# Moduli spaces of cubic hypersurfaces through a period map

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# Moduli spaces of cubic hypersurfaces through a period map

## Moduliruimten van kubische hyperoppervlakken via een periodenafbeelding

(met een samenvatting in het Nederlands)

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# Contents

# Introduction

It is always difficult to pinpoint the start of a development. My particular corner of modern geometry emerged at the beginning of the $19^{th}$ century, curiously prompted by a question that had fascinated me early on in my mathematical education.

At high school I had learned to solve integrals of the form

$$\int \frac{dx}{\sqrt{x^2 + ax + b}}$$

by using the $\arcsin(x)$ and $\arccos(x)$ as primitives. My whole high school career I thought this was the hardest mathematics would ever get. Until, somewhere during my third year at university, the power 2 became a 3 and a new, unforeseen world opened up behind this problem[1].

Integrals of the form $\int \frac{dx}{\sqrt{x^3+ax+b}}$ first appeared in the question of determining the arc length along an ellipse, and were studied in this form by the Norwegian mathematician Niels Henrik Abel and his contemporaries. Their first step was to introduce a new variable, and write the integral as

$$\int \frac{dx}{y}$$

over the curve $C$ is given by $y^2 = x^3 + ax + b$. The cubic plane curve which arises here came to be known as an *elliptic curve.*

Abel discovered that the value of this integral depends not only on the endpoints of the path, but also on the path itself. This in contrast to the case of the complex plane, where the value a holomorphic integral is independent of the path. In particular, the integral around a closed loop could be non-zero. He showed that the collection of such non-trivial loops could be spanned—up to "homotopy"—by two elements. These were dubbed the *periods*, and the "lattice" of paths they generated the *period lattice.* Most importantly, Abel found that the integral was well-defined *up to periods.*

[1] It is said that in mathematics, the most difficult step is often that from 2 to 3. In this case that is certainly true!

The form $\frac{dx}{y}$ can be written as a Fourier series $f(\tau)d\tau$, where $\tau$ is a coordinate of the complex upper half-plane and $f(\tau)$ is a power series constant on the period lattice. In this way the study of elliptic integrals became intertwined with the study of doubly-periodic functions. We mention this, although we will not be considering this perspective here.

The truly remarkable part of this discovery is in fact the implied map from an elliptic curve $C$ to an automorphic form. This map, known as the *period map*, in this case turns out to be an isomorphism. This isomorphism was conjectured by the Italian mathematician Torelli, who died tragically in the First World War; his name is applied to a class of theorems relating algebraic and arithmetic data using such a period map. What makes this map so remarkable is that it bridges two different worlds: it maps from an algebraic-geometric setting to a complex-analytic one. Whenever such a map exists, we can use properties of either world to understand the other.

### Geometric invariant theory

A question that arose at the end of the $19^{th}$ century (and is one of the motivating questions of this thesis) is how to meaningfully divide out a group action on a variety. Particularly in moduli problems, often one wants to divide out a group of symmetries, a choice of basis for example. One also wants to understand the quotient object *as a variety*, in a geometric way. It was Arthur Cayley[2] who proposed considering the set of functions left invariant by the group. Around the turn of the century David Hilbert showed that in the case of $\mathrm{SL}_n$ this set is a finitely-generated ring, and hence potentially the function ring associated to some quotient variety.

The modern answer, as given by David Mumford in the 1950's, came to be known as *Geometric Invariant Theory*, or GIT, after his book with that title. The idea can be most easily described in the case of an affine variety $A = \mathrm{Spec}(R)$. If there is a group $G$ acting on $A$, this action induces an action on $R$ by $(g \cdot f)(a) = f(g^{-1}a)$ for $g \in G, f \in R, a \in A$. We can ask which functions in $R$ are invariant under this action, and write $R^G$ for these. This $R^G$ is clearly a sub-algebra of $R$, and we can consider its spectrum. A point on $\mathrm{Spec}(R^G)$ corresponds to an orbit of $G$ on the original $A$, and so forms a good candidate for the quotient variety, written $A//G$.

Problems arise, however, when the group does not act freely–it leaves points fixed–or is not proper. In GIT this is approached by removing the bad points, taking the "good" quotient, and later compactifying this quotient space in a geometrically meaningful way. However, to do this compactification

[2]Incidentally, Cayley was also one of the first to study cubic surfaces, another chapter of this thesis.

explicitly is hard work: the case analysis of boundary strata can be very messy. For this reason it has been done explicitly only in a few low-dimensional cases; binary forms, curves and surfaces of low degree, and configurations of linear subspaces of a vector space spring to mind. The modern state of affairs described by Shah [**37**], [**36**] (for sextic curves and especially for quartic surfaces) in the 1980's, Allcock [**2**] (for cubic surfaces and threefolds) in the 1990's, and recently Laza [**21**] (for cubic fourfolds), have all involved a "tour de force", sometimes with the assistance of a computer.

It is for this reason that our interest in the period map, and the period domain, was aroused. In cases where a period map exists that can be shown to be a local isomorphism, we can bypass these GIT difficulties by studying the automorphic forms in the image.

## The period map for K3 surfaces

The period map for elliptic curves is well-understood, but for the purpose of this thesis it is not the most helpful example. As an illustration and motivation of the next chapters, we here summarize some results on K3 surfaces. Early in the 1980's, K3 surfaces attracted a lot of interest precisely because they permitted such a clear period map.

A K3 surface $X$ is a compact, complex surface which is regular—i.e. $H^1(X, \mathcal{O}_X) = 0$–and supports a nowhere-vanishing holomorphic 2-form. This means that a surface $X$ comes with a nowhere-zero section $\omega_X \in H^0(X, \Omega^2_X)$. This space is one-dimensional, so together with its conjugate $\overline{\omega}_X \in H^2(X, \mathcal{O}_X)$, the form in fact determines the Hodge decomposition of

$$H^2(X, \mathbb{C}) = H^0(X, \Omega^2_X) \oplus H^1(X, \Omega^1_X) \oplus H^2(X, \mathcal{O}_X).$$

On $H^2(X, \mathbb{C})$ there is a bilinear form (the cup product) coming from the intersection form on homology, which we denote by $(\cdot)$. It corresponds to a lattice $\Lambda = 3U \oplus 2E_8(-1)$, where $U$ is the standard hyperbolic lattice and $E_8$ is the unimodular lattice of rank 8. We see that $(\omega_X \cdot \omega_X)$ must be zero for this form, but the associated Hermitian form $\omega_X \cdot \overline{\omega_X}$ must be positive, since $(\omega_X \wedge \overline{\omega_X})$ is compatible with the orientation form of $X$ (as a real 4-manifold).

We use these two conditions to define

$$H_+ = \{x \in \Lambda \otimes \mathbb{C} | (x \cdot x) = 0, (x \cdot \overline{x}) > 0\}.$$

This space has two connected components, which are interchanged by complex conjugation. We write $H'_+$ for one of these components, so that $H_+ = H'_+ \cup \overline{H'_+}$. Since $\omega_X$ is only unique up to scalar, we are in fact interested only in $\mathbb{D}_{K3} = \mathbb{P}(H'_+)$, which we call the *period domain*.

The *period map* (for K3 surfaces) is then defined as

$$P(X) = [\omega_X] \in \mathbb{D}_{K3}$$

from a family of K3 surfaces to the period domain,

The surprising fact about this setup is that the period map turns out to be a local isomorphism, a result due to André Weil. More recently, the "global" version of this isomorphism was also proved. That is, if we fix the "marking"—to identify $\Lambda$ with $H^2(X,\mathbb{Z})$—and consider the universal family, then every K3 surface is determined uniquely by its period! As mentioned above, this result is referred to as the *strong Torelli theorem* for K3 surfaces. See [**6**] for a detailed discussion of K3 surfaces and the period map there.

## Automorphic forms

In the cases we will consider in this thesis, the image of a period map lies in a (quotient of a) symmetric space. In such space all points look the same; it is defined by the property that geodesic symmetries of a point—fixing the point and reversing geodesics through it—are isometric. A good reference is Helgason [**18**]. We will be interested in two cases, where the spaces are *Hermitian* and allow for an action by a reflection group. Necessarily these spaces have totally geodesically embedded hypersurfaces which are also locally symmetric, to act as the mirrors of the group. Firstly there is the case of a symmetric space of type I that can be constructed from a hyperbolic vector space—with a form of signature $(n, 1)$. This is also referred to as a *(complex) ball*. The type IV case lies in a space with signature $(n, 2)$, this resulting space is said to be of *type IV*. For such symmetric domains there exists a natural compactification which has come to be known as the *Baily-Borel compactification*. We briefly discuss it below, in the easier case of a ball.

Let us suppose we have an $n+1$-dimensional complex vector space $V$ with a Hermitian form of signature $(n, 1)$ on it. We will write $V_+$ for those elements where the Hermitian form is positive. Suppose further we have a lattice acting on $V$, that is a discrete group $\Gamma \subset U(V)$ of finite covolume (i.e. $U(V)/\Gamma$ has finite volume). This group preserves the form and hence also acts on $V_+$. We are interested in the quotient of this action, $\Gamma\backslash V_+$, a so-called *ball quotient*.

An automorphic form of degree $d$ is a $\Gamma$-invariant function $f : V_+ \to \mathbb{C}$ such that $f(\lambda v) = \lambda^{-d} f(v)$. In the case that $V_+$ has dimension 1, we can think of the modular group $\mathrm{PGL}(2,\mathbb{Z}) \subset U(V)$, and we remark that the form $\frac{dx}{y}$ is an example of an automorphic form; it is of degree 1. Strictly speaking, in this $n = 1$ case there are also requirements on its growth near the origin.

Write $A^{\Gamma}_{d}$ for the space of degree-$d$ $\Gamma$-forms. Taking the direct sum over all $d$, we see that the space $A^{\Gamma}_{\bullet}$ forms a graded algebra over $\mathbb{C}$.

By considering local (as opposed to global) $\Gamma$-invariant sections we can define the *automorphic line bundle* of holomorphic forms. We state the main theorem of Baily-Borel, its statements will be explained below.

THEOREM (Baily-Borel). *The algebra $A^{\Gamma}_{\bullet}$ is finitely generated, and $A^{\Gamma}_{-k}$ vanish for $k > 0$. The algebra separates $\Gamma$-orbits so we have maps*

$$\Gamma\backslash V_{+} \to \mathrm{Spec}(A^{\Gamma}_{\bullet}), \qquad \Gamma\backslash\mathbb{P}(V_{+}) \to \mathrm{Proj}(A^{\Gamma}_{\bullet}).$$

*There exist extensions $\widehat{V}_{+} \supset V_{+}$ and $\mathbb{P}(\widehat{V}_{+}) \supset \mathbb{P}(V_{+})$, by addition of some $\Gamma$-orbits, so that*

$$\Gamma\backslash\widehat{V}_{+} \cong \mathrm{Spec}(A^{\Gamma}_{\bullet}), \qquad \Gamma\backslash\mathbb{P}(\widehat{V}_{+}) \cong \mathrm{Proj}(A^{\Gamma}_{\bullet}).$$

*as homeomorphisms. This extension is now known as the* Baily-Borel compactification.

The Baily-Borel compactification works by associating boundary strata to isotropic subspaces $J$ of $V$. Here we consider the origin as an isotropic subspace. All others are of dimension 1, in which case $\mathbb{P}(J)$ is just a single point, on the boundary of $\mathbb{P}(V_{+})$. These points are referred to as *cusps*. Note that since the boundary consists only of isolated points, when $n > 1$ we do not have to specify growth conditions on our automorphic forms near the boundary—points are then of themselves of codimension at least 2. We can now give the compactification as a set:

$$\widehat{V}_{+} = V_{+} \sqcup \left( \bigsqcup_{J \text{isotropic}} V_{+}/J^{\perp} \right) \tag{1}$$

Writing the compactification in this way is slightly superfluous, but it gives a canonical way to name each boundary component. To make this into a topological space, the Satake (or 'horocyclic') topology is added. All we will say about this is that it extends the ordinary Hausdorff topology of $V_{+}$: a basis for this topology is given by a (Haussdorf) basis of $V_{+}$ with additionally sets centered on boundary points, restricted to $\widehat{V}_{+}$.

Remark that the simplest instance of this compactification is the $j$-line of elliptic curves. The linear group $\mathrm{PGL}(2,\mathbb{Z})$ acts on the upper half plane, and we can identify the quotient with the well-known fundamental domain. This domain is compactified by adding the $\mathrm{PGL}(2,\mathbb{Z})$-orbit of the origin; the only isotropic subspace of $\mathbb{C}$. The topology is defined as follows: open subsets are the (images of) intersections of interior disks, and disks centered at the origin—these are classically referred to as *horocycles*. We repeat that in this case automorphic forms are required to satisfy growth conditions.

Looijenga (in [**23, 24**]) has recently described a generalization of this construction, which gives a Baily-Borel type compactification for symmetric domains with a deleted $\Gamma$-arrangement of hyperplanes. These are of interest to us, since in the cases we will study, the period map lands in just such an arrangement complement in a symmetric space.

## This thesis

We can now briefly state the theme of this thesis. In several examples where there is a suitable period map, we use it to transfer the Baily-Borel compactification of the space of automorphic forms to a compactification of the moduli space under consideration. The aim is to bypass the unwieldy GIT analysis. We particularly focus on the period map for cubic fourfolds, using the work of Voisin in her thesis [**40**] proving the Torelli theorem for cubic fourfolds. This indicates that the period map for cubic fourfolds is a local isomorphism, and we use it to produce a period map for cubic surfaces and threefolds.

The rest of this thesis is structured as follows. We start with an introductory chapter, to recall fundamental results from geometric invariant theory and the GIT and Baily-Borel compactifications, and some general remarks about period maps and their limiting behavior. We take the opportunity to establish notation for the rest of this thesis; a table is provided at the end of the chapter.

The second chapter is a theoretical one, and develops a general framework for the period map if there is a type IV Hodge structure; it is strongly motivated by the example of K3 surfaces, where the period domain is a type IV symmetric space. This chapter gives an overview of work already presented in [**26**].

In the third chapter we describe a residue construction, which provides us with a period map for the algebraic varieties which we study in the remainder of this thesis. We apply it to the case of cubic fourfolds, and discuss its properties as a period map.

The theory is strongly motivated by specific examples, and the last chapters of this book will illustrate and apply this theory. Using our knowledge of the period map for fourfolds we will be able to describe the moduli space of cubic threefolds in the fourth chapter, without detailed knowledge of the GIT.

The final chapter is dedicated to cubic surfaces. With all the powerful machinery developed in the earlier chapters, we are able to provide a new perspective on the moduli of cubic surfaces in a few short pages.

We emphasize that many of the results have already been published, some (particularly about cubic surfaces) have been around for a long time. We have opted to repeat these, to illustrate the theory.

CHAPTER 1

# Preliminaries

The purpose of this section is to treat the subjects touched on in the introduction in more detail. We will follow same order: first we recall invariant theory, then we give some background on general period maps, and finally we take a closer look at Baily-Borel compactifications. We take the opportunity to establish notation which will reappear throughout the thesis. A table of notation can be found at the very end of this section. Throughout, we will assume our algebraic varieties to be defined over $\mathbb{C}$, the algebraically closed field of complex numbers.

## 1. Geometric invariant theory

We return first to the field of invariant theory. That is, we wish to define a quotient for the action of an algebraic group $G$ on an algebraic variety $X$, and to understand the quotient as an algebraic variety again. Unless stated otherwise, in this section we will assume all our groups and varieties to be algebraic.

Let us begin by trying to be more explicit what we mean by a quotient. Essentially we are looking for a variety whose points correspond to $G$-orbits of $X$. This is not always possible, as orbits of an action can be of geometrically very different nature.

In accordance with the literature, we make a distinction between a *categorical quotient* and a *geometric quotient*, which we define below.

Definition 1.1. A *categorical quotient* of a variety $X$ by $G$ is a morphism $\pi : X \to Y$ which is constant on $G$-orbits and with the following universal property: if $\psi : X \to Z$ is another $G$-invariant morphism, then $\psi$ factors over $\pi : X \to Y$.

Note that this definition can be applied to any suitable category, it does not use the fact that we are interested in geometry. This is added in the following notion.

Definition 1.2. A categorical quotient $\pi : X \to Y$ is called a *geometric quotient* if in addition we have that the fibres $\pi^{-1}(y)$ are $G$-orbits. If furthermore the pullback $\pi^* : \mathcal{O}_Y(U) \to (\pi^*\mathcal{O}_X(U))^G$ is an isomorphism, it is said to be a *good geometric quotient.*

Note that the quotient map $\pi : X \to Y$ is continuous by definition. Since the fibres $\pi^{-1}(y)$ are non-empty $G$-orbits, it is also seen to be surjective.

Categorical quotients are usually denoted by $\pi : X \to X/G$, whereas geometric quotients are given a double line $\pi : X \to X//G$.

**1.1. Affine quotients.** Let us suppose that both $G$ and $X = \mathrm{Spec}(R)$ are affine. Suppose also that we have a collection $\{f_i\} \in R^G = \mathbb{C}[X]^G$ of $G$-invariant functions, that is $g^* f_i = f_i$ for all $g \in G$ and all $i$. Then the map

$$\phi : X \to \mathbb{A}^n \qquad x \mapsto (f_1(x), \ldots, f_n(x))$$

is constant on $G$-orbits, that is it does not distinguish between $x$ and $g \cdot x$. This means it sends each $G$-orbit to a single point, and hence that the image variety $\phi(X)$ is a candidate for the quotient variety. Of course we wish to ensure that different orbits are separated by the map. Therefore the idea for the construction of a quotient is to find 'enough' invariants to make this work. The first condition we encounter is that the ring of all invariant functions $R^G \subset R$ is finitely generated. This condition is a property of the group $G$.

Definition 1.3. We say that an algebraic group $G$ is *linearly reductive* if for every surjective $G$-equivariant map $\phi : V \to W$ of $G$-representations, the induced map on invariants $\phi^G : V^G \to W^G$ is also surjective.

One of Hilbert's great results was to show that this property suffices to guarantee that the ring of invariants is finitely generated. Hilbert was also responsible for the theorem that the action of $\mathrm{SL}(V)$ on a (complex) vector space $V$ is linearly reductive, hence $\mathbb{C}[V]^{\mathrm{SL}(V)}$ is finitely generated.

If this is the case, we can find a finite generating set of invariants—not necessarily independent—to construct the map. However, it still can occur that different orbits are mapped to the same point in the quotient space. This happens if two orbits $Gx$ and $Gx'$ have a common point in their closure. Of course, if they intersected in the interior, they would form one orbit. When the orbit-closures meet, i.e. when we have a $z \in \overline{Gx} \cap \overline{Gx'} \neq \emptyset$, we encounter that $f_i(x) = f_i(z) = f_i(x')$ for all invariants, so *closure-equivalent* orbits cannot be separated by invariants. Therefore all we can hope for by considering invariants, is to separate orbits into closure-equivalence classes. David Mumford showed that the situation does not get worse than this: considering these equivalence classes gives us a geometric quotient.

THEOREM 1.4 (Mumford, *GIT*). *If $G$ is a linearly reductive group acting on an affine variety $X = \operatorname{Spec}(R)$, then the quotient map*

$$\Phi : X \to X//G = \operatorname{Spec}(R^G)$$

*is surjective and gives a bijection between points of $X//G$ and closure-equivalence classes of $G$-orbits in $X$.*

Furthermore, $\Phi$ sends $G$-invariant closed subsets $Z \subset X$ to closed varieties $\Phi(Z) \subset X//G$.

A last comment to make is that this quotient construction is relatively easy on points lying in a closed orbit.

DEFINITION 1.5. A point $x \in X$ is said to be *stable* if it satisfies these two conditions:

(1) the orbit $Gx \subset X$ is a closed set,
(2) the stabilizer subgroup $G_x = \{g \in G | gx = x\}$ is finite.

We refer to the set of all stable points for the $G$-action as $X^s \subset X$.

These conditions ensure that the map $g \to g\cdot x$, which sends $G$ to $Gx \subset X$, is a *proper* morphism when $x$ is stable.

**1.2. Projective quotients.** The above construction is not yet satisfactory, as is illustrated by the following example. Consider the construction of (complex) projective space, i.e. let $V$ be a (complex) vector space and let $\mathbb{C}^*$ act on it by multiplication. The only invariants of $\mathbb{C}[V]$ under this action are the constants $\mathbb{C}$ itself, so we would be forced to write the quotient $V/\mathbb{C}^* = \operatorname{Spec}(\mathbb{C}) = \{0\}$, a single point! This is clearly not projective space. What has happened here is that the origin $0 \in V$ is in the closure of all the orbits, so invariants will not be able to distinguish between them. We know projective space can constructed by first deleting the origin, and then taking the quotient by $\mathbb{C}^*$. This is generalized by what is called the Proj-construction, which we describe in some detail.

Recall first that every projective space $\mathbb{P}(V)$ comes with a line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$, which locally describes coordinate functions. In an affine neighborhood $U_0$ where $z_0 \neq 0$, say, we have that $\mathcal{O}_{\mathbb{P}(V)}(1)|_{U_0} \cong \mathbb{C}[\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}]$. This line bundle can be restricted to any (projective) subvariety $X \subset \mathbb{P}(V)$ to define $\mathcal{O}_X(1)$. Conversely, if we are given any line bundle $\mathcal{L} \to X$, we can use this to describe a map $X \to \mathbb{P}(V)$, where $V = H^0(X, \mathcal{L})$ is the space of sections. If $\mathcal{L}$ is very ample, we find an isomorphism of $\mathcal{L} \cong \mathcal{O}_X(1)$, and the map becomes an embedding.

Suppose that instead of the (ample) line bundle $\mathcal{L} \to X$ itself, we are given a graded ring $A_\bullet$ of sections of the line bundle, so $A_k = H^0(X, \mathcal{L}^k)$. We see that $A_0 = \mathbb{C}$ are the constant sections and that $A_k = 0$ for all negative integers $k$. A ring graded over $\mathbb{Z}$ comes with an action by $A_0^*$ (in this case $\mathbb{C}^*$) which preserves the grading.

Now the spectrum of this ring $\mathrm{Spec}(A_\bullet)$ has the structure of a cone over $X$. This can be seen as follows. The algebra $A_\bullet$ contains a unique maximal ideal $A_+ = \bigoplus_{k\geq 1} A_k$ sometimes known as the *irrelevant ideal*. This $A_+$ is fixed by the multiplicative action of $A_0^*$, so the point $\mathrm{Spec}(A_+) \in \mathrm{Spec}(A_\bullet)$ becomes the vertex of the cone. All other maximal ideals of $A_\bullet$ correspond to $A_0^*$-orbits in $\mathrm{Spec}(A_\bullet) - \mathrm{Spec}(A_+)$. Taking the quotient by this action results in the underlying set of $\mathrm{Proj}(A_\bullet) = X$.

This gives a natural geometric way to recover $X$ out of the graded algebra: delete the vertex of $\mathrm{Spec}(A_\bullet)$, and contract the fibres by the $\mathbb{C}^*$ action. This construction can be extended to any graded ring $A$ that is (finitely) generated in degree 1, and the resulting variety is denoted $\mathrm{Proj}(A)$. As a set, $\mathrm{Proj}(A)$ consists of the prime ideals of $A$ that do not contain $A_+$, but it inherits the structure of a scheme from its construction in $\mathrm{Spec}(A)$. In the case we have been working in, when $A_0 = \mathbb{C}$ is algebraically closed, $\mathrm{Proj}(A)$ is projective, justifying the name.

Now let there be an action of a group $G$ on our projective $X$. The Proj allows us to construct a quotient in a similar manner as in the previous section. We aim to consider $G$-invariant sections of the line bundle over $X$, and take the Proj of this algebra. However, to do this we need to pay attention as to how the group acts on line bundle.

