Tjalling C. Koopmans Research Institute

Discussion Paper Series nr: 12-06

Network Disruption and the Common Enemy Effect

Britta Hoyer

Universiteit Utrecht

Utrecht School of Economics

Tjalling C. Koopmans Research Institute
Utrecht School of Economics
Utrecht University

Kriekenpitplein 21-22
3584 EC Utrecht
The Netherlands
telephone +31 30 253 9800
fax +31 30 253 7373
website www.koopmansinstitute.uu.nl

The Tjalling C. Koopmans Institute is the research institute and research school of Utrecht School of Economics.
It was founded in 2003, and named after Professor Tjalling C. Koopmans, Dutch-born Nobel Prize laureate in economics of 1975.

In the discussion papers series the Koopmans Institute publishes results of ongoing research for early dissemination of research results, and to enhance discussion with colleagues.

Please send any comments and suggestions on the Koopmans institute, or this series to J.M.vanDort@uu.nl

ontwerp voorblad: WRIK Utrecht

How to reach the authors

Britta Hoyer
Utrecht University
Utrecht School of Economics
Kriekenpitplein 21-22
3584 TC Utrecht
The Netherlands.
E-mail: b.hoyer@uu.nl

This paper can be downloaded at: http://www.uu.nl/rebo/economie/discussionpapers

Utrecht School of Economics
Tjalling C. Koopmans Research Institute
Discussion Paper Series 12-06

Network Disruption and the Common Enemy Effect

Britta Hoyer

Utrecht University School of Economics

March 2012

Abstract

"The enemy of my enemy is my friend."

This common adage, which seems to be adhered to in social interactions (e.g. high school cliques or work relationships) as well as in political alliances within countries and between countries, describes the ability of groups or people to work together when they face an opponent, although otherwise they have little in common. In social psychology this phenomenon has been termed the "common enemy effect". Such group behavior can be studied using networks to depict the players within a group and the relationships between them. In this paper we study the effect of a common enemy on a model of network formation, where self-interested, myopic players can use links to build a network, knowing that they are facing a common enemy who can disrupt the links within the network and whose goal it is to minimize the overall value of the network. We find that introducing such a common enemy can lead to the formation of stable and efficient networks which would not be stable without the threat of disruption. However, we also find that fragmented networks as well as the empty networks are also stable. While the common enemy can thus have a positive effect on the incentives of players to form an efficient network, it can also lead to fragmentation and disintegration of the network.

Keywords: strategic network disruption, strategic network design, non-cooperative network games

JEL classification: C72, D85

1. Introduction

Connections between individuals (or firms or countries for example) enable these individuals to communicate and exchange goods or information. However, as Jackson and Wolinsky (1996) have shown, if the links that facilitate this communication between players are costly, it is not always possible for a group of myopic self-interested rational players to form a stable network and even if such a network is formed that is stable, it is usually not an efficient network, from a welfare perspective. This is due to the myopic outlook of players and their inherent self-interest.

The analysis of Jackson and Wolinsky (1996) ends with finding stable networks. Yet, once stable networks are formed, they are often soon threatened by attacks from outsiders to this network. This threat of an outside attack can take place in numerous different situations. It can be anything from a computer network that is threatened by hackers, to transport networks that face a possible attack by terrorists to terrorist networks being observed by law enforcement. It can also simply be a network of highschool friends that is threatened by an outsider spreading rumors. In each of these cases, not only are the people within the network threatened by an attack but also the network as a whole is threatened to be destroyed, which from a welfare point of view might be worse than the fact that certain people will be disconnected from the network. Take for example a computer network. Hackers may be able to disconnect certain computers from that network by installing viruses on this computer, which is bad for those users but as long as these computers do not hold central positions is not so bad for the network as a whole. However, should these computers be very central within the network, then by disconnecting them from the rest of the network, not only is this bad for the particular user but the network as a whole may be disconnected.

Previous work on random and strategic disruption of networks (see for example Goyal and Vigier (2009), Hong (2009) or Albert et al. (2000), as well as in our own previous work - Hoyer and De Jaegher (2010)) has mainly focused on robust designs against disruption. Therefore this work has focussed on games between a network designer and a network disruptor, not taking into account the payoffs of individual nodes within the network but only the value of the network as a whole. Consequently, the effect of a network disruptor in this work has always been strictly negative, as the introduction of a network disruptor by definition will lead to lower value in the network if links and/or nodes are taken out of the network. Contrary to this analysis, in this paper we focus on the incentives of self-interested individual players to form a network when faced with a network disruptor. We find that facing a network disruptor might actually have a positive effect on the individual nodes' payoffs, since it allows for the formation of stable networks that would not be formed without the threat of disruption. This leads to a purely economic explanation to what has generally been termed the "common enemy effect". The common enemy effect refers to the often observed fact that groups manage to work together more efficiently (or at all) when facing a common enemy than when there is no such threat. Thus the enemy might actually enhance efficiency and cooperation although his goal is to do the exact opposite. Examples of this are plenty. Think for example of political coalition between parties that usually stand for opposite ideas, joint demonstrations of different social and political groups against xenophobia, racism and climate policies, or more recently the different groups that joined forces in the "occupy"-movement. The following examples will illustrate the effect in more detail.

- "By-Product Mutualism" is a term used in biology to explain situations in which "cooperative behavior is an incidental consequence of 'ordinary selfish behavior', but only if the environment is sufficiently adverse." (Mesterton-Gibbons and Dugatkin (1992)). Thus behavior that is induced by means of a common enemy. The paper by Mesterton-Gibbons and Dugatkin (1992) states that his behavior is for example observed in lions, where it is found that on average relatedness does not influence the composition of female hunting groups. Rather it is shown that it is the "common enemy", thus the size of the prey, that incentivizes lionesses to hunt jointly, rather than individually. While lions hunt small prey individually, as the chance of success is big, for hunting of larger prey, where the chance of success of any one hunter goes down, they form hunting groups. This fall in the chance of success, overcomes the participation costs of having to share the prey and thereby enabling group hunting.

- Foundations of international organizations that independent countries establish to further their common interest, such as the NATO, are not a natural or easy task. Olson and Zeckhauser (1966) state that while the formation of such an alliance might be beneficial for each country, the countries existed without the organization for years. However, at the emergence of a common enemy - Russia - the states manage to overcome their individual reservations to forming an alliance and jointly founded the NATO.

The groups of people, countries, animals or parties working together against such a common enemy, can quite readily be depicted by means of networks - using nodes to depict the players and links to show the relationship between the players within the group. The analysis of the efficiency, stability and formation of such networks has been studied quite extensively in economics and sociology literature. For a good overview see for example Goyal (2009) or Jackson (2008). Including positive as well as negative relations between people has been the main focus of literature on the social balance of networks in psychology as well as sociology (see for example Heider (1982) or Newcomb (1961)). It has been shown that networks indeed are only considered stable once they are balanced - thus once friends of friends are friends and enemies of friends are enemies. A dynamic analysis of how such networks may evolve can be found in Antal et al. (2006). They use the example of changing alliances in pre-World War 1 Europe to show that while such networks may be stable, they are not necessarily socially desirable. While these social balance studies obviously have the common enemy effect imbedded, they do not specifically refer to it. They also start the analysis from pre-existing networks, thus looking at how the common enemy effect and other friendship relations influence the formation of stable alliances within the network but not at how they influence the formation of such networks to begin with. However, looking at the aforementioned real-world examples, it seems natural that the common enemy effect not only affects how stable alliances are but also which networks may be formed. In this paper we thus study the impact the common enemy effect has purely on the formation of such networks and if it can be explained in economic terms - thus by the incentives provided to the individual players.[1] Accordingly, this paper studies the formation of a network of self-interested, myopic players, when they are facing a common enemy in form of a network disruptor whose goal it is to maximally decrease the efficiency of the network. To do so we use the basic case of the network formation model of Jackson and Wolinsky (1996) and introduce a network disruptor to this setting. Using the concept of pairwise stability that was introduced in their paper - thus assuming that a network is stable if and only if no player wants to delete a link and no pair of players wants to add a link - we investigate which networks will be stable and efficient when an attack is faced. We assume that each of the players owns one piece of non-rival information that can be accessed by the other players if they are linked. To access the information of other players, the players can form a network by adding or deleting links. The network disruptor observes the network and can then choose which links to disrupt and thereby try to cause maximal damage to the network. As we assume that he wants to minimize the overall value of the network, we model his preferences to be lexicographic such that he will prefer one large component and 2 single nodes to one large component and a pair of nodes. The interaction between the formation of the network and the disruption yields a residual network consisting of one or more components. This is the basis for the payoff of the players.

The principal contribution of this paper is to propose a tractable model of defense and attack of a network of self-interested and myopic players. By doing so, we build on - and add to - several strands of literature. The theoretical literature on networks so far has mostly focussed on the formation (Jackson and Wolinsky (1996) and Bala and Goyal (2000)) and the structure of networks. While using one of the models of network formation - the connections model - first and foremost, our paper adds to the part of networks research that deals with networks under attack. This problem has received more attention in physics and operational research (e.g. Albert et al. (2000), Bollobás and Riordan (2003), Lipsey (2006), and Taylor et al.

[1] We of course realize that once such a network is formed all sorts of psychological and other factors play a role in how players will behave and how successful such a network will be. What we look at, however, is purely an explanation of the common enemy effect in terms of incentives of the individual players.

(2006)) than in economics. However, whereas the focus in the physics research is on random attacks on graphs, we are here looking at intelligent network disruptors, thus shifting the focus of the research from designing a network that is proof against random attacks to one that is proof against targeted attacks. The literature in operational research, mostly focusses on survival analysis. Thus, on how to build a road network or a military network in such a way that there is still a viable route through the network even if there is a random attack (earth quake, flood, etc) or a targeted attack (terrorism). Consequently their focus is more on the distance between players, not on what happens if the network is separated completely into different components. In economics, interest in research of attacks on networks has only increased in recent years.

Relevant papers in economics focus mainly on criminal networks, such as in the work of Ballester et al. (2006), where on an exogenously given network criminal activity is coordinated. Because the network is exogenously given, the focus is only on optimal network disruption, and not on the behavior of the players within the network, as in our paper. In particular, the disruptor tries to find the "key player" which is shown to be the node with the highest degree of Bonacich centrality (a centrality measure used in social network analysis). Our focus is on network disruption by taking out links. Thus the centrality of any one player is not as important. Another example is the work by Hong (2009). In his paper security services try to stop terrorists carrying an explosive through an exogenously given transport network, modeled as a directed flow network, from reaching their destination. To do so, they try to shut down a minimal number of links. In our paper the focus is on undirected networks. Also the manner of defense is exactly opposite, as here links are added to defend the network and not deleted.

