Universiteit Utrecht

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Master Thesis

# Algebraic Aspects of the Berezinian

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# Abstract

Many concepts of linear algebra can be generalized to the $\mathbb{Z}_{/2}$-graded setting, leading to linear superalgebra. Often, a formulation in terms of category theory facilitates this passage, and this e.g. provides an invariant description of the supertrace of an endomorphism $T\colon V \longrightarrow V$ of a super vector space. However, it is not so straightforward to describe the superdeterminant, also known as Berezinian, in a basis-independent way.

In this thesis we look at a characterization of the Berezinian, given by Deligne and Morgan, in terms of homological algebra. It generalizes the description of the ordinary determinant via the induced action on the top exterior power of a vector space. After introducing super linear algebra, we explain the invariant description, and illustrate it by explicitly working it out for some examples.


**Keywords.** Berezinian, supercommutative algebra, homological algebra.

# Contents

# Preface

This thesis has grown out of an attempt to learn the basics of the mathematics behind supersymmetry in (quantum) field theory: supermathematics. The central problem comes from a single paragraph, §1.10(B), in Deligne and Morgan [3]. This paragraph contains an invariant description of the Berezinian (or superdeterminant) of a linear map of (free) supermodules over a superalgebra. Trying to understand this description eventually became the main focus of this thesis, and working it out for some concrete examples has led from the study of locally superringed spaces and other supergeometry back to linear superalgebra and on to the world of homological algebra.

**Background.** None of the material in the introductory Chapter 1 is new, but the exposition is. Our account of linear superalgebra in Chapter 2 mostly follows our main reference, Deligne and Morgan [3]. Other useful sources have been Sachse [11] and Varadarajan [13]. We have filled in several details and proofs in Section 2.3. For the general theory of homological algebra in Chapter 3, Eisenbud [4] and Davis and Kirk [2] have been helpful. The content of Chapter 4 is due to us, as is the proof in Chapter 5.

**Prerequisites.** We assume that you are familiar with (ordinary) algebra and with the basic notions of category theory. This (and much more) can be found in Lang [8] for algebra, and in Mac Lane [9] and on the nLab[1] for category theory.

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[1] See http://ncatlab.org/nlab/show/HomePage .

# Chapter 1

# Introduction

The prefix 'super–' comes from physics; in mathematics it is a hyperbole for '$\mathbb{Z}_{/2}$-graded', where $\mathbb{Z}_{/2} \coloneqq \mathbb{Z}/2\mathbb{Z}$ ; we denote its elements by $\bar{0}$ and $\bar{1}$. Many familiar notions from algebra and geometry have a 'superanalogue'. This thesis revolves around the *Berezinian*, the $\mathbb{Z}_{/2}$-graded version of the determinant, also known as the *superdeterminant*. Everything will be super: the Berezinian belongs to linear superalgebra, and is related to supergeometry and (integration over) supermanifolds in particular. Of course, all of this takes place in the world of supermathematics.

In Section 1.1 we motivate the central problem of this thesis. Therefore, the exposition won't be very precise, so that the general line of thought is not obscured by details that are not relevant for the remainder. We borrow some ideas from physics, look at calculus on Grassmann algebras (which, historically, was an important step towards supermathematics), and use some heuristics to arrive at our destination. In Section 1.2 we give an outline of this thesis.

## 1.1 Invitation

Before we get to the Berezinian we quickly review how the ordinary determinant can be defined in linear algebra.

Consider a vector space $V$ over $\mathbb{R}$, of (finite) dimension $p$, and an endomorphism $T\colon V \longrightarrow V$. Choosing a basis $\{e_i\}$ for $V$ gives an identification $V \cong \mathbb{R}^p$, and allows us to represent $T$ by a matrix $\operatorname{mat} T = (T^i{}_j) \in \mathrm{GL}(V)$. The familiar Leibniz formula

$$\det(\operatorname{mat} T) = \sum_{\sigma \in S_p} \operatorname{sign}\sigma \prod_{n=1}^{p} T^n{}_{\sigma(n)} \tag{1.1}$$

tells us how to compute the determinant of this matrix. Any other choice of basis $\{e_i'\}$ for $V$ is related to $\{e_i\}$ by an automorphism $S\colon \mathbb{R}^p \longrightarrow \mathbb{R}^p$, and the matrix $\operatorname{mat}' T$ of $T$ with respect to the other basis is related to the old matrix via $\operatorname{mat}' T = \operatorname{mat} S^{-1} \circ \operatorname{mat} T \circ \operatorname{mat} S$. Because formula (1.1) turns out to be multiplicative (see Footnote 1 below), the line

$$\det(\operatorname{mat}' T) = \det(\operatorname{mat} S)^{-1} \cdot \det(\operatorname{mat} T) \cdot \det(\operatorname{mat} S) = \det(\operatorname{mat} T)$$

shows us that we can happily define the determinant of the endomorphism $T$ by

$$\det T \coloneqq \det(\operatorname{mat} T) \tag{1.2}$$

without worrying about the choice of basis.

This description of the determinant gives a recipe for the computation of the determinant of $T$. On the other hand, it requires some work to show that this definition doesn't depend on the choice of basis. It would be nice if we could find an *invariant* way to describe the determinant.

**An invariant description.** The signs and permutations in (1.1) give the determinant an alternating character. This suggests to consider the exterior, or Grassmann, algebra of $V$,

$$\Lambda^\bullet V = \bigoplus_{n \geq 0} \Lambda^n V \ .$$

The summands of $\Lambda^\bullet V$ have dimension $\dim(\Lambda^n V) = \binom{p}{n}$, so there is a top exterior power: $\Lambda^p V$ has dimension one, and all higher exterior powers vanish. The map

$$(\Lambda^p T)(v_1 \wedge \cdots \wedge v_p) \coloneqq (T\, v_1) \wedge \cdots \wedge (T\, v_p)$$

induced by $T$ on $\Lambda^p V$ is linear, so it must act by multiplication with some scalar. A couple of computations for examples such as $V = \mathbb{R}^2$ suggest that the factor is equal to $\det T$. That this is true is proved as follows.

It can be shown that the function $\det\colon \operatorname{End} V \longrightarrow \mathbb{R}$ given by (1.2) and (1.1) is uniquely characterized by the following three axioms:[1]

i) linearity in each row (or column) of the matrix of $T$ with respect to some basis;

ii) viewed as a $p$-linear function on the rows (or columns) of $T$, it is alternating: we pick up a sign when two rows (or columns) are interchanged;

iii) $I_p \longmapsto 1$, where $I_p$ is the identity matrix on $\mathbb{R}^p$.

These axioms are also satisfied by the map

$$\begin{aligned} \operatorname{End} V &\longrightarrow \operatorname{End}(\Lambda^p V) \cong \mathbb{R} \ , \\ T &\longmapsto \Lambda^p T \ . \end{aligned}$$

Indeed, (i)–(ii) follow from the definition of the wedge product, and (iii) is immediate. Thus

$$(\Lambda^p T)(v_1 \wedge \cdots \wedge v_p) = \det T \cdot v_1 \wedge \cdots \wedge v_p \ .$$

This provides an alternative definition of the determinant which is manifestly invariant, and the top exterior power $\Lambda^p V$ of $V$ deserves to be called $\det V$. This description is useful for e.g. applications in differential geometry.

**On to the supercase.** A super vector space is just an ordinary vector space $V$ made up of two parts, $V = V_{\bar 0} \oplus V_{\bar 1}$. The first summand is called the *even* part, and $V_{\bar 1}$ is the *odd* part. What is the analogue of the determinant for an endomorphism of this super vector space?

The first thing we do is to check whether either of the above descriptions can be generalized in an obvious way. Unfortunately, there is no obvious candidate generalizing formula (1.1). Perhaps the invariant description offers more possibilities.

However, the exterior algebra $\Lambda^\bullet V$ of a super vector space behaves quite differently from the exterior algebra of ordinary vector spaces. It's not very surprising that the correct generalization of the exterior algebra of a super vector space is

$$\Lambda^\bullet V = \Lambda^\bullet V_{\bar 0} \underset{\mathbb{K}}{\otimes} \operatorname{Sym}^\bullet V_{\bar 1} \ . \tag{1.3}$$

The first summand is the exterior algebra of the ordinary vector space $V_{\bar 0}$, and $\operatorname{Sym}^\bullet V_{\bar 1}$ is the *symmetric* algebra of the odd subspace $V_{\bar 1}$. Intuitively, the latter arises because we already pick up a minus sign when the odd elements are moved past each other, and this sign cancels the additional sign we get from the construction of the exterior algebra (see Section 2.2.2 for more details).

Because the multiplication in the symmetric algebra $\operatorname{Sym}^\bullet V_{\bar 1}$ is *commutative*, there is no maximal symmetric power $\operatorname{Sym}^n V_{\bar 1}$. The upshot of this discussion is that there is no obvious way to generalize the invariant description of the determinant to the $\mathbb{Z}_{/2}$-graded case.

To get some clues for the right way to proceed we go back to the founder of supermathematics, the Russian mathematical physicist Felix Alexander Berezin (1931-1980). He was the first to write down the formula that we're after, and the name 'Berezinian' was given in his honour. More about Berezin and his work can be found e.g. in [6].

[1] In fact, these axioms imply that the determinant is multiplicative, as can be seen by fixing $S \in \operatorname{End} V$ and considering the assignment $\operatorname{mat} T \longmapsto \det(\operatorname{mat} S \cdot \operatorname{mat} T)$. Indeed, this map satisfies (i) and (ii), but takes the value $\det(\operatorname{mat} S)$ at the identity $I_p$. Thus the normalization requirement (iii) tells us that the assignment is given by the determinant times $\det(\operatorname{mat} S)$.

**A tiny bit of quantum physics.** In the 1960s, Berezin was working on the foundations of quantum field theory. To understand why this led him to invent calculus on Grassmann algebras we have to know a few things about quantum physics. Don't worry: we keep it brief. For more about the mathematical foundations of quantum mechanics we refer to e.g. Strocchi [12].

In quantum physics, the state of a system is described by its *wave function* $\Psi$, which is an element of some complex Hilbert space, such as the space $L^2(X,\mu)$ of square-integrable complex-valued functions on a measure space $X$. The modulus $|\Psi|^2$ gives rise to a probability density, which describes the distribution of all possible outcomes under measurements of physical quantities such as the position of a particle in the system.

A simple thought experiment shows that elementary particles can be divided into two classes, reflecting their *statistics*. Consider two identical elementary particles, so that the wave function is defined on the product space $X \times X$. What happens when the two particles are interchanged? Firstly, since the particles are elementary and identical, they are *indistinguishable*, and the physical probabilities must be invariant under swapping the particles. Secondly, interchanging twice is the identity. These two observations force the wave function $\Psi$ to be either symmetric or antisymmetric in its two arguments.

In the symmetric case, the corresponding elementary particles are called *bosons*, and in the antisymmetric case we talk about *fermions*. (As a neat physical corollary, note that fermions obey the *Pauli exclusion principle*: two identical fermions cannot be in the same state, as the antisymmetry forces the probability for this to happen to vanish. Ordinary solid matter is made up of fermions, and without the exclusion principle the chair on which you sit would collapse.)

Now suppose that we have a system of several identical fermions. The only thing we need from physics is the following: if we want to construct the wave function out of several single-particle wave functions, we have to take their *anti-symmetric* product.

Incidentally, the space of all wave functions is an (infinite dimensional) super vector space, whose even part contains wave functions describing bosons and whose odd part corresponds to fermions. For this reason physicists often use 'bosonic' for 'even', and 'fermionic' for 'odd'.

**Integration over Grassmann algebras.** We go back to the 1960s. Berezin wanted to find a formalism for quantum fields representing fermions. This formalism should look the same as the functional approach for bosonic fields that already existed. Since fermions anticommute with each other, it's not surprising that Berezin started by looking at finitely generated Grassmann algebras. First he had to set up calculus on such algebras.

For definiteness consider the Grassmann algebra $\Lambda_q \cong \Lambda^\bullet \mathbb{R}^q$ on $q$ generators $\theta_j$ forming a basis for $\mathbb{R}^q$. We use the convention that $\theta$'s denote *anticommuting* generators:

$$\theta_i \wedge \theta_j = -\theta_j \wedge \theta_i \ . \tag{1.4}$$

To streamline the notation we suppress the wedge products $\wedge$ in the remainder.

Defining integration over $\Lambda_q$ is not very straightforward. In brief, in the formalism for bosonic fields the linearity of ordinary integration, and invariance under constant shifts of the integration variables (allowing one to complete the square in exponential functions), are used very often. Thus, requiring linearity and invariance under shifts $\theta \longmapsto \theta + \eta$ for $\eta$ odd, Berezin found that integration over $\Lambda_q$ should be defined like *differentiation*:

$$\int \theta_i \, \mathrm{d}\theta_j = c\,\delta_{ij} \quad \text{and} \quad \int \mathrm{d}\theta_i = 0 \ ;$$

often, the normalization $c = 1$ is used. Multiple integrals are defined by repeated application of the above rule, for example using the convention that the innermost integral is performed first.

Now in quantum field theory, integrals of Gaussian functions are by far the most important integrals. (This comes about because the partition function, the generating function for physically interesting quantities, is of the form $e^{i\,S}$, and the action $S$ typically is a *quadratic* functional in the fields.) Recall that ordinary Gaussian integrals go like

$$\int_{\mathbb{R}^p} \exp\Big(-\tfrac{1}{2}\sum K_{ij}\, x_i x_j\Big)\, \mathrm{d}x_1 \cdots \mathrm{d}x_p = \frac{(2\pi)^{p/2}}{\sqrt{\det K}} \ . \tag{1.5}$$

(The factor with $2\pi$ is not important for our purposes.)

Because of (1.4), the generators $\theta^i$ are nilpotent. This means that all power series in $\theta$'s are finite, so that functions on $\Lambda_q$ can be defined by their usual Taylor expansion. For example,

$$\exp\left(\theta_i\theta_j + \theta_k\theta_\ell\right) = 1 + \theta_i\theta_j + \theta_k\theta_\ell + 2\cdot\tfrac{1}{2}\,\theta_i\theta_j\theta_k\theta_\ell \ .$$

We can now compute a typical Gaussian integral,

$$\int_{\Lambda_q} \exp\Big(-\tfrac{1}{2}\sum N_{ij}\,\theta_i\theta_j\Big)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_q \ .$$

Without loss of generality, we may take the coefficients to be antisymmetric, $N_{ij} = -N_{ij}$. Carefully keeping track of the signs[2] the result can be written as

$$\int_{\Lambda_q} \exp\Big(-\tfrac{1}{2}\sum N_{ij}\,\theta_i\theta_j\Big)\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_q = \sqrt{\det N} \ . \tag{1.6}$$

This is a curious feature of integration on Grassmann algebras: unlike the ordinary result (1.5), the determinant of $N$ now ends up in the *numerator*. On the other hand, the form of the two results is quite alike. This, and similar observations, led to the development of supermathematics in the 1970s:

> "A striking coincidence of basic formulas of operator calculus in Fermi and Bose variants of the second quantization was discovered in 1961. [...] These circumstances suggested possibility of a generalization of all the basic notions of analysis in such a way that the generators of a Grassmann algebra would play a role equal to that of real or complex variables". [1, p.2]

**A formula for the Berezinian.** Now we can get back to our question: what is the correct analogue of the determinant for an endomorphism of $V$? Let's look at a simple case and assume that $T$ leaves the subspaces $V_{\bar 0}$ and $V_{\bar 1}$ invariant: $T(V_{\bar\imath}) \subseteq V_{\bar\imath}$. Write $p = \dim V_{\bar 0}$ and $q = \dim V_{\bar 1}$.

If we ask our question to a physicist, he will tell us the following. Pick a basis $\{e_i\}$ for $V$, and order it such that the even elements are first and the odd ones come last. Our assumption now means that the matrix of $T$ is block-diagonal:

$$\operatorname{mat} T = \left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right) \ . \tag{1.7}$$

Here, $K \in \mathrm{GL}(V_{\bar 0})$ is a $p\times p$-matrix acting on the even subspace $V_{\bar 0} \cong \mathbb{R}^p$, and $N \in \mathrm{GL}(V_{\bar 1})$ has size $q \times q$.

The crucial observation is that the odd subspace $V_{\bar 1}$ is isomorphic to the degree-one part $\mathbb{R}\,\theta_1 \oplus \cdots \oplus \mathbb{R}\,\theta_q$ of the Grassmann algebra $\Lambda_q = \mathbb{R}[\theta_1,\cdots,\theta_q]$. Therefore, the formulas (1.5) and (1.6) hint how we can compute the superdeterminant: the answer is

$$\operatorname{Ber}\left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right) = \det K \cdot \det N^{-1} \ . \tag{1.8}$$

It is reasonable to require the Berezinian to be multiplicative. Then the decomposition

$$\left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) = \left(\begin{array}{c|c} I_p & L\,N^{-1} \\ \hline 0 & I_q \end{array}\right)\left(\begin{array}{c|c} K - L\,N^{-1}\,M & 0 \\ \hline 0 & N \end{array}\right)\left(\begin{array}{c|c} I_p & 0 \\ \hline N^{-1}\,M & I_q \end{array}\right)$$

of the matrix of a general endomorphism $T\colon V \longrightarrow V$, together with (1.8), suggests the following general formula

$$\operatorname{Ber}(\operatorname{mat} T) = \det(K - L\,N^{-1}\,M)\cdot \det N^{-1} \ .$$

[2] See e.g. §9.5 of [10].

Multiplicativity further tells us that this definition does not depend on the choice of basis, and we can define

$$\operatorname{Ber} T = \det(K - L\,N^{-1}\,M) \cdot \det N^{-1} \ . \tag{1.9}$$

This is indeed the formula generalizing the determinant to the supercase.

This thesis revolves around equation (1.9). The above 'derivation' is mostly heuristics. It's not directly clear from the formula that the result is multiplicative, and we still have to show that it doesn't depend on the choice of basis for $V$. In other words, provided the Berezinian satisfies (1.8) and is multiplicative, formula (1.9) proves its uniqueness, but we still have to establish its existence.

Moreover it is a bit unsatisfactory that the roles of the matrices $K$ and $N$ in (1.9) are quite different. Also, although we have the result that we were looking for, it's still far from clear how we can give an invariant description of the Berezinian. In this thesis we will shed some light on these questions.

Finally we remark that (1.9) involves the inverse of $N$. As we will see, $T\colon V \longrightarrow V$ has to be *invertible* in order to define the Berezinian of $T$, and in that case both $K$ and $N$ are invertible.

## 1.2 Outline

In the next chapter we set the scene and properly introduce (linear) superalgebra. We will see where equation (1.3) comes from, and give a more thorough derivation of (1.9). Since category theory offers a nice way to look at superalgebra we will also spend some time to reformulate everything in the categorical setting.

In Chapter 3 first we show that the Berezinian is uniquely determined by three axioms. Then we give an invariant description of the Berezinian, which leads us into the realm of homological algebra. Since the invariant formulation of the Berezinian is rather abstract, we explicitly work out some examples to get familiar with the invariant formulation in Chapter 4. Finally, in Chapter 5 we prove that the invariant description is equivalent to (1.9).

# Chapter 2

# Linear superalgebra

In the previous chapter we have 'derived' a formula for the Berezinian with the help of a bit of physics and some history. Along the way we have encountered super vector spaces and briefly talked about their symmetric and exterior algebras.

In this chapter we fill in the gaps. We begin with a proper introduction of superalgebra, and discuss super vector spaces and the like, along the lines of Deligne and Morgan [3]. As we will see, category theory allows for an elegant and unified treatment of these notions, and can help to find the correct superanalogues of concepts in ordinary algebra. Examples where this happens are symmetric and exterior algebras, but also the supertrace, which incidentally gives another justification for the formula for the Berezinian.

In particular, in this chapter we will

- introduce the setting in which our problem takes place;
- see how the symmetric monoidal structure of the category of super vector spaces implies that a super vector space does not have a maximal exterior power; and
- find out what the consequences of the braiding are for the supertranspose and supertrace of a linear map, and what the inverse of such a map looks like.

## 2.1 $\mathbb{Z}_{/2}$-graded algebra

There are many areas of mathematics in which *graded algebra* pops up. To name just one familiar instance: the space $\Omega^\bullet(M)$ of differential forms on a smooth manifold $M$ has a natural grading $\Omega^\bullet(M) = \bigoplus_{n\geq 0} \Omega^n(M)$ in terms of differential $n$-forms. This is an example of a $\mathbb{Z}$-graded algebra.

More generally, we can look at $G$-graded algebra for *any* abelian group $G$. A typical notion is that of a *$G$-graded ring*: this is a ring $R$ with an (additive) decomposition

$$R = \bigoplus_{g\in G} R_g$$

such that the multiplication is compatible with this $G$-grading:

$$R_g \times R_h \longrightarrow R_{gh} \ .$$

Taking inverses in $G$ plays no role in this definition, and graded algebra can indeed be further generalized to the case where $G$ is a commutative *monoid*. However, as for $\Omega(M)$, often we do actually have a $\mathbb{Z}$-grading. The quotient map $\mathbb{Z} \longrightarrow \mathbb{Z}_{/2}$ then allows us to view any $\mathbb{Z}$-graded notion as $\mathbb{Z}_{/2}$-graded. This leads us to the super-case in which we are interested.

### 2.1.1 Super vector spaces

Let $\mathbb{K}$ be your favourite field of characteristic zero. A *super vector space over* $\mathbb{K}$ is a $\mathbb{Z}_{/2}$-graded vector space over $\mathbb{K}$: it has a direct-sum decomposition

$$V = V_{\bar{0}} \oplus V_{\bar{1}} \ .$$

If $p = \dim V_{\bar{0}}$ and $q = \dim V_{\bar{1}}$ we say that $V$ has *dimension* $p|q$.

An element $v \in V_{\bar{\imath}}$ is called *homogeneous* and has *parity* $p(v) := \bar{\imath}$. To make our formulas more transparent we often abbreviate $\bar{v} := p(v)$. (Note that we do not assign a parity to non-homogeneous elements, so parity is a function $p\colon V_{\bar{0}} \cup V_{\bar{1}} \longrightarrow \mathbb{Z}_{/2}$.)

Thus, to construct a super vector space all we have to do is to take an ordinary vector space, split it into two, and decide which elements we call *even* ($v \in V_{\bar{0}}$) and which *odd* ($v \in V_{\bar{1}}$).

**Examples.** (i) A dull example of a super vector space is the ordinary vector space of dimension $p$, viewed as a super vector space of dimension $p|0$.

(ii) A bit more interesting is the purely odd super vector space of dimension $0|q$, which we denote by $\Pi\,\mathbb{K}^q := 0 \oplus \mathbb{K}^q$. This is a copy of $\mathbb{K}^q$ where we declare all elements $v \in \Pi\,\mathbb{K}^q$ to have parity $\bar{v} = \bar{1}$.

(iii) The purely even and purely odd examples are special cases of the prototype super vector space $\mathbb{K}^{p|q}$ of dimension $p|q$.

**Constructions on super vector spaces.** There are a couple of ways to produce a new super vector space out of old ones. We have to supplement the usual constructions for ordinary vector spaces with a parity function.

Given two super vector spaces $V$ and $W$, the *direct sum* $V \oplus W$ is defined by

$$(V \oplus W)_{\bar{\imath}} := V_{\bar{\imath}} \oplus W_{\bar{\imath}} \ , \tag{2.1}$$

and the *tensor product* $V \otimes W$ by

$$(V \otimes W)_{\bar{k}} := \bigoplus_{\bar{\imath}+\bar{\jmath}=\bar{k}} V_{\bar{\imath}} \underset{\mathbb{K}}{\otimes} W_{\bar{\jmath}} \ ; \tag{2.2}$$

in other words, we have $p(v \otimes w) = p(v) + p(w)$ for homogeneous elements $v \in V$ and $w \in W$.

In Section 2.2.2 we describe two more ways to obtain new super vector spaces from a given super vector space: their symmetric and exterior algebra.

By definition, the set $\mathrm{Hom}(V, W)$ consists of all *parity-preserving* linear maps from $V$ to $W$, so $T(V_{\bar{\imath}}) \subseteq W_{\bar{\imath}}$ for any $T \in \mathrm{Hom}(V, W)$. This set forms an ordinary vector space. Closely related is the *super* vector space $\mathbf{Hom}(V, W)$ of *all* linear maps: $V \longrightarrow W$. Its even part is

$$\mathbf{Hom}(V, W)_{\bar{0}} = \mathrm{Hom}(V, W) \ ,$$

while the odd part consists of the parity *reversing* maps. We will get back to $\mathbf{Hom}(V, W)$ near the end of Section 2.1.2. In Section 2.2.1 we will take a closer look at the structure of linear maps between super vector spaces.

### 2.1.2 The category sVec of super vector spaces

The above can be nicely reformulated in terms of categories. Write Vec for the category of ordinary vector spaces and linear maps between them.[1] We are interested in the category sVec, which has super vector spaces over $\mathbb{K}$ as objects, and parity preserving linear maps as morphisms.