We are led to the introduction of so-called *linearizations* of actions on projective varieties. Let us consider a line bundle $\mathcal{L} \to X$ on our variety, and an action of an algebraic group $G$ on $X$. We will say that a *linearization* of $\mathcal{L}$ is an induced action of $G$ on $\mathcal{L}$, such that the zero section of $\mathcal{L}$ is $G$-invariant, and so that the diagram:

$$\begin{tikzcd} G \times \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\ G \times X \arrow[r] & X \end{tikzcd}$$

is commutative. It is a theorem that such a linearization exists in the case we are discussing. With such a linearization, we are able to define stability with respect to the bundle.

DEFINITION 1.6. Let $\mathcal{L}$ be an ample line bundle over a projective $X$, linearized with respect to a reductive algebraic group $G$.

- A point $x \in X$ is said to be *semi-stable* (with respect to $\mathcal{L}$) if there exists a $k > 0$ and a $G$-invariant section $s$ of $\mathcal{L}^k$ so that $s(x) \neq 0$.

- If in addition its stabilizer $G_x$ is finite, and the orbit $G \cdot x$ is closed, then $x$ is said to be *stable* (with respect to $\mathcal{L}$).
- A point that is not semi-stable is said to be *unstable.*

Recall our illustration of $\operatorname{Spec}(A_\bullet)$ as a cone over $X$ with vertex $\operatorname{Spec}(A_+)$, where $A_k = H^0(X, \mathcal{L}^k)$. To say that a point is unstable is to say that it cannot be distinguished from this vertex by $G$-invariant sections. When we take the $\operatorname{Proj}(A_\bullet^G)$, all unstable points will automatically be deleted, since they together make up the vertex $\operatorname{Spec}(A_+^G)$. The Proj deletes this unstable locus, and contracts the fibres by the $\mathbb{C}^*$-action defined by the linearization.

We denote the set of stable and semistable points by $X^s$ and $X^{ss}$, respectively. Along these lines we now are able to produce a categorical quotient of $X^{ss}$ by $G$.

Theorem 1.7 (Mumford). *Assume $X$ is projective and $\mathcal{L}$ is ample. There exists a categorical quotient*

$$\pi : X^{ss} \to X^{ss}/G,$$

*that is, the restriction of $\pi$ to $X^s$ defines a good geometric quotient $X^s//G$. We define*

$$A_\bullet = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n)$$

*as the algebra of sections of all powers of $\mathcal{L}$. Then $X^{ss}/G \cong \operatorname{Proj}(A_\bullet^G)$ ($G$-invariant sections) defines the categorical quotient. It is a projective variety. There exists an open subset $U \subset \operatorname{Proj}(A_\bullet^G)$ such that $\pi^{-1}(U)$ is identified with $X^s$.*

Therefore, apparently, the theory is finished. Yet nothing has been said about determining which points of $X$ are (semi-)stable. In applications where $X$ is some space parameterising a family of varieties, and $G$ some group of symmetries, often points $x \in X$ representing smooth members of the family will be *stable.* Singular degenerations can, however, be anything. There are criteria in many cases to determine the (semi-)stability of such orbits; however, the explicit construction of a quotient often requires a lot of work and in-depth, *ad hoc* analysis.

## 2. Period maps and monodromy

We now consider the next topic, period maps, in more detail. The concept of a variation of Hodge structure is needed to understand the period map, we will assume the reader is familiar with ordinary Hodge theory. A good reference for what follows is the proceedings of a workshop on trancendental methods, edited by Griffiths [**12**].

As an introduction, let $\mathcal{X} \to S$ be a family of polarized algebraic varieties. The most general kind of period map associates to a point $s \in S$ the Hodge structure of the fiber $X_s$ in a certain degree $k$. It may be ill-defined in an algebraic subset of $S$, so we write $S^\circ$ for its domain. However, even where it is defined we can only identify the cohomology lattices $H^k(X_s, \mathbb{Z})$ (modulo torsion) up to monodromy induced by $\pi_1(S)$. Therefore we first consider the map on a universal cover of $S^\circ$ to a target domain $\mathbb{D}$, and then reduce by the action of monodromy and its induced action by $\Gamma$ on $\mathbb{D}$ respectively. We then say that $P : S^\circ \to \Gamma\backslash\mathbb{D}$ is a *period map.*

What $\mathbb{D}$ means here was intentionally left a little vague. What is needed to define a Hodge structure is a *filtration* on $H$, a series of increasing linear subspaces in $H$. Therefore $\mathbb{D}$ sits inside some (large) flag variety—inside a product of Grassmannians—and is called the classifying space. There group $\Gamma$ is usually some group of symmetries of $\mathbb{D}$ induced by monodromy on $S$, that cannot be distinguished by the periods.

We quickly encounter situations where we wish to extend the period map to points $o \in S - S^\circ$. Let us consider this locally, so we suppose we have a family over a punctured disk, $\mathcal{X} \to \Delta^*$, having a de Rham bundle $\mathbb{H}$ which we want to extend nicely over the puncture. 'Nicely' here means that $\mathbb{H}$ has the same limit along all paths, particularly along paths looping around the puncture.

Consider a generic fiber $H$ of $\mathbb{H}$ over $p \in \Delta^*$. Any vector in this space can be transported along paths in $\Delta^*$. In particular, loops in $\Delta^*$ will transport such a vector to a potentially different vector in $H$. Thus the fundamental group $\pi_1(\Delta^*, p)$ gives rise to an action of a monodromy group $\Gamma$ on $H$. In particular, if we have a filtration of $H$, the monodromy actions will send it to a possibly different filtration. Therefore $\Gamma$ acts also on $\mathbb{D}(H)$, the space of Hodge filtrations. If we want to extend a period map across the puncture we can do so only up to monodromy, so we must find a description that takes this into account.

We denote the action of the canonical generator of the fundamental group (looping around the puncture counterclockwise) by $T$. We know that some power of $T$ is unipotent, and so by a suitable base change we may assume that already $T$ itself is unipotent. Hence $T - 1$ is nilpotent, and so too will be the "logarithm":

$$N = \log(T) = -\sum_{k \geq 1} \frac{1}{k+1}(1 - T)^k.$$

Since $(T - 1)$ is nilpotent, this is a finite sum.

Let $s$ be a coordinate for the disk $\Delta^*$ (after the base change). Its universal cover $\tilde{\Delta}$ is the upper half-plane $\mathfrak{h} = \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$, with covering map

$s = \pi(\tau) = e^{2\pi\sqrt{-1}\tau}$.

$$\begin{tikzcd} \mathfrak{h} = \tilde{\Delta} \arrow[r, "P"] \arrow[d, "\pi"'] & \mathbb{D}(H) \arrow[d] \\ \Delta^* \arrow[r, "\Gamma\backslash P"] & \Gamma\backslash\mathbb{D}(H) \end{tikzcd}$$

The period map now corresponds to a map $P : \mathfrak{h} \to \mathbb{D}(H)$ which will satisfy $P(\tau+1) = TP(\tau)$. This can now be reduced to a map from $\Delta^*$ by defining $\phi(\tau) = \exp(-\tau N)P(\tau)$, so that

$$\phi(\tau+1) = e^{-N\tau}\left(e^{-N}T\right)P(\tau) = \phi(\tau)$$

(since $e^N = T$) descends to a single-valued map $\phi : \Delta^* \to \mathbb{D}(H)$.

It was proved by Griffiths and Schmid [**14**] that this map extends holomorphically across $0 \in \Delta$, and so forms a candidate for the extension of the period map. However, $\phi(0)$ need not even lie in $\widehat{\mathbb{D}(H)}$. We will come back to this in the next chapter.

## 3. A Baily-Borel compactification for arrangement complements

As mentioned before, we aim to study period maps whose (smooth) image lies in some symmetric space as an open set. We therefore consider—in the spirit of [**23, 24**]—Baily-Borel-type compactifications of spaces that are the complement of hyperplane arrangements.

We will do this for the easier ball quotient case, to give an idea, and refer to the appendix of [**26**] for the more general case of type IV symmetric spaces. For the applications in this thesis, knowledge of ball quotients will suffice. Let us suppose therefore that $V$ is an $n+1$-dimensional complex vector space with a Hermitian form of signature $(n,1)$ on it. As in the introduction, write $V_+$ for those elements where the Hermitian form is positive. Suppose further we have an arithmetic group acting on $V$, that is a discrete group $\Gamma \subset U(V)$ of finite covolume (i.e. $U(V)/\Gamma$ has finite volume). This group preserves the form and hence also acts on $V_+$. Furthermore, the set $V_+$ is $\mathbb{C}^*$-invariant, and hence $\mathbb{P}(V_+)$ can be defined: it is this space that is commonly known as a *complex ball.*

Recall that an automorphic form of degree $d$ is a $\Gamma$-invariant function $f : V_+ \to \mathbb{C}$ such that $f(\lambda v) = \lambda^{-d}f(v)$. We write $A_d^\Gamma$ for the space of degree-$d$ $\Gamma$-forms, and $A_\bullet^\Gamma$ for the algebra obtained as the direct sum over all $d$. The Baily-Borel theory mentioned in the introduction 1 shows that $\mathrm{Proj}(A_\bullet^\Gamma) \cong \Gamma\backslash\mathbb{P}(\widehat{V}_+)$, where the space $\widehat{V}_+$ is produced by adding boundary strata to $V_+$ for isotropic subspaces $J \subset V$. The space $\widehat{V}_+$ looks like a cone with base $\mathbb{P}(\widehat{V}_+)$, with as vertex the boundary point corresponding to the origin.

Now consider a hyperplane $H \subset V$. There are two distinct possibilities: either $H$ is positive definite, with signature $(n, 0)$, or it has signature $(n-1, 1)$, in which case $H^\perp$ is positive definite. We wish to restrict to the second case, where $H$ has hyperbolic signature. However, it can also occur that $H$ is degenerate: that it contains an isotropic vector of $V$, and none of the vectors with product $\neq 0$ with it. In this case $H \subset H^\perp$, and $H^\perp/H$ is positive definite.

DEFINITION 3.1. A hyperplane $H \subset V$ is said to be $\Gamma$-*rational* if it has hyperbolic signature, is non-degenerate, and if its stabilizer $\Gamma_H \subset \Gamma$ is again an arithmetic subgroup of $U(H)$.

We can now be more explicit in what we mean by an arrangement.

DEFINITION 3.2. Suppose we have a set $\Theta$ of $\Gamma$-rational hyperplanes, permuted by the action of $\Gamma$. If this set consists of a finite union of $\Gamma$-orbits, we say $\Theta$ is a $\Gamma$-*arrangement*.

When we are presented with a $\Gamma$-arrangement $\Theta$, every element $H \in \Theta$ can be restricted to $H_+ = H \cap V_+$. Since the collection is locally finite, the union $\bigcup_{H\in\Theta} H_+$ is closed in $V_+$. We will denote the complement $V_+ - \bigcup_{H\in\Theta} H_+$ by $V_\Theta$, and refer to it as a $\Gamma$-*arrangement complement*. The deletion is done in such a way as to preserve the action of $\Gamma$, hence we can consider $\Gamma\backslash V_\Theta$ as a subspace of $\Gamma\backslash V_+$. Indeed, it is the complement of finitely many hyperplanes of $\Gamma\backslash V_+$.

With respect to such an arrangement we define $\Gamma$-meromorphic forms. These are a generalization of the automorphic forms described above.

DEFINITION 3.3. Let $f_d : V_\Theta \to \mathbb{C}$ be holomorphic, $\Gamma$-invariant and homogeneous of degree $-d$. If it has poles of order at most $d$ along the planes $H_+$ for all $H \in \Theta$, we say it is $\Gamma$-meromorphic on $V_\Theta$.

These forms define an algebra.

DEFINITION 3.4. The space of all $\Gamma$-meromorphic automorphic forms of degree $-d$ is denoted $A^\Gamma_{\Theta,d}$. Summing over all degrees gives the algebra $A^\Gamma_{\Theta,\bullet}$.

THEOREM 3.5 (Looijenga [**23, 24**], Heckman-Looijenga [**17**]). *This algebra $A^\Gamma_{\Theta,\bullet}$ is finitely generated by elements of positive degree. The algebra separates $\Gamma$-orbits so we have maps*

$$\Gamma\backslash V_\Theta \to \mathrm{Spec}(A^\Gamma_{\Theta,\bullet}), \qquad \Gamma\backslash\mathbb{P}(V_\Theta) \to \mathrm{Proj}(A^\Gamma_{\Theta,\bullet}).$$

*There exist extensions* $\widehat{V}_\Theta \supset V_\Theta$ *and* $\mathbb{P}(\widehat{V}_\Theta) \supset \mathbb{P}(V_\Theta)$*, by addition of some* $\Gamma$*-orbits, so that*

$$\Gamma\backslash\widehat{V}_\Theta \cong \mathrm{Spec}(A^\Gamma_{\Theta,\bullet}), \qquad \Gamma\backslash\mathbb{P}(\widehat{V}_\Theta) \cong \mathrm{Proj}(A^\Gamma_{\Theta,\bullet}).$$

*as homeomorphisms. These extensions come with a* $\Gamma$*-equivariant stratification by subvarieties.*

Let us see how the compactification arises. We need to take account of all intersections of hyperplanes in $V_+$, so we introduce $\mathcal{L}(\Theta)$ as the indexing set of linear subspaces arising as intersections of members of $\Theta$ meeting $V_+$. This includes $V$ itself as the 'trivial' intersection. We consider, for every $L \in \mathcal{L}(\Theta)$, isotropic subspaces $J \subset L$. We add to $V_\Theta$ a boundary stratum $V_\Theta/(L \cap J^\perp)$ for every isotropic $J$ (including the origin, which gives $V_\Theta/L$).

As a set, the new compactification is then:

$$\widehat{V}_\Theta = V_\Theta \sqcup \left( \bigsqcup_{L\in\mathcal{L}(\Theta), J\subset L \text{ isotropic}} V_\Theta/(L \cap J^\perp) \right).$$

This allows us to speak of a natural stratification of $\widehat{V}_\Theta$. The points formally adjoined to $V_\Theta$ are sometimes referred to as *cusps*. The strata they define are known as *boundary components*. The set $\widehat{V}_\Theta$ is given the Satake topology, as referred to in the introduction. Remark that when $\Theta$ is empty, this construction recovers precisely the ordinary Baily-Borel compactification of the introduction.

## 4. A few comments on singularities

The subject of singularity theory is a classic and well-developed one. We will not attempt to summarize it here, however, a few definitions are called for.

The word 'singularity' means many different things in different fields of mathematics. The first occurrence that is relevant to this thesis, is in complex analysis. Holomorphic functions that are undefined at certain points can be said to be singular there. The singularity is said to be a *pole* (sometimes a *regular singularity*) if the Laurent expansion of the function around the point is bounded from below, and this bound is called the *order* of the pole. If no such bound exists as in the case of $\exp[z^{-1}]$ around the origin, the singularity is called *essential*. We will exclusively be interested in poles of meromorphic functions, and will not refer to these as singularities to avoid confusion with the following.

Hypersurface singularities in our sense will be points on a variety where all partial derivatives of the defining form vanish. A lot has been said about the classification of hypersurface singularities (see the work of Arnol'd and many others), we will introduce two classes that will play an important role further on.

The first class of interest are the Kleinian singularities, also known as Du Val singularities or rational double points. These appear as the quotient of $\mathbb{C}^2$ by a finite subgroup of PSL(2), the resulting surface is singular at the origin. The remarkable confluence of group theory, representation theory and geometry around such singularities has given rise to a beautiful theory around the *McKay correspondence*, which has recently seen much development but which is far outside the scope of this thesis. Suffice it to say that by their very definition such singularities have a universal covering with a *finite* covering group. That is, they have a finite local fundamental group: the subgroup of PSL(2) used to construct the singularity. Taking suspensions (that is, raising the dimension in which they are embedded by adding squares of new variables) of such surface singularities does not change this property, which conversely characterizes this class of *simple singularities*. They are classified in the table below, with a representative form of the defining equation around the singularity. There is a bijective correspondence between finite groups of $\mathrm{PSL}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{R})$, we include the latter form of the groups.

| Name | function | Group in SO(3) |
|---|---|---|
| $A_n$: | $x^n + y^2 + z^2$ | cyclic |
| $D_n$: | $x^n + xy^2 + z^2$ | dihedral |
| $E_6$: | $x^4 + y^3 + z^2$ | tetrahedron |
| $E_7$: | $x^3 + xy^3 + z^2$ | octahedron |
| $E_8$: | $x^5 + y^3 + z^2$ | icosahedron |

They are given these names because the intersection graph of exceptional curves in the minimal resolution is a Dynkin diagram of that type. Hence such singularities are also sometimes referred to as singularities of *ADE type*. The monodromy group of each singularity is the Weyl group of given type. We emphasize that therefore simple singularities lead to finite monodromy groups, and are again characterized by this property.

The next class of singularities is known as the class of *simple-elliptic* singularities. These necessarily have infinite fundamental group, namely in the form of a Heisenberg group. The resolution of simple-elliptic singularities contains an elliptic curve as exceptional curve, therefore the $j$-function gives a modulus of the singularities, which appears as the parameter $\lambda$ in the classification below. We will use the same names for suspensions of such singularities.

$$\begin{array}{ll} \tilde{E}_6: & x^3 + y^3 + z^3 + \lambda xyz \\ \tilde{E}_7: & x^4 + xy^3 + z^2 + \lambda xyz \\ \tilde{E}_8: & x^6 + y^3 + z^2 + \lambda xyz \end{array}$$

We also note that these singularities have infinite monodromy, namely of the form of a Weyl group of affine type; that is, a Weyl group of finite type times an infinite space of translations. For example, $W(\tilde{E}_8) = W(E_8) \ltimes V$ where $V$ is a vector space of dimension $2\,\mathrm{rank}(E_8) = 16$.

Remark that both classes are examples of quasi-homogeneous singularities: that is singularities where the variables can be given various weights so as to make the function homogeneous. Simple singularities have weights that add to a value $> 1$ (i.e. $xyz$ has weight $> 1$), simple-elliptic singularities are weighted-homogeneous of degree $= 1$.

## 5. Final Remarks

This chapter has touched on several topics that will play an important part in the following, but it could not be exhaustive. Many others have written fascinating papers that have motivated or influenced this work, we mention but a few. Most of all the work by Allcock, Carlsen and Toledo [**3**] on the ball structure of the moduli space of cubic surfaces has set a guiding example. The work of Kondo on genus three curves [**20**] provided a motivation for some theory, the example is reproduced in Chapter 3. Recently, a survey has appeared by Dolgachev on reflection groups [**10**], which are everywhere present in the background of this thesis: his paper is a great overview with many different perspectives of the subject.

As mentioned in the introduction, we wish to study the period map when the target space is a symmetric domain. If we have a good understanding of the image, we can use the “Baily-Borel”-type compactification described above to produce a geometrical quotient without reference to GIT. From our geometrical viewpoint several questions arise. Is the period map well-defined? When is it a (local) isomorphism? When can we extend it to the whole family $S$? We will be especially interested in the case where the map is an open embedding—the image is dense in the target—so that it is already “almost” an isomorphism. These questions are the focus of the following chapter.

## Notation

The following notation will appear throughout this thesis.

| | |
|---|---|
| $\mathcal{X} \to S$ | a family of algebraic varieties |
| $j : S^\circ \subset S$ | the set of points above which the fiber $X_s$ is smooth |
| $\mathbb{H}$ | a (Hodge) sheaf or vector bundle over parameter space $S^\circ$ |
| $H$ | a stalk of $\mathbb{H}$ at a general point $s \in S^\circ$, which is often a vector space of signature $(m, 2)$ |
| $q$ | the quadratic form on $H$ |
| $x \cdot y$ | the associated bilinear form |
| $H_+ = H'_+ \amalg \overline{H'_+}$ | points in $H$ positive for $q$, its two connected components interchanged by complex conjugation |
| $\mathcal{F}^\bullet(\mathbb{H})$ | a Hodge filtration on $\mathbb{H}$ |
| $\mathcal{F}$ | $\mathcal{F}^1(\mathbb{H})$, a line bundle over $S^\circ$ |
| $\mathbb{D} = \mathbb{P}(H'_+)$ | a period domain, $H_+ = \{x \in H \vert x \cdot x = 0, x \cdot \overline{x} < 0\}$ |
| $\mathcal{L}$ | the automorphic line bundle $\mathcal{O}_{\mathbb{D}}(-1)^\Gamma$ |
| $\widehat{\mathbb{D}}$ | the Baily-Borel compactification of $\mathbb{D}$ |
| $\chi : A \to \mathbb{C}^*$ | a multiplicative character of a finite group $A$ |
| $\tilde{S} \to S$ | the universal cover of $S$ |
| $\Gamma$ | the monodromy group associated to this cover or the isomorphic group acting on $\mathbb{D}$ |
| $P : S \to \Gamma \backslash \mathbb{D}$ | the period map, $P(s) = \mathbb{P}(\mathbb{F}_s) \in \mathbb{D}$ |

Below is a brief list of terms not encountered yet.
For reference in future chapters, we include them here.

| | |
|---|---|
| $Y$ | a cubic fourfold |
| $X$ | a cubic threefold |
| $X_C$ | the chordal cubic threefold, defined in Chapter 4 |
| $\eta_Y$ | square of the hyperplane class of $Y$, $\eta \cdot \eta = 3$ |
| $S_0$ | the cubic surface with three cusps (pictured on the cover!) |
| $\mu_3$ | (the) group with three elements |
| $G$ | the group $\mu_3 \times \mu_3 \times \mu_3$ |
| $\mathcal{E}$ | the ring of Eisenstein integers |
| $U$ | the (unimodular) hyperbolic lattice of rank 2 |
| $\Lambda_k$ | a family of lattices with $\mathcal{E}$-module structure |

CHAPTER 2

# Period maps that are open embeddings

We again consider the general period map. Before we dive into our theory, let us first establish notation and describe the intuitive framework with which we will work.

Let $\mathcal{X} \to S$ be a family of polarized algebraic varieties. In general the period map $P : S \to \Gamma\backslash\mathbb{D}$ associates to a point $s \in S$ the Hodge structure of the fiber $X_s$ in a certain degree, where $\mathbb{D}$ is some classifying space and $\Gamma$ some arithmetic group of symmetries. We are interested in a more specific geometric setting. Therefore assume that the general variety $X_t$ of our family has a K3-like cohomology, that is in some even degree $2k$ it has Hodge numbers $(\ldots, 0, 1, m, 1, 0, \ldots)$. We perform a Tate twist to ensure that this Hodge structure has weight 0. Write $S^\circ \subset S$ for the points of the family where the fibres are smooth, and $j$ for the inclusion map. Let $\mathbb{H}$ be the de Rham bundle: the vector bundle over $S^\circ$ that has fiber $H^{2k}(X_s, \mathbb{C})$ over $s$. Sometimes we will abuse notation and write $\mathbb{H}$ also for $j_*\mathbb{H}$, the de Rham bundle pushed forward to $S$ itself. Each of the fibres of $\mathbb{H}$ comes with its Hodge filtration, we will write $\mathcal{F}(\mathbb{H})$ for the family of these. An important role is reserved for the (Hodge) line bundle $\mathcal{F}^1(\mathbb{H}) \to S^\circ$: it is one-dimensional because of our assumptions on the Hodge numbers. This $\mathcal{F}^1(\mathbb{H})$ is clearly a subbundle of $\mathbb{H}$, we abbreviate the notation to $\mathcal{F}$.