Enders and Su (2007) and Enders and Jindapon (2010) take a game-theoretic approach similar to ours, where more ties allow more information to be produced in the network. While in our paper we focus on sharing as much information as possible in the network, which the network disruptor tries to stop, in their work the main purpose of the network is to keep information secret, whereas the purpose of the network disruptor is to learn as much of this information as possible. Baccara and Bar-Isaac (2008) study networks of power relations. A disruptor to this network can take out nodes and thereby will also disconnect the nodes connected to that player. They focus however, solely on the design problem. Goyal and Vigier (2009)'s model has a similar focus on network defense against a disruptor as our model does. Unlike in our paper, in their approach, network defense takes the form of the creation of a "firewall" around key nodes; we focus instead on a network that is defended by adding links that would otherwise be redundant. Another key difference to their paper is that in our model the nodes are self-interested utility maximizing players, whereas they are using a game between a network designer and an attacker. A similar approach to the one of Goyal and Vigier is found in Hong (2008).[2] In Hoyer and De Jaegher (2010), we use a similar approach to Goyal and Vigier, by modeling a game between a network designer and a network disruptor, treating link deletion and node deletion. In contrast to the present paper however, there is a central network designer. In the present paper, we study the stability and efficiency of networks when facing an attack by an intelligent network disruptor, where the networks are made up of self-interested players. To the best of our knowledge, this is the first paper that studies this problem using not a network designer (as in for example Goyal and Vigier (2009) or Hoyer and De Jaegher (2010)) but a network of self-interested utility maximizing players.

Apart from the literature on networks, we also add to the literature on the "common enemy" effect. The term and the observation of this effect is by no means a new concept. It has been used as early as 1911 to explain the psychology of punitive justice in Wright (1911). Currently, the common enemy effect is used to explain everything, from the formation of terrorist groups (see for example Metz (2003)), to the formation of alliances amongst countries or against terrorism (e.g. Olson and Zeckhauser (1966), Maoz et al. (2007) or Das and Roy Chowdhury (2008)), from the cooperation among competing firms (Hamel and Prahalad (1993)) to explaining why lions are hunting in groups (Mesterton-Gibbons and Dugatkin (1992)). However,

[2]Kovenock and Roberson (2010) recently look in a similar way at network defense; yet their paper is less relevant to our model, as network structure is not taken into account. Instead, in their paper network vulnerability arises because of the production function generated by the nodes in the network, where in one extreme one node suffices to obtain full production and in the other case all nodes are necessary to produce.

most of the literature so far focusses on the empirical observation of this effect in human and animal life and gives psychological reasons for the behavior. What we are adding to this literature is a theoretical basis for this behavior that is not rooted in psychology but solely in incentives of the players.

The rest of the paper is organized as follows. Section 2 presents the model of network formation and disruption. Section 3 introduces the benchmark case and sections 4 and 5 focus on the analysis of stable networks, where we first introduce a complete characterization for the case of a disruption budget of one and then move on to more general results. Section 6 focusses on the efficiency analysis and section 7 concludes.

2. Model

We study a two stage game between n self-interested, myopic players and a disruptor. Modeling the structure of such a game can be achieved by using an undirected network model, consisting of links and nodes. The disruptor is modeled as a network disruptor, who has a disruption budget consisting of x links that he can destroy within the network. In the game the players move first and build a network by adding costly links bilaterally and deleting them unilaterally. The players know that an attack will ensue in the following stage of the game and they also know the size of the network disruptors disruption budget. Once the players are satisfied with the network - thus once they are in a stable network, the network is observed by the disruptor who can then choose which links to attack. The payoffs are based on the network that remains after the attack.

2.1. Graphs

For the model of network formation we use the connections model without decay, as presented in Jackson and Wolinsky (1996). There is a finite set of $N = \{1, ..., n\}$ players. These players are also commonly referred to as nodes, as the relations between these players are formally represented by a graph g, whose links capture the pairwise relations between players. Let g_{ij} denote a link between any two players i and j. We assume that $g_{ij} = 1$ if there is a link and $g_{ij} = 0$ if there is no link. Links are undirected, thus $g_{ij} = g_{ji}$, and costly. There exists a path between any two nodes i_1 and i_k if there exists a sequence $i_1, ..., i_k$, such that $g_{i_1,i_2} = 1 = ... = g_{i_{k-1},i_k}$. Only if there is a path between any two nodes we refer to them as connected. The degree of connectivity of any node is denoted by η. A component of the graph g is a maximally connected subset. A graph g plus link g_{ij}, is denoted as $g + g_{ij}$. A graph without this link is denoted as $g - g_{ij}$. We will now introduce some special graphs, which we use quite extensively in the paper.

Special Graphs - The Empty Network

The empty network is denoted by g^e and defined as a graph in which each node is connected of degree $\eta = 0$. There are thus no links within such a graph.

Special Graphs - The Star Network

The star network is denoted g^* and is a network consisting of one central node that is connected of degree $\eta = (n-1)$ and has $(n-1)$ end-players. End-players are such players that by definition are connected of degree $\eta = 1$. There are thus exactly $(n-1)$ links within such a network.

Special Graphs - Minimally Connected Networks

A network is minimally connected if and only if it consists of n nodes which are all connected in one component using exactly $(n-1)$ links. Thus by definition there is no cycle within such a network.

Special Graphs - The Cycle Network

The cycle network consists of n nodes connected by exactly n links in such a way that each node is connected of degree $\eta = 2$.

Special Graphs - r-Regular Networks

A network is r-regular if and only if it consist of n nodes which are all connected in one component where each node is connected of degree $\eta = r$. A special case of such a network is the cycle network.

2.2. The Nodes

In the network there are n individual self-interested players. Each player i has one unit of non-rival information w_i. For ease of exposition, we assume that $w_i = w_j = 1$. To obtain the piece of information node j has, node i can form a link to j. However, node j needs to accept the link. We assume that there is a "two-way" flow of information (thus each node can access the information of the nodes it is linked to) and both nodes have to pay costs c_{ij} to establish the link. We assume that $c_{ij} = c_{kl} = c$. By accessing the information of node j, node i also gains access to all the information j has gathered through his links from other nodes and all those nodes gain access to i's information. The payoff for node i can then be defined as follows: $u_i(g) = w_{ii} + \sum_{j \in \mathcal{N}_i} w_{ij} - \sum_{g_{ik} \in g} c_{ik}$, where $\mathcal{N}_i$ is the set of all nodes $j \neq i$ for which there is a path in the network between i and j and where k denotes all direct neighbors of i. Thus the payoff of node i is derived by the value of its own information, the information it obtains from the members of the component it belongs to minus the cost of its direct links. Since we assume that $w_i = w_j = 1$, we can rewrite this as $u_i(g) = y - \sum_{g_{ik} \in g} c_{ik}$, where $y \leq n$ denotes the size of the component that i belongs to. The goal of each individual player is to maximize his own payoff within the game.

2.3. Stability

To define which networks will be formed, we use the concept of pairwise stability, as introduced in Jackson and Wolinsky (1996). This means that nodes have the freedom to add or delete links. However, while the decision on link deletion is unilateral, for link formation both players need to consent. Thus, a network is pairwise stable, if and only if, no player unilaterally wants to delete a link and no pair of players wants to add a link. This can be stated as: A network g is pairwise stable iff

1. for all $ij \in g, u_i(g) \geq u_i(g - g_{ij})$ and $u_j(g) \geq u_j(g - g_{ij})$, and
2. for all $ij \notin g, u_i(g + g_{ij}) > u_i(g)$ then $u_j(g + g_{ij}) < u_j(g)$.

Thus at least one player needs to strictly prefer forming the link over not forming a link, when the other player is indifferent.

2.4. Value

Following Myerson (1977) and the assumption in the connections model in Jackson and Wolinsky (1996), we assume that the value of a network is the sum of all the individual utilities of the players. Therefore we assume that $v(g) = \sum_{i \in N} u_i(g)$. Given our assumptions on $u_i(g)$, this means that given the same number of links, a network with more nodes connected in one component, will always have a higher value than a network with less nodes. Given that what we model is essentially a communications network and the information is non-rival, this assumption is reasonable.

2.5. The Network Disruptor

At stage two, the network disruptor observes the network and can then choose to disrupt it. The network disruptor has a disruption budget x, where $x \geq 0$, which refers to the number of links the disruptor can take out of the network. The main goal of the network disruptor is to minimize the overall value $v(g)$ of the network. To model this, we assume that he has lexicographic preferences. Lexicographic preferences can be used to model this since they imply that the network disruptor prefers two small components rather than one single component and a large component plus two singleton nodes to a large component plus a connected pair. Given the way we model the value of the network, these preferences coincide with minimizing the overall value of the network.

The strategy of the network disruptor for any given network can be described as follows.[3] The network disruptor starts by looking at the k-connectivity of the network. A graph is k-connected if $\eta(i,j) \geq k$ for any two distinct nodes i and j holds. If $x < k$, the network disruptor cannot disconnect any node from the network. For a proof of this see the undirected link version of Menger's Theorem, as for example introduced in Bondy and Murty (2008) in Theorem 9.7.[4] If $x \geq k$, the network disruptor follows the following algorithm. He starts by considering all the possible partitions within a network where each partition consists only of one node and then considers the cardinality of the cut link set between these partitions. By a cut link set we mean a set of links whose removal will lead to the disconnection of the whole partition.[5] If the cardinality is smaller or equal to the network disruptors budget, he will disrupt. Consequently, he can cut the network into all disconnected single nodes. If this is not the case, he will look for all partitions where the weakly largest connected partition is a pair and again consider the cardinality of the link cut set between these partitions. If this cardinality is smaller or equal to his disruption budget, he will delete those links, if not, he will look for all partitions where the weakly largest connected partition includes 3 nodes and so on. Since $k \leq x$, there needs to be at least one link cut set. Thus he can at least disconnect one single node from the remaining network of size $(n-1)$. So in short this means that if you consider all the partitions of the n nodes with a link cut set with cardinality x or less, the network disruptor's best response is to look for such a partition where the weakly largest remaining component will be as small as possible.

3. Benchmark Case

We will begin the analysis by looking at a benchmark case in which there is no network disruptor. Since in the benchmark case the main distinction is between low linking costs, thus linking costs below 1, and high linking costs, thus costs above 1, we will use this distinction in the further analysis. Before we begin with a general analysis we will introduce the benchmark case and then give a full analysis of the case where the disruption budget equals one. Then we will continue with analyzing the general case.