The boring example (i) above describes the inclusion of categories $\mathsf{Vec} \hookrightarrow \mathsf{sVec}$.

Example (ii) indeed gives something a bit more interesting: the assignment $\mathbb{K} \longmapsto \Pi\,\mathbb{K}$ generalizes to a map that takes an arbitrary super vector space $V$ and turns it into a new super vector space $\Pi\,V$ via

$$(\Pi\,V)_{\bar{0}} := V_{\bar{1}} \ , \quad (\Pi\,V)_{\bar{1}} := V_{\bar{0}} \ . \tag{2.3}$$

[1]Note that, as with $\mathrm{Hom}(V, W)$, we omit any reference to the field to keep the notation light; more proper notation would be $\mathrm{Hom}_{\mathbb{K}}(V, W)$ and $\mathsf{Vec}_{\mathbb{K}}$. When the context asks for it, e.g. when we discuss modules over various rings in Section 2.2.1, we will be more precise.

This defines a functor $\Pi\colon \mathsf{sVec} \longrightarrow \mathsf{sVec}$ which is called the *parity reversing functor*. To practice notation: we have

$$\Pi\, V = \mathbb{K}^{0|1} \otimes V \ . \tag{2.4}$$

**Abelian structure.** The direct sum yields a bifunctor $\oplus\colon \mathsf{sVec} \times \mathsf{sVec} \longrightarrow \mathsf{sVec}$ sending pairs of super vector spaces to their direct sum (2.1) and pairs of morphisms $T\colon V \longrightarrow W$ and $T'\colon V' \longrightarrow W'$ to

$$\begin{aligned} T \oplus T' \colon V \oplus V' &\longrightarrow W \oplus W' \ , \\ v \oplus v' &\longmapsto T(v) \oplus T'(v') \ . \end{aligned}$$

In this way $\mathsf{sVec}$ gets the structure of an *abelian category*. Let's spell out what this means in three steps.

Firstly, the category $\mathsf{sVec}$ is *additive*:

i) The trivial super vector space $\mathbb{K}^{0|0} = 0$ is the *zero object*: the hom sets $\mathrm{Hom}(0, V)$ and $\mathrm{Hom}(V, 0)$ have precisely one element.

ii) We have already seen that the category $\mathsf{sVec}$ is *enriched over* $\mathsf{Vec}$: each hom set $\mathrm{Hom}(V, W)$ is a vector space, and the composition of morphisms

$$\mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(U, W)$$

is bilinear. Since a vector space is in particular an abelian group under addition, this implies that $\mathsf{sVec}$ is enriched over the category of abelian groups.

iii) All *finite products* and *finite coproducts* exist: these two notions coincide and are just repeated direct sums of super vector spaces.

Next, the *kernel* of a morphism $T\colon V \longrightarrow W$ can be described as the equalizer of the pair of arrows $0, T\colon V \rightrightarrows W$. Dually, the coequalizer of this pair is called the *cokernel* of $T$. The following property is clear for $\mathsf{sVec}$.

iv) Every morphism in $\mathsf{sVec}$ has a kernel and a cokernel. Concretely, for $T\colon V \longrightarrow W$, these are given by

$$\ker T\colon T^{-1}(0) \longrightarrow V \quad \text{and} \quad \operatorname{coker} T\colon W \longrightarrow W/T(V) \ .$$

The last property may look a bit strange at first sight:

v) If the kernel of $V \xrightarrow{T} W$ is zero, then $T$ is the kernel of its cokernel; and if its cokernel is zero, $T$ is the cokernel of its kernel.

To see what this entails, let's think of a general category $\mathsf{C}$ satisfying (i)–(v). Property (i) in particular means that for any two objects $C, D$ of $\mathsf{C}$ there is a (unique) *zero morphism* $0\colon C \longrightarrow D$, obtained as the composition $C \longrightarrow 0 \longrightarrow D$. According to (ii), $\mathrm{Hom}_{\mathsf{C}}(C, D)$ is an abelian group, and $0\colon C \longrightarrow D$ is the unit of this group. Property (iv) allows us to talk about the kernel and cokernel of any morphism in $\mathsf{C}$. Now property (v) in particular implies that if such a morphism is both monic (has kernel zero) and epi (has cokernel zero), it is an *iso*morphism. (For this reason, the morphisms of abelian categories are usually called *homomorphisms*.)

**Monoidal structure.** The tensor product (2.2) yields another bifunctor on $\mathsf{sVec}$, sending the pairs $T \in \mathrm{Hom}(V, W)$ and $T' \in \mathrm{Hom}(V', W')$ to

$$\begin{aligned} T \otimes T' \colon V \otimes V' &\longrightarrow W \otimes W' \ , \\ v \otimes v' &\longmapsto T(v) \otimes T'(v') \ . \end{aligned}$$

The tensor product further gives sVec the structure of a *monoidal category.* This means that

i) The tensor product is *associative* up to the natural isomorphism $u\otimes(v\otimes w) \longmapsto (u\otimes v)\otimes w$;

ii) There is a *unit* object $I := \mathbb{K}^{1|0}$ satisfying $I \otimes V \cong V \cong V \otimes I$, where the isomorphisms are again natural;

iii) The pentagon diagram for the associativity isomorphisms

$$\begin{tikzcd}
 & T \otimes (U \otimes (V \otimes W)) \arrow[dl] \arrow[dr] & \\
T \otimes ((U \otimes V) \otimes W)) \arrow[d] & & (T \otimes U) \otimes (V \otimes W) \arrow[d] \\
(T \otimes (U \otimes V)) \otimes W \arrow[rr] & & ((T \otimes U) \otimes V) \otimes W
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[r] \arrow[dr] & V \otimes (I \otimes W) \arrow[d] \\
 & V \otimes W
\end{tikzcd}$$

both commute.

The monoidal structure of sVec allows us to define *monoids* in sVec. This is an object $A$ with morphisms $\mu \in \mathrm{Hom}_{\mathbb{K}}(A \otimes A, A)$ and $\eta \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{1|0}, A)$ such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{K}^{1|0} \otimes A \arrow[r, "\eta \otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K}^{1|0} \arrow[l, "1 \otimes \eta"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

(The associativity isomorphisms are suppressed.) Of course $\mu$ is just the multiplication, which is associative according the first diagram, and the second diagram tells us that the unit $\eta$ is the inclusion of the base field into $A$: a monoid in sVec is an (associative, unital) *superalgebra.*

The sets $\mathbf{Hom}(V, W)$ also have to do with the monoidal structure of sVec. They are defined via a bijection of hom sets, natural in $U$, $V$ and $W$:

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathbf{Hom}(V, W)) \ . \tag{2.5}$$

To see how this works, consider a map $T\colon U \otimes V \longrightarrow W$. If we fix a homogeneous element $u \in U_{\bar{\imath}}$ we get $T_u\colon V \longrightarrow W$ with $T_u(V_{\bar{\jmath}}) \subseteq W_{\bar{\imath}+\bar{\jmath}}$. This means that the parity of $T_u$ is equal to that of $u$, so the assignment $u \longmapsto T_u$ preserves parity. We conclude that $\mathbf{Hom}(V, W)$ can be described as

$$\mathbf{Hom}(V, W)_{\bar{\imath}} = \{\ T\colon V \longrightarrow W \mid \ \mathbb{K}\text{-linear, and } \ T(V_{\bar{\jmath}}) \subseteq W_{\bar{\imath}+\bar{\jmath}}\ \}\ ,$$

in agreement with our definition above. The sets $\mathbf{Hom}(V, W)$ are called *internal* (or *inner*) hom sets and are themselves objects of sVec. The definition (2.5) says that the functor $\mathbf{Hom}(V, -)$ is right adjoint to $- \otimes V$.

The inner hom sets provide a nice example of a superalgebra:composition turns $\mathbf{Hom}(V, V)$ into a superalgebra. This is an example of a *non*commutative superalgebra. However, we will mostly be interested in another type of superalgebras which we discuss next.

## 2.2 Introducing signs: supercommutative algebra

We have seen that super vector spaces aren't really that different from ordinary vector spaces; the two parts do not interact with each other. As usual we can replace the field of scalars by something more interesting and look at modules over rings and algebras. Let's go forth and multiply.

### 2.2.1 Supermodules and superalgebras

A *superring* is a $\mathbb{Z}_{/2}$-graded ring $R = R_{\bar{0}} \oplus R_{\bar{1}}$ such that the grading is compatible with the multiplication: $R_{\bar{\imath}} \cdot R_{\bar{\jmath}} \subseteq R_{\bar{\imath}+\bar{\jmath}}$. Supermodules over superrings are defined in a similar way. A (left) *supermodule* $M = M_{\bar{0}} \oplus M_{\bar{1}}$ over $R$ is a module over $R$ such that $R_{\bar{\imath}} \cdot M_{\bar{\jmath}} \subseteq M_{\bar{\imath}+\bar{\jmath}}$. In other words, the left action $r \otimes m \longmapsto r \cdot m$ preserves parity.

A familiar example of a supermodule is a super vector space, which is a module over the field $\mathbb{K}$ (viewed as the boring superring $\mathbb{K} \oplus 0$). More interesting are modules that are defined over objects with the following additional structure.

As we have seen at the end of Section 2.1.2, a superalgebra is a super vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ together with a (parity preserving) multiplication

$$\begin{aligned} \mu\colon A \otimes A &\longrightarrow A\ , \\ a \otimes b &\longmapsto a \cdot b\ . \end{aligned}$$

Of course, we will often write $ab$ instead of $a \cdot b$. We assume that our superalgebras are associative and unital.

Next, $A$ is *supercommutative* if

$$b \cdot a = (-1)^{\bar{a}\,\bar{b}}\, a \cdot b \tag{2.6}$$

for all homogeneous elements $a, b \in A$. (We'll often do this: write down relations for *homogeneous* elements, and implicitly extend by linearity.) Equation (2.6) is called the *sign rule* and is crucial for supermathematics. On the one hand, superalgebra is not a part of commutative algebra. On the other hand, the non-commutativity in superalgebra is rather mild as we only have some minus signs. This gives precisely enough freedom to get interesting new structures in superalgebra (and supergeometry), while being close enough to ordinary algebra (and geometry) to be able to generalize familiar notions to get such new structures.

**Examples.** (i) The cup product turns any cohomology group into a supercommutative ring. (Chapter 3 of Hatcher [5] contains a nice introduction to cup products.)

(ii) A familiar example of a supercommutative algebra is the exterior (or Grassmann) algebra $\Lambda^\bullet V$ of an ordinary vector space $V$, viewed as a $\mathbb{Z}_{/2}$-graded algebra. In other words, typical even elements are the unit $1 \in \mathbb{K} = \Lambda^0 V$ and $v_1 \wedge \cdots \wedge v_{2n}$, while elements such as $v_1 \wedge \cdots \wedge v_{2n+1}$ are odd. (This construction works for any $\mathbb{Z}$-graded algebra, and e.g. polynomial rings and tensor algebras give rise to many more examples of superalgebras that are not supercommutative.)

(iii) Since exterior algebras are supercommutative, so is the algebra $\Omega^\bullet(M)$ of differential forms over a manifold $M$, which are smooth sections of $\Lambda^\bullet(T^*M)$.

**Free supermodules.** Of course we can also define modules over superalgebras. For the moment we restrict our attention to right modules. As we will explicitly see in Section 2.3, this is convenient when we want to write down matrices for maps of supermodules. In Section 2.2.3 we will look at left modules too.

Fix a superalgebra $A$ and define the super vector space

$$A^{p|q} := \mathbb{K}^{p|q} \otimes A\ .$$

A choice of a *homogeneous* basis $\{e_i\}$ for $\mathbb{K}^{p|q}$ gives an isomorphism between $A^{p|q}$ and $p+q$ copies of $A$ with generator $e_i$. It is convenient to order the basis elements such that the parities are

$$p(e_i) = \begin{cases} \bar{0} & \text{for } 1 \leq i \leq p \ , \ \text{and} \\ \bar{1} & \text{for } p+1 \leq i \leq p+q \ , \end{cases} \tag{2.7}$$

The diagonal right $A$-action, where the same $a \in A$ acts from the right on each of the summands, turns $M = A^{p|q}$ into a right supermodule over $A$. Clearly $M$ is free as an ordinary right $A$-module: each $m \in M$ can be written as $m = \sum e_i \, m^i$ for unique coefficients $m^i \in A$. For this reason $A^{p|q}$ is called the *standard* free $A$-module of rank $p|q$.

In general, a supermodule $M$ is *free* over $A$ if it is isomorphic (as an $A$-module) to $A^{p|q}$ for some $p$ and $q$, and we define its *rank* as $p|q$. In Chapter 4 we will use the following object a lot. Define the *free commutative $A$-algebra* $A[t_1, \cdots, t_p|\theta_1, \cdots, \theta_q]$ as the algebra freely generated (from the right) by even elements $t_i$ and odd $\theta_j$. This generalizes the Grassmann algebra $\Lambda_q$ in $q$ generators that we encountered in Section 1.1:

$$A[t_1, \cdots, t_p|\theta_1, \cdots, \theta_q] = \mathbb{K}^p[t_1, \cdots, t_p] \underset{\mathbb{K}}{\otimes} \Lambda_q \underset{\mathbb{K}}{\otimes} A \ . \tag{2.8}$$

In words: elements of $A[t_1, \cdots, t_p|\theta_1, \cdots, \theta_q]$ are polynomial in the $t_i$ and exterior in the $\theta_j$.

### 2.2.2 More about sVec: symmetric monoidal structure

Let's translate the above discussion to category theory. The all-important sign rule (2.6) is encoded in the *braided monoidal* structure of sVec. This means that sVec has the following properties:

i) In Section 2.1.2 we have already seen that $(\mathsf{sVec}, \otimes, I)$ is a monoidal category with unit object $I = \mathbb{K}^{1|0}$;

ii) For each pair of objects $V$ and $W$, there is a natural isomorphism

$$\begin{aligned} \gamma_{V,W} \colon V \otimes W &\longrightarrow W \otimes V \ , \\ v \otimes w &\longmapsto (-1)^{\bar{v}\,\bar{w}} \, w \otimes v \ , \end{aligned}$$

called the *braiding*. We will often omit the subscripts and simply write $\gamma$.

iii) Omitting three associativity isomorphisms, the 'hexagon' diagram

$$\begin{tikzcd} U \otimes V \otimes W \arrow[r, "\gamma_{U,V} \otimes 1"] \arrow[dr, "\gamma_{U,V\otimes W}"'] & V \otimes U \otimes W \arrow[d, "1 \otimes \gamma_{U,W}"] \\ & U \otimes V \otimes W \end{tikzcd}$$

commutes.

A superalgebra $A$ is supercommutative if $\mu \circ \gamma_{A,A} = \mu$. The tensor product of two superalgebras also uses the braiding: the product in $A \otimes B$ is defined via

$$\begin{tikzcd} A \otimes B \, \otimes \, A \otimes B \arrow[r, "1 \otimes \gamma_{B,A} \otimes 1"] \arrow[dr, dotted, "\mu_{A\otimes B}"'] & A \otimes A \, \otimes \, B \otimes B \arrow[d, "\mu_A \otimes \mu_B"] \\ & A \otimes B \end{tikzcd}$$

so $(a \otimes b) \cdot (a' \otimes b') = (-1)^{\bar{b}\,\bar{a}'} (a\,a') \otimes (b\,b')$.

Another place where the braiding is used is in the tensor product of inner hom sets in sVec. Given a pair $T \in \mathbf{Hom}(V, W)$ and $T' \in \mathbf{Hom}(V', W')$, we define

$$\begin{aligned} T \otimes T' \colon V \otimes V' &\longrightarrow W \otimes W' \ , \\ v \otimes v' &\longmapsto (-1)^{p(v)\,p(T')} \, T(v) \otimes T'(v') \ . \end{aligned} \tag{2.9}$$

The braiding gives rise to the minus sign in (2.9); in Section 2.3.1 we will give an elegant explanation of the sign in (2.9) (see (2.15)).

An important property of the braiding is that it is *symmetric*:

$$\gamma_{W,V} \circ \gamma_{V,W} = 1 \ .$$

Thus, $\mathsf{sVec}$ is a *symmetric monoidal category.* In fact, this pleasant property is one of the reasons that superalgebra is quite special from a mathematical point of view. Consider for a moment $\mathbb{Z}_{/n}$-graded algebra. As we have seen, graded vector spaces are not very interesting; hence, we want to include multiplicative structures. For this we need tensor products of objects, which leads us to braidings. Now the ordinary braiding $v \otimes u \longmapsto u \otimes v$ doesn't allow the multiplication to interact with the grading, so we'd like to find another braiding. However, the only case for which we get a *symmetric* braiding is $n = 2$: this is the supercase.

**Symmetric and exterior algebras.** We have already discussed direct sums and tensor products in $\mathsf{sVec}$. There are two more ways to construct new super vector spaces out of an old one: taking the symmetric and exterior algebra. They are constructed as usual, taking into account the braiding.

Consider the $n$-fold tensor product $V^{\otimes n}$ of copies of a super vector space $V$. The symmetric group $S_n$ acts on $V^{\otimes n}$ by permutation via the braiding:

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = (-1)^N \, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \ . \tag{2.10}$$

The integer $N$ determines the overall sign resulting from the braiding. For example, if $\sigma$ is a transposition $\sigma = (i, i+1)$ of two neighbours,

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) = (-1)^{p(v_i)\, p(v_{i+1})} \, v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n \ .$$

In general $N$ is given by the number of pairs $(i, j)$ such that $i < j$, $\bar{v_i} = \bar{v_j} = \bar{1}$, and $\sigma(i) > \sigma(j)$. In words: we pick up a sign whenever we move two odd elements past each other.[2]

Now take the quotient by the action of the permutation group:

$$\mathrm{Sym}^n V \coloneqq V^{\otimes n}/S_n = V^{\otimes n}/J_{\mathrm{sym}} \ ,$$

where $J_{\mathrm{sym}}$ denotes the ideal of $V^{\otimes n}$ generated by elements of the form $v \otimes v' - (-1)^{\bar{v}\,\bar{v'}} \, v' \otimes v$. In terms of the decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$ we have

$$\mathrm{Sym}^n V = \bigoplus_{k=0}^{n} \mathrm{Sym}^k V_{\bar{0}} \underset{\mathbb{K}}{\otimes} \Lambda^{n-k} V_{\bar{1}} \ ,$$

where the symmetric and exterior algebra on the right are the usual ones for ordinary vector spaces. The *symmetric algebra* is the direct sum

$$\mathrm{Sym}^\bullet V = \bigoplus_{n \geq 0} \mathrm{Sym}^n V \ .$$

Similarly, $\Lambda^n V$ is given by the quotient of the 'signed' action of $S_n$ on $V^{\otimes n}$:

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \operatorname{sign} \sigma \, (-1)^N \, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \ .$$

If $J_{\mathrm{ext}}$ is the ideal of $V^{\otimes n}$ generated by elements of the form $v \otimes v' + (-1)^{\bar{v}\,\bar{v'}} \, v' \otimes v$ we now define (cf. (1.3))

$$\Lambda^n V \coloneqq V^{\otimes n}/J_{\mathrm{ext}} = \bigoplus_{k=0}^{n} \Lambda^k V_{\bar{0}} \underset{\mathbb{K}}{\otimes} \mathrm{Sym}^{n-k} V_{\bar{1}} \ ,$$

[2] Clearly there is something to check here: the result should not depend on the precise way in which we get there. This can be done by induction on $n$, using that any permutation $\sigma \in S_n$ can be decomposed in terms of transpositions and that the braiding is symmetric. Details can be found in §3.1 of Varadarajan [13].

and the *exterior algebra* is given by

$$\Lambda^\bullet V = \bigoplus_{n\geq 0} \Lambda^n V \ .$$

Notice that this is an *infinite* direct sum since two odd elements of $V$ *commute* in $\Lambda^\bullet V$. This is the reason that we cannot simply generalize the description of the determinant as the scalar by which a linear map acts on the top exterior power.

Now we can give a rigorous characterization of the free commutative $A$-algebra:

$$A[t_1,\cdots,t_p|\theta_1,\cdots,\theta_q] \coloneqq \mathrm{Sym}^\bullet A^{p|q} = \mathrm{Sym}^\bullet \mathbb{K}^p \underset{\mathbb{K}}{\otimes} \Lambda^\bullet \mathbb{K}^q \underset{\mathbb{K}}{\otimes} A \ .$$

Since $\mathrm{Sym}^\bullet \mathbb{K}^p \cong \mathbb{K}[t_1,\cdots,t_p]$, and because $\Lambda^\bullet \mathbb{K}^q \cong \Lambda_q$ is the precise description of the Grassmann algebra in $q$ generators, we recover (2.8).

### 2.2.3 The categories $\mathsf{sMod}$ of left and right $A$-supermodules

Together with parity-preserving $A$-module maps, left supermodules over a superalgebra $A$ also form a category. We will denote this category by ${}_A\mathsf{sMod}$, and use the shorthand ${}_AM$ to indicate that $M$ is a left $A$-supermodule. Similarly we have a category $\mathsf{sMod}_A$ of right supermodules $M_A$.

Let $A$ be a supercommutative algebra. As in ordinary algebra, any $M_A$ can be turned into a left supermodule, but we have to use the braiding: if $M\otimes A \longrightarrow M$ is the action on $M_A$, the corresponding left supermodule ${}_AM$ with action $A\otimes M \longrightarrow M$ is defined via the commutative diagram

$$\begin{tikzcd} M\otimes A \arrow[r, "\gamma"] \arrow[dr] & A\otimes M \arrow[d] \\ & M \end{tikzcd}$$

Explicitly: $a\cdot m \coloneqq (-1)^{\bar m\bar a}\, m\cdot a$. This defines an action since $A$ is supercommutative, and gives an isomorphism of categories ${}_A\mathsf{sMod} \xrightarrow{\sim} \mathsf{sMod}_A$ because the braiding is symmetric. We can freely switch between left- and right modules as usual.

We can define the parity reversing functor $\Pi$ on $\mathsf{sMod}_A$ as in (2.3): given a right module with action $M\otimes A\longrightarrow M$ we have for the parity-reversed module $\Pi\, M = \mathbb{K}^{0|1}\otimes_{\mathbb{K}} M$ that

$$\mathrm{Hom}\Big((\mathbb{K}^{0|1}\otimes M)\otimes A\,,\,\Pi\, M\Big) \cong \mathrm{Hom}\Big(\mathbb{K}^{0|1}\otimes (M\otimes A), \Pi\, M\Big)$$

so there are no signs involved for right $A$-modules. However, for left modules we have to be a bit more careful and define $\Pi$ on ${}_AM$ by going via $\mathsf{sMod}_A$:

$$\begin{tikzcd} \mathsf{sMod}_A \arrow[r, "\sim"] \arrow[d, "\Pi"'] & {}_A\mathsf{sMod} \arrow[d, dotted, "\Pi"] \\ \mathsf{sMod}_A \arrow[r, "\sim"] & {}_A\mathsf{sMod} \end{tikzcd} \qquad \begin{tikzcd} (-1)^{\bar a\bar m}\, m\cdot a \arrow[d, mapsto] & a\cdot m \arrow[l, mapsto, "\gamma"'] \arrow[d, dotted, mapsto] \\ (-1)^{\bar a\bar m}\, \Pi(m)\cdot a \arrow[r, mapsto, "\gamma"] & (-1)^{\bar a}\, a\cdot \Pi(m) \end{tikzcd}$$

where we use $p\big(\Pi(m)\big) = \bar m + \bar 1$. Thus, the left action of $a\in A$ on $\Pi\, M$ is $(-1)^{\bar a}$ times its action on $M$.

The direct sums of two supermodules is just their direct sum (2.1) as super vector spaces. This turns ${}_A\mathsf{sMod}$ and $\mathsf{sMod}_A$ into abelian categories.

For the tensor product we have to make sure that the module-structure is compatible with the braiding. To avoid signs we define

$$M \underset{A}{\otimes} N \coloneqq (M\underset{\mathbb{K}}{\otimes} N)/J \ ,$$

where the ideal $J \subseteq M \otimes_{\mathbb{K}} N$ is generated by elements of the form $(m \cdot a) \otimes n - m \otimes (a \cdot n)$. In other words, the tensor product of two $A$-modules is defined as the tensor product over $A$ of $M_A$ and ${}_A N$. The unit object for this tensor product is the $A$-bimodule $A$. When $A$ is a bialgebra, ${}_A\mathsf{sMod}$ and $\mathsf{sMod}_A$ have a monoidal structure, and in case $A$ is a Hopf algebra, this monoidal structure is symmetric.