In analogy with K3 surfaces, the Hodge structure is determined by this line bundle. Therefore we can be more precise about the target space $\mathbb{D}$. We let $H$ be a general fiber of $\mathbb{H}$, it comes with a bilinear form of signature $(m, 2)$. We define the period domain $H_+$ to be the points of $H$ where the form is zero and the induced Hermitian form is negative $H_+ = \{x \in H | x \cdot x = 0, x \cdot \overline{x} < 0\}$. This has 2 connected components interchanged by complex conjugation, we single out one and write $\mathbb{D} = \mathbb{P}(H'_+)$ for its projectivization.

We can now finally define what we mean by the *(geometric) period map*: it is the map

$$\begin{aligned} P : S^\circ &\to \Gamma\backslash\mathbb{D}, \\ P(s) &= \mathbb{P}(\mathcal{F}_s) \in \mathbb{D}. \end{aligned}$$

In what follows, we will consider the situation where the period map defines an open embedding. When this is the case, we are naturally led to ask about the image of the map. This takes the form of a hyperplane arrangement, for which a compactification was described before.

## 1. Local considerations

We begin with a local study.

Suppose we start with a $\mathbb{Q}$-vector space $H$ with a quadratic form $q$ of signature $(m, 2)$. What does it mean to give such a space a (weight 0) Hodge structure? That is to choose a *complex* line $F \subset H_\mathbb{C}$ that is isotropic for the quadratic form $q_\mathbb{C}$ and (positive) definite for the accompanying Hermitian form. With such an $F$, we can uniquely determine a conjugate line $\overline{F}$, and subdivide $H \cong F \oplus (F \oplus \overline{F})^\perp \oplus \overline{F}$. Writing $(x \cdot y)$ for the bilinear form associated to $q_\mathbb{C}$, this means that for any nontrivial element $\alpha \in F$ we have $(\alpha \cdot \alpha) = 0$ and $(\alpha \cdot \overline{\alpha}) < 0$. Recognize that the subset $H_+$ of $H_\mathbb{C}$ defined by these two conditions separates into two connected components, complex conjugates of each other. Arbitrarily picking one of these components, we see that $\mathbb{D} = \mathbb{P}(H'_+)$ is a symmetric domain of type IV. The orthogonal group $O(H \otimes \mathbb{R})$ acts on $\mathbb{D}$, indeed $\mathbb{D}$ is the symmetric space of that group.

Definition 1.1. When we fix such a domain $\mathbb{D}$, we will refer to it as a *period domain.*

We wish to vary this situation in smooth families, so we introduce the following.

Definition 1.2. Let be given a family $\mathcal{X} \to S$, where $S$ is a smooth, complex manifold, and fibers of fixed dimension $n$. A *Hodge bundle of weight* $2k$ for this family is a sheaf $\mathbb{H}$ on $S$, such that the stalks $\mathbb{H}_s \cong H^{2k}(X_s, \mathbb{C})$ as vector spaces. Each of these stalks has a Hodge decomposition, which varies continuously. We will denote the Hodge fibration coming from this decomposition by $\mathcal{F}^\bullet(\mathbb{H})$.

The Hodge structure is said to be of *type IV* if it is of even weight $n = 2k$, polarized, and has Hodge numbers $h^{k-i,k+i} = 1$ for $|i| = 1$, and 0 for $|i| > 1$. In this case $\mathcal{F}^1(\mathbb{H})$ is a line bundle over $S$, which we abbreviate to $\mathcal{F}$.

Remark 1.3. The motivating example for all this notation is the case of K3 surfaces: in this case $\mathbb{H}_s$ would be a 22-dimensional vector space, $q$ the quadratic form of type $3U \oplus 2E_8(-1)$, and $\mathcal{F}^1(\mathbb{H}_s) = H^{2,0}(X_s)$ the line in $\mathbb{H}_t$ spanned by the holomorphic 2-form $\omega_t$. It is good to keep this picture in mind, although we are generalizing this example.

The following lemma shows what happens to the Hodge structure of the fibres when we let $s \in S^\circ$ approach a point in the boundary. It already encodes much of the structure that will be used in the rest of the chapter.

LEMMA 1.4 (Limiting lemma). *Suppose we are in the above geometric context, that is we have a space $S$ and a Zariski-open dense subset $S^\circ \subset S$ and a type IV variation of Hodge structure $\mathbb{H}$ on $S^\circ$. Furthermore, let $o \in S - S^\circ$ be a limit point, $v \in H^0(S^\circ, \mathbb{H})$ a smooth section of $\mathbb{H}$, and $\alpha$ a section of $\mathcal{F}^1(\mathbb{H})$ with the following properties:*

(1) *the function $s \in S^\circ \to v \cdot \alpha(s)$ extends holomorphically across $o$, and*
(2) *$\alpha(s) \cdot \overline{\alpha(s)} \to -\infty$ as $s \in S^\circ \to o$.*

*Then any limit of a family of lines $\lim_{s \to o} F^1(s)$ lies in $v^\perp$.*

Thus any such $v$ defines a linear constraint on the period map. We remark that the second property is already satisfied if $\alpha$ is defined at $o$ but $|\alpha(o)|^2 = -\infty$.

PROOF. For some simply connected open $U \subset S^\circ$ we complement the section $v$ to a basis $\{v, e_2 \dots e_N\}$ of $\mathbb{H}|_U$, in such a way that all the $e_i$ are perpendicular to $v$. Hence the section $\alpha$ can be described as $\alpha(s) = \alpha_v(s)v + \sum \alpha_i(s)e_i$, with all the coefficient functions holomorphic on $S^\circ$. The first condition from the lemma says that $\alpha_v(s)$ extends holomorphically across $o$ (and so is finite), the second that as $t \to o, |\alpha_v|^2 + \sum |\alpha_i|^2 \to \infty$. Hence at least one of the $\alpha_i$ goes to infinity, so in our projective representation $[\alpha_v(s), \alpha_2(s), \dots, \alpha_N(s)]$ we can divide by this "large" coordinate and see that any accumulation point has first coordinate equal to zero. $\square$

It is easily seen that linearly independent sections $v$ give linearly independent conditions for the period map, so the above lemma generalizes to higher codimensions.

We are of course interested in which sections $v$ give rise to conditions on the image of the period map. We denote the linear space of all sections with this property by $K$. We remark that this space may be degenerate for $q$, that is that $q|_K$ is no longer definite. We can therefore define $K_0 \subset K$ as the degeneracy locus of $q|_K$: vectors which are $q|_K$-orthogonal to all other sections in $K$. This can have at most dimension 2, since $q$ and our domain $\mathbb{D}$ have signature $(m, 2)$.

This leads to the following enumeration, with several easy consequences. The numbering perhaps surprising at first reading, but will become clear below. Note that we write $\widehat{\mathbb{D}}$ for the Baily-Borel closure of $\mathbb{D}$.

Lemma 1.5. *Write $K \subset H$ for the space of all sections satisfying the conditions of 1.4, and $K_0 \subset K$ for the degeneracy locus of $q|_K$. We have three cases:*

(1) $\dim K_0 = 0$. *Then $K$ is positive definite, and $K^\perp$ has signature $(\dim K^\perp - 2, 2)$. Also $\mathbb{P}(K^\perp) \cap \mathbb{D}$ is a symmetric submanifold of $\mathbb{D}$.*
(2) $\dim K_0 = 2$. *Then $K$ is positive semidefinite, and $K^\perp$ is negative semidefinite, and $\mathbb{P}(K_0) \cap \widehat{\mathbb{D}} = \mathbb{P}(K^\perp) \cap \widehat{\mathbb{D}}$.*
(3) $\dim K_0 = 1$. *In this final case $\mathbb{P}(K^\perp) \cap \widehat{\mathbb{D}}$ is a union of boundary components of $\mathbb{D}$, whose closures contain $\mathbb{P}(K_0)$.*

Notice that in the last two cases, $\mathbb{P}(K^\perp)$ does not meet $\mathbb{D}$, only its closure. Supposing we have a family of lines $\mathbb{F} \in H_+$ converging to $\mathbb{F}_{lim}$ as in lemma 1.4: then $\mathbb{F}_{lim} \perp K$, so that $[\mathbb{F}_{lim}] \in \mathbb{P}(K^\perp) \cap \widehat{\mathbb{D}}$. In the first case of this lemma, we have that $[\mathbb{F}_{lim}]$ is in the interior of $\mathbb{D}$, while in the other cases it lies in a boundary component. However, the limit on approaching a boundary depends on the path so we must take into account the monodromy.

Consider this locally, so that our family of lines lies over a disk $\Delta$, and write $H$ for a generic fiber of $\mathbb{H}$ over $p \in \Delta^*$. Any vector in this space can be transported along paths in $\Delta^*$. In particular, loops in $\Delta^*$ will transport such a vector to a potentially different vector in $H$. Thus the fundamental group $\pi_1(\Delta^*, p)$ gives rise to an action of a group $\Gamma$ on $H$.

As in the previous chapter, Section 2, we denote the action of the canonical generator of the fundamental group (looping around the puncture counterclockwise) by $T$. We know that some power of $T$ is unipotent, by a suitable base change we may assume that already $T$ itself is unipotent. Hence $T - 1$ is nilpotent, and so too will be the "logarithm":

$$N = \log(T) = -\sum_{k \geq 1} \frac{1}{k+1}(1-T)^k.$$

Since $(T-1)$ is nilpotent, this is a finite sum. Notice that the operation $N$ lies in the Lie algebra associated to the Lie group $O(H)$ of endomorphisms preserving the bilinear form. Therefore $N$ must satisfy the relation $Nx \cdot y + x \cdot Ny = 0$ for all $x, y \in H$.

Since $N$ is nilpotent, it defines a weight filtration on $H$ as follows. We abbreviate $N^{(p,q)} = \mathrm{Im}(N^p) \cap \ker(N^q)$, and we can set

$$W_k = \bigoplus_{q-p \geq k} N^{p,q}$$

(for $p, q$ both non-negative) as the spaces of weight $k$. Equivalently, we write

$$W_k = \{x \in H | N^{k+i}x \in N^{k+2i}H, \forall i \geq 0\}$$
$$W_{-k} = \{x \in H | N^{i}x \in N^{k+2i}H, \forall i \geq 0\}.$$

We can check that $N(W_k) \subset W_{k-2}$. Furthermore if $n$ is the nilpotency order of $N$ (the first integer so that $N^n = 0$), then $W_{-n} = 0$ and $W_n = H$.

This $N$ can of course be the zero transformation. When it is not identically zero, we can use the signature of $H$ and the relation $Nx \cdot y = -x \cdot Ny$ to find 2 nonzero vectors $a, b \in H$ with $(a \cdot a) = (a \cdot b) = 0$ and such that

$$N(x) = (x \cdot a)b - (x \cdot b)a.$$

We see that $a \in N^{(1,1)} \subset W_0$, and $b \in N^{(1,2)} \subset W_1$. It follows that $N^2(x) = (x \cdot a)(b \cdot b)a$ and $N^3 = 0$. Summarizing, we have three possible cases:

(1) $N = 0$,
(2) $N^2 = 0 \neq N$, i.e. $(b \cdot b) = 0$,
(3) $N^3 = 0 \neq N^2$, i.e. $(b \cdot b) \neq 0$.

Let us write $J$ for the span of $a$ and $b$, and $J_0$ for the nilspace of the form when restricted to $J$. That is to say, in the three cases above $J_0$ equals $0, J$, and $\mathbb{C}a$ respectively. Remark that $\mathrm{Ker}(N) = J^{\perp} \supset K$.

Write $s$ for a coordinate for the punctured disk $\Delta^*$, and $\tau$ for a coordinate of the upper half plane—its universal cover—with covering map $s = \pi(\tau) = e^{2\pi\sqrt{-1}\tau}$. The period map on the upper half plane will satisfy $P(\tau+1) = TP(\tau)$. We can reduce this to a map $\phi : \Delta^* \to H$ by defining $\phi(\tau) = \exp(-2\pi\sqrt{-1}\tau N)\tilde{P}(\tau)$, so that

$$\phi(\tau+1) = e^{-2\pi\sqrt{-1}N(\tau+1)}\tilde{P}(\tau+1)$$
$$= e^{-2\pi\sqrt{-1}N\tau}T^{-1} \cdot T\,\tilde{P}(\tau) = \phi(\tau)$$

descends to a well-defined map $\phi : \Delta^* \to H_+$. As already mentioned in the preceding chapter, this map extends across 0. Considering once again our line bundle $\mathbb{F} \to \Delta^*$, this means that the line $\mathbb{F}_{lim}$ defines a Hodge filtration, with weight filtration defined by $N$. This means that $N$ is a morphism of mixed Hodge structures on $H$, so that $H/J$ has a mixed Hodge structure induced by the image $\pi_J(\mathbb{F}_{lim})$, where $\pi_J : H \to H/J$ is the quotient map. It follows that if we take limits,

$$\lim \pi_J(\mathbb{F}_\tau) = \pi_J(\mathbb{F}_{lim})$$

so that the Hodge structure induced on $H/J$ can be understood as the limit of the Hodge structures on smooth fibres.

We can now go through the three cases for the nilpotency order of $N$:

order(N)=1 Here $N$ is the zero map, so the monodromy operator $T$ is the identity. Then the filtration is trivial, $\mathbb{F}_{lim}$ is the ordinary Hodge line $\mathcal{F}$ and has no additional meaning.

order(N)=2 Here we get a weight filtration $0 \subset J \subset J^{\perp} \subset H$. The image $\pi_{J^{\perp}}(\mathbb{F}_{lim})$ of the limit line $\mathbb{F}_{lim}$ is non-trivial in $H/J^{\perp} = J^*$. When $J = K_0$ is the whole degeneracy locus, then $(K/K_0)^*$ is pure of bi-degree (0,0) and $H/K_0 = K_0^*$ is of weight one. Otherwise $\mathbb{F}_{lim}$ exists, but does not imply a Hodge structure for which $H/J \to K^*$ is a morphism.

order(N)=3 The filtration here looks like $0 \subset J_0 \subset J_0 \subset J_0^{\perp} \subset J_0^{\perp} \subset H$. In this case $K^*$ inherits a Hodge structure which makes $(K/K_0)^*$ pure of bi-degree (0,0), and $H/K_0 = K_0^*$ bidegree (1,1).

It is worthwile to summarize the above discussion in the following proposition. Henceforth we will refer to the three types of monodromy with roman numerals (I,II,III).

Proposition 1.6. *Assume we are in the situation of lemma 1.4, that is we have a limit point $o \in S - S^{\circ}$, a space $K \subset j_*(\mathbb{H}_o)$ and an $\alpha \in K$ whose Hermitian norm has* $\lim_{s \to o} \alpha \cdot \overline{\alpha} = -\infty$. *Then $K$ is positive semidefinite, and letting $K_0$ denote the degeneracy locus, then $F = \mathbb{C}\alpha$ determines the Hodge structure on $K^*$: we give $(K/K_0)^*$ pure bi-degree (0,0) and $K_0^*$ pure weight equal to its dimension, and $F^1(K) = F$. Base changes determine morphisms of mixed Hodge structures $K \to \mathbb{H}_o$ unless* $\dim K_0 = 1$ *and the base change is of type II.*

Definition 1.7. If we are in a situation that avoids the last case, we say that $K$ is a *mixed Hodge subspace.*

Remark 1.8. It is now clear that the enumeration of the lemma was not by the size of the degeneracy locus, but by the nilpotency of the monodromy operator. The classes correspond to a similar classification of semistable[1] degenerations of K3 surfaces due to Kulikov, Persson and Pinkham: the central fiber of a semistable degeneration is either a smooth K3 surface itself (Type I), a chain of elliptic surfaces (Type II), or a system of rational surfaces whose dual graph triangulates the sphere (Type III).

## 2. Global perspective

We can now consider the global repercussions of the above. Therefore we return to the situation sketched at the beginning of the chapter.

[1]This term is not related to the GIT-notion of stability.

Let $S$ be a normal variety, we can think of it as being some moduli space. Let $j : S^\circ \subset S$ be a Zariski-open, dense subset with defined on it a variation of Hodge structures $\mathbb{H}$. Denote by $S^f \subset S$ the region where $\mathbb{H}$ has finite monodromy. It is clear that $S^\circ \subset S^f$. We will write $S^\infty$ for $S - S^f$, i.e. points where $\mathbb{H}$ has infinite monodromy.

When we choose a base point $p \in S^\circ$, following a loop gives a monodromy representation $\pi_1(S^\circ, p) \to O(H_+)$, whose image is the *monodromy group* $\Gamma$ of $\mathbb{H}$. We can extend the (unramified) monodromy cover $\tilde{S^\circ} \to S^\circ$ to a (ramified) $\Gamma$-covering $\tilde{S}^f \to S^f$. We can furthermore lift the period map to a map $P : \tilde{S}^f \to \mathbb{D}$, in such a way that $P$ is $\Gamma$-equivariant.

Now the theory of Baily-Borel gives us that if $\Gamma$ is arithmetic, then $\Gamma\backslash\mathbb{D}$ has a natural completion whose boundary is Zariski-closed, i.e. $\Gamma\backslash\mathbb{D}$ is quasiprojective. With some assumptions we can ensure that the monodromy-group $\Gamma$ is arithmetic:

Lemma 2.1. *If $S$ is complete, $\mathbb{H}$ has regular (i.e. non-essential) singularities on $S - S^f$, and the map $P : \tilde{S}^f \to \mathbb{D}$ is open, then $\Gamma$ is arithmetic and $P$ descends to an open morphism $S^f \to \Gamma\backslash\mathbb{D}$ of quasiprojective varieties.*

Proof. We must see that $\Gamma$ is arithmetic. It preserves the quadratic form on each fibre of $\mathbb{H}$ and so it stabilizes a the corresponding lattice $\Lambda$, let $\Gamma' = O(\Lambda) \supset \Gamma$. Then $P$ determines an analytic map $P' : S^f \to \Gamma'\backslash\mathbb{D}$. We note that both sides have algebraic compactifications: $S$ and the Baily-Borel $\widehat{\Gamma'\backslash\mathbb{D}}$ respectively. That $\mathbb{H}$ has at most regular singularities implies that the map $P'$ has no essential singularities, only poles of some finite order. Using the fact that the graph $(S^f, P'(S^f)) \subset S \times \Gamma'\backslash\mathbb{D}$ has algebraic closure, we recover that $P'$ has some finite degree, and in particular, the index $[\Gamma' : \Gamma]$ is also finite. Therefore $\Lambda$ is arithmetic.

If $P$ is an open map and $\Gamma$ is arithmetic, then the reduced map $S^f \to \Gamma\backslash\mathbb{D}$ is also an open map of quasiprojective categories. $\square$

Definition 2.2. The (quasi-)projective variety $\Gamma\backslash\mathbb{D}$ comes with its automorphic line bundle $\mathcal{L}(1) = \mathcal{O}_\mathbb{D}(-1)^\Gamma$: sections are locally $\Gamma$-equivariant functions. This allows us to look at the algebra of automorphic (global) sections $\mathcal{A}_\bullet^\Gamma = \bigoplus_{d\geq 0} H^0(\mathbb{D}, \mathcal{L}^{\otimes d})^\Gamma$. The pullback of this line bundle along $S^f \to \Gamma\backslash\mathbb{D}$ defines a line bundle $\mathcal{F} \to S^f$.

It is now possible to define a concept fundamental to this thesis.

Definition 2.3. Suppose we have a sheaf $\mathbb{F}$ on $S$, such that $\mathbb{F}|_{S^f} \subset \mathbb{H}$. If its image under the sheaf map $\mathbb{F} \subset \mathbb{H} \to \mathcal{F}^1(\mathbb{H})$ is a line bundle (an invertible sheaf) such that the norm tends to zero on $S^\infty$, we say that $\mathbb{F}$ is a *boundary extension* of $(S, \mathbb{H})$.

If $\mathbb{H}$ itself is a boundary extension, we say it is *tight* on $S$.

Such a sheaf means what it says: it makes possible the extension of the period map to the boundary of the base $S$. On all stalks we have the results of proposition 1.6. We will refer to the line bundle associated to $\mathbb{F}$ as the Hodge bundle $\mathcal{F}$.

Assume now that we have a boundary extension $\mathbb{V}$ of $(S, \mathbb{H})$. For every $s \in S - S^f$, this extension determines a $\Gamma$-arrangement of hyperplanes in $H$, by lemma 1.4, which we denote by $\mathcal{K}_s$. It has a specified type (1,2, or 3), according to the distinction made in lemma 1.5. Write $\mathcal{K} = \bigcup_{s \in S - S^f} \mathcal{K}_s$ for the union of all these arrangements. We can stratify $\mathcal{K}$ by the type of arrangement, writing

$$\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{K}_3.$$

Recall that members of $\mathcal{K}_2$ and $\mathcal{K}_3$ do not meet $\mathbb{D}$ itself, but members of $\mathcal{K}_1$ do. We will denote the deletion of these hypersurfaces from the period domain using circles in the superscript

$$H_+^\circ = H_+ - \bigcup_{K \in \mathcal{K}_1} K \cap H_+ \qquad \mathbb{D}^\circ = \mathbb{D} - \bigcup_{K \in \mathcal{K}_1} \mathbb{P}(K \cap H_+)$$

both giving open subsets. When we take the quotient by the monodromy group, $\Gamma \backslash \mathbb{P}(K \cap H_+)$ is known to be a closed subvariety. Thus we can also say that the quotient $\Gamma \backslash \mathbb{D}^\circ$ is Zariski-open in $\Gamma \backslash \mathbb{D}$. Finally, we write $H_+^\diamond$ for the open set in $H_+$, where we delete only those hyperplanes $K \in \mathcal{K}_1$ which do not meet the image of the period map, so that

$$H_+^\diamond = H_+ - \bigcup_{\substack{K \in \mathcal{K}_1 \\ K \cap \mathrm{Im}(P) = \emptyset}} K \cap H_+$$

Clearly $H_+^\circ \subset H_+^\diamond \subset H_+$.

PROPOSITION 2.4. *Suppose we are in the described situation: $S$ complete, $\mathbb{H}$ has at most regular singularities, and $P : \tilde{S}^f \to \mathbb{D}$ is an open map. Then $P(\tilde{S}^f)$ contains $\mathbb{D}^\circ$, and $\mathbb{D}^\diamond - P(\tilde{S}^f)$ is everywhere of codimension $\geq 2$, i.e. the map misses no hyperplanes.*

*Now suppose furthermore $S^\infty$ is of codimension $\geq 2$ in $S$, $P$ is injective and $\mathcal{F}$ is ample. Then the algebra of automorphic forms $\mathcal{A}$ is finitely generated, $S$ can be identified with* $\mathrm{Proj}(\mathcal{A}_\bullet^\Gamma)$, *and $\mathbb{D}^\circ = \mathbb{D}^\diamond = P(\tilde{S}_f)$.*

PROOF. The previous lemma 2.1 ensures that the monodromy group $\Gamma$ is arithmetic, and the map $\Gamma \backslash P : S^f \to \Gamma \backslash \mathbb{D}$ is an open morphism. In particular, the whole image of the morphism is Zariski-open in $\Gamma \backslash \mathbb{D}$, and hence dense. The same is true of the cover. We first check that $P(\tilde{S}^f) \supset \mathbb{D}^\circ$. Therefore let $\alpha \in \mathbb{D}^\circ$: since $\alpha$ is in the closure of the image of $P$, there must exist a sequence

$(\tilde{s}_i \in \tilde{S}^f)$ such that $\lim P(\tilde{s}_i) = \alpha$. The sequence can be projected into $S^f$ itself, and there we find a convergent subsequence with limit point $s \in S$. However, we find that in fact $s \in S^f$ is a point of finite monodromy: were it not, we would find that $\alpha \in \mathbb{D} - \mathbb{D}^\circ$. We can therefore choose an $\tilde{s} \in \tilde{S}^f$ over $s$. There exist $\gamma_i \in \Gamma$ so that the sequence $(\gamma_i \tilde{s}_i)$ converges to $\tilde{s}$. We consider the sequence $(\gamma_i P(\tilde{s}_i))$: since $\Gamma$ acts properly discontinuously, it follows that (some subsequence of) $(\gamma_i)$ is stationary, with limit $\gamma$, say. Therefore

$$\alpha = \lim P(\tilde{s}_i) = P(\gamma^{-1}\tilde{s}) \in P(\tilde{S}^f)$$

and we are done.