In the benchmark case we assume that there is no network disruptor. If there is thus no threat of an attack on the network, we are in the case analyzed by Jackson and Wolinsky (1996) in the symmetric connections model. However, additionally we assume here that there is no decay within the network. While Jackson and Wolinsky (1996) do not treat the case without decay explicitly, it is straightforward to look at their results for the case of no decay, thus if $\delta = 1$. Then the main distinction in their results is based on the level of linking costs. They find that for costs below δ, where we assume here that $\delta = 1$, the only pairwise stable network is the star network. However, this is based on the fact that they only look at cases for $\delta < 1$. For $\delta = 1$, it is easy to see that the structure of the minimally connected networks does not matter in this case. Thus not only is the star pairwise stable, but also any other minimally connected network, as they all use the exact same number of links to connect the exact same number of players into one component. Without decay, the distance between the players does not matter and consequently the structure of the minimally connected network does not matter. In terms of efficiency, minimally connected networks are also the only efficient networks in this setting, as the benefit from being connected outweighs the costs of a link. For linking costs above $\delta = 1$ on the other hand, they find that the only pairwise stable network is the empty network, g^e. This is due to the fact that no player is willing to be linked to an end-player, since the value of the information of one player alone is lower than the costs of linking. Thus the marginal benefits of being linked to a player are negative. Being in a circle however, is not a best response either, since each player has an incentive to delete one link because he also gets all the information if he is at the end of a chain. By deleting a link though, the player becomes an end-player and the network will unravel. However, the empty network is not necessarily the most efficient network.

---

[3]Here the strategy coincides with the strategy of the network disruptor described in Hoyer and De Jaegher (2010) for the case of link deletion.

[4]This states that the maximum number of pairwise internally disjoint paths between any two nodes is equal to the minimum number of links in any link-cut set.

[5]For a formal definition see for example Bondy and Murty (2008) Section 2.5.

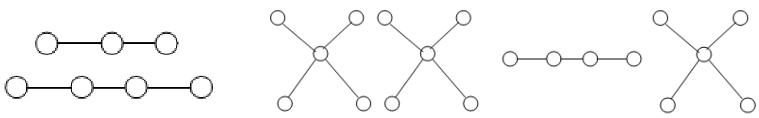
Depending on the size of the linking costs, connected networks will be more efficient, however, they are never stable. Take for example any minimally connected network. The value of the network $v(g^*)$ will then be $v(g^*) = n^2 - 2(n-1)c$. As compared to that the value of the empty network is $v(g^e) = n$. Comparing these two, we find that the star network is more efficient than the empty network if $\frac{n^2-n}{2n-2} > c > 1$. Take for example the case of $n = 12$. Then it holds that the star network is more efficient than the empty network for $6 > c > 1$. Thus, while the empty network is the only pairwise stable network it is not necessarily an efficient network. Accordingly the main tradeoff in the model by Jackson and Wolinsky (1996) is between stability and efficiency of a network, especially in the case of high linking costs.

4. Full Characterization for a disruption budget of $x = 1$

Before turning to a general analysis of the case where there is a disruptor present with a positive disruption budget, we will first provide a full characterization of the case where the disruption budget is equal to one, thus where it holds that $x = 1$. This will provide intuition for the analysis of the general case which will then follow. In line with the distinction along linking costs introduced in the benchmark case, we will first look at the case for low linking costs ($c < 1$) and then at the case for high linking costs ($c > 1$). To break the analysis down into parts we begin by looking at special classes of networks, starting with minimally connected networks and then looking at regular networks. Since minimally connected networks are the only networks that are pairwise stable for low linking costs in the benchmark case, they offer a good starting point for the analysis.

4.1. Low Linking Costs and $x = 1$ - Minimally Connected Networks

Minimally connected networks are all networks that connect n nodes into one single component using exactly $(n-1)$ links. There are, however, a multitude of network structures that fulfill this requirement for any given number of nodes. We will thus first consider two special cases of minimally connected networks, namely the line network and the star network. They are at opposite ends of the sphere of minimally connected networks in that they are the networks with the (most) least number of end players. In the line network there are exactly 2 end players, whereas in the star network there are $(n-1)$ end players. Next to looking at minimally connected networks where all nodes are connected into one component, we can also look at networks that consist of multiple separate components which are each minimally connected. Thus networks where there at least two nodes i and j for which holds that $g_{ij} = 0$ and at the same time there is also no path between the two nodes. Thus after having analyzed the case of minimally connected networks where all nodes are connected in one component, we will then also look at the case of having multiple components, which in themselves are minimally connect. Examples of networks that fulfill this criteria can be found in Figure 1.



(a) 2 lines (b) 2 stars (c) Line and Star

Figure 1: Networks consisting of multiple minimally connected components

Of course there are an immense number of additional ways to form minimally connected components, as they don't have to be a star or a line. Also such networks can consist of more than 2 separate components. However, just as for the single component networks these are good starting points, they are as well for networks consisting of multiple components.

4.1.1. The Line Network

The line network is the only minimally connected network that includes only two end players. Every other player in the network is connected of degree $\eta = 2$. The line network is only one link short of forming a cycle network, which for a disruption budget of $x = 1$, means that all players will remain connected in one component even after disruption. Therefore the line network is a good network to start with as it seems to be the network that is most likely **not** to be pairwise stable, since with a single added link a cycle can be built. Thus should the line network be pairwise stable, it would be straightforward to deduce that all minimally connected networks should be pairwise stable when facing a network disruptor with a disruption budget of $x = 1$.

To check for pairwise stability we need to consider the incentives for each player to add or delete a link to the network. Only if no pair of players would mutually benefit from adding a link and no single player would benefit from deleting a link, will the network be pairwise stable. By definition there are only two types of players in the line network - the two end-players that are connected of degree $\eta = 1$ and the remaining players that are connected of degree $\eta = 2$. The damage a network disruptor can cause to a line network is also straightforward to see. With a disruption budget of $x = 1$, the network disruptor can cut the network into two equally sized pieces.[6] Thus the payoff of any end-player i in a line network, when facing a network disruptor with a disruption budget of $x = 1$ is $u_i(g) = \frac{n}{2} - c$, whereas the payoff for any of the non end-players j is $u_j(g) = \frac{n}{2} - 2c$. It is straightforward to see that the line network will never be pairwise stable. Since it is enough that one of the stability conditions is not satisfied for the network to not be pairwise stable, here we only need to look at the incentives for the players to add a link. Both end-players and non end-players can benefit by forming the largest cycle possible. Take the end-player i, for example. By forming a link with the other end-player k, a cycle including all n nodes will be formed. The payoff for i is then $u_i(g + g_{ik}) = n - 2c$, since no node can be disconnected. Comparing this payoff to $u_i(g)$, it is straightforward that the line network is only pairwise stable if $c > n - \frac{n}{2}$. However, since by assumption for low linking costs $c < 1$, this is never the case. Thus the line network is not pairwise stable.

Looking at networks consisting of more than one component in line form, as in Figure 1(a), we have to analyze two cases. First the case, as in Figure 1(a) where the size of the components is unequal and then the case where the size of the components is equal. For unequally sized components, we have to distinguish between two subcases. In the first cases there is one strictly biggest component, however, this component is not bigger than any two of the smaller components combined. In the second subcase, the strictly biggest component is bigger than any combination of two smaller components. It is straight forward to see that in both subcases the network will not be pairwise stable. For subcase 1, the two smaller components are safe from disruption, they are thus pairwise stable. However, the biggest component, is then a line network on its own and can therefore be analyzed exactly as we did above. Therefore the network will not be pairwise stable. In the second subcase, the two smaller components will not be targeted even when combined, thus there is an incentive for players in each component to add a link between them. Thus the network is not pairwise stable. Looking at networks consisting of D separate equally sized line components, we can see on the other hand that such a network indeed may be pairwise stable. Take any network consisting of D separate equally sized line components. The payoff of any end-player i in such a network is given by $u_i(g) = \frac{1}{D} * \frac{n}{2D} + (1 - \frac{1}{D}) * \frac{n}{D} - c$. Adding a link to any other end-player j irrespective of which component this end-player is in, will lead to a payoff of $u_i(g + g_{ij}) = \frac{n}{D} - 2c$. It holds that $u_i(g) > u_i(g + g_{ij})$, thus the network is pairwise stable, if $c > \frac{n}{2D^2}$. This can be summarized in the following Lemma.

Lemma 1. *A line network consisting of one connected component is never pairwise stable. A network consisting of multiple separate line components is only pairwise stable if those components are equally sized and it holds that* $c > \frac{n}{2D^2}$.

4.1.2. The Star Network

The star network is the only minimally connected network that consists of $(n - 1)$ end-players. It is thus at the exact opposite of the line network in terms of number of end-players. In the star network

[6] Abstracting from divisibility issues.

the chance of being disconnected for any single player is therefore as low as possible in any minimally connected network. Therefore the incentives to add an additional link for any player is very low, giving this the best chances for pairwise stability of any minimally connected network. We can therefore deduce that if the star network is not pairwise stable when facing a network disruptor with a disruption budget of $x = 1$, then no other minimally connected network will be either.

To check for pairwise stability, we can focus on the end-players here, as the central player is already linked to everyone. The payoff of any end-player i in the star network will be $u_i(g^*) = \frac{1}{n-1} * 1 + (1 - \frac{1}{n-1})(n-1) - c$, as the chance for being the one node that will be disconnected is $\frac{1}{n-1}$. Should player i delete a link his payoff is simply $u_i(g^* - g_{ic}) = 1$. Comparing the two payoffs it is straightforward to see that for costs below 1, i never has an incentive to delete his link. Should he add a link to another end-player j his payoff will be $u_i(g^* + g_{ij}) = (n-1) - 2c$. Comparing this payoff with $u_i(g)$ we find that the only case for which the star network is pairwise stable is if $c > \frac{n-2}{n-1}$. Since by assumption for low linking costs $c < 1$, this indeed holds only for a very small cost range and for small n, since in the limit $\frac{n-2}{n-1}$ goes towards 1.

We begin our analysis of networks consisting of multiple star components by looking at Figure 1(b). Since we have seen that the star network is the only minimally connected network (consisting of one component) that is pairwise stable, it is a natural starting point to check the pairwise stability of networks consisting of multiple star components of equal size. We will then further discuss networks consisting of multiple star components of unequal size and show that like their counterpart consisting of only one component, these networks are generally not pairwise stable.