## 2.3 Maps between supermodules

Let $A$ be a supercommutative algebra. As for super vector spaces, given two $A$-supermodules $M$ and $N$ we write $\mathrm{Hom}(M, N)$ (or, more properly, $\mathrm{Hom}_A(M, N)$) for the set of $A$-module maps that are parity preserving, and $\mathbf{Hom}(M, N)$ for the super vector space of *all* $A$-module maps. In the $\mathbb{Z}_{/2}$-graded setting a homogeneous internal $A$-module map satisfies

$$T(a\, m) = (-1)^{\bar{T}\bar{a}}\, T(m) \ .$$

Both $\mathrm{Hom}(M, N)$ and $\mathbf{Hom}(M, N)$ are $A$-modules via $(a \cdot T)(m) \coloneqq a \cdot T(m)$.

To find out more about the structure of these maps let's take $M$ and $N$ free $A$-modules, $M = A^{p|q}$ and $N = A^{r|s}$, with bases $\{e_i\}$ and $\{f_j\}$, and let $T \in \mathbf{Hom}(M, N)$. It's practical to consider $M$ and $N$ as *right* modules over $A$ and expand elements and supermodule-maps in the unusual order

$$m = \sum e_i\, m^i \quad \text{and} \quad T(e_i) = \sum f_j\, T^j{}_i \ , \tag{2.11}$$

so that $m$ is represented by the column vector $m^i$ and $T$ by the $(r+s) \times (p+q)$ matrix $T^i{}_j$, with all coefficients in $A$. With this convention (2.11) we now have

$$T(m) = T\big(\sum e_i\, m^i\big) = \sum T(e_i)\, m^i = \sum f_j\, T^j{}_i\, m^i = \sum f_j\, (T\, m)^j \ ,$$

which says that the action of $T$ on $m$ is represented by the product of the components. Any other order in (2.11) would have forced us to keep track of signs already at this point. In addition, our convention works nicely for composition and gives $(S \circ T)^i{}_k = \sum S^i{}_j\, T^j{}_k$. For this reason it is convenient to consider with *right* modules when we want to represent maps by matrices.

By comparing the parities in the equations of (2.11) we see that $p(m_i) = p(m) + p(e_i)$ for each $i$. We can also tell something about the parities of the entries $T^i{}_j$. If $T$ is an even (parity-preserving) map in $\mathbf{Hom}(M, N)$, for the parities to match we see that its matrix has to be of the form

$$\operatorname{mat} T = \begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} r \\ s \end{array} & \left(\begin{array}{c|c} \text{even} & \text{odd} \\ \hline \text{odd} & \text{even} \end{array}\right) \end{array} \ , \tag{2.12}$$

where the size of the blocks is also indicated. For $T$ odd the parity of the blocks is reversed.

In particular, since super vector spaces are just supermodules over $A = \mathbb{K} \oplus 0$, this gives for homogeneous maps $T \in \mathbf{Hom}_{\mathbb{K}}(V, W)_{\bar{0}}$ and $S \in \mathbf{Hom}_{\mathbb{K}}(V, W)_{\bar{1}}$ matrices of the form

$$\operatorname{mat} T = \left(\begin{array}{c|c} \text{even} & 0 \\ \hline 0 & \text{even} \end{array}\right) \qquad \text{and} \qquad \operatorname{mat} S = \left(\begin{array}{c|c} 0 & \text{even} \\ \hline \text{even} & 0 \end{array}\right)$$

with respect to bases with (2.7). We already came across the matrix on the left in Section 1.1: in our heuristic derivation of the formula for the Berezinian we used such an even map of super vector spaces, cf. (1.7).

As in ordinary linear algebra, there are some operations we can perform on maps of $A$-modules. Again we have to be careful to take into account the signs we get by interchanging odd elements.

### 2.3.1 Duals and transposition

For $M$ a module over a superalgebra $A$, the *dual* of $M$ consists of the linear functionals on $M$:

$$M^\vee := \mathbf{Hom}_A(M, A).$$

Note that we use the inner hom: we also allow for odd functionals. For $M = A^{p|q}$ a standard free $A$-module this gives an identification $M^\vee \cong A^{p|q}$ by defining homogeneous basis elements $e^i$ for $M^\vee$ by $e^i(e_j) = \delta^i_j$. Notice that this in particular implies that the dual basis elements have parity $p(e^i) = p(e_i)$. In view of our convention (2.11) to expand $m \in M$ as $m = \sum e_i\, m^i$ we can use the opposite order $\omega = \sum \omega_i\, e^i$ for the expansion of $\omega \in M^\vee$: this ensures that $\omega(m) = \sum \omega_i\, m^i$ doesn't involve any signs. From this point of view, if $M$ is a right module, it's quite natural to consider $M^\vee$ as a left module. However, below we will see that it can be useful to view both $M$ and $M^\vee$ as right modules.

Given two $A$-modules $M$ and $N$ we have a map

$$\alpha\colon N \underset{A}{\otimes} M^\vee \longrightarrow \mathbf{Hom}_A(M, N)\ , \quad n \otimes \omega \longmapsto \Big[\ m \mapsto n\,\omega(m)\ \Big]\ . \tag{2.13}$$

In case $M$ is free and finitely generated, say $M = A^{p|q}$, this map has an inverse:

$$\alpha^{-1}\colon \mathbf{Hom}_A(M, N) \longrightarrow N \underset{A}{\otimes} M^\vee\ , \quad T \longmapsto \sum T(e_i) \otimes e^i\ . \tag{2.14}$$

If, in addition, $N = A^{r|s}$ with homogeneous basis $\{f_j\}$ we can further write this, using (2.11), as

$$T \overset{\delta}{\longmapsto} \sum f_j\, T^j{}_i \otimes e^i = \sum f_j \otimes T^j{}_i\, e^i\ .$$

The isomorphism $\alpha$ explains the minus sign in the tensor product (2.9) of two maps of super vector spaces: the action of the tensor product on two inner hom sets is given by

$$\begin{aligned}\mathbf{Hom}_{\mathbb{K}}(V, W) \otimes \mathbf{Hom}_{\mathbb{K}}(V', W') &\xrightarrow{\alpha^{-1}\otimes\alpha^{-1}} W \otimes V^\vee \otimes W' \otimes V'^\vee\\ &\xrightarrow{1\otimes\gamma\otimes 1} W \otimes W' \otimes V^\vee \otimes V'^\vee\\ &\xrightarrow{\ \cong\ } W \otimes W' \otimes (V \otimes V')^\vee\\ &\xrightarrow{\alpha\otimes\alpha} \mathbf{Hom}_{\mathbb{K}}(V \otimes V', W \otimes W')\end{aligned} \tag{2.15}$$

The sign is the result of the braiding used in the second line.

**Higher duals.** We can go on and consider the second dual $M^{\vee\vee} := (M^\vee)^\vee$ of $M = A^{p|q}$. The homogeneous basis $\{e'_i\}$ of $M^{\vee\vee}$ is defined by $e'_i(e^j) = \delta^j_i$; hence, $p(e'_i) = p(e_i)$. The map from $M$ into its double dual uses the braiding. First we compute

$$\begin{aligned} e_i \otimes e^j \ &\overset{\gamma}{\longmapsto}\ (-1)^{\bar{e_i}\,\bar{e_j}}\, e^j \otimes e_i\\ &\overset{\mathrm{ev}}{\longmapsto}\ (-1)^{\bar{e_i}\,\bar{e_j}}\, \delta^j_i \ = \ (-1)^{\bar{e_i}}\, \delta^j_i \ = \ (-1)^{\bar{e_i}}\, e'_i(e^j)\ ,\end{aligned}$$

where we use the evaluation map $M^\vee \otimes M \xrightarrow{\mathrm{ev}} A$ in the second step. For free, finitely generated modules, this calculation shows us that we should define the biduality isomorphism as

$$\beta\colon M \longrightarrow M^{\vee\vee}\ , \quad e_i \longmapsto (-1)^{\bar{e_i}}\, e'_i\ . \tag{2.16}$$

In super linear algebra, free supermodules of finite rank are isomorphic to their bidual, but the isomorphism involves a sign which does not arise in ordinary linear algebra.

**Transposition.** Given a map $T \in \mathbf{Hom}(M, N)$ of $A$-modules $M = A^{p|q}$ and $N = A^{r|s}$ as before we can form its transpose via the composition

$$\mathbf{Hom}_A(M, N) \xrightarrow{\gamma\circ\alpha^{-1}} M^\vee \otimes_A N \xrightarrow{1\otimes\beta} M^\vee \otimes_A N^{\vee\vee} \xrightarrow{\ \alpha\ } \mathbf{Hom}_A(N^\vee, M^\vee)\ .$$

At the start of this section we saw that when $M$ is a right $A$-module over a supercommutative algebra it is quite natural to view $M^\vee$ as a *left* $A$-module. On the other hand, we have also seen that *right* modules are more convenient when we want to write down matrices. This is what we will do.

First notice that the composition $\gamma \circ \alpha^{-1}$ acts on $T$ as

$$T \longmapsto \sum (-1)^{(\bar{T}+\bar{1})\,\bar{e_i}}\, e^i \otimes T(e_i) = \sum (-1)^{(\bar{T}+\bar{1})\,\bar{e_i}}\; e^i \otimes f_j\, T^j{}_i\ . \tag{2.17}$$

Here we use $p\big(T(e_i)\big) = p(T) + p(e_i)$, $p(e^i) = p(e_i)$ and $p(e_i)^2 = p(e_i)$.

Write $\langle\,\cdot\,,\,\cdot\,\rangle$ for the pairing between dual elements. We can find the relation between the matrix of $T^\vee$ and the matrix of $T$ as follows. On the one hand we have that

$$\big\langle e'_i, T^\vee(f^j)\big\rangle = \big\langle e'_i, \sum e^k\, (T^\vee)_k{}^j\big\rangle = \sum \big\langle e'_i, e^k\big\rangle\, (T^\vee)_k{}^j = (T^\vee)_i{}^j\ .$$

On the other hand, transposing $T$ and using $\beta$,

$$\begin{aligned}
\langle e'_i, T^\vee(f^j)\rangle &= (-1)^{\bar{e_i}}\,(-1)^{\bar{T}\,\bar{e_i}}\,(-1)^{(\bar{T}+\bar{e_i})\,\bar{f_j}}\,\big\langle f^j, T(e_i)\big\rangle \\
&= (-1)^{\bar{T}\,(\bar{e_i}+\bar{f_j})}\,(-1)^{\bar{e_i}\,(\bar{f_j}+\bar{1})}\,\big\langle f^j, \sum f_k\, T^k{}_i\big\rangle \\
&= (-1)^{\bar{T}\,(\bar{e_i}+\bar{f_j})}\,(-1)^{\bar{e_i}\,(\bar{f_j}+\bar{1})}\; T^j{}_i\ .
\end{aligned}$$

Therefore we find that the matrices of a map and its transpose are related by

$$(T^\vee)_i{}^j = (-1)^{\bar{T}\,(\bar{e_i}+\bar{f_j})}\,(-1)^{\bar{e_i}\,(\bar{f_j}+\bar{1})}\; T^j{}_i\ . \tag{2.18}$$

Comparing the parities on both sides we see that $p(T^\vee) = p(T)$.

To see what (2.18) boils down to concretely, write the matrix of $T$ in block form with respect to the bases $\{e_i\}$ and $\{f_j\}$:

$$\operatorname{mat} T = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right)\ .$$

Let superscript $t$ denote the usual transpose of a matrix. Since the index $j$ in (2.18) labels the columns of the matrix of $T^\vee$, the relation (2.18) translates to

$$\operatorname{mat}(T^\vee) = (\operatorname{mat} T)^{st} := \left(\begin{array}{c|c} K^t & (-1)^{\bar{T}}\, M^t \\ \hline -(-1)^{T}\, L^t & N^t \end{array}\right)\ , \tag{2.19}$$

where we have defined the *supertranspose* of the matrix of $T$. Thus, we find

$$(\operatorname{mat} T)^{st} = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right)^{st} = \begin{cases} \left(\begin{array}{c|c} K^t & M^t \\ \hline -L^t & N^t \end{array}\right) & \text{for } T \in \mathbf{Hom}(M,N)_{\bar{0}}\ , \\ \left(\begin{array}{c|c} K^t & -M^t \\ \hline L^t & N^t \end{array}\right) & \text{for } T \in \mathbf{Hom}(M,N)_{\bar{1}}\ . \end{cases}$$

Notice that supertransposition is an operation of order four. The identity $T^{\vee\vee} = T$ still holds, provided it is interpreted as saying that the diagram

$$\begin{tikzcd}
M \arrow[r, "T"] \arrow[d] & N \arrow[d] \\
M^{\vee\vee} \arrow[r, "T^{\vee\vee}"] & N^{\vee\vee}
\end{tikzcd}$$

commutes. The transpose of a composition involves a sign: $(S \circ T)^\vee = (-1)^{\bar{S}\bar{T}}\, T^\vee \circ S^\vee$.

### 2.3.2 Supertrace

Consider an endomorphism $T \in \mathbf{End}\, M := \mathbf{Hom}(M, M)$ of $M = A^{p|q}$. In ordinary linear algebra, the trace of an endomorphism of $M$ can be defined as the composition

$$\mathbf{End}_A\, M \xrightarrow{\alpha^{-1}} M \underset{A}{\otimes} M^\vee \xrightarrow{\cong} M^\vee \underset{A}{\otimes} M \xrightarrow{\text{ev}} A\ . \tag{2.20}$$

To pass to super linear algebra, all we have to do is to use the braiding. Thus, in terms of the basis $\{e_i\}$ for $M$, the first and second step are given by (2.17) with $N = M$ (so that $f_j = e_j$). Applying the evaluation map on the result we get the *supertrace*

$$\operatorname{str} T := \sum (-1)^{(\bar{T}+\bar{1})\,\bar{e}_i}\, T^i{}_i\ .$$

From the description (2.20) it is clear that the result does not depend on the choice of basis. In terms of the matrix

$$\operatorname{mat} T = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right)$$

the supertrace of $T$ is given by

$$\operatorname{str} T = \operatorname{tr} K - (-1)^{\bar{T}}\ \operatorname{tr} N\ , \tag{2.21}$$

where ‘tr’ denotes the ordinary trace. The supertrace is linear and satisfies $\operatorname{str}(T^\vee) = \operatorname{str} T$ and it has the cyclic property $\operatorname{str}(S \circ T) = (-1)^{\bar{S}\bar{T}} \operatorname{str}(T \circ S)$. Using the biduality isomorphism $\beta$ we can also write this as $\operatorname{str}(S \circ T) = \operatorname{str}(T \circ S^{\vee\vee})$.

The parities of the blocks of $T$ (cf. (2.12)) show that the supertrace is an even map of $A$-modules: it maps $\operatorname{str}(\mathbf{End}\, M)_{\bar{\imath}} \subseteq A_{\bar{\imath}}$.

### 2.3.3 Inversion

For $M$ an $A$-module, let $\mathbf{Aut}\, M \subseteq \mathbf{End}\, M$ be the group of internal automorphisms of $M$, so including odd invertible maps. Consider again an endomorphism $T \in \mathbf{End}\, M$ with block form

$$\operatorname{mat}_A T = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right)\ ,$$

where we have added the subscript $A$ to stress that the matrix components of $T$ have entries in $A$. To derive a criterion for the invertibility of $T$ in terms of its matrix components we take a closer look at the structure of $A$.

The subalgebra $J \subseteq A$ that is generated by the odd elements,

$$J := A_{\bar{1}}^2 \oplus A_{\bar{1}} \subseteq A_{\bar{0}} \oplus A_{\bar{1}}\ ,$$

is an nilpotent ideal of $A$. In particular this means that $J$ doesn’t contain the unit element of $A$, so it is a proper ideal of $A$. Taking the quotient of $A$ by $J$ we obtain an algebra $B := A/J$ which is also an $A$-module. We ‘extend scalars’ to $B$ by passing on to the $B$-module

$$_B M := {}_B B_A \underset{A}{\otimes} {}_A M\ ,$$

where we view $B$ as a $(B, A)$-bimodule. This gives a functor $_A\mathsf{sMod} \longrightarrow {}_B\mathsf{sMod}$ which sends $T \in \mathbf{Hom}_A(M, N)$ to $1 \otimes T \in \mathbf{Hom}_B(M, N)$.

Let’s go back to our inner endomorphism $T$ of $M$, and assume $T \in (\mathbf{End}\, M)_{\bar{0}}$ is even. Under the extension of scalars to $B$ the entries of its matrix are taken modulo $A_{\bar{1}}$. The parities (2.12) of the blocks imply that the resulting matrix is

$$\operatorname{mat}_B T = \left(\begin{array}{c|c} K \bmod A_{\bar{1}}^2 & 0 \\ \hline 0 & N \bmod A_{\bar{1}}^2 \end{array}\right)\ .$$

It is clear that this matrix is invertible when $T$ is so: its inverse is $(\mathrm{mat}_B\, T)^{-1} = \mathrm{mat}_B\,(T^{-1})$. In particular, $K$ and $N$ are invertible modulo $A_{\bar{1}}^2$.

Now suppose that $K$ and $N$ can be inverted modulo $A_{\bar{1}}^2$. Then $T$ is invertible modulo $J \subset A$, so there exists an $S \in \mathbf{End}\, M$ with $S \circ T = T \circ S = I_M + R$ for some $R \in \mathbf{End}\, M$ with entries in $J$. By the nilpotency of $J$ the series

$$I_M - R + R^2 - R^3 + \cdots$$

is finite and gives the inverse of $I_M + R$. This implies that $T$ is invertible with inverse given by $T^{-1} = (I_M + R)^{-1}\, S = S\,(I_M + R)^{-1}$. Moreover, the argument shows that the blocks $K$ and $N$ are invertible precisely when $T$ is.

**Proposition 2.1.** *Let $T \in (\mathbf{End}\, M)_{\bar{0}}$ be an even endomorphism of $M = A^{p|q}$ with block decomposition as before. Then $T$ is invertible if and only if the matrices $K$ and $N$ are invertible. Moreover, the inverse of $T$ is given by*

$$\mathrm{mat}_A\, T^{-1} = \left( \begin{array}{c|c} (K - L\,N^{-1}\,M)^{-1} & -K^{-1}\,L\,(N - M\,K^{-1}\,L)^{-1} \\ \hline -N^{-1}\,M\,(K - L\,N^{-1}\,M)^{-1} & (N - M\,K^{-1}\,L)^{-1} \end{array} \right) . \tag{2.22}$$

*Proof.* We have already established the first part in the preceding discussion. The second statement follows by inspection. Indeed, it is clear that (2.22) is the right inverse of the matrix of $T$; thus, it is also its left inverse. $\square$

Notice that, although $L$ and $M$ are not invertible, the combinations $K - L\,N^{-1}\,M$ and $N - M\,K^{-1}\,L$ are. For example,

$$(K - L\,N^{-1}\,M)^{-1} = K^{-1}\,(I_{M_{\bar{0}}} - L\,N^{-1}\,M\,K^{-1})^{-1} = K^{-1} \sum_{n \geq 0} (L\,N^{-1}\,M\,K^{-1})^n ,$$

with $I_{M_{\bar{0}}}$ the unit matrix on $M_{\bar{0}}$. The power series is finite by supercommutativity of $A$ and the nilpotency of the entries of $L$ and $M$. (This can also be used for a direct check that (2.22) is the left inverse of $T$.)

Equation (2.22) is very general and holds also for noncommutative algebra. Notice that, in case $M \cong \mathbb{K}^2$, when $K = k, \cdots, N = n$ are ordinary scalars, the equation can be written as

$$\mathrm{mat}_{\mathbb{K}}\, T^{-1} = \frac{1}{k\,n - l\,m} \begin{pmatrix} n & -l \\ -m & k \end{pmatrix} .$$

# Chapter 3

# The Berezinian

Having covered all necessary preliminaries from super linear algebra, we can turn to the Berezinian. Everything we have discussed in Chapter 2 is a rather straightforward generalization from ordinary linear algebra and commutative algebra, carefully keeping track of minus signs via braiding isomorphisms. This is not the case for the Berezinian.

In Chapter 1 we have seen how the determinant of an endomorphism of an ordinary vector space can be described in a manifestly basis-independent way via the induced action on the top exterior power of the vector space. As we have seen in the previous chapter, the exterior algebra of a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is given by

$$\Lambda^\bullet V = \Lambda^\bullet V_{\bar{0}} \underset{\mathbb{K}}{\otimes} \operatorname{Sym}^\bullet V_{\bar{1}} .$$

The symmetric algebra $\operatorname{Sym}^\bullet V_{\bar{1}}$ is infinitely generated (as $\mathbb{K}$-module), so that a super vector space has no maximal exterior power.

Nevertheless, the invariant description of the determinant can be generalized to supermathematics. It requires some familiarity with homological algebra to see how this works.

In this chapter we will

- characterize the formula for the Berezinian by three axioms and show its existence and uniqueness;
- give an invariant description of the Berezinian via the induced action on the Berezinian of a supermodule; and
- explain how the Berezinian of a supermodule can be computed using homological algebra.

In Chapter 4 we illustrate the methods that we develop in the present chapter, and in Chapter 5 we will prove that the invariant formulation really describes the Berezinian.

## 3.1 Definition of the Berezinian

In Chapter 1 we have motivated the formula for the Berezinian from the viewpoint of quantum physics. In this section we take a rigorous approach. We define the Berezinian by three axioms. We prove that these axioms uniquely determine the formula for the Berezinian. This proves the uniqueness of the Berezinian. Moreover, we demonstrate that the formula indeed satisfies the axioms defining the Berezinian, establishing its existence.

The set-up is as follows. Let $A$ be a supercommutative superalgebra, and let $M$ be a free supermodule over $A$ of rank $p|q$. Consider an *even*, *invertible* endomorphism $T \in \operatorname{End}_A(M)$. With respect to a homogeneous basis for $M$ the matrix for $T$ is given by

$$\operatorname{mat}_A T = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) . \tag{3.1}$$

### 3.1.1 Three axioms for the Berezinian

Shortly we will define the Berezinian by requiring three axioms. Before we give the definition we motivate these axioms.

The first axiom concerns automorphisms $T$ for which the decomposition (3.1) is block-diagonal:

$$\operatorname{mat}_A T = \left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right) .$$

Recall that in ordinary linear algebra, for $V$ a vector space over $\mathbb{K}$, the diagram

$$\begin{tikzcd} \mathrm{GL}\, V \arrow[r, "\det"] & \mathbb{K} \\ \operatorname{End} V \arrow[u, "\exp"] \arrow[r, "\mathrm{tr}"] & \mathbb{K} \arrow[u, "e^{\cdot}"'] \end{tikzcd}$$

commutes: we can write $\det K = e^{\mathrm{tr}\, \log K}$ for a matrix $K$. We want this relation to extend to superalgebra, so that the diagram

$$\begin{tikzcd} \mathrm{GL}\, M \arrow[r, "\mathrm{Ber}"] & A \\ \operatorname{End} M \arrow[u, "\exp"] \arrow[r, "\mathrm{str}"] & A \arrow[u, "e^{\cdot}"'] \end{tikzcd}$$

commutes for suitably defined vertical maps extending the ordinary exponential. For block-diagonal $T$ we have

$$\log(\operatorname{mat}_A T) = \left(\begin{array}{c|c} \log K & 0 \\ \hline 0 & \log N \end{array}\right) .$$

The expression $\mathrm{str}\, T = \mathrm{tr}\, K - \mathrm{tr}\, N$ for the supertrace of an even endomorphism (see (2.21)) gives

$$\mathrm{Ber}\,(\operatorname{mat} T) = e^{\mathrm{str}\, \log(\operatorname{mat} T)} = e^{\mathrm{tr}\, \log K} \left(e^{\mathrm{tr}\, \log N}\right)^{-1} = \det K \cdot \det N^{-1} .$$

Thus, if we want to maintain the relation between the (super)trace and the (super)determinant, this equation is forced by the categorical approach. This is the first axiom we require for the Berezinian.

Secondly, the Berezinian should give an invariant of $T$, so it must be independent of the choice of basis for $M$. This can be arranged by the natural requirement that the Berezinian be multiplicative. This is the second axiom.

As we will see in Section 3.1.2, for given rank $p|q$ the first two axioms already determine the formula for the Berezinian of an even automorphism of $M \cong A^{p|q}$. The third axiom connects the Berezinians for $A$-modules of different rank. It can be formulated as follows. Consider a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of $A$-modules, so that the map $M' \longrightarrow M$ is injective, $M \longrightarrow M''$ is surjective, and the two are related by $\ker(M \longrightarrow M'') = \operatorname{im}(M' \longrightarrow M)$. When the modules are free, exactness implies that the sequence splits, and $M = M' \oplus M''$. Consider an even automorphism

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "T'"] & M \arrow[r] \arrow[d, "T"] & M'' \arrow[r] \arrow[d, "T''"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

In other words, $T = T' \oplus T''$. The third requirement is

$$\operatorname{Ber} T = \operatorname{Ber} T' \cdot \operatorname{Ber} T'' \ .$$

This is a compatibility condition for the Berezinian and direct sums. See also the remark below.

