Graphs, Curves and Dynamics

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Graphs, Curves and Dynamics

Grafen, Krommen en Dynamica

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Proefschrift

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Voor de bomen van deze stad

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Chapter 1

Introduction

The title of this thesis is Graphs, Curves and Dynamics, and it is based on three publications. The first paper [28] touches on all three elements of the title, whereas the second paper [27] discusses graphs and curves. To make up for this appearent injustice, the third paper [49] is on dynamics only. In this introduction we discuss the results of these papers.

1.1 Measure-theoretic rigidity for Mumford curves

The celebrated measure-theoretic rigidity for hyperbolic Riemann surfaces specializes to the following statement (Mostow [62], Kuusalo [50]): if X_1 and X_2 are compact hyperbolic Riemann surfaces, written as quotients of the Poincaré disk Δ by Fuchsian groups Γ_1 and Γ_2 of the first kind, then X_1 and X_2 are isomorphic (viz., Γ_1 and Γ_2 are conjugate in $\mathrm{PSL}(2, \mathbf{R})$) if and only if the boundary map φ associated to Γ_1 and Γ_2 is absolutely continuous with respect to Lebesgue measure. Here, φ is a homeomorphism

$$\varphi\colon \partial\Delta = S^1 \to \partial\Delta = S^1$$

that is equivariant w.r.t. a chosen *group* isomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$ in the sense that

$$\varphi(\gamma_1 \cdot x) = \alpha(\gamma_1)\varphi(x)$$

for any $x \in S^1 \cong \mathbf{P}^1(\mathbf{R})$ (on which $\mathrm{PSL}(2, \mathbf{R})$ acts naturally by fractional linear transformations), and for any $\gamma_1 \in \Gamma_1$. For higher-dimensional hyperbolic manifolds, the absolute continuity is automatic, giving the celebrated rigidity theorem of Mostow.

There exist versions of this result for more general Fuchsian groups, i.e., Schottky groups, where, instead of Lebesgue measure, a more general Patterson–Sullivan measure on the boundary (=limit set) is used [65] [76]. There are also analogues for more general subgroups of isometry groups of hyperbolic space that, for example, apply to graphs, cf. Bowen [13], Coornaert [22], Hersonsky and Paulin [43], Tukia [81], Yue [87].

As is well-known, there is a p-adic, or more general non-Archimedean, analogue of uniformization of Riemann surfaces by Schottky groups, namely, Mumford's theory of uniformization by non-Archimedean Schottky groups (cf. for example, Mumford [63], [42]).

In exact terms, Mumford's uniformization theory by non-Archimedean Schottky groups comprises the following. A projective curve X of genus $g \geq 2$ over a complete algebraically closed non-Archimedean valued field K can be considered as a one-dimensional compact rigid analytic space (in the sense of Tate [78]), and this analytic space is a quotient $\Gamma\backslash(\mathbf{P}^1(K) - \Lambda_\Gamma)$ for some discrete subgroup $\Gamma \subseteq \mathrm{PGL}(2, k)$ with limit set $\Lambda_\Gamma \subseteq \mathbf{P}^1(k)$, cf. Gerritzen and Van der Put [42] prop 1.6.4 precisely if the special fiber of a mimimal model X over the ring of integers of k is totally split (i.e., a union of rational curves over the algebraic closure of the residue field). Here $k \subset K$ is a discrete valued complete non-Archimedean subfield.

To emphasize the analogy, let us briefly recall classical Schottky uniformization. A Riemann surface X can be described as a quotient $\Gamma\backslash(\mathbf{P}^1(\mathbf{C}) - \Lambda_\Gamma)$, where
$\Gamma \subseteq \mathrm{PGL}(2, \mathbf{C}) = \mathrm{Aut}(\mathbf{P}^1(\mathbf{C}))$ is a free discrete group of rank g (=genus of X) with limit set $\Lambda_\Gamma \subset \mathbf{P}^1(\mathbf{C})$. Manin and others have argued that the classical complex analogue of the special fibre should be the set of closed geodesics in the solid handlebody $\Gamma\backslash\mathbf{H}^3$ given by the quotient of real hyperbolic 3-space $\mathbf{H}^3$ by $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbf{C}) = \mathrm{Aut}(\mathbf{H}^3)$ (compare [55], [56]). Table 1.1 gives a tabular representation of the analogy.

	Complex case	Non-Archimedean case
Ground field	$\mathbf{C}$	k complete discrete non-Archimedean
		$k \subseteq K$ complete algebraically closed
Free group of rank g	F_g	F_g
Schottky group	$F_g \cong \Gamma \hookrightarrow \mathrm{PGL}(2, \mathbf{C})$	$F_g \cong \Gamma \hookrightarrow \mathrm{PGL}(2, k)$
Limit set	$\Lambda_\Gamma \subseteq \mathbf{P}^1(\mathbf{C})$	$\Lambda_\Gamma \subseteq \mathbf{P}^1(k) \subseteq \mathbf{P}^1(K)$
Curve	$X(\mathbf{C}) \cong \Gamma\backslash(\mathbf{P}^1(\mathbf{C}) - \Lambda_\Gamma)$	$X(K) \cong \Gamma\backslash(\mathbf{P}^1(K) - \Lambda_\Gamma)$
Hyperbolic space	$\mathbf{H}^3$	$\mathfrak{T}$ (Bruhat-Tits tree)
Boundary	$\partial\mathbf{H}^3 = \mathbf{P}^1(\mathbf{C})$	$\partial\mathfrak{T} = \mathbf{P}^1(k)$
Special fiber	$\{$closed geodesics in $\Gamma\backslash\mathbf{H}^3\}$	$\mathcal{X} \otimes \overline{k}$
		($\mathcal{X}$ = model of X over integers of k)
Subspace	$\mathbf{H}^3(\Lambda_\Gamma) =$	$\mathfrak{T}(\Lambda_\Gamma) =$
	$\{$geodesics ending on $\Lambda_\Gamma\}$	subtree spanned by limit set
Dual of special fiber	$\Gamma\backslash\mathbf{H}^3(\Lambda_\Gamma)$	$\Gamma\backslash\mathfrak{T}(\Lambda_\Gamma)$

Table 1.1: Analogy between classical and non-Archimedean Schottky uniformization

In chapter 2, we consider the analog of measure-theoretic rigidity for the case of Mumford curves: can one describe isomorphism of Mumford curves by measure-theoretic properties of an associated boundary map?

Let us first explain why an obvious analog of the complex analytic statement is false. A non-Archimedean Schottky group $\Gamma \subseteq \mathrm{PGL}(2,k)$ is also a group of automorphisms of the tree $T(\Lambda_\Gamma)$ spanned by the limit set as a subtree of the Bruhat-Tits tree over the residue field of k. This limit set indeed carries a Patterson-Sullivan measure of dimension $e(\Gamma)$, a constant depending on the group Γ, (see section 2.2), but absolute continuity of a boundary map for two such groups will only imply conjugacy of Schottky groups in the isometry group of the tree, *not* inside $\mathrm{PGL}(2,k)$. Now $\mathrm{Aut}(T(\Lambda_\Gamma))$ is much larger than its "linear cousin" $\mathrm{PGL}(2,k)$, and conjugacy of Schottky groups inside $\mathrm{Aut}(T(\Lambda_\Gamma))$ is, as we will see, only equivalent to an isomorphism of the *special fibers* of the two curves. There are many non-isomorphic curves with isomorphic special fibers—actually, for any given curve, uncountably many. Another manifestation of this fact is the statement that the group Γ has uncountably many conjugacy classes inside $\mathrm{PGL}(2,k)$, but only countably many inside $\mathrm{Isom}(T(\Lambda_\Gamma))$ (this observation was communicated to us by Lubotzky, compare [54]).

To remedy this problem, consider, in addition to Patterson–Sullivan measure, the so-called *harmonic measures of weight* ℓ on the boundary, in the sense of Schneider and Teitelbaum ([72], [79]). For $\ell = 2$ these are simply called harmonic measures, and denote the g-dimensional vector space of such by $\mathrm{C}_{\mathrm{har}}(\Gamma, 2)$. For $\ell \geq 3$, the harmonic measures of weight ℓ for Γ span a $(\ell-1)(g-1)$-dimensional vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, \ell)$. Denote by Ω_{X_Γ} the canonical bundle over the curve X_Γ. Teitelbaum's Poisson kernel theorem gives, for every even ℓ, explicit back and forth isomorphisms $(\mathrm{Poisson}(\ell), \mathrm{Res}(\ell))$ between the space of weight-ℓ harmonic measures and the space of holomorphic $\ell/2$-differential forms on the corresponding Mumford curve X_Γ (a.k.a. the space of weight ℓ modular forms for Γ):

$$\begin{tikzcd}
\mathrm{C}_{\mathrm{har}}(\Gamma, \ell) \arrow[rr, bend left, "\mathrm{Poisson}(\ell)"] & & H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes \ell/2}) \arrow[ll, bend left, "\mathrm{Res}(\ell)"]
\end{tikzcd}$$

Thus, we can define a *product* of harmonic measures: let $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_N)$ be a vector of strictly positive even integers, and define the product map $m_{\boldsymbol{\ell}}$ by the commutativity of the following diagram

$$\begin{tikzcd} \bigotimes_{i=1}^{N} \mathrm{C}_{\mathrm{har}}(\Gamma, \ell_i) \arrow[r, "m_{\boldsymbol{\ell}}"'] \arrow[d, "\bigotimes \mathrm{Poisson}(\ell_i)"'] & \mathrm{C}_{\mathrm{har}}(\Gamma, \sum_{i=1}^{N} \ell_i) \\ \bigotimes_{i=1}^{N} H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes \ell_i/2}) \arrow[r] & H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes \sum \ell_i/2}) \arrow[u, "\mathrm{Res}(\sum \ell_i)"'] \end{tikzcd} \tag{1.1}$$

in which the bottom arrow is just the usual product of differential forms. We use the shorthand notation $R_2(\Gamma) := \ker(m_{(2,2)})$, $R_3(\Gamma) := \ker(m_{(2,2,2)})$ and $R_4 := \ker(m_{(2,2,2,2)})$. These kernels describe the linear relations between products of degree two, three and four of harmonic measures.

Now our result is the following:

Theorem A. *Let k be a complete discrete valued non-Archimedean field, and let K denote a complete and algebraically closed field extension of k. Let $\Gamma_1, \Gamma_2 \leq \mathrm{PGL}(2,k)$ be Schottky groups of the same genus $g \geq 2$, defining Mumford curves X_1 and X_2, respectively over K. Let φ denote the boundary map associated to an isomorphism of Γ_1 with Γ_2.*

(i) *The intersection dual graphs of the special fibers of the minimal models of X_1 and X_2 are isomorphic if and only if there is an isometry*

$$F : T(\Lambda_{\Gamma_1}) \to T(\Lambda_{\Gamma_2})$$

which conjugates the Schottky groups Γ_i, if and only if the corresponding Patterson-Sullivan measures are of the same conformal dimension and the map φ is absolutely continuous for these measures.
If this is the case, then pullback by F^{-1} induces an isomorphism of the spaces of harmonic measures

$$F_*\colon\ \mathrm{C}_{\mathrm{har}}(\Gamma_1, 2) \xrightarrow{\sim} \mathrm{C}_{\mathrm{har}}(\Gamma_2, 2).$$

(ii) *Suppose that the curves X_i are not hyperelliptic. Then the curves X_1 and X_2 are isomorphic over K if and only if the corresponding Schottky groups Γ_i are conjugate in $\mathrm{PGL}(2,k)$, if and only if the map φ is absolutely continuous with respect to Patterson–Sullivan measure* and *F_* respects the linear relations of degree ≤ 4 between the harmonic boundary measures in the sense that*

$$F_*R_2(\Gamma_1) \subseteq R_2(\Gamma_2), F_*R_3(\Gamma_1) \subseteq R_3(\Gamma_2) \textit{ and } F_*R_4(\Gamma_1) \subseteq R_4(\Gamma_2).$$

The relations in R_4 are only necessary if $g = 3$. The relations in R_3 are only necessary if X is trigonal (i.e., admits a map $X \to \mathbf{P}^1$ of degree 3), or if X is isomorphic to a plane quintic.

The theorem says that when the usual measure-theoretic property of absolute continuity with respect to Patterson–Sullivan measure is enhanced to include preservation of linear relations in three finite-dimensional vector spaces of harmonic measures, then rigidity of Mumford curves follows. Note that Mumford proved in ([63], Corollary 4.11) that two Mumford curves with Schottky groups in $\mathrm{PGL}(2, k)$ are isomorphic over K if and only if the corresponding Schottky groups are conjugate over k.

We do not know at present how to deal with the hyperelliptic case.

The proof of the theorem uses the theorem of Petri et al. on the equations for the image of the canonical embedding of a non-hyperelliptic curve. The Poisson kernel theorem of Teitelbaum is used to translate this algebro-geometric statement back into a statement about harmonic measures.

1.2 A combinatorial Li–Yau inequality and rational points on curves

The gonality $\mathrm{gon}_K(X)$ of a projective curve X over a field K is the minimal degree of all rational maps $X \to \mathbf{P}^1(K)$. Abramovich [2] has proven a lower bound on the gonality of modular curves for congruence groups that is linear in the genus of the curves (or, what is the same, linear in the index of the group in the full modular group). The proof is an interesting combination of a lower bound on the gonality in terms of hyperbolic volume and the first eigenvalue of the Laplacian, established by Li and Yau [51], together with a lower bound on this eigenvalue in the arithmetic case, of which a sharp value is given by Selberg's $1/4$-conjecture; the current record seems to be at $975/4096 \approx 0.238037$, due to Kim and Sarnak [47]. In this chapter 3, we study a non-Archimedean analogue of the result, or rather, of the *method* used in [2]. The main intermediate results, i.e., a lower bound on the gonality of a curve in terms of gonality of intersection dual graphs of suitable semi-stable models and an analogue of the Li–Yau result for graphs, seem to be of independent interest.

The first result is an inequality between the (geometric) gonality $\mathrm{gon}_{\overline{k}}(X)$ of a curve X defined over a non-Archimedean valued field k (the field of fractions of an excellent discrete valuation ring R) and the "gonality" of the reduction graphs of suitable models of the curve. There are various complications, such as to establish a good theory for the reduction of a covering map

$X \to \mathbf{P}^1$. Such a map extends to the stable model, but not necessarily as a *finite* morphism. This can be remedied by choosing suitable semi-stable models. The problem was studied by Liu and Lorenzini [53], Coleman [21] and Liu [52], and more recently in [4]. In Section 3.1, we provide another (similar) solution, directly adapted to the applications that we have in mind.

Next, we relate the gonality of the special fiber to what we call the *stable gonality* of the intersection dual graph. For standard graph terminology, we refer to Section 3.2. We also need the notion of an (indexed) finite harmonic graph morphism, for which we refer to Definition 3.2.6. Given a graph G, then another graph G' is called a *refinement* of G if it can be obtained from G by performing subsequently finitely many times one of the two following operations: (a) subdivision of an edge; (b) addition of a *leaf*, i.e., the addition of an extra vertex and an edge between this vertex and a vertex of the already existing graph. The *stable gonality* of G, denoted $\mathrm{sgon}(G)$, is defined as the minimal degree of a finite harmonic morphism from any refinement of G to a tree. This relates to, but is different from previous notions of gonality for graphs as introduced by Baker, Norine, [9] and Caporaso [16].

Theorem B. *Let X be a geometrically connected projective smooth curve over k, and $\mathscr{X}$ the stable R-model of X. Let $\overline{k}$ be an algebraic closure of k. Let $\Delta(\mathscr{X}_0)$ denote the intersection dual graph of the special fiber $\mathscr{X}_0$. Then we have*

$$\mathrm{gon}_{\overline{k}}(X) \geq \mathrm{sgon}(\Delta(\mathscr{X}_0)).$$

Two examples (3.2.9 and 3.2.10) illustrate that both refinement operation are necessary. First, the "banana graph" B_n given by two vertices joined by $n > 1$ distinct edges has stable gonality 2, although the minimal degree of a finite harmonic graph morphism from B_n itself (without any refinement) to a tree is n. Secondly, the minimal degree of a finite harmonic graph morphism from any subdivision of the complete graph K_4 to a tree is 4. However, by adding leaves, the stable gonality can be shown to be 3.

We then prove an analogue for graphs of the upper bound on gonality from Brill-Noether theory for the gonality of curves over arbitrary fields (in this generality a theorem of Kleiman–Laksov [48]):

Theorem C. *For any graph G with first Betti number $g \geq 2$, we have an upper bound*

$$\mathrm{sgon}(G) \leq \lfloor \frac{g+3}{2} \rfloor.$$

For this, we check that stable gonality of graphs with first Betti number ≥ 2 is defined on equivalence classes given by refinements, and use the previous theorem.

The second main result is a spectral lower bound for the stable gonality of a graph. Let λ_G denote the first non-trivial eigenvalue of the Laplacian L_G of G, and let

$$\Delta_G := \max\{\deg(v) \colon v \in V(G)\}$$

denote the maximal vertex degree of G. Finally, let $|G|$ denote the number of vertices of G. Then we have

Theorem D. *The stable gonality of a graph G satisfies*

$$\operatorname{sgon}(G) \geq \left\lceil \frac{\lambda_G}{\lambda_G + 4(\Delta_G + 1)} |G| \right\rceil .$$

An attractive feature of the formula is that the lower bound depends on spectral data for the original graph, not on all possible refinements of the graph. Also, since stable gonality is defined up to equivalence, one may put on the right hand side of this bound the data corresponding to any graph in the same equivalence class as G.

A similar result can be proven using the normalized graph Laplacian, replacing $|G|$ by the "volume" of the graph, cf. Theorem 3.7.7.

The result can be seen as an analogue of the Li–Yau inequality in differential geometry [51], which states that the gonality $\operatorname{gon}(X)$ of a compact Riemann surface X (minimal degree of a conformal mapping from X to the Riemann sphere) is bounded below by

$$\operatorname{gon}(X) \geq \frac{1}{8\pi} \lambda_X \operatorname{vol}(X),$$

where λ_X is the first non-trivial eigenvalue of the Laplace-Beltrami operator of X, and $\operatorname{vol}(X)$ denotes the volume of X. In Remark 3.7.4, we will show that the strict graph theory analog of such a formula fails.

We then apply the two theorems above to Drinfeld modular curves over a general global function field K over a finite field with q elements, and we find the positive characteristic analogue of Abramovich's result. In the applications, we will write $|\,\mathfrak{n}\,|_\infty$ for the valuation corresponding to a fixed "infinite" place ∞ of degree δ of K, we denote by A the subring of K of elements that are regular outside ∞, and we let Y denote a rank-two A-lattice in the completion K_∞ of K at ∞. Up to equivalence, such lattice correspond to elements of $\operatorname{Pic}(A)$. Let H denote the maximal abelian extension of K inside $k = K_\infty$; then $\operatorname{Gal}(H/K) \cong \operatorname{Pic}(A)$. In the "standard" example where $K = \mathbf{F}_q(T)$ is the function field of $\mathbf{P}^1$ and $\infty = T^{-1}$, $Y = A \oplus A$ is unique up to equivalence, and $H = K$. Congruence subgroups Γ of $\Gamma(Y) := \operatorname{GL}(Y)$ (i.e., containing $\ker(\Gamma(Y) \to \mathrm{GL}(Y/\mathfrak{n}Y))$ for some non-trivial ideal $\mathfrak{n}$ of A) act by fractional linear transformations on the Drinfeld "upper half plane" Ω, and the quotient analytic space can be compactified into a smooth projective curve X_Γ by adding finitely many cusps.

Theorem E. *Let Γ denote a congruence subgroup of $\Gamma(Y)$. Then the gonality of the corresponding Drinfeld modular curve X_Γ satisfies*

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{q,\delta} \cdot [\Gamma(Y) : \Gamma]$$

where the constant $c_{q,\delta}$ is

$$c_{q,\delta} := \frac{q^\delta - 2\sqrt{q^\delta}}{5q^\delta - 2\sqrt{q^\delta} + 8} \cdot \frac{1}{q(q^2-1)}$$

This implies a linear lower bound in the genus of modular curves of the form

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c'_{K,\delta} \cdot (g(X_\Gamma) - 1),$$

where $c'_{K,\delta}$ is a bound that depends only on the function field K and the degree δ of ∞. If K is a rational function field and $\delta = 1$, then we can put $c'_{K,\delta} = 2c_{q,1}$.

In the proof, we use the structure of the reduction graph of the principal modular curve of level $\mathfrak{n}$ (or rather, its components $X(Y, \mathfrak{n})$ indexed by Y running through classes in $\mathrm{Pic}(A)$). Also used in the proof is a bound of the Laplace eigenvalue for this graph that follows from the Ramanujan conjecture, proven by Drinfeld (in combination with the Courant-Weyl inequalities). The proof of the genus bound is not entirely automatic, due to possible wild ramification. The constant $c_{q,\delta}$ is probably not optimal, and it would be interesting to know whether it can be replaced by an absolute constant, or at least a constant depending on q, but tending to an absolute non-zero constant as q increases. Also notice that the bound is vacuous (since a negative number) if $q^\delta < 4$, and that the general upper bound $(g+3)/2$ implies that any suitable constant $c_{K,\delta}$ should be smaller than $5/2$.

All previously known results on gonality of Drinfeld modular curves used point counting arguments modulo primes, rather than the above "geometric analysis" method. The best previously known bounds, due to Andreas Schweizer ([74], Thm. 2.4) are not linear in the index and are established for a rational function field $K = \mathbf{F}_q(T)$ only. An extra asset of the new method is that it works without much extra effort for a general function field K, rather than just a rational function field.

We give applications to rational points of bounded degree on various modular curves.

Theorem F. *With the same notations as in Theorem E, if X_Γ is defined over a finite extension K_Γ of K, then the set*

$$\bigcup_{[L:K_\Gamma]\leq\frac{1}{2}\left(c_{q,\delta}\cdot[\Gamma(1):\Gamma]-1\right)} X_\Gamma(L)$$

is finite.

As a final remark, there has recently been a surge in the use of gonality and graph theory in arithmetic, but mainly in characteristic zero; for example in the work of Ellenberg, Hall and Kowalski on generic large Galois image, coupling gonality to expander properties of Cayley graphs embedded in Riemann surfaces [34]. Also in our applications, in a rather different way, the graph expansion properties of the reduction graphs of Drinfeld modular curves seem to intervene in a crucial way in establishing interesting lower bound for their gonality (originally, over rational function fields, we deduced the bounds from natural bounded concentrator properties of subgraphs, as in the work of Morgenstern [61]). More generally, the stable gonality in a family of Ramanujan graphs with fixed regularity is bounded below linearly in the number of vertices (cf. Remark 3.7.3).

1.3 Dynamics measured in a non-Archimedean field

The study of dynamical systems using measure-theoretic methods has shown to be extremely useful. In almost all results real or complex valued measures are used. However, there exists theories in which measures take values in other fields, in particular non-Archimedean fields. In a series of papers [58] Monna started the study of non-Archimedean functional analysis. Integration theory using measures for non-Archimedean valued functions on locally compact topological zero dimensional spaces was developed by Monna and Springer [59] [60], and later generalized to all zero-dimensional topological spaces by Van Rooij and Schikhof [85]. A nice overview of non-Archimedean functional analysis, including measure theory, can be found in [84].

Recently, non-Archimedean analysis, and in particular measure theory, found several applications to theoretical physics [44], [46],[45]. In chapter 4 we elaborate on the natural translation of several notions, e.g., measure spaces probability measures, isomorphic transformations, entropy, from classical dynamical systems to a non-Archimedean setting.

Let us explain why it is expected that these notions behave differently. For a discrete complete non-Archimedean field K, we cannot expect that a K-valued Borel measure is σ-additive. On the contrary, any σ-additive K-valued function on the Borel algebra is trivial, i.e., such a function is a, possibly infinite, sum of Dirac measures ([84], lemma 4.19). To overcome this problem, instead of σ-algebras, separating covering rings are used. These rings form a basis for a zero-dimensional Hausdorff topological space.

A measure $\mu : \mathcal{R} \to K$ is then a additive map satisfying some boundedness and continuity condition (see 4.1.1 for the exact definition). It comes with a real valued function,

$$\| \cdot \|_\mu : \mathcal{R} \to \mathbf{R}, A \mapsto \sup\{|\mu(B)| : B \in \mathcal{R}, B \subset A\}.$$

Sets $A \in \mathcal{R}$ are *negligible* if $\|A\|_\mu = 0$. One of the most eye-catching distinctions between classical measures and these measures is that there exists a set $X_0(\mu)$, which is the *biggest* negligible set, i.e., A is negligible if and only if $A \subset X_0(\mu)$. The real-valued function $\| \cdot \|_\mu$ induces a seminorm on the space of K-valued functions on X:

$$\|f\|_\mu = \sup\{|f(x)|N_\mu(x) : x \in X\}$$

where $N_\mu(x) = \inf\{\|A\|_\mu : A \in \mathcal{R}, x \in A\}$. This seminorm is used to find the integrable functions with respect to μ. Let $S(X)$ be the K-vector space of step functions, i.e., the space of finite linear combinations of characteristic functions χ_A for $A \in \mathcal{R}$. Integration with respect to μ is defined as the unique functional such that for every $A \in \mathcal{R}$, $\int_X \chi_A d\mu = \mu(A)$. A function $f : X \to K$ is integrable if there exists a sequence $\{f_n\}$ of step functions in $S(X)$ such that $\lim_{n\to\infty} \|f_n - f\|_\mu = 0$. This process leads to an extension of $\mathcal{R}$ to a covering ring $\mathcal{R}_\mu$ which contains all sets for which the characteristic function is integrable. The set is of integrable functions restricted to $X^+ = X \backslash X_0(\mu)$ is a Banach space for the induced norm $\| \cdot \|_\mu$, denoted by $L^1(\mu)$. A triple $(X, \mathcal{R}_\mu, \mu)$ is called a *probability space* if $\mathcal{R}_\mu$ is an algebra, and if $\mu(X) = 1$.

The aim of chapter 4 is to develop the theory of dynamical systems on these probability spaces. We call a measurable map $T : X \to X$ *measure preserving* if for any $A \in \mathcal{R}$, $\mu(T^{-1}A) = \mu(A)$, and we call a four-tuple $(X, \mathcal{R}_\mu, \mu, T)$ a *dynamical system*. The first noticeable property is that the biggest negligible set $X_0(\mu)$ is invariant under any measure preserving transformation. This is very different from the classical situation: we can totally neglect $X_0(\mu)$ by restricting to $(X^+, \mathcal{R}_\mu^+, \mu, T)$, where $\mathcal{R}_\mu^+$ is the ring $\{A \cap X^+ : A \in \mathcal{R}_\mu\}$. A consequence is that the direct analog of the notion of ergodicity in the classical sense is not very useful; it reduces in this setting to the statement that any T-invariant subset of X is negligible or contains the full set X^+.

The notion of isomorphic dynamical systems, however, is still useful. In fact, neglecting $X_0(\mu)$ is also what happens in our definition; two dynamical systems $(X, \mathcal{R}_\mu, \mu, T)$ and $(Y, \mathcal{R}_\nu, \nu, S)$ are called *isomorphic* if there is a measure preserving $(\mathcal{R}_\mu, \mathcal{R}_\nu)$ homeomorphism $\phi : X^+ \to Y^+$ such that $\phi \circ T = S \circ \phi$. Completely analogous to the classical setting we call these two dynamical systems *conjugate* if there is a measure algebra isomorphism $\Phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ such that $\Phi^{-1} \circ S^{-1} = T^{-1} \circ \Phi^{-1}$. In the classical theory isomorphy implies conjugacy, but not conversely.

Theorem G. *Two non-Archimedean measure preserving transformations are isomorphic if and only if they are conjugate.*

A measure preserving transformation induces a linear map $U_T : L^1(\mu) \to L^1(\mu), f \to f \circ T$ which is an isometry if T is invertible. The definition of *spectral isomorphy* of two invertible measure preserving transformations $T : X \to X$ and $S : Y \to Y$ comprises an isometry $W : L^1(\mu) \to L^1(\nu)$ such that $U_S \circ W = W \circ U_T$. If two dynamical systems are isomorphic, then they are also spectral isomorphic. We give some conditions under which spectral isomorphy implies conjugacy, and hence isomorphy.

Theorem H. *Let $W : L^1(\mu) \to L^1(\nu)$ be an isometry. If for all bounded functions $f, g \in L^1(\mu)$, $W(fg) = W(f)W(g)$, and if for all bounded functions $f \in L^1(\mu)$,*

$$\int_X f d\mu = \int_Y W(f) d\nu,$$

then there exists a measure algebra isomorphism $\phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ such that $W = U_\phi$.

