INTERPOLATION IN A FRAGMENT OF INTUITIONISTIC PROPOSITIONAL LOGIC

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ABSTRACT

Let NNIL (No Nestings of Implication to the Left) be the fragment of IpL (intuitionistic propositional logic) in which the antecedent of any implication is always prime. The following strong interpolation theorem is proved: if IpL $\vdash A\rightarrow B$ and A <u>or</u> B is in NNIL, then there is an interpolant I in NNIL. The proof consists in constructing I from a proof of $A\rightarrow B$ in a sequent calculus system by means of a variant of a method devised by K. Schütte. This settles a question posed by A. Visser.

1. INTRODUCTION

1.1. Let NNIL (No Nestings of Implication to the Left) be the following fragment of IpL (intuitionistic propositional logic):

i) $\top,\bot$ and all propositional variables are in NNIL;

ii) NNIL is closed under $\wedge$ and $\vee$;

iii) if P is a propositional variable and $A\in$NNIL, then $P\to A\in$NNIL.

In this note, we prove the following:

1.2. Theorem (strong interpolation for NNIL).

Let A,B be formulae of IpL such that $\vdash A\to B$ (i.e. IpL proves $A\to B$). If $A\in$NNIL or $B\in$NNIL, then there is an interpolant $I\in$NNIL for $A\to B$, i.e.

i) $\vdash A\to I$, $\vdash I\to B$;

ii) all propositional variables of I occur both in A and in B.

The proof consists in constructing the interpolant I from a (cut-free) proof of $A\to B$ in a sequent calculus system.

This settles a question posed by A. Visser in [V85]. In that paper, he obtains (among other things) an indirect proof for the more difficult part of 1.2. (the case $A\in$NNIL; see 1.1.4. in [V85]), and asks whether a direct proof can be given.

2. PRELIMINARIES

2.1. Notation.

All formulae are in intuitionistic propositional logic, with $\wedge$, $\vee$, $\to$ as connectives and the constants $\top$ and $\bot$. P,Q,... are propositional variables; A,B,C,... are formulae; Γ, Δ , Γ',... are finite (possibly empty) sets of formulae. $\mathcal{P}$ is the set of propositional variables. We write Γ,Δ for the conjunction of Γ and Δ; Γ,A stands for $\Gamma,\{A\}$.

$[A]^+$ ($[A]^-$) is the set of all positively (negatively) occurring propositional variables in A, defined by

$[\top]^+=[\top]^-=[\bot]^+=[\bot]^-=\phi$

$[P]^+=\{P\}$

$[P]^-=\phi$

$[A\wedge B]^+=[A\vee B]^+=[A]^+\cup[B]^+$

$[A\wedge B]^- = [A\vee B]^- = [A]^- \cup [B]^-$

$[A\to B]^+ = [B\to A]^- = [A]^- \cup [B]^+$

$[A] := [A]^+ \cup [A]^-$ is the set of <u>all</u> propositional variables in A.

2.2. The derivation system.

We use the following sequent calculus:

axioms:

(P) $\Gamma, P \vdash P$

($\perp$) $\Gamma, \perp \vdash P$

(T) $\Gamma \vdash T$

rules:

($\wedge$R) $\dfrac{\Gamma\vdash A \quad \Gamma\vdash B}{\Gamma\vdash A\wedge B}$ ($\wedge$L) $\dfrac{\Gamma, A, B\vdash C}{\Gamma, A\wedge B\vdash C}$

($\vee$R) $\dfrac{\Gamma\vdash A_i}{\Gamma\vdash A_1\wedge A_2}$ (i=1,2) ($\vee$L) $\dfrac{\Gamma, A\vdash C \quad \Gamma, B\vdash C}{\Gamma, A\vee B\vdash C}$

($\to$R) $\dfrac{\Gamma, A\vdash B}{\Gamma\vdash A\to B}$ ($\to$L) $\dfrac{\Gamma\vdash A \quad \Gamma, B\vdash C}{\Gamma, A\to B\vdash C}$

Related systems can be found in [S62], [T75]. All these systems satisfy the Cut Elimination property:

(*) if $\Gamma\vdash A$ and $\Gamma, A\vdash B$, then $\Gamma\vdash B$.