**Definition.** The *Berezinian* $\operatorname{Ber}\colon \operatorname{Aut}_A M \longrightarrow A_{\bar{0}}$ by definition satisfies the following axioms:

i) For block-diagonal matrices we have

$$\operatorname{Ber} \begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix} = \det K \cdot \det N^{-1} \ ; \tag{3.2}$$

ii) Multiplicativity: given $T, S \in \operatorname{Aut} M$ we have

$$\operatorname{Ber}(S \circ T) = \operatorname{Ber} S \cdot \operatorname{Ber} T \ ;$$

iii) Compatibility with direct sums: for an automorphism of the short exact sequence of free $A$-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ we have the relation

$$\operatorname{Ber} T = \operatorname{Ber} T' \cdot \operatorname{Ber} T'' \ . \tag{3.3}$$

We have to show that this definition is not vacuous, and completely determines the Berezinian.

**Remark.** There is some redundancy in the axioms. As stated above, the first two axioms must hold for all $p|q$. The proofs of Section 3.1.2 will show that, these axioms completely determine the formula of the Berezinian, and in Section 3.1.3 we will see that this formula implies that axiom (iii) holds. Thus, a more minimal approach would be to remove axiom (iii). On the other hand, in Chapter 5 we will prove that the invariant formulation describes the Berezinian by first showing it satisfies axioms (ii) and (iii), and use this to verify axiom (i). The definition above is not incorrect, and we will stick with it.

### 3.1.2 Uniqueness

Our first task is to show that axioms (i)–(iii) uniquely characterize the Berezinian.

**Theorem 3.1** (Uniqueness of the Berezinian.)**.** *Let $M$ be a free $A$-supermodule and consider an even automorphism $T\colon M \longrightarrow M$ of $M$. Then axioms (i) and (ii) of the Berezinian imply that*

$$\operatorname{Ber} T = \det(K - L\,N^{-1}\,M) \cdot \det N^{-1} \in A_{\bar{0}} \ , \tag{3.4}$$

*where the matrices in the determinants are the block components of the matrix (3.1) of $T$ with respect to some homogeneous basis for $M$.*

A few comments are in order before we prove the theorem. Firstly, in Proposition 2.1 we have seen that $T$ is an automorphism of $M$ if and only if both $K$ and $N$ are invertible. This allows us to take the inverse of $N$ in the above. Secondly, seems like we do not need the inverse of $K$, so the roles of $K$ and in (3.4) appear to be rather different — but see Corollary 3.3.

The proof of Theorem 3.1 involves the following special case, which we treat first.

**Lemma 3.2.** *For block triangular matrices representing an even automorphism, axioms (i) and (ii) of the Berezinian imply*

$$\operatorname{Ber} \left( \begin{array}{c|c} K & L \\ \hline 0 & N \end{array} \right) = \operatorname{Ber} \left( \begin{array}{c|c} K & 0 \\ \hline M & N \end{array} \right) = \det K \cdot \det N^{-1} \ .$$

*Proof.* We give the proof for the block upper-triangular case; the other case is obtained by transposition. Notice that we can write the matrix as a product

$$\left(\begin{array}{c|c} K & L \\ \hline 0 & N \end{array}\right) = \left(\begin{array}{c|c} K & 0 \\ \hline 0 & -N \end{array}\right)\left(\begin{array}{c|c} I_p & K^{-1}L \\ \hline 0 & -I_q \end{array}\right) ,$$

and we further have

$$\begin{aligned}\left(\begin{array}{c|c} I_p & K^{-1}L \\ \hline 0 & -I_q \end{array}\right) &= \left(\begin{array}{c|c} I_p & -\frac{1}{2}\,K^{-1}L \\ \hline 0 & I_q \end{array}\right)\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right)\left(\begin{array}{c|c} I_p & +\frac{1}{2}\,K^{-1}L \\ \hline 0 & I_q \end{array}\right) \\ &= \left(\begin{array}{c|c} I_p & -\frac{1}{2}\,K^{-1}L \\ \hline 0 & I_q \end{array}\right)\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right)\left(\begin{array}{c|c} I_p & -\frac{1}{2}\,K^{-1}L \\ \hline 0 & I_q \end{array}\right)^{-1}\end{aligned}$$

By multiplicativity, the product of the Berezinian of the conjugating matrices is $1 \in A$, so that

$$\operatorname{Ber}\left(\begin{array}{c|c} K & L \\ \hline 0 & N \end{array}\right) = \operatorname{Ber}\left(\begin{array}{c|c} K & 0 \\ \hline 0 & -N \end{array}\right) \operatorname{Ber}\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) = \operatorname{Ber}\left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right) .$$

Axiom (i) gives the desired result. □

*Proof of Theorem 3.1.* Together with the previous lemma and axioms (i) and (ii), the decomposition

$$\left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) = \left(\begin{array}{c|c} I_p & L\,N^{-1} \\ \hline 0 & I_q \end{array}\right)\left(\begin{array}{c|c} K - L\,N^{-1}\,M & 0 \\ \hline 0 & N \end{array}\right)\left(\begin{array}{c|c} I_p & 0 \\ \hline N^{-1}\,M & I_q \end{array}\right) ,$$

shows that

$$\operatorname{Ber}\left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) = \det(K - L\,N^{-1}\,M) \cdot \det N^{-1} .$$

Since the matrix of $T$ with respect to another homogeneous basis of $M$ is related by a similarity transformation via an even automorphism of $M$, the result does not depend on the choice of homogeneous basis. □

Applying the proof of Theorem 3.1 to the decomposition

$$\left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) = \left(\begin{array}{c|c} I_p & 0 \\ \hline M\,K^{-1} & I_q \end{array}\right)\left(\begin{array}{c|c} K & 0 \\ \hline 0 & N - M\,K^{-1}\,L \end{array}\right)\left(\begin{array}{c|c} I_p & K^{-1}L \\ \hline 0 & I_q \end{array}\right)$$

we find an alternative formula for the Berezinian of $T$:

**Corollary 3.3.** *In the set-up of Theorem 3.1, axioms (i) and (ii) of the Berezinian also lead to the formula*

$$\operatorname{Ber} T = \det(K) \cdot \det(N - M\,K^{-1}\,L)^{-1} . \tag{3.5}$$

### 3.1.3 Existence

Our second task is to prove that (3.4) satisfies axioms (i)–(iii) of the Berezinian.

**Theorem 3.4** (Existence of the Berezinian.)**.** *With the same set-up as in Theorem 3.1, the formula* (3.4) *determines a map* $\operatorname{Aut}_A M \longrightarrow A_{\bar{0}}$ *satisfying the three axioms defining the Berezinian.*

*Proof.* To see that the formula yields values in the even part $A_{\bar{0}}$ of $A$, notice that, since $T$ is even, we only take the determinant of matrices with coefficients in $A_{\bar{0}}$. The Leibniz formula (1.1) expresses the determinants as polynomials in the coefficients of these matrices; since multiplication preserves parity, we find that $\operatorname{Ber} T \in A_{\bar{0}}$. (In fact, the values of the Berezinian are units of $A_{\bar{0}}$ as follows from the multiplicativity of the Berezinian and the invertibility of $T$.)

Write $\mathrm{GL}_{p|q}\,A$ for the (multiplicative) group of *invertible*, *even* matrices with the usual block decomposition

$$X = \begin{array}{c c} & \begin{array}{cc} p & q \end{array} \\ \begin{array}{c} p \\ q \end{array} & \left(\begin{array}{c|c} K & M \\ \hline L & N \end{array}\right) \end{array} ;$$

from Proposition 2.1 we know that $X \in \mathrm{GL}_{p|q}\,A$ if and only if $K$ and $N$ are invertible. Define the map

$$\begin{aligned} \mathrm{B}\colon \mathrm{GL}_{p|q}\,A &\longrightarrow A_{\bar{0}} \\ X &\longmapsto \det(K - L\,N^{-1}\,M) \cdot \det N^{-1} \ . \end{aligned}$$

We have to show that this maps satisfies axioms (i)–(iii).

Axiom (i) is obvious. Next we verify that axioms (iii) holds. Consider $X' \in \mathrm{GL}_{p'|q'}\,A$ and $X'' \in \mathrm{GL}_{p''|q''}\,A$ given by

$$X' = \left(\begin{array}{c|c} K' & L' \\ \hline M' & N' \end{array}\right) , \quad X'' = \left(\begin{array}{c|c} K'' & L'' \\ \hline M'' & N'' \end{array}\right) .$$

We can arrange that the matrix of $X = X' \oplus X''$ is of the block-form

$$\left(\begin{array}{c|c} \text{even} & \text{odd} \\ \hline \text{odd} & \text{even} \end{array}\right)$$

by ordering the basis of $A^{p|q} \cong A^{p'|q'} \oplus A^{p''|q''}$ as follows: first we take the even ordered basis elements of $A^{p'|q'}$, then those of $A^{p''|q''}$, and then the odd elements of $A^{p'|q'}$ and finally those of $A^{p''|q''}$. With respect to this basis, $X$ is given by

$$X = \left(\begin{array}{cc|cc} K' & 0 & L' & 0 \\ 0 & K'' & 0 & L'' \\ \hline M' & 0 & N' & 0 \\ 0 & M'' & 0 & N'' \end{array}\right) .$$

The third axiom now follows from a direct calculation:

$$\begin{aligned} \mathrm{B}\,X &= \det\left( \begin{pmatrix} K' & 0 \\ 0 & K'' \end{pmatrix} - \begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix} \begin{pmatrix} N' & 0 \\ 0 & N'' \end{pmatrix}^{-1} \begin{pmatrix} L' & 0 \\ 0 & L'' \end{pmatrix} \right) \\ &\qquad \cdot \det \begin{pmatrix} N' & 0 \\ 0 & N'' \end{pmatrix}^{-1} \\ &= \det(K' - M'\,N'^{-1} L')\,\det(K'' - M''\,N''^{-1} L'')\,\det(N')^{-1}\,\det(N'')^{-1} \\ &= \mathrm{B}\,X' \cdot \mathrm{B}\,X'' \ . \end{aligned}$$

It is more work to show that the formula is multiplicative. We will follow the proof in §3.6 of Varadarajan [13]. It is convenient to introduce some notation. Write $G \coloneqq \mathrm{GL}_{p|q}\,A$. Consider the subgroups $G^+$, $G^0$ and $G^-$ of $G$ consisting of elements of the form

$$X^+ = \left(\begin{array}{c|c} I_p & L \\ \hline 0 & I_q \end{array}\right) , \quad X^0 = \left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right) , \quad \text{and } X^- \left(\begin{array}{c|c} I_p & 0 \\ \hline M & I_q \end{array}\right) ,$$

respectively. The proof of Theorem 3.1 shows that $G = G^+G^0G^-$. We will prove multiplicativity in six steps.

*Step one.* The first step is to notice that B is multiplicative on each of $G^+$, $G^0$ and $G^-$ separately. In fact, B is equal to $1 \in A_{\bar{0}}$ on $G^{\pm}$.

*Step two.* In terms of the matrices defined above,

$$\mathrm{B}(X^+X^0X^-) = \mathrm{B}\left(\begin{array}{c|c} K + L\,N\,M & LN \\ \hline NM & N \end{array}\right) = \det K \cdot \det N^{-1} \ ,$$

so that $\mathrm{B}(X^+X^0X^-) = 1 \cdot \mathrm{B}\,X^0 \cdot 1 = \mathrm{B}\,X^+ \cdot \mathrm{B}\,X^0 \cdot \mathrm{B}X^-$. Therefore, for $Y^+ \in G^+$ and $X \in G$ we obtain $\mathrm{B}(Y^+X) = \mathrm{B}\,Y^+ \cdot \mathrm{B}\,X$.

*Step three.* Next, from the calculation

$$\left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right)\left(\begin{array}{c|c} I_p & L \\ \hline 0 & I_q \end{array}\right) = \left(\begin{array}{c|c} I_p & K\,L\,N^{-1} \\ \hline 0 & I_q \end{array}\right)\left(\begin{array}{c|c} K & 0 \\ \hline 0 & N \end{array}\right)$$

we see that $G^0\,G^+ = G^+\,G^0$; hence for $Y^0 \in G^0$ our map further satisfies $\mathrm{B}(Y^0X) = \mathrm{B}\,Y^0 \cdot \mathrm{B}\,X$.

So far we have shown that $\mathrm{B}(YX) = \mathrm{B}\,Y \cdot \mathrm{B}\,X$ for all $X \in G$ and all $Y \in G^+G^0$. Indeed, it is not hard to verify this by direct computation. The key point of the proof is to show that $\mathrm{B}(YX) = \mathrm{B}\,Y \cdot \mathrm{B}\,X$ also holds for $Y \in G^-$.

*Step four.* It suffices to take $X \in G^+$: for general $X = X^+X^0X^-$ we then have

$$\mathrm{B}(YX) = \mathrm{B}(YX^+X^0X^-) = \mathrm{B}(YX^+) \cdot \mathrm{B}(X^0X^-) = \mathrm{B}\,Y \cdot \mathrm{B}\,X^+ \cdot \mathrm{B}(X^0X^-) = \mathrm{B}\,Y \cdot \mathrm{B}\,X \ .$$

Thus, we may restrict ourselves to

$$Y = \left(\begin{array}{c|c} I_p & 0 \\ \hline M & I_q \end{array}\right) \ , \quad X = \left(\begin{array}{c|c} I_p & L \\ \hline 0 & I_q \end{array}\right) \ .$$

*Step five.* Let $H$ be the additive group of $p \times q$ matrices with coefficients in $A_{\bar{1}}$. Our problem is further reduced by noticing that the map $H \longrightarrow G^+$ given by

$$L \longmapsto \left(\begin{array}{c|c} I_p & L \\ \hline 0 & I_q \end{array}\right)$$

is a homomorphism. This implies that we may further assume that $L$ has only one nonzero entry $\lambda \in A_{\bar{1}}$.

*Step six.* Since all entries of the product $M\,L$ are proportional to $\lambda$, $(M\,L)^2 = (L\,M)^2 = 0$. Therefore, $I_q + M\,L$ is invertible with inverse $(I_q + M\,L)^{-1} = I_q - M\,L$, and

$$I_p - L\,(I_p + L\,M)^{-1}M = I_p - L\,(I_p - L\,M)M = I_p - L\,M \ .$$

This yields

$$\begin{aligned} \mathrm{B}(Y\,X) &= \mathrm{B}\left(\begin{array}{c|c} I_p & L \\ \hline M & I_q + M\,L \end{array}\right) \\ &= \det(I_p - L\,(I_p + L\,M)^{-1}M)\ \det(I_q + M\,L)^{-1} \\ &= \det(I_p - L\,M)\ \det(I_q - M\,L) \ . \end{aligned}$$

Once more using $(M\,L)^2 = (L\,M)^2 = 0$ we can further write this as

$$\begin{aligned} \mathrm{B}(Y\,X) &= (1 - \mathrm{tr}\,L\,M)\,(1 - \mathrm{tr}M\,L) \\ &= 1 - \mathrm{tr}\,L\,M - \mathrm{tr}\,M\,L \ . \end{aligned}$$

Since both matrices $L$ and $M$ are odd, using the cyclic property of this (ordinary) trace leads to a minus sign, so that we conclude that

$$\mathrm{B}(Y\,X) = 1 = \mathrm{B}\,Y \cdot \mathrm{B}\,X \ .$$

This finishes the proof of the theorem. □

## 3.2 Alternative definition of the Berezinian

Recall the set-up from the previous section. Let $A$ be a supercommutative superalgebra, and let $M$ be a free supermodule over $A$ of rank $p|q$. Consider an *even*, *invertible* endomorphism $T\colon M \longrightarrow M$. With respect to a homogeneous basis for $M$ the matrix for $T$ is given by

$$\operatorname{mat}_A T = \left(\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right) .$$

In Section 3.1 we have seen that the Berezinian can be characterized as the unique map $\operatorname{Ber}\colon \operatorname{Aut}_A M \longrightarrow A_{\bar{0}}$ satisfying three characterizing axioms. The first axiom requires the use of a homogeneous basis for $M$.

Theorems 3.1 and 3.4 tell us that these axioms are equivalent to the formula

$$\operatorname{Ber} T = \det(K - L\, N^{-1}\, M) \cdot \det N^{-1}$$

for the Berezinian of $T$. Although the multiplicativity implies that the outcome of this formula does not depend on the choice of basis yielding the matrix for $T$, we would like to find a manifestly invariant formulation of the Berezinian. Our approach comes from §1.10(B) of Deligne and Morgan [3].

The main idea is to extend the invariant description of the determinant as the induced action of a linear map on the top exterior power of the space on which it acts. Indeed, if $V$ is a $p$-dimensional vector space over $\mathbb{K}$, we have $\det V \coloneqq \Lambda^p\, V$, and the determinant is given by

$$\begin{aligned} \operatorname{End} V &\longrightarrow \operatorname{End}(\det V) \cong \mathbb{R} \ , \\ T &\longmapsto \Lambda^p\, T = \det T \cdot \mathrm{id}_{\det V} \ . \end{aligned}$$

To generalize this to the supersetting, we have define the Berezinian of a free supermodule.

**A first look at the Berezinian of a supermodule.** Let $M$ be a free supermodule of rank $p|q$ over a supercommutative algebra $A$. The Berezinian $\operatorname{Ber} M$ of $M$ can be defined using methods from homological algebra. Perhaps a bit surprisingly, it involves the *symmetric* algebra on $M$: define the superalgebra

$$R \coloneqq \operatorname{Sym}^\bullet (M^\vee) \ .$$

Here $M^\vee$ is the dual of $M$ (see Section 2.3.1). To see what this means concretely, let's pick a homogeneous basis for $M$, and denote the *dual* basis for $M^\vee$ by $t_1, \cdots, t_p, \theta_1, \cdots, \theta_q$. Then, using the notation from Section 2.2.2, $R$ is given by the free supercommutative algebra

$$R = A[t_1, \cdots, t_p | \theta_1, \cdots, \theta_q] \ .$$

In terms of this algebra, the Berezinian of $M$ is given by the following mysterious formula:

$$\operatorname{Ber} M \coloneqq \mathbf{Ext}^p_R(A, R) \ . \tag{3.6}$$

At this point, this definition may not mean anything to you. In Section 3.3 we will discuss the prerequisites from homological algebra we need to understand (3.6). Nevertheless, even then the definition of $\operatorname{Ber} M$ may still appear rather opaque. In Chapter 4 we will remedy this by explicitly computing the Berezinian for some concrete examples; for instance, we will see that in the case of a purely even supermodule, where $q = 0$, we recover the top exterior power.

Let us start by examining the constituents of (3.6). We have already defined $R$ in terms of operations we know from the previous chapter. Clearly, $R$ is an $A$-module; in general, when $p \neq 0$, it is infinitely generated. The direct sum decomposition of $R$ starts as follows:

$$R = \bigoplus_{n \geq 0} \operatorname{Sym}^n (M^\vee) = \underset{n=0}{A} \oplus \underset{n=1}{M^\vee} \oplus \underset{n \geq 2}{\cdots} \ ,$$

where we have indicated the degrees. Reversely, the superalgebra $A$ can be viewed as an $R$-module via the *augmentation map* $\epsilon\colon R \longrightarrow A$. This is an algebra homomorphism, so it maps the unit $1 \in R$ to the unit $1 \in A$, and it suffices to describe its action on the generators $t_i$ and $\theta_j$ of $R$:

$$\begin{aligned} \epsilon\colon R &\longrightarrow A \ , \\ t_i &\longmapsto 0 \ , \\ \theta_j &\longmapsto 0 \ . \end{aligned} \tag{3.7}$$

Viewing $A$ as an $R$-module via augmentation, the Berezinian (3.6) of $M$ can be computed. More generally, one can calculate the 'inner Ext' $\mathbf{Ext}^n_R(A, R)$ for any $n \in \mathbb{N}$. In Section 3.3 we will explain how this is done. As we will see in Chapter 5, the result is

$$\mathbf{Ext}^n_R(A, R) \cong \begin{cases} 0 & \text{if } n \neq p \ ; \\ A^{1|0} & \text{if } n = p \text{ and } q \text{ is even} \ ; \\ A^{0|1} & \text{if } n = p \text{ and } q \text{ is odd} \ . \end{cases}$$

Thus, $\mathbf{Ext}^n_R(A, R)$ is concentrated in degree $n = p$, and

$$\operatorname{Ber} M \cong \begin{cases} A^{1|0} & \text{if } q \text{ is even} \ ; \\ A^{0|1} & \text{if } q \text{ is odd} \ . \end{cases} \tag{3.8}$$

The rank $q$ of the odd part of $M$ determines the parity of the elements in the Berezinian. Notice that the Berezinian is an $A$-supermodule of rank one ($1|0$ or $0|1$).

In particular, for $q = 0$ the Berezinian has degree $p$ and is isomorphic to $A$. For an ordinary vector space $V$ over a field $\mathbb{K}$ the Berezinian is isomorphic to $\mathbb{K}$. In Section 4.2 we will see that this coincides with the top exterior power. Thus, in the purely even case, this description is a more sophisticated version of the invariant formulation of the determinant — one that *does* directly generalize to the supercase.

Of course, having defined the Berezinian of a supermodule, we have to know how the induced action of an even automorphism $T\colon M \longrightarrow M$ can be computed. We will come back to this in Chapter 4 and work out the induced action of $T$ for some explicit examples.

## 3.3 Homological algebra

In this section we discuss the things we need from homological algebra to understand and work with the definition (3.6) of the Berezinian of a supermodule. In Sections 2.1.2 and 2.2.3 we have seen that ${}_R\mathsf{sMod}$ is an abelian category. This allows us to set up homological algebra. We will keep it brief; for more background we refer to Appendix A.3 of Eisenbud [4], Chapters XX and XXI of Lang [8], or Chapter 2 of Davis and Kirk [2].

### 3.3.1 Chain complexes

The following concepts are at the basis of homological algebra. A *chain complex* $\mathcal{E} = (E_\bullet\,, d_\bullet\,)$ is a sequence of $R$-supermodules $E_n$ ($n \in \mathbb{Z}$) and module maps $d_n \in \operatorname{Hom}_R(E_{n+1}, E_n)$ such that the composition of two successive maps vanishes: $d_{n+1} \circ d_n = 0$ for all $n$. Chain complexes are often depicted as

$$E_\bullet : \qquad \cdots \longrightarrow E_{n+1} \xrightarrow{d_{n+1}} E_n \xrightarrow{d_n} E_{n-1} \longrightarrow \cdots \ .$$

The maps $d_n$ are called *differentials*, and $E_n$ is said to have *degree* $n$.

The condition $d_n \circ d_{n+1} = 0$ means that the image of each $d_n$ is contained in the kernel of the following $d_{n+1}$. The *homology* of $\mathcal{E}$ at $E_n$ is defined as the quotient

$$H_n\,\mathcal{E} := \ker d_n \,/\, \operatorname{im} d_{n+1} \ .$$

In the category ${}_R\mathsf{sMod}$ the $H_n\,\mathcal{E}$ are themselves $R$-supermodules. We say that $\mathcal{E}$ is *exact at* $E_n$ if its $n$th homology supermodule vanishes. The chain complex $\mathcal{E}$ is *exact* when it is exact in each degree, so that its *(total) homology supermodule*

$$H_\bullet\,\mathcal{E} := \bigoplus_n H_n\,\mathcal{E}$$

is zero.

Alternatively we can think of $\mathcal{E}$ as *graded* $R$-supermodule $E_\bullet = \bigoplus_n E_n$ with an endomorphism $d\colon E \longrightarrow E$ of degree $-1$ satisfying $d^2 = 0$.

**Maps of chain complexes.** Now consider two chain complexes $\mathcal{E} = (E_\bullet\,, d)$ and $\mathcal{E}' = (E'_\bullet\,, d')$. A *map of chain complexes* is a map $f = f_\bullet$ consisting of (supermodule) maps $f_n\colon E_n \longrightarrow E'_n$ such that the squares

$$\begin{tikzcd}
E_\bullet : \arrow[d, "f_\bullet"'] & \cdots \arrow[r] & E_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & E_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
E'_\bullet : & \cdots \arrow[r] & E'_n \arrow[r, "d'_n"'] & E'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commute.

The definition of a map $f$ of chain complexes implies that $f_n(\ker d_n) \subseteq \ker d'_n$, and likewise for the images of the differentials. This, in turn, means that $f$ descends to a map on homology. We denote this induced map by $f$ as well: $f_\bullet : H_\bullet\,\mathcal{E} \longrightarrow H_\bullet\,\mathcal{E}'$.