If we look at a network consisting of two star networks that are of an equal size, we know that the incentives for players in one of the stars are the same as those of players in the other star. Thus we only need to look at one of the two. For such a network to be pairwise stable, it needs to hold than that no node wants to delete a link and no pair of nodes wants to add a link. The analysis runs parallel to that of the discussion of the single star, however, now we need to take into account that the chance of each one of the components to be targeted is only $\frac{1}{2}$. Thus the payoff for any node i in one of the components will be $u_i(g) = \frac{1}{2}[\frac{2}{n-2} * 1 + (1 - \frac{2}{n-2})(\frac{n-2}{2})] + \frac{1}{2} * \frac{n}{2} - c$. As long as it holds that $u_i(g) > 1$, no node wants to delete a link. This inequality holds in all cases, since by assumption $c < 1$ and $n \geq 6$ since otherwise we cannot build 2 star components of equal size. A node will not add a link to a node j in the other components, as long as $u_i(g) > \frac{n}{2} - 2c$, since by adding this link they ensure that the network disruptor will target this link. This inequality holds as long as $c > \frac{n-3}{n-2}$ is satisfied. Since $c < 1$, this is only satisfied for a very small cost range again. In the analysis of the simple star, we did not have to take the center nodes into account, since they never had an incentive to add or delete a link. Here however, since each component will only be targeted with a probability of $\frac{1}{2}$, the center node's best response is no longer as straightforward, as by deleting one of his link, the component becomes strictly smaller, thus the network disruptor will always target the other component. Looking at the payoff of the center node we find that it is $u_c(g) = \frac{1}{2} * \frac{n-2}{2} + \frac{1}{2} * \frac{n}{2} - \frac{n-2}{2} * c$. Should the center node delete his link to node i, his payoff will be $u_c(g - g_{ci}) = \frac{n-2}{2} - \frac{n-2}{2}c$. Comparing these two we find indeed that the network is only pairwise stable if $\frac{1}{2} > c$. Combining this with the restriction we found on node i, we can see that the network is never pairwise stable, as $\frac{1}{2} > c > \frac{n-3}{n-2}$ never holds. In Lemma 2 we generalize this result for networks consisting of more than 2 equal sized star components.

Next to networks consisting of equally sized star components we can also look at networks consisting of star components of unequal size. Here we need to distinguish between two cases, namely one where the biggest component is larger than any combination of two of the smaller components together and one where this is not the case. We will start by looking at the case where the biggest component is smaller than the combination of two smaller ones. In this case it is straightforward to see that the two smaller components will never be targeted. Thus as in the benchmark case, those networks are pairwise stable. They do not have an incentive to add a link to connect the components, as by assumption this component would be larger than the biggest component, therefore the link would be targeted by the network disruptor. The analysis of the biggest component is then exactly the same as that for the simple star network. The network will thus be pairwise stable for $c > \frac{y-2}{y-1}$, where y is a subset of n that denotes the size of the largest component. Now looking at the case where the size of the biggest component does exceed the size of the combination of two of the smaller components it can directly be seen that in this case the network will never be pairwise stable, as then nodes in the smaller components have an incentive to add a link between them, since they will still not be targeted.

Lemma 2. *A star network consisting of one connected component is only pairwise stable if* $c > \frac{n-1}{n-2}$*. A network consisting of multiple unconnected star networks is only pairwise stable if there is one strictly largest component, where the size of the largest component is not larger than the combination of two of the smaller components, and it holds that* $c > \frac{y-2}{y-1}$*, where* y *is a subset of* n *that denotes the size of the largest component.*

Proof For the part on the simple star network, see the discussion above.
For a network consisting of multiple equally sized star components we can generalize the discussion above. Assume that there are D separate star components. The payoff to any end-node i will then be $u_i(g) = \frac{1}{D}[\frac{D}{n-D} * 1 + (1 - \frac{D}{n-D}) * \frac{n-D}{D}] + \frac{D-1}{D} * \frac{n}{D} - c$. This can be simplified to $u_i(g) = \frac{3D-nD+n^2-2n}{D(n-D)} > c$. No end-player wants to delete a link as it always holds that $u_i(g) > 1$. This can be verified by substituting n with $3D$, which is the minimal size of n, as there are D components and they all at least have to be of size 3, otherwise they would not qualify as stars. The payoff for adding a link to any node j in a different component will be $u_i(g + g_{ij}) = \frac{n}{D} - 2c$. As in the example for $D = 2$, here except for in a very small cost range the network will not be pairwise stable. The network is only pairwise stable if $c > \frac{Dn^2-nD^2-3D+nD-n^2+2n}{D(n-D)}$ holds. Again knowing that $n \geq 3D$, this equation is true for small cost ranges. However, looking at the payoff for the central node c, we see that $u_c(g) = \frac{1}{D} * \frac{n-D}{D} + \frac{D-1}{D} * \frac{n}{D} - \frac{n-D}{D} * c$ is larger than $u_i(g - g_{ic}) = \frac{n-D}{D} - n - DDc$ only if $\frac{D-1}{D} > c$. This condition on c and the condition found by looking at the incentives of i to add a link, however, can never be satisfied at the same time. Thus such networks can never be pairwise stable.
For the part on unequally sized star networks, see the discussion above. □

In this section, we have thus seen that unlike in the benchmark case, neither the star nor the line network are generally pairwise stable for the case of low linking costs. Indeed, only for a very small cost range are there cases when the star network is pairwise stable at all. Looking at networks consisting of multiple separated star or line components, we have seen that such networks can be pairwise stable for certain cost ranges, if the components are equally sized. Thus if there is not one definite target for the network disruptor but a multitude of different possible targets. We therefore have to look also at other minimally connected networks to see if they might be pairwise stable. In the following section we will do this and show that if the star network is not pairwise stable, no other minimally connected network can be pairwise stable in this setting.

4.1.3. Minimally Connected Networks in general

We can directly exclude any minimally connected network from the analysis that is asymmetric - thus any network in which there is one link that will definitely be targeted by the network disruptor because there is one strictly largest part of the network that can be disconnected by the network disruptor if he disrupts one link. In such a network, it is straightforward that the nodes that are adjacent to the link that will be disrupted will benefit from deleting the link. At the same time, any node in the component that will be disconnected benefits from adding a link to the rest of the network, such that a circle is formed and the node cannot be disconnected. Consider an asymmetric network, where there is one link that the network disruptor strictly prefers to disrupt. By disrupting this link, he can disconnect a set of m nodes from the network. A player can then either be in the set of m nodes or in the remaining set of $(n - m)$ nodes. If a player i is in the set of m nodes, his payoff will be $u_i(g) = m - s_i c$, where s_i denotes the number of links player i currently has. If a player j is not in the set of m nodes, his payoff will be $u_j(g) = n - m - s_j c$, where s_j denotes the number of links player j currently maintains. Note that no player in set m has any incentive to link to another player within this set, as the network disruptor's strategy would remain the same. He would disrupt the link between set m and the remaining network. Thus adding a link within the set m

would only lead to higher costs for player i. By the same reasoning, no player within the remaining network would link to another player in the remaining network. Consequently the only case we need to consider is where player i in m would link to a player j not in m. Supposing that adding one link means that x_i extra nodes are connected, a player i in the set with m nodes wants to add a link iff $m - s_i c < m + x_i - (s_i + 1)c$, which holds iff $c < x_i$. Supposing that the adding of one link means x_j extra nodes are connected, a player j in the set with $n - m$ nodes will add a link iff $(n - m) - s_j c < (n - m) + x_j - (s_j + 1)c$, which holds iff $c < x_j$. Since by assumption x_i and x_j are at least one, these conditions are always met for low linking costs. It follows that no asymmetric minimally connected network can be pairwise stable for low linking costs. Accordingly, the only minimally connected networks that may be pairwise stable are those where the individual player is uncertain about the outcome - i.e. where the network disruptor is indifferent between targeting a number of links. Thus the network needs to contain some degree of symmetry.

So assume that there are D components including m nodes that can be separated from the rest of the network by taking out 1 link. Accordingly the network disruptor is then indifferent between disrupting any one of the D links, which form link cuts. Then any node j included in the $(n - Dm)$ nodes that cannot be disconnected, will never form a link with a node in one of the D components, as his payoff will remain the same, since the disruptor will simply disconnect one of the other components including m nodes. Thus the only link that a node i in any of the components can form is to another node k in one of the other components including m nodes. For the analysis, we consider two cases: $D = 2$ and $D \geq 3$. For both cases we look at the payoff of a player i in one of the D components of size m. To check for pairwise stability we first look at the payoff of player i without added links and then compare this to the payoff he would achieve by linking to a player k in one of the other components of size m. We have already shown above that no node in the component that cannot be disrupted will accept a link so we need not consider this case. By linking to any node k, player i ensures that he is part of a component that will not be disconnected, since he will form a cycle with node k. We then go on to proof that such networks will indeed never be pairwise stable for low linking costs when facing a network disruptor with a disruption budget of $x = 1$.

Theorem 1. *For $x = 1$ and $c < 1$ no minimally connected network except for the star network is pairwise stable.*

Proof For asymmetric networks see the discussion above.
For $D = 2$, consider a non-line network that is minimally connected. Assume player i is in one of the 2 components of size m, then player i's payoff is $u_i g) = \frac{1}{2}m + \frac{1}{2}(n - m) - s_i c$, where s_i denotes the number of links of player i. Adding a link to node k, he will be part of a component of at least size $n - (m - 1)$ and his payoff will be $u_i(g + g_{ik}) = n - (m - 1) - (s_i + 1)c$. The network is not pairwise stable iff $u_i(g) < u_i(g + g_{ij})$, which holds iff $c < \frac{n+2-2m}{2}$. Given that n needs to be at least of size $(2m + 2)$ since size $(2m + 1)$ would denote a line network and we have already excluded line networks, we can see that the network will not be pairwise stable since substituting $(2m + 2)$ for n would lead to $c < 2$ and by assumption for low linking costs $c < 1$.
For $D \geq 3$ node i's payoff will be $u_i(g) = \frac{1}{D}m + (1 - \frac{1}{D})(n - m) - s_i c$. Adding a link to node k will lead to a payoff of $u_i(g + g_{ik}) = n - m - (s_i + 1)c$. The network is not pairwise stable iff $u_i(g) < u_i(g + g_{ik})$, which holds iff $c < \frac{n-2m}{D}$. Given that n needs to be at least of size $(Dm + 1)$, we can see that the network will not be pairwise stable since substituting $(Dm + 1)$ for n would lead to $c < \frac{1+m(D-2)}{D}$. Since $D \geq 3$, we can see that the right hand side of this equation will be larger than 1 for all cases if $m \geq 2$. Since $m = 1$ would denote the star network and we have already analyzed this above, we can thus assume that $m \geq 2$.
□

Thus unlike in the benchmark case, where minimally connected networks were stable and efficient, we have shown here that with the exception of the star network (and there only for a very small range of linking costs, namely $1 > c > \frac{n-2}{n-1}$, minimally connected networks are not stable when facing a network disruptor. Looking at networks consisting of multiple minimally connected components, we have seen in Lemmata 1 and 2 that there are indeed cases when such networks may be pairwise stable, even if, as in the case of the line network, they are not pairwise stable once the network only consists of one component. The bottom line here is, that such networks need to have more than one obvious link that the network disruptor may decide to target. Then there can be networks consisting of multiple separate minimally connected components of any shape that can be pairwise stable. As soon as there is only one link that the network disruptor may choose to target the network will not be pairwise stable. Thus in general, with the exception of the star, such networks need to consist of multiple equally sized components to be pairwise stable. For the star network this does not hold. In the star network, while there are multiple links that can be deleted, as soon as one of the components includes one node less, it will no longer be a target at all. Therefore the incentive of the center player, who bears most of the costs in the network, to delete a link is too big for such a network to be stable.