2.3. Schütte's interpolation method.

Schütte gives in [S62] a method to build an interpolant from a derivation of $\vdash A\to B$. This method can be rendered as follows:

(iP1) $\Gamma(T)\,\Delta, P\vdash P$ (iP2) $\Gamma, P(P)\,\Delta\vdash P$

(i$\perp$1) $\Gamma(T)\,\Delta, \perp\vdash P$ (i$\perp$2) $\Gamma, \perp(\perp)\,\Delta\vdash P$

(iT) $\Gamma(T)\,\Delta\vdash T$

(i$\wedge$R) $\dfrac{\Gamma(I_1)\,\Delta\vdash A \qquad \Gamma(I_2)\,\Delta\vdash B}{\Gamma\,(I_1\wedge I_2)\,\Delta\vdash A\wedge B}$

(i$\vee$L1) $\dfrac{\Gamma(I_1)\,A, \Delta\vdash C \qquad \Gamma(I_2)\,B, \Delta\vdash C}{\Gamma(I_1\wedge I_2)\;A\vee B, \Delta\vdash C}$

$$(i\vee L2)\quad \frac{\Gamma, A\ \ (I_1)\ \Delta\vdash C \qquad \Gamma, B\ \ (I_2)\ \Delta\vdash C}{\Gamma, A\vee B\ \ (I_1 \vee I_2)\ \Delta\vdash C}$$

$$(i\rightarrow L1)\quad \frac{\Gamma(I_1)\ \Delta\vdash A \qquad \Gamma(I_2)\ B, \Delta\vdash C}{\Gamma(I_1 \wedge I_2)\ A\rightarrow B, \Delta\vdash C}$$

$$(i\rightarrow L2)\quad \frac{\Delta\ (I_1)\ \Gamma\vdash A \qquad \Gamma, B\ (I_2)\ \Delta\vdash C}{\Gamma, A\rightarrow B\ (I_1 \rightarrow I_2)\ \Delta\vdash C}$$

We explain this notation with an example.

$(i\wedge R)$ means:

$$\begin{array}{lll} \text{if } \Gamma\vdash I_1 & , & I_1, \Delta\vdash A \\ \text{and } \Gamma\vdash I_2 & , & I_2, \Delta\vdash B, \\ \text{then } \Gamma\vdash I_1\wedge I_2, & & I_1\wedge I_2, \Delta\vdash A\wedge B; \end{array}$$

so $(i\wedge R)$ indicates how an interpolant for $\Gamma \rightarrow (\Delta \rightarrow A\wedge B)$ can be obtained from interpolants for $\Gamma\rightarrow(\Delta \rightarrow A)$ and $\Gamma \rightarrow(\Delta \rightarrow B)$.
(Here $\Gamma\rightarrow A$ stands for $(\bigwedge\Gamma) \rightarrow A$.)

For rules not mentioned here ($(\wedge L)$, $(\vee R)$, $(\rightarrow R)$), the interpolant for the conclusion is the same as for the premiss.

Now the Interpolation Theorem is proved as follows. Assume $\vdash A\rightarrow B$; then there is a derivation of $A\vdash B$ in the derivation system defined in 2.2. With induction over the length of the derivation it is shown (using Schütte's method) that any partition $\Gamma, \Delta\vdash A$ of a sequent in the derivation has an interpolant I, i.e. $\Gamma\vdash I$, $I, \Delta\vdash A$ and $[I]\subset[\Gamma]\cap[\Delta, A]$. Hence $A\rightarrow B$ has an interpolant.