To see that $\mathbb{D}^\diamond - P(\tilde{S}^f)$ is of codimension at least 2, let $K \in \mathcal{K}_1$ be such that $\mathbb{P}(K_+) \cap P(\tilde{S}^f)$ is a nonempty open subset of $\mathbb{P}(K_+)$. Then the image of the reduced map $\Gamma\backslash P$ also meets $\mathbb{P}(K_+)$ in a nonempty open subset. Using either the Hausdorff or the Zariski topology, we obtain our assertion.

If $S^\infty$ is of codimension 2, cohomology will not distinguish between $S$ and $S^f$. Remark that the line bundle $\mathcal{F}$ is the pull-back of the automorphic line bundle $\mathcal{L}$ on $\Gamma\backslash\mathbb{D}$, so we have an injection:

$$H^0(\mathbb{D}^\diamond, \mathcal{L}^{\otimes d})^\Gamma \hookrightarrow H^0(S^f, \mathcal{F}^{\otimes d}) \cong H^0(S, \mathcal{F}^{\otimes d}).$$

If furthermore the map $P$ is injective and $\mathcal{F}$ is ample, then the injection is in fact an isomorphism and

$$S = \operatorname{Proj}\left(\bigoplus_{d\geq 0} H^0(S, \mathcal{F}^{\otimes d})\right) = \operatorname{Proj}\mathcal{A}^\Gamma_\bullet$$

describes a projective completion of $\Gamma\backslash\mathbb{D}^\diamond$. So $\Gamma\backslash P$ is in fact an *isomorphism* from $S^f$ to $\Gamma\backslash\mathbb{D}^\diamond$ and in particular $\mathbb{D}^\circ = \mathbb{D}^\diamond$. $\square$

We have discussed in the previous chapter the Baily-Borel compactification of a variety of the form $\Gamma\backslash\mathbb{D}^\diamond$: a symmetric domain minus an arrangement of hypersurfaces. This compactification has the additional property that the automorphic line bundle $\mathcal{L}$ over $\Gamma\backslash\mathbb{D}^\circ$ extends as an ample line bundle over $\widehat{\Gamma\backslash\mathbb{D}^\circ}$, so that the completion can even be understood to be projective.

We use this compactification to construct the following. This summarizes all the considerations above.

Proposition 2.5. *Assume $S$ is complete, $\mathbb{H}$ has at most regular singularities, $\mathcal{F}$ is ample and $S^\infty$ has codimension at least 2 in $S$, and finally that $P : \tilde{S}^f \to \mathbb{D}$ is an open embedding.*

*Assume furthermore that* $\dim S \geq 3$ *and the Hermitian form on $H$ is positive on 2-dimensional intersections coming from $\mathcal{K}_1$. Then we can say that* *the isomorphism* $S^f \cong \Gamma\backslash\mathbb{D}^\circ$ *(from proposition 2.4) extends to an isomorphism of* $S$ *with the compactification* $\widehat{\Gamma\backslash\mathbb{D}^\circ}$.

Proof. The condition that $\dim S \geq 3$ implies that $\dim H = \dim \mathbb{D} + 2 \geq 5$. The condition on the intersections of members of $\mathcal{K}_1$ gives that the compactification $\widehat{\mathbb{D}^\circ}$ has no codimension 1 stratum, and together with the previous fact, ensures that the boundary points of $\widehat{\Gamma\backslash\mathbb{D}^\circ}$ are of codimension $\geq 2$ everywhere.

This means that the automorphic line bundle $\mathcal{L}$ extends as an ample line bundle over the boundary, so that

$$H^0(\mathbb{D}^\circ, \mathcal{L}^{\otimes d})^\Gamma = H^0(\widehat{\mathbb{D}^\circ}, \mathcal{L}^{\otimes d})^\Gamma$$

from which it follows that

$$\begin{aligned} S &= \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0(S, \mathcal{F}^{\otimes d})\right) \\ &= \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0(\mathbb{D}^\circ, \mathcal{L}^{\otimes d})^\Gamma\right) \\ &= \mathrm{Proj}\left(\bigoplus_{d\geq 0} H^0(\widehat{\mathbb{D}^\circ}, \mathcal{L}^{\otimes d})^\Gamma\right). \end{aligned}$$

Therefore $P : S \to \widehat{\Gamma\backslash\mathbb{D}^\circ}$ in fact defines an isomorphism. □

## 3. Additional action by finite groups

For the purpose of applications, it is worthwile to consider the situation when we have an additional action of a finite group on the period domain. In this way we will be able study the image of period maps constrained by certain symmetries. The theory developed above can be easily modified to accommodate this situation, so we will state them without proof.

Therefore assume again that we have a vector space $V$ with a quadratic form of signature $(m, 2)$, but now with a finite group $A$ which acts on $V$ respecting the form. We assume further a complex character $\chi : A \to \mathbb{C}^*$ with non-trivial eigenspaces $V_\chi \neq V_{\overline{\chi}}$.

We can observe that $V_\chi$ is isotropic for the quadratic form: namely for $v \in V_\chi$ and $g \in A$ such that $\chi(g)^2 \neq \pm 1$ we have $q(v) = q(gv) = \chi(g)^2 q(v)$ and hence $q(v) = 0$.

We also assume that $V_\chi$ intersects $H_+$ (where the Hermitian form is negative). This implies that both $V_\chi$ and $V_{\overline{\chi}}$ have hyperbolic signature $(1, k)$ (for some $k < \frac{1}{2}m$). We denote the intersection $H_+ \cap V_\chi$ by $H_{\chi,+}$. In this case $\mathbb{D}_\chi$ (and also $\mathbb{D}_{\overline{\chi}}$) are complex balls. We emphasize that this fact, although almost trivial, considerably simplifies the applications in which it appears. More will be said about this in the last chapters.

This can now be applied to the situation of proposition 2.1 and proposition 2.5. The results are immediate.

PROPOSITION 3.1. *If $S$ is complete, $\mathbb{H}$ has regular singularities on $S - S^f$ and comes with the action of some finite group $A$ with character $\chi : A \to \mathbb{C}^*$, and the map $P : \tilde{S}^f \to \mathbb{D}_\chi$ is open, then $\Gamma$ is arithmetic and $P$ descends to an open morphism $S^f \to \Gamma \backslash \mathbb{D}_\chi$ of quasiprojective varieties.*

PROPOSITION 3.2. *Assume $S$ is complete, $\mathbb{H}$ has at most regular singularities, $\mathcal{F}$ is ample and $S^\infty$ has codimension at least 2 in $S$, and finally that $P : \tilde{S}^f \to \mathbb{D}_\chi$ is an open embedding. Then $P(\tilde{S}^f)$ is all of $\mathbb{D}^\circ_\chi = \mathbb{D}^\diamond_\chi$.*

*Now assume furthermore that the Hermitian form on $H$ is positive on 2-dimensional intersections $K_\chi$ coming from $\mathcal{K}_1$. Then we can say that the isomorphism $S^f \cong \Gamma \backslash \mathbb{D}^\circ_\chi$ extends to an isomorphism of $S$ with the compactification $\widehat{\Gamma \backslash \mathbb{D}^\circ}_\chi$.*

## 4. Geometric implications

The above theory gives powerful tools for studying moduli spaces of objects which permit a period map.

Suppose we are given a family $\mathcal{X} \to S$ where general fibres are smooth, and a fiberwise action of an algebraic group $G$. Suppose further we have a $G$-equivariant ample line bundle $\mathcal{F}$ on $S$, which we use to define stability (since it is a linearization of the action). Let $\mathbb{H}$ be the local system of $R^k f_* \mathbb{Q}_X$, or a subquotient of this.

We insist that the following properties hold:

- The bundle $\mathbb{H}$ has a type IV structure $\mathcal{F}^\bullet(\mathbb{H})$
- The points $s \in S$ where $\mathbb{H}$ has finite monodromy are all stable (w.r.t. $\mathcal{F}$)
- The set $S^\infty$ is of codimension $\geq 2$ in $S^{ss}$
- $\mathcal{F}|_{S^{ss}}$ can be given the structure of a geometric Hodge bundle, such that it has proper norm. Locally this reproduces the conditions of Lemma 1.4.
- We possibly include an additional action of a finite group $A$ on $\mathbb{H}$, acting on fibres. We assume it acts by a character $\chi : A \to \mathbb{C}^*$.

In this situation, the period map $P : \tilde{S}^f \to \mathbb{D}$ is defined on the monodromy cover of $S^f$. We finally require this map to be an open embedding. This leads to our main result:

THEOREM 4.1. *Assume we have all of the above. In this situation, the monodromy group $\Gamma$ of $\mathbb{H}_\chi$ is arithmetic. For $s \in S^{ss} - S^f$, the hyperplanes of type I make up a $\Gamma$-arrangement $\mathcal{H}$ in $\mathbb{D}_\chi$ which are exactly the points missed by the period map. We thus produce an isomorphism $A\backslash S^f \cong \Gamma\backslash\mathbb{D}^\circ_\chi$. If $\dim S > 3$, the associated boundary extension is tight.*

*This results in an isomorphism $A\backslash S^f \cong \Gamma\backslash(\mathbb{D} - \mathcal{H})$. If $\dim S \geq 3$, then the boundary extension of $\mathbb{H}$ over $S^{ss}$ is tight.*

PROOF. All the pieces are now in place. The variation of Hodge structures here is known to have at most regular singularities. We know that $G\backslash\backslash S^{ss} = \mathrm{Proj}(\bigoplus_{d\geq 0} H^0(S, \mathcal{L}^{\otimes d}))$ is a projective variety. The hypotheses of propositions 2.4 and 3.2 are satisfied by the line bundle $\mathcal{F}/G$ (taking the quotient as orbifolds), and hence the conclusion follows. $\square$

In practice, it is usually only the condition that the bundle has proper norm on $S^{ss}$ that requires any checking. This leads us to make the following fundamental definition.

DEFINITION 4.2. Suppose we have an $m$-dimensional projective variety $X \subset \mathbb{P}^N$. Assume furthermore that there exists a smoothing of $X$ such that general fibres of the smoothing have type IV cohomology in degree $2k$, for some $k \leq m$. If there exists a line $F \subset H^{2k}(X_{reg}, \mathbb{C})$, such that every smoothing of $X$ has a Hodge-bundle with $F$ as a closed fiber, then we say $F$ is a *residual Hodge line*. If such Hodge bundles have proper norm, we shall say $X$ is a *boundary variety (in dimension $2k$)* and refer to the couple $(X, F)$ as a *boundary pair*.

Such a pair will have an intrinsic type, independent of the chosen smoothing. This type depends only on the monodromy type of the hypersurface determined by the pair.

We are now equipped with a good understanding of the general (type IV) period map, and its image, in the case of an open embedding. In the next chapter we examine certain algebraic cases, when the period map can be constructed using residues. The strategy will be this: we try to find a period map on a space $P : S^\circ \to \mathbb{D}$ parametrize certain smooth varieties, such that this map has poles on limit points $o \in S - S^\circ$. We identify the *type* of the hyperplanes in $\Gamma\backslash\mathbb{D}$ determined by these degenerations, and produce the compactification $\widehat{\Gamma\backslash\mathbb{D}^\circ}$ described in theorem 4.1.

CHAPTER 3

# Period maps from residues

The theory of the preceding chapter is powerful, but requires illustration. In this chapter we will construct a period map by introducing a residue map for certain algebraic hypersurfaces. We show that this map satisfies the conditions to be a good period map, and can easily read off geometrically interesting results from the theorems. We illustrate with some well-known examples. The last part of this chapter describes the period map in the case of cubic fourfolds and sets the stage for later chapters.

## 1. Residues: a motivation

Residues first appeared in the theory of complex functions in the first half of the $19^{th}$ century. Undergraduates will be familiar with Cauchy's integral theorem for holomorphic functions:

$$\int_{C_\epsilon} \frac{f(z)}{z} dz = 2\pi\sqrt{-1} f(0)$$

if $C_\epsilon$ is any small circular path around the origin (followed anticlockwise). The complex-analytic function $\frac{f(z)}{z}$ is undefined at the origin, but outside of this it is holomorphic. At the origin it is tame enough to construct a *residue* as above, it is said to have a pole of order 1. The residue encodes all relevant information about the form $f(z)\frac{dz}{z}$; integrating this along any path will result in $2\pi\sqrt{-1}f(0)$ times the number of times the path encircles the origin, taking into account the orientation.

The concept of a residue of a function on $\mathbb{C}$ can be generalized first to any Riemann surface, and then to complex algebraic varieties of any dimension. The construction is similar: if we have a meromorphic form (i.e. holomorphic except at codimension 1 poles) we can integrate it along a tubular neighborhood of the poles to produce a holomorphic form on the polar variety. This construction is known as the *Poincaré residue map*. Write $\Omega_X$ for the sheaf of holomorphic forms on some complex algebraic variety $X$, and $\Omega_X(V)$ for meromorphic forms on $X$ with poles (of order 1) in a subvariety $V$ only. The residue construction defines a surjective morphism $\Omega_X(V) \to \Omega_V$, and all holomorphic forms on $X$ lie in the kernel of this map. So the Poincaré map produces a short exact sequence of sheaves

$$0 \to \Omega_X \to \Omega_X(V) \to \Omega_V \to 0.$$

The associated long exact sequence of cohomology involves elements of the Hodge decomposition, and we use this to suggest a period map. We first illustrate this with our favorite example.

Recall from the introduction the "invariant form" $\frac{dx}{y}$ of an elliptic curve given by $y^2 = f(x)$. This form is rather miraculous for it was found to be invariant under the action of $\mathrm{PGL}(2, \mathbb{Z})$, the famous Möbius transformations. We will give a construction which makes this observation almost trivial.

Write $F$ for the homogenized version of the defining equation of the curve $E$ (it is of degree three) in $\mathbb{P}^2$. Taking a step back to three-dimensional affine space, we have there a holomorphic volume form $dx \wedge dy \wedge dz$. This form is of weight 3 for the action of $\mathbb{C}^*$, dividing by our relation $F$ we get a form of weight 0. This means we can take its residue on $\mathbb{P}^2$, for example along the plane $\{z = 0\}$. Furthermore, we can take the residue along the hypersurface where $F = 0$, and we are left with a class on the elliptic curve:

$$\omega = \mathrm{Res}_{F=0} \mathrm{Res}_{\mathbb{P}^2} \frac{dx \wedge dy \wedge dz}{F}.$$

We remark that PSL(2) acts trivially on the numerator, and although the denominator is not fixed its orbit-class does not change. Since the residue only depends on this class, we conclude $\omega$ is invariant under PSL(2). It is not hard to calculate

$$\omega = \mathrm{Res}_{F=0} \mathrm{Res}_{\mathbb{P}^2} \frac{dx \wedge dy \wedge dz}{F} = \mathrm{Res}_E \frac{dx \wedge dy}{y^2 - f(x)} = \frac{dx}{y}$$

and we see *why* our "invariant form" from the introduction is so special! We caution however that this 1-form is determined only up to addition by an exact form; it does however represent a uniquely determined class in cohomology $[\omega] \in H^1(E, \mathbb{C})$.

This construction begs to be generalized. This is clearly possible when we have an algebraic hypersurface in $\mathbb{P}^n$, whose degree is some factor of $n + 1$.

Lemma 1.1 (Griffiths [**15**]). *Let $X$ be a smooth hypersurface in $\mathbb{P}^n$, defined by a polynomial $F$. Suppose $F$ has degree $d$ such that $n+1 = d \cdot k$, we construct the following residue:*

$$\eta = \mathrm{Res}_{F=0} \mathrm{Res}_{\mathbb{P}^n} \frac{dz_0 \wedge \ldots \wedge dz_n}{F^k}.$$

*This form defines a class in $[\eta] \in H^{n-1}(X, \mathbb{C})$. We even know its Hodge type, namely that $[\eta] \in H^{n-k,k-1}(X)$.*

Example 1.2. An immediate example, after cubic curves, is that of quartic surfaces. Let $V$ be a 4-dimensional vector space over $\mathbb{C}$, let $\mu$ be its volume form and $F \in \mathrm{Sym}^4(V)$ characterize a smooth quartic surface. Then the form $\eta = \mathrm{Res}_{F=0} \mathrm{Res}_{\mathbb{P}^3} \frac{\mu}{F}$ is a nowhere-vanishing $(2,0)$-form. This is holomorphic, and so the surface in $\mathbb{P}^3$ defined by $F$ is a K3 surface.

The idea of the Griffiths' proof is to use "partial integration" of forms. Write $\omega$ for a (holomorphic) differential form, and $F$ for a homogeneous polynomial. Consider the meromorphic forms with fixed pole order along $\{F = 0\}$. Remark that

$$d\left(\frac{\omega}{F^k}\right) = \frac{d\omega}{F^k} - \frac{kdF \wedge \omega}{F^{k+1}}$$

is exact, so that up to an *exact form* of pole order $k$, the two terms on the right are equivalent. This means we can lower the pole order of the form $\eta$ (up to an exact form of lower order) if we choose $\omega$ such that $(k-1)dF \wedge \omega = \mu := dz_0 \wedge \ldots \wedge dz_n$, i.e. when $\omega = \frac{1}{k-1} i_{dF}\mu$ is the contraction of $\mu$ with $dF$. Repeating this process allows us to reduce the pole order to 1, where it becomes possible to take the Poincaré residue. The effect on the Hodge type is also illustrated by this process: every step to lower the pole order shifts the Hodge type by one.

## 2. An extended example: quartic curves

As an illustration of the above, and as motivation for some of what follows, we discuss the case of quartic curves. The work of Kondo [**20**] forms part of our inspiration for this research. The example is archetypical for later work, and has seen much attention from Heckman (private communications) and ourselves, see [**26**].

The GIT of quartic plane curves is due to Mumford [**30**]. To summarize, a quartic curve $Q$ is stable if it contains only double points and cusps (locally given by $y^2 = x^2$ and $y^2 = x^3$ respectively), and is strictly semistable if it contains a tacnode (locally $y^2 = x^4$). The only way a quartic curve can have a tacnode is if it is the union of two conics, tangent at a point. The *minimal* strictly semistable locus can then be seen to be the 1-parameter family of conics tangent at two points, given in local coordinates by $(z_1^2 - \lambda z_0 z_2)(z_1^2 - \mu z_0 z_2)$ for $[\lambda : \mu] \in \mathbb{P}^1$. At either $[0 : 1]$ or $[1 : 0]$ (when $\frac{\lambda}{\mu} = 0$ or $\infty$) we still have two tacnodes—a smooth conic with two tangent lines. The most degenerate case, however, is when $\lambda = \mu$ and the two conics are not only tangent but coincident! We will consider these two cases–tangent conics and the double conic–separately.

We can use this information to study the moduli space by considering the following period map. When given a three-dimensional vector space $V$

and a polynomial $f \in \mathrm{Sym}^4(V)$ representing a curve, we may consider the surface which is the 4-fold cover of the plane, branched along the curve. The action of $\mathrm{PGL}(V)$ on $\mathrm{Sym}^4(V)$ extends to an action of $\mathrm{PGL}(V) \times \mathbb{C}^*$, which we denote by $G$ on $\mathrm{Sym}^4(V \oplus \mathbb{C})$. We will denote this space by $S$, and by $S^0, S^{st}, S^{ss}, S^f$ respectively the locus of smooth, stable, semi-stable and finite-monodromy surfaces. Such a surface is described as the zero-locus of $\{F = w^4 + f(x,y,z)\}$, and is a K3 surface $X_F$. It is clearly of degree 4, and carries an automorphism of degree 4 which fixes the curve. We write $\chi$ for a character of the automorphism group. When the surface is smooth, the eigenspace $H^2(X_F, \mathbb{C})_\chi$ of the automorphism is hyperbolic of dimension 7, as shown by Kondo in [**20**].

We remark that the fourfold cover branched along a tacnode is a singularity of type $\tilde{E}_6$. Hence we see that surfaces of finite monodromy are all stable.

We perform the residue construction using this K3 surface, writing $\mu$ for a volume form of $V \oplus \mathbb{C}$ and $\omega_F = \mathrm{Res}_{\mathbb{P}^3,F}(w^4 + f(x,y,z))^{-1}\mu$, defined where $f$ (and hence $F$) is smooth.

The residue construction provides a map $P : S^\circ \to \mathbb{D}_{K3}$ from the moduli space of smooth quartic curves $S^\circ$ to the K3 period domain. It does more: it gives an isomorphism of line bundles between $\mathcal{O}_{S^{ss}}(1)$ with the automorphic line bundle $\mathcal{O}_{\mathbb{D}_{K3}}(-1)^G$. Notice that the map inverts the degree of a section, since it sends a form $F$ to a fraction with $F$ in the denominator. As in the case of cubic curves, it is invariant under the action of $G$. Hence taking $\mathcal{F} = \mathcal{O}_S(1)$, we have all the pieces in place to apply theorem 4.1.

Proposition 2.1. *Application of theorem 4.1 to $\mathcal{F} = \mathcal{O}_S(1)$ gives us an isomorphism $G\backslash\backslash S^{ss} \cong \widehat{\Gamma\backslash\mathbb{D}^\circ}_\chi$. Here $\mathbb{D}_\chi = \mathbb{P}(H_{\chi,+})$ where $H_\chi$ is a vector space of dimension 7 over $\mathbb{Q}(\sqrt{-1})$, with a hyperbolic Hermitian form. The hyperplanes to be deleted from $\mathbb{D}_\chi$ form a single $\Gamma$-orbit. We have that $S^f = S^s$ and $S^{ss} - S^s$ consists of 2 strata: (i) where the curve consists of two tangent conics, of type II, and (ii) where the curve is a double conic, of type I, and hence not in the image of the extended period map.*

Proof. Much of this has already been discussed above. The conditions of theorem 4.1 can all be easily checked in this situation, only that the norm of $\mathcal{F}$ goes to zero as it approaches the boundary needs to be considered.

We verify this along the boundaries. Let us first consider the semistable case when the curve consists of two tangent conics, at least one of which is smooth. Here the covering surface $X$ will have (two) simple-elliptic singularities, locally of the form $w^4 + x^4 + y^2 + z^2 = 0$. This will be an application of proposition 3.2, but we can consider this example more closely. The two simple-elliptic singularities on $X$ can be resolved maintaining the $\mu_4$-action as $\tilde{X} \to X$, which has an elliptic curve in the exceptional locus above each singularity. We can consider $\tilde{X} \cong \mathbb{P}^1 \times E$, where $E$ is an elliptic curve with a $\mu_4$-action, isomorphic to $\mathbb{C}/\mathbb{Z}[\sqrt{-1}]$. We denote the exceptional locus as $E_0$ and $E_\infty$. Let $z$ be an affine coordinate of $\mathbb{P}^1$, and $\alpha_E$ a non-vanishing holomorphic differential on $E$. The form $z^{-1}dz \wedge \alpha_E$ defines a meromorphic 2-form on $\tilde{X}$, with poles on $E_0$ and $E_\infty$, which is the pullback of a unique section $\alpha$ on $X$ generating $\omega_X$. This $\alpha \in H^2(X_{reg}, \mathbb{C})$ is non-square-integrable and non-zero, this case is of type II.