Summarizing, we have not found a lot of networks that are pairwise stable for low linking costs, when facing a network disruptor with a disruption budget of $x = 1$. This is possibly the case because these networks are not "equal" enough, in that there are always links that will be targeted with a higher likelihood than others. Consequently we need to turn to other classes of networks to find pairwise stable networks in this setting, and those networks should divide cost and benefit more equally between players. We will therefore now move to another frequently studied class of networks, namely regular networks.

4.2. Low Linking Costs and $x = 1$ - Regular Networks

In a r-regular network, each node receives exactly r links. One example of such a network is the 2-regular network, which is the cycle.[7] Next to being regular, the cycle network is also symmetric in the graph theoretic sense of the word. This means that not only do all nodes receive the same amount of links but in a symmetric network also every node has exactly N_x nodes at distance x, with $x = 1, 2, ..., d_{max}$, where d_{max} denotes the maximal distance between any two nodes in a network. Whereas the cycle network is both regular and symmetric, in general regular networks are not necessarily symmetric. However, for each degree of regularity symmetric networks exist. For a proof of this see Hoyer and De Jaegher (2010). By construction, in a symmetric network that is regular of degree $x + 1$, where x denotes the size of the disruption budget of the network disruptor, no single node nor group of nodes can be disconnected from the network. Thus as we have already discussed above, when facing a network disruptor with a disruption budget of $x = 1$, in a cycle (which is a 2-regular symmetric network) no single node nor group of nodes can be removed from the connected component. Thus any node i will have a payoff of $u_i(g) = n - 2c$. There is no incentive for any node to add a link, since they are already safe within the network. There is also no incentive for any node to delete a link, as they would then move to being an end-player within the line network.

Lemma 3. *The cycle network is pairwise stable for low linking costs when facing a network disruptor with a disruption budget of $x = 1$, for any cost range.*

Proof The payoff of any node in the cycle is $u_i(g) = n - 2c$. The payoff should a node decide to delete a link will be $u_i(g - g_{ij}) = \frac{n}{2} - c$. For pairwise stability it needs to hold that:

$$\begin{aligned} u_i(g) &> u_i(g - g_{ij}) \\ n - 2c &> \frac{n}{2} - c \\ \frac{n}{2} &> c \end{aligned}$$

Since by assumption $c < 1$, this condition always holds and thus the cycle network is pairwise stable. □

However, we can also look at networks that are not exactly 2-regular, but regular and symmetric of a higher order. Take a network that is $x + 2$-regular, thus 3-regular in our current case. Every node will have

[7] The cycle is in so far a special case of regular networks since it is the only 2-regular network. For all other degrees of regularity there are more than one way to build a regular network.

an incentive to delete a link, since they are then still part of a cycle and therefore safe within the network, but they will have to pay only for two links instead of three. Thus networks that are regular of a degree higher than $(x+1)$ are not pairwise stable. Regular networks of a lower order that include one connected component only, do not exist for the case of $x = 1$, thus they play no role here.

4.3. Low Linking Costs and $x = 1$ - General Results

For a disruption budget of $x = 1$ when dealing with low linking costs, we can use the results we obtained in the previous sections to state some general results. Before we move on to generalizing these results, however, we will first have a look at the empty network, g^e. Whereas in the benchmark case each player has an incentive to form a link, here this one link will become an automatic target for the network disruptor and will thus definitely be taken out. Therefore there is no incentive for players to add links and the empty network is pairwise stable.

Lemma 4. *For the case of low linking costs, the empty network, g^e, is pairwise stable when facing a network disruptor with a disruption budget of $x = 1$.*

Proof The payoff for any node i in the empty network is $u_i(g^e) = 1$. Should node i decide to add a link this link will be the only link in the network and therefore the automatic target of the network disruptor. Thus the payoff for node i would be $u_i(g^e + g_{ij}) = 1 - c$. Since $u_i(g^e) < u_i(g^e + g_{ij})$, the empty network is pairwise stable. □

This result, while very straightforward, is also a marked distinction between this analysis and the benchmark case, since in the benchmark case the empty network is only pairwise stable for high linking costs.

Now turning to generalizing the results we have introduced before. We have seen that the star network can be pairwise stable, although only for a very limited cost range and that if the star is not pairwise stable, then no other minimally connected network will be. We have shown that networks consisting of more than one minimally connected component can be pairwise stable for certain cost ranges if they leave multiple targets for the network disruptor, and we have also seen that the cycle network is pairwise stable for any costs below 1. Of course there are still numerous other networks to look at. However, most can be dismissed directly. First we can dismiss any network that is not minimally connected but includes end-players. They will not be pairwise stable, as we have already seen in the discussion of minimally connected networks that the star network is the only one that might be pairwise stable, and there chances of being disconnected are lowest. If there are now networks that include a core network that is interlinked and some periphery nodes that are end-players the chances of being disconnected are automatically increased because there are less nodes that may be disconnected. If, for example, the core consists of 3 nodes, there can maximally be $n - 3$ end-players. Therefore even if everything else is as in the star network, the network will not be pairwise stable.

Next we can look at any connected network that does contain a cycle spanning all nodes and no end-players. Any network that includes a cycle spanning all nodes but also includes some additional links (i.e. links not necessary to form the cycle) will not be pairwise stable, as the players will have an incentive to delete this superfluous links because they do not add any value, only costs. However, we need to look at not only the simple cycle network (the 2-regular network discussed above) but also at networks include one cycle spanning all nodes but also have one or more central nodes. For examples of such a network see Figure 2. These two networks only differ in the number of nodes that are in each petal.

Such networks are also pairwise stable, for any cost range when looking at low linking costs. Thus we can conclude that the only networks that consist of one single connected component that are pairwise stable for low linking costs when facing a network disruptor with a disruption budget of $x = 1$ are the star network and networks including a cycle spanning all nodes and no additional links.

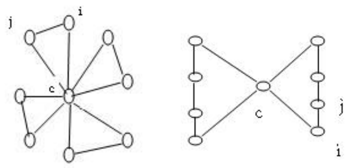
Theorem 2. *For the case of low linking costs when facing a network disruptor with a disruption budget of $x = 1$ the only pairwise stable networks consisting of only one connected component are the star network and the cycle spanning all nodes and no additional links.*

Proof For the proof on the star network see Theorem 1 and Section 4.1.3.
Any network that includes a cycle spanning all nodes, consists either of a simple cycle or a cycle going through one central node. Any additional links in such a network only increase costs but not the value of any one player, therefore they cannot be pairwise stable. For a proof that the simple cycle is pairwise stable, see Lemma 3. For the cycle going through a central node, we can define the payoff for any one of the players in the petals as $u_i(g) = n - 2c$. There is no incentive to add a link, as they are all already safe within the network. Should the player delete a link, his payoff will depend on the size of the petal and on whether the deleted link is to the central node or another node in the petal. We can show that in either case the network is pairwise stable for any cost range for $c < 1$. To do so we have to look at 3 different cases.

1. A link between the petal and the central node is deleted
   - There is never an incentive to delete a link for the central node (see analysis of the star network)
   - If player i within the petal deletes his link to the central node, he will become part of a disconnected petal. In the best case scenario this would lead to a payoff of $u_i(g - g_{ic}) = \frac{n}{2} - c$. Comparing this payoff to $u_i(g)$, we see that $u_i(g) > u_i(g - g_{ic})$ if $\frac{n}{2} > c$. Since by assumption $c < 1$ this holds for any c if $n > 2$.
2. A link within the petal is deleted such that the network disruptor is indifferent between which parts of the petal to disconnect.
   - This can only be the case if node i deletes a link to node j within a petal with an even number of nodes where i and j are exactly in the middle.
   - Assume that the size of the petal is y. The payoff of node i is then $u_i(g - g_{ij}) = \frac{1}{2}(n - \frac{1}{2}y) + \frac{1}{2}(\frac{1}{2}y) - c$. Where the left part denotes the payoff if node i is not disconnected and the right part denotes the payoff if it is disconnected.
   - Comparing this payoff with $u_i(g)$, we can see that $u_i(g) > u_i(g - g_{ij})$ if $\frac{n}{2} > c$. Since by assumption $c < 1$ this holds for any c if $n > 2$.
3. A link within the petal is deleted by node i such that node i will not be disconnected.
   - This can only be the case if node i deletes a link to node j within a petal of size y in such a way that the network disruptor can disconnect a larger part of the petal by disconnecting the part of the petal including node j.
   - The payoff of node i in this case depends on the size of the petal. However, it can maximally be $u_i(g) - u_i(g - g_{ij}) = n - 2 - c$, if the network disruptor can only disconnect 2 nodes from the network.[8]
   - Comparing this payoff with $u_i(g)$, we can see that $u_i(g) > u_i(g - g_{ij})$ if $2 > c$. Since by assumption $c < 1$, this holds in any case.

Thus the cycle network going through a central node is pairwise stable for any $c < 1$ if $n > 2$. □

[8] Only 1 nodes does not work because then he would be indifferent between disconnecting j and i.



(a) Flower (b) Bow-Tie

Figure 2: Cycles with a central node

Finally we still need to look at cycle networks that include more than one component, as we did for the minimally connected networks. Since we have described the components needed for this already in the previous sections, we can summarize the results here as follows.

