,\{c_i\}(s)}\\ &\qquad\qquad \longrightarrow z\in W_{m,y}\vee\bigvee_{j\in\chi} z\in W_{j,y}\vee z\in B_y\Big)\Big].\end{aligned}$$

If any $\{c_i\}(s)$ with $i\in\alpha$ diverges, then $\{h(\kappa,\lambda,\alpha,(c_i)_{i\in\alpha},\chi,m)\}(s)$ is left undefined.

At the $(n+1)$st stage we find out effectively in $\underline{p}'$ whether

$$\exists\kappa\subseteq\underline{p}\ \exists\lambda\subseteq\omega-\underline{p}\ \big(\{h(\kappa,\lambda,\alpha_n,(b_i)_{i\in\alpha_n},\chi_n,n)\}\text{ is total}\big).$$

If κ and λ exist, find such a pair (effectively in $\underline{p}$ by Lemma 4.1), and put

$$\alpha_{n+1}=\alpha_n\cup\{n\},\quad b_n=h(\kappa,\lambda,\alpha_n,(b_i)_{i\in\alpha_n},\chi_n,n),\quad\text{and}\quad \chi_{n+1}=\chi_n.$$

Otherwise, $\alpha_{n+1}=\alpha_n$ and $\chi_{n+1}=\chi_n\cup\{n\}$.

4.25. Proof of Proposition 4.23. As in 4.21, $\boldsymbol{x}$ is non-standard in any model of any T_n.

To show T_0 consistent, take any countable model $M_{\boldsymbol{x}}\models\mathrm{TA}$ with $\mathrm{Th}_{\Sigma_1}M_{\boldsymbol{x}}=p$. Just as in the proof of Corollary 3.33(a), we may select an $\boldsymbol{s}\geq\boldsymbol{x}$ such that $M_{\boldsymbol{x}}\models\boldsymbol{x}\in W_e\to\boldsymbol{x}\in W_{e,\boldsymbol{s}}$ for each $e\in\omega$. Since $A\in p$, we have $M_{\boldsymbol{x}}\models\boldsymbol{x}\in A_{\boldsymbol{s}}$. Theorem 3.30 then constructs a model $(K_{\boldsymbol{x}},\boldsymbol{s})\models\mathrm{TA}+\boldsymbol{x}\in Q_{f(\boldsymbol{s})}+\boldsymbol{x}\notin B$. Since we have assumed the enumeration $(W_{e,t})_{e,t\in\omega}$ to be Δ_0 and the $L'(\boldsymbol{x})$-structures $[0,\boldsymbol{s}]_K$ and $[0,\boldsymbol{s}]_M$ are isomorphic by Theorem 3.30, formulas of the form $\boldsymbol{x}\in W_{i,\boldsymbol{s}}$ are absolute between $(K_{\boldsymbol{x}},\boldsymbol{s})$ and $(M_{\boldsymbol{x}},\boldsymbol{s})$ when $i\in\omega$. Thus $(K_{\boldsymbol{x}},\boldsymbol{s})\models T_0$.

Assuming T_n consistent, consider T_{n+1}:

If there are $\kappa\subseteq\underline{p}$ and $\lambda\subseteq\omega-\underline{p}$ such that $\{b_n\}=\{h(\kappa,\lambda,\alpha_n,(b_i)_{i\in\alpha_n},\chi_n,n)\}$ is total, then $T_n\vdash\boldsymbol{x}\in W_{n,\{b_n\}(\boldsymbol{s})}$ by the construction of h, so that T_{n+1} is entailed by T_n.

If κ and λ as above fail to exist and $T_{n+1}=T_n+\boldsymbol{x}\notin W_n$ were inconsistent, compactness provides finite sets $\kappa'\subseteq\underline{p}$ and $\lambda'\subseteq\omega-\underline{p}$ such that

$$\begin{aligned}&\forall x,s\,\Big(\mathop{\&}_{k\in\kappa'} x\in W_{k,s}\ \&\ \mathop{\&}_{\ell\in\lambda'} x\notin W_{\ell,s}\ \&\ x\in Q_{f(s)}\ \&\ \mathop{\&}_{i\in\alpha_n} x\in W_{i,\{b_i\}(s)}\\ &\qquad\qquad\longrightarrow x\in W_n\vee\bigvee_{j\in\chi_n} x\in W_j\vee x\in B\Big).\end{aligned}$$

Which brings about a contradiction, for $\{h(\kappa',\lambda',\alpha_n,(b_i)_{i\in\alpha_n},\chi_n,n)\}$ is then total.