We say that $f, g\colon \mathcal{E} \longrightarrow \mathcal{E}'$ are *homotopy equivalent* or *homotopic* if there exists an $R$-module map $h = h_\bullet : \mathcal{E} \longrightarrow \mathcal{E}'$ of degree one

$$\begin{tikzcd}
\cdots \arrow[r] & E_{n+1} \arrow[r, "d_{n+1}"] \arrow[d] \arrow[dl] & E_n \arrow[r, "d_n"] \arrow[d] \arrow[dl, "h_n"'] & E_{n-1} \arrow[r] \arrow[d] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & E'_{n+1} \arrow[r, "d'_{n+1}"'] & E'_n \arrow[r, "d'_n"'] & E'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

with the property that $f - g = d' \circ h + h \circ d$. (Notice, therefore, that the above diagram does *not* commute.) We claim that such $f$ and $g$ give rise to the same map on homology.

*Proof.* It suffices to show that if $f\colon \mathcal{E} \longrightarrow \mathcal{E}'$ is homotopy equivalent to the zero map then it descends to the zero map on homology. Let $x \in \ker d_n$ represent an element of $H_n\,\mathcal{E}$. Compute

$$f_n(x) = \big(d'_{n+1} \circ h_n\big)(x) + \big(h_{n-1} \circ d_n\big)(x) = d'_{n+1}\big(h_n(x)\big) + h_{n-1}(0)\ .$$

Since $h$ is a map of $R$-modules, $h(0) = 0$, and we see that the image of $f$ lies in the kernel of $d'_{n+1}$. This shows that the image of $f_n$ is $(0) \subseteq H_n\,\mathcal{E}'$. □

**Tensor products; higher complexes.** The tensor product $\mathcal{E}' \otimes \mathcal{E}''$ of two chain complexes $\mathcal{E}' = (E'_\bullet, d')$ and $\mathcal{E}'' = (E''_\bullet, d'')$ is defined as follows. The $R$-module in degree $n$ is given by $(E' \otimes E'')_n := \bigoplus_{m=0}^n E'_m \otimes_R E''_{n-m}$. This differential $d_n\colon (E' \otimes E'')_n \longrightarrow (E' \otimes E'')_{n-1}$ is defined on $x' \otimes x'' \in E'_m \otimes E''_{n-m}$ as

$$d_n(x' \otimes x'') := (d'_m\, x') \otimes x'' + (-1)^m\, x' \otimes (d''_{n-m}\, x'')\ . \tag{3.9}$$

The sign arises since the differential $d''_{n-m}$, which has degree $-1$, moves past $x' \in E'_m$. Direct computation shows that this map indeed satisfies $d_{n-1} \circ d_n = 0$ for all $n$.

A *double complex* or *bicomplex* $(E_{\bullet\bullet}, d)$ is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & E_{m,n+1} \arrow[r, "d_{\mathrm{hor}}"] \arrow[d, "d_{\mathrm{ver}}"'] & E_{m,n} \arrow[r] \arrow[d, "d_{\mathrm{ver}}"] & \cdots \\
\cdots \arrow[r] & E_{m-1,n+1} \arrow[r, "d_{\mathrm{hor}}"'] \arrow[d] & E_{m-1,n} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd} \tag{3.10}$$

whose rows and columns are chain complexes. Higher complexes can be defined similarly.

An example of a double complex is given by a map of complexes, for which there are only two nonzero rows. Another way to get a double complex is via a second tensor product of chain complexes $\mathcal{E}$ and $\mathcal{E}'$, which we denote by $\mathcal{E} \boxtimes \mathcal{E}'$. It is defined by $(E \boxtimes E')_{mn} := E_m \otimes_R E'_n$ and has differentials $d_{\mathrm{hor}} := 1 \otimes d'$ and $d_{\mathrm{ver}} := d \otimes 1$ :

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & E_m \otimes E'_{n+1} \arrow[r, "1 \otimes d'_{n+1}"] \arrow[d, "d_m \otimes 1"'] & E_m \otimes E'_n \arrow[r] \arrow[d, "d_m \otimes 1"] & \cdots \\
\cdots \arrow[r] & E_{m-1} \otimes E'_{n+1} \arrow[r, "1 \otimes d'_{n+1}"'] \arrow[d] & E_{m-1} \otimes E'_n \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

It's clear that the squares commute. By iterating this process, taking repeated $\boxtimes$'s, we obtain higher complexes.

Reversely, given a double complex $(E_{\bullet\bullet}, d)$ we can construct an ordinary chain complex, the associated *total complex* $\mathrm{Tot}\,\mathcal{E}$, as follows. The $R$-supermodule in degree $n$ is

$$(\mathrm{Tot}\,E)_n := \bigoplus_{m=0}^{n} E_{m,n-m} \ . \tag{3.11}$$

Thus, the part of degree $n$ is the direct sum of the modules on the anti-diagonals in the double complex (3.10). Writing the expansion of the direct sum in (3.11) vertically, we can represent the way the differential acts on them in a diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & (\mathrm{Tot}\,E)_{m+n+1} \arrow[r, "d_{m+n+1}"] & (\mathrm{Tot}\,E)_{m+n} \arrow[r] & \cdots \\
 & \vdots & \vdots & \\
{} \arrow[dr] & \oplus & {} \arrow[dr] \oplus & \\
\cdots \arrow[r] & E_{m,n+1} \arrow[r, "(-1)^m\, d_{\mathrm{hor}}"] \arrow[drr, "d_{\mathrm{ver}}"'] & E_{m,n} \arrow[r] \arrow[drr, "d_{\mathrm{ver}}"] & \cdots \\
 & \oplus & \oplus & \\
 & \vdots \arrow[r] & E_{m-1,n+1} \arrow[r, "(-1)^{m-1}\, d_{\mathrm{hor}}"'] \arrow[dr] & E_{m-1,n} \\
 & & \oplus & {} \\
 & & \vdots &
\end{tikzcd}$$

The signs in the definition of the differentials of $\operatorname{Tot}\mathcal{E}_{\bullet\bullet}$ ensures that each 'square' in this diagram 'anticommutes'. An element of e.g. $E_{m,n+1}$ is mapped to $E_{m-1,n}$ in two ways, and the result of the two routes differ by a sign, so they add up to zero.

Notice that $\operatorname{Tot}(\mathcal{E}' \boxtimes \mathcal{E}'') = \mathcal{E}' \otimes \mathcal{E}''$. This offers a nice point of view to understand '$\otimes$'; in particular, the above diagram offers an elegant description of the differential (3.9). For the 'square' shown in the diagram, $d_{\mathrm{hor}} = d''_{n+1}$ and $d_{\mathrm{ver}} = d'_m$.

We will only need the tensor product '$\boxtimes$' for this alternative viewpoint of the tensor product '$\otimes$'.

### 3.3.2 Free resolutions

Consider the $R$-supermodule $A$, viewed as module over $R$ via the augmentation map. We can use homological algebra to capture the $R$-module structure of $A$ in terms of chain complexes. Of course we can view $A$ as a complex:

$$A_\bullet : \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots ,$$

where the only nonzero term is $A$ itself, which we have put in degree zero. (Often on both sides all but one of the zeros are omitted in the notation: $0 \longrightarrow A \longrightarrow 0$.)

However, there is a closely related chain complex consisting of nice $R$-modules which also captures the $R$-supermodule structure of $A$: its *free resolution*. This is a chain complex consisting of *free* $R$-supermodules

$$F_\bullet : \qquad \cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots ,$$

which is exact in all degrees $n \geq 1$.[1]

**Existence.** There is a concrete recipe for the construction of a free resolution of a given $R$-module $A$. The key observation is that for every $R$-module $S$ there exists a *free* $R$-module that surjects onto $S$: simply take the module freely generated on the (homogeneous) generators of $S$.

Using the augmentation map, starting with $S = A$ the first step is easy: we can simply take $F_0$ equal to the free $R$-module $R^{1|0} \cong R$. The surjection can be written as an exact sequence

$$F_0 \xrightarrow{\pi_0} A \longrightarrow 0 ;$$

of course, $\pi_0 = \epsilon$ is just the augmentation map. $F_0$ is the first supermodule in the free resolution of $A$.

The difference between $F_0$ and $A$ is given by the kernel $K_1 \coloneqq \ker \pi_0 \subseteq F_0$. Denoting the inclusion by $i_1$ we get an exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_0 \xrightarrow{\pi_0} A \longrightarrow 0 . \tag{3.12}$$

$K_1$ is itself an $R$-supermodule, but it need not be free. However, there is again a free $R$-supermodule $F_1$ surjecting onto $S = K_1$:

$$F_1 \xrightarrow{\pi_1} K_1 \longrightarrow 0 . \tag{3.13}$$

We can combine (3.13) and (3.12) to get an exact sequence by *splicing* (3.13) on (3.12): we

[1] Similarly, other resolutions can be defined, e.g. projective resolutions. Although $\mathbf{Ext}^n_R(A, R)$ is usually defined in terms of projective resolutions, we will always be able to construct free resolutions. Since these are easier to work with, and any free module is in particular a projective module, we will restrict ourselves to free resolutions. For more about projective resolutions see e.g. the references at the start of Section 3.3.

define $d_1$ via a commuting triangle

$$
\begin{tikzcd}
 & & & A \\
F_1 \arrow[rr, dotted, "d_1"] \arrow[dr, "\pi_1"'] & & F_0 \arrow[ur] \arrow[rr] & & 0 \\
 & K_1 \arrow[ur, "i_1"'] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}
\tag{3.14}
$$

The horizontal sequence is going to be the right-hand side of our free resolution, with differential $d_1 = i_1 \circ \pi_1$.

Now we can construct a free resolution of $A$ by induction on the degree. Suppose that we have already found the first $n$ steps of the resolution:

$$
F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0 \ . \tag{3.15}
$$

To get the free $R$-supermodule in degree $n+1$, define $K_{n+1} \coloneqq \ker d_n \xrightarrow{i_{n+1}} F_n$. As before, let $F_{n+1}$ be a free module that surjects onto $K_{n+1}$:

$$
F_{n+1} \xrightarrow{\pi_{n+1}} K_{n+1} \longrightarrow 0 \ .
$$

Splice this sequence on (3.15) to get the next step of the free resolution of $A$:

$$
\begin{tikzcd}
 & & & F_{n-1} & & & & & \\
F_{n+1} \arrow[rr, dotted, "d_{n+1}"] \arrow[dr, "\pi_{n+1}"'] & & F_n \arrow[ur, "d_n"] \arrow[rr, "d_n"] & & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & 0 \\
 & K_{n+1} \arrow[ur, "i_{n+1}"'] \arrow[dr] & & & & & & & \\
0 \arrow[ur] & & 0 & & & & & &
\end{tikzcd}
$$

By construction, $d_n \circ d_{n+1} = 0$. Continuing in this way we obtain a chain complex. If at any step the kernel $K_{n+1}$ is already a free $R$-module, we may take $\pi_{n+1}$ to be an isomorphism, and we are done: in that case the horizontal sequence in

$$
\begin{tikzcd}
0 \arrow[dr] & & & & F_{n-1} & & & & & \\
0 \arrow[r] & F_{n+1} \arrow[rr, dotted, "d_{n+1}"] \arrow[dr, "\pi_{n+1}"'] & & F_n \arrow[ur, "d_n"] \arrow[rr, "d_n"] & & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & 0 \\
 & & K_{n+1} \arrow[ur, "i_{n+1}"'] \arrow[dr] & & & & & & & \\
 & 0 \arrow[ur] & & 0 & & & & & &
\end{tikzcd}
$$

is a free resolution of $A$. Such a resolution is called *finite* of *length* $n+1$. As we will see in Section 4.2 this happens for supermodules of rank $p|0$. For rank $p|q$ with $q \geq 1$ the free resolution will be infinite, but after a while it will start repeating itself (see Section 4.3).

Observe that $A$ can be recovered as the cokernel of the first differential:

$$
A = \operatorname{coker} d_1 = F_0 / \operatorname{im} d_1 = H_0 \, \mathcal{F} \ .
$$

Our construction further ensures that $F_\bullet$ is exact at all higher degrees, so we have indeed constructed a free resolution of $A$. Its total homology supermodule is given by

$$
H_\bullet \, \mathcal{F} = H_0 \, \mathcal{F} = A \ .
$$

**Uniqueness.** In general, free resolutions are highly non-unique: if

$$F_\bullet : \qquad \cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is a free resolution of $A$, then for any free $R$-module $F'$,

$$\cdots \longrightarrow F_{n+1} \longrightarrow F_n \oplus F' \xrightarrow{d_n \oplus 1} F_{n-1} \oplus F' \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

is also a free resolution of $A$. Nevertheless, if, at each step, we define $F_{n+1}$ as the free module on the generators of $K_{n+1}$, we get the *minimal free resolution* of $A$.

Now consider two $R$-supermodules $S$ and $S'$ with free resolutions $F_\bullet$ and $F'_\bullet$. A module map $f\colon S \longrightarrow S'$ can always be lifted to a map $f_\bullet$ of chain complexes, and such lifts are unique up to homotopy equivalence. In other words, there is a bijection between module maps and homotopy classes of the corresponding free resolutions. (See e.g. §A.3.6 of [4] or §2.5 of [2].) This result is sometimes referred to as the *fundamental lemma of homological algebra*.

In particular, by lifting the identity map of $S$, we see that there exists a map of chain complexes $f_\bullet : F_\bullet \longrightarrow F'_\bullet$ between any two free resolutions of $S$. Reversing the roles of the two resolutions, we also get a map of chain complexes $g_\bullet : F'_\bullet \longrightarrow F_\bullet$. Further, since both compositions $f \circ g$ and $g \circ f$ descend to the identity on $S$, they are each others inverse up to homotopy equivalence. Thus, any two free resolutions of $S$ are homotopy equivalent.

**Induced maps.** Let $A$ be a supercommutative algebra, and $M$ a right module over $A$ of rank $p|q$. Let $t_1, \cdots, t_p, \theta_1, \cdots, \theta_q$ be a homogeneous basis for $M^\vee$. Suppose that we have found a free resolution of $A$ considered as a module over

$$R = \operatorname{Sym}^\bullet (M^\vee) \cong A[t_1, \cdots, t_p | \theta_1, \cdots, \theta_q]$$

via the augmentation map (3.7):

$$F_\bullet : \qquad \cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0\ .$$

Consider an even automorphism $T\colon M \longrightarrow M$. To calculate the Berezinian of $T$ we will have to compute the map induced by $T$ on $F_\bullet$. Let's write $T$ going downwards:

$$\begin{tikzcd} M \arrow[d, "T"'] \\ M \end{tikzcd} \tag{3.16}$$

If we take the dual, we get a map going up:

$$\begin{tikzcd} M^\vee \\ M^\vee \arrow[u, "T^\vee"] \end{tikzcd} \tag{3.17}$$

Its matrix is given by the supertranspose of the matrix of $T$ (cf. Section 2.3.1). The algebra $R$ contains $M^\vee$ as its degree-one part. This allows us to lift $T^\vee$ to an $A$-linear algebra homomorphism which we will denote by $T_R$:

$$\begin{tikzcd} R & = & A & \oplus & M^\vee & \oplus & \cdots \\ R \arrow[u, "T_R"] & = & A \arrow[u, "1"] & \oplus & M^\vee \arrow[u, "T^\vee"] & \oplus & \cdots \end{tikzcd} \tag{3.18}$$

In the construction of a free resolution of $A$ we have seen that $F_\bullet$ starts with $F_0 = R^{1|0} \cong R$. This makes it possible to further lift $T_R$ to a map

$$\begin{tikzcd} F_\bullet : & \cdots \arrow[r] & F_n \arrow[r, "d_n"] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, equal] & R \arrow[r] & 0 \\ F_\bullet : \arrow[u, "\tau_\bullet"] & \cdots \arrow[r] & F_n \arrow[r, "d_n"] \arrow[u, "\tau_n"] & F_{n-1} \arrow[r] \arrow[u, "\tau_{n-1}"] & \cdots \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[u, "\tau_1"] & F_0 \arrow[r, equal] \arrow[u, "\tau_0"] & R \arrow[r] \arrow[u, "T_R"] & 0 \end{tikzcd} \tag{3.19}$$

acting on degree zero by $\tau_0 = T_R$. The lift $\tau_{n-1}$ is not unique, but may differ by elements in the image of the differential $d_n$. Moreover, in our examples, the induced map $\tau$ on the free resolution will not be $R$-linear: $\tau$ will not be a map of chain complexes. Nevertheless, $\tau$ will be $A$-linear, so that it will yield an $A$-linear map at the level of the Berezinian of $M$. This suffices for our purposes.

### 3.3.3 Ext

Now we are ready to discuss the meaning of the definition $\operatorname{Ber} M = \mathbf{Ext}^p_R(A, R)$ of the Berezinian of a supermodule. There are three steps in its construction. The first one we have already covered: obtain a free resolution of $A$ viewed as an $R$-supermodule via augmentation.

To motivate the second step, notice that in (3.16) the map $T$ goes down, while the induced maps in (3.17)–(3.19) all go up. We have to take some kind of dual once more to get a map going down. This is done as follows: starting with

$$F_\bullet : \quad \cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0 \ ,$$

we can apply the inner hom functor $\mathbf{Hom}_R(-, R)$ to get a complex going the opposite direction. Writing $E^* := \mathbf{Hom}_R(E, R)$ to distinguish it from the notation '$^\vee$' for duals of $A$-supermodules, we get

$$F_\bullet^* : \quad \cdots \longleftarrow F_n^* \xleftarrow{d_n^*} F_{n-1}^* \longleftarrow \cdots \longleftarrow F_1^* \xleftarrow{d_1^*} F_0^* \longleftarrow 0 \ .$$

Here, $d_n^*$ is the pull back along $d_n$:

$$\begin{tikzcd} F_n \arrow[r, "d_n"] \arrow[dr, dotted] & F_{n-1} \arrow[d] \\ & R \end{tikzcd}$$

Since $d_n$ is an $R$-linear map, and '$*$' denotes the dual of $R$-supermodules, the matrix of $d_n^*$ is given by the supertranspose of the matrix of $d_n$ with respect to the generators of the free $R$-supermodules $F_n$ and $F_{n-1}$.

Notice that

$$d_n^* \circ d_{n-1}^* = (d_{n-1} \circ d_n)^* = 0^* = 0 \ .$$

This shows that the complex $F_\bullet^*$ is an example of a *cochain complex*: a sequence of $R$-supermodules with differentials of degree $+1$. Usually a cochain complex is written with upper indices: we can define $E^n := F_n^*$ and $d^n := d_n^*$, so that we have

$$E^\bullet : \quad \cdots \longleftarrow E^n \xleftarrow{d^n} E^{n-1} \longleftarrow \cdots \longleftarrow E^1 \xleftarrow{d^1} E^0 \longleftarrow 0 \ .$$

A cochain complex $(E^\bullet, d^\bullet)$ is said to be *exact* in degree $n$ if the *cohomology at* $E_n$,

$$H^n\mathcal{E} := \ker d^{n+1} / \operatorname{im} d^n \ ,$$

vanishes. This is the third and final step in the computation of $\mathbf{Ext}^n_R$: taking cohomology.

We can summarize the above discussion in the following formula:

$$\mathbf{Ext}^n_R(A,R) \coloneqq H^n\,\mathbf{Hom}_R(F_\bullet\,,R)\ . \tag{3.20}$$

Here, $\mathcal{F}$ is a free resolution of $A$, and $\mathbf{Ext}^n_R$ computes the $n$th cohomology module of the cochain complex $\mathcal{F}^* \coloneqq (F^*_\bullet, d^*_\bullet)$, which is the $R$-dual of a free resolution $\mathcal{F}$ of the $R$-supermodule $A$ (via augmentation). As we have seen, the fundamental lemma of homological algebra implies that (3.20) does not depend on the particular free resolution $F_\bullet$.

We conclude our discussion of homological algebra by mentioning that $\mathbf{Ext}^n_R$ is an example of a *derived functor*; see §A.3.9-A.3.11 and §A.3.14 of [4] or Chapter 6, §6, of [8].

# Chapter 4

# Explicit computations using Ext

As before, let $A$ be a supercommutative algebra, and $M$ be a (right) $A$-supermodule that is free of rank $p|q$. Consider an even automorphism $T: M \longrightarrow M$. In the previous chapter we have seen that the Berezinian can be characterized by three axioms, which uniquely determine a formula for the Berezinian of $T$. In Section 3.2 we have stated that the Berezinian of $T$ can be described in a manifestly basis-independent way as the induced action of $T$ on the Berezinian of $M$. In order to define the latter we use homological algebra, which we introduced in Section 3.3. It's time to get a better feeling for what is happening and apply the abstract machinery from Section 3.3 to some concrete examples.

In this chapter we will

- swiftly formulate the general procedure that we use to tackle the examples;
- work out what happens for the purely even case, where $M \cong A^{p|0}$, recovering the determinant via Koszul complexes;
- find out why $T$ has to invertible in order to extract the Berezinian, and see how we get the inverse power of the determinant in the purely odd case, with $M \cong A^{0|q}$; and
- treat the intermediate case $M \cong A^{1|1}$ to see how the calculation works in general.

Along the way we will collect some useful facts that we will need in Chapter 5, where we prove that the invariant formulation really is equivalent to the formula of the Berezinian.

## 4.1 Plan of attack

Given a free $A$-supermodule $M$ of rank $p|q$, the calculation of the Berezinian goes as follows. Form the symmetric algebra $R = \mathrm{Sym}^\bullet(M^\vee)$ of the ($A$-)dual module $M^\vee$. If $t_1, \cdots, t_p, \theta_1, \cdots, \theta_q$ is a homogeneous basis for $M^\vee$ we have $R =\cong A[t_1, \cdots, t_p|\theta_1, \cdots, \theta_q]$.

The first step is to find the minimal free resolution of $A$ viewed as an $R$-supermodule via the augmentation map (3.7); for this we use the construction showing the existence of free resolutions from Section 3.3.2.

The second step is to take the $R$-dual and compute the cohomology modules to get $\mathbf{Ext}^n_R(A, R) = H^n\, \mathbf{Hom}_R(F_\bullet, R)$. This yields the Berezinian $\operatorname{Ber} M$ of $M$.

The third and final step is to compute the induced action of an even automorphism $T \in \operatorname{Aut} M$. In Section 3.3.2 we have seen how $T$ can be lifted to the free resolution $F_\bullet$. Recall that we mentioned that the resulting map will be unique up to elements in the image of the differentials, and that it won't be a map of chain complexes, but that it will be $A$-linear. Before we move on to the examples, we sketch in which way we can find the induced map on the $R$-dual cochain complex $F_\bullet^*$.

**Induced maps revisited.** Suppose we have lifted $T$ to a map $T_R$ on $R$, and to the map $\tau_\bullet$ on $F_\bullet$. We can find the induced map of $F_n^* = \mathbf{Hom}_R(F_n, R)$ as follows. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_{n+1} \arrow[r, "d_{n+1}"] & F_n \arrow[rr, "d_n"] \arrow[dr, dashed] & & F_{n-1} \arrow[r] & \cdots \\
& & & R & & \\
\cdots \arrow[r] & F_{n+1} \arrow[r, "d_{n+1}"'] \arrow[uu, "\tau_{n+1}"] & F_n \arrow[rr, "d_{n-1}"'] \arrow[uu, "\tau_n"] \arrow[dr, dotted] & & F_{n-1} \arrow[r] \arrow[uu, "\tau_{n-1}"] & \cdots \\
& & & R \arrow[uu, "T_R"] & &
\end{tikzcd}$$

The dashed and dotted arrows represent elements of $F_n^*$. The induced map on the $R$-dual modules goes down (in Section 4.2.1 we will see why this must be the case). To see how this map is defined consider for a moment the following more general setting.

Let $S$ and $\tilde{S}$ be isomorphic rings. Consider modules $E$ and $E'$ over $S$, and $\tilde{E}$ and $\tilde{E}'$ over $\tilde{S}$, and suppose that we have further isomorphisms $\phi\colon \tilde{E} \longrightarrow E$ and $\psi\colon \tilde{E}' \longrightarrow E'$. Then the map

$$\begin{tikzcd}
\mathrm{Hom}_S(E, E') \arrow[d] & \ni & f \arrow[d, mapsto] \\
\mathrm{Hom}_{\tilde{S}}(\tilde{E}, \tilde{E}') & \ni & \tilde{f}
\end{tikzcd}$$

is defined as the composition

$$\begin{tikzcd}
E \arrow[dr, dashed, "f"] & \\
& E' \\
\tilde{E} \arrow[uu, "\phi"] \arrow[dr, dotted, "\tilde{f}"'] & \\
& \tilde{E}' \arrow[uu, "\psi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi(\tilde{e}) \arrow[dr, dashed, mapsto] & \\
& (f \circ \phi)(\tilde{e}) \arrow[dd, mapsto] \\
\tilde{e} \arrow[uu, mapsto] \arrow[dr, dotted, mapsto] & \\
& (\psi^{-1} \circ f \circ \phi)(\tilde{e})
\end{tikzcd}$$

Now let $e_i$ be a homogeneous basis for $F_n$, and denote the ($R$-)dual basis for $F_n^*$ by $e^i$. The map induced by $T$ is determined by its action on the $e^i$. We proceed as above:

$$\begin{tikzcd}
F_n \arrow[dr, dashed, "e^i"] & \\
& R \\
F_n \arrow[uu, "\tau_n"] \arrow[dr, dotted] & \\
& R \arrow[uu, "T_R"']
\end{tikzcd}$$

We map $e^i$ to the element of $F_n^*$ downstairs that is determined by requiring the diagram to commute. This description determines the induced action of $T$ on $F_n^*$ up to elements in the image of the differential $F_n^* \longleftarrow F_{n-1}^*$. The induced map at the level of cohomology, in which we are ultimately interested, will be uniquely determined.