It is an interesting problem whether these are necessary conditions, too.

In the last section we develop a notion of non-Archimedean *measure entropy*, which is an invariant under isomorphisms. Let α be a partition of X by elements of $\mathcal{R}_\mu$ and let $M(\alpha)$ be the number of elements of $\mathcal{M}(\alpha) = \{A \in \alpha : \|A\| > 0\}$. The measure entropy is the defined by $H_\mu(\alpha) = \min\{\|A\| \log M(\alpha) : A \in \mathcal{M}(\alpha)\}$. This measure entropy is connected to the topological entropy, for the topology induced by $\mathcal{R}_\mu$. For compact X^+ and measures μ such that for all nonempty $A \in \mathcal{R}_\mu^+$, $|\mu(A)| = 1$ the measure entropy equals the topological entropy.

Theorem I. *Let $(X, \mathcal{R}_\mu, \mu)$ be a compact probability space satisfying $\|X\| = 1$, and let $T : X \to X$ be a measure preserving transformation, then*

$$h_\mu(T) \leq h_{top}(T),$$

and equality holds if $X_0 = \emptyset$ and if for any nonempty set $A \in \mathcal{R}_\mu$, $\|A\| = 1$.

Let us also digress on some notions from dynamical systems of which we don't know how to translate them to the non-Archimedean setting; Poincaré recurrence and the Birkhoff ergodic theorem. Poincaré recurrence states that in a probability space for any non-negligible measurable set A the subset of A of elements which are not recurrent under a measure preserving T is negligible. The classical proof relies heavily on the measurability of the set of non-recurrent points. The non-Archimedean measures, however, are in general not σ-additive. In fact, it is possible to construct examples for which the recurrent set is not measurable and indeed not negligible.

The Birkhoff ergodic theorem assures the convergence in L^1 of the average

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)),$$

to a T-invariant function in L^1. However, in a non-Archimedean field there is not a notion of average. In particular, the sequence $\{\frac{1}{n}\}_{n\in\mathbf{N}}$ is not convergent.

Chapter 2

Measure-theoretic rigidity for Mumford curves

Onder de eik bleef de olifant staan.
'Denk je dat ik door de eik heen kan lopen?' vroeg hij.
'Nee' zei de eekhoorn 'Hij is te dik.'
'Dik is geen reden,' zei de olifant. 'Mist is soms heel dik. En molmsoep.'

Toon Tellegen [80]

One can describe isomorphism of two compact hyperbolic Riemann surfaces of the same genus by a measure-theoretic property: a chosen isomorphism of their fundamental groups corresponds to a homeomorphism on the boundary of the Poincaré disc that is absolutely continuous for Lebesgue measure if and only if the surfaces are isomorphic.

In this chapter, we find the corresponding statement for Mumford curves, a non-Archimedean analog of Riemann surfaces. In this case, the mere absolute continuity of the boundary map (for Schottky uniformization and the corresponding Patterson–Sullivan measure) only implies isomorphism of the special fibers of the minimimal models of Mumford curves, and the absolute continuity needs to be enhanced by a finite list of conditions on the harmonic measures on the boundary (certain non-Archimedean distributions constructed by Schneider and Teitelbaum) to guarantee an isomorphism of the Mumford curves.

The proof combines a generalization of a rigidity theorem for trees due to Coornaert, the existence of a boundary map by a method of Floyd, with a classical theorem of Babbage–Enriques–Petri on equations for the canonical embedding of a curve.

This chapter is based on [28], 'Measure-theoretic rigidity of Mumford curves', joint work with Gunther Cornelissen.

2.1 Introduction

In this chapter we prove:

Theorem 2.1.1 (=Theorem A). *Let k be a complete discrete non-Archimedean valued field, and let K denote a complete and algebraically closed field extension of k. Let $\Gamma_1, \Gamma_2 \leq \mathrm{PGL}(2,k)$ be Schottky groups of the same genus $g \geq 2$, defining Mumford curves X_1 and X_2, respectively over K. Let φ denote the boundary map associated to an isomorphism of Γ_1 with Γ_2.*

(i) *The intersection dual graphs of the special fibers of the minimal models of X_1 and X_2 are isomorphic if and only if there is an isometry*

$$F : T(\Lambda_{\Gamma_1}) \to T(\Lambda_{\Gamma_2})$$

which conjugates the Schottky groups Γ_i, if and only if the corresponding Patterson-Sullivan measures are of the same conformal dimension and the map φ is absolutely continuous for these measures.
If this is the case, then pullback by F^{-1} induces an isomorphism of the spaces of harmonic measures

$$F_* : \ \mathrm{C}_{\mathrm{har}}(\Gamma_1, 2) \xrightarrow{\sim} \mathrm{C}_{\mathrm{har}}(\Gamma_2, 2).$$

(ii) *Suppose that the curves X_i are not hyperelliptic. Then the curves X_1 and X_2 are isomorphic over K if and only if the corresponding Schottky groups Γ_i are conjugate in $\mathrm{PGL}(2,k)$, if and only if the map φ is absolutely continuous with respect to Patterson–Sullivan measure* and *F_* respects the linear relations of degree ≤ 4 between the harmonic boundary measures in the sense that*

$$F_* R_2(\Gamma_1) \subseteq R_2(\Gamma_2), F_* R_3(\Gamma_1) \subseteq R_3(\Gamma_2) \text{ and } F_* R_4(\Gamma_1) \subseteq R_4(\Gamma_2).$$

The relations in R_4 are only necessary if $g = 3$. The relations in R_3 are only necessary if X is trigonal (i.e., admits a map $X \to \mathbf{P}^1$ of degree 3), or if X is isomorphic to a plane quintic.

In the first part, we prove a version of Coornaert's ergodic rigidity theorem for two different trees (the original theorem was for one tree), and apply it to the intersection dual graph of the special fiber of the minimal models of two Mumford curves—this will prove the first part of the theorem. Then we give the proof of the second part of the theorem.

Example

Let us give an intuitive example of a Mumford curve in positive characteristic p to illustrate all the ingredients. Let $\mathbf{F}_q$ be the finite field with $q = p^n$ elements, let $k = \mathbf{F}_q((T))$ be the valued field of power series in T, and let K be a complete algebraically closed extension of k. Let $\lambda \in k$ be such that $0 < |\lambda| < 1$, i.e., let $\lambda = \sum_{i=N}^{\infty} a_i T^i$, with $a_i \in \mathbf{F}_q$ and $N \geq 1$. Then the following equation describes a so called Artin-Schreier-Mumford curve over K:

$$X_\lambda = \{(x,y)|(x^q - x)(y^q - y) = \lambda\} \subset \mathbf{P}^1(K) \times \mathbf{P}^1(K).$$

The reduction of this curve, i.e., the curve modulo T over the residue field, which in this example is the finite field $\mathbf{F}_q[[T]]/(T) \cong \mathbf{F}_q$, is given by

$$\{(x,y)|(x^q - x)(y^q - y) = 0\} \subset \mathbf{P}^1(\mathbf{F}_q) \times \mathbf{P}^1(\mathbf{F}_q).$$

The solution of this equation forms a "chess board" of q horizontal an q vertical copies of $\mathbf{P}^1(\mathbf{F}_q)$. This object is what is called the special fibre of the curve. The very same Mumford curve can also be obtained by a uniformization process. The uniformizing space is the nonarchimedean analytic space $\mathbf{P}^1(K) - \Lambda_\Gamma$, where Λ_Γ is the limit set of a Schottky group $\Gamma \leq \mathrm{PGL}(2,k)$, which acts by Möbius transformations properly discontinuously on $\mathbf{P}^1(K) - \mathbf{P}^1(k)$, having a limit set contained in $\mathbf{P}^1(k)$. A Schottky group corresponding to the curve X_λ can be constructed as follows: define

$$\epsilon_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}, \epsilon'_u = \tau\epsilon_u\tau,$$

where $u \in \mathbf{F}_q$ and $t \in k^*$ such that $0 < |t| < 1$. Let $\Gamma(t)$ be the group generated by the commutators $[\epsilon_u, \epsilon'_v]$. Then there is a t such that $X_\lambda = \Gamma(t)\backslash(\mathbf{P}^1(K) - \Lambda_{\Gamma(t)})$. The relation between λ and t is transcendental and studied in [25]. The group $\Gamma(t)$ also acts properly discontinuously on the Bruhat-Tits tree, having limit points in the boundary of the tree. The boundary of the tree is canonically isomorphic to $\mathbf{P}^1(k)$ and the set of limit points is exactly the same set Λ_Γ as above. Denote by $\mathcal{T}(\Lambda_\Gamma)$ the tree spanned by the limit set. The quotient $\Gamma\backslash\mathcal{T}(\Lambda_\Gamma)$ is a finite graph which is in fact the intersection graph of the special fibre. In this example this is the regular bi-partite graph with $2q$ vertices; a vertex for each copy of $\mathbf{P}^1(\mathbf{F}_q)$.

2.2 Measure-theoretic rigidity of trees and special fiber isomorphism

Throughout, we only consider locally finite trees T without ends, that is, trees such that for any vertex the valency is finite and the complement is disconnected. We equip the set of vertices of T with a metric d in which all edges are of length one. All graphs will be oriented. For a graph G, we denote the set of vertices with $\mathrm{V} = \mathrm{V}(G)$ and the set of edges with $\mathrm{E} = \mathrm{E}(G)$. We denote the set of edges from $x \in \mathrm{V}(G)$ to $y \in \mathrm{V}(G)$ with $\mathrm{E}(x, y)$. A *basic subset* of a tree is a set containing all vertices and edges which are on the target side of a given edge, i.e., an edge $e \in \mathrm{E}(x, y)$ defines the basic set $A = A(e)$ where $v \in A$ if $d(v, y) < d(v, x)$.

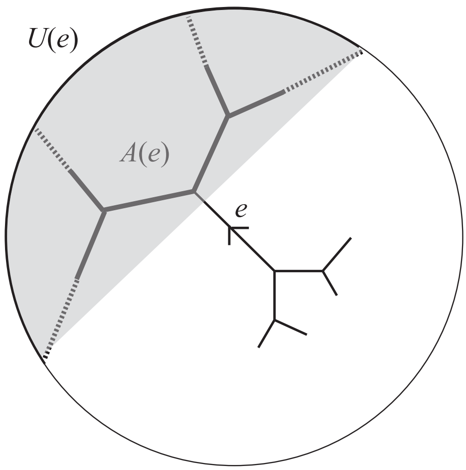


Figure 2.1: Basic set $A(e)$ and compact open set $U(e)$

Automorphism groups

Let $G = \mathrm{Aut}(T)$ be the automorphism group of a tree T. Equip G with the topology in which the stabilizers of the vertices form the compact open neighborhoods of the identity element. Denote by G^+ the subgroup of G of automorphisms which act without inversion, i.e., automorphisms which do not switch the orientation of any edge. The group G^+ can be partitioned into so-called *elliptic automorphisms* (which have a fixed vertex), and *hyperbolic automorphisms* (which do not fix any vertex).

We are concerned with certain discrete subgroups of G^+ which are called tree-Schottky subgroups.

Definition 2.2.1. A subgroup Γ of G^+ is called a *tree-Schottky group* if it is finitely generated, and all non-identity elements are hyperbolic.

In particular, a tree-Schottky group is a discrete and free group (see Serre [75] §I.3.3). The rank of the group is called the *genus* of the group.

Remark 2.2.2. Tree-Schottky groups satisfy remarkable geometric properties, which can be used to give the following second equivalent definition (for a proof of the equivalence see Lubotzky [54], section 1). A tree-Schottky group Γ of rank g is a free subgroup of G^+ such that for a choice of generators $\Gamma = \langle \gamma_1, ...\gamma_g \rangle$ there exist distinct basic sets $A_i, B_i \subset T$ for $1 \leq i \leq g$ satisfying:

$$\gamma_i(A_i) = T - B_i$$
$$\gamma_i^{-1}(B_i) = T - A_i.$$

Patterson–Sullivan measures

The *boundary* ∂T of the tree T is defined as the space of geodesic rays $r : \mathbf{Z}_{\geq 0} \to \mathrm{V}(T)$ modulo the following equivalence relation: two rays r, r' are considered equivalent if there are numbers t_0, s, such that for all $t > t_0$, $r(t) = r'(t+s)$. The metric on ∂T with respect to a fixed base point $x_0 \in T$ is defined as follows. Every boundary point $\xi \in \partial T$, has a unique representative ray $r : \mathbf{Z}_{\geq 0} \to E(T)$ with initial point of $r(0)$ equal to x_0. The image of r is denoted by $[x_0, \xi]$.

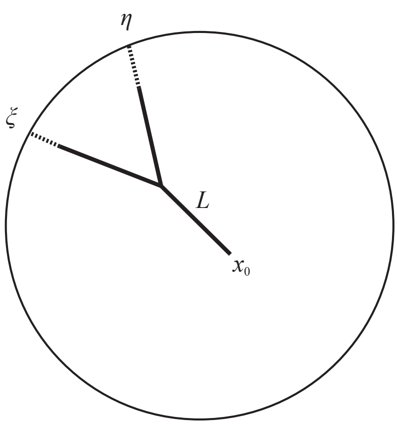


Figure 2.2: Distance between boundary points

The *distance between two boundary points* $\eta, \xi \in \partial T$ is given by

$$\rho_{x_0}(\eta, \xi) = e^{-L},$$

where L is the length of $[x_0, \eta] \cap [x_0, \xi]$. We fix x_0 from now on and leave it out of the notation.

This metric induces a topology on the boundary. The set of all boundary points that have a representative ray that is entirely contained in a basic set $A(e)$ is denoted by $U(e)$. One can check that $U(e)$ are compact open subsets of the boundary.

An isometry γ of T induces a homeomorphism of ∂T to itself. The map measuring locally the dilatation by γ is

$$j_\gamma(\xi) = e^{d(x_0,p)-d(p,\gamma^{-1}x_0)},$$

where p is the projection of ξ on $[x_0, \gamma^{-1}x_0]$. A discussion on this map can in ([23], §3).

Let Γ be a group of isometries acting properly discontinuously on T. A Borel measure μ on ∂T is called Γ-conformal of dimension d, if for all $\gamma \in \Gamma$ $\gamma^*\mu = j_\gamma^d \mu$, where $\gamma^*\mu$ is defined by $\gamma^*\mu(A) = \mu(\gamma(A))$. In [23], Coornaert constructs such a measure completely analogously to the Patterson–Sullivan measures on the limit set of a Fuchsian group. That is

$$\mu(A) = \lim_{s \to e(\Gamma)} \frac{\sum_{y \in \Gamma(x)} e^{-sd(y,x_0)} \operatorname{dirac}(y)(A)}{\sum_{y \in \Gamma(x)} e^{-sd(y,x_0)}},$$

here $e(\Gamma)$ is the critical exponent for the Poincaré series (which is the denominator in the above formula). In more concrete terms, denote with $n_Y(R)$ the number of points in a orbit Y within a distance R from the basepoint x_0. Then

$$e(\Gamma) = \limsup_{R \to \infty} \frac{1}{R} \log n_Y(R).$$

The critical exponent does neither depend on the orbit Y, nor on the basepoint x_0, and the measure μ is Γ-conformal of dimension $e(\Gamma)$. This measure however, does depend on the base point x_0, but measures with different base points are absolutely continuous with respect to each other. For a Γ-conformal measure μ of dimension d on ∂T, a Γ-invariant measure ν on

$$\partial^2 T = \{(\xi, \eta) | \xi, \eta \in \partial T, \xi \neq \eta\}$$

is given by the following formula:

$$\nu = \frac{\mu \times \mu}{\rho(\xi, \eta)^{2d}}.$$

In particular, ν is supported on the limit set $\Lambda_\Gamma \times \Lambda_\Gamma$, and is ergodic with respect to the group action on the limit set, which is in particular the case if T/Γ is a finite graph (cf. [22]). In contrast to μ, the measure ν does not depend on the base point.

Rigidity theorem for trees

These measures are used in the following rigidity theorem, which is a slight adaptation of the main theorem of Coornaert [22] to the setting of two (a priori different) trees.

Theorem 2.2.3. *For $i = 1, 2$, let T_i be a complete locally compact tree without endpoints, and let Γ_i be an isometry group acting properly discontinuously on T_i. Let μ_1 and μ_2 be Γ_1- resp. Γ_2-conformal measures on ∂T_1 resp. ∂T_2, of the same dimension $d \geq 0$, depending on reference vertices x_0, y_0 in T_1 an T_2 respectively. If there is a measurable bijection $\varphi : \partial T_1 \to \partial T_2$ and a bijection $\alpha : \Gamma_1 \to \Gamma_2$ satisfying the conditions:*

equivariance *for μ_1 almost all $\xi \in \partial T_1$ and for all $\gamma_1 \in \Gamma_1$,*

$$\varphi(\gamma_1 \xi) = \alpha(\gamma_1)\varphi(\xi),$$

non-singular *for all Borel sets $A \subset \partial T_1$,*

$$\mu_1(A) = 0 \text{ if and only if } \mu_2(\varphi(A)) = 0,$$

ergodic *the action of Γ_1 on $(\partial^2 T_1, \nu_1)$ is ergodic, where $\nu_1 = \frac{\mu_1 \times \mu_1}{\rho_{x_0}(\xi,\eta)^{2d}}$,*

support *the support of μ_1 is ∂T_1,*

then there exists a bijective isometry $F \colon T_1 \to T_2$, such that $\varphi(\xi) = F(\xi)$ for μ_1-almost all ξ in ∂T_1. Moreover, α is a group isomorphism with $\alpha(\gamma) = F \circ \gamma \circ F^{-1}$ for all γ in Γ_1.

Proof. The main ingredient of the proof is the invariance of the *cross-ratio* under φ. Recall that the cross-ratio $c(\xi_1, \xi_2, \xi_3, \xi_4)$ of four pairwise disjoint elements of ∂T_1 is the quotient

$$c(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{\rho(\xi_3, \xi_1) \cdot \rho(\xi_4, \xi_2)}{\rho(\xi_3, \xi_2) \cdot \rho(\xi_4, \xi_1)}. \tag{2.1}$$

This ratio is independent of x_0, and is in fact the same as

$$c(\xi_1, \xi_2, \xi_3, \xi_4) = e^{-L}, \tag{2.2}$$

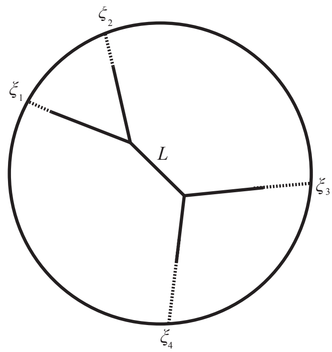


Figure 2.3: Cross-ratio $c(\xi_1, \xi_2, \xi_3, \xi_4) = e^{-L}$

where L is the signed length of the intersection $[\xi_1, \xi_2] \cap [\xi_3, \xi_4]$, where the sign depends on the orientation of the geodesics.

Note that Γ_2 acts ergodically on $(\partial^2 T_2, \nu_2)$; indeed let $B \subset \partial^2 T_2$ be Γ_2-invariant, then $(\varphi \times \varphi)^{-1}(B)$ is Γ_1-invariant, and hence, since ν_1 is **ergodic**;

$$\nu_1((\varphi \times \varphi)^{-1}(B)) = 0 \text{ or } \nu_1(\partial^2 T_1 - (\varphi \times \varphi)^{-1}(B)) = 0.$$

So in particular,

$$\mu_1 \times \mu_1((\varphi \times \varphi)^{-1}(B)) = 0 \text{ or } \mu_1 \times \mu_1(\partial^2 T_1 - (\varphi \times \varphi)^{-1}(B)) = 0.$$

Using **non-singular**, we get

$$\mu_2 \times \mu_2(B) = 0 \text{ or } \mu_2 \times \mu_2(\partial^2 T_2 - B) = 0,$$

and hence

$$\nu_2(B) = 0 \text{ or } \nu_2(\partial^2 T_2 - B) = 0.$$

By **non-singular** the measure $(\varphi \times \varphi)_* \nu_1$ on $\partial^2 T_2$ is absolutely continuous with respect to ν_2. Therefore, there exists a measurable function h, the Radon-Nikodym derivative, such that $(\varphi \times \varphi)_* \nu_1 = h\nu_2$. The function h is Γ_2-invariant, and hence by the ergodicity of ν_2, the function h is almost everywhere constant with respect to ν_2. Likewise, let g be a function on ∂T_1 such that $\varphi_*^{-1} \mu_2 = g\mu_1$. Now let A be any Borel set in $\partial^2 T_1$, and let

$B = (\varphi \times \varphi)(A) \subset \partial^2 T_2$, then

$$\begin{aligned}
\frac{\mu_1 \times \mu_1}{\rho_{x_0}(\xi,\eta)^{2d}}(A) &= \nu_1(A) \\
&= (\varphi \times \varphi)_* \nu_1(B) \\
&= h \cdot \nu_2(B) \\
&= h \cdot \frac{\mu_2 \times \mu_2}{\rho_{y_0}(\varphi(\xi),\varphi(\eta))^{2d}}(B) \\
&= g^2 h \cdot \frac{\mu_1 \times \mu_1}{\rho_{y_0}(\varphi(\xi),\varphi(\eta))^{2d}}(A).
\end{aligned}$$

We find a function $f = (g^2 h)^{1/2d}$ such that the following equation holds ν_1 almost everywhere:

$$\rho_{y_0}(\varphi(\xi_1),\varphi(\xi_2)) = f(\xi_1) f(\xi_2) \rho_{x_0}(\xi_1,\xi_2).$$

It follows that the cross-ratio is invariant under φ almost everywhere, i.e., there is a subset $W \subset \partial T_1$ of full μ_1 measure on which φ is well defined and preserves the cross-ratio. The tree T_1 is without ends and the support of μ_1 is ∂T_1. We will first define the map F for vertices $v \in \mathrm{V}(T_1)$ for which there exists a triple ξ_1, ξ_2, ξ_3 of pairwise disjoint elements of W such that v is the center of this triple, i.e.,

$$v = [\xi_1,\xi_2] \cap [\xi_1,\xi_3] \cap [\xi_2,\xi_3].$$

Define $F(v)$ to be:

$$F(v) = [\varphi(\xi_1),\varphi(\xi_2)] \cap [\varphi(\xi_1),\varphi(\xi_3)] \cap [\varphi(\xi_2),\varphi(\xi_3)].$$

Extend the map F uniquely to a map $T_1 \to T_2$ by linear interpolation along the rays. It is necessary to prove that F is well defined. Let $\xi_4 \in W$ be such that v is also the center of ξ_1, ξ_2, ξ_4. Then the projection of ξ_4 on

$$[\xi_1,\xi_2] \cup [\xi_1,\xi_3] \cup [\xi_2,\xi_3]$$

lies on $[v,\xi_3]$. The invariance of the cross ratio shows that the projection of $\varphi(\xi_4)$ lies on $[F(v),\varphi(\xi_3)]$.

The map F is an isometry; given $v, w \in \mathrm{V}(T_1)$ there are two pair of points in W, say, ξ_1, ξ_2 and ξ_3, ξ_4 such that $[v,w] = [\xi_1,\xi_2] \cap [\xi_3,\xi_4]$, and then the distance between v and w is given by

$$\begin{aligned}
d(v,w) &= \ln(c(\xi_1,\xi_2,\xi_3,\xi_4)) \\
&= \ln(c(\varphi(\xi_1),\varphi(\xi_2),\varphi(\xi_3),\varphi(\xi_4))) \\
&= d(F(v),F(w)).
\end{aligned}$$

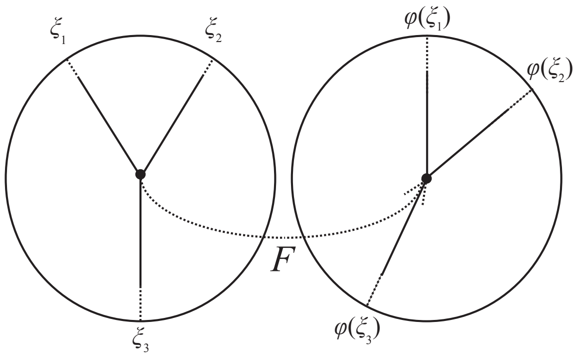


Figure 2.4: Definition of isometry F

For $\xi \in W$, and hence μ_1 almost everywhere it holds that $F(\xi) = \varphi(\xi)$. Moreover, the isometry of T_2 given by $F \circ \gamma_1 \circ F^{-1}$ coincides with $\alpha(\gamma_1)$, for all $\gamma_1 \in \Gamma_1$. □

Existence of boundary morphism in the tree-Schottky case

In this section, we prove that for two tree-Schottky groups of the same genus g there is always an equivariant homeomorphism between the limit sets (i.e., maps φ and α satisfying (1) as in Theorem 2.2.3). This is a consequence of the existence of a canonical identification between the limit set of a Schottky group and the Floyd boundary of the free group F_g, the latter being independent of the realisation of F_g as Schottky group.

First, recall the following definitions: A metric space (X, d_X) is called hyperbolic if there exists a $\delta \geq 0$ such that for any base point x_0 in X,

$$(x \cdot y) \geq \min((x \cdot y)(y \cdot z)) - \delta \text{ for any } x, y, z \in X.$$

Here, $(x \cdot y)$ denotes the Gromov product with respect to x_0 defined as

$$(x \cdot y) = (d_X(x, x_0) + d_X(y, x_0) - d_X(x, y))/2.$$

This implies in particular that trees are hyperbolic spaces (with $\delta = 0$). Secondly, a finitely generated group is called hyperbolic if its Cayley graph relative to a finite system of generators is hyperbolic. In particular, free groups are hyperbolic. The *boundary ∂G of a group G* is defined by Floyd in [37]: it is the complement of the Cayley graph of G inside its completion, by considering it as a metric space for the "Floyd metric" that assigns to two adjacent words w_1 and w_2, (i.e., with $|w_1^{-1}w_2| = 1$), the length

$$1/\max\{|w_1|, |w_2|\}^2$$

(with $|\cdot|$ the usual word length, and the empty word having length zero). Then we have:

Theorem 2.2.4 ([24], Theorem 4.1)**.** *Let X be a proper geodesic space and let Γ be an isometry group of X which acts properly and discontinuously on X, and let the quotient of this action be compact. Then Γ is hyperbolic if and only if X is hyperbolic. Moreover, if Γ is hyperbolic there is a canonical homeomorphism $\partial\Gamma \to \partial X$.*

Remark 2.2.5. In our case, let F_g denote a free group of rank g, let $A = \{A_1, ..., A_g\}$ denote a set of generators, and let

$$\rho : F_g \to \mathrm{Aut}^+(T)$$

be a faithful representation, such that the image $\rho(F_g) = \Gamma$ is a tree-Schottky group for the tree T. Denote by $\mathrm{Cay}(F_g, A)$ the Cayley graph with respect to the given generating set A, pick a base point x_0 and define

$$\varphi : \mathrm{Cay}(F_g, A) \to T, w \mapsto \rho(w)(x_0).$$

The boundary morphism is obtained by extending this map to the completions on both sides, and then restricting to the respective boundaries, giving a map

$$\varphi\colon \partial F_g \xrightarrow{\sim} \partial T.$$

The above theorem indeed implies our claim:

Corollary 2.2.6. *For two tree-Schottky groups Γ_1, Γ_2 of the same rank $g \geq 2$, and acting on two trees T_1, T_2, there is a group isomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$ and a homeomorphism $\varphi : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ between the respective limit sets, that is equivariant w.r.t. α.*

Proof. Let T_{Γ_i} be the subtree of the T_i containing all geodesics connecting the elements of the limit set Λ_{Γ_i}. Let $\rho_i\colon F_g \hookrightarrow \mathrm{Aut}^+(T_i)$ denote the representation of the free group F_g on g letters; then $\alpha := \rho_2 \circ \rho_1^{-1}$ is a homomorphism between Γ_1 and Γ_2.

The quotient graph $\Gamma_i \backslash T_{\Gamma_i}$ is finite, and hence compact. Hence all requirements of Theorem 2.2.4 are fulfilled, and therefore we obtain two homeomorphisms $\varphi_i : \partial F_g \to \Lambda_{\Gamma_i}$. One can then set $\varphi = \varphi_2 \circ \varphi_1^{-1}$, and this is equivariant w.r.t. α by construction.

$$\begin{tikzcd}
 & \Lambda_{\Gamma_1} \arrow[dd, "\varphi"] \\
\partial F_g \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
 & \Lambda_{\Gamma_2}
\end{tikzcd}$$

□

2.2.1 Rigidity theorem for tree-Schottky groups

We end this section with a theorem on the rigidity of tree-Schottky groups.