It follows that $T=\bigcup_{n\in\omega}T_n$ is consistent. Hence $q=T|_{\Sigma_1(\boldsymbol{x})}$ is a prime, where $\underline{q}=\bigcup_{n\in\omega}\alpha_n$. We clearly have $p<q\in Q^\star-B^\star$. In any model $(N_{\boldsymbol{x}},\boldsymbol{s})\models T$, if $\boldsymbol{x}\in Q_{\mathrm{at}\,t}$ then $t\leq f(\boldsymbol{s})$ and $t\gg\boldsymbol{s}$ because $Q\notin p=\{\,W_i\mid(N_{\boldsymbol{x}},\boldsymbol{s})\models\boldsymbol{x}\in W_{i,\boldsymbol{s}}\,\}$. Since for each $W_i\in q$ we have $(N_{\boldsymbol{x}},\boldsymbol{s})\models\boldsymbol{x}\in W_{i,\{b_i\}(\boldsymbol{s})}$ where $\{b_i\}$ is total, we also have $(N_{\boldsymbol{x}},\boldsymbol{s})\models\boldsymbol{x}\in W_i\searrow_{g_i}Q$ for some total recursive g_i. Thus $W_i\searrow_{g_i}Q\in\mathrm{Th}_{\Sigma_1}N_{\boldsymbol{x}}=q$, which shows that q hinges on Q.

Suppose $p\leq r<q$. Then by Proposition 2.13 there are models $I_{\boldsymbol{x}}\subseteq_{\mathrm{e}}N_{\boldsymbol{x}}$ such that $(N_{\boldsymbol{x}},\boldsymbol{s})\models T$ for some $\boldsymbol{s}\in N$, and $\mathrm{Th}_{\Sigma_1}I_{\boldsymbol{x}}=r$. Note that $\boldsymbol{s}\in I$ cannot happen, for then $I_{\boldsymbol{x}}\models\boldsymbol{x}\in Q$, while Q, being a hinge for q, cannot belong to $\mathrm{Th}_{\Sigma_1}I_{\boldsymbol{x}}=r<q$. Therefore $\mathrm{Th}_{\Sigma_1}I_{\boldsymbol{x}}\subseteq\{\,W_i\mid(N_{\boldsymbol{x}},\boldsymbol{s})\models\boldsymbol{x}\in W_{i,\boldsymbol{s}}\,\}=p$. Thus $r=p$, establishing that $p\lessdot q$. ∎

5. Questions

In this section we present a small selection of open questions about the structure of $(\mathcal{E}^*)^\star$. Unfortunately, I cannot guarantee that all of these questions are difficult.

5.A. Automorphisms of $\mathcal{E}^*$

Every automorphism φ of $\mathcal{E}^*$ gives rise to an order-autohomeomorphism $\varphi^\star$ of $(\mathcal{E}^*)^\star$:

$$\varphi^\star(p) = \{\,\text{r.e. } X \mid \varphi(X) \in p\,\}.$$

Shore [37, Lemma 4] shows that each automorphism of $\mathcal{E}^*$ is uniquely determined by its action on $\mathcal{R}^*$, the sublattice of recursive sets. It follows that each automorphism is also determined by the action of its dual $\varphi^\star$ on $\min(\mathcal{E}^*)^\star$, as well as by the action on $\max(\mathcal{E}^*)^\star$. This suggests that the main direction of the action of $\varphi^\star$ is horizontal.

5.1. Question. Is there an automorphism φ such that $\varphi^\star(p) > p$ for some prime p ?

We have seen in Proposition 4.19 and Corollary 4.22 that the multi-/single-sky distinction among minimal primes affects their position in $(\mathcal{E}^*)^\star$.

5.2. Question. Is the collection of single-sky minimal primes invariant under automorphisms?

5.B. Minimal primes

For minimal primes, we have in section 4 established some implications between the multi-sky property, cylindricity, and existence of dense germs. It is natural to ask if any of these can be reversed:

5.3. Questions. (a) Can a single-sky minimal prime be cylindric? In particular, can the heel of an r-maximal set or a minimax prime be cylindric?