Applying Schütte's method to derivations of $A\vdash B$ with e.g. $A\in$NNIL does not always yield an $I\in$NNIL:

$$\begin{array}{l}
(\rightarrow L)\ \underline{P\ (P)\ \vdash P \qquad Q\ (Q)\ P\vdash Q} \\
\qquad (\rightarrow L)\ \underline{P\rightarrow Q\ \ (P\rightarrow Q)\ \ P\vdash Q \qquad P, R(R)\ P\rightarrow Q\vdash R} \\
\qquad\qquad (\wedge L)\ \underline{P, Q\rightarrow R\ \ ((P\rightarrow Q)\rightarrow R)\ \ \ P\rightarrow Q\vdash R} \\
\qquad\qquad (\rightarrow R)\ \underline{P\wedge(Q\rightarrow R)\ \ ((P\rightarrow Q)\rightarrow R)\ \ \ P\rightarrow Q\vdash R} \\
\qquad\qquad\qquad P\wedge(Q\rightarrow R)\ \ ((P\rightarrow Q)\rightarrow R)\vdash (P\rightarrow Q)\rightarrow R
\end{array}$$

It turns out that $(i\rightarrow L_2)$, the only place where an $\rightarrow$ is added to I, has to be modified. This will be done in the next section. First we state some properties of the derivation system:

2.4. Lemma

i) $\Gamma \vdash A \Rightarrow \Gamma, \Delta \vdash A$;

ii) $\Gamma \vdash A \Rightarrow \Gamma[P:=B] \vdash A[P:=B]$;

iii) $P \notin [A]^{-} \Rightarrow A, P \to B \vdash A[P:=B]$.

Proof.

i) Induction over the length of a derivation of $\Gamma \vdash A$.

ii) Induction over the length of a derivation of $\Gamma \vdash A$. For the case $\Gamma \vdash A$ is an axiom (P) or ($\perp$), we need to show that $\Gamma, A \vdash A$ and $\Gamma, \perp \vdash A$ are derivable, which is easy.

iii) Simultaneous induction over the logical complexity of A, together with the statement

$$P \notin [A]^{+} \Rightarrow A[P:=B], P \to B \vdash A. \qquad \square$$

3. THE PROOF

3.1. We split the proof of 1.2. in two parts, according to whether $A \in$ NNIL or $B \in$ NNIL.

3.2. Lemma. Assume $\Gamma, \Delta \vdash C$ is derivable and $\Gamma \subset$ NNIL. Then there is an I with:

i) $\Gamma \vdash I, \quad I, \Delta \vdash C$;

ii) $[I] \subset [\Gamma] \cap [\Delta, C]$;

iii) $I \in$ NNIL;

iv) if C is a propositional variable and $C \notin [\Delta]$ then $C \notin [I]^{-}$.

Proof. Induction over the length of a derivation of $\Gamma, \Delta \vdash C$.

$\Gamma, \Delta \vdash C$ is an axiom or the conclusion of a rule different from ($\to$L): apply Schütte's interpolation method (2.3). (i)-(iv) follow directly by induction.

$\Gamma, \Delta \vdash C$ is the conclusion of ($\to$L): we have two cases.

Case 1: $A \to B \in \Gamma$ ($A \to B$ is the 'new' formula in the conclusion of ($\to$L)). Then $A \to B \in$ NNIL, so A is a propositional variable, say P.

By the induction hypothesis, we have a Γ' with $\Gamma', P \to B = \Gamma$ and I_1, I_2 with:

a) $\Gamma' \vdash I_1$; $I_1, \Delta \vdash P$; $\Gamma', B \vdash I_2$; $I_2, \Delta \vdash C$;

b) $[I_1] \subset [\Gamma'] \cap [\Delta, P]$; $[I_2] \subset [\Gamma', B] \cap [\Delta, C]$;

c) $I_1, I_2 \in$ NNIL;

d) $P \notin [\Delta] \Rightarrow P \notin [I_1]^{-}$; if C is a propositional variable and $C \notin [\Delta]$ then $C \notin [I_2]^{-}$.

Now we must find an I and show that (i)-(iv) hold. We consider three subcases:

Subcase 1A: $C = P$.

Put $I:=I_1$.

i) $\Gamma\vdash I$ follows from $\Gamma'\vdash I_1$ (by (a)), $\Gamma'\subseteq\Gamma$ and 2.4(i).

ii), iii): evident.

iv) follows from (d), first part.