Theorem 2.2.7. *Let T_i be trees $(i = 1, 2)$ and let Γ_i be corresponding tree-Schottky groups, both of genus g, such that $\Lambda_{\Gamma_i} = \partial T_i$. Then the following are equivalent:*

1. *The quotient graphs $\Gamma_i \backslash T_i$ are isomorphic.*
2. *There is an isomorphism $\alpha \colon \Gamma_1 \to \Gamma_2$ and a bijective isometry*
$$F \colon T_1 \to T_2 \text{ such that for all } \gamma \in \Gamma_1, F \circ \gamma = \alpha(\gamma) \circ F$$
3. *There is an isomorphism $\alpha : \Gamma_1 \to \Gamma_2$ and an equivariant homeomorphism $\varphi : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}$ which is absolutely continuous with respect to the respective Patterson–Sullivan measures. Moreover, these measures are of the same dimension.*

Proof. Statement (3) implies (2): this is a direct consequence of Theorem 2.2.3. We check that all conditions of Theorem 2.2.3 are satisfied. The homeomorphism φ is equivariant and absolutely continuous by assumption. The Patterson–Sullivan measures are of the same dimension and, moreover, the Patterson–Sullivan measure of the first group is supported on $\Lambda_{\Gamma_1} = \partial T_1$. As noted above, the group Γ_1 acts ergodically on $\Lambda^2_{\Gamma_1} = \partial^2 T_1$.

Statement (2) implies (1): suppose there is an isometry as in (2), and let $y_1 \in \mathrm{V}(T_1)$. Then for any $y \in \Gamma y_1$ it holds that

$$F(y) = F(\gamma(y_1)) = \alpha(\gamma) F(y) \in \Gamma_2 F(y).$$

Therefore, F sends orbits of vertices to orbits of vertices. Moreover, because F is an isometry, it also sends orbits of edges to orbits of edges. Hence, F induces an isomorphism between the quotient graphs.

What is left to prove is that (1) implies (3). For this, denote $G_i = \Gamma_i \backslash T_i$, and let $f : G_1 \to G_2$ be the given isomorphism.

A graph G is called *combinatorial* if $\mathrm{E}(x, y)$ contains at most one edge and if $\mathrm{E}(x, x) = \emptyset$ for all $x \in \mathrm{V}(G)$. If G_1 is not combinatorial we make it combinatorial by replacing every edge by three edges while adding two new vertices, i.e.,

$$\bullet \!\!\longrightarrow\!\! \bullet \text{ is replaced by } \bullet - \bullet - \bullet - \bullet$$

We do this simultaneously with T_1 and denote the new graphs with $\overline{G}_1$, and $\overline{T}_1$. There is a unique way of extending the action of Γ_1 on T_1 to $\overline{T}_1$, and the quotient $\Gamma_1 \backslash \overline{T}_1$ is $\overline{G}_1$. Repeat the procedure with G_2, such that f extends to an isomorphism $\overline{f} : \overline{G}_1 \to \overline{G}_2$. From now on we will assume that the quotient graphs are already combinatorial. Let $P_1 \subset G_1$ be a maximal subtree with $2g$ endpoints. The tree T_1 is the universal covering of G_1, and hence there is a lift $\ell_1 : P_1 \hookrightarrow T_1$, which is unique once one point of the lift is chosen. Moreover, $\ell(P_1)$ is a fundamental domain for Γ_1, and therefore there are g group elements $\gamma_1, ..., \gamma_g$ of Γ_1 and there is a labeling $a_1, b_1, \ldots, a_g, b_g$ of the end points of P_1 such that for $1 \leq i \leq g$,

$$d(\gamma_i(\ell_1(a_i)), \ell_1(b_i)) = 1.$$

The group Γ_1 is generated by $\gamma_1, \ldots, \gamma_g$. Let $P_2 = f(P_1)$, which is a maximal subtree of G_2. Again, after the choice of one point there is a unique lift $\ell_2 : P_2 \hookrightarrow T_2$. Moreover, there are g elements $\delta_1, \ldots, \delta_g$ generating Γ_2 and satisfying

$$d(\delta_i(\ell_2 \circ f(a_i)), \ell_2 \circ f(b_i)) = 1.$$

There are representations

$$\rho_1 : F_g \to \mathrm{Isom}(T_1) \text{ and } \rho_2 : F_g \to \mathrm{Isom}(T_2),$$

such that

$$\rho_2 \circ \rho_1^{-1}(\gamma_i) = \alpha(\gamma_i) = \delta_i$$

induces the isomorphism $\alpha : \Gamma_1 \to \Gamma_2$. The corresponding boundary morphism is the continuation to the boundary of the map

$$\phi : T_1 \to T_2 : x \mapsto \alpha(w_x)(\ell_2 f \ell_1^{-1}(\widetilde{x})),$$

where $\widetilde{x}$ is the unique element in $\ell_1(P_1)$ for which there is an unique word w_x in the letters $\gamma_1, \gamma_1^{-1}, \ldots, \gamma_g, \gamma_g^{-1}$ such that $w_x(\widetilde{x}) = x$. The map ϕ is a surjective isometry, and therefore the induced boundary morphism is absolutely continuous with respect to the Patterson–Sullivan measures. Moreover, since ϕ sends orbits to orbits, it holds for all $R \geq 0$, that $n_Y(R) = n_{\phi(Y)}(R)$, and hence $e(\Gamma_1) = e(\Gamma_2)$. □

2.3 Mumford curves

We now turn our attention to the setting of algebraic curves. Let k be a complee discrete valued field, with valuation v_k, and denote with $\mathcal{O}_k$ the ring of integers of k. Let π be a uniformizer and let $\overline{k}$ be the residue field. The field K is a complete algebraically closed field extension of k.

The group $G = \mathrm{PGL}(2, k)$ acts on $\mathbf{P}^1(K)$ through linear fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},$$

leaving the space of k-rational points $\mathbf{P}^1(k)$ invariant. A group element $g \in G$ is called *hyperbolic* if the two eigenvalues have different valuation. Hyperbolic group elements have two fixed points, both lying in $\mathbf{P}^1(k)$.

Schottky groups

Analogous to tree-Schottky groups we will define Schottky groups, which are certain subgroups of $\mathrm{PGL}(2, k)$.

Definition 2.3.1. A subgroup Γ of $\mathrm{PGL}(2, k)$ is called a Schottky group if

1. Γ is finitely generated;
2. all non-identity elements $\gamma \in \Gamma$ are hyperbolic.

Definition 2.3.2. The quotient space $X_\Gamma = \Gamma \backslash (\mathbf{P}^1(K) - \Lambda_\Gamma)$ is a well-defined one-dimensional compact rigid analytic space, and as such, it is the unique analytification of an algebraic curve over K. We will not distinguish the analytic and algebraic curve, and call it simply a *Mumford curve*.

The tree $T(\Lambda_\Gamma)$

There is a tree $T(\Lambda_\Gamma)$ related to Λ_Γ in a natural way, cf. ([42], chapter 1, §2). First, we consider the Bruhat–Tits tree $\mathfrak{T}$ of $\mathrm{PGL}(2,k)$. This is a $q+1$-regular tree (with q the cardinality of the residue field of k), which has a natural action by $\mathrm{PGL}(2,k)$ induced by its interpretation as equivalence classes of rank two lattices over the ring of integers of k. In this way, the boundary of the Bruhat–Tits tree also becomes naturally identified with $\mathbf{P}^1(k)$.

Given a Schottky group Γ, we let $T(\Lambda_\Gamma)$ denote the subtree of $\mathfrak{T}$ *spanned* by the limit set in the following sense: the subtree is the union of all vertices and edges that occur in geodesics $[\xi_1,\xi_2]$ with $\xi_1 \in \Lambda_\Gamma$ and $\xi_2 \in \Lambda_\Gamma$. A map on the boundary that preserves the cross ratio (in the sense of formula (2.1)) defines an isometry of the tree by the construction from Figure 2.4. Conversely, an isometry of the tree preserves the cross ratio by formula (2.2). By construction, a matrix $\gamma \in \Gamma \subseteq \mathrm{PGL}(2,k)$ acts on the Bruhat–Tits tree by isometries (and its subtree $T(\Lambda_\Gamma)$), and this induces an action of γ on the boundary, which is the natural action on $\mathbf{P}^1(k)$.

Since the action of the group Γ is isometric, it is a tree-Schottky group for this action. The boundary of the tree $T(\Lambda_\Gamma)$ is Λ_Γ by construction. The action of Γ on $T(\Lambda_\Gamma)$ captures information about the curve $T(\Lambda_\Gamma)$ in the following sense. The quotient graph $\Gamma\backslash T(\Lambda_\Gamma)$ is the intersection graph of the special fiber of the minimal model of X_Γ, ([63] §3, [42], remark 2.12.3). In particular, the quotient is a finite graph.

The curve and the tree

As stated above, the curve X_Γ and the tree $T(\Lambda_\Gamma)$ are strongly related. Another instance of this was pointed out by Teitelbaum [79]: the vector spaces of rigid analytic modular forms on $\mathbf{P}^1(K) - \Lambda_\Gamma$ are isomorphic to spaces of harmonic cocycles on the edges of the tree $T(\Lambda_\Gamma)$. We recall the notions involved and then describe the isomorphism.

Definition 2.3.3. Let n be an even integer. Denote by $\mathrm{S}_n(\Gamma)$ the space of *rigid analytic modular forms* on $\mathbf{P}^1(K) - \Lambda_\Gamma$ of weight n for Γ over K, i.e., $\mathrm{S}_n(\Gamma)$ comprises the rigid analytic functions $f : \mathbf{P}^1(K) - \Lambda_\Gamma \to K$ such that

$$f(\gamma z) = \frac{(cz+d)^n}{(\det(\gamma))^{n/2}} f(z).$$

Remark 2.3.4. The space of weight two rigid analytic modular forms for a Schottky group Γ is isomorphic to the space of holomorphic 2-forms on the corresponding Mumford curve:

$$\mathrm{S}_2(\Gamma) \to H^0(X_\Gamma, \Omega_{X_\Gamma})$$

by mapping a modular form f to the differential form corresponding to $f(z)dz$ on $\mathbf{P}^1(K) - \Lambda_\Gamma$. More generally, the space of weight 2ℓ analytic modular forms is isomorphic to the space of holomorphic ℓ-forms:

$$\mathrm{S}_{2\ell}(\Gamma) \to H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes \ell}),$$

with the isomorphism induced by $f \mapsto f dz^{\otimes \ell}$.

Definition 2.3.5. Let n be a positive even integer. Denote with $P_n[X] \subset K[X]$ the vector space of polynomials of degree at most n. A $P_n[X]$-valued function c on the edges of the tree $T(\Lambda_\Gamma)$ is called a Γ-*harmonic cocycle* if

1. for all edges e, $c(e) = -c(\overline{e})$, where $\overline{e}$ is the inverted orientation of e.
2. for all vertices v, $\sum_{e \mapsto v} c(e) = 0$, where the sum is over the oriented edges with target v.
3. c is equivariant with respect to the Γ-action, i.e.,

$$c(\gamma e)(X) = \gamma \cdot c(e)(X).$$

Here $\gamma \in \Gamma$ acts on $p \in P_n[X]$ by:

$$\gamma \cdot p(X) = \frac{(bX+d)^n}{(\det(\gamma))^{n/2}} \cdot p(\frac{aX+c}{bX+d}).$$

The set of harmonic Γ-cocycles with values in $P_{n-2}[X]$ is a vector space over K denoted by $\mathrm{C}_{\mathrm{har}}(\Gamma, n)$.

In [72] (page 228), Peter Schneider constructed distributions on the boundary of the tree based on these harmonic cocycles. Recall that a distribution is a finitely additive function on the compact opens. Let $c \in \mathrm{C}_{\mathrm{har}}(\Gamma, 2)$. Then the corresponding distribution μ_c is defined by the following fundamental relation:

$$\mu_c(U(e)) \quad = \quad c(e),$$

where $U(e)$ is the compact open corresponding to the basic set $A(e)$, as in Figure 2.1.

These distributions are only finitely additive, because no convergence over infinite sums can be guaranteed. They can be seen as measures of finite unions of sets of the form $U(e)$. In particular, by property (2), the "measure" of the entire space is $\mu_c(\Lambda_\Gamma) = 0$. It was found by Teitelbaum that one can use the integration theory of Amice-Vélu and Vishik to integrate with respect to such distributions, and to construct a Poisson kernel for modular forms:

Theorem 2.3.6 (Teitelbaum [79]). *For any Schottky group* Γ *in* $\mathrm{PGL}(2,k)$ *and any even positive integer* n*, there are explicit back and forth isomorphisms* $(\mathrm{Poisson}(n), \mathrm{Res}(n))$*:*

$$\begin{tikzcd} \mathrm{C}_{\mathrm{har}}(\Gamma, n) \arrow[r, bend left, "\mathrm{Poisson(n)}"] & H^0(X_\Gamma, \Omega_{X_\Gamma}^{\otimes n/2}) \arrow[l, bend left, "\mathrm{Res}(n)"] \end{tikzcd}$$

of K*-vector spaces.*

The maps are described explicitly as follows: in ([72], p. 221, [79] p. 397) Schneider and Teitelbaum define:

$$\mathrm{S}_{2n}(\Gamma) \to \mathrm{C}_{\mathrm{har}}(\Gamma, 2n) \colon f \mapsto c_f$$

with

$$c_f(e)(X) = \sum_{i=0}^{2n-2} \mathrm{Res}_e(z^i f dz) \binom{2n-2}{i} X^i,$$

where the *residue* $\mathrm{Res}_e(\omega)$ is the usual rigid analytic residue of a differential form along the annulus defined by the edge e. The inverse of this mapping is the Poisson integral of Teitelbaum in ([79], part 2):

$$f(z) = \int_{\Lambda_\Gamma} \frac{1}{(z-x)} d\mu_c(x).$$

We finish this section with the proof of the first part of our theorem.

Proof of Theorem A, part (i)*=Theorem 2.1.1, part* (i) *.* Let Γ_1 and Γ_2 be Schottky groups of the same genus. Let $\alpha : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ be a group isomorphism. By Corollary 2.2.6 there is an α-equivariant boundary morphism

$$\varphi : \Lambda_{\Gamma_1} \to \Lambda_{\Gamma_2}.$$

By Theorem 2.2.7, φ is absolutely continuous with respect to the respective Patterson–Sullivan measures if and only if the special fibers of the minimal models, that is the quotient graphs $\Gamma_1 \backslash T(\Lambda_{\Gamma_1})$ and $\Gamma_2 \backslash T(\Lambda_{\Gamma_2})$, are isomorphic. Moreover, there is then an isometry $F : T(\Lambda_{\Gamma_1}) \to T(\Lambda_{\Gamma_2})$. This isometry induces a K-linear isomorphism by pullback via F^{-1}:

$$F_* : \mathrm{C}_{\mathrm{har}}(\Gamma_1, 2) \to \mathrm{C}_{\mathrm{har}}(\Gamma_2, 2), c \mapsto c \circ F^{-1}.$$

□

Remark 2.3.7. It is *not* true that pullback by F^{-1} induces an isomorphism of higher weight harmonic cocycles for Γ_1 and Γ_2 — the problem is in the equivariance in general weight ℓ, which means invariance in weight 2. As we will see in the next section, this is exactly the key to formulating a measure-theoretic criterion for isomorphism.

2.4 Rigidity for Mumford curves

An important consequence of the theorem of Teitelbaum is that we can define a *product* of harmonic cocycles, simply by postulating the commutativity of the diagram (1.1) from the introduction. This multiplication can be used to translate multiplicative properties of modular forms / differential forms into those of harmonic measures.

The multiplicative properties of differential forms and how they relate to the isomorphism type of the curve are the subject of classical theorems of Noether, Babbage [7], Enriques-Chisini and Petri [66] (see [30] for a detailed historical overview). We refer to Saint-Donat [71] or [6] III.3 for a modern proof over arbitrary algebraically closed ground fields (which is the setting that we will need).

Suppose now that $X = X_\Gamma$ is not hyperelliptic. Then Ω_X is normally generated, by a theorem of Noether, so the natural multiplication map

$$\mu_n\colon \mathrm{Sym}^n H^0(X,\Omega_X) \to H^0(X,\Omega_X^{\otimes n})\colon f_1\otimes\cdots\otimes f_k \mapsto f_1\cdot\dots\cdot f_k$$

is surjective for all $n \geq 1$; Ω_X is base point free and we have the canonical embedding

$$\varphi_X\colon X \to \mathbf{P}|\Omega_X| = \mathbf{P}^{g-1}.$$

Now $\ker(\mu_n)$ (for varying n) generate the ideal I_{Ω_X} that defines a canonical embedding of X, and the theorem of Babbage–Chisini–Enriques–Petri and Saint-Donat says the following:

Theorem 2.4.1. *For non-hyperelliptic X, the ideal I_{Ω_X} is spanned by $\ker(\mu_n)$ for $n = 2$, unless one of the following cases occurs:*

(a) *if $g = 3$, the ideal is generated by the quartic that generates $\ker(\mu_4)$;*
(b) *if $g = 6$ and the curve is isomorphic to a plane quintic, the ideal is generated by $\ker(\mu_n)$ for $n \leq 3$, but is not generated by $\ker(\mu_2)$ only;*
(c) *if X is trigonal (i.e., admits a degree 3 morphism to $\mathbf{P}^1$), then the ideal is generated by $\ker(\mu_n)$ for $n \leq 3$, but is not generated by $\ker(\mu_2)$ only.* □

Proof of Theorem A, part (ii)=*Theorem 2.1.1, part* (ii). Suppose that we have two Schottky groups Γ_1 and Γ_2, corresponding to non-hyperelliptic Mumford curves X_1 and X_2. From part (i) of the theorem, we have a K-linear isomorphism of vector spaces

$$F_*\colon \ \mathrm{C}_{\mathrm{har}}(\Gamma_1, 2) \to \mathrm{C}_{\mathrm{har}}(\Gamma_2, 2).$$

Recalling the isomorphisms of K-vector spaces

$$H^0(X_1, \Omega_{X_1}) \cong \mathrm{S}_2(\Gamma) \cong \mathrm{C}_{\mathrm{har}}(\Gamma, 2)$$

for $\Gamma = \Gamma_i$, the map F_* can be considered as a linear isomorphism

$$H^0(X_1, \Omega_{X_1}) \xrightarrow{\sim} H^0(X_2, \Omega_{X_2}),$$

which extends to a map of the symmetric algebra

$$F_*\colon \bigoplus_{n\geq 0} \mathrm{Sym}^n H^0(X_1, \Omega_{X_1}) \xrightarrow{\sim} \bigoplus_{n\geq 0} \mathrm{Sym}^n H^0(X_2, \Omega_{X_2}),$$

and hence to an isomorphism

$$\mathbf{P}|\Omega_{X_1}| \to \mathbf{P}|\Omega_{X_2}|.$$

The map induces an isomorphism between X_1 and X_2 precisely if it respects the ideal of relations in the canonical embeddings, i.e., if

$$F_*(\ker(\mu_n^1)) \subseteq \ker(\mu_n^2) \text{ for all weights } n.$$

Using the Poisson kernel Theorem 2.3.6 of Teitelbaum, we can translate this condition to a condition on harmonic measures, as follows: for any n and $\Gamma = \Gamma_i$, consider the map

$$\begin{aligned} \mu_n'(\Gamma) : \mathrm{Sym}^n\, \mathrm{C}_{\mathrm{har}}(\Gamma, 2) &\overset{\mathrm{Poisson}(2)^{\otimes n}}{\longrightarrow} \mathrm{Sym}^n H^0(X, \Omega_X) \\ &\overset{\mu_n}{\to} H^0(X, \Omega_X^{\otimes n}) \overset{\mathrm{Res(n)}}{\longrightarrow} \mathrm{C}_{\mathrm{har}}(\Gamma, 2n). \end{aligned} \tag{2.3}$$

Then the isomorphism of X_1 and X_2 is equivalent to

$$F_*(\ker(\mu_n'(\Gamma_1))) = \ker(\mu_n'(\Gamma_2)) \text{ for all } n \geq 0.$$

Now Theorem 2.4.1 implies that we only need to require this for the given n in the given cases. Since X_1 an X_2 are isomorphic over K and defined by Schottky groups Γ_1, Γ_2 in $\mathrm{PGL}(2, k)$, the groups Γ_1 and Γ_2 are conjugate in $\mathrm{PGL}(2, k)$ (Mumford, [63] Corollary 4.11). $\square$

Remark 2.4.2. We do not know how to deal with the case where the curves are hyperelliptic. The reason for this is that the measure-theoretic property from tree-rigidity only gives a natural isomorphism on the space of harmonic cocycles of weight two, but for a hyperelliptic curve, the image of the canonical map is $\mathbf{P}^1$. Trying to use a pluricanonical embedding does not help, since we cannot conclude from tree-rigidity that the tree isometry induces an isomorphism of the corresponding spaces of higher weight harmonic measures.

Remark 2.4.3. One may try to translate the property of being trigonal into a property of the reduction graph or the Schottky group. Matthew Baker [8] has introduced and studied the concept of gonality of a graph, and his results imply in our case that the gonality of the reduction graph of X (for a semistable model in which all components are smooth) bounds above the K-gonality of X. Compare also chapter 3.

Remark 2.4.4. One may wonder how constructive this theorem can be made, in the following sense. Suppose given two Schottky groups of the same genus, given by finite sets of explicit generating matrices in $\mathrm{PGL}(2, k)$. Are the corresponding curves isomorphic? From the point of view of traditional rigidity, the question is equivalent to the Schottky group being conjugate in $\mathrm{PGL}(2, k)$; but we do not know an efficient algorithm for k an infinite (recursive) field, and the general subgroup conjugacy problem is believed to be computationally hard (compare e.g., [68]). From our point of view of measure theoretic rigidity, one would have to construct a basis for the spaces of harmonic measures, find the matrix that represents F on them, and then compute kernels of multiplication maps. The computability of the latter depends on the computability of the maps $(\mathrm{Poisson}, \mathrm{Res})$.

Remark 2.4.5. It is well-known that two sets of four distinct points in $\mathbf{P}^1(k)$ can be mapped to each other by a fractional linear transformation if and only if they have the same cross ratio. Now given two sets S, Q of $2g$ ($g \geq 2$) distinct points in $\mathbf{P}^1(k)$, is it possible to find an element $g \in \mathrm{PGL}(2, k)$ such that $g(S) = Q$? Our theorem relates to this problem in the following way. Make a partition P_S of the set S into pairs (s_i^+, s_i^-) for $1 \leq i \leq g$. For every pair, choose a hyperbolic element $\gamma_i \in \mathrm{PGL}(2, k)$ of amplitude 1 and for which s_i^+ (respectively s_i^-) is the attracting (respectively the repelling) fixed point of γ_i. Choose minimal exponents $(n_1, ..., n_g)$ such that

$$\Gamma(\mathrm{P}_S) = \langle \gamma_1^{n_1}, ..., \gamma_g^{n_g} \rangle,$$

is a Schottky group (to find these minimal exponents, one may for example use Remark 2.2.2). Since all data needed is of purely metric nature on the tree, these minimal exponents are invariant under conjugation. Now the following holds: for S and Q, there is an element $g \in \mathrm{PGL}(2,k)$ such that $g(S) = Q$ if and only if there is a partition P_S and a partition P_Q such that $g\Gamma(\mathrm{P}_S)g^{-1} = \Gamma(\mathrm{P}_Q)$, which means

$$X_{\Gamma(\mathrm{P}_S)} \cong X_{\Gamma(\mathrm{P}_Q)}.$$

For this, our main theorem gives a purely measure theoretic reformulation (to be checked on the finitely many choices of partitions).

Chapter 3

A combinatorial Li–Yau inequality and rational points on curves.

'Er moet een reden zijn' kreunde hij. 'Dat moet!'
De eekhoorn zei niets meer.
'Ik ga net zo lang denken totdat ik een reden weet,' riep de olifant.
'En als ik geen reden kan bedenken dan probeer ik het nog een keer en dan loop ik dwars door die eik heen let maar op!'

Toon Tellegen [80]

We present a method to control gonality of non-Archimedean curves based on graph theory.

Let k denote the fraction field of an excellent discrete valuation ring. We first prove a lower bound for the gonality of a curve over the algebraic closure of k in terms of the minimal degree of a class of graph maps, namely: one should minimize over all so-called finite harmonic graph morphisms to trees, that originate from any *refinement* of the dual graph of the stable model of the curve.

Next comes our main result: we prove a lower bound for the degree of such a graph morphism in terms of the first eigenvalue of the Laplacian and some "volume" of the original graph; this can be seen as a substitute for graphs of the Li–Yau inequality from differential geometry, although we also prove that the strict analogue of this conjecture fails for general graphs.

Finally, we apply the results to give a lower bound for the gonality of arbitrary Drinfeld modular curves over finite fields and for general congruence subgroups Γ of $\Gamma(1)$ that is linear in the index $[\Gamma(1) : \Gamma]$, with a constant that only depends on the residue field degree and the degree of the chosen "infinite" place. This is a function field analogue of a theorem of Abramovich for classical modular curves.

This chapter is based on [27], 'A Combinatorial Li–Yau inequality and rational points on curves', joint work with Gunther Cornelissen and Fumiharu Kato

3.1 Extension of covering maps

3.1.1. Let R be an excellent discrete valuation ring with uniformizer π and residue field of characteristic $p \geq 0$; and let $k = \mathrm{Frac}(R)$ denote its field of fractions. For any R-scheme $\mathscr{X}$ we denote by $\mathscr{X}_\eta$ (respectively $\mathscr{X}_0$) the generic fiber (respectively the closed fiber). We denote by $\mathscr{X}_{\mathrm{sing}}$ the singular locus of $\mathscr{X}$.

3.1.2. Let X be a geometrically connected projective smooth curve over k. An R-*model* of X is a pair $\mathscr{X} = (\mathscr{X}, \phi)$ consisting of an integral normal scheme $\mathscr{X}$ that is projective and flat over R and a k-isomorphism $\phi\colon \mathscr{X}_\eta \xrightarrow{\sim} X$. An R-model $\mathscr{X}$ of X is said to be *semi-stable* if its special fiber $\mathscr{X}_0$ is reduced with only ordinary double points as singularities. Such a model is called *stable* if any irreducible component of the special fiber has a finite automorphism group as a marked curve, where the marking is given by its intersection points with other components.

3.1.3. As was shown by Liu and Lorenzini in [53], every finite morphism $f\colon X \to Y$ between geometrically connected projective smooth curves over k extends to a morphism between the stable models of X and Y, but the resulting map is *not necessarily finite*. Similar problems were already encountered and studied by Abhyankar in [1]. This problem occurs “in nature”: for example, it follows from the results in Edixhoven’s thesis [32] that the natural map of modular curves $X_0(p^2) \to X_0(p)$ cannot be extended to a finite morphism of stable models. However, there exists a semi-stable model admitting an extension of the map that is a finite morphism, as was shown by Coleman [21] and Liu [52]. We need a slightly different statement, that we prove along similar lines as Liu:

Theorem 3.1.4. *Let $f\colon X \to Y$ be a finite morphism between geometrically connected projective smooth curves over k, and $\mathscr{X}$ an R-model of X. Then there exist a finite separable field extension k'/k, semi-stable R'-models $\mathscr{X}'$ and $\mathscr{Y}'$ of $X_{k'}$ and $Y_{k'}$, respectively, over the integral closure R' of R in k', and an R'-morphism $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ such that the following conditions are satisfied*:

(a) *$\mathscr{X}'$ dominates $\mathscr{X}_{R'}$*;
(b) *φ is finite, surjective, and extends $f_{k'}$*;
(c) *the induced morphism $\varphi_0\colon \mathscr{X}'_0 \to \mathscr{Y}'_0$ satisfies*

$$\varphi_0^{-1}((\mathscr{Y}'_0)_{\mathrm{sing}}) = (\mathscr{X}'_0)_{\mathrm{sing}}.$$

Proof. The proof is a slight modification of the proof of Proposition 3.8 in [52]. We first prove the theorem in the special case where f is a finite Galois covering. Let G be the Galois group of f. Then, replacing k by a finite separable extension if necessary, X has a semi-stable model $\mathscr{X}''$ that dominates $\mathscr{X}$ and admits an extension of the G-action (see Corollary 2.5 in [52]). We want to modify this to a semi-stable model with *inversion-free* action, as follows. Suppose an element $\sigma \in G$ of order two interchanges two components C_1 and C_2 (possibly $C_1 = C_2$) intersecting at a node u. Then we blow-up $\mathscr{X}''$ at the closed point u; we do this at all such nodes. The exceptional curves have multiplicity two. Then we replace k by a ramified quadratic extension k', and take the normalization to obtain a model $\mathscr{X}'$ of $X_{k'}$; it is clear that the G-action extends to $\mathscr{X}'$. The quotient $\mathscr{Y}' = \mathscr{X}'/G$ is a semi-stable model of $Y_{k'}$ (see Proposition 1.6 in [53]), and the quotient map $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ has the desired properties; we postpone the verification of property (c).