Now consider the case where the conics coincide. Say the conic is represented by $f \in \mathrm{Sym}^2(\mathbb{C}^3)$. Then the covering surface $X$, given by $F = w^4 + f^2 = 0$, splits as a union of two quadric surfaces $S_\pm = \{w^2 \pm \sqrt{-1}f = 0\}$, which intersect along the conic $f = 0$. Each quadric can be identified with $\mathbb{P}^1 \times \mathbb{P}^1$, where their intersection becomes the diagonal $D \subset \mathbb{P}^1 \times \mathbb{P}^1$. If we let $u$ and $v$ be affine coordinates of distinct copies of $\mathbb{P}^1$ we can let

$$\zeta = \frac{du \wedge dv}{(u-v)^2}$$

represent a 2-form that extends to $\mathbb{P}^1 \times \mathbb{P}^1$, with a pole of order 2 along $D$. We claim that the 2-form $\alpha|_{S_\pm} = \pm\zeta$ spans a limit line $[\omega_X]$ for $H^2(X, \mathbb{C})$.

We must check that $\alpha$ is not square-integrable and defines a non-zero class in $H^2(X_{reg}, \mathbb{C}) = H^2(\mathbb{P}^1 \times \mathbb{P}^1 - D, \mathbb{C})$. The former is clear. To check the latter, we integrate $\alpha$ along a cycle defined by a small tubular neighborhood of $D$. If we let $[u]$ and $[v]$ represent the two different rulings of the quadric surface, we can find a (non-algebraic) cycle $\gamma_\epsilon$ on $\mathbb{P}^1 \times \mathbb{P}^1 - D$ homologous to $[u] - [v]$, which avoids $D$. We write $T_\epsilon$ for the tube around $D$, with coordinates $u - v = e^{\sqrt{-1}\phi}$ and $u + v = te^{\sqrt{-1}\phi}$. Then we have

$$\int_{\gamma_\epsilon} \zeta = \int_{T_\epsilon} \frac{du \wedge dv}{(u-v)^2} = \int \frac{de^{\sqrt{-1}\phi} \wedge dte^{\sqrt{-1}\phi}}{2e^{2\sqrt{-1}\phi}} = \int_0^{2\pi}\int_{-1}^{1} \sqrt{-1}\frac{d\phi \wedge dt}{2} = 2\pi\sqrt{-1}$$

nonzero.

Now let $\gamma \in H_2(X_{reg})$ be the image of $[u] - [v] \in H_2(\mathbb{P}^1 \times \mathbb{P}^1 - D)$ lifted to $H_2(X_{reg})$. Writing $g \in \mu_4$ for a generator, we get that $g^2\gamma = -\gamma$ and $H_2(X_{reg})$ is freely generated by $\gamma$ and $g\gamma$, and hence forms a $\mu_4$-module isomorphic to the Gaussian integers $\mathbb{Z}[\sqrt{-1}]$. Since $\gamma^2 = ([u] - [v])^2 = -2$, the quadratic form on $H_2(X_{reg})$ is given by $-2||z||^2$, hence negative definite. Since this lattice has no overlattices, it is primitively and $\mu_4$-equivariantly embedded in the homology of nearby smooth quadric surfaces. Hence the stratum of double conics defines a negative definite hyperplane in $\mathbb{D}_\chi$, of type I. $\square$

We summarize briefly. The residue map maps the moduli space of stable quartic curves to the quotient of a 6-ball. This is an open embedding, i.e. the image is all but an arrangement of hyperplanes. This arrangement consists of 2 $\Gamma$-orbits, the first of which corresponds to semistable quartics (with tacnodes, type II), the second to double conics (type I). The residue map can be extended to the first orbit, but not across the hyperelliptic locus. The map thus produces an isomorphism $G\backslash\backslash S^{ss} \cong \widehat{\Gamma\backslash\mathbb{D}^{\circ}}_{\chi}$ between the moduli space of semistable quartic curves and the complement of the hyperelliptic locus in the 6-ball.

Remark 2.2. One can ask: how does a single quartic curve (the double conic) account for a whole hyperplane? It is because this hyperplane consists of all hyperelliptic curves, all of which lie in the orbit of the double conic. To see how this occurs, suppose we deform the double conic in the direction of a general quartic. The quartic and the conic will intersect in 8 points, and using the (Cremona) isomorphism between a conic and $\mathbb{P}^1$, we can interpret the deformation as a double cover of $\mathbb{P}^1$, branched above 8 points, i.e. a hyperelliptic curve of genus 3. These curves have a moduli space of dimension 5 usually denoted $\mathcal{H}_5$, which sits inside the 6-ball as a hyperball[1]. The period map on the hyperelliptic locus can further be described in terms of the Deligne-Mostow map [**9**] (see also our remark 4.4 in Chapter 4).

## 3. The residue map for cubic fourfolds

The first example to arise where the resulting form is not holomorphic is the case of cubic fourfolds. The map will be an important component of the following chapters, we study it in depth here.

Let $V$ be a complex 6-dimensional vector space, with some fixed volume form $\mu$, and let $F \in \mathrm{Sym}^3(V^*)$ be a cubic form. We let $Y_F$ be the cubic fourfold defined by $F$, and define

$$\alpha_F = \mathrm{Res}_{Y_F} \mathrm{Res}_{\mathbb{P}^5} \frac{\mu}{F^2}.$$

and see that it defines a $(3,1)$-form on $Y_F$, when $F$ is smooth.

Now it was shown by Beauville and Donagi [**7**] that the Hodge numbers of a smooth cubic fourfold closely resemble those of a K3 surface: $h^{4,0} = h^{0,4} = 0, h^{3,1} = h^{1,3} = 1, h^{2,2} = 21$. In fact a cubic fourfold is deformation equivalent to a symmetric square of K3 surfaces. What is interesting for us is that $h^{3,1} = 1$, so the image of our residue construction gives a generator for this space. Again comparing this with what we know of K3 surfaces, we see that $\alpha_F$ determines the Hodge structure of $Y_F$.

In fact, for smooth fourfolds we can go further and describe the integral cohomology. The hyperplane class of $Y$ defines a class $\eta_Y \in H^2(Y,\mathbb{Z})$ such that $\eta_Y^2 \in H^4(Y,\mathbb{Z})$ determines a class with square $\eta_Y^2 \cdot \eta_Y^2 = \eta_Y^4 = 3$. Lefschetz

[1] This becomes a *Hegner divisor* in the quotient by $\Gamma$.

theory tells us that the orthogonal complement of $\eta_Y^2$ is generated by vanishing cycles of self-intersection 2, hence is an even lattice. This complement has the same discriminant as the line spanned by $\eta_Y^2$, namely 3, and can be shown to have signature $(20, 2)$.

Lattice theory allows us to identify this lattice

$$H_0^4(Y, \mathbb{Z}) \cong 2E_8 \oplus 2U \oplus A_2.$$

To see this, remark that we can choose a description for $H^4(Y, \mathbb{Z})$ (non-primitive) such that $\eta_Y^2 = e_1 + e_2 + e_3$, where the $e_i$ are orthogonal elements with unit length. This allows us to split off an $A_2$ summand in $\bigoplus_i \mathbb{Z}\langle e_i \rangle$, the elements spanned by $e_1 - e_2$ and $e_2 - e_3$, orthogonal to $\eta_Y^2$. The remaining part of the lattice is even, unimodular, and of signature $(18, 2)$. The work of Serre [**35**] or Nikulin [**32**] tells us this must decompose as a sum of lattices of type $E_8$ and $U$, and we find the multiplicities by comparing the signature.

We use this knowledge to set up the requirements for theorem 4.1 of Chapter 2. Suppose $\Lambda$ is an abstract lattice of signature $(21, 2)$, $V = \Lambda \otimes \mathbb{C}$ a vector space with the same signature, and some element $\eta \in \Lambda$ with $\eta \cdot \eta = 3$. The group $O(V)$ acts transitively on the lattice, so that all such elements are equivalent. Write $\Gamma \subset O(V)$ for the group of isometries that leave our chosen $\eta$ fixed.

We write $\Lambda_0 = \eta^\perp$ for the lattice orthogonal to $\eta$, we know that abstractly $\Lambda_0 \cong 2E_8 \oplus 2U \oplus A_2$. Let $H = \Lambda_0 \otimes \mathbb{C}$ for short, and denote by $H_+$ the points isotropic for the bilinear form and positive for the Hermitian form. Finally write $\mathbb{D} = \mathbb{P}(H_+)$ for what will become the period domain (after a choice of connected component).

The group $\Gamma$ acts on $\mathbb{D}$, and on the line bundle $\mathcal{O}_\mathbb{D}(-1)$, which is the restriction to $\mathbb{D}$ of $\mathcal{O}_{\mathbb{P}(V)}(-1)$. This may seem odd as the latter bundle has no global sections, but the restriction to $\mathbb{D}$ does allow for global sections. We now consider $\Gamma$-invariant sections of $\mathcal{O}_\mathbb{D}(-1)$, and write $\mathcal{L}$ for this *automorphic line bundle*.

**3.1. The period map for fourfolds.** The residue map can be well-understood in a long exact sequence of cohomology. Suppose our cubic fourfold is singular, but that the singular locus $Y_{sg}$ is (at worst) a nonsingular curve. Write $Y_{reg} = Y - Y_{sg}$ for the regular points of $Y$. Excision of $Y \subset \mathbb{P}^5$ leads to the long sequence, of which we show the relevant part (with coefficients in $\mathbb{C}$)

$$H^5(\mathbb{P}^5) \to H^5(\mathbb{P}^5 - Y) \to H^6_Y(\mathbb{P}^5) \to H^6(\mathbb{P}^5) \to H^6(\mathbb{P}^5 - Y)$$

where $H^6_Y(\mathbb{P}^5)$ denotes the relative cohomology isomorphic to $H^6(\mathbb{P}^5, \mathbb{P}^5 - Y)$. We know $\mathbb{P}^5$ has cohomology only in even degree, and $\mathbb{P}^5 - Y$ is affine and so has no cohomology in degree higher than 5 (its dimension). Thus the extremal terms are zero, and $H^5(\mathbb{P}^5 - Y)$ is isomorphic to the primitive part of $H^6_Y(\mathbb{P}^5)$.

We also look at the sequence of cohomology of the series of inclusions $Y_{sg} \subset Y \subset \mathbb{P}^5$ which gives the following sequence:

$$H^6_{Y_{sg}}(\mathbb{P}^5) \to H^6_Y(\mathbb{P}^5) \to H^4(Y_{reg})(-1) \to H^7_{Y_{sg}}(\mathbb{P}^5).$$

Again the extremal terms are zero, now because the singular locus has (complex) codimension $\geq 4$ in $\mathbb{P}^5$. Substituting this isomorphism in the first sequence, and using $H^6(\mathbb{P}^5) \cong \mathbb{Z}$ we have the following result.

Lemma 3.1. *If $Y \subset \mathbb{P}^5$ is a cubic fourfold with singular set $Y_{sg}$ smooth of dimension $\leq 1$, then*

$$0 \to H^5(\mathbb{P}^5 - Y) \stackrel{\mathrm{Res}}{\to} H^4(Y_{reg})(-1) \to \mathbb{Z} \to 0$$

*is exact, where the first arrow is the residue map.*

Hence we see that classes of $Y_{reg}$ can be obtained from "residues" of classes on $\mathbb{P}^5 - Y$. We use this to define our a period map. First write $S^\circ$ for the space of *smooth* cubic fourfolds. The residue construction produces a period map $P : S^\circ \to \Gamma\backslash\mathbb{D}$, that identifies $\mathcal{O}_{\mathbb{D}}(-1)^\Gamma$ with $\mathcal{O}_{S^\circ}(2)$. These line bundles have a distinct geometric interpretation. The line $\mathcal{O}_{S^\circ}(2)_F$ above a polynomial $F \in S^\circ$ is spanned by $F^2$, while the line $\mathcal{O}_{\mathbb{D}}(-1)^\Gamma_\alpha$ above a $\Gamma$-automorphic form $\alpha$ is spanned by $\alpha$ itself. Here $P$ sends a line spanned by an equation $[F]$ to the line spanned by its residue $[\alpha_F]$.

A class of isolated singularities known as *simple singularities* is easily accounted for. These singularities occur as suspensions of the surface singularities of ADE-type (or rational double points). Simple singularities have a finite monodromy cover—the suspension does not change that. The domain of the period map can be extended from smooth varieties to finite-monodromy varieties (as discussed in the previous chapter) so simple singularities essentially do not obstruct the definition of the period map.

The next class of isolated singularities, the simple elliptic singularities, present a degeneration of type II.

Proposition 3.2. *Let $Y$ be a cubic fourfold (defined by a cubic form $F$), with simple elliptic singularities (of type $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$). Then writing $\alpha_F = \mathrm{Res}\, F^{-2}\mu = P(F)$, we can say that $(Y, P(F))$ defines a boundary pair of type II.*

Proof. We must check that $\alpha_F$ is not square-integrable, and that $H_4(Y_{reg})$ contains two independent isotropic vectors that are embedded by $\alpha_F$ into $\mathbb{C}$. Essentially this is a result from singularity theory, which we illustrate in the case a singularity of type $\tilde{E}_8$ (the other cases are similar). So we can choose coordinates $(z_1, \ldots, z_5)$ around a singular point $o$ such that locally $Y$ is given by $f(z) = z_1^6 + z_2^3 + \lambda z_1 z_2 z_3 + z_4^2 + z_5^2 = 0$ (such that $o$ is the points where all coordinates vanish). This $f$ is weighted homogeneous with weights $(1, 2, 3, 3, 3)$. Away from the point $o$, the residue $\alpha$ of $f^{-2} dz_1 \wedge \ldots \wedge dz_5$ is invariant under the action of $\mathbb{C}^*$; both the numerator and the denominator have weight 12 with respect to multiplication. Therefore the form $\alpha \wedge \overline{\alpha}$ is also $\mathbb{C}^*$-invariant, and since it is also positive it can not be square-integrable near $o$.

The link $L$ of such a singularity is known to have the property that $H_4(L)$ is free of rank 2, and the cycles on it come from cycles on the elliptic curve defined by $f$ in suitably-weighted projective space. These cycles have self-intersection zero. Under the induced maps in the Milnor-sequence, they are mapped isometrically from $H_4(L) \to H_4(Y_{reg})$ and so they span an isotropic lattice of rank two there. $\square$

**3.2. The period lattice of cubic fourfolds.** We take a moment to analyze the period lattice in some detail, for reference in the next chapters. This reproduces Looijenga's recent work from [**25**]. We can use this lattice to find all different strata of type II degenerations, by finding orbits with distinct isotropic planes.

Let us briefly repeat the abstract set-up here. Suppose $\Lambda$ is an abstract unimodular lattice (over $\mathbb{Z}$) of signature $(21, 2)$, $V = \Lambda \otimes \mathbb{C}$ a vector space, and $\eta \in \Lambda$ some element with $\eta \cdot \eta = 3$. Let $\Gamma \subset O(V)$ be defined as the group that leaves $\eta$ fixed. We write $\Lambda_0 = \eta^\perp$ for the lattice orthogonal to $\eta$, we know that abstractly $\Lambda_0 \cong 2E_8 \oplus 2U \oplus A_2$. It has discriminant three.

Definition 3.3. When we need to explicitly identify the cohomology $H^4(Y, \mathbb{Z})$, for a certain fourfold $Y$, with the abstract lattice $\Lambda$, we will achieve this by a *marking*, that is an isometry which sends the square of the hyperplane class to $\eta$.

Lemma 3.4. *All primitive isotropic elements of* $\Lambda_0$ *are conjugate under* $\Gamma$.

Vinberg has described an algorithm for finding the fundamental polyhedron of a discrete reflection group $W(\Lambda)$, starting from a point $x_0$. This algorithm, if it terminates, produces a finite collection of roots $\Sigma(W)$ that span $\Lambda^*$ (either over $\mathbb{Z}_{\geq 0}$ or over $\mathbb{Z}_{\leq 0}$) and such that their inner product is always negative. Note that this collection will not be linearly independent over $\mathbb{Q}$, as it augments the root basis for $\Lambda$. A result of this algorithm is the theorem that the fundamental polyhedron is of finite volume if and only if every *affine* subdiagram of $\Sigma(W)$ is a connected component of an affine diagram of corank 1. These affine subdiagrams represent possible degenerations of the period lattice.

We perform the algorithm starting from an isotropic element. Since in our case all isotropic elements are conjugate, the algorithm has the same result no matter which is chosen. The Dynkin diagram of the result is shown in the figure below.

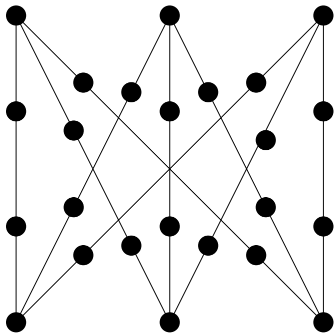


FIGURE 1. The diagram resulting from the algorithm, with emphasis on its $K_{3,3}$ structure.

REMARK 3.5. We remark that this diagram resembles $K_{3,3}$, the complete graph on two sets of three vertices, with extra internal vertices. It and $K_5$ (the complete graph on 5 vertices) are fundamental examples of nonplanar graphs. Another remark to be made is that Vinberg in [**39**] found a singular $K3$ surface (with Picard number 20) having this period lattice, and was dubbed one of the "most algebraic" $K3$ surfaces.

By considering the maximal affine subdiagrams in this figure, one finds 6 special orbits, whose closures meet in a single point. Each of these subdiagrams corresponds to a stratum of type II, since each orbit contains an isotropic plane. We find the following subdiagrams in Figure 1 by some entertaining puzzle-work.

$$3\tilde{E}_6, \tilde{A}_{11} \perp \tilde{D}_7, \tilde{E}_7 \perp \tilde{D}_{10}, \tilde{A}_{17}, 2\tilde{E}_8, \text{ and } \tilde{D}_{16} \tag{2}$$

These diagrams are not all of corank 1, but since the polyhedron is finite, Vinberg's theorem says they can be completed to such. The point where the closures of these orbits meet corresponds to the fourfold obtained as the secant variety of the Veronese surface.

This list classifies the possible types of primitive rank 2 isotropic lattices in $\Lambda_0$. The Baily-Borel compactification of the space $\mathbb{D}$ associated to this lattice will therefore have six types of boundary strata, after the orbit of the Veronese secant is removed.

**3.3. Conclusion.** We summarize these results in the following theorem. Note that no reference has been made to GIT results, for the following chapters we will not need a complete description of the moduli space of cubic fourfolds. This description is available, based on the techniques described here, see Looijenga's work in [**25**].

THEOREM 3.6. *Let $V$ be a complex vector space of dimension 6 and write $S = \mathrm{Sym}^3(V^*)]^{\mathrm{SL}(V)}$ for the space of cubic fourfolds. When we restrict the period map to fourfolds containing at most isolated simple singularities $P : S^f \to \Gamma\backslash\mathbb{D}$, we have an open embedding. The automorphic line bundle identification $\mathcal{L} = \mathcal{O}_{\mathbb{D}}(-1)^\Gamma = \mathcal{O}_{S^\circ}(2)$ gives an isomorphism*

$$\bigoplus_k H^0(\mathbb{D}, \mathcal{L}^{\otimes k})^\Gamma \to \mathbb{C}[\mathrm{Sym}^3(V^*)]^{\mathrm{SL}(V)}$$

*which multiplies degrees by 2.*

CHAPTER 4

# Cubic Threefolds

We now have the apparatus to study two specific examples, namely cubic surfaces and threefolds.

The period map we have seen for cubic fourfolds is very similar to that of K3 surfaces, except that the image form is of Hodge type $(3,1)$ instead of $(2,0)$. Therefore, like in our example of quartic curves, the situation lends itself to a study of the moduli space of cubic threefolds and surfaces using this period map. This is *not* the traditional period map for cubic threefolds which uses *intermediate Jacobians*, as for instance in [**4**] and [**8**].

Given a vectors space $V$ of dimension 5 and cubic threefold in $X \subset \mathbb{P}(V)$, we can consider a cyclic cover over $\mathbb{P}(V)$ branched along $X$: if $X$ is the zero-set of a (cubic) polynomial $F \in \mathrm{Sym}^3(V)$, this cover can be described by $\{x^3 + F = 0\}$. This gives a cubic fourfold $Y$ (with automorphism of degree 3), and we can construct the residue map

$$P : \mathrm{Sym}^3(V) \to \mathbb{D}, \qquad P(F) = \mathrm{Res}\,\frac{dx \wedge \mu_V}{F + x^3}$$

as in Chapter 3.

Similarly, given a cubic surface $S \subset \mathbb{P}^3$, we can repeat the operation to produce a fourfold described by $\{x^3 + y^3 + f(x_0, \ldots, x_3) = 0\}$ (where $f \in \mathrm{Sym}^3(\mathbb{C}^4)$ describes the surface). The fourfold thus obtained has two separate automorphisms, which we must take into account to construct the surface period map. Surfaces will be the topic of the next chapter.

We stress that in both cases, there is an action of a finite group on the fourfolds: a $\mu_3$ action (in coordinates: $w \to e^{2\pi i/3}w$) in the case of threefolds, and a $\mu_3 \times \mu_3$ action (similarly, on $u$ and $v$) in the case of surfaces. We have seen in Section 3 of Chapter 2 that an irreducible representation of such an action is a ball quotient—it has signature $(m,1)$ for some $m$—and so the compactification is easier to perform than in the case of cubic fourfolds themselves. We mention that the study of fourfolds has already been performed by others [**25, 21**], we do not include it here.

## 1. Eisenstein lattices

We begin by introducing a concept that appears in both cases. Many readers will be familiar with the ring of *Gaussian integers*, usually denoted by $\mathbb{Z}[i] \cong \mathbb{Z}[T]/(T^2+1)$, which has an automorphism of order 4. The integers that will be of interest to us appear as the integers in the field $\mathbb{Q}(\zeta_3)$, where $\zeta_3 = \frac{1}{2}(\sqrt{-3}-1)$ is a third root of unity, and are known as the *Eisenstein integers*. This ring carries an automorphism of order 3. We will write $\mathcal{E}$ for this ring, and note that $\mathcal{E} = \mathbb{Z}[\zeta_3] \cong \mathbb{Z}[T]/(T^2+T+1)$: the action is then given by multiplication with $T$.

Suppose we have a lattice $\Lambda$ with a symmetric bilinear form $(\cdot)$ which we suppose to be even, and that the lattice has an action by $\mu_3$ preserving the form. Then we can understand $\Lambda$ as an $\mathcal{E}$-module in the following way. We consider the form defined by

$$\phi(x,y) = (x \cdot Ty) - \zeta_3(x,y).$$

and remark that this is skew Hermitian. That is,

$$\begin{gathered} \phi(y,x) = (y \cdot Tx) - \zeta_3(y \cdot x) = \\ (T^2 y \cdot x) - \zeta_3(y \cdot x) = -(Ty \cdot x) - (y \cdot x) - \zeta_3(y \cdot x) = \\ -(x \cdot Ty) + \zeta_3^2(x \cdot y) = -\overline{\phi(x,y)} \end{gathered}$$

which relies on the relations $T^2+T+1=0$ in $\mathcal{E}$, and $\zeta_3^2+\zeta_3+1=0$ in $\mathbb{C}$. Multiplying this by the purely imaginary term $\sqrt{-3}$ turns it into a Hermitian form:

$$h(x,y) = \sqrt{-3}\phi(x,y).$$

From the definition we see that it is linear in the first term, and anti-linear in the second. Notice furthermore that because

$$2\phi(x,x) = (x \cdot Tx) + (Tx \cdot x) - 2\zeta_3(x \cdot x) = (1-2\zeta_3)(x \cdot x) = -\sqrt{-3}(x \cdot x)$$

so the forms have the property that $h(x,x) = \sqrt{-3}\phi(x,x) = \frac{3}{2}(x \cdot x)$. A lattice with this structure will be referred to as an *Eisenstein lattice.*

Certain Eisenstein lattices are of special interest, and we define them here, following the presentation in the appendix of [**17**]. Let $\Lambda_k$ be a free $\mathcal{E}$-module, with generators $r_1, \dots, r_k$, and Hermitian form given by:

$$h(r_i, r_j) = \begin{cases} 3 & \text{if } j = i \\ \sqrt{-3} & \text{if } j = i+1 \\ 0 & \text{if } j > i+1 \end{cases}$$

Equivalently we can demand $(r_i \cdot r_i) = 2, (r_i \cdot Tr_{i+1}) = -1$ and all other pairings are 0.