Subcase 1B: $C\neq P$, $P\in[\Delta]$. Put $I:=I_1\wedge(P\to I_2)$.

i)

$$2.4(i)\ \dfrac{\dfrac{\Gamma'\vdash I_1\ \ (a)}{\Gamma',P\to B\vdash I_1} \qquad \dfrac{\dfrac{\Gamma',P\vdash P \qquad \Gamma',B\vdash I_2\ \ (a)}{\Gamma',P\to B,P\vdash I_2}}{\Gamma',P\to B\vdash P\to I_2}}{\Gamma',P\to B\vdash I_1\wedge(P\to I_2),\ \text{i.e. } \Gamma\vdash I.}$$

$$\dfrac{\dfrac{(a)\ I_1,\Delta\vdash P \qquad \dfrac{I_2,\Delta\vdash C\ \ (a)}{I_1,I_2,\Delta\vdash C}}{I_1,P\to I_2,\Delta\vdash C}}{I_1\wedge(P\to I_2),\Delta\vdash C \quad \text{i.e.}\quad I,\Delta\vdash C.}$$

ii) $[I_1, I_2]\subseteq[\Gamma]\cap[\Delta,C]$ is easy; as $\Gamma=\Gamma',P\to B$ and $P\in[\Delta]$ we also have $P\in[\Gamma]\cap[\Delta,C]$, so $[I]\subseteq[\Gamma]\cap[\Delta,C]$.

iii) I_1, $I_2\in$NNIL (by (c)), so $I_1\wedge(P\to I_2)\in$NNIL by definition of NNIL.

iv) Assume C is a propositional variable and $C\notin[\Delta]$. Then $C\neq[I_2]^-$ (by (d)). As $C\neq P$, we also have $C\notin[\Delta,P]$, so (by (b)) $C\notin[I_1]$. Hence $C\notin[I]^-$.

Subcase 1C: $C\neq P$, $P\notin[\Delta]$. Put $I:=I_1[P:=I_2]$.

i) As in subcase 1B, we have $\Gamma',P\to B\vdash I_1\wedge(P\to I_2)$; by (d) we have $P\notin[I_1]^-$, so $I_1,P\to I_2\vdash I$ (by 2.4.iii); now apply (*) and we get $\Gamma',P\to B\vdash I$, i.e. $\Gamma\vdash I$.

$$(*)\ \dfrac{P\notin[\Delta]\ \dfrac{2.4.ii\ \dfrac{I_1,\Delta\vdash P}{I_1[P:=I_2],\Delta[P:=I_2]\vdash I_2}}{I_1[P:=I_2],\Delta\vdash I_2} \qquad I_2,\Delta\vdash C}{I_1[P:=I_2],\Delta\vdash C \qquad \text{i.e. } I,\Delta\vdash C.}$$

ii) By (b), $([I_1]\backslash\{P\}) \cup [I_2] \subset [\Gamma',B] \cap [\Delta,C]$; as $[I]=([I_1]\backslash\{P\}) \cup [I_2]$, we have $[I] \subset [\Gamma] \cap [\Delta,C]$.

iii) $I \in$ NNIL follows from (c) and $P \notin [I_1]^-$ (by (d)).

iv) Assume C is a propositional variable, $C \notin [\Delta]$. Then $C \notin [I_1]$ by (b), and $C \notin [I_2]^-$ by (d); also $P \notin [I_1]^-$, by (d). Hence $C \notin [I]^-$.

Case 2: $A \rightarrow B \in \Delta$. Apply $(i \rightarrow L1)$ of Schütte's method: it yields an interpolant $I=I_1 \wedge I_2$.

(i)-(iv) follow directly. □

3.3. Corollary. If $\vdash A \rightarrow B$, $A \in$ NNIL, then there is an interpolant $I \in$ NNIL of $A \rightarrow B$. □

Now we prove the other half of 1.2.:

3.4. Lemma. Assume $\Gamma, \Delta \vdash C$ is derivable.

Then there is an I with:

i) $\Gamma \vdash I$; $I, \Delta \vdash C$;

ii) $[I] \subset [\Gamma] \cap [\Delta,C]$;

iii) if $C \in$ NNIL and $\Delta \subset \mathcal{P}$ then $I \in$ NNIL;

iv) if $\Gamma \subset \mathcal{P}$, then I is the conjunction of propositional variables.