Next, we treat the case where f is separable. Let $\widetilde{X}$ denote the Galois closure of $f\colon X \to Y$. Then, replacing k by a finite separable extension if necessary, we may assume that $\widetilde{X}$ is smooth over k. As in the proof of Proposition 3.8 in [52], replacing k furthermore by a finite separable extension if necessary, one has a semi-stable model $\widetilde{\mathscr{X}}$ of $\widetilde{X}$ that dominates $\mathscr{X}$ and admits an extension of the action of $G = \mathrm{Gal}(\widetilde{X}/Y)$. As in the first part, we modify $\widetilde{\mathscr{X}}$ to an inversion-free semi-stable model $\widetilde{\mathscr{X}'}$ (after replacing K by a finite separable extension). Then the obvious map

$$\varphi\colon \mathscr{X}' = \widetilde{\mathscr{X}'}/H \to \mathscr{Y}' = \widetilde{\mathscr{X}'}/G,$$

where $H = \mathrm{Gal}(\widetilde{X}/X)$, gives the desired model of f, as we will see soon below.

In general, we decompose f into a finite separable $X \to Z$ followed by a purely inseparable Frobenius map $Z \to Y \cong Z^{(p^r)}$(see Proposition 3.5 in [52]). The first part $X \to Z$ of the decomposition has an R'-model $\mathscr{X}' \to \mathscr{Z}'$ obtained as above. Setting $\mathscr{Y}' = \mathscr{Z}'^{(p^r)}$, we find that the composite map

$$\varphi\colon \mathscr{X}' \to \mathscr{Z}' \to \mathscr{Y}'$$

gives the answer.

The R'-morphism $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ thus obtained has properties (a) and (b). In order to show that (c) holds, it suffices to show that neither of the following two situations occurs:

(i) there exists a double point u of $\mathscr{X}'_0$ that is mapped to a smooth point of $\mathscr{Y}'_0$;

(ii) there exists a smooth point u of $\mathscr{X}_0'$ that is mapped to a double point of $\mathscr{Y}_0'$.

One can see from the construction (due to the 'inversion-free' nature) above that the situation (i) does not occur. Finally, situation (ii) is also excluded due to Proposition 1.6 in [53]. □

3.2 Graphs and their stable gonality

3.2.1. Let G be a connected finite graph. In this chapter, a graph can have multiple edges (this is sometimes called a "multigraph", but we will not use this terminology). We denote the sets of vertices and edges by $\mathrm{V} = \mathrm{V}(G)$ and $\mathrm{E} = \mathrm{E}(G)$, respectively. We denote by $|G|$ the cardinality $|\,\mathrm{V}(G)|$ of the vertex set. By $\mathrm{E}(x, y)$ we denote the set of edges connecting two vertices $x, y \in \mathrm{V}(G)$, and more generally, for two subsets $A, B \subseteq \mathrm{V}$, we denote by $\mathrm{E}(A, B)$ the set of edges in G that connect elements from A to elements from B:

$$\mathrm{E}(A, B) = \bigcup_{x \in A \wedge y \in B} \mathrm{E}(x, y).$$

Our graphs are, unless clearly indicated, undirected, i.e., $\mathrm{E}(x, y) = \mathrm{E}(y, x)$. In case we have an oriented edge we will write (x, y) for an edge with source x and target y. The set of edges incident to a given vertex x is denoted by E_x. The number of edges in E_x where edges in $\mathrm{E}(x, x)$ are counted with multiplicity two is called the *degree* or *valancy* of x, and is written d_x. A graph is called *k-regular* if $d_x = k$ for all $x \in \mathrm{V}$. A graph is called *loopless* if $|\,\mathrm{E}(x, x)| = 0$ for all $x \in V$. Two vertices x, y are called *adjacent* if $|E(x, y)| \geq 1$, and we denote it by $x \sim y$. For a subset $S \subset V$ the *volume* is defined to be

$$\mathrm{vol}(S) = \sum_{v \in S} d_v.$$

In particular, $\mathrm{vol}(G) = 2 \cdot |\,\mathrm{E}\,|$.

Another important invariant of a graphs is the *genus*, by which we mean the first Betti number $g(G) = |\,\mathrm{E}\,| - |\,\mathrm{V}\,| + 1$. Note that this differs from another convention in graph theory in which "genus" means the minimal genus of a Riemann surface in which the graph can be embedded without self-intersection. A graph of genus 0 is called a *tree*.

Functions $f : \mathrm{V} \to \mathbf{R}$, are simply called "functions on G". These form a finite dimensional vector space, equipped with the standard inner product

$$\langle f, g \rangle = \sum_{v \in \mathrm{V}(G)} f(v)g(v).$$

3.2.2. Denote by $A = A_G$ the adjacency matrix of a connected graph G (of which the (x, y)-entry is $|E(x, y)|$) and with $D = D_G$ the diagonal matrix with the degrees of the vertices on the diagonal. Then the Laplace operator is defined by $L = L_G = D - A$.

For any graph, L_G is a real symmetric positive-semidefinite matrix, and therefore has non-negative real eigenvalues. The function $\mathbb{1}$, defined as being identically equal to 1 on V, is an eigenfunction of L_G with eigenvalue 0. The other eigenvalues are positive. We order the eigenvalues

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 ... \leq \lambda_{n-1},$$

where n is the number of vertices of the graph. It is the first non-zero eigenvalue which is important for us; we denote it by $\lambda_G := \lambda_1$.

3.2.3. Sometimes, one uses the *normalized Laplacian* of G, defined as

$$L_G^{\sim} = D_G^{-1/2} L_G D_G^{1/2}$$

weighted by vertex degrees (compare Chung [19]).

Definition 3.2.4. A graph G is called *stable* if all vertices have degree at least 3. A graph G' is called a *refinement* of G if it can be obtained from G by performing subsequently finitely many times one of the two following operations:

1. subdivision of an edge,
2. addition of a *leaf*, i.e., the addition of an extra vertex and an edge between this vertex and a vertex of the already existing graph.

Remark 3.2.5. One of the main tools in this chapter is the notion of harmonic morphisms of graphs as developed by Urakawa [82] and Baker and Norine [9], and later generalized to harmonic indexed morphisms by Caporaso [16]. We will use a terminology that is compatible with that of [4], and different from [16], and we will only consider "unweighted" graphs in the sense of [16]. In the appendix, we will discuss the relations between different notions of gonality for graphs.

Definition 3.2.6. Let G, G' be two loopless graphs.

1. A *finite morphism between G and G'* (denoted by $\varphi : G \to G'$) is a map

$$\varphi\colon\ \mathrm{V}(G) \cup \mathrm{E}(G) \to \mathrm{V}(G') \cup \mathrm{E}(G')$$

such that $\varphi(\mathrm{V}(G)) \subset \mathrm{V}(G')$ and for every edge $e \in \mathrm{E}(x, y)$, $\varphi(e) \in \mathrm{E}(\varphi(x), \varphi(y))$, together with, for every $e \in \mathrm{E}(G)$, a positive integer $r_\varphi(e)$, the *index* of φ at e.
2. A finite morphism is called *harmonic* if for every $v \in \mathrm{V}(G)$ there exists a well-defined number, $m_\varphi(v)$, such that for every $e' \in \mathrm{E}_{\varphi(v)}(G')$ we have

$$m_\varphi(v) = \sum_{e \in \mathrm{E}_v, \varphi(e) = e'} r_\varphi(e).$$

3. For a finite harmonic morphism the following number, which is called the *degree* of φ, is independent of $v' \in \mathrm{V}(G')$ or $e \in \mathrm{E}(G')$:

$$\deg \varphi = \sum_{v \in \varphi^{-1}(v')} m_\varphi(v) = \sum_{e \in \varphi^{-1}(e')} r_\varphi(e).$$

From the perspective of this chapter, it is natural to define the following notion of gonality (this is different from existing notions of gonality, but we will discuss these in the appendix).

Definition 3.2.7. A graph G is called *stably d-gonal* if it has a refinement that allows a degree d finite harmonic morphism to a tree. The *stable gonality* of a graph G is defined to be

$$\operatorname{sgon}(G) = \min\{\deg \varphi \mid \varphi\colon G' \to T\}$$

with G' a refinement of G and φ a finite harmonic morphism to a tree T.

Remark 3.2.8. Although finite harmonic morphisms are defined only for loopless graphs, stable gonality is defined for all graphs, as loops can be "refined away" by subdividing the loop edges.

Example 3.2.9. The "banana graph" B_n (see Figure 3.1) given by two vertices joined by $n > 1$ distinct edges is the intersection dual graph of two rational curves intersecting in n points. The minimal degree of a finite harmonic morphism from B_n to a tree is n. However, if we subdivide each edge once, the resulting graph admits such a finite harmonic morphism of degree 2 to a tree, which is a vertex with n edges sticking out (by identifying the two original vertices). Hence the banana graph has stable gonality equal to 2. This is compatible with the fact that the banana graph can be the intersection dual graph of both hyperelliptic and non-hyperelliptic (if $n > 3$) curves, and these are not distinguished by all subdivisions of their reduction graph.

This example occurs in nature as the stable reduction of the modular curve $X_0(p)$ over $\mathbf{Q}_p$, where n is then the number of supersingular elliptic curves modulo p. One should observe ([8], 3.6) that stable reduction graphs are naturally *metric* graphs, and as such, the stable reduction graph of $X_0(p)$ is only equal to the (unit-length metrized) banana graph for $p = 1 \bmod 12$ with $n = (p-1)/12$.

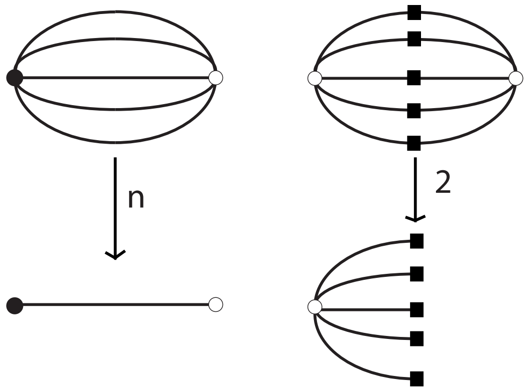


Figure 3.1: A banana graph B_n with a finite harmonic morphism of (minimal) degree n, and its subdivision, with a finite harmonic morphism of degree 2 (all indices are 1).

Example 3.2.10. The minimal degree of a finite harmonic morphism from the complete graph K_4 to a tree is 4, but by adding leaves, such a morphism of degree 3 can be constructed, see Figure 3.2.

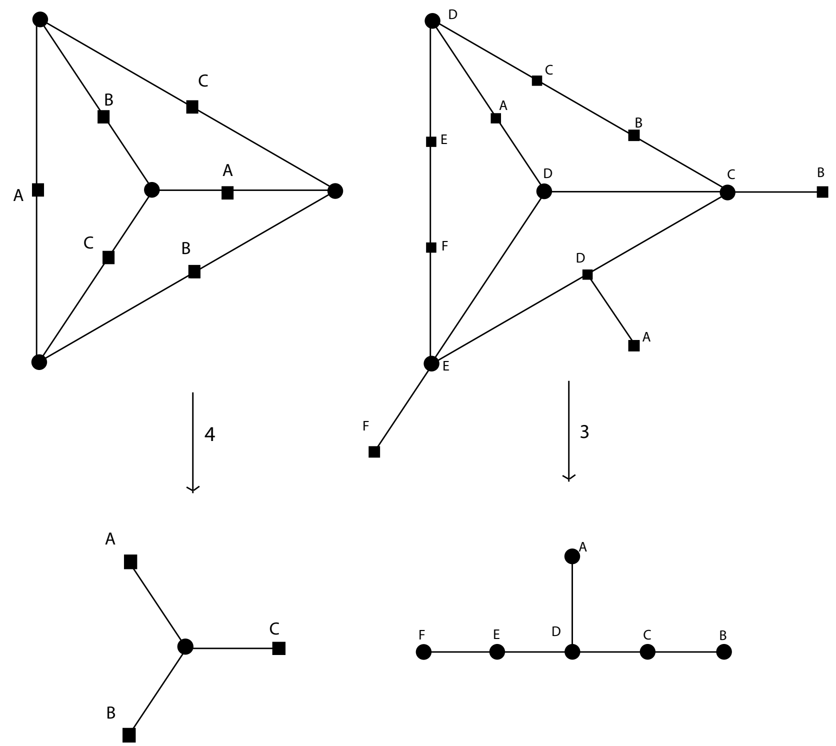


Figure 3.2: A subdivision of K_4 with a finite harmonic morphism of degree 4, and a refinement (with leaves) of K_4 with a finite harmonic morphism of degree 3 (all indices are 1).

3.3 Comparing curve gonality and graph gonality

3.3.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be R-models of geometrically connected projective smooth curves over k, and $\varphi\colon \mathscr{X} \to \mathscr{Y}$ an R-morphism. Let us say φ is *inversion-free semi-stable* if the following conditions are satisfied:

(a) $\mathscr{X}$ and $\mathscr{Y}$ are semi-stable;
(b) φ is finite and surjective;
(c) $\varphi_0^{-1}((\mathscr{Y}_0)_{\mathrm{sing}}) = (\mathscr{X}_0)_{\mathrm{sing}}$.

Theorem 3.1.4 says that any finite cover $f\colon X \to Y$ between geometrically connected projective smooth curves over k admits, after replacing k by a finite separable extension, an inversion-free semi-stable model f; moreover, given an arbitrary R-model $\mathscr{X}$ of X, we can take such an R-model $\varphi\colon \mathscr{X}' \to \mathscr{Y}'$ of f such that $\mathscr{X}'$ dominates $\mathscr{X}$.

3.3.2. Let $\varphi\colon \mathscr{X} \to \mathscr{Y}$ be an inversion-free semi-stable model of f. Consider the dual graphs $\Delta := \Delta(\mathscr{X}_0)$ and $\Gamma = \Delta(\mathscr{Y}_0)$ of the special fibers of $\mathscr{X}_0$ and $\mathscr{Y}_0$, respectively. The vertices of Δ (respectively Γ) correspond to irreducible components of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$), and two of them are connected by an edge if and only if they intersect.

The morphism φ induces the following two set-theoretic maps:

- since φ is finite, it maps each component of $\mathscr{X}_0$ surjectively onto a component of $\mathscr{Y}_0$; in particular, it induces a map $\mathrm{V}(\Delta) \to \mathrm{V}(\Gamma)$ between the sets of vertices of the graphs;
- due to the condition (c) above, each double point of $\mathscr{X}_0$ is mapped to a double point of $\mathscr{Y}_0$; that is, we have the map $\mathrm{E}(\Delta) \to \mathrm{E}(\Gamma)$ between the sets of edges.

Thus we obtain a graph map $\phi\colon \Delta \longrightarrow \Gamma$.

3.3.3. We now assume that f is *separable*, and define the index r_ϕ for such f. Let $e \in \mathrm{E}(\Delta)$ be an edge with extremities $v, v' \in \mathrm{V}(\Delta)$. Let C, C' (respectively D, D') be the components of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$) corresponding to v, v' (respectively $\phi(v), \phi(v')$), respectively. The maps $C \to D$ and $C' \to D'$ ramify at the intersection point u with the same decomposition group; then define $r_\phi(e)$ to be the order of this group. In this way, ϕ becomes a finite morphism of graphs in the sense of Definition 3.2.6.

Proposition 3.3.4. *For a separable $f\colon X \to Y$ that admits an inversion-free semi-stable model $\varphi\colon \mathscr{X} \to \mathscr{Y}$, the finite graph morphism $\phi\colon \Delta \to \Gamma$ constructed above is harmonic of degree* $\deg(f)$ (*in the sense of Definition 3.2.6*).

Proof. Let $v \in \mathrm{V}(\Delta)$ be a vertex, and C (respectively D) the component of $\mathscr{X}_0$ (respectively $\mathscr{Y}_0$) corresponding to v (respectively $\phi(v)$). Let $m_\phi(v)$ be the degree of the covering map $C \to D$. Then for any edge $e' \in \mathrm{E}(\Gamma)$ emanating from $\phi(v)$, we have

$$m_\phi(v) = \sum_{\phi(e)=e'} r_\phi(e),$$

and hence ϕ is harmonic of degree $\deg(f)$. □

Corollary 3.3.5 (=Theorem B)**.** *Let X be a geometrically connected projective smooth curve over k, and $\mathscr{X}$ the stable R-model of X, and let $\Delta(\mathscr{X}_0)$*

denote the intersection dual graph of the special fiber. Let $\overline{k}$ be an algebraic closure of k. Then we have

$$\operatorname{gon}_{\overline{k}}(X) \geq \operatorname{sgon}(\Delta(\mathscr{X}_0)).$$

Proof. Gonality is the minimal degree of a map $f\colon X \to \mathbf{P}^1$. Since we work over an algebraically closed field, we can decompose such a map into a separable part $f\colon X \to Z$ and a purely inseparable part $Z \to Z^{(p^r)} \cong \mathbf{P}^1$. Since the genus is preserved by the purely inseparable part, we find that $Z \cong \mathbf{P}^1$, too, and hence the separable part of a general map is a map of lower degree to $\mathbf{P}^1$. Hence we can restrict to bounding the degree of a separable f.

The assertion now follows from Proposition 3.3.4 and the following auxiliary observations.

(1) By Theorem 3.1.4, for any given finite cover $f\colon X \to \mathbf{P}^1_k$, replacing k by a finite separable extension, one has an inversion-free semi-stable model $\varphi\colon \mathscr{X}' \to \mathscr{P}'$ of f such that $\mathscr{X}'$ dominates $\mathscr{X}$. In particular, $\Delta(\mathscr{X}_0')$ gives a graph that arises from $\Delta(\mathscr{X}_0)$ by subdividing some edges (corresponding to blowing up nodes) and/or adding some leaves (corresponding to blowing up smooth points) — this is exactly the notion of refinement as we have defined it.

(2) By replacing the base field k by an arbitrary finite extension k', the base-change $\mathscr{X}_{R'}$, where R' is the integral closure of R in k', is a semi-stable model of $X_{k'}$ (see Section 1.5 in [53]), which obviously gives the same dual graph as $\Delta(\mathscr{X}_0)$. $\square$

3.4 The Brill–Noether bound for stable gonality of graphs

One can use this comparison theorem to prove the analog for stable gonality of graphs of the upper bound for the gonality of curves given by Brill–Noether theory: a curve of genus g over an algebraically closed field has gonality bounded above by $\lfloor (g+3)/2 \rfloor$; this was proven in general by Kleiman and Laksov [48]. To prove this for graphs, we first show that finite harmonic morphisms can be "refined", in a sense to be made precise.

Definition 3.4.1. For any two refinements G_1 and G_2 of a graph G let $G_1 \vee G_2$ be the set of all common refinements of G_1 and G_2.

Definition 3.4.2. A refinement G' of a graph G induces refinements of all of its subgraphs. If $e \in \mathrm{E}(v,w)$ is an edge in G that connect two vertices $v, w \in \mathrm{V}(G)$, denote by $[e]$ the subgraph of G consisting of the vertices v

and w joined by the edge e. Similarly, we denote for a vertex v, with $[v]$ the subgraph which consists of v only. Denote with $G'[x]$ the refinement of $[x]$ in G', and for an edge $e \in \mathrm{E}(v, w)$ denote with $RG'[e]$ the restricted refinement:

$$RG'[e] = G'[e] - (G'[v] - [v]) - (G'[w] - [w]).$$

Definition 3.4.3. A *refinement of a finite harmonic morphism* $\varphi : G \to T$ is a finite morphism

$$\varphi' : G' \to T'$$

such that G' (respectively T') is a refinement of G (respectively T), and such that

1. for all $v \in \mathrm{V}(G)$, $\varphi'(v) = \varphi(v)$;
2. for any $v, w \in \mathrm{V}(G)$ and any edge $e \in \mathrm{E}(v, w)$, every refinement of $[e]$ in G' is mapped to the refinement of $[\varphi(e)]$ in T', viz.,

$$\varphi(G'[e]) = T'[\varphi(e)];$$

3. for every $e \in \mathrm{E}(G)$ and for all $e' \in RG'[e]$, the index $r_{\varphi'}(e') = r_\varphi(e)$;
4. for every $v \in \mathrm{V}(G)$ and for all $e' \in G'[v]$, the index $r_{\varphi'}(e') = m_\varphi(v)$;

It follows that φ is a finite harmonic morphism and $\deg \varphi = \deg \varphi'$.

Lemma 3.4.4. *Let $\varphi : G \to T$ be a finite harmonic morphism. Then*

(i) *for any refinement T' of T, there exists a refinement $\varphi' : G' \to T'$ of φ;*
(ii) *for any refinement H of G, there exists a refinement $\varphi' : G' \to T'$ of φ, such that G' is a refinement of H, and T' is a refinement of T.*

Proof. For part (i), use the following recipe:

1. replace every edge e in G by $T'[\varphi(e)]$;
2. put indices such that conditions (3) and (4) in Definition 3.4.3 are satisfied;
3. extend φ in the obvious way.

For part (ii), first choose for every edge e_0 in T an element in

$$\bigvee_{e \in \varphi^{-1}(e_0)} H([e]),$$

and replace e_0 by this common refinement. Call the resulting new tree T', and then apply part (i). □

Example 3.4.5. In Figure 3.3, one sees a finite harmonic morphism $G \to T$ on the left, where all edges have index 1, except the indicated edge that has index two. The middle picture is a refinement H of the original graph G, and the right hand picture shows the refinement $G' \to T'$ as constructed in Lemma 3.4.4. Both morphisms have degree 3.

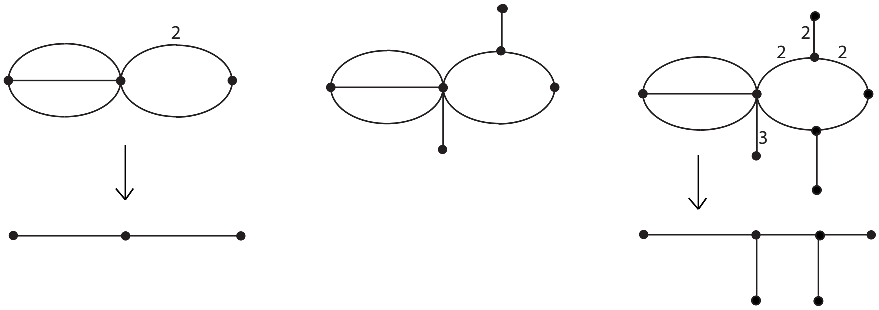


Figure 3.3: Left: a morphism $G \to T$; middle: a refinement H of G; right: a refinement $G' \to T'$ with G' a refinement of H (only indices > 1 are depicted)

Corollary 3.4.6. *Call two graphs* equivalent *if they are refinements of the same stable graph. This defines an equivalence relation on the set of all graphs of genus at least* 2*. The map* sgon *is defined on equivalence classes of graphs.*

Proof. Let G' be a refinement of G. It follows directly from the definition that $\mathrm{sgon}(G') \geq \mathrm{sgon}(G)$. Since refinement of morphisms preserves degree, the previous lemmas imply that the other inequality $\mathrm{sgon}(G') \leq \mathrm{sgon}(G)$ also holds. □

Theorem 3.4.7 (= Theorem C)**.** *For any graph G of genus $g \geq 2$, the Brill–Noether bound holds:*

$$\mathrm{sgon}(G) \leq \lfloor \frac{g+3}{2} \rfloor.$$

Proof. Since sgon is defined on the equivalence classes of graphs it is sufficient to prove the bound for one representative of each equivalence class. It is sufficient to show that any stable graph G of genus $g \geq 2$ admits a refinement G' such that there exists a curve X such that G' is the dual graph of the minimal model of X. Indeed, since the genus of X equals the genus of G', which equals g (since the genus of a graph doesn't change under refinement), the classical bound $\mathrm{gon}_{\overline{k}}(X) \leq \lfloor (g+2)/3 \rfloor$ holds (cf. Kleiman–Laksov [48]). The result follows from $\mathrm{sgon}(G') \leq \mathrm{gon}_{\overline{k}}(X)$ (Theorem B).

We now show the existence of such a refinement. Let G be a stable graph of genus $g \geq 2$ and let $\Delta_G = \max\{d_x | x \in \mathrm{V}(G)\}$. Choose g edges $e_1, ..., e_g$ of G such that $G - \{e_1, ..., e_g\}$ is a tree. Replace each edge e_i (connecting two vertices x_i and y_i) by two edges $[x_i, v_i]$ and $[w_i, y_i]$, where v_i and w_i are new vertices not connected to any other vertex. In this way, G is replaced by a tree T_G. Choose an embedding of T_G in the Bruhat-Tits tree $\mathcal{T}$ for $k = \mathbf{F}_q((t))$, where q satisfies $q + 1 \geq \Delta_G$. Denote the images of v_i and w_i in $\mathcal{T}$ by the same letters. Now choose hyperbolic elements $\gamma_1, ..., \gamma_g$ in $\mathrm{PGL}(2, k)$ such that each γ_i acts as translation along a geodesic through v_i and w_i, and $\gamma_i(v_i) = w_i$. Then $\Gamma = \langle \gamma_1, ..., \gamma_g \rangle$ is a Schottky group. Denote by $\mathcal{T}_\Gamma$ the subtree of $\mathcal{T}$ spanned by the limit set of Γ. Then $G' \simeq \Gamma \backslash \mathcal{T}_\Gamma$, where G' is the refinement of G given by subdividing each of the edges $e_1, ..., e_g$ once. Also, G' is the intersection dual graph of the minimal model of the Mumford curve corresponding to Γ ([63], page 164). $\square$

It would be interesting to have a purely graph theoretical proof of the above result.

Remark 3.4.8. More general lifting results, such as Lemma 6.3 in [70] or Corollary B.3 from [8] with Theorem 3.1.4, also imply the existence of the refinement and the curve.

3.5 A spectral lower bound for the stable gonality of a graph

In this section, we prove an analogue of the Li–Yau bound, viz., a spectral lower bound for the stable gonality of a graph. The basic philosophy of the proof is to find a lower bound for the first Laplace eigenvalue using its variational characterization, in terms of the degree of a finite harmonic morphism $\varphi \colon G' \to T$ and the minimal "size" of the inverse image of the two parts in which the tree gets cut by removing one of its edges. Then a dichotomy occurs: either the minimal such size is large, or there is a vertex with a large inverse image. Initially, "large" depends on the maximal vertex degree of the tree T, but if this degree is too large, we change the refinement and morphism to produce a lower bound that only depends on the original graph, not the morphism or tree itself.

We start by studying such "sizes" on trees abstractly:

Definition 3.5.1. A *measured tree* (T, ν) is a connected tree T with a probability measure ν on $\mathrm{V}(T)$. For an edge $e \in \mathrm{E}(T)$, we decompose $T - e$ into its two connected components $T_1(e)$ and $T_2(e)$:

$$T - e = T_1(e) \bigsqcup T_2(e).$$

We define the *size* of an edge $e \in \mathrm{E}(T)$ (w.r.t. ν) by

$$\mathrm{size}_\nu(e) := \min\{\nu(T_1(e)), \nu(T_2(e))\}.$$

Let $c > 0$. Call a measured tree (T, ν) *c-thick* if for every vertex $x \in \mathrm{V}(T)$, the graph $T - x$ has a connected component of measure at least c.

Lemma 3.5.2. *A c-thick measured tree (T, ν) has an edge of size at least c.*

Proof. For any vertex $x \in V(T)$, choose a connected component C_x of $T - x$ of measure at least c. Orient the unique edge that connects x to a vertex in C_x in the direction of x. By doing this for each vertex, $|T|$ different orientations are assigned to the $|T| - 1$ edges of T. Hence at least one edge of T is oriented in both directions, and such an edge has size at least c. □

Remark 3.5.3. If (T, ν) is not c-thick, then there exists a vertex $x \in \mathrm{V}(T)$ with $\nu(x) > 1 - c\deg(x)$. Indeed, since $T - x$ has $\deg(x)$ components, all of measure less than c, we find that $1 - \nu(x) = \nu(T - x) < c\deg(x)$.