Lemma 5. *For the case of low linking costs and a disruption budget of $x = 1$, networks consisting of more than one not minimally connected component are pairwise stable if:*

- *each component is pairwise stable in itself and*
- *the cost range for pairwise stability of these components also holds for lower n*

Proof In any network that consists of multiple in itself pairwise stable components, where none of the components is minimally connected, and thus none of the components includes an end-players (we have shown above that no pairwise stable network that is not minimally connected can include an end-player), no player has an incentive to delete a link by definition. There is also no incentive to add a link to a node in the other pairwise stable component, since this link will be an automatic target for the network disruptor and therefore will be taken out, meaning that only costs will be increased for the players but not the value. Thus pairwise stable networks can consist of multiple cycles or cycles and single nodes. □

We can thus conclude the analysis of the case of low linking costs and a disruption budget of $x = 1$, where we have found that minimally connected networks are in general not pairwise stable, with the exception of the star for a very small range of linking costs. Instead, networks that are regular and symmetric and therefore distribute benefits and costs of the network very equally are pairwise stable for all cost ranges. At the same time, networks that are separated into multiple components may also be pairwise stable, which is a marked difference from the benchmark case. Thus for the case of low linking costs, the presence of a network disruptor has on the one hand a cohesiveness inducing effect, by making players link more than without a disruptor, on the other hand, however, it can also lead to the separation of different connected components up to the occurence of the empty network as an equilibrium.

4.4. *High Linking Costs and $x = 1$*

Next we analyze the case of high linking costs, thus $c > 1$, when facing a network disruptor with a disruption budget of $x = 1$. We have seen in the benchmark case that for the case of high linking costs the only network that is pairwise stable is the empty network. This is due to fact that it is not worth to be linked to one node for the value of this one node alone. We will see in the analysis below that when facing a network disruptor this changes. We find that connected networks can be pairwise stable even for high linking costs. While for low linking costs, the fact that pairwise stable networks are more dense - thus include more links - when facing a network disruptor than in a game without a disruptor can be easily accepted since it makes the network structure more robust, it seems highly counterintuitive at first glance that the introduction of a network disruptor will lead to connected networks being pairwise stable for high linking costs, when without a disruptor no connected network is pairwise stable. Here comes the common enemy effect that we have referred to in the introduction in to play. While without the threat of disruption cooperation among the players is impossible if linking costs are high, the threat of disruption enables the purely selfish players to cooperate, while their behavior remains selfish. Thus the incentives change in such a way that what is best for each single player is also best for the group. While this phenomenon has often been explained with psychological characteristics of persons, what we show here is that there are also purely economic reasons that can explain this effect. As the analysis below (and in Section 5.2 for the case of a general disruption budget) will show, the introduction of a network disruptor changes the incentives of the players in such a way that while the empty network remains pairwise stable, also connected networks as well as networks consisting of multiple separate components may be pairwise stable.

For the simple case of $x = 1$ the discussion of the case of high linking costs is a straightforward extension of the analysis of low linking costs. Therefore, before starting the main part of the analysis for this setting, we can already dismiss a number of networks. We can dismiss all minimally connected networks as well as any network including end-players, since the value of each node is 1 and costs are above 1, thus you will no longer link to any one player just for the value of this player alone.The possible candidates for pairwise stable networks are thus only networks that do not include end-players. Remember that in the benchmark case the only network that is stable for high linking costs is the empty network. Therefore we will start our analysis with this network as well. The reasoning is the same as in Lemma 4. Thus we can directly deduce that the empty network is also pairwise stable for the case of high linking costs.

In the discussion on low linking costs, we have seen that the simple cycle network is pairwise stable for $\frac{n}{2} > c$. Thus the cycle network is also pairwise stable for high linking costs as long as this condition is fulfilled. The difference to the case of low linking costs is then that here the pairwise stability is not automatically fulfilled but depends on the number of players within the network. Equally affected by the change to high linking costs are networks consisting of more than one component. The same holds for the cycle networks with a central node. Here the cost range for pairwise stability now depends on the number of players within the petals. Again take into account Figure 2. Both networks are pairwise stable for certain cost ranges, as we will show below. Take the flower network for example. The payoff for node i will be $u_i(g) = n - 2c$. If node i deletes the link to node j the payoff will be $u_i(g - g_{ij}) = \frac{1}{2}(n-1) + \frac{1}{2} * 1 - c$. The payoff of the original network is larger for $\frac{1}{2}n > c$. If node i deletes the link to player c, the payoff will be $u_i(g - g_{ic}) = 2 - c$, which is less attractive than deleting the link to node j. Node c's incentive to delete a link, however, is somewhat higher, as node c's payoff for the original network is only $u_c(g) = n - (n-1)c$, whereas deleting the link to node i, the payoff will be $u_c(g - g_{ci}) = n - 2 - (n-2)c$. The only cost range for which this network remains pairwise stable is then for $2 > c > 1$, since otherwise the central node has an incentive to cut a link.

Now the network that looks like a bow-tie, is like a flower network with only two petals. In the network the payoff to node i will be $u_i(g) = n - 2c$. If node i deletes the link to node j, it's payoff will be $u_i(g - g_{ij}) = n - 3 - c$. Thus the network is pairwise stable only for $3 > c > 1$. The central node only has an incentive to cut a link if $c < 4$. Thus the condition on node i is binding.

Both networks are thus pairwise stable for limited cost ranges. The size of these cost ranges does not depend so much on the number of nodes in the network as it depends on the size of the 'petals'. From the point of view of the central player, it is easy to see that the larger the petal, the lower his incentive to delete a link. To see this, compare the flower network in Figure 2(a) with the bow-tie network in Figure 2(b). Both networks use $n = 9$ players, but whereas in the flower network, the petals are only of order 2, in the bow-tie network they are of order 4. We have shown above that from the point of view of the central player, the flower network is pairwise stable, if $2 > c$, where the 2 is only due to the fact that there are only 2 players in the petal. In the bow-tie network, there are 4 players in the petal and it is pairwise stable from the point of view of the central player if $4 > c$, where again the 4 is only due to the fact that there are 4 players in the petal. Thus, from the point of view of the central player, the incentive to delete a link decreases in the size of the petals. Accordingly, the network that is pairwise stable for the largest cost range from the point of view of the central player has the largest possible petal, which is of size $n - 3$.[9] However, contrary to the

[9]Otherwise there cannot at least be 2 petals.

central player's incentives, are the incentives of player i, who is part of the petal but directly linked to the central player. Player i's incentives to delete the link to the central player increase in the size of the petal. To see this, again look at the flower network introduced in Figure 2(a) versus the bow-tie network in Figure 2(b). Player i only has an incentive to delete the link to c if $c > n - 2$ in the flower network. However, in the bow-tie network his incentives to delete the link increase such that he will delete the link if $c > n - 4$. Again this is solely due to the size of the petals. Thus taking these two contrary incentives into account, it is easy to see that the largest possible cost range for such a network to be pairwise stable is only achieved, if one petal is of size $\frac{n}{2}$. In that case the constraint on pairwise stability is the same for the central player and player i. The network is then pairwise stable if $\frac{n}{2} > c > 1$, which is the same constraint on pairwise stability as for the cycle network.

Lemma 6. *For $c > 1$ and facing a network disruptor with a disruption budget of $x = 1$, the only pairwise stable network for any cost range is the empty network, g^e. The (simple) cycle networks and networks consisting of more than one in itself pairwise stable component are also pairwise stable, however, only for a maximal cost range of $\frac{n}{2} > c$.*

Proof The payoff of node i in the empty network is $u_i(g^e) = 1$, by adding a link to node j this payoff will decrease to $u_i(g^e + g_{ij}) = 1 - c$. For $c > 0$ it holds that $u_i(g^e) < u_i(g^e + g_{ij})$, so the empty network is pairwise stable for any cost range. We have shown in the section above that the simple cycle network is pairwise stable for $\frac{n}{2} > c$ and that cycle networks with a central node can be stable for maximally the same cost range. The same holds for networks consisting of more than one in itself pairwise stable component, as here as well the pairwise stability depends on the number of nodes within each component. □

This concludes the complete characterization of the case of a network of self interested players facing a network disruptor with a disruption budget of $x = 1$. We can summarize the results in the following theorem.

Theorem 3. *For the case of a network of self-interested players facing a network disruptor with a disruption budget of $x = 1$, we found that*

- *the empty network is pairwise stable irrespective of the level linking costs c*
- *the star network is pairwise stable for $c > \frac{n-2}{n-1}$*
- *the simple cycle network is pairwise stable for $\frac{n}{2} > c$ and cycle networks with a central node are pairwise stable maximally for the same cost range*
- *networks consisting of more than one component are pairwise stable if each of the components is in itself pairwise stable. The cost range for pairwise stability, however, is decreasing in the size of the components as the pairwise stability then depends on the size of the component*

Proof This follows from Theorem 1 and Theorem 2 and from Lemmata 3, 4, 5 and 6. □

5. General Disruption Budget

After having given a complete characterization for the case of $x = 1$, we will now move to give more extensive results for a general disruption budget. Here we cannot provide a full characterization of all networks that are pairwise stable for all cost ranges and number of nodes, due to the multitude of possible networks. Nevertheless, we can provide results showing that what we have shown in the section on a disruption budget of $x = 1$ in some cases holds also for a general disruption budget (such as the pairwise stability of regular networks and the empty network), whereas in other cases such as the pairwise stability of the star network, the results are clearly special to the case of $x = 1$. As before, we will start with an analysis of the case of low linking costs and then move to high linking costs.

5.1. Low Linking Costs

Just as for the case of $x = 1$ we will first look at two well studied classes of networks, namely minimally connected networks and regular networks before making more general statements about low linking costs.