## 4.2 Purely even case: Koszul complexes

First we look at the purely even case, where $M$ has rank $p|0$. From ordinary linear algebra we already know what the answer should be. We will introduce some notation and conventions for graphical representations of the (co)chain complexes.

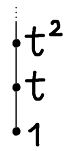
Since we will work with matrices, all supermodules will be viewed as right modules.

### 4.2.1 The case $p = 1$

Let's start off very gently and consider the case where $M$ is a free (right) $A$-supermodule of rank $1|0$. The dual $M^\vee$ is also isomorphic to $A^{1|0}$; we denote its generator by $t$: $M^\vee \cong t\,A$. Thus, $R$ is the algebra of polynomials in $t$ with coefficients in $A$:

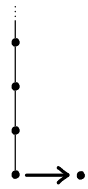
$$R = \operatorname{Sym}^\bullet(M^\vee) \cong A[t] = A \oplus t\,A \oplus t^2\,A \oplus \cdots \ .$$

The last expression shows the structure of $R$ as a ($\mathbb{Z}$-graded) $A$-module. We can draw this structure as follows:




Each node corresponds to a copy of $A$, and we have written the generator (as an $A$-module) next to the node. The degree increases as we go higher.

Reversely, $A$ is an $R$-module via the augmentation map $\epsilon\colon R \longrightarrow A$, which acts as the identity on the degree-zero part of $R$, and kills all elements of degree one or higher. We indicate this as



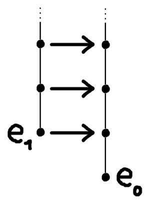
**Free resolution.** Let $F_0 \coloneqq R^{1|0}$ and $\pi_0 = \epsilon$, and write $e_0$ for the generator of $F_0$ as $R$-module. To get the minimal free resolution of $A$ we use the kernel $K_1 = \ker \pi_0 = (e_0\,t) \subseteq F_0$: this is the ideal of $F_0 \cong R$ generated by the element $e_0\,t \in F_0$ as an $R$-module. Let $i_1\colon K_1 \longrightarrow F_0$ be the inclusion. Since $K_1$ is a *free* $R$-module, we are already done: take $F_1 \coloneqq R^{1|0}$ with generator $e_1$, and splice

$$\begin{aligned} \pi_1\colon F_1 &\xrightarrow{\cong} K_1 \ , \\ e_1 &\longmapsto e_0\,t \ , \end{aligned}$$

on $0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ (cf. (3.14)) to get the free resolution:

$$F_\bullet : \quad 0 \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0 \ .$$

We denote this free resolution by $\mathcal{F}_{1|0} = (F_\bullet\,, d_\bullet\,)$, where the subscript reminds us that it comes from $M \cong A^{1|0}$. It looks like




From this picture we can see that the differential $d_1$ is given by multiplication with $t \in R$. Indeed, $d_1(e_1) = (i_1 \circ \pi_1)(e_1) = i_1(e_0\,t) = e_0\,t$. In other words, with respect to the bases $e_n$

of $F_n$, the differential $d_1$ has matrix that we write as[1]

$$\operatorname{mat} d_1 = (\,\underline{t}\rfloor) := \left(\begin{array}{c|c} t & \emptyset \\ \hline \emptyset & \emptyset \end{array}\right)$$

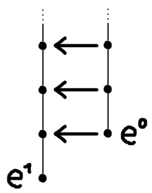
**The Berezinian of $M$.** Apply $\mathbf{Hom}_R(-,R) = \mathrm{Hom}_R(-,R)$ to get the $R$-dual cochain complex

$$\begin{tikzcd} F_\bullet^*: & 0 & F_1^* \arrow[l] & F_0^* \arrow[l, "d_1^*"'] & 0 \arrow[l] \\ K^\bullet(t): & 0 & R^{1|0} \arrow[l] \arrow[u, "\cong"] & R^{1|0} \arrow[l, "d^1"] \arrow[u, "\cong"] & 0 \arrow[l] \end{tikzcd}$$

The vertical maps are the isomorphisms recognizing that the $R$-dual free $R$-modules $F_n^*$ are also free $R$-modules of the same rank, with generators $e^n$ dual to the $e_n$. The cochain complex $K^\bullet(t)$ on the bottom is called the *Koszul complex*. With respect to the dual generators, the matrix of its differential $d^1$ is given by the supertranspose of the matrix of $d_1$:

$$\operatorname{mat} d^1 = (\,\underline{t}\rfloor)^{\mathrm{st}} = (\,\underline{t}\rfloor) \ .$$

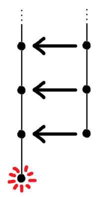
Graphically we have




Taking cohomology we find the values of $\mathbf{Ext}^n_R(A,R) = \mathrm{Ext}^n_R(A,R)$:

$$\mathbf{Ext}^0_R(A,R) = H^0(\mathcal{F}^*_{1|0}) = \ker d^1 / \operatorname{im}(F_0^* \longleftarrow 0) = 0 \ ,$$
$$\mathbf{Ext}^1_R(A,R) = H^1(\mathcal{F}^*_{1|0}) = \ker(0 \longleftarrow F_1^*)/\operatorname{im} d^1 = R^{1|0}/(e_0\, t) \cong A^{1|0} \ .$$

This can also be seen in the picture:



(Recall that there are zeros at both ends of the resolution; any node without an arrow going out is mapped to zero.)

Thus, $\operatorname{Ber} M \cong A^{1|0}$ has $A$-generator $[e^1] \in H^1(\mathcal{F}^*_{1|0})$ living in degree one.

**The Berezinian of $T$.** Our next task is to find the action induced by an even automorphism

$$\begin{tikzcd} M \arrow[d, "T"'] \\ M \end{tikzcd}$$

[1] The matrix on the right is the (even) block matrix of $d_n$ viewed as a map of supermodules. At this point, the distinction is quite pedantic, but when we include odd rank, it pays out to be careful (especially for the differentials). We have chosen to be consistent and use the same notation throughout.

whose matrix is given by $\operatorname{mat} T = \left(\begin{array}{c|} k \\ \hline \end{array}\right)$, for $k \in A_{\bar 0}$ invertible. In fact, since we want to recover the determinant of $T$, the calculation should not depend on the invertibility of $k$.

Supertransposition gives the induced map

$$\begin{tikzcd} M^\vee \arrow[r, "\cong"] & t\,A \\ M^\vee \arrow[u, "T^\vee"] \arrow[r, "\cong"] & t\,A \arrow[u, "\left(\begin{array}{c|} k \\ \hline \end{array}\right)^{\mathrm{st}} = \left(\begin{array}{c|} k \\ \hline \end{array}\right)"'] \end{tikzcd}$$

This map lifts to an $A$-linear algebra homomorphism:

$$\begin{tikzcd}[column sep=small] R \arrow[r, phantom, "="] & A \arrow[r, phantom, "\oplus"] & M^\vee \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & t\,A \arrow[r, phantom, "\oplus"] & t^2\,A \arrow[r, phantom, "\oplus"] & \cdots \\ R \arrow[u, "T_R"] \arrow[r, phantom, "="] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & M^\vee \arrow[u, "T^\vee"] \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\cong"] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & t\,A \arrow[u, "\left(\begin{array}{c|} k \\ \hline \end{array}\right)"] \arrow[r, phantom, "\oplus"] & t^2\,A \arrow[u, "\left(\begin{array}{c|} k^2 \\ \hline \end{array}\right)"] \arrow[r, phantom, "\oplus"] & \cdots \end{tikzcd}$$

where we have used that $T_R(t^2) = T_R(t)^2 = (t\,k)^2 = t^2\,k^2$. Since we will not have to transpose this map, we will not have to be this careful in our notation, and indicate this map instead by

$$\begin{tikzcd}[column sep=small] R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & t\,A \arrow[r, phantom, "\oplus"] & t^2\,A \arrow[r, phantom, "\oplus"] & \cdots \\ R \arrow[u, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & t\,A \arrow[u, "k"] \arrow[r, phantom, "\oplus"] & t^2\,A \arrow[u, "k^2"] \arrow[r, phantom, "\oplus"] & \cdots \end{tikzcd}$$

In turn, $T_R$ has an $A$-linear lift to $\mathcal{F}_{1|0}$:

$$\begin{tikzcd} F_\bullet : & 0 \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, equal] & R \arrow[r] & 0 \\ F_\bullet : \arrow[u, "\tau_\bullet"] & 0 \arrow[r] & F_1 \arrow[u, dotted, "\tau_1"] \arrow[r, "d_1"] & F_0 \arrow[u, "\tau_0"] \arrow[r, equal] & R \arrow[u, "T_R"] \arrow[r] & 0 \end{tikzcd}$$

Requiring the squares to commute, and can find $\tau_1$ by a simple diagram chase:

$$\begin{tikzcd} R^{1|0} \arrow[r, "\left(\begin{array}{c|} t \\ \hline \end{array}\right)"] & R^{1|0} \\ R^{1|0} \arrow[u, dotted, "\tau_1"] \arrow[r, "\left(\begin{array}{c|} t \\ \hline \end{array}\right)"] & R^{1|0} \arrow[u, "\tau_0 = T_R"'] \end{tikzcd} \qquad \begin{tikzcd} e_1\,k \arrow[r, mapsto] & e_0\,(k\,t) = e_0\,(t\,k) \\ e_1 \arrow[u, dotted, mapsto] \arrow[r, mapsto] & e_0\,t \arrow[u, mapsto] \end{tikzcd}$$

Thus, we see that $\tau_1(e_1) = e_1\,k$. Since $\tau_0 = T_R$ is a homomorphism of $A$-modules, $\tau_1$ inherits similar properties. For example, it is easy to check that

$$\tau_1(e_1\,t) = e_1\,t\,k^2 = \tau_1(e_1)\,\tau_0(t) \ .$$

At any rate, we only want to know the induced action on the Berezinian, for which it suffices to know $\tau_1(e_1) = e_1\,k$.

The next step is to find the induced action on the $R$-dual cochain complex $\mathcal{F}^*_{1|0}$:

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[rr, "d_1"] \arrow[dr, dashed] & & F_0 \arrow[r] & 0 \\ & & R & & \\ 0 \arrow[r] & F_1 \arrow[uu, "\tau_1"] \arrow[rr, "d_1"] \arrow[dr, dashed] & & F_0 \arrow[uu, "\tau_0"'] \arrow[r] & 0 \\ & & R \arrow[uu, "T_R"] & & \end{tikzcd}$$

Let's take a closer look at that part of the diagram involving the dashed lines. Since $F_1 \cong R^{1|0}$, we have

$$\begin{tikzcd}[column sep=tiny]
A & \oplus & t\,A & \oplus & \cdots & \cong & F_1 \arrow[dr, dashed] & & & & & & & \\
& & & & & & & R & \cong & A & \oplus & t\,A & \oplus & \cdots \\
A \arrow[uu, "k"] & \oplus & t\,A \arrow[uu, "k^2"] & \oplus & \cdots & \cong & F_1 \arrow[uu, "\tau_1"] \arrow[dr, dashed] & & & & & & & \\
& & & & & & & R \arrow[uu, "T_R"] & \cong & A \arrow[uu, "1"] & \oplus & t\,A \arrow[uu, "k"] & \oplus & \cdots
\end{tikzcd}$$

Now we have to make a choice: should the map induced by $T$ go up or down? This ambiguity is fixed by remembering that, in the present case, we want to obtain the determinant of $T$. Our calculation should also work when $T$ is *not* invertible, so when $k = 0$. From the diagram we see that $\tau_1$ does not have an inverse in this case, whilst $T_R$ is invertible in the lowest degree. In other words, we have to go as follows:

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[rr, "d_1"] \arrow[dr, dashed, "e^1"] & & F_0 \arrow[r] & 0 \\
& & R & & \\
0 \arrow[r] & F_1 \arrow[uu, "\tau_1"] \arrow[rr, "d_1"] \arrow[dr, dotted] & & F_0 \arrow[uu, "\tau_0"] \arrow[r] & 0 \\
& & R \arrow[uu, "T_R"] & &
\end{tikzcd}$$

To see what the dotted arrow does we follow the diagram

$$\begin{tikzcd}
e_1\,k \arrow[dr, dashed, mapsto] & \\
& k \arrow[dd, mapsto] \\
e_1 \arrow[uu, mapsto] \arrow[dr, dotted, mapsto] & \\
& k
\end{tikzcd}$$

Since the (lower) map $e_1 \longmapsto k$ is given by $e^1\,k$, we find

$$\begin{tikzcd}
e^1 \arrow[d, mapsto] \\
e^1\,k
\end{tikzcd}
\quad \text{and therefore} \quad
\begin{tikzcd}
[e^1] \arrow[d, mapsto] \\
k
\end{tikzcd}$$

The induced map on cohomology is given by multiplication by $k$, so $\operatorname{Ber} T = \operatorname{Ber}\left(\begin{array}{c|} k \\ \hline \end{array}\right) = k$.

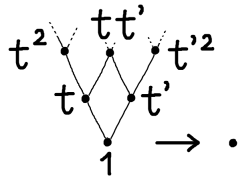
Although the result is far from spectacular, it does agree with the invariant approach of the determinant via the top exterior power. Moreover, we have fixed the only choice that arises in the calculation by requiring that the Berezinian coincides with the determinant in the purely even case: the map induced by $T$ should go in the same direction as $T$ itself.

### 4.2.2 The case $p = 2$

To see what happens for higher rank, we also work out the case $M = A^{2|0}$. Denote the generators of the dual by $t$ and $t'$: $M^\vee \cong t\,A \oplus t'\,A$. Then

$$R = \operatorname{Sym}^\bullet(M^\vee) \cong A[t, t'] = A \oplus t\,A \oplus t'\,A \oplus \cdots \ .$$

As before, we can draw the $A$-module structure of $R$; we also include the augmentation map $\epsilon\colon R \longrightarrow A$:

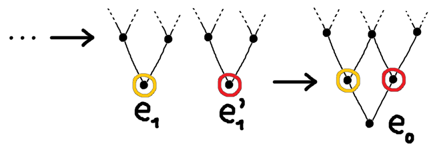



**Free resolution.** As before let $F_0 \coloneqq R^{1|0}$ and $\pi_0 = \epsilon$, and let $e_0$ be the generator of the $R$-module $F_0$. The kernel $K_1 = \ker \pi_0 = (e_0\, t, e_0\, t') \xrightarrow{i_1} F_0$ now has two generators over $R$. However, $K_1$ is not freely generated. This can be clearly seen in terms of the free $R$-module $F_1 \coloneqq R^{2|0}$ on two generators $e_1$ and $e_1'$ surjecting to $K_1$ via

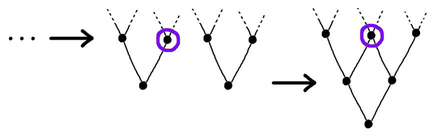
$$\begin{aligned} \pi_1\colon F_1 &\longrightarrow K_1\ , \\ e_1 &\longmapsto e_0\, t\ , \quad e_1' \longmapsto e_0\, t'\ . \end{aligned}$$

Indeed, $\pi_1(e_1\, t') = e_0\, t\, t' = e_0\, t'\, t = \pi_1(e_1'\, t)$ shows the relation between the generators of $K_1$. This will be accounted for in the second step of the free resolution.

Splicing the surjection $F_1 \longrightarrow K_1 \longrightarrow 0$ on $0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$ we find the first degree of the free resolution:




It is clear that it's no longer feasible to include all maps at all degrees; instead we have indicated what happens with the generators $e_1$ and $e_1'$ of $F_1$. Since the differential $d_1 \coloneqq i_1 \otimes \pi_1$ is $R$-linear, it is easy to read off what happens in general. For example:

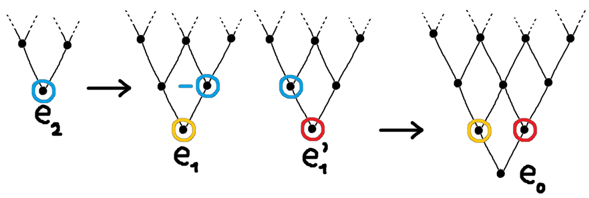


Also notice that this is consistent with diagram representing the augmentation map above. In that case, we also show what happens with the generator, but since there isn't any room on the right for any of the nodes of $R$ that correspond to degree one or higher, those nodes are all mapped to zero.

The differential $d_1$ has matrix

$$\operatorname{mat} d_1 = \begin{pmatrix} t\,, & t' \end{pmatrix}\ . \tag{4.1}$$

To find the second degree of the free resolution, let $K_2 \coloneqq \ker\, d_1 = (-e_1 t' + e_1' t) \xrightarrow{i_2} F_1$. This is a free module on one generator, so we take $F_2 \coloneqq R^{1|0}$ with generator $e_2$, and $\pi_2\colon F_2 \longrightarrow K_2$ is an isomorphism. Splicing the surjection on $0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$ we obtain our free resolution $\mathcal{F}_{2|0}$:



Here we have also indicated the minus sign from $\pi_2(e_2) = e_1\, t' - e_1'\, t$. The differential $d_2$ has matrix

$$\operatorname{mat} d_2 = \left(\begin{array}{c|} -t' \\ t \\ \hline \end{array}\right) .$$

As a check, notice that the composition with (4.1) gives zero, as should be the case.

Throwing in some zeros we can also display the $R$-module structure of $\mathcal{F}_{2|0}$ as

$$\begin{tikzcd}
0 \arrow[r] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & 0 \\
0 \arrow[r] & R^{1|0} \arrow[r, "-t'"] \arrow[dr, "t"'] & R^{1|0} \arrow[r] \arrow[dr, "t"] & 0 & \\
& 0 \arrow[r] & R^{1|0} \arrow[r, "t'"'] \arrow[u, phantom, "\oplus"] & R^{1|0} \arrow[r] & 0
\end{tikzcd}$$

This might look familiar. There is another way to get this complex: it's the total complex of the double complex

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R^{1|0} \otimes_R R^{1|0} \arrow[r, "1 \otimes t'"] \arrow[d, "t \otimes 1"'] & R^{1|0} \otimes_R R^{1|0} \arrow[r] \arrow[d, "t \otimes 1"] & 0 \\
0 \arrow[r] & R^{1|0} \otimes_R R^{1|0} \arrow[r, "1 \otimes t'"'] \arrow[d] & R^{1|0} \otimes_R R^{1|0} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 &
\end{tikzcd}$$

But this is precisely $\mathcal{F}_{1|0} \boxtimes \mathcal{F}_{1|0}$! Our conclusion is that

$$\mathcal{F}_{2|0} = \operatorname{Tot}\left(\mathcal{F}_{1|0} \boxtimes \mathcal{F}_{1|0}\right) = \mathcal{F}_{1|0} \otimes \mathcal{F}_{1|0} .$$

**The Berezinian of $M$.** The $R$-dual cochain complex $F_\bullet^*$ has generators $e^0$, $e^1$, $e'^1$ and $e^2$. The cochain complex is again isomorphic to a Koszul complex, whose differentials have matrix given by the supertransposes of the matrices of $d_1$ and $d_2$:

$$\begin{tikzcd}
F_\bullet^* \colon & 0 & F_2^* \arrow[l] & F_1^* \arrow[l, "d_2^*"'] & F_0^* \arrow[l, "d_1^*"'] & 0 \arrow[l] \\
K^\bullet(t,t') \colon & 0 & R^{1|0} \arrow[l] \arrow[u, "\cong"] & R^{2|0} \arrow[l, "\left(\begin{array}{cc|} -t' , & t \\ \hline \end{array}\right)"] \arrow[u, "\cong"] & R^{1|0} \arrow[l, "\left(\begin{array}{c|} t \\ t' \\ \hline \end{array}\right)"] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

We have $K^\bullet(t,t') \cong K^\bullet(t) \otimes K^\bullet(t')$, where the isomorphism changes some signs.

The cohomology is concentrated in degree two:

$$\begin{aligned}
\mathbf{Ext}_R^0(A,R) &= H^0(\mathcal{F}_{2|0}^*) = (0)\,/(0) = 0 \ , \\
\mathbf{Ext}_R^1(A,R) &= H^1(\mathcal{F}_{2|0}^*) = (e^1\, t' + e'^1\, t)\,/(e^1\, t' + e'^1\, t) \cong 0 \ , \\
\mathbf{Ext}_R^2(A,R) &= H^2(\mathcal{F}_{2|0}^*) = R^{1|0}\,/(t_1,t_2) \cong A^{1|0} \ .
\end{aligned}$$

Hence $\operatorname{Ber} M \cong A^{1|0}$ is generated over $A$ by $[e^2] \in H^2(\mathcal{F}_{2|0}^*)$.

**The Berezinian of $T$.** An even automorphism

$$\begin{tikzcd} M \arrow[d, "T"'] \\ M \end{tikzcd} \quad \text{with} \quad \operatorname{mat} T = \left(\begin{array}{cc|} k_{11} & k_{12} \\ k_{21} & k_{22} \\ \hline \end{array}\right) \,,$$

induces a map

$$\begin{tikzcd} M^\vee \\ M^\vee \arrow[u, "T^\vee"] \end{tikzcd} \quad \text{with} \quad \operatorname{mat} T^\vee = \left(\begin{array}{cc|} k_{11} & k_{21} \\ k_{12} & k_{22} \\ \hline \end{array}\right) \,.$$

This map is the degree-one part of the lift $T_R$ of $T^\vee$ to $R$:

$$\begin{tikzcd} R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & t\,A \arrow[r, phantom, "\oplus"] & t'\,A \arrow[r, phantom, "\oplus"] & \cdots \\ R \arrow[u, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & t\,A \arrow[u, "k_{11}"] \arrow[ur, "k_{12}"] \arrow[r, phantom, "\oplus"] & t'\,A \arrow[u, "k_{22}"'] \arrow[ul, "k_{21}"'] \arrow[r, phantom, "\oplus"] & \cdots \end{tikzcd}$$

Now we have to lift $T_A$ to a map on $\mathcal{F}_{2|0}$:

$$\begin{tikzcd} F_\bullet : & 0 \arrow[r] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, equal] & R \arrow[r] & 0 \\ F_\bullet : \arrow[u, "\tau_\bullet"] & 0 \arrow[r] & F_2 \arrow[u, dotted, "\tau_2"] \arrow[r, "d_2"] & F_1 \arrow[u, dotted, "\tau_1"] \arrow[r, "d_1"] & F_0 \arrow[u, "\tau_0"] \arrow[r, equal] & R \arrow[u, "T_R"] \arrow[r] & 0 \end{tikzcd}$$

We start with the square in the middle:

$$\begin{tikzcd} R^{2|0} \arrow[r, "\left(\begin{array}{cc|} t\,, & t' \\ \hline \end{array}\right)"] & R^{1|0} \\ R^{2|0} \arrow[u, dotted, "\tau_1"] \arrow[r, "\left(\begin{array}{cc|} t\,, & t' \\ \hline \end{array}\right)"'] & R^{1|0} \arrow[u, "\tau_0 = T_R"'] \end{tikzcd}$$

To find $\tau_1$ have to compute its action on both generators:

$$\begin{tikzcd} e_1\,k_{11} + e_1'\,k_{12} \arrow[r, mapsto] & e_0\,(t\,k_{11} + t'\,k_{12}) \\ e_1 \arrow[u, dotted, mapsto] \arrow[r, mapsto] & e_0\,t \arrow[u, mapsto] \end{tikzcd} \qquad \begin{tikzcd} e_1\,k_{21} + e_1'\,k_{22} \arrow[r, mapsto] & e_0\,(t\,k_{21} + t'\,k_{22}) \\ e_1' \arrow[u, dotted, mapsto] \arrow[r, mapsto] & e_0\,t' \arrow[u, mapsto] \end{tikzcd}$$

Using this, we can find $\tau_2$:

$$\begin{tikzcd} R^{1|0} \arrow[r, "\left(\begin{array}{c|} -t' \\ t \\ \hline \end{array}\right)"] & R^{2|0} \\ R^{1|0} \arrow[u, dotted, "\tau_2"] \arrow[r, "\left(\begin{array}{c|} -t' \\ t \\ \hline \end{array}\right)"'] & R^{2|0} \arrow[u, "\tau_1"'] \end{tikzcd} \qquad \begin{tikzcd} e_2\,(k_{11}\,k_{22} - k_{12}\,k_{21}) \arrow[r, mapsto] & (-e_1\,t' + e_1'\,t)\,(k_{11}\,k_{22} - k_{12}\,k_{21}) \\ e_2 \arrow[u, dotted, mapsto] \arrow[r, mapsto] & -e_1\,t' + e_1'\,t \arrow[u, mapsto] \end{tikzcd}$$

This looks very promising (the factor is precisely the determinant of $T$!) but we are not done yet. We still have to find the induced action on the generator $e^2$ in the cochain complex $\mathcal{F}^*_{2|0}$. Since $F_2 \cong R^{1|0}$ we have

$$\begin{tikzcd}[column sep=tiny]
A & \oplus & tA & \oplus & t'A & \oplus & \cdots & \cong & F_2 \arrow[dr, dashed, "e^2"] & & & & & & & & & \\
 & & & & & & & & & R & \cong & A & \oplus & tA & \oplus & t'A & \oplus & \cdots \\
A \arrow[uu, mapsto, "{\substack{k_{11}\,k_{22}\\ -k_{12}\,k_{21}}}"] & \oplus & tA \arrow[uu, "\cdots"'] & \oplus & t'A \arrow[uu, "\cdots"'] & \oplus & \cdots & \cong & F_2 \arrow[uu, "\tau_1"] \arrow[dr, dotted] & & & & & & & & & \\
 & & & & & & & & & R \arrow[uu, "\tau_R"] & \cong & A \arrow[uu, "1"] & \oplus & tA \arrow[uu, "k_{11}"] \arrow[uurr, "k_{12}"'] & \oplus & t'A \arrow[uu, "k_{22}"'] \arrow[uull, "k_{21}"] & \oplus & \cdots
\end{tikzcd}$$

As we have seen in the case $p = 1$ we have to go as follows:

$$\begin{tikzcd}
e_2\,(k_{11}\,k_{22} - k_{12}\,k_{21}) \arrow[dr, dashed, mapsto] & \\
 & k_{11}\,k_{22} - k_{12}\,k_{21} \arrow[dd, mapsto] \\
e_2 \arrow[uu, mapsto] \arrow[dr, dotted, mapsto] & \\
 & k_{11}\,k_{22} - k_{12}\,k_{21}
\end{tikzcd}$$

Hence we obtain

$$\begin{tikzcd}
e^2 \arrow[d, mapsto] \\
e^2\,(k_{11}\,k_{22} - k_{12}\,k_{21})
\end{tikzcd}$$

The induced map on cohomology multiplies $[e^2]$ by $k_{11}\,k_{22} - k_{12}\,k_{21}$, so we conclude

$$\operatorname{Ber} T = \operatorname{Ber}\left(\begin{array}{cc|} k_{11} & k_{12} \\ k_{21} & k_{22} \\ \hline \end{array}\right) = k_{11}\,k_{22} - k_{12}\,k_{21} = \det\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} .$$

### 4.2.3 Arbitrary $p$

The above computation is representative for what happens in the purely even case. Of course it is not an efficient way to compute the determinant, but we see that we do get the correct result. Moreover, it is not hard to see how the computation of the Berezinian of $M$ works out in general.