The measure we will use counts vertices of G' that belong to the original graph G:

Definition 3.5.4. Let G denote a graph, and G' a refinement of G. The probability measure μ_G on $\mathrm{V}(G')$ is defined by

$$\mu_G(A) := \frac{|A \cap \mathrm{V}(G)|}{|G|} \text{ for } A \subseteq \mathrm{V}(G').$$

Lemma 3.5.5. *If G' is a refinement of a graph G, and $\varphi\colon G' \to T$ a finite harmonic morphism, then $(T, \varphi_*\mu_G)$ is a measured tree, and for any vertex $x \in \mathrm{V}(T)$, we have*

$$\deg\varphi \geq \varphi_*\mu_G(x) \cdot |G|.$$

Proof. It suffices to remark that

$$\varphi_*\mu_G(x) \cdot |G| = |\varphi^{-1}(x) \cap G| = \sum_{\substack{v \in G \\ \varphi(v) = x}} 1 \leq \sum_{\substack{v \in G \\ \varphi(v) = x}} m_\varphi(v) \leq \deg\varphi.$$

□

The next proposition says that size and degree controls the first eigenvalue of the Laplacian:

Proposition 3.5.6. *If G' is a refinement of a graph G, and $\varphi\colon G' \to T$ a finite harmonic morphism, then for any edge $e \in T$, we have an inequality*

$$\deg\varphi \geq \frac{1}{2}\cdot\lambda_G\cdot\mathrm{size}_{\varphi_*\mu_G}(e)\cdot|G|.$$

Proof. If we let $G_i := \mathrm{V}(G)\cap\varphi^{-1}(\mathrm{V}(T_i(e)))$ for $i = 1, 2$ then the statement to be proven is equivalent to

$$\frac{1}{2}\lambda_G\min(|G_1|, |G_2|) \leq \deg(\varphi).$$

First, note that the inequality is trivial if $\min(|G_1|, |G_2|) = 0$. Now assume the minimum is non-zero. The estimate follows from the variational characterization of λ_G via the Rayleigh-quotient,

$$\lambda_G = \inf_{f\perp\mathbb{1}}\frac{\langle f, Lf\rangle}{\langle f, f\rangle} = \inf_{f\perp\mathbb{1}}\frac{\sum\limits_{u\sim v}(f(u)-f(v))^2}{\sum_v f(v)^2},$$

where notations are as in 3.2.1. We construct an appropriate function f based on the finite harmonic morphism $\varphi\colon G' \to T$ and the removed edge $e \in T$, as follows:

$$f(v) = \begin{cases} \frac{1}{|G_1|} & \text{if } v \in G_1, \\ -\frac{1}{|G_2|} & \text{if } v \in G_2. \end{cases}$$

It is easy to check that $f \perp \mathbb{1}$, and therefore,

$$\begin{aligned}
\lambda_G &\leq \frac{\sum\limits_{u\sim v}(f(u)-f(v))^2}{\sum\limits_v f(v)^2} \\
&= |\mathrm{E}(G_1, G_2)|\left(\frac{1}{|G_1|}+\frac{1}{|G_2|}\right) \\
&\leq \frac{2|\mathrm{E}(G_1, G_2)|}{\min(|G_1|, |G_2|)}
\end{aligned}$$

We finish the proof by showing that $\deg(\varphi) \geq |\mathrm{E}(G_1, G_2)|$. Suppose an edge $e \in \mathrm{E}(G_1, G_2)$ is replaced in G' by a path, possibly of length 1. Let us describe this path as a series of edges $e_1, \ldots, e_n \in \mathrm{E}(G')$, such that e_1 is incident to a vertex in G_1, and e_n is incident to a vertex in G_2. Then for at least one of the e_i it holds that $\varphi(e_i) = e$. The desired inequality follows. $\square$

A lower bound for the degree of φ now follows easily if the map gives a c-thick measured tree:

Corollary 3.5.7. *If G' is a refinement of a graph G, and $\varphi\colon G' \to T$ a finite harmonic morphism such that $(T, \varphi_* \mu_G)$ is c-thick, then*

$$\deg \varphi \geq \frac{c}{2} \cdot \lambda_G \cdot |G|.$$

Proof. Immediate from Lemma 3.5.2 and Proposition 3.5.6. □

Remark 3.5.8. If the tree is *not* c-thick, we know from Remark 3.5.3 that the tree has a vertex x with "large" measure: $\varphi_* \mu_G(x) \geq 1 - c \deg_T(x)$. Putting, for example, $c = 1/(\Delta_T + 1)$, the previous results gives a non-trivial lower bound on the degree of φ in terms of λ_G, $|G|$ and Δ_T of the form

$$\deg(\varphi) \geq \min \left\{ \frac{\lambda_G}{2}, 1 \right\} \frac{|G|}{\Delta_T + 1}.$$

However, we want to find a bound that solely depends on G, not on T. For this, in the next proposition, we engineer another harmonic morphism from a different refinement of G, but whose degree is controlled by that of φ.

Proposition 3.5.9. *Let G denote a graph with maximal degree Δ_G. Let $A, B, C > 0$ be constants such that $A + B + C \leq 1$. If G' is a refinement of G, and $\varphi\colon G' \to T$ a finite harmonic morphism such that*

(i) *$(T, \varphi_* \mu_G)$ is not $(C/2)$-thick; and*
(ii) *all vertices $x \in T$ have measure $\varphi_* \mu_G(x) < B$,*

then there exists a refinement $G^\#$ of G, a tree $T^\#$, and a finite harmonic morphism $\varphi^\# : G^\# \to T^\#$ such that

(a) *$\deg \varphi^\# \leq \Delta_G \deg \varphi$; and*
(b) *there exists an edge $e^\#$ of $T^\#$ with $\mathrm{size}_{\varphi^\#_* \mu_G}(e^\#) \geq A/2$.*

We postpone the proof to the next section, and first discuss the main corollary.

Corollary 3.5.10 (= Theorem D)**.** *Let G be a graph with maximal vertex degree Δ_G and first Laplace eigenvalue λ_G. The stable gonality of G is bounded from below by*

$$\mathrm{sgon}(G) \geq \left\lceil \frac{\lambda_G}{\lambda_G + 4(\Delta_G + 1)} |G| \right\rceil.$$

Proof. Let G be a graph and let $\varphi : G' \to T$ be a finite harmonic morphism, and let A, B, C be as in Proposition 3.5.9. If $(T, \varphi_* \mu_G)$ is $C/2$-thick, then by Corollary 3.5.7, we find

$$\deg \varphi \geq \gamma \cdot |G| \text{ with } \gamma := \frac{C\lambda_G}{4}.$$

On the other hand, if there is a vertex $x \in T$ with $\varphi_* \mu_G(x) \geq B$, then by Lemma 3.5.5, we get

$$\deg \varphi \geq \beta \cdot |G| \text{ with } \beta := B.$$

In the remaining case, Proposition 3.5.9 implies that

$$\deg \varphi \geq \frac{1}{\Delta_G} \deg \varphi^{\#} \geq \frac{\lambda_G}{2\Delta_G} \operatorname{size}_{\varphi_*^{\#} \mu_G}(e^{\#})|G| \geq \alpha \cdot |G| \text{ with } \alpha := \frac{A\lambda_G}{4\Delta_G},$$

where the second inequality follows from Proposition 3.5.6.

Translating the constraints $A, B, C > 0$ and $A + B + C \leq 1$, we conclude that it always holds that

$$\frac{\deg \varphi}{|G|} \geq \max_{\substack{\alpha,\beta,\gamma>0 \\ a\alpha+b\beta+c\gamma\leq 1}} \min(\alpha, \beta, \gamma). \tag{3.1}$$

with $a = 4\Delta_G/\lambda_G$, $b = 1$ and $c = 4/\lambda_G$. The maximum in (3.1) is achieved for $\alpha = \beta = \gamma$ and $a\alpha + b\beta + c\gamma = 1$, and plugging this back into (3.1) gives the result. □

3.6 Proof of Proposition 3.5.9

In the proof of Proposition 3.5.9 we will use the following concept several times to construct new graphs from old:

Definition 3.6.1. Let $H_1, \ldots, H_n$ denote n different graphs and let

$$(v_1^1, \ldots, v_1^n), \ldots, (v_m^1, \ldots, v_m^n) \in \mathrm{V}(H_1) \times \cdots \times \mathrm{V}(H_n)$$

denote m-tuples of their vertices. The graph H obtained by *gluing* H_i along these vertices is defined to be

$$H := \bigsqcup_{i=1}^{n} H_i \Big/ \langle v_1^1 = \cdots = v_1^n, \ldots, v_m^1 = \cdots = v_m^n \rangle.$$

Lemma 3.6.2. *Let $\varphi : G \to T$ be a finite harmonic morphism and let $T_0 \subset T$ be a connected subtree. Then the restriction of φ to any of the connected components of $\varphi^{-1}(T_0)$ is a finite harmonic morphism to T_0.*

Proof. Let T_0 be such a subtree and let $\varphi_0 : G_0 \to T_0$ be the restiction of φ to one of the connected components G_0 of $\varphi^{-1}(T_0)$. Then $m_{\varphi_0}(v)$ is well-defined for all $v \in V(G_0)$, namely, all edges $e \in \mathrm{E}_v(G)$ which are mapped to an edge $e' \in E_{\varphi(v)}(T_0)$ are contained in $\mathrm{E}_v(G_0)$, so that in particular $m_{\varphi_0}(v) = m_\varphi(v)$. $\square$

Proof of Proposition 3.5.9. Write $\nu = \varphi_* \mu_G$. Since (T, ν) is not $C/2$-thick, we can choose a vertex $x_0 \in \mathrm{V}(T)$ such that all components of $T - x_0$ have measure $< C/2$. Let $G^s \subset G'$ be the refinement of G which only comprises the subdivided edges of G in G', and let T^s denote its image $\varphi(G^s)$. Observe that $x_0 \in \mathrm{V}(T^s)$, and let d denote the degree of x_0 in T^s. Also note that the maximal degree of G^s is the same as that of G, i.e., $\Delta_{G^s} = \Delta_G$. Denote by $T_1^s, \ldots, T_d^s$ the connected components of $T^s - x_0$. We divide the connected components into two sets of approximately the same measure. Since $\nu(x_0) < B$ and $\nu(T_i) < C/2$ for all $i = 1, \ldots, d$, it is possible to find a partition $I_L \cup I_R = \{1, \ldots, d\}$, such that

$$\min \left\{ \nu(\bigcup_{i \in I_L} T_i^s), \nu(\bigcup_{i \in I_R} T_i^s) \right\} \geq \frac{1 - \nu(x_0)}{2} - \frac{C}{2} > \frac{1 - B}{2} - \frac{C}{2} \geq \frac{A}{2}.$$

We now show how to construct the different pieces of the new map

$$\varphi^\# : G^\# \to T^\#.$$

The construction of $T^\#$. For each $i = 1, \ldots, d$ let $y_i \in \mathrm{V}(T_i^s)$ be the unique vertex which is adjacent to x_0 in T^s. A new graph $S^\#$ is obtained by gluing all T_i^s together at the y_i, and adding a leaf at the image of the y_i. Call x the new vertex of the added leaf. Take two copies $(S^{\#,1}, x_1)$ and $(S^{\#,2}, x_2)$ of the pair $(S^\#, x)$, and glue them together at x_1 and x_2 to obtain a tree $T^\#$. Let X_0 be the image of x_1 and x_2 in $T^\#$, and call the image of $S^{\#,1}$ in $T^\#$ the "left part" $T_L^\#$, and the image of $S^{\#,2}$ in $T^\#$ the "right part" $T_R^\#$.

The construction of $G^\#$. For each $i = 1, \ldots, d$, let S_i be the subtree of T^s obtained by adding to T_i^s the (unique) edge in $\mathrm{E}(x_0, y_i)$. The subgraph $\varphi^{-1}(S_i) \subset G'$ might be disconnected; let G_i'' be the union of the connected components of $\varphi^{-1}(S_i)$ for which the set of edges has a non-empty intersection with $\mathrm{E}(G^s)$. By Lemma 3.6.2 the restriction

$$\varphi_i'' := \varphi|_{G_i''} : G_i'' \to S_i.$$

is finite harmonic.

Now observe that $S^{\#}$ is a refinement of S_i, so by Lemma 3.4.4 there exists a finite harmonic refinement morphism (in the sense of Definition 3.4.3)

$$\varphi_i^{\#} : G_i^{\#} \to S^{\#},$$

with an inclusion map $\iota_i : G_i'' \to G_i^{\#}$.

Define, for any $v \in \varphi^{-1}(x_0) \cap G^{\#}$, the integer

$$d^{\#}(v) := \sum_{i \in I_L} \left(\sum_{e \in \mathrm{E}_v(G^{\#}) \cap \mathrm{E}(G_i^{\#})} r_{\varphi_i^{\#}}(e) \right) - \sum_{i \in I_R} \left(\sum_{e \in \mathrm{E}_v(G^{\#}) \cap \mathrm{E}(G_i^{\#})} r_{\varphi_i^{\#}}(e) \right).$$

The graph $G^{\#}$ is obtained by gluing all $G_i^{\#}$ together at $\iota_i(v)$ for all $v \in \varphi^{-1}(x_0) \cap G_i''$, and, for any v with $d^{\#}(v) \neq 0$, gluing an additional copy $(S^{\#,v}, x_v)$ of $(S^{\#}, x)$ at v.

The construction of $\varphi^{\#}$. To define $\varphi^{\#}$, it suffices to define its restriction to $G_i^{\#}$ for $i = 1, \ldots, d$, and its restriction to $S^{\#,v}$, and show that these are compatible on intersections. Define the restrictions as follows:

1. If $i \in I_L$, set $\varphi^{\#}|_{G_i^{\#}} : G_i^{\#} \xrightarrow{\varphi_i^{\#}} S^{\#} \xrightarrow{\sim} T_L^{\#}$ with index $r_{\varphi^{\#}}(e) = r_{\varphi_i^{\#}}(e)$ for all $e \in G_i^{\#}$;
2. If $i \in I_R$, set $\varphi^{\#}|_{G_i^{\#}} : G_i^{\#} \xrightarrow{\varphi_i^{\#}} S^{\#} \xrightarrow{\sim} T_R^{\#}$ with index $r_{\varphi^{\#}}(e) = r_{\varphi_i^{\#}}(e)$ for all $e \in G_i^{\#}$;
3. If $v \in \varphi^{-1}(x_0)$ with $d^{\#}(v) > 0$, set $\varphi^{\#}|_{S^{\#,v}} : S^{\#,v} \xrightarrow{\sim} T_R^{\#}$ with index $r_{\varphi^{\#}}(e) = d^{\#}(v)$ for all $e \in S^{\#,v}$;
4. If $v \in \varphi^{-1}(x_0)$ with $d^{\#}(v) < 0$, set $\varphi^{\#}|_{S^{\#,v}} : S^{\#,v} \xrightarrow{\sim} T_L^{\#}$ with index $r_{\varphi^{\#}}(e) = -d^{\#}(v)$ for all $e \in S^{\#,v}$.

One checks that this glues together correctly to a finite graph morphism

$$\varphi^{\#} : G^{\#} \to T^{\#}.$$

The finite morphism $\varphi^{\#}$ is harmonic. We check that $m_{\varphi^{\#}}(v)$ is well-defined for all $v \in \mathrm{V}(G^{\#})$. For all $v \notin \varphi^{\#-1}(X_0)$ there is either a unique $i = 1, \dots, d$ such that $v \in \mathrm{V}(G_i^{\#})$ and then $m_{\varphi^{\#}}(v) = m_{\varphi_i''}(v)$, or there is a unique $w \in \varphi^{\#-1}(X_0)$ such that $v \in \mathrm{V}(S^{\#,x_w})$ and then $m_{\varphi^{\#}}(v) = |d_{\varphi^{\#}}(w)|$. For all $v \in \varphi^{\#-1}(X_0)$, it holds that

$$m_{\varphi^{\#}}(v) = \max\left\{ \sum_{i\in I_L} \left(\sum_{e\in \mathrm{E}_v(G^{\#})\cap \mathrm{E}(G_i^{\#})} r_{\varphi^{\#}}(e) \right), \sum_{i\in I_R} \left(\sum_{e\in \mathrm{E}_v(G^{\#})\cap \mathrm{E}(G_i^{\#})} r_{\varphi^{\#}}(e) \right) \right\}.$$

The edge $e^{\#}$. Any of the two edges $e^{\#} \in \mathrm{E}_{X_0}(T^{\#})$ satisfies

$$\mathrm{size}_{\varphi_*^{\#}\mu_G}(e^{\#}) \geq \min\left\{ \nu(\bigcup_{i\in I_L} T_i''), \nu(\bigcup_{i\in I_R} T_i'') \right\} > \frac{A}{2}.$$

The degree of $\varphi^{\#}$. Consider a vertex $v \in \mathrm{V}(G^{\#}) \cap \varphi^{-1}(x_0)$: if v belongs to G^s, then it belongs to at most Δ_G different $G_i^{\#}$ for $i = 1, \dots, d$; and if $v \notin G^s$, then there is a unique $i = 1, \dots, d$ such that the unique path from v to G^s is contained in G_i''. It follows that for each $v \in \mathrm{V}(G^{\#}) \cap \varphi^{-1}(x_0)$, at most Δ_G of the neighboring $G_i^{\#}$ are either all send to $T_L^{\#}$, or all to $T_R^{\#}$. Hence

$$m_{\varphi^{\#}}(v) \leq \Delta_G m_{\varphi}(v),$$

and this implies $\deg \varphi^{\#} \leq \Delta_G \deg \varphi$. ☐

Remark 3.6.3. The point x_0 (used in the proof) with the property that all components of $T - x_0$ have measure $< C/2$ is in fact *unique*. Indeed, if there are two such vertices, say, x_0 and x_1, then let e denote any edge on a path between x_0 and x_1. One component of $T - x$ contains $T_1(e)$ and one component of $T - y$ contains $T_2(e)$, and hence by assumption, $\nu(T_1(e)) < C/2$ and $\nu(T_2(e)) < C/2$ but $\nu(T_1(e)) + \nu(T_2(e)) = 1$. Hence $C > 1$, but this is impossible with $C \leq 1 - A - B$ and $A, B > 0$.

3.7 Discussion of the spectral lower bound on stable gonality

We now give some examples that illustrate the bound.

Example 3.7.1. For the banana graph B_n, we have $\Delta_{B_n} = n$, $|B_n| = 2$ and $\lambda_{B_n} = 2n$, so the lower bound is trivial: $\mathrm{sgon}(B_n) \geq 1$. However, the stable gonality of B_n (for $n \geq 2$), equals 2. See Figure 3.1 for such a map of degree 2.

Example 3.7.2. For the complete bipartite graph $K_{n,n}$, we have $\Delta_{K_{n,n}} = n$, $|K_{n,n}| = 2n$ and $\lambda_{K_{n,n}} = n$. If n is even, then the lower bound is

$$\mathrm{sgon}(K_{n,n}) \geq \left\lceil \frac{2n^2}{5n+4} \right\rceil,$$

We expect that the stable gonality of $K_{n,n}$ equals n. A morphism which attains degree n is given by mapping $K_{n,n}$ to the star with one central vertex and n emanating edges in the obvious way. For $n = p^r + 1$ (p prime), the graph $K_{n,n}$ occurs as stable reduction graph of the curve

$$X_{\lambda,r} \colon (x^{p^r} - x)(y^{p^r} - y) = \lambda$$

(seen in $\mathbf{P}^1 \times \mathbf{P}^1$) with $|\lambda| < 1$, over a valued field $(k, |\cdot|)$ of characteristic p, which, as a fiber product of two projective lines, admits an obvious morphism of degree $p^r + 1$ to $\mathbf{P}^1$. The stable reduction itself consist of two transversally intersecting families of p^r+1 rational curves ("check board with p^{2r} squares"). For more details on these curves, see for example [26].

Example 3.7.3. If X_n is a family of Ramanujan graphs with (fixed) regularity d and n vertices (n increasing), then by the Alon-Boppana bound, we get inequalities

$$\sqrt{d-1} - o(1) \leq \lambda_{X_n}/2 \leq \sqrt{d-1},$$

so we find a lower bound of the form

$$\mathrm{sgon}(X_n) \geq \kappa_d \cdot n$$

for n sufficiently large with κ_d a constant only depending on d. In any family of Mumford curves whose stable reduction graphs are d-regular Ramanujan graphs, the gonality goes to infinity as the number of components of the stable reduction does so.

Remark 3.7.4. The famous Li–Yau inequality from differential geometry [51] states that the gonality $\mathrm{gon}(X)$ of a compact Riemann surface X (minimal degree of a conformal mapping φ of X to the Riemann sphere) is bounded below by

$$\mathrm{gon}(X) \geq \frac{1}{8\pi} \lambda_X \mathrm{vol}(X),$$

where λ_X is the first non-trivial eigenvalue of the Laplace-Beltrami operator of X, and $\mathrm{vol}(X)$ denotes the volume of X.

For graphs G with any Laplacian (normalized or not), an inequality of the form

$$\text{“}\mathrm{sgon}(G) \geq \kappa \cdot \lambda_G \cdot \mathrm{vol}(G)\text{”} \quad (*)$$

for some constant κ fails. A counterexample is given by the complete graph K_n, which has stable gonality $n-1$. However, a lower bound of the form $(*)$ would be $\kappa \cdot n^2(n-1)$ for the usual Laplacian, and $\kappa \cdot n^2$ for the normalized Laplacian (see Table 3.A in the appendix for the data; one deduces that the analog of the Li–Yau inequality also fails if one uses any of the other notions of gonality from the existing literature and are outlined in the appendix.)

One sees from our result that in a graph, the constant κ needs to be roughly divided by the maximal edge degree for such an inequality to hold.

As we have seen in Corollary 3.4.6, stable gonality is defined on equivalence classes of graphs, in the sense that two graphs G and G' are equivalent (notation $G \sim G'$) if they are refinements of the same stable graph. Hence the result also implies that

Corollary 3.7.5. *For any graph G with $g \geq 2$, we have*

$$\operatorname{sgon}(G) \geq \max_{G' \sim G} \left\lceil \frac{\lambda_{G'}}{\lambda_{G'} + 4(\Delta_{G'} + 1)} |G'| \right\rceil .$$

□

Remark 3.7.6. It is tempting to consider the limiting value for the lower bound in this theorem when the graph is further and further refined. Whereas it is clear how the number of vertices and the maximal vertex degree change under refinements, the change of the eigenvalue under refinements is not so well-understood (apart from regular graphs). For applications in solid state physics, Eichinger and Martin have developed an algorithm that computes the change in eigenvalues under refinement by applying only linear algebra to the original Laplace matrix [33]. Examples (such as the banana graph) suggest that (iterated) refinement might worsen the lower bound.

There is a similar result for the normalized Laplacian. Denote with $\widetilde{\lambda}_G$ the first non-trivial eigenvalue of $L_G^{\sim}$.

Theorem 3.7.7. *Let G be a graph with maximal degree Δ_G and first normalized Laplace eigenvalue $\widetilde{\lambda}_G$. The stable gonality of G is bounded from below by*

$$\operatorname{sgon}(G) \geq \frac{\widetilde{\lambda}_G}{\Delta_G \widetilde{\lambda}_G + 4(\Delta_G + 1)} \operatorname{vol}(G).$$

Sketch of proof. The proof is virtually the same as Theorem D, so instead of providing all details, we briefly outline the differences. Instead of μ_G, we use the measure η_G on $V(G')$ defined for $A \subset V(G')$ by

$$\eta_G(A) := \frac{\sum\limits_{v \in A \cap G} d_v^G}{\mathrm{vol}(G)},$$

where d_v^G is the degree of v in G. Lemma 3.5.2 is valid for all probability measures, and therefore also for $\varphi_* \eta_G$. Since $\varphi_* \eta_G(x)$ counts the number of edges instead of vertices, the conclusion of Lemma 3.5.5 changes to

$$\deg \varphi \geq \frac{\varphi_* \eta_G(x)}{\Delta_G} \mathrm{vol}(G).$$

The analogue of Proposition 3.5.6 can be derived by using the test function

$$f(v) = \begin{cases} \frac{1}{\mathrm{vol}(G_1)} & \text{if } v \in G_1, \\ -\frac{1}{\mathrm{vol}(G_2)} & \text{if } v \in G_2. \end{cases}$$

Proposition 3.5.9 does not change for this new measure. We conclude that the proof of Corollary 3.5.10 only changes in the step where Lemma 3.5.5 is used. □

Remark 3.7.8. For k-regular graphs

$$k\widetilde{\lambda}_G = \lambda_G \text{ and } \mathrm{vol}(G)/k = |G|,$$

and hence the lower bounds are identical for the two different Laplace operators.

Remark 3.7.9. Generically (in the sense of algebraic geometry), the gonality of a curve attains the Brill–Noether bound (cf. for example the references in the Appendix of [67]). However, curves of fixed genus and fixed stable reduction graph can have widely varying gonality (e.g., the banana graph B_n has stable gonality 2, but its subdivisions can occur as stable reduction graph of curves whose gonality takes on all the values $2, \dots, n$); in particular, the gonality of the curve can be much higher than the stable gonality of the reduction graph. One may try to find the most probably stable gonality of a random connected (multi-)graph and compare it to the most probable value of the Brill–Noether bound. In this remark, we compute something much simpler: the difference between the expected value of the Brill–Noether bound and the lower bound in our theorem, for the Erdős–Rényi random graph model with a specific connection probability in the non-sparse region.

For a random graph model $G = G_{n,p}$ of Erdős-Rényi type [35] with n vertices and edge probability $p = p(n) = n^{-\delta}$ for some $0 < \delta < 1$, the threshold for almost sure connectivity holds, the expected number of edges is $n(n-1)p/2$, so the first Betti number of $G_{p,n}$ is $g = \frac{1}{2}n^{1-\delta}(n-1) - n + 1$ almost surely. Chung, Lu and Vu [18] have shown that (for a class of function including these $p(n)$) the normalized eigenvalue $\widetilde{\lambda}$ tends to 1 with high probability. Also, the given assumptions imply that $\Delta_{G_{n,p}} = pn(1+o(1))$ in probability ([10], 3.14).

Hence the lower bound tends with high probability to

$$\approx \frac{n}{5} \approx \frac{1}{5}\sqrt[2-\delta]{2g},$$

which is sublinear in g (and for $\delta \to 0$, tends to $\sqrt{2g}/5$, up to a constant the actual value of the stable gonality $n-1 = \sqrt{2g}$ for the complete graph K_n of genus $g = (n-1)^2/2$), whereas the Brill–Noether bound is linear in g (which happens if $\delta \to 1$).

There are at least two ways to interpret this heuristic observation: either the lower bound is asymptotically bad for random graphs; or stable gonality of random graphs is significantly lower than generic gonality of curves.

3.8 A linear lower bound on the gonality of Drinfeld modular curves

We recall the main concepts and notations from the theory of general Drinfeld modular curves, cf. [39], [41].

3.8.1. Let K denote a global function field of a smooth projective curve X over a finite field $k = \mathbf{F}_q$ with q elements and characteristic $p > 0$, and ∞ a place of degree δ of K. Let A denote the subring of K of elements that are regular outside ∞.

3.8.2. Let Y denote a rank-two A-lattice in the completion K_∞ of K at ∞. Such lattices are classified up to isomorphism by their determinant, so they are isomorphic to $A \oplus I$, where I runs through a set of representatives of $\mathrm{Pic}(A)$, the ideal class group of A.

Let $\mathrm{GL}(Y)$ denote the automorphism group of the lattice Y:

$$\mathrm{GL}(Y) = \{\gamma \in \mathrm{GL}_2(K) \colon \gamma Y = Y\},$$

and let Γ denote a congruence subgroup of $\Gamma(Y) := \mathrm{GL}(Y)$. This means that Γ contains a principal congruence group $\Gamma(Y, \mathfrak{n})$ as a finite index subgroup, where

$$\Gamma(Y, \mathfrak{n}) = \ker\left(\Gamma(Y) \to \mathrm{GL}(Y/\,\mathfrak{n}Y)\right),$$

for $\mathfrak{n}$ an ideal in A. Let $Z \cong \mathbf{F}_q^*$ denote the center of $\mathrm{GL}(Y)$.