5.1.1. Minimally Connected Networks

We have seen that even for a disruption budget of $x = 1$ the only minimally connected network that maybe stable is the star network. Consequently this is the only network we need to look at for higher disruption budgets. We know from the discussion of $x = 1$ that the central player does not have any incentive to delete a link in the case of low linking costs. Therefore from the point of view of the central player the star network is pairwise stable. Looking at the end-players, we know that the chance for any end-player to be disconnected in a star network for a disruption budget of x is $\frac{x}{n-1}$. Accordingly the payoff to any end-player i will be $u_i(g^*) = \frac{x}{n-1} * 1 + (1 - \frac{x}{n-1})(n - x) - c$. To check for pairwise stability, we need to verify that no player would like to delete a link and no pair of players would benefit from adding a link. By deleting its link, the payoff for player i will be $u_i(g^* - g_{ic}) = 1$. The condition on pairwise stability is then $\frac{x}{n-1} * 1 + (1 - \frac{x}{n-1})(n - x) - c > 1$. This can be rearranged to: $\frac{n^2+x^2+2x-2n-2xn+1}{n-1} > c$. Simplifying this we get $n - 1 - 2x + \frac{x^2}{n-1} > c$, which can indeed be satisfied. We thus have to check the incentives an end-player may have to add a link. We have seen that for the case of $x = 1$, adding one link meant that player i was ensured to remain within the network. For $x > 1$ this is no longer the case, but still, due to the lexicographic preferences of the network disruptor, adding a link will ensure the player to remain within the network since the network disruptor will rather disconnect two single nodes than one pair of nodes. Consequently the payoff for player i when adding a link will be $u_i(g^* + g_{ij}) = n - x - 2c$. Comparing this payoff with $u_i(g)$, we can see that the network is only pairwise stable if $c > x - \frac{x^2}{n-1}$ holds. This is the case if $\frac{x^2}{n-1} > n - 1$ holds. For $x \to \infty$ the left hand side of this equation goes towards 1. Thus it reaches its largest value for $x = 2$, namely 4. Therefore, we can conclude that the star network is never pairwise stable, as we only look at cases for $n > 4$. From this we can conclude the following lemma:

Lemma 7. *For $c < 1$ and $x > 1$, no minimally connected network is pairwise stable.*

Proof We have already seen for the case of $x = 1$ that the only network that may be pairwise stable is the star network. Since by assumption $n > 4$, we have shown above that the star network is not pairwise stable for $x > 1$ and thus no minimally connected network will be pairwise stable for $x > 1$. □

5.1.2. Regular Networks

Next to minimally connected networks we have seen for the case of $x = 1$ that $x + 1$-regular networks (namely the cycle) are pairwise stable. This indeed also holds true for $x > 1$. However, unlike for the case of $x = 1$, where the only $x + 1$-regular network was the cycle for all other numbers of x there are more than one way to build an $x + 1$-regular network. In Figure 3, you can find 3 examples of a 3-regular network with 16 nodes.

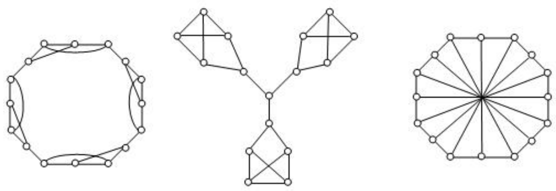


Figure 3: 3 regular networks

The type of network we will concern ourselves with here, is the one to the right. This network is not only 3-regular, it is also symmetric (in the graph theoretic sense of the word).[10] As we've explained in

[10]For a formal definition see e.g. Chartrand (1977) and our previous paper (Hoyer and De Jaegher, 2010).

section 4.2 this means that the network is build in such a way that not only do all nodes receive the same amount of links but in a symmetric network also every node has exactly N_x nodes at distance x, with $x = 1, 2, ..., d_{max}$, where d_{max} denotes the maximal distance between any two nodes in a network. This construction implies that a network disruptor with a disruption budget of $Dis = x$, where $x < r$, can not disconnect a single node from the network irrespective of which links he chooses to delete. In both of the other networks, the network disruptor can disconnect a number of nodes by taking out 2 links or even only 1 link (in the middle network). Neither the left nor the middle network is pairwise stable against a disruption budget of 2, as can easily be verified, whereas the right network is pairwise stable against a disruption budget of 2. We will therefore now treat the case of symmetric r-regular networks, now in more detail.

As has been shown in our previous paper, in a symmetric r-regular network, by construction, a network disruptor with a disruption budget of x, where $x = r - 1$ cannot disconnect a single node, nor a group of nodes, from the network. Thus any node i in such a network will have a payoff of $u_i(g) = n - r * c$. There is no incentive for any node to add a link, because they are already safe within the network and adding a link can only increase their costs. Deleting a link, will automatically make the node that deletes the link, as well as the one to whom the link is deleted a target of the network disruptor, as they then only have x links. By the way these symmetric networks are constructed, the network disruptor can then take out exactly node i who deleted the link or node j to whom the link used to be, but no other nodes. Therefore the expected payoff to node i and j is $u_i(g - g_{ij}) = \frac{1}{2} * 1 + \frac{1}{2}(n-1) - (r-1) * c$. Thus the network will be pairwise stable, if $n - r * c > \frac{1}{2} * 1 + \frac{1}{2}(n-1) - (r-1) * c$ holds. This is the case if $\frac{n}{2} > c$, which always holds, since we look only at networks of at least $n = 4$. Thus the symmetric r-regular network is pairwise stable against network disruption with a disruption budget of $x = r - 1$.

As we have discussed for the case of $x = 1$, r-regular networks need not necessarily be $x + 1$-regular. They can also be $x + y$-regular. We have shown in section 4.2 that if $y > 1$ the network will never be pairwise stable since each node has an incentive to delete a link. The same holds true here. For $y < 1$ we will first look at the case for $y = 0$ and then at $y < 0$. For a network with $y = 0$, we have to differentiate between the cases of $x = 2$ and $x > 2$. For $x = 2$, the only x-regular network is the cycle. The payoff for any node i in the cycle when faced with a network disruptor with a disruption budget of 2, is $u_i(g) = \frac{n}{2} - 2c$. If node i adds a link to a node j in such a way that this link spans exactly half of the network, the payoff for node i will be $u_i(g + g_{ij}) = \frac{n}{2} + 1 - 3c$. Thus the network will only be pairwise stable if $c > 1$. Since by assumption $c < 1$, the cycle network is never pairwise stable when facing a network disruptor with a disruption budget of $x = 2$. Now for the cases of $x > 2$, we know that the payoff to any node in a x-regular network will be $u_i(g) = \frac{1}{n} * 1 + (1 - \frac{1}{n})(n-1) - xc$. If player i adds a link to player k, his payoff will be $u_i(g + g_{ik}) = n - 1 - (x+1)c$. Thus for the x-regular network to be pairwise stable, the following condition needs to hold. $\frac{1}{n} * 1 + (1 - \frac{1}{n})(n-1) - xc > n - 1 - (x+1)c$. This only holds if $c > \frac{n-2}{n}$. However, since we assume that $1 > c$, this condition is only fulfilled for very high costs of linking. The higher the number of players, the smaller the cost range between 1 and $\frac{n-2}{n}$ for which these networks will actually be pairwise stable. Lastly we need to look at the case for y being negative. Again these networks will typically not be pairwise stable. However, general statements about this case cannot be made as it largely depends on issues of divisibility if the networks will remain pairwise stable or not.

Lemma 8. *For $c < 1$ and $x > 1$ the $x + 1$-regular network is pairwise stable. The $x + y$-regular network*

- *is never pairwise stable if $y > 1$*
- *is pairwise stable if $\frac{n-2}{n}$ for $y = 0$ and $x \neq 2$*
- *is never pairwise stable for $y = 0$ if $x = 2$*

Proof See discussion in the section above. □

5.1.3. General Results

Although we cannot define all networks that are pairwise stable for $c < 1$ and $x > 1$, we can state some general conditions that need to be fulfilled to achieve pairwise stability. We have already seen in the case for $x = 1$ that the empty network is pairwise stable for any cost range and any disruption budget, so this holds true here as well. The same applies to the reasoning behind networks consisting of more than one component. The condition that each component needs to be pairwise stable in itself also needs to hold here.

Unlike for the case of $x = 1$ we cannot make a general statement about the existence of end-players within pairwise stable networks. For $x = 1$, end-players were always automatic targets. Consequently, adding one link would ensure the position of each player within the network. In cases where $x > 1$, because of the lexicographic preferences of the network disruptor, links to end-players are not necessarily automatic targets. For an example see Figure 4.

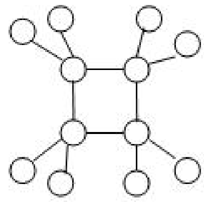


Figure 4: Pairwise Stable Networks with End-Players

In Figure 4 the network includes end-players. If you assume a disruption budget of $x = 2$, the target links are going to be within the closed cycle because then the network can be cut in half, instead of disconnecting only 2 end-players. Therefore, the network may be stable even when including end-players.

We can summarize the results in the following theorem:

Theorem 4. *For $c < 1$ and $x > 1$ conditions for pairwise stability are fulfilled if:*

- *the network is the empty network*
- *the network is $x + 1$-regular*
- *the network is $x + y$-regular and $y = 0$ and $x \neq 2$ if $\frac{n-2}{n}$*
- *the network consists of multiple components that are in itself pairwise stable*

For $c < 1$ and $x > 1$ conditions for pairwise stability are not *fulfilled if:*

- *the network is minimally connected*
- *the network is $x + y$-regular and $y > 1$ or $y = 0$ and $x = 2$*
- *the network consists of more than one component and at least one is not pairwise stable in itself*

Proof For the proof on when the conditions for pairwise stability are fulfilled, see Theorem 2 and Lemmata 4, 5 and 8.

For the proof on when the conditions for pairwise stability are not fulfilled, see Lemmata 7 and 8 Theorem 4. □

5.2. High Linking Costs

As for the case of $x = 1$, here we will also continue our analysis by looking at the case for $c > 1$. In this case it is not worth to be linked to one node just for the value of this one node alone. Consequently, in the benchmark case, the only pairwise stable network is the empty network. As in the previous cases, of course the empty network is also stable in this case. However, as for the case of $x = 1$, we find that when a network disruptor is introduced into this setting, we can find networks that are connected and are still pairwise stable. We have also already discussed that $x+1$-regular networks remain pairwise stable, also for high linking costs. However, they are only pairwise stable up to a cost level of $\frac{n}{2} > c$. The same holds for networks consisting of more than one component. They can remain pairwise stable, however, the cost range for pairwise stability will be lower, since it depends on the size of the components now.

While defining all networks that will be pairwise stable for any given disruption budget for high linking costs is analytically untractable, we can define certain criteria that all pairwise stable networks need to fulfill.

Theorem 5. *For $c > 1$ and $x > 1$ conditions for pairwise stability are fulfilled if:*

- *the network is the empty network*
- *the network is $x+1$-regular and $\frac{n}{2} > c$*
- *the network consists of multiple components that are in itself pairwise stable for a cost range depending on the size of the components*

Additionally, for $c > 1$ and $x > 1$, any network that is pairwise stable needs to fulfill the following criteria:

- *there are no end-players in the network*
- *the network is not $x+y$-regular if $y > 1$*

Proof For the proof on when the conditions for pairwise stability are fulfilled, see Theorem 2 and Lemmata 4 and 5.