The standard $E_8$ lattice has a Coxeter element $h$ of order 30, letting $T$ act as $h^{10}$ gives this lattice a $\mu_3$ action which identifies it with $\Lambda_4$. Likewise defining actions on $E_6$, $D_4$, and $A_2$ by $h^4$, $h^2$, and $h$ (where $h$ is the Coxeter element of order 12, 6, and 3 respectively) gives these lattices the structure of $\Lambda_3$, $\Lambda_2$, and $\Lambda_1$. In the case of the $A_2$ lattice this is just the rotation in the plane (of order 3). The hyperbolic Eisenstein lattice $U_{\mathcal{E}}$, finally, is spanned over $\mathcal{E}$ by two isotropic vectors with inner product $\sqrt{-3}$.

We remark finally that it is known that $\Lambda_{10}$ is isomorphic to $\Lambda_4 \perp U_{\mathcal{E}} \perp \Lambda_4$ (see [**1**]).

## 2. The Eisenstein lattice of Cubic Threefolds

When we use the residue map described at the beginning of the chapter to construct the period map, we need to understand cubic fourfolds with a $\mu_3$ action that fixes a threefold. We make the following observation.

Remark 2.1. If we are considering threefolds, and varying these in degenerating families, then the fourfolds in our construction will always carry the covering $\mu_3$-action, also above the strictly semistable threefolds. One can observe from the list of degenerating fourfolds (equation **??** in the previous chapter) that the lattices allowing a $\mu_3$ action—i.e. those containing only $A_2, D_4, E_6$ or $E_8$ diagrams—are precisely $3\tilde{E}_6$ and $2\tilde{E}_8$. This means that the fourfolds appearing as cyclic covers of strictly semistable threefolds (or surfaces, in the next chapter) must have one of these as a period lattice. The underlying threefolds (surfaces) will have a period lattice that appears as the fixed part under the action of $\mu_3$ (or $\mu_3 \times \mu_3$).

Consider one such semistable case, namely where the threefold $X$ has 2 singularities of type $A_5$. There is a one-parameter family of such threefolds which can be described in coordinates found by Allcock [**2**]:

$$\alpha x_2^3 + x_0 x_3^2 + x_1^2 x_4 - x_0 x_2 x_4 - 2\beta x_1 x_2 x_3$$

The parameter can be expressed as $\frac{\alpha}{\beta^2} \in \mathbb{P}^1$. There is one member $X_0$ of this family (with $\frac{\alpha}{\beta^2} = 0$), that has an extra $A_1$ singularity. The $A_5$ singularities can be seen to appear at points $[1:0:0:0:0]$ and $[0:0:0:0:1]$, and the extra $A_1$ occurs at $[0:0:1:0:0]$. Taking the three-cyclic cover as in our construction gives a fourfold $Y_0$ with two singularities of type $\tilde{E}_8$, and one of type $A_2$. Smoothing $Y_0$, we see that the lattice of vanishing cycles corresponding to $\tilde{E}_8 \amalg \tilde{E}_8 \perp A_2$ must lie in the homology $H_4(Y_s)$.

However, both copies of $\tilde{E}_8$ contain an isotropic lattice. This lattice is the same in both cases, since this threefold defines a type II degeneration. Choose two generators for the lattice $I = \langle e_1, e_2 \rangle$ of shortest length. (Since $I$

can be embedded in $\mathbb{C}$, it inherits the complex norm so this statement makes sense.) The lattice $I$ is a free $\mathcal{E}$-module, so has an action on it by a group of order 3. We are free to choose the generators so that the operation acts as $T : e_1 \mapsto e_2, T : e_2 \mapsto -e_1 - e_2$. Since these elements are chosen to be 'shortest', no other vectors can lie inside the (unit) circle of the following figure. If a vector lies on the unit circle, so must its difference with $e_1$ and $e_2$ since it lies in the lattice, this is only possible at the vertices of the hexagon. We conclude that $I$ embeds in $\mathbb{C}$ as the lattice of Eisenstein integers.

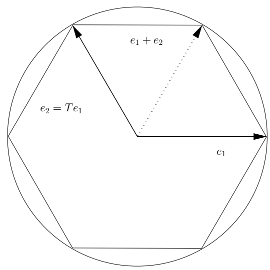



FIGURE 2. The isotropic lattice $I$, with its order 3 action $T$.

To embed this isotropic plane in a non-degenerate lattice, we must include vectors that intersect it nontrivially. We know $H_4(Y_s) \subset \eta^\perp \subset \mathbb{Z}^{(21,2)}$, where $\eta$ is a vector of length 3, and $\mathbb{Z}^{(21,2)}$ is the standard unimodular lattice of signature $(21, 2)$. Since $e_1$ lies in this unimodular lattice, there must exist $\phi \in (\mathbb{Z}^{(21,2)})^* = \mathbb{Z}^{(21,2)}$ such that $\phi(e_1) = 1, \phi(e_2) = 0$. Since each copy of $E_8$ is unimodular, we can choose $\phi$ orthogonal to these. Furthermore, since $A_2 \perp \eta \subset \mathbb{Z}^3$ has a unimodular overlattice, we can also choose $\phi$ orthogonal to this overlattice, and so to $A_2$ and $\eta$. Hence we have found one hyperbolic summand.

Now acting on $\phi$ with the $\mu_3$ action gives $T^*\phi(e_1) = \phi(e_2) = 0, T^*\phi(e_2) = \phi(-e_1 - e_2) = -1$. Therefore $-T^*\phi$ can be used to identify a second hyperbolic summand. We emphasize that we have done this in a $\mu_3$-equivariant fashion. In this way, we have found $\mathcal{E}^* \subset \eta^\perp$ so that $\mathcal{E} \oplus \mathcal{E}^* \cong_{\mathcal{E}} U_{\mathcal{E}} \cong_{\mathbb{Z}} U^2$. Therefore the lattice $E_8 \perp E_8 \perp U \perp U \perp A_2$ carries a $\mu_3$ action that identifies it with the Eisenstein lattice $\Lambda_4 \perp \Lambda_4 \perp U_{\mathcal{E}} \perp \Lambda_1 \cong \Lambda_{10} \perp \Lambda_1$ of signature (10,1) over $\mathcal{E}$.

We have shown the following.

PROPOSITION 2.2. *Let $X_0 \subset \mathbb{P}(V)$ be this threefold with 2 singularities of type $A_5$ and 1 of type $A_1$. Let $\mathcal{X} \to \Delta$ be a smoothing of $X_0$, and let $X$ be a general smooth fiber. Denote by $Y_0$, $\mathcal{Y} \to \Delta$, and $Y$ the corresponding cubic* *fourfolds. Then the kernel of the natural map $H_4(Y,\mathcal{E}) \to H_4(Y_0,\mathcal{E})$—given its intersection pairing and $\mu_3$-action—is isomorphic to $\Lambda_{10} \perp \Lambda_1$.*

## 3. Intermezzo: the chordal cubic

There is one exceptional semistable threefold: the secant variety of a rational normal quartic curve $C$ in $\mathbb{P}^4$. We will show below that this is indeed a cubic threefold. We will refer to this object as the *chordal cubic* and denote it $X_C$. It is singular along the curve $C$, with transversal singularities of type $A_1$ there. The fourfold that is obtained by the 3-cyclic cover then has $A_2$ singularities along the curve.

We shall make this more concrete. The normal quartic can be given as the image of the diagonal map

$$\mathbb{P}(W) \to \mathbb{P}(\mathrm{Sym}^4(W))$$

where $W$ is vector spaces of complex dimension 2. In coordinates this sends $[s:t] \mapsto [s^4 : s^3t : s^2t^2 : st^3 : t^4]$. The image of this map can be described as the locus where the matrix

$$A = \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}$$

of coordinates has rank 1, i.e. all $2 \times 2$ minors vanish.

The chordal cubic $X_C$ can then be defined by the (cubic) equation $\det(A) = 0$.

Remark 3.1. Another viewpoint of the chordal cubic is provided by representation theory. The equation for the chordal cubic is what is known classically as the *catalecticant* for a binary quartic. Suppose $W$ is a vector space of dimension 2, and $f_4 = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4$ is a quartic in $\mathrm{Sym}^4(W)^*$. Then the determinant $\det(A)$ of

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}$$

is invariant under $\mathrm{SL}(W)$. It generates $\mathrm{Sym}^4(W)^{\mathrm{SL}(W)}$, since the configuration space of 4 points on a line has one modulus.

A binary quartic $f_4 \in \mathrm{Sym}^4(W)^*$ can be considered as a pairing on $\mathrm{Sym}^2(W)$, by composing

$$\mathrm{Sym}^2(W) \times \mathrm{Sym}^2(W) \to \mathrm{Sym}^4(W) \xrightarrow{f_4} \mathbb{C}.$$

and the matrix above becomes the intersection matrix of this pairing. If the determinant vanishes as in our definition of the chordal cubic, this pairing is degenerate.

Conversely, a point on the chordal cubic threefold $X_C$ corresponds to a secant of the normal quartic curve $C$. A point on $C$ in turn corresponds to the fourth power $u^4$ of a coordinate $u$ of $\mathbb{P}(W)$; a secant between distinct points of $C$ is then described by a linear combination $\lambda u^4 + \mu v^4$. This quartic determines a degenerate pairing. When we choose $[\lambda : \mu] = [1 : -1]$ to represent the secant, the four points become the fourth roots of unity $\pm 1, \pm\sqrt{-1}$ in the *harmonic* cross-ratio $-1$.

The singular locus of the threefold $X_C$ is the curve $C$. To see that the singularity is of type $A_1$ requires a short argument. Consider the affine neighborhood $z_0 \neq 0$ of the point $[s : t] = [1 : 0]$ of C, here the tangent to the curve is in the direction of $z_1$, so the transversal part $X_C$ is obtained by setting $z_1 = 0$. The equation then reduces to

$$\det(A)|_{z_0=1, z_1=0} = z_2 z_4 - z_3^2 - z_2^3 = z_2(z_4 - z_2^2) - z_3^2 \cong uv - w^2$$

in suitable coordinates. The associated fourfold covering of $\mathbb{P}^4$, ramified along $X_C$, will be denoted $Y_C$ with equation $w^3 + \det(A)$. We see this to be singular of type $A_2$ along $C$.

We have yet to determine the type of this threefold.

PROPOSITION 3.2. *Let $X_C \subset \mathbb{P}(V)$ denote the chordal cubic obtained in this manner, and write $X_{reg} = X_C - C$ for its regular part. Write $Y_C$ for the cubic fourfold obtained as a 3-cyclic cover of $\mathbb{P}(V)$ ramified along $X_C$, and $Y_{reg}$ for its regular part. Then the $\mu_3$-invariant part of the homology of the smooth part is of rank 4, $H_\bullet(Y_{reg}, \mathbb{C})_\chi = H_4(Y_{reg}, \mathbb{C})_\chi$ and is of dimension 1. The intersection pairing from Poincaré duality defines a* positive *Hermitian form so that the pair $(X_C, P(X_C))$ defines a boundary pair of type I.*

The proof is contained in the last section of [**27**]. We reproduce it below.

Notice first that the threefold $X_C$ (and the fourfold $Y_C$) are special because permit an action by $\mathrm{PGL}(W)$. This arises from the (4-uple) embedding of $C$ into $X_C$, a symmetry of $C$ will lead to a symmetry of of $X_C$. Therefore we can ask what the $\mathrm{PGL}(W)$-orbits are that $X_C$ will fall into. Each point $y \in X_C$ corresponds to a secant of $C$ by definition, meeting $C$ in at most 2 points. Since $C$ is a quartic $y \cap C$ will be an effective rank 4 divisor—a formal sum of points on $C$, whose coefficients are positive and sum to 4. It is clear that the action of $\mathrm{PGL}(W)$ will not disturb the configuration of these points. This leaves us with 3 orbits of $\mathrm{PGL}(W)$: the curve $C$ itself which parametrises divisors of the type $4p$; secants triply tangent to $C$ having divisors of type

$3p+q$, the orbit of which we denote $T_C - C$; and divisors of type $2p+2q$ denoted by $X_C - T_C$.

Proof. The proof will use two subsidiary lemmas, proved below. The fourfold $Y_C$ associated to the chordal cubic threefold $X_C$ allows for a Poincaré duality: a nondegenerate pairing $H^k(Y_C,\mathbb{C})_\chi \times H^{8-k}(Y_C,\mathbb{C})_{\overline{\chi}} \to \mathbb{C}$. In particular, there is a nondegenerate Hermitian form on $H^4(Y_C,\mathbb{C})_\chi$.

We first make a short calculation of Euler characteristics, using the PGL$(W)$-orbit decomposition. The curve $C$ is rational, and has $e(C)=2$. The tangent bundle $T_C$ has characteristic $e(T_C)=e(C)\cdot e(\mathbb{P}^1)=4$. The regular orbit $X_C - T_C$ can be considered as a $\mathbb{C}^*$ fiber-bundle and so has $e(X_C-T_C)=0$. Therefore $e(X_C)=4$ and consequently $e(\mathbb{P}(V)-X_C)=e(\mathbb{P}(V))-e(X_C)=5-4=1$. Since the cover $Y_C\to\mathbb{P}(V)$ is of order 3 and totally ramified along $X_C$, we get $e(Y)=3e(\mathbb{P}(V)-X_C)+e(X_C)=3+4=7$. The $\mu_3$ orbit space of $Y_C$ can be identified with $\mathbb{P}(V)$, so that finally $e(Y_C)_\chi = e(Y_C)_{\overline{\chi}} = \frac{1}{2}(7-5)=1$. Because of this fact, to prove the proposition we must show that $H^k(Y_C,\mathbb{C})_\chi = 0$ for $k\le 3$.

Choose a Lefschetz pencil $L$ of hyperplanes for $X_C$. Such a pencil is defined by a generic plane $A\subset V$ giving a line $\mathbb{P}(A)\subset\mathbb{P}(V)$ which lies on all hyperplanes. Since it is assumed generic, this line avoids $C$ and meets $X_C$ transversally at 3 points. A hyperplane $H$ in the pencil meets $X_C$ in a cubic surface $X_H$, with 4 singular points at $C\cap H$. Since this intersection is transversal, the singularities are of type $A_1$ and Lemma 3.3 says that in this case the equivariant cohomology is one-dimensional, concentrated in degree 3.

Write $\tilde{X_C}\subset\mathbb{P}(V)\times L$ for the Lefschetz pencil, and $\tilde{Y_C}\subset\mathbb{P}(V\oplus\mathbb{C})\times L$ for the corresponding pencil of fourfolds. The map $H^\bullet(Y_C,\mathbb{C})\to H^\bullet(\tilde{Y_C},\mathbb{C})$ is an injection, since the projection $\tilde{Y_C}\to Y_C$ is a contraction of $Y_C\times L$ along its projection onto $L$. Therefore it suffices to show that the last space vanishes for $k\le 3$.

Therefore consider the $\chi$-Leray spectral sequence of the projection $\pi:\tilde{Y}\to L$. The derived sheaf $R^q\pi_*\mathbb{C}_{\tilde{Y}}$ vanishes unless $q=3$ by Lemma 3.3, so the sequence degenerates and we are left with isomorphisms

$$H^k(\tilde{Y},\mathbb{C})_\chi = H^{k-3}(L,(R^q\pi_*\mathbb{C}_{\tilde{Y}})_\chi).$$

This shows that cohomology of $\tilde{Y}$ vanishes for $k\le 2$ ($L$ is a line), and Lemma 3.4 implies that if a hyperplane $H$ in the pencil is tangent to a regular point in $X_C - T_C$, then the stalk $(R^3\pi_*\mathbb{C}_{\tilde{Y}})_\chi$ vanishes. Such hyperplanes occur in the pencil, therefore we conclude that $H^3(\tilde{X},\mathbb{C})_\chi = H^0(L,(R^3\pi_*\mathbb{C}_{\tilde{Y}})_\chi)$ vanishes also.

The rest of the statement in the proposition is now a consequence of the more general result Lemma 4.1 given in the next section. □

LEMMA 3.3. *Let $S_0 \subset \mathbb{P}^3$ be the (Cayley) cubic surface with four nodes, and $X_0$ be the cubic threefold that is the normal 3-cyclic cover of $\mathbb{P}^3$ ramified along $S_0$. Then $X_0$ has $\mu_3$-invariant cohomology only in degree 3 and $H^3(X_0, \mathbb{C})_\chi$ is of dimension 1.*

PROOF. This accomplished by a dimension count. The four nodes (type $A_1$) on $S_0$ lead to four cusps (type $A_2$) on $X_0$. The vanishing homology of these cusps is a lattice of rank 2, the threefold with 4 such singularities must have a vanishing lattice of rank 8. Since for a smooth cubic threefold $X$, $H^3(X, \mathbb{C})$ has dimension 10, it follows that $H^3(X_0)$ has dimension 2, and is split into two one-dimensional eigenspaces $H^3(X_0)_\chi$ and $H^3(X_0)_{\overline{\chi}}$. □

LEMMA 3.4. *Let $p \in X_C - T_C$ is a regular point of the chordal cubic, and $H \subset T(X_C)$ is the tangent hyperplane at that point. The threefold $X_H$ produced by cyclically covering the surface $X_C \cap H$ has no $\mu_3$-invariant cohomology at all. Consequently, the same holds for the associated fourfold $Y_H$.*

PROOF. The action of $\mathrm{PGL}(W)$ is transitive on $X_C - T_C$, so we can suffice by showing this for any point. We use the description given above to represent the chordal threefold by $\det(A) = 0$, and regard the point $p = [1 : 0 : 0 : 0 : 1]$. The tangent hyperplane is there described by $\{z_2 = 0\}$, so the equation for $X_C \cap H_p$ reduces to $\{z_0 z_3^2 + z_1^2 z_4 = 0\}$, and the triple cover $X_H$ is given by $\{y^3 + z_0 z_3^2 + z_1^2 z_4 = 0\}$. This threefold is singular along the line $l$ given by $\{y = z_1 = z_3 = 0\}$. Considering the $\mathbb{C}^*$ on $X_H$ defined by

$$\lambda[z_0 : z_1 : z_3 : z_4; y] = [z_0 : \lambda^3 z_1 : \lambda^3 z_3 : z_4; \lambda^2 y]$$

(which preserves the defining equation) we note that the fixed point set of the action consists of the lines $l$ and $l' = \{y = z_1 = Z_4 = 0\}$. This observation gives us a contraction of $X_H - l'$ onto $l$ so that $X_H - l'$ and $l$ are homotopically equivalent. We see that $H^\bullet(l', \mathbb{C})_\chi = 0 = H^\bullet(l, \mathbb{C})_\chi = H^\bullet(X_H - l', \mathbb{C})_\chi$. Since $l'$ lies in the smooth part of $X_H$, we can use the Gysin sequence of the pair $(X_H, l')$

$$H^{k-1}(l')_\chi \to H^k(X_H)_\chi \to H^k(l')_\chi \to H^{k-2}(l')_\chi$$

to conclude that $H^\bullet(X_H, \mathbb{C})_\chi = 0$. □

## 4. The period map for cubic threefolds

As mentioned at the start of this chapter, to describe the moduli space of cubic threefolds we can consider cubic fourfolds with a suitable automorphism of degree 3. Therefore assume $Y$ is a cubic fourfold, with an automorphism $\mu_3$

leaving invariant a cubic threefold. If this is the case we can find coordinates so that $Y$ is given by an equation of the form $w^3 + f(z) = 0$, where $f$ is a cubic form in 5 variables describing the invariant variety.

We implement the residue map of the previous chapter as the period map: whenever $Y_F$ is a smooth cubic fourfold with equation $F$, then the form $\alpha_F = \mathrm{Res}_{\mathbb{P}^5, Y_F} F^{-2}\mu \in H^4(Y_{reg}, \mathbb{C})$ is a generator of $H^{3,1}(Y_{reg})$. The residue map can be extended beyond smooth fourfolds, but we are interested only in the case when its image in cohomology is non-zero, and so generates $H^{3,1}(Y_{reg})$: if this is the case will denote the resulting line by $P(Y)$. This is a point of the period domain.

We have already seen in Section 3 of Chapter 3 that the period map can be extended without effort across simple singularities (with finite monodromy), and that fourfolds with isolated simple elliptic singularities form boundary varieties of type II. We have not yet considered the case where the singular locus on the fourfold has dimension greater than 0.

LEMMA 4.1. *Let $Y$ be a cubic fourfold such that $Y_{sg}$ contains a component of dimension 1, and such that the transversal type of the singularity is $A_2$ (or worse). If the primitive homology $H_4^0(Y_{reg})$ has nontrivial intersection pairing, then $(Y, P(Y))$ is a boundary pair of type I and $H_4^0(Y_{reg})$ is positive semi-definite.*

PROOF. Locally around a point on the singular locus, $Y$ is given in coordinates by $f(z) = z_1^3 + z_2^2 + z_3^2 + z_4^2$ (in the affine neighborhood $z_0 \neq 0$ and letting the curve be parametrized by $z_5$). This $f$ is weighted homogeneous of degree six if variables are weighted $(2, 3, 3, 3)$, so the form $\alpha = f^{-2} dz_1 \wedge \ldots \wedge dz_5$ is homogeneous of degree $-1$, hence $\alpha \wedge \overline{\alpha}$ is not integrable around the origin.

Now consider a 4-cycle $Z \in H_4(Y_{reg})$ perpendicular to the squared hyperplane class, and of non-zero selfintersection. This class will extend over any smoothing $\mathcal{Y} \to \Delta$ of $Y$, so that $\int_{Z_t} \alpha(Y_t)$ can be defined: clearly it is bounded. However, $\int_{Y_t} \alpha(Y_t) \wedge \overline{\alpha(Y_t)}$ goes to infinity as $t \to 0 \in \Delta$. Thus the conditions of lemma 1.4 are satisfied: therefore any limit point of the line $[\alpha(Y_t)] \subset H^4(Y_t, \mathbb{C})$ lies in the hyperplane defined by $[Z_t]$. Since $Z_t \cdot Z_t \neq 0$, the hyperplane must be of type I. □

REMARK 4.2. It is *a priori* unclear whether the image of the chordal cubic under the residue map spans a line in cohomology; that is, that the resulting form is not exact. If it were clear, we would have been able to omit the detailed analysis of the intermezzo 3.

Let $V$ be a complex vector space of dimension 5. Write $S = \mathrm{Sym}^3(V)$ for brevity. Let $S^\circ$ denote the smooth part of $\mathrm{Sym}^3(V)/SL(V)$ parametrising smooth cubic threefolds, and $\mathbb{D} = \mathbb{P}(H^3(X, \mathbb{C})_+)$ (for a smooth threefold

$X$) define a 10-dimensional ball. Here we have identified that $H^3(X,\mathbb{C}) \cong H^4(Y_{X^\chi},\mathbb{C}) = H^4(Y_X,\mathbb{C})_\chi = \Lambda_{10} \perp \Lambda_1$.