Let $M \cong A^{p|0}$, yielding an $A$-module

$$R = \operatorname{Sym}^\bullet(M^\vee) \cong A[t_1, \cdots, t_p]$$

which is polynomial in $p$ variables. Resolving $A$ as an $R$-module gives a minimal resolution that can be written as the $p$-fold tensor product

$$\mathcal{F}_{p|0} = \operatorname{Tot}\left({\mathcal{F}_{1|0}}^{\boxtimes p}\right) = {\mathcal{F}_{1|0}}^{\otimes p}$$

of length $p$. The cochain complex $\mathcal{F}^*_{p|0}$ obtained by taking the $R$-dual of $\mathcal{F}_{p|0}$. The total cohomology module is

$$H^\bullet(\mathcal{F}^*_{p|0}) \cong H^\bullet(\mathcal{F}^*_{1|0} \otimes \cdots \otimes \mathcal{F}^*_{1|0}) \cong H^\bullet(\mathcal{F}^*_{1|0}) \otimes \cdots \otimes H^\bullet(\mathcal{F}^*_{1|0}) .$$

Since, as we have seen, $H^n(\mathcal{F}^*_{1|0}) \cong \delta^n_0\, A^{1|0}$, we find

$$H^n(\mathcal{F}^*_{p|0}) \cong \bigoplus_{n_1+\cdots+n_p=n} H^{n_1}(\mathcal{F}^*_{1|0}) \underset{A}{\otimes} \cdots \underset{A}{\otimes} H^{n_p}(\mathcal{F}^*_{1|0}) \cong \delta^n_p\, A^{1|0}$$

Thus, the cohomology is concentrated in degree $p$, and isomorphic to $A^{1|0}$. Let's formulate this result in a proposition.

**Proposition 4.1.** *Let $M$ be a free supermodule of rank $p|0$ over a supercommutative algebra $A$, and let $R$ be the symmetric algebra on the dual of $M$ as above. Then we have*

$$\mathbf{Ext}^n_R(A, R) \cong \begin{cases} A^{1|0} & \text{if } n = p\ ; \\ 0 & \text{if } n \neq p\ , \end{cases}$$

*where $A$ is considered as an $R$-module via augmentation.*

*Moreover, $\mathbf{Ext}^p_R(A, R)$ is generated over $A$ by $[e^p]$, where $e^p$ is the element dual to the generator of $F_p \cong R^{1|0}$.*

Another way to see this is to notice that the $R$-dual complex $\mathcal{F}^*_{p|0}$ is isomorphic to the Koszul complex $K^\bullet(t_1, \cdots, t_p)$. As before,

$$K^\bullet(t_1, \cdots, t_p) \cong K^\bullet(t_1) \otimes \cdots K^\bullet(t_p)\ ;$$

where the isomorphism changes around some of the signs. The Koszul complex $K^\bullet(t_1, \cdots, t_p)$ is closely related to the exterior algebra $\Lambda^\bullet M$ of $M$, and the free $R$-module of rank $1|0$ is isomorphic to the maximal exterior power of $M$. This shows that the invariant description of the Berezinian really is an extension of the invariant formulation of the determinant, and that the induced action of $T$ on the Berezinian is given by multiplication with the determinant of $T$. For more about the relation between the determinant of a module (and of maps of modules) and Koszul complexes, see e.g. Chapter 17 of Eisenbud [4] or §XXI.4 of Lang [8].

## 4.3 Purely odd case: getting the inverse power

Next we investigate the purely odd case, for which $M \cong A^{0|q}$. As we will find out shortly, in this case, the free resolutions are infinite, but nice enough to allow us to find out everything we need for the computation of the Berezinian of an even automorphism of $M$.

The calculations will show us why the Berezinian of $M$ is isomorphic to $A^{1|0}$ when $q$ is even, and to $A^{0|1}$ when $q$ is odd. We will also see why the even endomorphism $T$ of $M$ has to be invertible, from the point of view offered by the invariant description, in order to extract the Berezinian. In addition we will see that we do indeed get the inverse power of the determinant of the matrix of $T$.

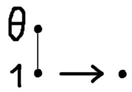
Again we start with the easiest situation.

### 4.3.1 The case $q = 1$

Let $M = A^{0|1}$ and let $\theta$ be the (odd) generator of the dual, so that $M^\vee \cong \theta\, A$ and

$$R = \mathrm{Sym}^\bullet(M^\vee) \cong A[\theta] = A \oplus \theta\, A\ .$$

Since $\theta^2 = 0$, this is a finitely generated algebra over $A$. Together with the augmentation map $\epsilon\colon R \longrightarrow A$ it looks like

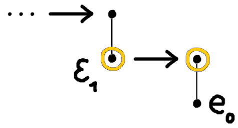



Since we perform explicit calculations it is convenient to denote odd elements by Greek symbols so that we do not forget their parity.

**Free resolution.** Again we start with $F_0 \coloneqq R^{1|0}$ with (even) generator $e_0$ (as $R$-supermodule) and projection $\pi_0 = \epsilon$. The kernel $K_1 = \ker \pi_0 = (e_0\,\theta) \cong e_0\,\theta\,A \xrightarrow{i_1} F_0$ has one odd generator, and is not free ($\theta$ times the generator is zero). Thus we take $F_1 \coloneqq R^{0|1}$ on one odd generators $\varepsilon_1$ surjecting onto $K_1$ via the (even) projection

$$\begin{aligned} \pi_1 \colon F_1 &\longrightarrow K_1 \ , \\ \varepsilon_1 &\longmapsto e_0\,\theta \ . \end{aligned}$$

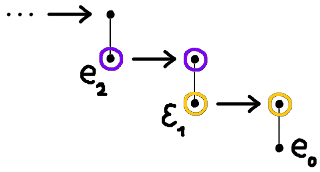
By splicing this map on $0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow S \longrightarrow 0$ we find the first degree of the free resolution:




The differential $d_1$ is an *even* map of $R$-supermodules and has matrix

$$\operatorname{mat} d_1 = \left( \lfloor\underline{\,\theta\,} \right) \coloneqq \left( \begin{array}{c|c} \emptyset & \theta \\ \hline \emptyset & \emptyset \end{array} \right) \ . \tag{4.2}$$

Next we set $K_2 \coloneqq \ker\, d_1 = (\varepsilon_1\,\theta) \xrightarrow{i_2} F_1$. This is a module on one even generator, so we take $F_2 \coloneqq R^{1|0}$ with even generator $e_2$ and projection $\pi_2 \colon F_2 \longrightarrow K_2$. The kernel $K_2$ is not a free $R$-module, for the same reason as before. By splicing we obtain the next degree of our free resolution:

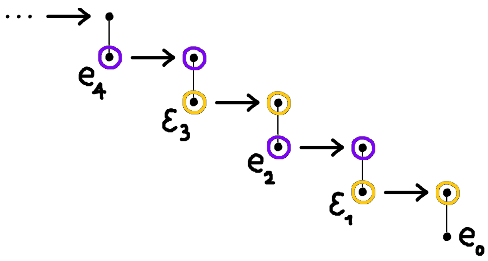



The second differential has matrix

$$\operatorname{mat} d_2 = \left( \overline{\,\theta\,}\rceil \right) \ , \tag{4.3}$$

which is again even. As a check, note that the composition with (4.2) gives zero.

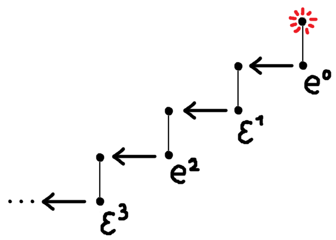
To find the higher degrees, observe that the kernel $K_3 = (e_2\,\theta)$ of $d_2$ is isomorphic to $K_1$. The next free module $F_3 \coloneqq R^{0|1}$ with generator $\varepsilon_3$ is isomorphic to $F_1$, etcetera: at degree three the free resolution looks the same as at degree one. But then degree four is as degree two, and so on. The free resolution of $A$ is infinite, and repeats itself with period two:




**The Berezinian of $M$.** The $R$-dual cochain complex $F_\bullet^*$ has generators $e^0$, $\varepsilon^1$, $e^2$, $\varepsilon^3$ and so forth. The differentials $d^n$ of the cochain complex have matrix given by the supertransposes (2.19) of the matrices of the $d_n$:

$$\begin{tikzcd}
\cdots & F_3^* \arrow[l] & F_2^* \arrow[l, "d_3^*"'] & F_1^* \arrow[l, "d_2^*"'] & F_0^* \arrow[l, "d_1^*"'] & 0 \arrow[l] \\
\cdots & R^{0|1} \arrow[l] \arrow[u, "\cong"] & R^{1|0} \arrow[l, "(\overline{\,-\theta\,}\rceil)"'] \arrow[u, "\cong"] & R^{0|1} \arrow[l, "(\lfloor\underline{\,\theta\,}\;)"'] \arrow[u, "\cong"] & R^{1|0} \arrow[l, "(\overline{\,-\theta\,}\rceil)"'] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

The periodicity is preserved under taking the $R$-dual. Graphically:




As this suggests, the only nonvanishing cohomology module sits in degree zero, and is generated (as $A$-module) by the odd generator $[e^0\,\theta]$. Therefore we find $\operatorname{Ber} M = \mathbf{Ext}^0_R(A, R) \cong A^{0|1}$.

**The Berezinian of $T$.** Consider an even automorphism

$$\begin{tikzcd} M \arrow[d, "T"'] \\ M \end{tikzcd} \qquad \text{with} \quad \operatorname{mat} T = (\lceil\overline{\,n\,}\;)\ .$$

The dual map

$$\begin{tikzcd} M^\vee \\ M^\vee \arrow[u, "T^\vee"] \end{tikzcd} \qquad \text{with} \quad \operatorname{mat} T^\vee = (\lceil\overline{\,n\,}\;)$$

lifts to an (even) map

$$\begin{tikzcd}
R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & \theta A \\
R \arrow[u, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & \theta A \arrow[u, "n"]
\end{tikzcd}$$

Since the generator of the Berezinian lives in degree zero of the cochain complex, we do not have to lift this map to $\mathcal{F}_{0|1}$: we are only interested in $\tau_0 = T_R$. To obtain the induced action on the generator $e^0\,\theta$ in the cochain complex $\mathcal{F}^*_{0|1}$ we proceed as before:

$$\begin{tikzcd}
A \arrow[r, phantom, "\oplus"] & A\theta \arrow[r, phantom, "\cong"] & F_0 \arrow[dr, dashed, "e^0\,\theta"] & & & \\
 & & & R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & \theta A \\
A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & A\theta \arrow[uu, "n"] \arrow[r, phantom, "\cong"] & F_0 \arrow[uu, "\tau_0"] \arrow[dr, dotted] & & & \\
 & & & R \arrow[uu, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & \theta A \arrow[uu, "n"]
\end{tikzcd}
\qquad
\begin{tikzcd}
e_0 \arrow[dr, mapsto, dashed] & \\
 & \theta \arrow[dd, mapsto] \\
e_0 \arrow[uu, mapsto] \arrow[dr, mapsto, dotted] & \\
 & n^{-1}\,\theta
\end{tikzcd}$$

On the right of both 'squares' we use the inverse of the dual map $T^\vee$, which exists precisely $T$ is an automorphism: here we see explicitly that the Berezinian can only be defined for *invertible* even maps, at least when the rank $p|q$ of $M$ satisfies $q > 0$.

Hence we obtain

$$\begin{array}{c} e^0\,\theta \\ \big\downarrow \\ e^0\,\theta\, n^{-1} \end{array}$$

Thus, $T$ acts on cohomology by multiplication by $n^{-1}$, so that

$$\mathrm{Ber}\, T = \mathrm{Ber}\,\left(\begin{array}{|c}\hline n\end{array}\right) = n^{-1} = \det\left(\ n\ \right)^{-1} \ .$$

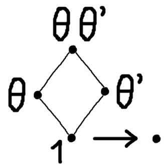
We have recovered the inverse power of the determinant.

### 4.3.2 The case $q = 2$

To get a feeling for what happens when $q > 1$ we also compute the Berezinian of an even automorphism of $M = A^{0|2}$. Denote the odd generators of $M^\vee$ by $\theta$ and $\theta'$. Then

$$R = \mathrm{Sym}^\bullet\,(M^\vee) \cong A[\theta, \theta'] = A \oplus \theta\, A \oplus \theta'\, A \oplus \theta\,\theta'\, A \ .$$

Pictorially this structure is

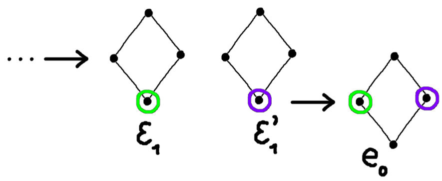



where the augmentation map has also been included.

**Free resolution.** Set $F_0 \coloneqq R^{1|0}$, with generator $e_0$ and projection map $\pi_0 = \epsilon$. The kernel $K_1 = \ker \pi_0 = (e_0\,\theta,\ e_0\,\theta') \xrightarrow{i_1} F_0$ now has two odd generators, and is not free. Hence we take $F_1 \coloneqq R^{0|2}$ with generators $\varepsilon_1$ and $\varepsilon_1'$ and projection

$$\begin{aligned} \pi_1 \colon F_1 &\longrightarrow K_1 \ , \\ \varepsilon_1 &\longmapsto e_0\,\theta \ , \quad \varepsilon_1' \longmapsto e_0\,\theta' \ . \end{aligned}$$

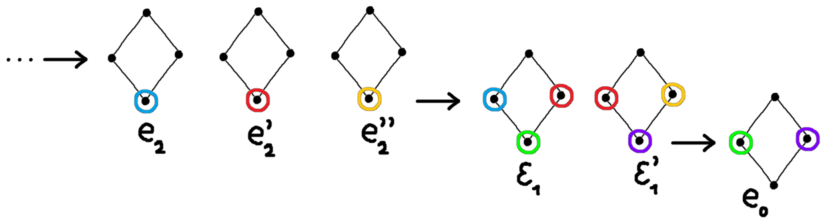
Thus, the first degree of the free resolution is




The differential $d_1$ has matrix

$$\mathrm{mat}\, d_1 = \left(\begin{array}{|cc} \theta\,, & \theta' \\ \hline \end{array}\right) \ . \tag{4.4}$$

To find degree two, let $K_2 \coloneqq \ker d_1 = (\varepsilon_1\,\theta, \varepsilon_1\,\theta' + \varepsilon_1'\,\theta, \varepsilon_1'\,\theta') \xrightarrow{i_2} F_1$. From this we see that $F_2 \coloneqq R^{3|0} \xrightarrow{\pi_2} K_2$ has three even generators $e_2$, $e_2'$ and $e_2''$, and the next degree of the free resolution is




and $d_2$ is given by

$$\operatorname{mat} d_2 = \left( \begin{array}{ccc|} \hline \theta & \theta' & 0 \\ 0 & \theta & \theta' \end{array} \right) \ . \tag{4.5}$$

The $R$-module structure of what we have obtained so far is

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow 0$$

$$\begin{tikzcd}
\ddots \arrow[dr] & & & \\
\cdots \arrow[r] \arrow[ddr] & R^{1|0} \arrow[ddr, "\theta"] & & \\
 & \oplus & & \\
\cdots \arrow[r] \arrow[ddr] & R^{1|0} \arrow[r, "\theta'"] \arrow[ddr, "\theta"'] & R^{0|1} \arrow[ddr, "\theta"] & \\
 & \oplus & \oplus & \\
\cdots \arrow[r] & R^{1|0} \arrow[r, "\theta'"'] & R^{0|1} \arrow[r, "\theta'"'] & R^{1|0}
\end{tikzcd}$$

From our experience of the purely even case we may expect this to be the lowest degrees of the tensor product of two copies of $\mathcal{F}_{0|1}$. However, if that is the case, wouldn't we get a minus sign in front of the differentials of every other diagonal? To see what happens we first construct the bicomplex $\mathcal{F}_{0|1} \boxtimes \mathcal{F}_{0|1}$:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & R^{0|1} \otimes_R R^{0|1} \arrow[r, "-1 \otimes \theta'"] \arrow[d, "\theta \otimes 1"'] & R^{0|1} \otimes_R R^{1|0} \arrow[r] \arrow[d, "\theta \otimes 1"] & 0 \\
\cdots \arrow[r] & R^{1|0} \otimes_R R^{0|1} \arrow[r, "1 \otimes \theta'"'] \arrow[d] & R^{1|0} \otimes_R R^{1|0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Notice that the differential at the top has a sign: this takes into account that we have moved the odd element $\theta'$ acting on the right copy of $R^{0|1}$ past the left copy of $R^{0|1}$, which is an odd $R$-module. In other words, $\boxtimes$ is a tensor product of $\mathbb{Z}_{/2}$-graded chain complexes. Notice that the sign ensures that the square commutes:

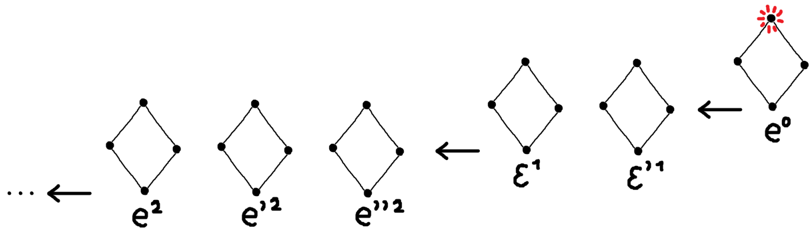
$$(1 \otimes \theta')\,(\theta \otimes 1) = -\theta \otimes \theta' = (\theta \otimes 1)\,(-1 \otimes \theta') \ .$$

If we now take the total complex of the double complex $\mathcal{F}_{0|1} \boxtimes \mathcal{F}_{0|1}$ we get our free resolution:

$$\mathcal{F}_{0|2} = \operatorname{Tot}\left(\mathcal{F}_{0|1} \boxtimes \mathcal{F}_{0|1}\right) = \mathcal{F}_{0|1} \otimes \mathcal{F}_{0|1} \ .$$

Thus, our observation from the purely even case does apply to the purely odd case as well, provided that we are careful and treat the tensor products $\boxtimes$ and $\otimes$ as monoidal structures on the category of graded chain complexes.

**The Berezinian of $M$.** Taking the $R$-dual of $\mathcal{F}_{0|2}$ we get the cochain complex $\mathcal{F}^*_{0|2}$



Again the only nonvanishing cohomology module lives in degree zero; now its generator $[e^0\,\theta\,\theta']$ is even. Hence $\operatorname{Ber} M = \mathbf{Ext}^0_R(A,R) \cong A^{1|0}$.

**The Berezinian of $T$.** Let $T$ be an automorphism

$$\begin{array}{c} M \\ {\scriptstyle T}\big\downarrow \\ M \end{array} \qquad \text{with} \quad \operatorname{mat} T = \left( \begin{array}{|cc} \hline n_{11} & n_{12} \\ n_{21} & n_{22} \end{array} \right) .$$

Its dual $T^\vee$ lifts to an algebra homomorphism map

$$\begin{tikzcd}
R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & \theta A \arrow[r, phantom, "\oplus"] & \theta' A \arrow[r, phantom, "\oplus"] & \theta\theta' A \\
R \arrow[u, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[u, "1"] \arrow[r, phantom, "\oplus"] & \theta A \arrow[u, "n_{11}"] \arrow[ur] \arrow[r, phantom, "n_{12}\, n_{21}"] & \theta' A \arrow[u, "n_{22}"'] \arrow[ul] \arrow[r, phantom, "\oplus"] & \theta\theta' A \arrow[u, "n_{11}\, n_{22} - n_{12}\, n_{21}"']
\end{tikzcd}$$

We see the determinant of the matrix of $T$ appearing in the highest degree of $R$:

$$T_R(\theta\,\theta') = T_R(\theta)\,T_R(\theta') = (\theta\,n_{11} + \theta'\,n_{22})(\theta\,n_{21} + \theta'\,n_{12}) = \theta\,\theta'\,(n_{11}\,n_{22} - n_{12}\,n_{21}) \ .$$

The rest of the computation is as for $q = 1$:

$$\begin{tikzcd}[column sep=small]
A \arrow[r, phantom, "\oplus"] & \theta A \arrow[r, phantom, "\oplus"] & \theta' A \arrow[r, phantom, "\oplus"] & \theta\theta' A \arrow[r, phantom, "\cong"] & F_0 \arrow[dr, dashed, mapsto, "e^0\,\theta\,\theta'"] & & & & & \\
& & & & & R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & \theta A \arrow[r, phantom, "\oplus"] & \theta' A \arrow[r, phantom, "\oplus"] & \theta\theta' A \\
A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & \theta A \arrow[uu, "n_{11}"] \arrow[uur] \arrow[r, phantom, "n_{12}\, n_{21}"] & \theta' A \arrow[uu, "n_{22}"'] \arrow[uul] \arrow[r, phantom, "\oplus"] & \theta\theta' A \arrow[uu, "\substack{n_{11}\, n_{22} \\ -n_{12}\, n_{21}}"] \arrow[r, phantom, "\cong"] & F_0 \arrow[uu, "\tau_0"'] \arrow[dr, dotted, mapsto] & & & & & \\
& & & & & R \arrow[uu, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & \theta A \arrow[uu, "n_{11}"] \arrow[uur] \arrow[r, phantom, "n_{12}\, n_{21}"] & \theta' A \arrow[uu, "n_{22}"'] \arrow[uul] \arrow[r, phantom, "\oplus"] & \theta\theta' A \arrow[uu, "\substack{n_{11}\, n_{22} \\ -n_{12}\, n_{21}}"]
\end{tikzcd}$$

so

$$\begin{tikzcd}
e_0 \arrow[dr, dashed, mapsto] & \\
& \theta\,\theta' \arrow[dd, mapsto] \\
e_0 \arrow[uu, mapsto] \arrow[dr, dotted, mapsto] & \\
& \theta\,\theta'\,(n_{11}\,n_{22} - n_{12}\,n_{21})^{-1}
\end{tikzcd}$$

Therefore, $T$ acts on cohomology by multiplication by the inverse power of its determinant, whence

$$\operatorname{Ber} T = \det \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}^{-1} .$$

### 4.3.3 Arbitrary $q$

As in the purely even case, it is not hard to generalize the construction of a free resolution, and the computation of the Berezinian of $M$, to the general odd case with $M \cong A^{0|q}$. The symmetric algebra in the dual of $M$ is

$$R = \operatorname{Sym}^{\bullet}(M^{\vee}) \cong A[\theta_1, \cdots, \theta_p] \ ;$$

it is exterior in each of the odd generators $\theta_i$ of $M^{\vee}$. It is a finitely generated $A$-module, with $2^q$ generators $1, \theta_i, \theta_i\,\theta_l, \cdots, \theta_1 \,\cdots \theta_q$.