No end-players can be in the network, since it is not worth to be linked to any player for the value of that one player alone. Thus any player that is linked to an end-player has an incentive to delete that link, irrespective of the threat of disruption. That the network can not be $x+y$-regular and $y > 1$ has been shown in Lemma 8. □

6. Value and Efficiency

In the model described in Jackson and Wolinsky (1996), which we have used as a benchmark case in our analysis, there is a tradeoff between stability and efficiency in the case of high linking costs. For low linking costs in the model without decay, all minimally connected networks are pairwise stable and equally efficient, since they connect all nodes in one component using the minimal number of links to do so. The value of the network can thus easily be calculated as $v(g) = n * n - 2(n-1)c$. That this is larger than the value of the same network with one additional link is obvious, since this link will only add costs and no benefits. The payoff if a link is removed, also has to be lower, since the cost of one link is by definition lower than the value of one node. Thus for low linking costs are minimally connected networks not only stable but also efficient in the benchmark case. For high linking costs in the benchmark case the only pairwise stable network is the empty network, which has a value of $v(g^e) = n$. It is easy to see that other networks might have a higher value. Take for example the minimally connected value. Comparing the values, we find that $v(g) > v(g^e)$ holds if $\frac{n^2-n}{2n-1} > c$. The numerator as well as the denominator move towards infinity for increasing values of n. The numerator does so more quickly though because of the squared value of n. Therefore this holds for a large cost rance of c, especially for high values of n.[11] Thus for the case of high linking costs the only pairwise stable network is not efficient.

We have shown, however, that when introducing a network disruptor, minimally connected networks are generally not pairwise stable (with the exception of the star network for $x = 1$ and $c > \frac{n-2}{n-1}$). Instead we have shown that $(x+1)$-regular networks, as well as networks consisting of separated components,

[11] Even for $n = 4$, it already holds for any value of $c < 4$.

where each component is pairwise stable in itself, and the empty network are pairwise stable, for certain cost ranges. The question is now, are these networks also more efficient than the empty network. The value of the $(x+1)$-regular network can be calculated as $v(g^{x+1}) = n^2 - (x+1)c$. Comparing this to the value of the empty network, we find that the $(x+1)$-regular network is more efficient than the empty network if $\frac{n-1}{x+1} > c$. This means that for low linking costs, the $(x+1)$-regular network is always more efficient than the empty network. For high linking costs, it depends on the value of x and the level of linking costs. For the case of $x = 1$, we can see that the cycle network is pairwise stable if $\frac{n}{2} > c$ and it is more efficient than the empty network in almost all these cases (except for the extreme case where $c = \frac{n-1}{2}$). For an increasing x it is no longer the case though that whenever the $(x+1)$-regular network is stable it is also more efficient than the empty network. However, since by assumption $x < n-2$, there are always cases also for high linking cost where the $(x+1)$-regular network is not only stable but also more efficient than the empty network.

The value of pairwise stable networks that consist of multiple in itself pairwise stable components will always be lower than that of any pairwise stable network consisting of only one component. This is due to the fact that this way the nodes only have access to the information of the nodes in their own component. Therefore, while using the same amount of links as if it was one pairwise stable component encompassing all n nodes, the value generated in each component is less than that generated in the network as a whole.

Theorem 6. *For low linking costs, connected pairwise stable networks are always more efficient than the empty network, where networks connecting all nodes in one component are the most efficient. For high linking costs, there are for every combination of n and x cost ranges for which pairwise stable networks which connect all nodes in one connected component are more efficient than the empty network.*

Proof Comparing $v(g^e)$ with $v(g^{x+1})$ we find that $v(g^{x+1}) > v(g^e)$ holds if $\frac{n-1}{x+1} > c$. For $c < 1$ this condition is always satisfied. For $c > 1$, there are always cases for which this condition is satisfied, since by assumption $x < n-2$, thus $\frac{n-1}{x+1} > 1$. For networks consisting of multiple components, the same amount of links is used as in a pairwise stable network of the same form consisting of only one component. However, the value generated within one component consisting of less than n nodes is always lower than that generated in one component consisting of n nodes. Thus the value of the network as a whole is lower than that of a pairwise stable network encompassing all nodes in one connected component. □

7. Conclusion

In this paper we looked at the implications for stability and efficiency of networks made up of self-interested players when they are facing a network disruptor in the setting of a pairwise stability model. We have found that for low as well as high costs of linking there are connected networks that are pairwise stable as well as more efficient than the unconnected empty network. This is an interesting finding since in the model without a network disruptor, the only network that is pairwise stable for high linking costs is the highly inefficient empty network. What we have shown here is that when facing an outside force that aims to destroy the value of the network as a whole, it is possible for a group of self-interested, myopic players to build a network that is stable and actually more efficient than they would be able to do without this outside threat. We have thus given a purely economic explanation for the psychological concept of a common enemy effect. We have also shown that once players are caught in the empty network or in a network consisting of multiple components, they are caught in these inefficient equilibria, because they are also stable.

This paper suggests that an effect rooted in psychology can also be explained by purely economic means. Future research should consequently aim at exploring the economic incentives for the common enemy effect more thoroughly in this setting by adding heterogeneous players, information asymmetry or looking at 'coalition'-stability, where groups of more than 2 players can deviate at the same time. Additionally, in the pairwise stability model, the focus lies purely on which networks are stable but not on how players can actually reach such networks. Future research should therefore focus on implementing actual mechanisms on how such a network can be formed (e.g. in an economic laboratory experiment) and if myopic players can reach all equilibrium networks. Another avenue of research focusses on the farsighted behavior of players. Theoretical work on networks assuming farsightedness of players (see for example the work by Morbitzer and Buskens (2009) or Herings et al. (2009)) shows that a concept of (perfect) farsighted stability can be established. Network experiments (see for example the work by Mantovani et al. (2011)) on the same topic show that players do tend to behave more farsightedly than myopic in such experiments. Future research may analyze how the assumption of farsightedness would change the results we have found here.

Bibliography

Albert, R., Jeong, H., and Barabasi, A.-L. (2000). Error and attack tolerance of complex networks. *Nature*, 406(6794):378–382. M3: 10.1038/35019019; 10.1038/35019019.

Antal, T., Krapivsky, P. L., and Redner, S. (2006). Social balance on networks: The dynamics of friendship and enmity. *Physica D Nonlinear Phenomena*, 224:130–136.

Baccara, M. and Bar-Isaac, H. (2008). How to organize crime. *The Review of Economic Studies*, 75(4):1039–1067.

Bala, V. and Goyal, S. (2000). A noncooperative model of network formation. *Econometrica*, 68(5):1181–1229.

Ballester, C., Calvo-Armengol, A., and Zenou, Y. (2006). Who's who in networks. wanted: The key player. *Econometrica*, 74(5):1403–1417.

Bollobás, B. and Riordan, O. (2003). Robustness and vulnerability of scale-free random graphs. *Internet Mathematics*, 1(1):1–35.

Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*. Springer, London, 2 edition.

Chartrand, G. (1977). *Introductory Graph Theory*. Dover Publication, New York.

Das, S. P. and Roy Chowdhury, P. (2008). Deterrence, preemption and panic: A common-enemy problem of terrorism. *MPRA Paper*.

Enders, W. and Jindapon, P. (2010). Network externalities and the structure of terror networks. *Journal of Conflict Resolution*, 54(2):262–280.

Enders, W. and Su, X. (2007). Rational terrorists and optimal network structure. *Journal of Conflict Resolution*, 51(1):33–57.

Goyal, S. (2009). *Connections: An Introduction to the Economics of Networks*. Princeton University Press, Princeton.

Goyal, S. and Vigier, A. (2009). Robust networks. *Working Paper, Cambridge University*.

Hamel, G. and Prahalad, C. (1993). Strategy as stretch and leverage. *Harvard Business Review*, 71(2):75–84.

Heider, F. (1982). *The psychology of interpersonal relations*. Lawrence Erlbaum.

Herings, P., Mauleon, A., and Vannetelbosch, V. (2009). Farsightedly stable networks. *Games and Economic Behavior*, 67(2):526–541.

Hong, S. (2008). Hacking-proofness and stability in a model of information security networks. *Working Paper, Vanderbilt University*.

Hong, S. (2009). Enhancing transportation security against terrorist attacks. *Working Paper, Vanderbilt University*.

Hoyer, B. and De Jaegher, K. (2010). Strategic network disruption and defense. *TKI Discussion Paper Series*, (10-13).

Jackson, M. O. (2008). *Social and Economic Networks*. Princeton University Press, Princeton.

Jackson, M. O. and Wolinsky, A. (1996). A strategic model of social and economic networks. *Journal of Economic Theory*, 71(1):44–74.

Kovenock, D. and Roberson, B. (2010). The optimal defense of networks of targets. *Purdue University, Working Paper*, page 14. Oct. 2010.

Lipsey, R. A. (2006). Network warfare operations: Unleashing the potential. *Mimeo Center for Strategy and Technology, Air War College, Air University*.

Mantovani, M., Kirchsteiger, G., Mauleon, A., and Vannetelbosch, V. (2011). Myopic or farsighted? an experiment on network formation. *FEEM Working Papers 45*, (45).

Maoz, Z., Terris, L. G., Kuperman, R. D., and Talmud, I. (2007). What is the enemy of my enemy? causes and consequences of imbalanced international relations, 1816?2001. *Journal of Politics*, 69(1):100–115.

Mesterton-Gibbons, M. and Dugatkin, L. A. (1992). Cooperation among unrelated individuals: Evolutionary factors. *The Quarterly review of biology*, 67(3):pp. 267–281.

Metz, S. (2003). Insurgency and counterinsurgency in iraq. *The Washington Quarterly*, 27(1):25–36.

Morbitzer, D. and Buskens, V. (2009). Network formation with limited forward-looking actors.

Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research*, 2(3):pp. 225–229.

Newcomb, T. (1961). *The acquaintance process*. Holt, Rinehart & Winston.

Olson, M. and Zeckhauser, R. (1966). An economic theory of alliances. *The review of economics and statistics*, 48(3):pp. 266–279.

Taylor, M., Sekhar, S., and D'Este, G. (2006). Application of accessibility based methods for vulnerability analysis of strategic road networks. *Networks and Spatial Economics*, 6(3):267–291. M3: 10.1007/s11067-006-9284-9.

Wright, W. K. (1911). The psychology of punitive justice. *The Philosophical Review*, 20(6):pp. 622–635.