We define the residue map $P : S^\circ \to \Gamma\backslash\mathbb{D}$ as stated.

We recover the following description of the GIT-boundary of $S^\circ$.

- When $F \in \mathrm{Sym}^3(V)$ has isolated simple singularities, the period map extends by making a base change. Thus the map extends to a map $P : S^f \to \Gamma\backslash\mathbb{D}$ to all points with finite monodromy; this is all as in the previous chapter.
- When $G \in \mathrm{Sym}^3(V)$ has isolated simple elliptic singularities, the monodromy is infinite but the image of $\alpha(G)$ determines a type II hyperplane in $\mathbb{D}$. The period map extends $w^3 + G$ by mapping these points to cusps of $\mathbb{D}$.

  This situation occurs in two cases. The first is the isolated point where $G = x_0^3 + x_4^3 + x_1x_2x_3$ has three $D_4$ singularities, in which case $Y(G)$ has three $\tilde{E}_6$ singularities.

  The second is the 1-parameter family when $G = x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 - 2\beta x_1x_2x_3 + \alpha x_2^3$ has two $A_5$ singularities [**2**], $Y(G)$ will then have two $\tilde{E}_8$ singularities. When $\frac{\alpha}{\beta^2} = 0$ an extra $A_1$ singularity appears on the threefold.
- Finally, when $Y$ is given by $w^3 + \det(A)$ is the fourfold corresponding to the chordal cubic, it has a curve of $A_1$ singularities and defines a hyperplane of type I in $\mathbb{D}$. It must be deleted to make a meaningful compactification. This fourfold is a limiting case of the family above.

We write $\mathbb{D}^\circ$ for the complement of the orbit of this last type I hyperplane. Write $\mathcal{L} \cong O_{\mathbb{D}^\circ}(-1)^\Gamma$ for the automorphic line bundle. We sum up our discussion in the theorem below.

THEOREM 4.3. *The residue map defines an isomorphism from the moduli space of* stable *cubic threefolds to the ball quotient* $P : \mathrm{SL}(V)\backslash S^f \to \Gamma\backslash\mathbb{D}^\circ$. *Furthermore,* $P$ *induces a* $\mathbb{C}$*-algebra isomorphism* $p : \bigoplus H^0(\mathbb{D}^\circ, \mathcal{L}^{\otimes k})^\Gamma \to \mathbb{C}[\mathrm{Sym}^3(V^*)]^{\mathrm{SL}(V)}$. *This means that* $H^0(\mathbb{D}^\circ, \mathcal{L}^{\otimes k})^\Gamma = 0$ *for* $k < 0$, *and taking the* Proj *of these algebras allows us to identify the GIT and our Baily-Borel-type compactifications*

$$\mathrm{SL}(V)\backslash\backslash \mathrm{Sym}^3(V)^{ss} \cong \mathrm{Proj} \bigoplus_{k\geq 0} H^0(\mathbb{D}^\circ, \mathcal{L}^{\otimes k})^\Gamma$$

*showing that the isomorphism* $P$ *extends properly to the first two degenerate cases.*

PROOF. The chordal cubic defines a boundary pair of type I, as we have said. Such a pair determines a $\Gamma$-orbit $\mathcal{K}_1$ of hyperplanes of $\mathbb{D}$. At the other boundary points of $S^\circ$, the degeneration is of type II and so the period map is *proper* over $\mathbb{D}^\circ$.

The period map

$$P : \mathrm{SL}(V)\backslash S^{st} \to \Gamma\backslash\mathbb{D}$$

is a local isomorphism, by the local Torelli theorem for fourfolds. Even stronger, it is of degree one as a result of Voisin's global Torelli theorem for fourfolds [**40**]. We conclude that the period map is a local isomorphism of degree one, and so an open embedding.

We must still check that the image of the map is disjoint with $\mathcal{K}_1$. If this were not the case, the map $p : \bigoplus H^0(\mathbb{D}, \mathcal{L}^{\otimes k})^\Gamma \to \mathbb{C}[\mathrm{Sym}^3(V^*)]^{\mathrm{SL}(V)}$ would define an isomorphism between the GIT-compactification $\mathrm{SL}(V)\backslash\backslash S^{ss}$ and the Baily-Borel compactification $\Gamma\backslash\mathbb{D}$. However, this is not possible as the first has one-dimensional (strictly) semistable locus, and the second only finitely many cusps. The isomorphism $P : \mathrm{SL}(V)\backslash S^{st} \to \Gamma\backslash\mathbb{D}^\circ$ lifts to an isomorphism of line bundles: $\mathcal{L}^3|_{\mathbb{D}^\circ}$ is identified with $\mathcal{O}_{\mathbb{P}(S)}(1) \to \mathrm{SL}(V)\backslash S^{st}$. Taking the Proj of both sides yields the statement of the theorem. □

REMARK 4.4. Again we may ask ourselves how a single orbit accounts for a hyperplane in the moduli space. The answer is the same as in the case of quartic curves studied in the previous chapter. Consider a smoothing of the chordal cubic $\mathcal{X} \to \Delta$, given by the equation $\det(A) + tG = 0$. The singular (quartic) curve $C$ will meet the smooth cubic hypersurface $\{G = 0\}$ in 12 points. The configuration of these 12 points of $C \cong \mathbb{P}^1$ has $12 \dim \mathbb{P}^1 - |\,\mathrm{PGL}_2\,| = 9$ moduli, and give a hyperelliptic curve of genus 5 $\mathcal{H}_5$. Therefore the chordal cubic alone accounts for all the "hyperelliptic locus".

Again, the work of Deligne and Mostow [**9**] appears. There the period map from $\mathcal{H}_5$ to the quotient of the 9-ball is described by

$$y^6 = \prod_1^{12}(x - z_i) \mapsto [\int_{z_1}^{z_2} \omega : \int_{z_2}^{z_3} \omega : \ldots] \text{ where } \omega = \frac{dx}{y}$$

(modulo ordering). They go on to describe the stability of degenerations where different $z_i = z_j$ coincide. The interested reader is referred to their fundamental paper.

CHAPTER 5

# Cubic Surfaces

An analysis similar to that discussed in the previous chapter can be performed on cubic surfaces. In the same way, the period map for cubic fourfolds is used to describe the moduli space. In this case, the stratification is easier because it consists only of isolated cusps, while understanding the action of the stabilizing group $\mu_3 \times \mu_3$ is more involved. We sketch the theme of this chapter.

Let $V$ be a complex vector space of dimension 4, and $\mathrm{Sym}^3(V)^*$ the space of cubic forms. Consider a six-dimensional space $V \oplus \mathbb{C}^2$, where $\mathbb{C}^2$ is given coordinates $x$ and $y$. Write $\mu_V$ for a volume form of $V$. The residue map is defined to send

$$f \in \mathrm{Sym}^3(V)^* \mapsto \mathrm{Res}_Y \,\mathrm{Res}_{\mathbb{P}(V\oplus\mathbb{C}^2)} \frac{dx \wedge dy \wedge \mu_V}{(x^3+y^3+f)^2}$$

where $Y$ is the fourfold in $\mathbb{P}(V \oplus \mathbb{C}^2)$ defined by the vanishing of $x^3+y^3+f$. As in the previous cases, in smooth cases it maps a surface to a (3,1)-form on a fourfold with two separate $\mu_3$ actions. In this case, the fourfolds permitting two such actions form a 4-ball. We have already seen the important properties of this map in the previous chapters, the following comments are immediate. Surfaces with nodes (locally given by $f = x^2+y^2+z^2 = 0$) lead to fourfolds with singularities of type $D_4$, which have finite monodromy. Surfaces with cusps (locally $f = x^3+y^2+z^2 = 0$) lead to fourfolds with singularities of type $\tilde{E}_6$. There is one particular surface with three cusps, which is minimal in its orbit. It is represented by the equation $z^3 = uvw$, and we denote it by $S_0$. Since this is the minimal element in its orbit closure, this surface represents the orbit of all surfaces with cusps. This leads precisely to the stratum of fourfolds we referred to as $3\tilde{E}_6$, this is the only boundary stratum of fourfolds which allows three separate $\mu_3$ actions.

The GIT of cubic surfaces is well-studied and goes back to Hilbert [**19**]: exactly this surface with three cusps represents the strictly semistable locus. Smooth surfaces, and surfaces with nodes, are stable; reducible surfaces and those with higher singularities are unstable.

We need to spend some time understanding the locus of fourfolds with two $\mu_3$ actions leaving a surface invariant. To do this we concentrate on the singular case $3\tilde{E}_6$.

### 1. The singular case

We concentrate on the cubic fourfold $Y_0$ given by the equation $\{x^3 + y^3 + z^3 + uvw = 0\}$. This fourfold has three singularities of type $\tilde{E}_6$, one each at points where $u, v$ or $w$ are nonzero. The equation is very symmetric, in that we see several groups that preserve the form of the equation. The symmetric group $S_3$ acts on triples $\{x, y, z\}$ and $\{u, v, w\}$ while remaining in the orbit. More importantly for our purposes we have three $\mu_3$ actions sending for example $x \to \zeta_3 x$ (where $\zeta_3 = e^{2\pi\sqrt{-1}/3}$). We write $G = (\mu_3)^3$ for this group, and note that it is abelian and finite (of order 27).

Let us first look locally, around one of the singularities. Let $\mathcal{Y} \to \Delta$ be a smoothing of the fourfold. If we ignore two of the singularities, a general fiber $Y_t$ has a defining equation that can be written in the form $\{x^3 + y^3 + z^3 + u^2 + v^2 = t\}$. We consider the homology of the Milnor fiber $\partial Y_t$, it is the first line of the following diagram. We can consider this $\partial Y_t$ as a cone over the elliptic curve $C$ in $\mathbb{P}^2$ given by $\{x^3 + y^3 + z^3 = 0\}$. Remark that this is the curve with $j$-invariant 0, and with an automorphism of order 3 (multiplying any variable by a third root of unity). We write the Gysin sequence of this $\mathbb{P}^1$-bundle in the second line of the diagram.

$$\begin{tikzcd}
 & H_4(\partial Y_t) \arrow[r, hook] \arrow[d, equal] & H_4(Y_t) \arrow[r] & H_4(Y_t, \partial Y_t) \arrow[r, two heads] & H_3(\partial Y_t) \arrow[d, equal] & \\
H_1(C) \arrow[r, hook] & H_4(\partial Y_t) \arrow[r, "0"] & H_2(C) \arrow[r, "\cdot 3"] & H_0(C) \arrow[r] & H_3(\partial Y_t) \arrow[r, two heads] & H_1(C)
\end{tikzcd}$$

Here we know $H_1(C)$ is a lattice of rank 2 over $\mathbb{Z}$. Because of the $\mu_3$ action on it, we can consider it also as a free module of rank 1 over the Eisenstein integers $\mathcal{E}$. We are interested in understanding the actions of $\mu_3$ on $H_4(Y_t)$.

Notice that $Y_t$, locally given by a quasihomogeneous equation $\{x^3 + y^3 + z^3 + u^2 + v^2 = t\}$, is what is called a *Brieskorn variety.* That is, it is quasihomogeneous with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$. The work of Pham [**33**] on such quasihomogeneous varieties tells us that the homology of $Y_t$ has the form:

$$H_k(Y_t) \cong \begin{cases} \mathbb{Z} & \text{when } k = 0, \\ \mathbb{Z}[x, y, z]/(1+x+x^2, 1+y+y^2, 1+z+z^2) & \text{when } k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Here we have used the terms $x, y$ and $z$ as shorthand for the actions of $\mu_3$ on those coordinates. Remark that $H_4(Y_t) = \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{E} \otimes_{\mathbb{Z}} \mathcal{E}$ is the tensor of three copies of $\mathcal{E}$. We write $R$ for this ring. The ring $\mathcal{E}$ can be diagonally embedded into this, so that $H_4(Y_t)$ has a diagonal action of $\mathcal{E}$ on it: the multiplication by $xyz$.

REMARK 1.1. Pham's work is more precise, in that it also describes the intersection form on this homology. We will sketch it here, but use the equivalent description below.

Let us write $G = \mu_3^3$ for the group and $\mathbb{Z}[G] = \mathbb{Z}[x, y, z]/(x^3-1, y^3-1, z^3-1)$ for the group ring. This ring contains the ideal $\langle(1-x)(1-y)(1-z)\rangle$, which can be identified with the ring $R$ described above. We denote the generating element of this ideal by $e$. Both the homology $H_4(Y_t)$ and cohomology $H^4(Y_t)$ can be identified with this ideal. For the homology this is done by sending $u \in \mathcal{E}^3 \mapsto ue \in \mathbb{Z}[G]$ so that $R \cong \mathbb{Z}[G]/(e)$. The 'Poincaré duality' is accomplished by mapping $j : H_4(Y_t) \to H^4(Y_t), j(u) = (1 - xyz)u$.

There is a (perfect) pairing $\langle|\rangle : H_4(Y_t) \times H^4(Y_t) \to \mathbb{Z}$ described by $\langle u|v\rangle = c_1(ue\overline{v})$. Here $c_1(w)$ denotes the integer coefficient of the unit element of $\mathbb{Z}[G]$, $ue\overline{v}$ is the product of elements in $\mathbb{Z}[G]$ and $\overline{v}$ is the involution of the group ring. Every group ring has such an involution defined on generators as $\overline{e}_g = e_{g^{-1}}$. In this case the group is abelian and the involution is an automorphism: it is the involution which interchanges $x$ and $x^2$ (and similarly for the other variables). If we consider $\mathcal{E} \subset \mathbb{C}$ it is indeed complex conjugation.

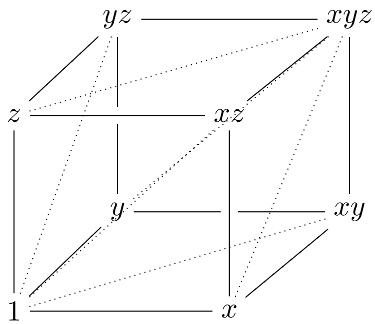
We can turn this pairing into an intersection form on $H_4(Y_t)$, by sending $(u, v)$ to $\langle u|j(v)\rangle$. This intersection form has a kernel, the kernel of $j$. This kernel is spanned by $1 + xyz + x^2y^2z^2$ and is of rank 2 over $\mathbb{Z}$. We denote it by $I$, for it consists of the elements which are *isotropic* for the intersection form. We have the exact sequence

$$0 \to I \to H^4(Y_t) \stackrel{1-\sigma_3}{\to} H^4(Y_t) \to \dots$$

where we can understand the image of $(1-\sigma_3)$ as the lattice of $E_6$, as we shall explain below.

The intersection form on $H_4(Y_t)$ was also calculated by Gabrielov [**11**] and is exhibited by the cube on the next page. Here the monomials are a basis over $\mathbb{Z}$ of the ring homology ring $R$. The vertices in the diagram have self-intersection 2, dotted lines imply an intersection 1, and continuous lines an intersection $-1$. Vertices that are not connected have intersection 0. Although it is not clear in the picture, there is a continuous line along the main diagonal between vertices 1 and $xyz$, implying that $\langle 1, xyz\rangle = -1$.

Observe that if we write $\sigma_i$ for the elementary symmetric polynomials of degree $i$ in $x, y$, and $z$, then the three dependent vectors given by $-(\sigma_1+\sigma_2)$, $1+\sigma_1-\sigma_3$ and $\sigma_3+\sigma_2-1$ are isotropic for the form, orthogonal to the generators, and sum to zero. Additionally they are invariant under permutation of the variables.




Furthermore, these isotropic elements are cyclically permuted by the multiplication with either $x, y$, or $z$, and so are left fixed by $\sigma_3 = xyz$. In this way we see explicitly that this plane forms the kernel of $(1 - \sigma_3)$, as noted by Pham [**33**] and in the remark above.

Gabrielov further supplies an algorithm to determine what is termed a *distinguished basis* of $H_4(Y_t)$. We follow a slightly different construction and produce the extended Dynkin diagram pictured below. This shows that the $\tilde{E}_6$ lattice can be decomposed as the sum of an ordinary lattice of type $E_6$ and an isotropic plane $I$, spanned by isotropic vectors indicated above.

REMARK 1.2. When we need an explicit description of $E_6 \subset \tilde{E}_6$, we will choose the basis $\langle 1, x, y, z, \sigma_3 x, \sigma_3 y\rangle$.

The dotted lines indicate that $\langle 1, \sigma_3 - \sigma_1\rangle = 2$. We find the isotropic vectors as the two copies of the "usual" isotropic vector for the affine $\tilde{E}_6$ diagram, schematically given by $\begin{pmatrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \\ & & 1 & & \end{pmatrix}$, and (one third of) their difference.

This picture makes it clear how the group $G = (\mu_3)^3$ acts on $H_4(Y_t)$. Since each of $x, y$ and $z$ are of order three, the group $G$ acts by multiplying with one of these. In the coordinate-free description, the action is perhaps even clearer: $R = \mathcal{E}^3$ is evidently acted upon by $G$.

## 2. Embedding $3E_6$ in the fourfold period lattice

Now we study how three copies of the diagram above are embedded in $\Lambda = \mathbb{Z}^{21,2}$, the lattice of cubic fourfolds (without polarization). Each of the $\tilde{E}_6$-singularities maps to the fourfold lattice, but this is only possible if the images of the isotropic plane from each of the copies coincide. Therefore, a priori, we have in $\Lambda$ three copies of the ordinary $E_6$ lattice, one isotropic plane $I$, and one polarizing vector $\eta$ of square three, orthogonal to the rest. Therefore $\Lambda \supset 3E_6 \perp I \perp \eta$, we are left to find two isotropic vectors that

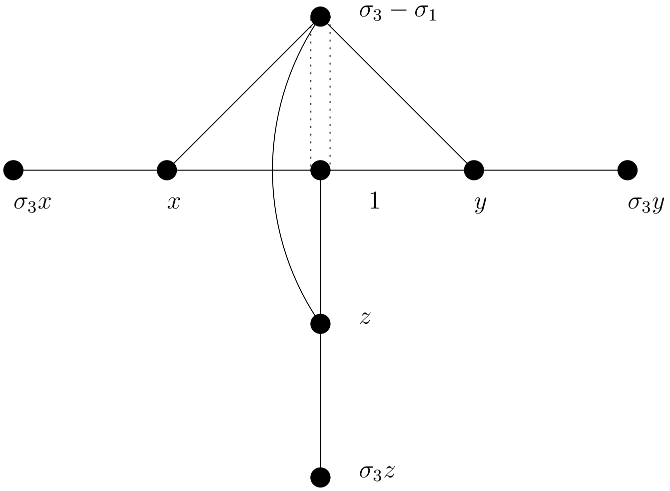


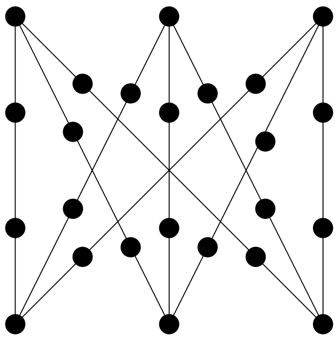

FIGURE 3. A distinguished basis for the $\tilde{E}_6$ lattice

extend the action of $G$ which we have on the singularities, in such a way as to complete $I$ to a nondegenerate, hyperbolic lattice.

We reproduce the diagram of Figure 1 in Chapter 3 below for easy reference. This was constructed from $\Lambda$ by first taking any isotropic element $e$ of $\eta^\perp \subset \Lambda$, and completing the root basis of $\Lambda_0 = e^\perp/\mathbb{Z}e$ to a fundamental polyhedron in $\eta^\perp$. These roots span an even lattice of discriminant 3 (the discriminant of $\eta$). To understand the period domain of cubic surfaces, we must understand how the three copies of $E_6$ lie in $\Lambda$ and how the group $G$ acts on it.

We make a series of remarks to set the scene.

First of all we recall that the $E_6$ lattice itself is of discriminant three. The discriminant group $|E_6^*/E_6|$ has two nontrivial elements which can be



represented by elements dual to the extremities of the 'arms' of the Dynkin diagram. Schematically, we can represent these by

$$\varpi = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 \\ & & 3 & & \end{pmatrix}$$

and

$$\varpi' = \frac{1}{3}\begin{pmatrix} 2 & 4 & 6 & 5 & 4 \\ & & 3 & & \end{pmatrix}.$$

We see that the sum $\varpi + \varpi'$ has integer coefficients and so lies in the lattice $E_6$, we say $\varpi + \varpi' \equiv 0 \pmod{E_6}$.

LEMMA 2.1. *Let $\varpi \in E_6^*$ be a generator of the discriminant group. Then the lattice $K$, which contains $E_6 \oplus E_6 \oplus E_6$ and $(\varpi, \varpi, \varpi)$, is even and has discriminant 3.*

PROOF. The lattice $3E_6 = E_6 \oplus E_6 \oplus E_6$ has discriminant $|E_6^*/E_6|^3 = 27$. If $K$ is any proper overlattice, then the square of the index $[K : 3E_6]$ must divide 27. Therefore $K$ can have only index 3 or 1 (which we excluded), and must have discriminant 3. Each copy of $|E_6^*/E_6|$ has two nontrivial elemants elements $\varpi$ and $\varpi'$ which each generate the discriminant group, these are represented by the vectors dual to the extremities of the 'arms' of the $E_6$ diagram. These both have square length $\frac{4}{3}$.

We write $\xi = (\varpi, \varpi, \varpi) \in (E_6^*)^3$ specifying a choice in each component. Note that $\xi \cdot \xi = 3 \times \frac{4}{3} = 4$. Now define $K = E_6^3 \cup \mathbb{Z}\xi$: this lattice is integral, even, and strictly contains $E_6^3$. Its index $[K : E_6^3]$ cannot equal 1, therefore it must have discriminant 3.

Note that there are multiple choices for $\xi$: we could have chosen $\xi' = (\pm\varpi, \pm\varpi, \pm\varpi)$ in 8 ways, however this would only determine 4 groups. The only one preserved under interchange of the singularities is the one we have described—choosing pluses everywhere. $\square$

The lemma below has already been remarked upon.

LEMMA 2.2. *The isotropic lattice $I \subset \tilde{E}_6$ has an action of $G$, which makes it into a free module of rank 1 over the Eisenstein ring $\mathcal{E}$.*

PROOF. We have seen that the basis for $I$ is symmetric in $x, y$, and $z$, and multiplication with any of these has the same result. This means that $G = (\mu_3)^3$ acts through $[(T^a, T^b, T^c), e] \mapsto T^{a+b+c}e$, which induces the $\mathcal{E}$ module structure. $\square$

The dual lattice $I^*$ has and induced action by $G$, preserving the natural form $I \oplus I^* \to \mathcal{E}, (u, \nu) = \nu(u)$.

Recall that $Y_0$ denotes the fourfold with three $\tilde{E}_6$ singularities, and $Y_t$ is a general fiber of a smoothing $\mathcal{Y} \to \Delta$ of this singular fourfold. As in the previous chapters we write $\Lambda \cong \mathbb{Z}^{21,2}$ for the period lattice of cubic fourfolds. We let $\eta \in \Lambda$ be a vector with square $\eta^2 = 3$. Its dual element $\eta^*$ is equal to $\eta/3$, with square $\frac{1}{3}$. We write $O(\Lambda)_\eta$ for isometries of $\Lambda$ which fix $\eta$. With the above results we turn to the embedding of $H_4(Y_0) \hookrightarrow H_4(Y_t) \cong \eta^\perp \subset \Lambda$.