If $Y = A \oplus A$ is the "standard" lattice, we revert to the standard notations $\Gamma(1) := \Gamma(A \oplus A)$ and $\Gamma(\mathfrak{n}) := \Gamma(A \oplus A, \mathfrak{n})$.

3.8.3. The groups Γ act by fractional transformations on the Drinfeld space $\Omega = \mathbf{C}_\infty - K_\infty$, where $\mathbf{C}_\infty$ is the completion of an algebraic closure of K_∞. The quotient $\Gamma\backslash\Omega$ is an analytic smooth one-dimensional space, and is the analytification of a smooth affine algebraic curve Y_Γ, that can be defined over a finite abelian extension of K inside K_∞. It can be compactified to a Drinfeld modular curve X_Γ by adding finitely many points, called cusps.

The $\mathbf{C}_\infty$-points of the (coarse) moduli scheme $M(\mathfrak{n})$ of rank-two Drinfeld A-modules with full level $\mathfrak{n}$-structure (i.e., an isomorphism of $(A/\,\mathfrak{n})^2$ with the torsion of the Drinfeld module) can be described as

$$M(\mathfrak{n})(\mathbf{C}_\infty) = \bigsqcup_{Y \in \mathrm{Pic}(A)} \Gamma(Y, \mathfrak{n})\backslash\Omega.$$

We denote such a component by $Y(Y, \mathfrak{n}) := \Gamma(Y, \mathfrak{n})\backslash\Omega$, and its compactification by $X(Y, \mathfrak{n})$.

3.8.4. The groups Γ also act by automorphisms on the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$ [75]. The quotient $\Gamma\backslash\mathcal{T}$ is the union of a finite graph $G_\Gamma := (\Gamma\backslash\mathcal{T})^0$ and a finite number of half lines in correspondence with the cusps of X_Γ, and the curve X_Γ is a Mumford curve over K_∞ [63] such that the intersection dual graph of the reduction, which is a finite union of rational curves over $\mathbf{F}_{q^\delta}$ intersecting transversally in $\mathbf{F}_{q^\delta}$-rational points, equals the finite graph G_Γ.

Theorem 3.8.5 (= Theorem E)**.** *Let Γ denote a congruence subgroup of $\Gamma(Y)$. Then the gonality of the corresponding Drinfeld modular curve X_Γ satisfies*

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_{q,\delta} \cdot [\Gamma(Y) : \Gamma]$$

where the constant $c_{q,\delta}$ is

$$c_{q,\delta} := \frac{q^\delta - 2\sqrt{q^\delta}}{5q^\delta - 2\sqrt{q^\delta} + 8} \cdot \frac{1}{q(q^2-1)}$$

This implies a linear lower bound in the genus of modular curves of the form

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c'_{K,\delta} \cdot (g(X_\Gamma) - 1),$$

where $c_{K,\delta}$ is a bound that depends only on the function field K. If K is a rational function field and $\delta = 1$, then we can put $c'_{K,\delta} = 2c_{q,1}$.

Proof. First observe that $\mathrm{gon}_{\overline{K}}(X) = \mathrm{gon}_{\overline{K_\infty}}(X)$, (see, e.g., the appendix of [67]) so we now consider X_Γ as a curve over $k = K_\infty$ and are in a set up where we can apply our previous results. The remainder of the proof has various parts.

Reduction to principal congruence groups. First of all, we observe that it suffices to prove the bound for the groups $\Gamma(Y, \mathfrak{n})$. Indeed, if $\varphi\colon X_\Gamma \to \mathbf{P}^1$ is a morphism, then from the inclusion $\Gamma(Y, \mathfrak{n}) \leq \Gamma$ we get a composed morphism

$$X_{\Gamma(Y,\mathfrak{n})} \to X_\Gamma \to \mathbf{P}^1 \tag{3.2}$$

of degree

$$\frac{[\Gamma : \Gamma(Y, \mathfrak{n})]}{|\Gamma \cap Z|} \cdot \deg \varphi,$$

and hence

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq \mathrm{gon}_{\overline{K}}(X(Y, \mathfrak{n}))/[\Gamma : \Gamma(Y, \mathfrak{n})]. \tag{3.3}$$

Therefore, the desired inequality

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c_q[\Gamma(Y) : \Gamma]$$

follows from

$$\mathrm{gon}_{\overline{K}}(X(Y, \mathfrak{n})) \geq c_q[\Gamma(Y) : \Gamma(Y, \mathfrak{n})].$$

We now prove the gonality bound by invoking Theorem D for the reduction graph of the Drinfeld modular curve $X(Y, \mathfrak{n})$. We first compute the necessary spectral data from the covering $G := G_{\Gamma(Y,\mathfrak{n})} \to G_{\Gamma(Y)}$.

A lower bound on the number of vertices. Both of these graphs are the finite parts of quotients of the Bruhat–Tits tree $\mathcal{T}$ of $\mathrm{PGL}(2, K_\infty)$, in which every vertex is $(q^\delta + 1)$-regular. Let us consider the special vertex of $\mathcal{T}$ corresponding to the class of the trivial rank-two vector bundle $[\mathcal{O}_\infty \oplus \mathcal{O}_\infty]$ on X, and let v_0 denote the corresponding vertex in $G_{\Gamma(Y)}$. The stabilizer of this vertex is precisely $\mathrm{PGL}(2, \mathbf{F}_{q^\delta})$ (namely, an element of the stabilizer induces an automorphism of the "star" of the vertex, which is given by $\mathbf{P}^1(\mathbf{F}_{q^\delta})$.) The stabilizer intersects $\Gamma(Y)/Z$ (where Z is the center) in the "constant group" $\mathrm{PGL}(2, \mathbf{F}_q)$, and the group $\Gamma(Y, \mathfrak{n})$ (for $\mathfrak{n} \neq 1$) in the trivial group. We conclude that

$$|G| \geq \frac{1}{q(q^2-1)} \cdot [\Gamma(Y) : \Gamma(Y, \mathfrak{n})], \tag{3.4}$$

since the right hand side is the number of vertices in $\Gamma(Y, \mathfrak{n})$ above v_0, and $\mathrm{PGL}(2, \mathbf{F}_q)$ has cardinality $q(q^2 - 1)$.

Remark 3.8.6. This estimate for the number of vertices of $G_{\Gamma(Y,\mathfrak{n})}$ will be enough for our purposes, since it differs from the index only by a constant in q. But one might also count the total number of vertices of the graph. For a rational function field $K = \mathbf{F}_q(T)$ with a place ∞ of degree one, this is easily done, the result being

$$|G_{\Gamma(\mathfrak{n})}| = \frac{2q^{\deg(\mathfrak{n})+1} - q - 1}{q^{\deg(\mathfrak{n})+1}(q^2-1)(q-1)}[\Gamma(1) : \Gamma(\mathfrak{n})];$$

compare also with computations in [61] (cf. [17], [69]) and [40]. It seems another proof of the lower bound on the gonality is possible by using Morgenstern's result that there is a perfect matching between a very large (constant fraction depending only on q, not on $\deg(\mathfrak{n})$) subset of the vertices above v_0 in $G_{\Gamma(\mathfrak{n})}$ and vertices in the complement, but we did not pursue this, since it would give a less general and worse result.

Remark 3.8.7. The gonality is *not* always realized by the obvious map $X_\Gamma \to X(1) \cong \mathbf{P}^1$. For example, set $q = 2$ and let $\mathfrak{p}$ denote an prime of degree 3; then the modular curve $X_0(\mathfrak{p})$ is hyperelliptic, but the map $X_0(\mathfrak{p}) \to X(1)$ has degree 9. Also notice that for a general base field K, the modular curve $X(1)$ is not even itself a rational curve.

Remark 3.8.8. Counting the number of cusps (so the number of vertices above a vertex in $G_{\Gamma(Y)}$ corresponding to a split bundle of high degree) is not enough to get a linear estimate in the index, since the cusps have rather large stabilizers (of size roughly the third root of the index).

The maximal degree of a vertex Since the tree $\mathcal{T}$ is $(q^\delta + 1)$-valent, the maximal valency of a vertex in any graph $G_{\Gamma(Y,\mathfrak{n})}$ is bounded above by $q^\delta + 1$. Actually, this bound is attained at any vertex above v_0, since the vertex and edge stabilizers in $G = G_{\Gamma(Y,\mathfrak{n})}$ are trivial; the edge stabilizer in $\mathrm{PGL}(2, K_\infty)$ of any edge emanating from $v_0 \in \mathcal{T}$ is the Iwahori group (by contrast, edges in the cusps have non-trivial stabilizers) [75]. We conclude

$$\Delta_G = q^\delta + 1. \tag{3.5}$$

The first eigenvalue of the Laplace operator Since all stabilizers of vertices and edges in the finite graph $G := G_{\Gamma(Y,\mathfrak{n})}$ are trivial, its adjacency operator is the (unweighted) projection of the Hecke operator T_∞ on $\mathcal{T}$ corresponding to the characteristic function of the place ∞. The Ramanujan-Petersson conjecture for global function fields, proven by Drinfeld [31] implies that the eigenvalues of the operator T_∞ are bounded below by $2\sqrt{q^\delta}$.

Therefore, the operator

$$L'_G := (q^\delta + 1)\mathbf{1} - T_{\infty|G}$$

has non-zero eigenvalues

$$\lambda'_i \geq q^\delta + 1 - 2\sqrt{q^\delta}.$$

This operator is a perturbation of the Laplace operator L_G of G. More precisely, let B denote the diagonal matrix which has

$$B_v := \begin{cases} 1 & \text{if } v \text{ is adjacent to a cusp in } \Gamma(Y,\mathfrak{n})\backslash\mathfrak{T}, \\ 0 & \text{otherwise;} \end{cases}$$

then

$$L_G + B = L'_G.$$

Now the Courant-Weyl inequalities (e.g., Theorem 2.1 in [29]) imply that λ_G is larger than the first eigenvalue of L'_G minus the largest eigenvalue of B, leading to

$$\lambda_G \geq q^\delta - 2\sqrt{q^\delta}. \tag{3.6}$$

Conclusion of the proof of the main bound. Since the function

$$\lambda \mapsto \frac{\lambda}{\lambda + 4(\Delta + 1)}$$

is monotonously increasing in λ, we find the result by plugging the data from equations (3.4), (3.5) and (3.6) in the lower bound from Theorem D.

Linear lower bound in the genus. We now show how to convert the lower bound on the gonality of X_Γ in terms of the index $[\Gamma(Y) : \Gamma]$ into a lower bound that is linear in the genus, of the form

$$\mathrm{gon}_{\overline{K}}(X_\Gamma) \geq c'_{K,\delta}(g(X_\Gamma) - 1),$$

for c_K a constant depending only on the ground field K and the degree δ of ∞. This is not entirely obvious in positive characteristic, due to wild ramification.

First of all, it is again enough to establish such a bound for a principal congruence subgroup $\Gamma(Y,\mathfrak{n})$. Indeed, from the Riemann-Hurwitz formula (see e.g. [64]) for the (Galois) cover (3.2) and formula (3.3), it follows that

$$\begin{aligned}
\mathrm{gon}_{\overline{K}}(X_\Gamma) &\geq \frac{\mathrm{gon}_{\overline{K}}(X(Y,\mathfrak{n}))}{[\Gamma:\Gamma(Y,\mathfrak{n})]}|\Gamma \cap Z| \\
&\geq \frac{c'_{K,\delta}(g(X(Y,\mathfrak{n}))-1)}{[\Gamma:\Gamma(Y,\mathfrak{n})]}|\Gamma \cap Z| \\
&\geq c'_{K,\delta}(g(X_\Gamma)-1+\frac{r}{2}) \\
&\geq c'_{K,\delta}(g(X_\Gamma)-1),
\end{aligned}$$

where we have assumed that the desired bound holds for $\Gamma(Y,\mathfrak{n})$, and r is the ramification term in the Riemann-Hurwitz formula for the cover (3.2).

We now establish the bound for $X(Y,\mathfrak{n})$. If this curve has genus zero or one, the required bound for the gonality holds trivially. Therefore, we can assume $g(X(Y,\mathfrak{n})) \geq 2$. The Riemann-Hurwitz formula for the cover $X(Y,\mathfrak{n}) \to X(Y)$ implies a relation of the form

$$[\Gamma(Y):\Gamma(Y,\mathfrak{n})] = (g(X(Y,\mathfrak{n}))-1)\cdot \frac{2(q-1)}{2g(X(Y))-2+R},$$

where R is the degree of the ramification divisor for this cover. Hence to prove our result, it suffices to prove a lower bound of the form

$$2g(X(Y))-2+R \leq c''_{K,\delta}$$

for some constant $c''_{K,\delta}$ depending only on K and δ.

We recall some information about the ramification number R and the genus $g(X(Y))$ from [39] (There, the formulae are worked out for the principal component $Y = A \oplus A$ only, but hold in general). First of all, the genus of $X(Y)$ depends only on K and δ. Secondly, ramification takes place above elliptic points and cusps of $X(Y)$. Let us write $R = R_e + R_c$ with R_e the contribution from elliptic points, and R_c the contribution from cusps. The ramification above elliptic points is tame; and the number of elliptic points depends only on K and δ. Hence R_e is a constant in K and δ.

The ramification above the cusps is wild, but *weak*; this means that the second ramification groups are trivial, and the first ramification group is just the p-Sylow group of the stabilizer of the cusp (this follows, for example, from the fact that $X(Y)$ are Mumford curves, hence ordinary—since their Jacobian admits a Tate uniformization, and hence has maximal p-rank—, by applying a result of Nakajima [64]). In the end, we need a lower bound on

$$R_c = \frac{q^{d+1}-2}{(q-1)q^d}$$

where $d = \deg(\mathfrak{n}) \geq 1$, that is independent of d; for example,

$$R_c \geq \frac{1}{q-1}$$

will do, and this finishes the proof. $\square$

Remark 3.8.9. In the "standard" case of a rational function field $K = \mathbf{F}_q(T)$ with a place ∞ of degree one, one can make all data explicit. The cover $X(\mathfrak{n}) \to X(1) \cong \mathbf{P}^1$ is ramified tamely at the unique elliptic point, of order $q+1$, and at the unique cusp, of order $q^d(q-1)$, where $d = \deg(\mathfrak{n})$. Hence the Riemann-Hurwitz formula becomes

$$\begin{aligned} 2(g(X(\mathfrak{n})) - 1) &= [\Gamma(1) : \Gamma(\mathfrak{n})] \left(1 - \frac{1}{q+1} - \frac{1}{q^d(q-1)} - \frac{1}{q^d}\right) \\ &\leq [\Gamma(1) : \Gamma(\mathfrak{n})], \end{aligned}$$

and it follows that one can set $c'_{K,\delta} = 2c_q$ in this case.

Remark 3.8.10. The previous best (non-linear) bounds were due to Schweizer ([74], Thm. 2.4), who showed that if K is a rational function field, then one has, for example,

$$\mathrm{gon}_{\overline{\mathbf{F}_q(T)}} X_0(\mathfrak{n}) \geq \frac{1}{\sqrt{(q^2+1)(q+1)}} \cdot [\Gamma(1) : \Gamma_0(\mathfrak{n})]^{\frac{q-1}{2q}}.$$

3.9 Rational points of higher degree on Drinfeld modular curves

We first quote the positive characteristic analogue of a theorem of Frey [38]:

Proposition 3.9.1. *Let X denote a curve over a global function field K, such that its Jacobian does not admit a $\overline{K}$-morphism to a curve defined over a finite field. If d is an integer such that $2d+1 \leq \mathrm{gon}_{\overline{K}}(X)$, then the set of points of degree d on X is finite, i.e.,*

$$\left| \bigcup_{[K':K] \leq d} X(K') \right| \leq \infty.$$

Remark 3.9.2. The result was proven in [73] (Theorem 2.1) under the assumption that X has a K-rational point (similar to a hypothesis of Frey), but Clark has shown that this hypothesis is unnecessary, cf. [20], Theorem 5.

Remark 3.9.3. This result has now been improved into a quantitative statement over more general fields by Cadoret and Tamagawa [14], as follows: recall that gonality may alternatively be defined as the minimal d for which there exists a non-constant morphism from a $\mathbf{P}^1$ to the d-th symmetric power $X^{(d)}$ of the curve X. Define the *isogonality* $\mathrm{isogon}_K(X)$ of X as the minimal d for which there exists a non-constant morphism from a K-isotrivial curve to the d-th symmetric power $X^{(d)}$ of the curve X. Then the result from [14] says: *for any finitely generated field K of positive characteristic $p > 0$, and any smooth geometrically integral curve X over K, if d is a natural number with $2d + 1 \leq \mathrm{gon}_{\overline{K}}(X)$ and $d + 1 \leq \mathrm{isogon}_{\overline{K}}(X)$, then the set of points of degree $\leq d$ on X is finite.*

Remark 3.9.4. Suppose that K is a finitely generated field, and k the fraction field of an excellent discrete valuation ring, with $K \subseteq k$; for example, K is a global function field and $k = K_\infty$ is the completion of K at a place ∞. For any curve X over K, $\mathrm{gon}_{\overline{K}}(X) = \mathrm{gon}_{\overline{k}}(X)$ ([67], Appendix), so the lower bound that we obtained for $\mathrm{gon}_{\overline{k}}(X)$ from the stable gonality of its reduction graph applies equally well to $\mathrm{gon}_{\overline{K}}(X)$.

Remark 3.9.5. If X/K is a Mumford curve over a valued field $k \supseteq K$, then its Jacobian has split reduction, and hence it admits no map to an isotrivial curve, and $\mathrm{isogon}_{\overline{K}}(X) = \mathrm{gon}_{\overline{K}}(X)$.

Example 3.9.6. Consider the curve $X_{\lambda,r}$ from Example 3.7.2, with $\lambda = \lambda(T) \in K := \mathbf{F}_p(T)$ of negative degree in T. Observe that $X_{\lambda,r}$ is a Mumford curve over $k = \mathbf{F}_p((T^{-1}))$. Our gonality bound from Example 3.7.2 implies for example that if $p^r > 14$, then the set of points on $X_{\lambda,r}$ of degree $\leq p^r/6$ is finite. Note that the set of points of degree $p^r + 1$ is infinite, so the result is best up to a constant (for varying p and r).

Theorem 3.9.7 (= Theorem F)**.** *If X_Γ is defined over a finite extension K_Γ of K, then the set*

$$\bigcup_{[L:K_\Gamma] \leq \frac{1}{2}\left(c_{q,\delta}\cdot[\Gamma(1):\Gamma]-1\right)} X_\Gamma(L)$$

is finite.

Proof. The curves X_Γ are Mumford curves for the ∞-valuation. Therefore, the conditions to apply Proposition 3.9.1 are satisfied by $X = X_\Gamma$ and $K = K_\Gamma$. □

Remark 3.9.8. Since Γ is a congruence group, the curve X_Γ is covered by some $X(Y, \mathfrak{n})$, and hence the curve X_Γ is defined over H. Hence one may always choose $K_\Gamma = H$, but K_Γ might be chosen smaller. Also, Drinfeld modular curves always have K-rational points, namely, the cusps, so the refinement of result 3.9.1 by Clark is not necessary for this application.

3.A Appendix: Other notions of gonality from the literature

In this appendix, we describe various other notions of graph gonality from the literature, and discuss the relation of stable gonality to these alternatives.

3.A.1. We first recall the notion of graph gonality from Caporaso [16], but we change the terminology to be compatible with [4] and the current thesis. For the convenience of the reader, we include a dictionary between the terminology in [16] and this paper in Table 3.1.

A *morphism* between two loopless graphs G and G' (denoted by $\varphi : G \to G'$) is a map

$$\varphi\colon\ \mathrm{V}(G) \cup \mathrm{E}(G) \to \mathrm{V}(G') \cup \mathrm{E}(G')$$

such that $\varphi(\mathrm{V}(G)) \subset \mathrm{V}(G')$, and for every edge $e \in \mathrm{E}(x, y)$, either $\varphi(e) \in \mathrm{E}(\varphi(x), \varphi(y))$ or $\varphi(e) \in \mathrm{V}(G')$ and $\varphi(x) = \varphi(y) = \varphi(e)$; together with, for every $e \in \mathrm{E}(G)$, a non-negative integer $r_\varphi(e)$, the *index* of φ at e, such that $r_\varphi(e) = 0$ if and only if $\varphi(e) \in \mathrm{V}(G')$.

Previously, in Definition 3.2.6, we only considered *finite* morphisms, which are morphisms that map edges to edges. The notions of harmonicity and degree that we introduced in Definition 3.2.6 make sense for morphisms, even if they are not finite. A harmonic morphism is called *non-degenerate* if $m_\varphi(v) \geq 1$ for every $v \in \mathrm{V}(G)$ (this is automatic if it is finite).

Table 3.1: Small dictionary of terminology

Terminology in [16]	Terminology in this thesis
indexed morphism	morphism
homomorphism	finite morphism
stable refinement	refinement
pseudo-harmonic	harmonic

3.A.2. The *gonality* of a graph is defined to be

$$\mathrm{gon}(G) = \min\{\deg \varphi | \varphi \text{ a non-degenerate harmonic morphism to a tree } T\}.$$

Obviously, for any graph G, we have an inequality $\mathrm{sgon}(G) \geq \mathrm{gon}(G)$. Caporaso proves that *the gonality of a complex nodal curve is bounded below by the gonality of any refinement of its intersection dual graph.*

Lemma 3.A.3. *The stable gonality of a graph G is equal to the minimal gonality of any of its refinements.*

Proof. It suffices to prove that any non-degenerate harmonic morphism $\varphi\colon G \mapsto T$ from a graph G to a tree T admits a refinement $\varphi'\colon G' \to T'$ that is a *finite* harmonic morphism of the same degree as φ. Thus, let $e = (v_1, v_2) \in G$ denote an edge that is mapped to a vertex $\varphi(e) = x \in \mathrm{V}(T)$. Add an extra leaf ℓ to T at x , subdivide e into two edges (v_1, m) and (m, v_2), and map both e_1 and e_2 to ℓ. Set $r_{\varphi'}(e_i) = m_\varphi(v_i)$ for $i = 1, 2$. Finally, add a leaf ℓ_w to all $w \in \varphi^{-1}(x)$, map them all to ℓ, and set $r_{\varphi'}(\ell_w) = m_\varphi(w)$. $\square$

The following elementary fact, a "trivial" spectral bound on the gonality, does not seem to have been observed before:

Proposition 3.A.4. *The gonality of a graph G is bounded below by the edge-connectivity (viz., the number of edges that need to be removed from the graph in order to disconnect it):*

$$\mathrm{gon}(G) \geq \eta(G).$$

If G is a simple graph (i.e., without multiple edges), unequal to a complete graph, then

$$\mathrm{gon}(G) \geq \lambda_G.$$

Proof. Let $\varphi\colon G \mapsto T$ denote a harmonic non-degenerate morphism. Choose any edge $e \in \mathrm{E}(T)$. Since removing e from T disconnects it, $\varphi^{-1}(e)$ is a set of edges of G whose removal disconnects G. Hence

$$\mathrm{gon}(G) \geq |\varphi^{-1}(e)| \geq \eta(G).$$

For a simple graph which is not complete, the bound

$$\eta(G) \geq \lambda_G$$

is one of the inequalities of Fiedler [36] (4.1 & 4.2). $\square$

Remark 3.A.5. The "trivial" spectral bound in the above proposition is not very useful in practice, since it does not contain a "volume" term (like the Li–Yau inequality). Also, since every graph acquires edge connectivity two or one by refinements, the lower bound in the proposition trivializes under refinements (which are required by the reduction theory of morphisms).

3.A.6. Another notion of gonality of graphs G and, more generally, of *metric* graphs Γ was introduced by Baker in [8], defined as the minimal degree d for which there is a g^1_d on Γ (in analogy to the definition from algebraic geometry). Following Caporaso, we call this gonality of graphs *divisorial gonality*. In [15], Caporaso has proven a Brill–Noether upper bound for divisorial gonality. For a fixed unmetrized graph, there is in general no relation (inequality either side) between gonality and divisorial gonality, cf. [16] Remark 2.7; one example is given by the graph on line 6 of Table 3.A, in which one of the middle edges is given index 2. However, if that edge is subdivided, then the divisorial gonality decreases; see line 7 of Table 3.A.

Since the reduction of a stable curve is naturally a metric graph (cf. [8]), one should not ignore the metric in connection with gonality of curves. Baker has proven that *the gonality of a curve X is larger than or equal to the divisorial gonality of its metric reduction graph* ([8], Cor. 3.2).

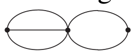
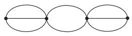
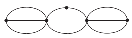
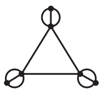
The stable gonality of a graph is larger than or equal to its divisorial gonality, but the inequality can be strict. This can be seen from the last line of Table 3.A, which shows an example of Luo Ye (taken from [4]) of stable gonality 4 but divisorial gonality 3 (for the metrization in which all edges have unit length).

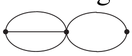
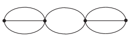
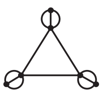
The banana graph B_n has divisorial and stable gonality 2 but edge connectivity n (cf. Table 3.A), showing that an equality analogous to the one in Proposition 3.A.4 cannot hold for divisorial or stable gonality. Dion Gijswijt remarked that $\mathrm{dgon}(G) \geq \min\{|G|, \eta(G)\}$. With Josse van Dobben de Bruyn, he has also proven that the divisorial gonality of a graph is larger than or equal to its treewidth (unpublished, but some preliminaries can be found in [83]), but the entries in Table 3.A show that the inequality can be strict. Lower bounds on treewidth imply such bounds on divisorial gonality (e.g., [11], [77]).

3.A.7. It seems that our notion of stable gonality of a graph coincides with the notion of gonality introduced in [4] from the viewpoint of tropical geometry. The connection between tropical curves and metric graphs can already be found in Mikhalkin [57], and the notion of harmonic morphism of metric graphs in Anand [5].

3.A.8. We have collected some sample values in Table 3.A. As above, λ_G is the first eigenvalue of L_G, and $\lambda_G^{\sim}$ is the first eigenvalue of the normalized Laplacian $L_G^{\sim}$; $\eta(G)$ is the edge connectivity, Δ_G the maximal vertex degree, $\mathrm{vol}(G)$ is the volume of the graph, $\mathrm{tw}(G)$ its treewidth; $\mathrm{gon}(G)$ is the gonality, $\mathrm{dgon}(G)$ is the divisorial gonality, and $\mathrm{sgon}(G)$ is the stable gonality of G. We leave out the lengthy but elementary calculations (for the divisorial gonality of K_n, we refer to [8], 3.3).

Table 3.2: Some graphs and their invariants, including gonalities

Graph G	$\mathrm{sgon}(G)$	$\mathrm{gon}(G)$	$\mathrm{dgon}(G)$	$\eta(G)$	$\mathrm{tw}(G)$
Complete graph K_n	$n-1$	$n-1$	$n-1$	$n-1$	$n-1$
Cycle graph C_n	2	2	2	2	2
Utility graph $K_{3,3}$	3	3	3	3	3
Banana graph B_n	2	n	2	n	2
	3	3	3	2	2
	3	3	4	2	2
	3	$?$	3	2	2
	4	> 3	3	2	2

Graph G	Δ_G	λ_G	$\lvert G\rvert$	$\lambda_G^{\sim}$	$\mathrm{vol}(G)$
Complete graph K_n	$n-1$	n	n	$\frac{n}{n-1}$	$n(n-1)$
Cycle graph C_n	2	$4\sin^2(\frac{\pi}{n})$	n	$2\sin^2(\frac{\pi}{n})$	$2n$
Utility graph $K_{3,3}$	3	3	6	1	18
Banana graph B_n	n	$2n$	2	2	$2n$
	5	$5-\sqrt{7}\approx 2.35$	3	λ_G	10
	5	$5-\sqrt{13}\approx 1.93$	4	$\frac{2}{5}$	16
	5	$\frac{3}{2}(5-\sqrt{13})\approx 1.15$	5	$\frac{11-\sqrt{61}}{10}\approx 0.32$	18
	5	$\frac{9-3\sqrt{5}}{2}\approx 1.46$	6	$\frac{11-\sqrt{61}}{10}\approx 0.32$	24

Chapter 4

Dynamics measured in a non-Archimedean field

'Dus' zei de olifant, 'kan ik dwars door de eik heen lopen.'
Hij nam een aanloop en liep in volle vaart tegen de eik op.
De eik trilde tot in zijn wortels. Versuft lag de olifant aan zijn voet.