Proof. Induction over the length of a derivation of $\Gamma, \Delta \vdash C$.

$\Gamma, \Delta \vdash C$ is an axiom or the conclusion of a rule different from $(\rightarrow L)$: apply Schütte's method.

$\Gamma, \Delta \vdash C$ is the conclusion of $(\rightarrow L)$: we have three cases.

Case 1: $A \rightarrow B \in \Gamma$, $C \in$ NNIL, $\Delta \subset \mathcal{P}$. Here $(i \rightarrow L2)$ of Schütte's method prescribes the interpolant $I_1 \rightarrow I_2$ (see 2.3). This interpolant satisfies (i) and (ii), but not (iii) in general (only if $I_1 \in \mathcal{P}$). However, I_1 is the conjunction of propositional variables, for it is the interpolant of $\Delta, \Gamma' \vdash A$ (with $\Gamma=\Gamma'$, $A \rightarrow B$) and $\Delta \subset \mathcal{P}$. Let $I_1=P_1 \wedge \ldots \wedge P_n$, then we put $I:=P_1 \rightarrow (\ldots\ (P_n \rightarrow I_2)\ \ldots\ )$. Now $I \equiv I_1 \rightarrow I_2$ so I satisfies (i), (ii); also $I \in$ NNIL, for (by ind. hyp.) $I_2 \in$ NNIL. (iv) is trivially satisfied (for $A \rightarrow B \in \Gamma$).

Case 2: $A \rightarrow B \in \Gamma$, ($C \notin$ NNIL or $\Delta \not\subset \mathcal{P}$. Now follow $(i \rightarrow L2)$ of Schütte's method: then (i),(ii) are satisfied, and (iii),(iv) are trivially true.

Case 3: $A \rightarrow B \in \Delta$.

Follow $(i \rightarrow L1)$ of Schütte's method: this yields an interpolant $I=I_1 \wedge I_2$ for which (i),(ii) hold. (iii) is trivially true (for $A \rightarrow B \in \Delta$) and (iv) follows by ind. hyp. □

3.5. Corollary. If $\vdash A \rightarrow B$, $B \in NNIL$, then there is an interpolant $I \in NNIL$ of $A \rightarrow B$. □

4. CONCLUDING REMARKS.

4.1. Uniform Interpolation.

Let $NNIL_n$ be the set of all formulae A of NNIL with $[A] \subset \{P_1, \ldots, P_n\}$. It is not hard to show that, for any n, $NNIL_n$ is finite modulo provable equivalence (induction over n: use $P \rightarrow A \equiv P \rightarrow A[P:=T]$).

A consequence of this is the following uniform version of 1.2.:

i) if $A \in NNIL, \vartheta \subset [A]$, then there is an $I \in NNIL$ with:
   - $[I] \subset \vartheta$;
   - $\vdash A \rightarrow I$
   - for any B with $[A] \cap [B] \subset \vartheta$ and $\vdash A \rightarrow B$ we have $\vdash I \rightarrow B$.

ii) if $\vartheta \subset [A]$, then there is an $I \in NNIL$ with
   - $[I] \subset \vartheta$;
   - $\vdash A \rightarrow I$;
   - for any $B \in NNIL$ with $[A] \cap [B] \subset \vartheta$ and $\vdash A \rightarrow B$ we have $\vdash I \rightarrow B$.

4.2. Positive and negative occurrence.

Schütte's method (2.3.) yields an interpolant I for $\vdash A \rightarrow B$ with

$$(\pm) \qquad [I]^+ \subset [A]^+ \cap [B]^+, \quad [I]^- \subset [A]^- \cap [B]^-.$$

However, our adaptation of Schütte's method used in 3.2. does not respect $(\pm)$: e.g. in subcase 1B we have $P \in [I]^-$, but $P \in [\Delta \rightarrow C]^-$ is not excluded. So we conclude with the following open problem:

does 1.2. hold if $(\pm)$ is added?

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