THEOREM 2.3. *We can find an even lattice $Q$ in the primitive lattice of cubic fourfolds $\eta^\perp \subset \Lambda$ with the following properties:*

- *The lattice $Q$ has rank 22 and discriminant 3,*
- *It contains $3E_6 \perp 2U$ (of discriminant 27) with index 3,*
- *The group $G = (\mu_3)^3$ acts on $Q$, and this action reduces to the action on $\tilde{E}_6$ we have described above when restricted to each of the components.*

*This lattice is uniquely determined up to an isometry in $O(\Lambda)_\eta$.*

PROOF. Notice that a priori $H_4(Y_0)$ is a lattice with a copy of the $\tilde{E}_6$-lattice for each of the three singularities. The fourfold is of type II so there is a unique isotropic plane $I \subset H_4(Y_0)$. The isotropic parts of each $\tilde{E}_6$ copy must hence be identified in the embedding. This proof will be 'bottom up', in that we will complete our lattice of known vanishing cycles $3E_6 \perp I \subset H_4(Y_0)$ to a lattice with $G$ action. We call this $Q$, and show it to be unique.

Recall that each $\tilde{E}_6$ as described in figure 3 has three separate actions by $\mu_3$, which we denoted by multiplication with $x, y$, and $z$. Any such multiplication cyclically permutes basis vectors as of the isotropic plane as $T$, one can calculate that for instance

$$\cdot x : -\sigma_1 - \sigma_2 \mapsto 1 + \sigma_1 - \sigma_3 \mapsto -1 + \sigma_2 + \sigma_3$$

with $y$ and $z$ acting similarly (see Lemma 2.2). This then gives the action on $I^*$ as well. We write $e = -\sigma_1 - \sigma_2$ and $e' = Te = 1 + \sigma_1 - \sigma_3$. This $I$ lies in the nondegenerate $\Lambda$, so we can find an isotropic element $f$ to complete a hyperbolic summand. Given such an isotropic $f$ with $e \cdot f = 1$, we can act by $T$ to find an $f' = Tf$ such that $e' \cdot f' = Te \cdot Tf = 1$ to complete the second hyperbolic summand.

We distinguish elements in different copies of $E_6$ by means of subscripts. Let $\alpha_i$ be the longest root of each copy of $E_6$, given schematically by $\left( {}^{1\ 2\ \substack{3\\2}\ 2\ 1} \right)$. The isotropic parts of each $\tilde{E}_6$ copy are identified. This $I$ has an action of order three on it, which we denote by $T$.

Let $e$ and $e' = Te$ be isotropic lattice elements in $I$, such that they are 'shortest' generators (for the norm in $\mathbb{C}$). Now taking our three copies of $E_6$

and adjoining the following six elements:

$$\begin{array}{ll} -\alpha_1 - e, & e+f \\ -\alpha_2 - e, & -\varpi_1 - \varpi_2 - \varpi_3 - e + f \\ -\alpha_3 - e, & -\varpi_1' - \varpi_2' - \varpi_3' - e + f \end{array}$$

we obtain again the diagram of Figure 1. In particular, the elements in the second row are seen to be orthogonal, with square 2, and intersect with the copies of $E_6$ in such a way as to fashion the three verteces at the bottom of the diagram.

It requires a calculation to see that the $\mu_3^3$ action indeed preserves this lattice. The action on each of the $E_6$ components is described by figure 3, and this action extends to the $-\alpha_i - e$. We are left to verify that the image of $-\varpi_1 - \varpi_2 - \varpi_3 - e + f$ lies in the lattice. To that end, consider the description of $E_6$ as in figure 3 and denote $\varpi = \frac{1}{3}(4\sigma_3 x + 5x + 6 + 4y + 2\sigma_3 y + 3z)$. We act on this by multiplication with $x$ (other multiplications are similar) and perform the following clumsy calculation:

$$\begin{aligned} x \cdot \varpi &= \frac{1}{3}(4\sigma_3 x^2 + 5x^2 + 6x + 4xy + 2\sigma_3 xy + 3xz) \\ &= \frac{1}{3}(4yz - 5 - 5x + 6x + 4xy + 2z(1+x)(1+y) + 3xz) \\ &= \frac{1}{3}(-5 + x + 2z + 4xy + 6yz + 5xz + 2xyz) \\ &= \frac{1}{3}\left([-6 - 4x - 5y - 3z - 2\sigma_3 x - \sigma_3 y] + 4(\sigma_1 + \sigma_2) + (1 + \sigma_1 - \sigma_3)\right) \\ &= -\varpi' + \sigma_3 y - \frac{4}{3}e + \frac{1}{3}e' \end{aligned}$$

where we have used for instance $\sigma_3 x = x^2 yz = -(yz + xyz)$. Note furthermore that, modulo the lattice $E_6$, we have $\varpi' \equiv -\varpi$. This action is analogous in the three $E_6$ components: note we act by $x_i$ in each component, and the isotropic elements are identified. Therefore we conclude:

$$x \cdot (-\varpi_1 - \varpi_2 - \varpi_3 - e + f) = \left(\sum -\varpi_i - (\sigma_3 y)_i\right) - 4e + e' - e' + f'$$

and the lattice is preserved.

When we consider two isotropic elements $e, e' = Te$ in $Q$, and look at $(e^\perp \cap e'^\perp)/(\mathbb{Z}e + \mathbb{Z}e')$ we are deleting the two hyperbolic summands and we retrieve the lattice $K$ from Lemma 2.1. This shows it is even, of discriminant 3, and of the correct rank and signature.

That this $Q$ is unique follows from the symmetry of the group. The choices of isotropic elements $e$ and $f$ are arbitrary, the lattice $I \oplus I^*$ they span with

$Te$ and $Tf$ is unimodular and determined uniquely. When we wish the elements to be unchanged under renumbering the indices—that is reordering the singularities—we see that our choices displayed above are the only possibilities. $\square$

Using this lattice we can describe the fourfolds which appear as covers of surfaces. Such fourfolds must have two separate $\mu_3$ actions. We therefore take the description of $Q$ above and study the sublattice fixed under two actions of $\mu_3$. We can do this by studying multiplicative characters $\chi : G \to \mathbb{C}^*$.

Representation theory tells us there are as many irreducible characters of a group as there are group elements. The vector space $Q \otimes \mathbb{C}$ is a representation of the group $G$, so we can decompose it into irreducible representations

$$Q \otimes \mathbb{C} = \bigoplus_{\chi} (Q \otimes \mathbb{C})_{\chi}$$

where $\chi$ runs over all irreducible characters and $(Q \otimes \mathbb{C})_{\chi}$ denotes the subspace where the group acts by that character. Any form in the lattice which comes from a fourfold with a nontrivial $\mu_3$ action on it will lie in such an eigenspace. For example, let $\omega = \operatorname{Res} \frac{dx \wedge \mu}{x^3 + f}$ be the form associated to the fourfold covering the threefold defined by $f$. If $g = (T^a, T^b, T^c) \in G$ and $\chi_x : G \to \mathbb{C}^*$ is the 'tautological character' of the first term $\chi_x(g) = \zeta_3^a$, we see that acting by $g$ sends

$$g : \omega \mapsto \operatorname{Res} \frac{d(\chi(g))x \wedge \mu}{(\chi(g)x)^3 + f} = \chi(g)\omega$$

and we see that this $\omega \in (Q \otimes \mathbb{C})_{\chi}$ for this character.

Now write $G_0 \subset G$ for those elements of the group that fix $z$. Clearly $G_0 \cong \mu_3 \times \mu_3$. By the comments above, we can identify the lattice invariant under $G_0$ by simply listing the spaces with prescribed characters.

Corollary 2.4. *Let $\psi : G_0 \to \mathbb{C}^*$ be the character that sends $g = (T^a, T^b) \mapsto \zeta_3^{a+b}$. The lattice of fourfolds allowing two distinct $\mu_3$ actions is $Q_{\psi} \cong 3\Lambda_1 \perp U_{\mathcal{E}}$; this is the unique unimodular Eisenstein lattice of signature $(4, 1)$.*

Proof. First consider the space of isotropic elements $I \times I^*$. As noted in Lemma 2.2, $x$ and $y$ (and even $z$) act as multiplication by $\zeta_3$ on $I$. This gives $I \times I^*$ an $\mathcal{E}$-module structure, and identifies it as the hyperbolic plane $U_{\mathcal{E}}$ over the Eisenstein ring.

Restricting then to each $E_6$-lattice, we find only the roots denoted by 1 and $z$ to be invariant for this character. Together these span an $A_2$ lattice over $\mathbb{Z}$, or a $\Lambda_1$ lattice over $\mathcal{E}$. In this way we find three copies of $\Lambda_1$ and the hyperbolic Eisenstein lattice $U_{\mathcal{E}}$ in the invariant lattice $Q_{\psi}$. $\square$

REMARK 2.5. Equivalently, this space is the sum of three eigenspaces of characters of $G$ itself: the trivial character, the character $\chi_z : (T^a, T^b, T^c) \mapsto \zeta_3^c$ and its conjugate $\overline{chi}_z : (T^a, T^b, T^c) \mapsto \zeta_3^{2c}$.

This is the lattice $\mathcal{E}^5$ with quadratic form $-x_1^2 + x_2^2 + \ldots + x_5^2$ that appears in Allcock, Carlson and Toledo [**3**].

We can therefore define the target space of the period map as the subspace of this $H_+ = \{x \in Q_\psi | x \cdot x = 0, x \cdot \overline{x} > 0\}$, with two components $H_+ = H'_+ \oplus \overline{H'}_+$, and $\Gamma\backslash\mathbb{D}_\psi = \mathbb{P}(H'_+)$, where $\Gamma \subset O(\Lambda)$ consists of $\mu_3 \times \mu_3$-automorphisms fixing the polarizing $\eta$. Note that $\Gamma\backslash\mathbb{D}_\psi$ is a 4-ball.

REMARK 2.6. If we restrict the group further and consider the case of only one $\mu_3$ action by a character $\phi : \mu_3 \to \mathbb{C}^*, \phi(T^a) = \zeta_3^a$, then $Q_\phi$ must be the lattice $\Lambda_{10} \perp \Lambda_1$ we found in the previous chapter, Proposition 2.2.

## 3. The period map for cubic surfaces

We are now in a position to combine our results in the following theorem. Let $V$ be a complex vector space of dimension 4, $T = \mathrm{Sym}^3(V)*$ the space of cubic forms, and $S = \mathrm{SL}(V)\backslash\backslash T$ the moduli space of cubic surfaces. The residue construction gives us a period map on the smooth surfaces $P : S^\circ \to \Gamma\backslash\mathbb{D}_\psi$, which extends as seen to surfaces with isolated singularities of finite monodromy $S^f$. The unique semistable point $o \in S^{ss} - S^f$ is represented by the surface with three cusps.

This map is a priori an open embedding, and using Voisin's Torelli result it is a local isomorphism. At smooth points the period map is injective, so we can state that the map $P$ has degree 1.

PROPOSITION 3.1. *The period map $P : S^f \to \Gamma\backslash\mathbb{D}_\psi$ is a proper map. As a result, it is seen to be a global isomorphism.*

PROOF. To see that $P$ is a proper map, we check that the preimage of every point in $\Gamma\backslash\mathbb{D}_\psi$ is compact. This is already true almost everywhere: since $P$ has degree 1, for almost all points in the image there is exactly one point in the preimage. This is already true for all smooth cubic surfaces, for instance.

The important observation is that there are *no* type I hyperplanes to be deleted from $\mathbb{D}_\psi$, since this subspace of $\mathbb{D} = \mathbb{P}(\{x \in \Lambda \otimes \mathbb{C} | x \cdot x = 0, x \cdot \overline{x} > 0\})$ is disjoint from the unique orbit (in $\mathbb{D}$) of type I hyperplanes. Put simply, there is no way to rewrite the equation of the chordal cubic threefold $\det(A) = 0$ (with $A$ as in Section 3) so as to split off a cubic term. There is no surface which has the chordal cubic as covering threefold (or fourfold). Therefore $P$ is one-to-one to its whole image $\Gamma\backslash\mathbb{D}_\psi$.

Now $\Gamma\backslash\mathbb{D}_\psi$ which is a contractible space—hence its own universal covering. Since $P$ is a covering map, and proper, it must be a global isomorphism $S^f \cong \Gamma\backslash\mathbb{D}_\psi$. $\square$

Remark 3.2. We emphasize how this perspective differs from the construction in Allcock, Carlsen and Toledo [**3**]. They study cubic surfaces by covering them to a cubic threefold, and then use a map to *intermediate Jacobians* to define a period map. However, in their construction, it is by no means clear that the map is proper. Hence the corresponding proposition is more difficult to prove.

Our surface period map furthermore identifies the line bundle $\mathcal{O}_{S^f}(1)$ with the automorphic line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{D}}(-1)^\Gamma$. The period map extends over the unique semistable point to give an isomorphism of $S^{ss}$ with the Baily-Borel compactification $\widehat{\Gamma\backslash\mathbb{D}_\psi}$, as explained in the following theorem.

Theorem 3.3 (Moduli space of cubic surfaces). *The residue construction provides a period map from the moduli space of* stable *cubic surfaces to the ball quotient $P : \mathrm{SL}(V)\backslash T^f \to \Gamma\backslash\mathbb{D}_\psi$. Furthermore, $P$ induces a $\mathbb{C}$-algebra isomorphism $p : \bigoplus H^0(\mathbb{D}_\psi, \mathcal{L}^{\otimes k})^\Gamma \to \mathbb{C}[\mathrm{Sym}^3(V^*)]^{\mathrm{SL}(V)}$. This means that $H^0(\mathbb{D}_\psi, \mathcal{L}^{\otimes k})^\Gamma = 0$ for $k < 0$, and taking the* Proj *of these algebras allows us to identify the GIT and Baily-Borel compactifications*

$$\mathrm{SL}(V)\backslash\backslash \mathrm{Sym}^3(V)^{ss} \cong \mathrm{Proj} \bigoplus_{k\geq 0} H^0(\mathbb{D}_\psi, \mathcal{L}^{\otimes k})^\Gamma$$

*showing that the isomorphism $P$ extends properly to the sole semistable case, that of the surface with three cusps.*

Remark 3.4. In the course of this chapter we have also reproduced—in a rather unwieldy fashion—the period map for cubic curves. Instead the map described in the introduction

$$P : f \in \mathrm{Sym}^3(V) \mapsto \mathrm{Res}_{\mathbb{P}(V), f=0} \frac{\mu}{f}$$

for $V$ 3-dimensional and $\mu$ its volume form, we take the map

$$P_2 : f \in \mathrm{Sym}^3(V) \mapsto \mathrm{Res}_{\mathbb{P}(V\oplus V'), F=0} \frac{\mu'}{(f + u^3 + v^3 + w^3)^2}.$$

where $V'$ is also 3-dimensional so $\mu'$ is now of degree 6, and $F = f + u^3 + v^3 + w^3$. This map sends a cubic form in three variables to the (3,1) form of a cubic fourfold permitting *three* actions by $\mu_3$. Let $\phi$ be the diagonal character that sends $(T^a, T^b, T^c) \mapsto \zeta_3^{a+b+c}$. Referring to the proof of Corollary 2.4, we find that only the isotropic vectors lie in this eigenspace. Thus fourfolds with three different actions are rare and their lattice $Q_\phi = U_{\mathcal{E}}$ forms only a 1-ball! A node on the curve is sent to an $\tilde{E}_6$-singularity on the fourfold, the minimal orbit represented by $\{xyz = 0\}$ is sent to the unique boundary point $3\tilde{E}_6$ studied in detail above. Therefore the period map $P_2$ reproduces the isomorphism of the moduli space of cubic curves with a quotient of a 1-dimensional ball (or half-space) with one boundary point: this is precisely the $j$-line.

In this example the construction is obviously superfluous, but it closes the circle of this thesis quite nicely.

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# Samenvatting in het Nederlands

Om te beginnen moet ik de woorden uit de titel verklaren. Kubische oppervlakken, drievouden en viervouden zijn meetkundige weergaven van veeltermen. Dat ze *kubisch* zijn geeft aan dat alle termen in deze veeltermen precies graad 3 hebben (bijvoorbeeld een derdemacht $x^3$, een kwadraat maal een lineair stuk $x^2(y+z)$, of drie lineaire stukken $(x-y)(y-z)(x-z)$). De woorden *oppervlakken, drie-* en *viervouden* geven aan uit hoeveel variabelen de veelterm bestaat. Van de middelbare school kennen we vooral relaties met 2 variabelen $x$ en $y$, de genoemde objecten worden beschreven door respectievelijk 4, 5, en 6 variabelen.

Als wiskundigen zo'n object tegenkomen, willen ze weten hoeveel verschillende soorten ervan zijn. Vaak wordt er dan gezocht naar een ruimte van parameters die de verschillende objecten beschrijven. Zo'n parameter heette in de 19e eeuw een *modulus*, vandaar dat de ruimte van alle parameters een *moduli-ruimte* heet. Veel problemen uit de meetkunde kunnen teruggevoerd worden op het begrijpen van de moduli-ruimte van bepaalde objecten. Er is al veel naar algemene moduli-theorie gekeken en de bekendste moderne theorie voor algebraisch-meetkundige objecten is de *meetkundige invariantentheorie* van David Mumford, in het Engels afgekort tot *GIT*. Deze beschrijft hoe de moduli-ruimte zelf te begrijpen is als een meetkundig object, maar kan slecht omgaan met singuliere randgevallen. Het doel van dit proefschrift was om deze randgevallen (in de genoemde gevallen) beter te begrijpen, via een andere structuur.

Om uit te leggen wat de periodenafbeelding is, geef ik hier een voorbeeld van dat dicht bij huis ligt. Stel dat de objecten die we willen classificeren fietsbanden zijn[1]. En stel dat we alleen interesse hebben in de *vorm* van de band, en niet kijken naar de grootte. Op een fietsband liggen twee belangrijke cykels (cirkels): een grote langs het wiel, en een kleine dwars op de buis.

[1]Dit klinkt misschien vergezocht, maar de vorm van een fietsband is ook de vorm van een elliptische kromme. Dit voorbeeld is wiskundig van belang, en is ook in meer wiskundige termen uit te leggen. Bovendien weet iedereen die wel eens een te kleine fietsband heeft proberen te monteren dat er ook praktisch belang is.

Deze cirkels hoeven niet dwars op elkaar te staan, de hoek ertussen is ook interessant. Als je de band langs deze cirkels zou openknippen, krijg je een rubberen ruit met zijden gelijk aan de omtrek van de cirkels, en met correcte hoeken. Als de grootte van de band er niet toe doet, dan is de *verhouding* van de zijden samen met de basis-hoek genoeg om de ruit (en dus de band) te reconstrueren. Deze verhouding en hoek kunnen samengepakt worden in een complex getal, wat een complexe structuur definieert. Dit complexe getal heet de *periode* van de band en bepaalt een rooster: het rooster dat je krijgt door kopieën van de ruit naast elkaar te leggen. Nu volgt de wiskundige uitspraak: de moduli-ruimte van elliptische krommen is dezelfde is als de ruimte van roosters in het complexe vlak. Problemen over elliptische krommen kunnen worden beantwoord met resultaten uit de theorie van roosters, en andersom. Dit kleine wonder vormt de aanzet tot dit proefschrift.

Een voorbeeld dat zich laat uitleggen aan de hand van geknipte fietsbanden, noemen wiskundigen 'triviaal'. De periodenafbeelding wordt dan ook interessanter bij het voorbeeld van K3 oppervlakken. Dit zijn nu 4-dimensionale objecten met daarop 22 cykels—die nu boloppervlakken zijn. Hoewel dit niet meer te visualiseren is, hebben K3 oppervlakken nog steeds de eigenschap dat een complexe structuur op de ruimte van cykels een uniek bijbehorend K3 oppervlak bepaalt. De moduli-ruimte van K3 oppervlakken is dus hetzelfde als de ruimte van complexe structuren op een vaste ruimte. De afbeelding van een K3 oppervlak naar deze structuur heet een *periodenafbeelding*, en is in dit geval een isomorfisme. Dit resultaat staat bekend als een Torelli stelling, naar de Italiaanse wiskundige die deze stelling voor elliptische krommen formuleerde.

Het bestaan van zo'n periodenafbeelding geeft dus een nieuw perspectief op een moduliprobleem: men kan naar de resulterende roosters kijken, en die proberen te classificeren. Het tweede hoofdstuk van dit proefschrift beschrijft dit, en zoekt naar de voorwaarden waaronder de afbeelding uitbreidt naar singuliere gevallen.

Periodenafbeeldingen bestaan veel algemener dan hier genoemd, maar dat ze een isomorfisme vormen is zeldzaam. Naast de voorbeelden van elliptische krommen en K3 oppervlakken ontdekte Claire Voisin in 1985 dat de periodenafbeelding voor kubische viervouden injectief is. Dat wil zeggen dat niet alle roosters voorkomen in het beeld, maar als ze dat doen dan hoort er slechts één kubisch viervoud bij. Verder lijkt de periodenruimte van een kubisch viervoud erg op dat van een K3 oppervlak. Het fundament is nu aanwezig voor de toepassing van de theorie, dit gebeurt in de latere hoofdstukken. Uit een kubisch oppervlak of een drievoud kan een viervoud worden gemaakt, en zo kan ook voor deze objecten een periodenafbeelding tot stand worden gebracht.

De resulterende periodenroosters beschrijven de moduliruimten van kubische oppervlakken en drievouden: dit zijn kleine aanpassingen van complexe ballen van dimensie 4 respectievelijk 10. Inmiddels heeft mijn promotor Eduard Looijenga ook het algemenere probleem voor kubische viervouden zelf met deze methode aangepakt.

Dan nog de vraag die elke lezer beziggehouden heeft vanaf het moment dat hij het proefschrift in handen kreeg: wat is de figuur op de kaft? Dat is een reële weergave van het kubisch oppervlak dat een speciale rol speelt in dit proefschrift, namelijk het oppervlak met drie kuspen[2]. Dit is een singulier randgeval van de kubische oppervlakken, en het is zelfs in zekere zin het enige. In de studie van kubsiche drievouden en viervouden komen overdekkingen van dit voorbeeld steeds terug, vandaar dat het een plek op de kaft wel verdient.

[2]In de tekst wordt dit weegegeven door de homogene vergelijking $\{w^3 = xyz\}$, om het in $\mathbb{R}^3$ weer te geven is een andere vorm gebruikt.

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Een proefschrift wordt zelden door maar één persoon geschreven, deze zeker niet. Er zijn veel mensen die ik wil bedanken voor wat zij bijgedragen hebben, direct of indirect. Ik kan natuurlijk niet iedereen noemen in het detail dat ze verdienen, vergeef me de omissies.

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Als laatste lieve Cécile: er is niets dat ik hier kan zeggen dat voldoende uitdrukt wat je voor mij betekent. Ik laat het maar zo.

# Curriculum Vitae

Rogier Willem Swierstra werd op 12 april 1980 geboren in Salvador de Bahia de Todo os Santos (kortweg Salvador) in Brazilië. Hij wist al snel dat hij brandweerman zou worden. Na omzwervingen door Brazilië (Saõ Paulo), de VS (Atlanta en Pittsburg, ridder), Saoedi-Arabië (Riad, piraat) en Tsjechië (Praag, wetenschapper), behaalde hij in 1997 in Hilversum het *International Baccalaureate* aan het Alberdink Thijm College.

Hij studeerde wis- en natuurkunde aan de Universiteit Utrecht, en studeerde in augustus 2003 in beide disciplines af. De maandag erop begon hij aan zijn promotie in de meetkunde bij Eduard Looijenga, waarvan dit proefschrift het tastbare resultaat is.