(b) Can a non-maximal acylindric minimal prime have a dense germ? In particular, can a prime in the shadow of a D-hhsimple set have a dense germ?

(c) Suppose a minimal non-maximal prime only has successor germs. Must it lie outside some hhsimple set?

Known non-maximal examples of single-sky minimal primes come from D-hhsimple and r-maximal sets. We know that in the shadow of an appropriate hhsimple set one already finds $2^{\aleph_0}$ many single-sky primes, whereas all r-maximal sets can only account for countably many heels. What we do not know is whether the heels of r-maximal sets contribute anything new compared to the shadows of D-hhsimple sets. Accordingly, we ask:

5.4. Question. Is there an r-maximal set whose heel does not lie in the shadow of any D-hhsimple set?

5.C. Order types of branches

We conjecture that the minimax primes are responsible for the only difference between the collections of order types of branches through the E-tree (Shavrukov & Solovay [36]) and those of branches through $(\mathcal{E}^*)^\star$. Recall that it is sufficient to describe the hinged skeleta of such branches.

5.5. Conjecture. The order types of hinged skeleta of branches through $(\mathcal{E}^*)^\star$ are singleton order types (given by minimax primes) together with all countable linear ordertypes that have a least point, have non-trivial densely ordered subsegments occur cofinally often, and support a jump-the-gap ladder.

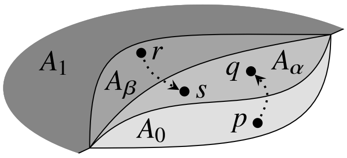
A positive answer to some variant of the next question could likely be a step towards verification of the conjecture, taking care of densely ordered subsegments.

5.6. Question. Let A_1 be a major subset of A_0. Does there exist a uniformly r.e. family $(A_\alpha)_{\alpha\in\mathbb{Q}\cap(0,1)}$ such that for all $\alpha < \beta$ where $\alpha, \beta \in \mathbb{Q} \cap (0, 1)$ we have

(i) $A_1 \subset_\infty A_\beta \subset_\infty A_\alpha \subset_\infty A_0$;

(ii) for each prime $p \in A_0^\star - A_\alpha^\star$ there is a $q > p$ such that $q \in A_\alpha^\star - A_\beta^\star$;

(iii) for each prime $r \in A_\beta^\star - A_1^\star$ there is an $s < r$ such that $s \in A_\alpha^\star - A_\beta^\star$?



This really is a question about $\mathcal{M}^*$, for its answer does not depend on the particular position within $\mathcal{E}^*$ of the individual instance of the major inerval.

Starting with an r.e. non-recursive A_0 and letting A_{i+1} be a small major subset of A_i, we obtain an ω-sequence with properties similar to (i)–(iii), but it is not sufficiently dense. Theorem 1.4.6 in Stob [44] offers a mechanism for subdivision, but clause (iii) then becomes problematic.

In Σ_1/T, the analogue of an individual r.e. set A_α satisfying (i)–(iii) is a Σ_1 sentence which is doubly conservative in the enveloping interval (see Lindström [22, Theorem 5.3(a)]). Dense effective sequences of Σ_1 sentences with properties (i)–(iii) are produced in Shavrukov & Solovay [36] and Lindström [23]. Rather than use subdivision, however, both constructions proceed as a single operation. In Σ_1/T, the role of the major interval $[A_1, A_0]$ can be played by any non-trivial interval.

Another ingredient to a hypothetical positive answer to Conjecture 5.5 are successor germs. While we may infer from Corollary 3.33(a) and results of Maass [26] that the π-closure of the set of primes that have successor germs includes at least all non-minimal primes, we still want more.

5.7. Question. Does every hinged non-maximal prime sprout a successor germ? Is there a workable class of r.e. sets that satisfy Corollary 3.33(a) which is larger than the class of sets promptly simple in a given interval?

5.D. RK

In conclusion, we would just like to spell out the definition of a semi-effective miniaturization of the Rudin–Keisler pre-ordering for elements of $(\mathscr{E}^*)^\star$:

5.8. Definition. $p \leq_{\mathrm{rk}} q$ if there is a partial recursive function f with $q \ni \operatorname{dom} f$ such that for each r.e. X one has $X \in p \Leftrightarrow f^{-1}[X] \in q$.

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