Toon Tellegen. [80]

In this chapter, we study dynamical systems using measures taking values in a non-Archimedean field. The underlying space for such measure is a zero-dimensional topological space. We elaborate on the natural translation of several notions, e.g., probability measures, isomorphic transformations, entropy, from classical dynamical systems to a non-Archimedean setting.

This chapter is based on 'Dynamics measured in a non-Archimedean field' [49].

4.1 Non-Archimedean measures and integration theory

Measures

We start by explaining the main set up, as described in ([84], chapter 7). Let K be a complete discrete valued non-Archimedean field. Instead of σ-algebras, separating covering rings are used. For any set X, denote by $\mathcal{P}(X)$ its power set.

Definition 4.1.1. A collection $\mathcal{R} \subseteq \mathcal{P}(X)$ is called a *covering ring* if it has the following properties:

1. if $A, B \in \mathcal{R}$ then $A \cap B$, $A \cup B$ and $A \backslash B$ are in $\mathcal{R}$.
2. $\mathcal{R}$ covers X.

Such a ring $\mathcal{R}$ is called *separating* if for any distinct $x, y \in X$, there is an $A \in \mathcal{R}$ such that $x \in A$, and $y \notin A$. A covering ring is an *algebra* if $X \in \mathcal{R}$.

A covering ring $\mathcal{R}$ is the basis of a zero dimensional topology–in which the elements of $\mathcal{R}$ are closed and open. This so called $\mathcal{R}$-topology is Hausdorff if and only if $\mathcal{R}$ is separating. In the text below all covering rings are separating. A subcollection $\mathcal{A} \subset \mathcal{R}$ is called *shrinking* if the intersection of any two elements of $\mathcal{A}$ contains an element of $\mathcal{A}$.

Definition 4.1.2. A map $\mu : \mathcal{R} \to K$ is called a *measure*, if it is

additive for disjoint $A, B \in \mathcal{R}$, $\mu(A \cup B) = \mu(A) + \mu(B)$,
bounded for all $A \in \mathcal{R}$ the set $\{\mu(B) : B \in \mathcal{R}, B \subset A\}$ is bounded,
continuous if $\mathcal{A}$ is shrinking and $\bigcap_{A \in \mathcal{A}} A = \emptyset$, then $\lim_{A \in \mathcal{A}} \mu(A) = 0$.

The latter limit is defined as follows: for every $\epsilon > 0$ there is an $A_0 \in \mathcal{A}$, such that for every $A \in \mathcal{A}$ contained in A_0, $|\mu(A)| < \epsilon$.

The continuity property of the measure is the replacement for σ-additivity.

Lemma 4.1.3. *Let $\{B_n\}_{n \in \mathbf{N}} \subset \mathcal{R}$ a collection of disjoint sets such $\bigcup_{n \in \mathbf{N}} B_n \in \mathcal{R}$, then $\mu(\bigcup_{n \in \mathbf{N}} B_n) = \sum_n \mu(B_n)$.*

Proof. Define $A_n = \bigcup_{i=n}^{\infty} B_i$, then $\{A_i\}_{i \in \mathbf{N}}$ forms a shrinking collection with empty intersection. Moreover, $B_i = A_i \backslash A_{i+1}$ and $\mu(B_i) = \mu(A_i) - \mu(A_{i+1})$. Hence

$$\mu(\bigcup_i^{\infty} B_i) = \mu(A_1) = \lim_{i \to \infty} \sum \mu(A_i) - \mu(A_{i+1})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \mu(B_i) = \sum_n \mu(B_i). \qquad \square$$

Let us illustrate these measures with some examples, and also a non-example to illustrate the necessity of the continuity condition.

Example 4.1.4. In our first example $X = \mathbf{Q}_p$, the field of p-adic numbers. It is a valued field, and the valuation v_p induces a metric $|x-y|_p = p^{-v_p(x-y)}$. This metric is non-Archimedean, and, therefore, balls of the form $B_{p^n}(x) = \{y \in X : |x-y| < p^n\}$ are both open and closed. The collection $B_c(X)$ of all compact clopen subsets of X is a covering ring. Let q be a prime number. There is a unique additive function $\mu : B_c(X) \to \mathbf{Q}_q$ such that $\mu(B_{p^n}(x)) = p^n$. If $q \neq p$, then μ is a measure, which takes values in $\mathbf{Z}_q^* \cup \{0\}$, where $\mathbf{Z}_q^*$ denotes the ring of invertible q-adic integers. If $p = q$, then μ is not a measure, because the boundedness condition for measures is violated. For instance, the values of μ of the sequence of sets

$$B_{p^0}(0) \supset B_{p^{-1}}(0) \supset .... \supset B_{p^{-n}}(0) \supset ...,$$

form a sequence with increasing, and even unbounded, absolute values.

Example 4.1.5. Let X be any set, and let $\mathcal{R}$ be the ring which consists of all finite subsets of X. Let $h : X \to K$ be any function. Then $\kappa : \mathcal{R} \to K, A \mapsto \sum_{a \in A} h(a)$ is a measure. More generally, let $h : X \to K$ be any function, and let $\mathcal{R}_h$ be the ring which consists of all subsets A of X for which $\sum_{a \in A} h(a)$ converges. Define the measure κ as above.

Non-example 4.1.6. This example is a non-example in the following sense. The map described below is additive and bounded, but we will later prove that it lacks the continuity condition. Let p be a prime and let $\mathfrak{X}_p^k = \{\frac{n}{p^k} : 0 \leq n \leq p^k\}$ and $\mathfrak{X}_p = \cup_{k \geq 0} \mathfrak{X}_p^k$. Finally define $X_p = [0,1] \backslash \mathfrak{X}_p$. For $r, s \in \mathfrak{X}_p, r < s$ let $I_{r,s}$ be the open interval $(r,s) \subset [0,1]$. Then the collection $J_{r,s} = I_{r,s} \cap X_p$ generates a covering algebra $\mathcal{J}$ of X_p. Define

$$\upsilon : \mathcal{J} \to \mathbf{Q}_p, \upsilon(J_{r,s}) = \begin{cases} \frac{1}{s} - \frac{1}{r} & \text{if } r \neq 0 \\ \frac{1}{s} & \text{if } r = 0 \end{cases}$$

The map υ is bounded because $v_p(\frac{1}{x}) \geq 0$ for all $x \in \mathfrak{X}_p$, and therefore $v_p(\upsilon(J_{r,s})) \geq 0$ for all $r, s \in \mathfrak{X}_p$.

One of the main losses of using K-valued measures is that K is not ordered, and therefore, the measure doesn't order sets into "bigger" or "smaller". Moreover, it could very well happen that there are sets of measure zero which contain sets of non-zero measure. To overcome these problems, a measure comes with a real valued function:

$$\| \cdot \|_\mu : \mathcal{R} \to \mathbf{R} : A \mapsto \sup\{|\mu(B)| : B \in \mathcal{R}, B \subset A\}.$$

In case confusion is unlikely we will suppress the subscript μ. This $\mathbf{R}$-valued function is not at all like a real valued measure in the classical sense; it possesses the following properties:

monotonicity if $A \subset B$, then $\|A\| \leq \|B\|$,
convexity for any $A, B \in \mathcal{R}$, $\|A \cup B\| \leq \max(\|A\|, \|B\|)$.
minimum for $\|A \cap B\| \leq \min(\|A\|, \|B\|)$

Lemma 4.1.7 ([84], page 249)**.** *The continuity property of a measure is equivalent to the following assertion. If $\mathcal{A} \subset \mathcal{R}$ is shrinking and $\bigcap\limits_{A \in \mathcal{A}} A = \emptyset$, then*

$$\lim_{A \in \mathcal{A}} \|A\| = 0.$$

Define the *norm function* $N_\mu : X \to [0, \infty)$ by

$$N_\mu(x) = \inf\{\|U\|_\mu : U \in \mathcal{R}, x \in U\}.$$

The reason that this function is called the norm function is because it is used to define a seminorm on the space of K-valued functions on X. For $f : X \to K$ define

$$\|f\|_\mu = \sup_{x \in X} |f(x)| N_\mu(x).$$

This apparent abuse of notation is justified by the following lemma.

Lemma 4.1.8 ([84], Lemma 7.2)**.** *For the indicator function χ_B of any $B \in \mathcal{R}$, $\|\chi_B\| = \|B\|$.*

We denote the level set $\{x \in X : N_\mu(x) = 0\}$ by $X_0(\mu)$.

Definition 4.1.9. Subsets of $X_0(\mu)$ are called *negligible*.

Lemma 4.1.10. *For any $A \in \mathcal{R}$ the following properties are equivalent.*

1. *A is negligible,*
2. *$\|A\| = 0$,*
3. *for all $B \in \mathcal{R}$, $\mu(A \cap B) = \mu(A)$.*

Proof. We first prove that (1)⇔(2). Let A be a negligible set. Then by lemma (4.1.8) $\|A\| = \|\chi_A\| = \sup_{x \in A} N_\mu(x)$, hence $\|A\| = 0$ if and only if A is negligible.

Next we prove (2)⇔(3). Suppose that $\|A\| = 0$, then $\mu(A) = 0$ and for all $B \in \mathcal{R}$, $A \cap B \subset A$, and hence $\mu(A \cap B) = 0$. Conversely, for any $B \in \mathcal{R}$ with $B \subset A$, $B = B \cap A$, and since $\emptyset \in \mathcal{R}$, $\mu(B) = \mu(A) = \mu(\emptyset) = 0$. It follows that $\|A\| = 0$. □

Example 4.1.11. Let us have a closer look to the examples discussed above. In example 4.1.4 we have $X_0(\mu) = \emptyset$, while in example 4.1.5, $X_0(\kappa) = \{x \in X : h(x) = 0\}$; in fact, $N_\kappa(x) = |h(x)|$.

Non-example 4.1.12. We determine $N_\upsilon(x)$ on X_p in non-example 4.1.6. Any element $x \in [0, 1]$ can be represented by

$$x = \sum_{i=1}^{\infty} \frac{a_i}{p^i}, \text{ where } a_i \in \{0, ..., p-1\}.$$

It is well known that such representation are sometimes not unique. Elements with representations with coordinates that are constant 0 or $p-1$ eventually, are in $\mathfrak{X}_p$. Let $x \in X_p$ with an expansion $x = \sum_{i=1}^{\infty} a_i/p^i$. Then for any $n \in \mathbf{N}$,

$$\sum_{i=1}^{n} \frac{a_i}{p^i} < x < \frac{1}{p^n} + \sum_{i=1}^{n} \frac{a_i}{p^i}.$$

We call this interval $J_n(x)$, and calculate $\upsilon(J_n(x))$ for an n for which $a_n \neq 0$. Denote $r = \sum_{i=1}^{n} \frac{a_i}{p^i}, t = \frac{1}{p^n}$, and let v_p be the p-adic valuation. As $v_p(r) = v_p(t)$ we find

$$v_p(\upsilon(J_n)) = v_p(\frac{1}{r+t} - \frac{1}{t}) = v_p(r) - (v_p(t) + v_p(r+t)) = -v_p(r+t).$$

Define $k_n = -v_p(r+t) = \max\{k : a_k \neq p-1, k \leq n\}$. Since the coordinates of x can not be constant $p-1$ eventually, it follows that

$$\lim_{n\to\infty} \upsilon(J_n(x)) = \lim_{n\to\infty} p^{k_n} = 0.$$

In particular, this shows that $N_\upsilon(x) = 0$ for all $x \in X_p$, i.e., the entire set X_p is negligible. This finishes the proof that υ does not satisfy the continuity condition, because, for instance,

$$\|\chi_{X_p}\|_\upsilon = \sup_{x \in X_p} N_\upsilon(x) = 0 \neq \|X_p\|_\upsilon,$$

which contradicts lemma 4.1.8.

Integration

Analogous to the classical integration theory, integrals with respect to a measure are defined by approximation by step functions.

Definition 4.1.13. An $\mathcal{R}$-*step-function* is a finite K-linear combination of indicator functions χ_A of elements in $A \in \mathcal{R}$.

Note that step functions can be written as a finite linear combination of indicator functions of disjoint elements of $\mathcal{R}$. The step functions $S(X)$ form a K-vector space. The integral is the unique linear functional $S(X) \to K$ for which

$$\int_X \chi_A(x) d\mu(x) = \mu(A).$$

for any $A \in \mathcal{R}$. It satisfies the inequality

$$|\int_X f(x) d\mu(x)| \leq \|f\|_\mu. \tag{4.1}$$

A function $f : X \to K$ is called μ-*integrable* if there exists a sequence $\{f_n\}_{n\in N}$ of step functions such that $\lim_{n\to\infty} \|f_n - f\|_\mu \to 0$. The space of integrable functions is a vector space, denoted by $L(\mu)$. The integration functional is extended to $L(\mu)$ by continuity. Inequality (4.1) holds for this extension.

A set $A \subset X$ is *measurable* if its indicator function χ_A is in $L(\mu)$. The collection $\mathcal{R}_\mu$ of measurable sets is characterized by the following lemma.

Lemma 4.1.14 ([84], Lemma 7.3)**.** *A set $A \in \mathcal{P}(X)$ is an element of $\mathcal{R}_\mu$ if and only if for every $\epsilon > 0$ there exists a $B \in \mathcal{R}$ such that on the symmetric difference $A\Delta B$, $N_\mu \leq \epsilon$.*

It follows that $\mathcal{R}_\mu$ itself again forms a covering ring. Clearly, $\mathcal{R}$ is contained in $\mathcal{R}_\mu$, and the $\mathcal{R}_\mu$-topology is finer than the $\mathcal{R}$-topology. In particular all subsets of $X_0(\mu)$ are measurable. The ring $\mathcal{R}_\mu$ is maximal in the following sense: repetition of the procedure above would lead again to $\mathcal{R}_\mu$, i.e, $(\mathcal{R}_\mu)_\mu = \mathcal{R}_\mu$.

Definition 4.1.15. A triple $(X, \mathcal{R}_\mu, \mu)$ is called a *measure space*.

Two functions f, g are said to be equal μ-almost everywhere, if $f(x) = g(x)$ for all $x \in X$ except maybe on a subset of X_0. Being equal μ-almost everywhere defines an equivalence relation on $L(\mu)$ denoted by $\sim_\mu$.

Definition 4.1.16. The space $L^1(\mu)$ is defeined to be $L(\mu)$ modulo the relation $\sim_\mu$, equipped with the norm induced by the seminorm $\|\cdot\|_\mu$ on $L(\mu)$.

4.2 Measure preserving transformations

We study the dynamics on measure spaces, where μ is a probability measure.

Definition 4.2.1. A measure $\mu : \mathcal{R} \to K$ is called a *probability measure* if

1. the covering ring $\mathcal{R}$ is an algebra, i.e., it is a covering ring such that $X \in \mathcal{R}$,
2. $\mu(X) = 1$.

A measure space $(X, \mathcal{R}_\mu, \mu)$ is called a *probability space* if μ is a probability measure.

Definition 4.2.2. Let $T : (X, \mathcal{R}_\mu, \mu) \to (Y, \mathcal{R}_\nu, \nu)$ be a map. It is called *measurable* if T is continuous relative to the $\mathcal{R}_\mu$ and $\mathcal{R}_\nu$-topologies. It is called *measure preserving* if it is measurable and if for any $B \in \mathcal{R}_\nu$, $\mu(T^{-1}B) = \nu(B)$. A measure preserving $T : (X, \mathcal{R}_\mu, \mu) \to (X, \mathcal{R}_\mu, \mu)$ is called a *measure preserving transformation*. A measure preserving transformation is *invertible* if it is a homeomorphism, and if for all $B \in \mathcal{R}_\mu$, $\mu(TB) = \mu(B)$.

Definition 4.2.3. A map $\Phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ between two measure algebras is called a *measure algebra isomorphism* if Φ is a bijection which preserves complements and unions and $\nu(\Phi(B)) = \mu(B)$ for all $B \in \mathcal{R}_\mu$.

Clearly, a measure algebra isomorphism also preserves inclusions and intersections.

Lemma 4.2.4. *Let $\Phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\mu, \mu)$ be a measure algebra isomorphism. The map $\|\cdot\|_\mu$ is invariant under Φ.*

Proof. Let Φ be a measure algebra isomorphism and let $F \in \mathcal{R}_\mu$ then

$$\begin{aligned}\|\Phi^{-1}F\| &= \sup\{|\mu(B)| : B \in \mathcal{R}_\mu, B \subset \Phi^{-1}F\} \\ &= \sup\{|\mu(\Phi^{-1}B')| : B' \in \mathcal{R}_\mu, B' \subset F\} \\ &= \sup\{|\mu(B')| : B' \in \mathcal{R}_\mu, B' \subset F\} = \|F\|. \end{aligned}$$ □

Corollary 4.2.5. *The biggest negligible set $X_0(\mu)$, is invariant under measure preserving transformations.*

Proof. Recall from lemma 4.1.10 that $X_0(\mu)$ is the maximal set A for which $\|A\|_\mu = 0$. □

Lemma 4.2.6. *The norm map N_μ is invariant under any* invertible *measure preserving transformation $T : (X, \mathcal{R}_\mu, \mu) \to (X, \mathcal{R}_\mu, \mu)$.*

Proof.

$$\begin{aligned}N_\mu(x) &= \inf_{U\in\mathcal{R}_\mu, x\in U} \|U\| \\ &= \inf_{TU\in\mathcal{R}_\mu, x\in U} \|U\| \text{ (rename } TU = W) \\ &= \inf_{W\in\mathcal{R}_\mu, x\in T^{-1}W} \|T^{-1}W\| \\ &= \inf_{W\in\mathcal{R}_\mu, x\in T^{-1}W} \|W\| \\ &= \inf_{W\in\mathcal{R}_\mu, Tx\in W} \|W\| = N_\mu(Tx).\end{aligned}$$ □

Let us give an example of a (non-invertible) measure preserving map, for which $N_\mu(x)$ is not invariant.

Example 4.2.7. Let $\Omega = \{0, ..., p-1\}^{\mathbf{N}}$ be the space of one sided infinite words $w = w_0w_1...$ in the alphabet $\{0, ..., p-1\}$. The map

$$\mathbf{Z}_p \to \Omega : \sum_{i\geq 0} a_i p^i \mapsto w = a_0a_1a_2...$$

is a bijection. The restriction to $\mathbf{Z}_p$ of the measure μ with values in $\mathbf{Q}_q$ from the first example (4.1.4) induces a measure μ_q on Ω. Let us describe this measure. Let $\omega = a_0...a_{n-1}$ be a finite word in the same alphabet. Define

$$U_\omega = \{w \in \Omega : w_0 = a_0, ..., w_{n-1} = a_{n-1}\}.$$

The set U_ω is called a *cylindrical set*, and it is the image of $\sum_{i=0}^{n-1} a_i p^i + p^n\mathbf{Z}_p$ of the bijection above. We find, $\mu_q(U_\omega) = p^n$. Of course, Ω is very similar to the classical one sided shift space. In fact the shift map

$$\sigma : \Omega \to \Omega, w_0w_1w_2... \mapsto w_1w_2w_3...,$$

is measure preserving. Clearly, σ is not injective, and therefore, not invertible. Again, very similar to the classical Bernoulli-shift, more shift invariant measures can be found. Let $\mathbf{q} = (q_0, ..., q_{p-1})$ be a vector in $\mathbf{Q}_q^p$, such that $\sum_{i=0}^{p-1} q_i = 1$, and for all i, $|q_i|_q \leq 1$. Let $\mu_{\mathbf{q}}$ be the measure on Ω such that

$$\mu_{\mathbf{q}}(U_\omega) = q_{a_0}...q_{a_n}.$$

This measure is invariant under the shift, for

$$\sigma^{-1}(U_\omega) = \bigcup_{i=0}^{p-1} U_{i\omega},$$

and

$$\mu_{\mathbf{q}}(\bigcup_{i=0}^{p-1} U_{i\omega}) = \sum_{i=o}^{p-1} q_i \mu_{\mathbf{q}}(U_\omega) = \mu_{\mathbf{q}}(U_\omega).$$

Now choose $p = 2, q = 3$, and let $\mathbf{q} = (-2, 3)$. Let U_ω be a cylindrical set in Ω, as for any cylindrical set

$$\begin{aligned} U_{\omega'} &\subset U_\omega, \\ |\mu_{\mathbf{q}}(U_{\omega'})| &\leq |\mu_{\mathbf{q}}(U_{\omega'})|, \\ \|U_\omega\| = |\mu_{\mathbf{q}}(U_\omega)| &= 3^{-\#\{i:\omega_i=1\}}. \end{aligned}$$

The biggest negligible set is given by $\Omega_0(\mu_{\mathbf{q}}) = \{w \in \Omega : \#\{i : w_i = 1\} = \infty\}$. For $w \notin \Omega_0(\mu_{\mathbf{q}})$, $N_{\mu_{\mathbf{q}}}(w) = 3^{-\#\{i:w_i=1\}}$. The function $N_{\mu_{\mathbf{q}}}$ is not invariant under σ; if w is such that $w_0 = 1$ then $N_{\mu_{\mathbf{q}}}(\sigma w) = 3N_{\mu_{\mathbf{q}}}(w)$.

4.3 Isomorphisms and spectral isomorphisms

In classical dynamics, several equivalence relations on the collection of dynamical systems are distinguished, for instance, isomorphy, conjugacy and spectral isomorphy. We discuss the analogs of these notions, and, remarkably, we will show that isomorphy and conjugacy turn out to be the same.

Definition 4.3.1. Let $(X_1, \mu, \mathcal{R}_\mu)$ and $(X_2, \nu, \mathcal{R}_\nu)$ be probability spaces, and let there be measure preserving transformations $T_i : X_i \to X_i$, $i = 1, 2$. Then T_1 and T_2 are called *isomorphic*, $T_1 \cong T_2$, if there are sets $M_i \subset X_i$, such that $X_i \backslash M_i$ is negligible and $T_i M_i = M_i$ for $i = 1, 2$, and if there is a measure preserving transformation $\phi : M_1 \to M_2$ such that $\phi \circ T_1(x) = T_2 \circ \phi(x)$ for all $x \in M_1$. Equivalently, T_1 and T_2 are isomorphic if there is a measure preserving transformation $\phi : X_1^+ \to X_2^+$ such that $\phi \circ T_1 = T_2 \circ \phi$.

Definition 4.3.2. Two measure preserving transformations $T_1 : (X_1, \mathcal{R}_\mu, \mu) \to (X_1, \mathcal{R}_\mu, \mu)$ and $T_2 : (X_2, \mathcal{R}_\nu, \nu) \to (X_2, \mathcal{R}_\nu, \nu)$ are called *conjugate*, $T_1 \sim T_2$, if there is a measure algebra isomorphism $\Phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ such that $\Phi^{-1} \circ T_2^{-1} = T_1^{-1} \circ \Phi^{-1}$.

Theorem 4.3.3 (=Theorem G)**.** *Two non-Archimedean measure preserving transformations are isomorphic if and only if they are conjugate.*

Proof. An isomorphism ϕ induces a measure algebra isomorphism Φ directly; define for $A \in \mathcal{R}_\mu$, $\Phi(A) = \phi(A)$.

Conversely, since for any $A \in \mathcal{R}_\mu$, $\|A\|_\mu = \|\Phi(A)\|_\nu$, we may restrict ourselves to X^+. Choose $x \in X^+$. Since X^+ equipped with the $\mathcal{R}_\mu$-topology is a Hausdorff space, the singleton $\{x\}$ is closed. In particular, $X\backslash\{x\}$ is open, and therefore, there is a sequence $\{A_i\}_{i\in\mathbf{N}} \subset \mathcal{R}_\mu$ such that $X\backslash\{x\} = \cup_i A_i$. Then

$$B_n = X\backslash \bigcup_{i=1}^{n} A_i$$

defines a descending sequence in $\mathcal{R}_\mu$, such that $\bigcap_n B_n = \{x\}$. Because Φ preserves intersections $\{\Phi(B_n)\}_{n\in\mathbf{N}}$ is a descending sequence as well. We would like to define

$$\phi(x) = \bigcap_n \Phi(B_n),$$

however, we should check that this intersection is a singleton. To do so, first suppose it is empty, then $\mathcal{B} = \{\Phi(B_n)\}_{n\in\mathbf{N}}$ is a shrinking collection, and by the continuity condition in lemma 4.1.7 on ν $\lim_{\mathcal{B}} \|B\|_\nu = 0$, i.e., there is for any $\epsilon > 0$ a $N \in \mathbf{N}$ such that for any $n > N$,

$$\|(\Phi(B_n))\|_\nu = \|B_n\|_\mu < \epsilon.$$

So, in particular, also for $0 < \epsilon < N_\mu(x)$. But this contradicts that for all $n \in \mathbf{N}$ we have $N_\mu(x) \leq \|B_n\|_\mu$. Secondly, we check that $\cap_n \Phi(B_n)$ contains at most one element. Suppose $y \in \cap_n \Phi(B_n)$, let $U_y \in \mathcal{R}_\nu$ contain y. Then $\Phi^{-1}(U_y) \cap B_n \neq \emptyset$ for all $n \in \mathbf{N}$. Hence $x \in \Phi^{-1}(U_y)$. Now suppose that $y' \in \cap_n \Phi(B_n)$ and $y \neq y'$. Because $\mathcal{R}_\nu$ is a separable ring, it is possible to choose U_y, $U_{y'}$ containing y and y' respectively, such that $U_y \cap U_{y'} = \emptyset$, and hence $\Phi^{-1}(U_y) \cap \Phi^{-1}(U_{y'}) = \emptyset$. However, x is contained by both $\Phi^{-1}(U_y)$ and $\Phi^{-1}(U_{y'})$. It follows that ϕ is a well defined map.

It is left to check that ϕ is indeed an isomorphism. Let x and B_n be as above, then

$$T_1^{-1}(x) = T_1^{-1}(\bigcap_n B_n) = \bigcap_n (T_1^{-1} B_n),$$

and hence,

$$\begin{aligned} \phi(T_1^{-1}(x)) &= \bigcap_n (\Phi \circ T_1^{-1} B_n) = \bigcap_n (T_2^{-1} \circ \Phi B_n) \\ &= T_2^{-1}(\bigcap_n \Phi B_n) = T_2^{-1}(\phi(x)). \end{aligned}$$

Therefore, $\phi(T_1(x)) = T_2(\phi(x))$, for all $x \in X^+$. That ϕ is measure preserving follows since for any $A \in \mathcal{R}_\mu$, $\phi(A) = \Phi(A)$. □

A measure preserving transformation $T : X \to X$ induces an operator $U_T : L^1(\mu) \to L^1(\mu), f \mapsto f \circ T$. Classically, U_T is unitary if and only if T is invertible. The analogous statement here is weaker.

Lemma 4.3.4. *If T is an invertible transformation, then U_T preserves the norm on $L^1(\mu)$.*

Proof. Let $f \in L^1(\mu)$, then

$$\begin{aligned} \|U_T f\|_\mu &= \sup_{x \in X} |f(Tx)| N_\mu(x) \\ &= \sup_{x \in X} |f(Tx)| N_\mu(Tx) = \|f\|_\mu. \end{aligned}$$

□

Definition 4.3.5. Let $T : (X, \mathcal{R}_\mu, \mu) \to (X, \mathcal{R}_\mu, \mu)$ and $S : (Y, \mathcal{R}_\nu, \nu) \to (Y, \mathcal{R}_\nu, \nu)$ be two invertible measure preserving transformations. They are called *spectrally isomorphic* if there is an invertible linear isometry

$$W : L^1(\mu) \to L^1(\nu), \text{ such that } U_S \circ W = W \circ U_T.$$

Lemma 4.3.6. *Conjugacy implies spectral isomorphy.*

Proof. Let $\phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ be the measure algebra isomorphism. Recall that $S(X)$ is the space of step function on $\mathcal{R}_\mu$. Define a linear map

$$U_\phi : S(X) \to S(Y), \sum_i^N \alpha_i \chi_{A_i} \mapsto \sum_i^N \alpha_i \chi_{\phi(A_i)}.$$

The map U_ϕ is invertible, since ϕ is a bijection. Let $f = \sum \alpha_i \chi_{A_i}$ be an element of $S(X)$ such that $\cap_i A_i = \emptyset$, then

$$\begin{aligned} \|U_\phi(f)\|_\nu &= \sup_{y \in \cup_i \phi(A_i)} |\alpha_i| \cdot \inf\{\|U\|_\nu : U \in \mathcal{R}_\nu, y \in U\} \\ &= \sup_{x \in \cup_i A_i} |\alpha_i| \cdot \inf\{\|\phi(U)\|_\nu : U \in \mathcal{R}_\nu, x \in U\} \\ &= \sup_{x \in \cup_i A_i} |\alpha_i| \cdot \inf\{\|U\|_\nu : U \in \mathcal{R}_\mu, x \in U\} = \|f\|_\mu. \end{aligned}$$

So U_ϕ is an isometry. Extend U_ϕ by continuity. Because S and T are conjugate it follows that $U_S \circ U_\phi = U_\phi \circ U_T$. □

We do not know if spectral isomorphy induces conjugacy. However, the following lemma gives some conditions on a spectral isomorphism to come from a measure algebra isomorphism.

