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★**Geometry of algebraic curves. Volume II.**

With a contribution by Joseph Daniel Harris.

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Here, after a quarter of a century, is finally the sequel to Volume I [E. Arbarello et al., *Geometry of algebraic curves. Vol. I*, Grundlehren Math. Wiss., 267, Springer, New York, 1985; MR0770932 (86h:14019)]. That volume, essentially devoted to properties of a single curve, appeared as number 267 of the Springer Grundlehren series. The present volume has its focus on their moduli and one might say that its series number, 268, attests to the confidence of the publisher in its eventual appearance.

As in Volume I, curves are assumed to be defined over the complex field and, as is well known, a nonsingular complex projective curve is essentially the same thing as a compact Riemann surface. This classical, basic fact accounts in part for the richness of this subject, especially when one is dealing with moduli, for one can approach it then from several angles, each of which has something interesting to say that helps our understanding from other viewpoints. The authors have evidently taken this to heart; witness the chapters devoted to Teichmüller theory and the combinatorial aspects of the moduli spaces of curves. But choices had to be made, for the field is vast and, despite its 19th-century origins, still so much in flux that waiting for the dust to settle might have required another quarter of a century. The reviewer thinks that the authors made their choices wisely.

As is explained in the book's introduction, its focus shifted towards moduli as it evolved. This must have had a lot to do with the appearance of D. B. Mumford's 1983 paper [in *Arithmetic and geometry, Vol. II*, 271–328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983; MR0717614 (85j:14046)], which opened a whole new window on the subject and was at the beginning of many fascinating developments, among them the remarkable interaction with string theory that involved intersection numbers on moduli spaces of curves. Mumford's paper was, in turn, built on his construction with Deligne of the moduli stack of stable curves, of which Mumford, Knudsen and Gieseker proved in the late 1970's that the underlying variety (and its pointed version) are projective. Since their introduction, these fundamental moduli spaces have been central to most developments in this area and it is therefore appropriate that a substantial part (almost half) of the book is devoted to their construction and first properties.

The authors must have had in mind a reader who is conversant with a reasonable amount of working knowledge of algebraic geometry, without necessarily being completely familiar with the contents of Volume I or being at ease with Hilbert schemes or general deformation theory. Indeed, the book begins with a general discussion of Hilbert schemes in Chapter IX (the chapter numbering of the two volumes is consecutive), then gets to the topology and algebraic geometry of degenerations into nodal curves, leading up to the construction of the Deligne-Mumford moduli stack of stable point curves in Chapter XII. The completeness of the Deligne-Mumford extension is established earlier in Chapter XII, a fact that should also be of interest to topologists (for it implies that the mapping class group acts cocompactly on the Harvey bordification of Teichmüller space). After an investigation of the Picard group of the Deligne-Mumford moduli stack in Chapter XIII, the proof that the underlying variety is projective is finished in Chapter XIV. The next two chapters discuss basic Teichmüller theory and the construction of smooth Galois covers of moduli spaces of stable pointed curves respectively and constitute a bridge to the goal that according to the authors (and the reviewer agrees) gives the book its eventual focus, namely to give a complete exposition of Kontsevich's proof of the Witten conjecture. This begins with a chapter introducing and identifying relations in the tautological Chow algebras that one has defined for these moduli spaces. Then there is a chapter that treats the triangulation of (thickened, extended) Teichmüller space by means of metrized ribbon graphs, followed by a chapter giving applications, among them Harer's bound for the virtual cohomological dimension of the mapping class group. Here we also encounter the volume computation that is the key to Kontsevich's proof of the Witten conjecture. The proof of the latter is then brought to completion in Chapter XX, both in the context of the KdV hierarchy and in the context of matrix models. The methods employed are also used to give, following Kontsevich, a quick derivation of the virtual Euler characteristic of the mapping class groups (originally due to Harer and Zagier).

Volume I came with many exercises, which not only helped the reader to absorb the material, but often also played an educational, 'Bourbaki-esque' role by pointing the reader to many interesting results that the authors presumably did not want to put in the main text. This volume also has plenty of exercises, but the prominence of this secondary function depends on the chapter. An extensive bibliography of about 700 items and a lengthy index make this monumental treatise very useful for quick reference.

To the best of the reviewer's knowledge, there is in this subject only one other book that covers similar ground and that is the one by J. D. Harris and I. L. Morrison [*Moduli of curves*, Grad. Texts in Math., 187, Springer, New York, 1998; MR1631825 (99g:14031)]. But the style, ambition and emphasis of that book are quite different, and although there is overlap, there is no inclusion. The book under review is very helpful for reference and for learning the details, whereas the book of Harris and Morrison is perhaps better suited for classroom use.

Summing up: every algebraic geometer should have a copy, while a Teichmüller person and a topologist should seriously consider getting one.

Reviewed by *E. Looijenga*