Lemma 4.3.7. *(=Theorem H) Let $W : L^1(\mu) \to L^1(\nu)$ be an isometry. If for all bounded functions $f, g \in L^1(\mu)$, $W(fg) = W(f)W(g)$, and if for all bounded functions $f \in L^1(\mu)$,*

$$\int_X f d\mu = \int_Y W(f) d\nu,$$

then there exists a measure algebra isomorphism $\phi : (\mathcal{R}_\mu, \mu) \to (\mathcal{R}_\nu, \nu)$ such that $W = U_\phi$.

Proof. For any $B \in \mathcal{R}_\mu$, $W(\chi_B^2) = W(\chi_B)W(\chi_B) = W(\chi_B)$, and therefore, $W(\chi_B)$ only takes values in $\{0, 1\}$. Because $W(\chi_B)$ is an integrable function in $L^1(\nu)$, there is a $A \in \mathcal{R}_\nu$ such that $W(\chi_B) = \chi_A$. It follows that W sends indicator functions to indicator functions. Define $\phi(B) = A$, with B and A as above. Because, $\mu(B) = \int \chi_B d\mu = \int \chi_A d\nu = \nu(A)$, it is only left to prove that ϕ preserves unions and complements. Take $B, C \in \mathcal{R}_\mu$, then

$$\chi_{B\cup C} = \chi_B + \chi_C - \chi_B \chi_C,$$

and hence,

$$\chi_{\phi(B\cup C)} = \chi_{\phi(B)} + \chi_{\phi(C)} - \chi_{\phi(B)}\chi_{\phi(C)} = \chi_{\phi(B)\cup\phi(C)}.$$

To prove that ϕ preserves complements we first prove that $\phi(X) = Y$. Suppose that $\phi(X) = A$, and that there is a nonempty $A' \subset Y\backslash A$ in $\mathcal{R}_\nu$. Then there is a nonempty $D \in \mathcal{R}_\mu$ such that $W^{-1}(\chi'_A) = \chi_D$. Because ϕ preserves unions, $\chi_A = W(\chi_D + \chi_{X\backslash D}) = \chi_{A'} + \chi_{\phi(X\backslash D)}$, and in particular, $A' \subset A$, which is a contradiction. Hence, $\phi(X) = Y$, and therefore for any $B \in \mathcal{R}_\mu$,

$$\chi_Y = \chi_B + \chi_{\phi(X\backslash B)},$$

and thus $\phi(X\backslash B) = Y\backslash\phi(B)$. □

4.4 Entropy

One of the most important invariants for dynamical systems is the entropy. We will discuss a version of measure-theoretic entropy for non-Archimedean measures. In some special cases, including some of our examples, this non-Archimedean entropy coincides with the topological entropy. Our treatment is based on that of Walters in [86]. All logarithms in this section are in base 2.

Partitions, subalgebras and entropy.

Let $(X, \mu, \mathcal{R}_\mu)$ be a probability space.

Definition 4.4.1. A *partition* of $(X, \mu, \mathcal{R}_\mu)$ is a collection of disjoint elements of $\mathcal{R}_\mu$ which cover X.

A partition α is called finite if it contains only finitely many elements. The set of partitions is a partial ordered space, where $\alpha \prec \beta$ means that each element of α is a union of elements of β. The collection which consists of all, possibly empty, unions of elements of α forms a finite subalgebra of $\mathcal{R}_\mu$. This algebra is denoted by $\mathcal{A}(\alpha)$. There is a one-to-one correspondence between finite subalgebras and partitions in the following way. Let $\mathcal{C} = \{C_1, ..., C_k\}$ be a finite subalgebra, then the nonempty intersections of the form $B_1 \cap B_2 \cap ... \cap B_k$ where $B_i = C_i$ or $B_i = X \backslash C_i$ is a partition denoted by $\alpha(\mathcal{C})$. This correspondence respects the partial order in the sense that $\mathcal{A}(\alpha) \subset \mathcal{A}(\beta)$ if and only if $\alpha \prec \beta$.

Definition 4.4.2. For two partitions $\alpha = (A_1, ..., A_k)$ and $\beta = (B_1, ..., B_n)$ define

$$\alpha \vee \beta = \{A_i \cap B_j : 1 \leq i \leq k, 1 \leq j \leq n\},$$

which itself is a partition.

The operation $\vee$ is defined similarly for finite subalgebras such that $\alpha(\mathcal{A} \vee \mathcal{C}) = \alpha(\mathcal{A}) \vee \alpha(\mathcal{C})$ and $\mathcal{A}(\alpha \vee \beta) = \mathcal{A}(\alpha) \vee \mathcal{A}(\beta)$. For a finite partition α we define the significant part $\mathcal{M}(\alpha) = \{A \in \alpha : \|A\| > 0\}$. Let $M(\alpha)$ be the cardinality of $\mathcal{M}(\alpha)$, this number is submultiplicative in the sense that $M(\alpha \vee \beta) \leq M(\alpha)M(\beta)$.

Definition 4.4.3. Let $\mathcal{A}$ be a finite subalgebra, and let $\alpha(\mathcal{A}) = (A_1, ..., A_n)$ be the corresponding partition, then the *measure entropy* with respect to $\mathcal{A}$ is defined by

$$H_\mu(\mathcal{A}) = \min_{A \in \mathcal{M}(\mathcal{A})} \|A\| \log M(\alpha).$$

Lemma 4.4.4. *The measure entropy possesses the following properties:*

1. $H_\mu(\mathcal{A} \vee \mathcal{B}) \leq H_\mu(\mathcal{A}) + H_\mu(\mathcal{B})$,
2. *for any measure preserving transformation* T, $H_\mu(\mathcal{A}) = H_\mu(T^{-1}\mathcal{A})$.

Proof.

$$\begin{aligned} 1.\ H_\mu(\mathcal{A} \vee \mathcal{B}) &= \min_{A_i \cap B_j \in \mathcal{M}(\alpha \vee \beta)} \|A_i \cap B_j\| \log M(\alpha \vee \beta) \\ &\leq \min_{A_i \cap B_j \in \mathcal{M}(\alpha \vee \beta)} (\|A_i\|, \|B_j\|)(\log M(\alpha) + \log M(\beta)) \\ &\leq H_\mu(\mathcal{A}) + H_\mu(\mathcal{B}) \end{aligned}$$

2. Both $\|\cdot\|$ and M are invariant under T. □

Definition 4.4.5. Let $T : X \to X$ be a measure preserving transformation, then the *measure entropy* of T with respect to a finite subalgebra $\mathcal{A}$ is

$$h_\mu(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{A} \vee T^{-1}\mathcal{A} \vee ... \vee T^{-(n-1)}\mathcal{A})$$

To prove that $h_\mu(T, \mathcal{A})$ exists we need the following lemma.

Lemma 4.4.6 ([86], Theorem 4.4)**.** *If $\{a_n\}_{n\in N}$ is a sequence in $\mathbf{R}$ which satisfies $a_n \geq 0$, $a_{n+m} \leq a_n + a_m$, then $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to* $\inf_n \frac{a_n}{n}$.

Proposition 4.4.7. *The limit $h_\mu(T, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{A} \vee T^{-1}\mathcal{A} \vee ... \vee T^{-(n-1)}\mathcal{A})$ exists.*

Proof. Let $a_n = H_\mu(\mathcal{A} \vee T^{-1}\mathcal{A} \vee ... \vee T^{-(n-1)}\mathcal{A})$, then by lemma (4.4.4)

$$\begin{aligned} a_{n+m} &= H_\mu(\mathcal{A} \vee T^{-1}\mathcal{A} \vee ... \vee T^{-(n-1)}\mathcal{A} \vee T^{-n}\mathcal{A} \vee ... T^{-(n+m-1)}) \\ &\leq a_n + H_\mu(T^{-n}\mathcal{A} \vee ... T^{-(n+m-1)}) \\ &= a_n + a_m. \end{aligned} \tag{4.2}$$

Then the result follows by application of lemma (4.4.6). □

Example 4.4.8. We consider $\mu_{(-2,-2,5)}$ with values in $\mathbf{Q}_5$ on $\{0,1,2\}^{\mathbf{N}}$ like in example (4.2.7), and compute the entropy with respect to several partitions. First, let $\alpha = \{U_0, U_1, U_2\}$. Then

$$H_\mu(\sigma, \mathcal{A}(\alpha)) = \lim_{n\to\infty} \frac{1}{n} 5^{-n} \log(3^n) = 0.$$

Second, let $\beta = \{A_0 = U_0, A_1 = U_1 \cup U_2\}$. Since elements of $U_{i_0...i_n}$ with $i_0...i_n \in \{0,1\}^{n+1}$ are contained in $T^{-n}A_{i_n} \vee ... \vee A_{i_0}$,

$$H_\mu(\sigma, \mathcal{A}(\beta)) = \lim_{n\to\infty} \frac{1}{n} \log(2^n) = \log(2).$$

Definition 4.4.9. The *measure-theoretic entropy* of a measure preserving transformation T is

$$h_\mu(T) = \sup_{\mathcal{A}} h_\mu(T, \mathcal{A}),$$

where the supremum is taken over all finite subalgebras.

Remark 4.4.10. The Kolmogorov-Sinai Theorem (e.g.[86], 4.9) for classical measure entropy states that if $\mathcal{A}$ is a finite algebra such that $\vee_{n=-\infty}^{\infty} T^n \mathcal{A} \doteq \mathcal{B}$, where $\mathcal{B}$ is the σ-algebra, then $h_\mu(T) = h_\mu(T, \mathcal{A})$. Here $\mathcal{C} \doteq \mathcal{B}$ means that for any $C \in \mathcal{C}$ there is a $B \in \mathcal{B}$ such that $\mu(C \triangle B) = 0$, and vice versa. Example 4.4.8 shows that such theorem is not true in this non-Archimedean setting. The partition α generates the covering ring, however, the entropy with respect to β is greater then the entropy with respect to α.

Proposition 4.4.11. *The measure-theoretic entropy is invariant under conjugacy.*

Proof. This follows because $\|\cdot\|$ is invariant under measure algebra isomorphism and moreover, for any finite subalgebra, $M(\alpha)$ is invariant. $\square$

Connections with topological entropy

An other form of entropy is topological entropy, introduced by Adler, Konheim and McAndrew [3]. In this section we study the topological entropy on the zero dimensional topology induced by a separating covering ring $\mathcal{R}_\mu$ and the connections between measure entropy and topological entropy.

The analog of a measure preserving transformation in topological dynamics is simply a homeomorphism $T : X \to X$. Two such homeomorphisms $T : X \to X$, $S : Y \to Y$ are called *topologically conjugate* if there is a homeomorphism $\phi : X \to Y$ such that $\phi \circ T = S \circ \phi$.

Let X be a compact space. Any open cover $\mathcal{U}$ of X has a finite subcover. Denote with $N(\mathcal{U})$ the least cardinality of all subcovers of $\mathcal{U}$.

Definition 4.4.12. The *topological* entropy with respect to an open cover $\mathcal{U}$ is

$$H_{top}(\mathcal{U}) = \log N(\mathcal{U}).$$

The collection of open covers of X behaves in many senses similar to the collection of partitions. For instance, it is partially ordered. For two open covers $\mathcal{U}$ and $\mathcal{W}$ we write that $\mathcal{U} < \mathcal{W}$ if every member of $\mathcal{W}$ is a subset of a member of $\mathcal{U}$, we say that $\mathcal{W}$ is a *refinement* of $\mathcal{U}$. If $\mathcal{U} < \mathcal{W}$, then

$$H_{top}(\mathcal{U}) \leq H_{top}(\mathcal{W}).$$

The *join* of two covers is defined by

$$\mathcal{U} \vee \mathcal{W} = \{U \cap W : U \in \mathcal{U}, W \in \mathcal{W}\}.$$

In particular, $\mathcal{U} < \mathcal{U} \vee \mathcal{W}$, and $N(\mathcal{U} \vee \mathcal{W}) \leq N(\mathcal{U})N(\mathcal{W})$ and it follows that

$$H_{top}(\mathcal{U} \vee \mathcal{W}) \leq H_{top}(\mathcal{U}) + H_{top}(\mathcal{W}).$$

Definition 4.4.13. Let $T : X \to X$ be a homeomorphism, then the *topological entropy of* T *with respect to* $\mathcal{U}$ is given by:

$$h_{top}(T,\mathcal{U}) = \lim_{n\to\infty} \frac{1}{n} H_{top}(\mathcal{U} \vee T^{-1}\mathcal{U} \vee ... \vee T^{-(n-1)}\mathcal{U}).$$

The proof that this limit exist is very similar to the proof of proposition (4.4.7) together with the observation that $H_{top}(T^{-1}\mathcal{U}) = H(\mathcal{U})$.

Definition 4.4.14. The *topological entropy* of a homeomorphism $T : X \to X$ is defined by:

$$h_{top}(T) = \sup_{\mathcal{U}} h_{top}(T,\mathcal{U}),$$

where the supremum is taken over all open covers of X.

Note that if $\mathcal{U}' \subset \mathcal{U}$ is a finite subcover, then $\mathcal{U} < \mathcal{U}'$ and $h_{top}(T,\mathcal{U}) \leq h_{top}(T,\mathcal{U}')$. Therefore, it is sufficient to take the supremum over all finite open covers of X.

Partitions and coverings

Let us consider a probability space $(X, \mu, \mathcal{R}_\mu)$. A partition α of X is itself an open cover, because elements of α are required to be elements of $\mathcal{R}_\mu$. The two partial orders coincide on collection of partitions, i.e., $\alpha \prec \beta$ if and only if $\alpha < \beta$. The other way round, given a finite open cover $\mathcal{U} = \{U_1, ..., U_k\}$ one can construct a partition $\alpha(\mathcal{U})$ by taking the nonempty sets of the form

$$A_1 \cap ... \cap A_k,$$

where A_i is U_i or $X \backslash U_i$.

Lemma 4.4.15. *Let* $\mathcal{U}, \mathcal{W}$ *be two finite open covers, then*

$$\alpha(\mathcal{U} \vee \mathcal{W}) = \alpha(\mathcal{U}) \vee \alpha(\mathcal{W}),$$
$$\textit{and if } \mathcal{U} < \mathcal{W} \textit{ then } \alpha(\mathcal{U}) \prec \alpha(\mathcal{W}).$$

Proof. The fist assertion follows from the identity $(U \cap W)\backslash(A \cup B) = U\backslash A \cap W\backslash B$. Note for the second assertion that, since $\alpha(\mathcal{U})$ and $\alpha(\mathcal{W})$ are both partitions, it sufficient to prove that any $B \in \alpha(\mathcal{W})$ is contained in an $A \in \alpha(\mathcal{U})$. This is clear, for let $B = B_1 \cap ... \cap B_l$ be a nonempty element of $\alpha(\mathcal{W})$, where B_i is W_i or $X\backslash W_i$. Then B is contained in at least one of the sets of the form $A_1 \cap ... \cap A_l$ where $B_i \subset A_i$. □

Theorem 4.4.16. *(=Theorem I) Let $(X, \mathcal{R}_\mu, \mu)$ be a compact probability space satisfying $\|X\| = 1$, and let $T : X \to X$ be a measure preserving transformation, then*

$$h_\mu(T) \leq h_{top}(T),$$

and equality holds if $X_0 = \emptyset$ and if for any nonempty set $A \in \mathcal{R}_\mu$, $\|A\| = 1$.

Proof. Note that for any finite open cover $\mathcal{U}$ with corresponding partition $\alpha(\mathcal{U})$

$$\begin{aligned} \min_{A \in \alpha(\mathcal{U})} \|A\| \log(M(\alpha(\mathcal{U}))) &\leq \log(N(\alpha(U))), \text{ and therefore,} \\ h_\mu(T, \alpha(\mathcal{U})) &\leq h_{top}(T, \alpha(\mathcal{U})). \end{aligned}$$

Therefore, $h_\mu(T) = \sup_{\mathcal{U}} h_\mu(T, \alpha(\mathcal{U})) \leq \sup_{\mathcal{U}} h_{top}(T, \alpha(\mathcal{U})) = h_{top}(T)$, where the last equality follows from the fact that all partitions are open covers. This proves the first part of the theorem. The second part follows, because if $X_0 = \emptyset$ and if for all $A \in \mathcal{R}_\mu$ $\|A\| = 1$, then all inequalities above are equalities. □

Remark 4.4.17. The shift map in example 4.2.7 satisfies the conditions of this theorem if one takes the probability vector $\mathbf{q} = (\mathbf{q_0}, ..., \mathbf{q_{p-1}})$ such that all $|q_i| = 1$.

Samenvatting voor niet-wiskundigen

Once a man was telling a story, it was
a very good story too, and it made him very
happy, but he told it so fast that nobody
understood it.

Louise Bourgeois. [12]

Als u wel eens een wiskundevoordracht heeft gehoord dan zal het niet verbazen dat Louise Bourgeois, voordat zij kunstenares werd, wiskunde studeerde. Het communiceren over wiskunde is extreem moeilijk, en vereist behalve een goede spreker een geconcentreerde luisteraar. Wees dus voorbereid.

Stelt u zich eens voor dat de volgende vergelijking moet worden opgelost:

$$(x^p - x)(y^p - y) = 0 \tag{4.3}$$

waar x en y de variabelen zijn en p een van tevoren vastgelegd priemgetal. Nu zijn er verscheidene dingen die u zich kunt afvragen. Zoals, waarom zou dat moeten, en, wat wiskundiger van aard, wat mogen de oplossingen zijn, bijvoorbeeld gehele getallen, breuken, complexe getallen of nog andere zaken. Het antwoord op de eerste vraag is eenvoudig: omdat ik met deze vergelijking mooi een methode kan illustreren die in dit proefschrift ontwikkeld is. Het antwoord op de tweede vraag is, naar mijn mening, relevanter en ga ik u nu geven. De getallen x en y moeten allebei een geheel maar niet negatief getal zijn kleiner dan p. De vermenigvuldiging en optelling van twee getallen gaat modulo p; dat betekent dat het gaat zoals u gewend bent, maar steeds als u klaar denkt te zijn met het sommetje moet u er nog net zo vaak p vanaf trekken totdat u weer een geheel getal in handen heeft dat kleiner is dan p. Deze verzameling getallen met deze vermenigvuldiging en optelling heet “een lichaam met p elementen” en noteren we met $\mathbf{F}_p$.

Laten we eens even een voorbeeld bekijken en $p = 5$ kiezen. We willen nu dus

$$(x^5 - x)(y^5 - y) = 0 \tag{4.4}$$

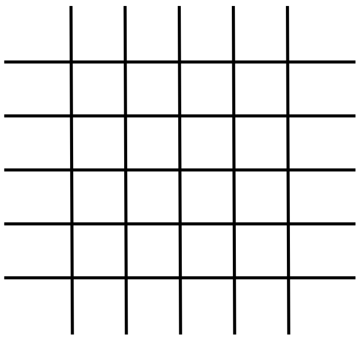
oplossen. We hoeven daarvoor alleen maar

$$x^5 - x = 0 \text{ of } y^5 - y = 0$$

op te lossen. Er zijn maar vijf niet-negatieve gehele getallen kleiner dan 5 dus we kunnen alle mogelijkheden opsommen.

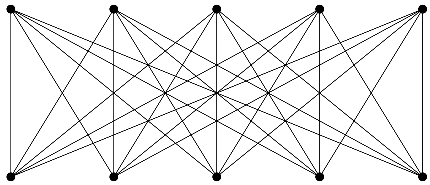
x	0	1	2	3	4
x^5	0	1	32	243	1024
x^5 modulo 5	0	1	2	3	4

Er valt direct op dat de eerste en de laatste rij identiek zijn en dat heeft grote gevolgen voor de oplossingen van vergelijking (4.4). Alle paren (x, y) zijn oplossingen van de vergelijking. Dit komt niet doordat 5 zo'n bijzonder priemgetal is; dit werkt voor alle priemgetallen. Grafisch kunnen we de oplossingen als volgt weergeven:



Figuur 4.1: De horizontale lijnen zijn de punten waarvoor $y^5 - y = 0$, terwijl de verticale lijnen de punten zijn waarvoor $x^5 - x = 0$.

Dit is een verzameling horizontale en verticale lijnen, die elkaar snijden. Dit kunnen we dan weer grafisch weergeven in een graaf, zie figuur 4.2. Dat doen we als volgt: voor iedere gladde kromme zetten we een punt neer. Dus in ons voorbeeld zetten we vijf punten neer voor de vijf horizontale lijnen en nog eens vijf punten voor de verticale lijnen. Twee punten verbinden we met elkaar als de twee lijnen elkaar snijden. Dus in figuur 4.2 zitten evenveel lijnen als snijpunten figuur in 4.1. In ons voorbeeld krijgen we dan dit:



Figuur 4.2: De graaf.

Dit is allemaal mooi en aardig, maar de vergelijking 4.4 hadden we al opgelost, dus waarom zouden we ons nog druk maken om deze graaf. Dat is omdat we eigenlijk geinteresseerd zijn in een iets moeilijker vraag. Het is natuurlijk niet zo dat in een (hedendaags) proefschrift alleen maar gezocht wordt naar de oplossingen van (4.3) in $\mathbf{F}_p$, maar het komt langs bij het oplossen van een andere vergelijking. In plaats van ons te focussen op $\mathbf{F}_p$ kijken we naar uitdrukkingen van de vorm:

$$a + bT + cT^2,$$

waarbij a, b, c weer elementen van $\mathbf{F}_p$ zijn. Of, sterker nog, we malen niet over de eindigheid van de som, maar kijken zelfs naar uitdrukkingen van de vorm

$$\sum_{i=n}^{\infty} a_i T^i = a_n T^n + a_{n+1} T^{n+1} + \cdots ,$$

waarbij alle a_i weer in $\mathbf{F}_p$ zitten en n een geheel getal is, dat eventueel zelfs negatief mag zijn. De verzameling van al deze uitdrukkingen noteren we met $\mathbf{F}_p(T)$ en is een voorbeeld van een lichaamsuitbreiding. Laten we nu eens terug gaan naar het geval $p = 5$ en de vergelijking iets moeilijker maken. We willen nu bijvoorbeeld het volgende oplossen:

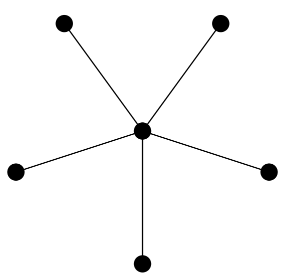
$$(x^5 - x)(y^5 - y) = 4T^2 + 3T^6 + T^{10}, \tag{4.5}$$

waarbij de oplossingen uitdrukkingen mogen zijn in $\mathbf{F}_p(T)$. Dit is veel moeilijker. Het is in eerste instantie in het geheel niet duidelijk dat er überhaubt oplossingen zijn, of aan de andere kant, dat er maar eindig veel zijn. Nu is het leuke dat je met behulp van de getekende graaf iets zinnigs kunt zeggen over het aantal oplossingen. Ik kan wel verklappen dat er in $\mathbf{F}_p(T)$ maar eindig veel zijn. Maar wat als we het aantal toegestane oplossingen nog wat groter maken en het lichaam $\mathbf{F}_p(T)$ nog wat verder uitbreiden? Elk zo'n uitbreiding komt met een getal waarmee we kunnen meten hoeveel groter het lichaam is geworden. We noemen dit de “graad” van de uitbreiding.

Ik zal het nut van de graaf uitleggen met behulp van pakpapier. Alle oplossingen van (4.4) in alle (nou ja, alle, we kunnen hier ook gaan voor “heel veel”) lichaamsuitbreidingen van $\mathbf{F}_p(T)$ vormen samen een “kromme”, en die zijn heel goed te vergelijken met pakpapier, maar wel een beetje onhandig pakpapier.

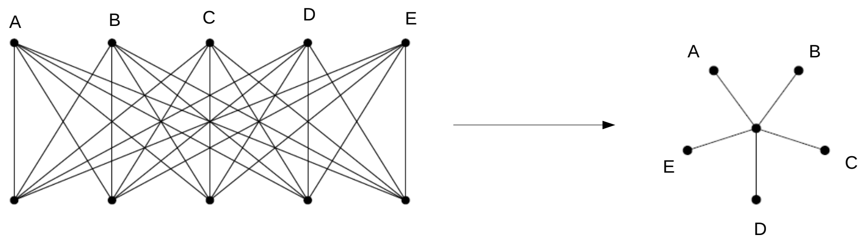
Stel, u wil een bal inpakken en één van die eeuwige creativelingen in uw omgeving blijkt net te hebben gespeeld met het pakpapier en daar in zitten knippen en plakken tot een grote, ingewikkelde en onoverzichtelijke constructie onstond. U heeft haast dus u besluit dit gewoon te gebruiken. Tegen de tijd dat de bal is ingepakt blijkt dat overal op de bal vijf lagen papier zit. Het minimale aantal lagen dat u kon bereiken met dit pakpapier heet de *gonaliteit* van het pakpapier. Zoiets kunnen we ook met de kromme hierboven doen; we wikkelen hem zo handig mogelijk om een bol heen en het aantal lagen kromme op de bol heet de gonaliteit van de kromme. Nu geldt er *hoe groter de gonaliteit van de kromme is, des te groter de graad van de lichaamsuitbreiding mag zijn zodat er toch nog maar eindig veel oplossingen zijn.* Het is dus interessant te weten hoe groot de de gonaliteit van de kromme is. Maar het uitrekenen van de gonaliteit van een kromme is helemaal niet makkelijk.

Nu blijkt de graaf handig te zijn. De gonaliteit van de kromme is groter dan wat we de gonaliteit van de graaf noemen. De gonaliteit van een graaf is net anders gedefinieerd dan de gonaliteit van een kromme. In plaats van met een kromme een bol in te pakken gaan we een graaf op een boom leggen. Een boom is een graaf waarin je geen rondjes kunt lopen. Bijvoorbeeld:



Figuur 4.3: Een voorbeeld van een boom (vergelijk ook de omslag van dit proefschrift).

Een methode om de graaf boven op deze boom te leggen is als volgt:



Figuur 4.4: De gelabelde punten A, B, C, D, E links worden respectievelijk gelegd op de gelabelde punten A, B, C, D, E rechts. De ongelabelde punten links worden allen gelegd op het ongelabelde punt rechts. Iedere lijn links tussen A en het ongelabelde punt wordt rechts neergelegd op de lijn tussen A en het ongelabelde punt. De overige lijnen worden op een vergelijkbare manier neergelegd.

Merk op dat er op ieder lijntje van de boom, vijf lagen lijntjes van de graaf komen te liggen. De gonaliteit van deze graaf is dan ook 5. Nu is het vervelende dat de gonaliteit van een graaf eigenlijk ook niet zo heel veel gemakkelijker uit te rekenen is. Het is wel mogelijk om een ondergrens te geven, waarbij, grappig genoeg, de eigenschappen van de graaf als elektrisch netwerk worden gebruikt.

In dit proefschrift is onder andere deze methode voor het begrenzen van de gonaliteit van krommen ontwikkeld.

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Curriculum Vitae

Janne Kool behaalde in 2006 haar bachelor diploma wiskunde aan de Universiteit Utrecht. In het academisch jaar 2005-2006 studeerde zij via het Erasmus programma aan Helsinki Yliopisto in Finland. Na terugkomst begon zij haar master wiskunde in Utrecht. Hiervoor studeerde zij in 2008 cum laude af met haar onderzoek over symplectische vezelbundels onderbeleiding van Marius Crainic. Bovendien behaalde zij het propedeutisch examen automome kunst van de Hogeschool van de Kunsten Utrecht. Sinds 2008 is zij promovenda in Utrecht onder begeleiding van Gunther Cornelissen. In 2011 organiseerde zij samen met drie andere promovendi de zomerschool van de European Women in Mathematics te Leiden. Vanaf september 2013 zal zij als postdoc gaan werken aan het Max Planck instituut voor Wiskunde in Bonn.

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