Viewing $A$ as an $R$-module via augmentation, we get a free resolution of $A$ by taking the $q$-fold tensor product of $\mathbb{Z}_{/2}$-graded chain complexes $\mathcal{F}_{0|q} = \operatorname{Tot}\left(\mathcal{F}_{0|1}{}^{\boxtimes q}\right) = \mathcal{F}_{0|1}{}^{\otimes q}$. The total cohomology of the $R$-dual cochain complex is

$$H^n(\mathcal{F}^*_{q|0}) \cong \bigoplus_{n_1+\cdots+n_q=n} H^{n_1}(\mathcal{F}^*_{0|1}) \underset{A}{\otimes} \cdots \underset{A}{\otimes} H^{n_p}(\mathcal{F}^*_{0|1}) \cong \delta_0^{n_1}\, A^{0|1} \underset{A}{\otimes} \cdots \underset{A}{\otimes} \delta_0^{n_q}\, A^{0|1} \ .$$

For any $q$ it is concentrated in degree zero, and has generator $[e^0\,\theta_1 \,\cdots \theta_q]$. This shows that the Berezinian of $M$ is isomorphic to $A^{1|0}$ when $q$ is even, and to $A^{0|1}$ when $q$ is odd.

We can also find the induced action of an even automorphism $T$ with matrix $\left(\begin{array}{c|c} & \\ \hline & N \end{array}\right)$ of $M$. Indeed, we have

$$\begin{tikzcd}
F_0 \arrow[dr, dashed, "e^0\,\theta_1\cdots\theta_q"] & \\
& R \\
F_0 \arrow[uu, "\tau_0"] \arrow[dr, dotted] & \\
& R \arrow[uu, "T_R"']
\end{tikzcd}$$

Now notice that $\tau_0(e_0) = e_0$. Moreover, since $T^{\vee}$ acts on the generators $\theta_i$ of $M^{\vee}$ with the transposed matrix $N^t$, and the $\theta_i$ anticommute, it is clear that $T_R \colon R \longrightarrow R$ acts on the highest degree $\theta_1 \cdots \theta_q\, A$ by multiplication with $\det N^t = \det N$. Hence we have

$$\begin{tikzcd}
e_0 \arrow[dr, dashed, mapsto] & \\
& \theta_1\cdots\theta_q \arrow[dd, mapsto] \\
e_0 \arrow[uu, mapsto] \arrow[dr, dotted, mapsto] & \\
& \theta_1\cdots\theta_q\ \det N^{-1}
\end{tikzcd}$$

Taking cohomology, we obtain the following

**Proposition 4.2.** *Let $M$ be a free supermodule of rank $0|q$ over a supercommutative algebra $A$, and let $R$ be as above. Then*

$$\mathbf{Ext}^n_R(A, R) \cong \begin{cases} A^{1|0} & \text{if } n = 0 \text{ and } q \text{ is even} \ ; \\ A^{0|1} & \text{if } n = 0 \text{ and } q \text{ is odd} \ ; \\ 0 & \text{if } n \neq 0 \ , \end{cases}$$

*where $A$ is considered as an $R$-module via augmentation. The nonvanishing $A$-module $\mathbf{Ext}^p_R(A, R)$ is generated by $[e^0\,\theta_1 \,\cdots \theta_q]$, and the action induced by $T \in \operatorname{Aut} M$ on this generator is given by multiplication by $\det(\operatorname{mat} T)$.*

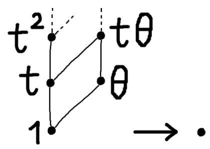
## 4.4 An intermediate case

In general some subtleties arise because the matrix of $T$ now has odd entries as well. To illustrate how this can be dealt with we perform one last calculation, with $M = A^{1|1}$. This is also the first case where we see that we really have to look at the induced map *at the level of cohomology*; in all of the examples above, taking cohomology didn't affect the induced maps.

The dual module $M^\vee$ is free of rank $1|1$; let $t$ be the even generator, and $\theta$ the odd generator. Set

$$R = \mathrm{Sym}^\bullet(M^\vee) \cong A[t|\theta] = A \oplus t\,A \oplus \theta\,A \oplus \cdots \ .$$

Including the augmentation map, this looks like




**Free resolution.** We have enough experience with the construction of free resolutions to take a quicker route this time: we just use $\mathcal{F}_{1|1} = \mathrm{Tot}\,\mathcal{F}_{1|0} \boxtimes \mathcal{F}_{0|1}$. First we construct the double complex $\mathcal{F}_{1|0} \boxtimes \mathcal{F}_{0|1}$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & R^{1|0} \otimes_R R^{0|1} \arrow[r, "1 \otimes \theta"] \arrow[d, "t \otimes 1"'] & R^{1|0} \otimes_R R^{1|0} \arrow[r, "1 \otimes \theta"] \arrow[d, "t \otimes 1"'] & R^{1|0} \otimes_R R^{0|1} \arrow[r, "1 \otimes \theta"] \arrow[d, "t \otimes 1"'] & R^{1|0} \otimes_R R^{1|0} \arrow[r] \arrow[d, "t \otimes 1"'] & 0 \\
\cdots \arrow[r] & R^{1|0} \otimes_R R^{0|1} \arrow[r, "1 \otimes \theta"'] \arrow[d] & R^{1|0} \otimes_R R^{1|0} \arrow[r, "1 \otimes \theta"'] \arrow[d] & R^{1|0} \otimes_R R^{0|1} \arrow[r, "1 \otimes \theta"'] \arrow[d] & R^{1|0} \otimes_R R^{1|0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Taking the total complex we obtain $\mathcal{F}_{1|1}$:

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & R^{1|0} \arrow[r, "-\theta"] \arrow[dr, "t"] & R^{0|1} \arrow[r, "-\theta"] \arrow[dr, "t"] & R^{1|0} \arrow[dr, "t"] & \\
 & \oplus & \oplus & \oplus & \\
\cdots \arrow[r] & R^{0|1} \arrow[r, "\theta"'] & R^{1|0} \arrow[r, "\theta"'] & R^{0|1} \arrow[r, "\theta"'] & R^{1|0}
\end{tikzcd}$$

Hence $F_0$ has a single even generator $e_0$, and the $F_n$ for $n \geq 1$ have two generators; as always, we order them such that the even generator comes first: $e_n, \varepsilon_n$. Taking this into account, we can read off the differentials. The first one is

$$\mathrm{mat}\, d_1 = \left( \begin{array}{c|c} t & \theta \\ \hline \end{array} \right) \ ,$$

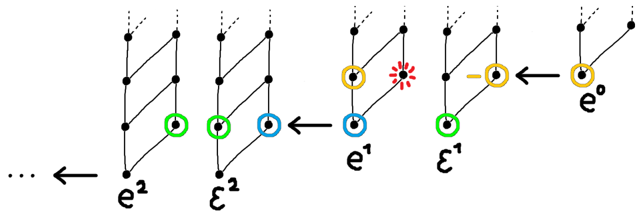
and for higher degree we get a repeating pattern:

$$\mathrm{mat}\, d_{2n} = \left( \begin{array}{c|c} 0 & -\theta \\ \hline \theta & t \end{array} \right) \ , \quad \mathrm{mat}\, d_{2n+1} = \left( \begin{array}{c|c} t & \theta \\ \hline -\theta & 0 \end{array} \right) \ .$$

**The Berezinian of $M$.** The $R$-dual cochain complex $\mathcal{F}^*_{1|1}$ is obtained by supertransposition:

$$\begin{tikzcd}
\cdots & F_3^* \arrow[l] & F_2^* \arrow[l, "d_3^*"'] & F_1^* \arrow[l, "d_2^*"'] & F_0^* \arrow[l, "d_1^*"'] & 0 \arrow[l] \\
\cdots & R^{1|1} \arrow[l] \arrow[u, "\cong"] & R^{1|1} \arrow[l, "\left(\begin{array}{c|c} t & -\theta \\ \hline -\theta & 0 \end{array}\right)"] \arrow[u, "\cong"] & R^{1|1} \arrow[l, "\left(\begin{array}{c|c} 0 & \theta \\ \hline \theta & t \end{array}\right)"] \arrow[u, "\cong"] & R^{1|0} \arrow[l, "\left(\begin{array}{c|c} t & \\ \hline -\theta & \end{array}\right)"] \arrow[u, "\cong"] & 0 \arrow[l]
\end{tikzcd}$$

The cohomology is concentrated at degree one:




We find that $\operatorname{Ber} M = \mathbf{Ext}^1_R(A, R) \cong A^{0|1}$ with generator $[e^1\,\theta]$.

**The Berezinian of $T$.** Consider an automorphism

$$\begin{tikzcd} M \arrow[d, "T"'] \\ M \end{tikzcd} \quad \text{with} \quad \operatorname{mat} T = \left(\begin{array}{c|c} k & \lambda \\ \hline \mu & n \end{array}\right) .$$

The dual map

$$\begin{tikzcd} M^\vee \\ M^\vee \arrow[u, "T^\vee"] \end{tikzcd} \quad \text{with} \quad \operatorname{mat} T^\vee = \left(\begin{array}{c|c} k & \lambda \\ \hline \mu & n \end{array}\right)^{st} = \left(\begin{array}{c|c} k & \mu \\ \hline -\lambda & n \end{array}\right) .$$

lifts to an algebra homomorphism

$$\begin{tikzcd}
R & \cong & A & \oplus & t\,A & \oplus & \theta\,A & \oplus & \cdots \\
R \arrow[u, "T_R"] & \cong & A \arrow[u, "1"] & \oplus & t\,A \arrow[u, "k"] \arrow[urr, "\mu"'] & \oplus & \theta\,A \arrow[u, "n"'] \arrow[ull, "-\lambda"] & \oplus & \cdots
\end{tikzcd}$$

We have to find a lift to $T_1$:

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[r, equal] & R \arrow[r] & 0 \\
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[u, dotted, "\tau_1"] & F_0 \arrow[r, equal] \arrow[u, "\tau_0"] & R \arrow[r] \arrow[u, "T_R"] & 0
\end{tikzcd}$$

The lift $\tau_1$ to degree one is determined by

$$\begin{tikzcd}
e_1\,k + \varepsilon_1\,\lambda \arrow[r] & e_0\,t\,k + e_0\,\theta\,\lambda & e_1\,\mu + \varepsilon_1\,n \arrow[r, mapsto] & e_0\,t\,\mu + e_0\,\theta\,n \\
e_1 \arrow[r] \arrow[u, dotted] & e_0\,t \arrow[u] & \varepsilon_1 \arrow[r, mapsto] \arrow[u, dotted] & e_0\,\theta \arrow[u, mapsto]
\end{tikzcd}$$

Next we compute the induced action on $e^1\,\theta \in T_1^*$:

$$
\begin{tikzcd}[column sep=small]
A \arrow[r, phantom, "\oplus"] & t\,A \arrow[r, phantom, "\oplus"] & \theta\,A \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\cong"] & F_0 \arrow[dr, dashed, "e^1\,\theta"] \\
& & & & & R \arrow[r, phantom, "\cong"] & A \arrow[r, phantom, "\oplus"] & t\,A \arrow[r, phantom, "\oplus"] & \theta\,A \arrow[r, phantom, "\oplus"] & \cdots \\
A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & t\,A \arrow[uu, "k"] \arrow[uur, "\lambda"] \arrow[r, phantom, "\oplus"] & \theta\,A \arrow[uu, "n"'] \arrow[uul, "\mu"'] \arrow[r, phantom, "\oplus"] & \cdots \arrow[r, phantom, "\cong"] & F_0 \arrow[uu, "\tau_0"] \arrow[dr, dotted] \\
& & & & & R \arrow[uu, "T_R"] \arrow[r, phantom, "\cong"] & A \arrow[uu, "1"] \arrow[r, phantom, "\oplus"] & t\,A \arrow[uu, "n"] \arrow[uur, "-\lambda"] \arrow[r, phantom, "\oplus"] & \theta\,A \arrow[uu, "n"'] \arrow[uul, "\mu"'] \arrow[r, phantom, "\oplus"] & \cdots
\end{tikzcd}
$$

On the right we need the inverse of the matrix with which $T_R$ acts in degree one (i.e. the matrix of $T^\vee$). From Proposition 2.1 we know that it is given by

$$
\begin{aligned}
\operatorname{mat}(T^\vee)^{-1} &= \left(\begin{array}{c|c} (k - \mu\, n^{-1} \cdot -\lambda)^{-1} & -k^{-1}\,\mu\,(n - -\lambda\, k^{-1}\,\mu)^{-1} \\ \hline -n^{-1} \cdot -\lambda\,(k - \mu\, n^{-1} \cdot -\lambda)^{-1} & (n - -\lambda\, k^{-1}\,\mu)^{-1} \end{array}\right) \\
&= \left(\begin{array}{c|c} (k + \mu\, n^{-1}\,\lambda)^{-1} & -k^{-1}\,\mu\, n^{-1} \\ \hline n^{-1}\,\lambda\, k^{-1} & (n + \lambda\, k^{-1}\,\mu)^{-1} \end{array}\right)
\end{aligned}
$$

Therefore

$$
\begin{tikzcd}
e_1\,k + \varepsilon_1\,\lambda \arrow[rr, mapsto, dashed, "e^1\,\theta"] & & \theta\,k \arrow[d, mapsto] \\
e_1 \arrow[u, mapsto] \arrow[r, mapsto, dotted] & t\,k \cdot -k^{-1}\,\mu\, n^{-1} + \theta\,k\,(n + \lambda\, k^{-1}\,\mu)^{-1} & {}
\end{tikzcd}
$$

and likewise

$$
\begin{tikzcd}
e_1\,\mu + \varepsilon_1\,n \arrow[rr, mapsto, dashed, "e^1\,\theta"] & & \theta\,\mu \arrow[d, mapsto] \\
\varepsilon_1 \arrow[u, mapsto] \arrow[r, mapsto, dotted] & t \cdot -k^{-1}\,\mu\, n^{-1} \cdot \mu + \theta\,(n + \lambda\, k^{-1}\,\mu)^{-1}\,\mu & {}
\end{tikzcd}
$$

This means that

$$
\begin{tikzcd}
e^1\,\theta \arrow[d, mapsto] \\
-e^1\,t\,\mu\, n^{-1} + e^1\,\theta\,k\,(n + \lambda\, k^{-1}\,\mu)^{-1} + \varepsilon^1\,\theta\, n^{-1}\,\mu
\end{tikzcd}
$$

Now something happens that we have not seen in any of the previous examples. Notice that the combination $e^1\,t - \varepsilon^1\,\theta$ lies in the image of the differential $d_1^*$ of the cochain complex $\mathcal{F}_{1|1}$. Thus, the induced map on cohomology simplifies and is given by

$$
\begin{tikzcd}
{[e^1\,\theta]} \arrow[d, mapsto] \\
{[e^1\,\theta]\,k\,(n + \lambda\, k^{-1}\,\mu)^{-1}}
\end{tikzcd}
$$

We conclude that

$$\operatorname{Ber} T = k\,(n + \lambda\, k^{-1}\, \mu)^{-1} = k\,(n - \mu\, k^{-1}\, \lambda)^{-1}\ ,$$

which agrees with the alternative formula (3.5) for the Berezinian from Corollary 3.3.

# Chapter 5

# Equivalence of the descriptions

In this final chapter we prove that the invariant formulation really describes the Berezinian. Since in Section 3.1 we have shown that the Berezinian is uniquely characterized by three axioms, it suffices to prove that the induced map on the Berezinian of a supermodule satisfies these three axioms.

We need the following lemma.

**Lemma 5.1.** *Let $R'$ and $R''$ be supermodules over a supercommutative algebra $A$ and form the tensor product $R := R' \otimes_A R''$. Viewing $A$ as a module over $R'$, $R''$ and $R$ via augmentation, there is an isomorphism of $\mathbb{Z}$-graded $A$-supermodules:*

$$\mathbf{Ext}^{\bullet}_{R}(A, R) \cong \mathbf{Ext}^{\bullet}_{R'}(A, R') \otimes \mathbf{Ext}^{\bullet}_{R'}(A, R') \ .$$

*That is, in degree $n$ we have*

$$\mathbf{Ext}^{n}_{R}(A, R) \cong \bigoplus_{m=0}^{n} \mathbf{Ext}^{m}_{R'}(A, R') \underset{A}{\otimes} \mathbf{Ext}^{n-m}_{R'}(A, R') \ .$$

*Proof.* In Section 3.3.3 we have seen that

$$\mathbf{Ext}^{n}_{R'}(A, R') = H^n \, \mathbf{Hom}_{R'}(F'_{\bullet}, R') \ ,$$

where $F'_{\bullet}$ is a finitely generated, free resolution of $A$ as $R'$-supermodule: the homology of $F'_{\bullet}$ is concentrated in degree zero and isomorphic to $A$,

$$H_n \, \mathcal{F}' \cong \delta_{n,0} \, A \ ,$$

and each $F'_n$ is a free $R'$-supermodule of finite rank. Similarly, let $\mathcal{F}''$ be a free resolution of $A$ viewed as $R''$-modules.

We claim that $\mathcal{F} = \mathcal{F}' \otimes \mathcal{F}''$ is a free resolution of $A$ considered as $R$-module. Indeed, the $n$th cohomology supermodule of $\mathcal{F}$ is given by

$$H_n \, \mathcal{F} \cong \bigoplus_{m=0}^{n} (H_m \, \mathcal{F}') \underset{A}{\otimes} (H_{n-m} \, \mathcal{F}'') \cong \bigoplus_{m=0}^{n} \delta_{m,0} \, A \underset{A}{\otimes} \delta_{n-m,0} \, A \cong \delta_{n,0} A \ .$$

Moreover, all $\bigoplus_m F'_m \otimes_A F''_{n-m}$ are free and finitely generated over $R$ since each of the $F'_m$ ($F''_{n-m}$) is so over $R'$ (and $R''$, respectively). This establishes the claim.

Since the $F'_m$ and $F''_{n-m}$ are finitely generated, we further have an isomorphism of $\mathbb{Z}_{/2}$-graded chain complexes:

$$\mathbf{Hom}_R(F_{\bullet}, R) \cong \mathbf{Hom}_{R'}(F'_{\bullet}, R') \otimes \mathbf{Hom}_{R''}(F''_{\bullet}, R'') \ .$$

Taking cohomology we arrive at the desired result. ∎

With this lemma it is not hard to prove the next result, which is central for the definition of the Berezinian of a supermodule as we have already seen in Section 3.2.

**Proposition 5.2.** *Let $M$ be a free supermodule of rank $p|q$ over a supercommutative algebra $A$. Define the superalgebra $R \coloneqq \operatorname{Sym}^\bullet(M^\vee)$, and view $A$ as an $R$-module via augmentation. Then*

$$\mathbf{Ext}^n_R(A,R) \cong \begin{cases} A^{1|0} & \text{if } n = p \text{ and } q \text{ is even ;} \\ A^{0|1} & \text{if } n = p \text{ and } q \text{ is odd ;} \\ 0 & \text{if } n \neq p \text{ .} \end{cases}$$

*Proof.* In Propositions 4.1 and 4.2 we have seen that the result holds when either $p = 0$ or $q = 0$. Let $M'$ be a free $A$-supermodule of rank $p|0$, and take $R' \coloneqq \operatorname{Sym}^\bullet(M'^\vee)$. Likewise, let $M''$ free of rank $0|q$ and set $R'' \coloneqq \operatorname{Sym}^\bullet(M''^\vee)$. In terms of these $A$-modules we can rewrite the content of Propositions 4.1 and 4.2 as

$$\mathbf{Ext}^n_{R'}(A,R') \cong \delta^n_p\, A^{1|0}\ , \quad \text{and} \quad \mathbf{Ext}^n_{R''}(A,R'') \cong \delta^n_0 \left(A^{0|1}\right)^{\otimes q}\ .$$

Notice that $M \cong A^{p|q} \cong A^{p|0} \oplus A^{0|q} \cong M' \oplus M''$, and

$$R \coloneqq \operatorname{Sym}^\bullet(M^\vee) \cong \operatorname{Sym}^\bullet(M'^\vee \oplus M''^\vee) \cong R' \underset{A}{\otimes} R''\ .$$

Thus, we may apply Lemma 5.1:

$$\begin{aligned} \mathbf{Ext}^n_R(A,R) &\cong \bigoplus_{m=0}^{n} \mathbf{Ext}^m_{R'}(A,R') \underset{A}{\otimes} \mathbf{Ext}^{n-m}_{R'}(A,R') \\ &\cong \bigoplus_{m=0}^{n} \delta^m_p\, A^{1|0} \underset{A}{\otimes} \delta^{n-m}_0 \left(A^{0|1}\right)^{\otimes q} \\ &\cong \delta^n_p \left(A^{0|1}\right)^{\otimes q}\ . \end{aligned}$$

This is just a more compact notation for what we want to show. □

Thus we set $\operatorname{Ber} M = \mathbf{Ext}^p_R(A,R)$, cf. (3.6). Lemma 5.1 directly implies

**Corollary 5.3.** *Let $M \cong M' \oplus M'$ be a supermodule over a supercommutative algebra $A$. Then* $\operatorname{Ber} M = \operatorname{Ber} M' \otimes \operatorname{Ber} M''$.

Now consider an even automorphism $T$ of $M$. Our final task is to show that the invariant formulation really does describe the Berezinian of $T$.

**Theorem 5.4.** *Let $M$ be a free supermodule over a supercommutative algebra $A$, and let $T \in$ $\operatorname{Aut}_A M$ be an even, invertible endomorphism of $M$. Set $R \coloneqq \operatorname{Sym}^\bullet(M^\vee)$, and define*

$$\operatorname{Ber} M = \mathbf{Ext}^p_R(A,R)$$

*Then the induced action of $T$ on* $\operatorname{Ber} M$ *is given by multiplication by the Berezinian.*

*Proof.* We will show that the action induced by $T$ satisfies the three axioms that uniquely characterize the Berezinian. Recall that these axioms are

i) The Berezinian of a block-diagonal matrix is given by

$$\operatorname{Ber}\begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix} = \det K \cdot \det N^{-1}\ ; \tag{5.1}$$

ii) Multiplicativity: if $T, S \in \operatorname{Aut} M$ then

$$\operatorname{Ber}(S \circ T) = \operatorname{Ber} S \cdot \operatorname{Ber} T\ ;$$

iii) Compatibility with direct sums: for an automorphism $T = T' \oplus T''$ of the short exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ we have

$$\operatorname{Ber} T = \operatorname{Ber} T' \cdot \operatorname{Ber} T'' \ . \tag{5.2}$$

Unlike the formula for the Berezinian, it is not hard to see that the invariant description of the Berezinian is multiplicative: for $S, T \in \operatorname{Aut} M$ the diagram

$$\begin{tikzcd} \operatorname{Ber} M \arrow[r, "\operatorname{Ber} T"] \arrow[dr, "\operatorname{Ber}(S \circ T)"'] & \operatorname{Ber} M \arrow[d, "\operatorname{Ber} S"] \\ & \operatorname{Ber} M \end{tikzcd}$$

commutes since $\operatorname{Ber} M$ has rank one ($1|0$ or $0|1$) according to Proposition 5.2.

The third axiom follows from Corollary 5.3: if $T' \in \operatorname{Aut} M'$ and $T'' \in \operatorname{Aut} M''$ then the map induced on $\operatorname{Ber} M$ by $T = T' \oplus T'$ is multiplication by

$$\operatorname{Ber} T = \operatorname{Ber} T' \underset{A}{\otimes} \operatorname{Ber} T'' = \operatorname{Ber} T' \ \cdot \ \operatorname{Ber} T'' \ ,$$

where we again use Proposition 5.2 telling us that the Berezinian module has rank one.

It remains to verify axiom (i). Axiom (iii) greatly simplifies our task: we may restrict ourselves to two separate cases, considering automorphisms $T$ with block matrix

$$\left(\ \underline{\ K\,}\!\big|\ \right) \quad \text{or} \quad \left(\big|\overline{\ N\ }\ \right)$$

where we use the notation from Sections 4.2 and 4.3, $K$ is a $p \times p$ matrix, and $N$ has size $q \times q$. But for these cases, we have already verified in (the discussion below) Proposition 4.1, and in Proposition 4.2. $\square$

# Conclusion

## Summary

Supercommutative algebra deals with $\mathbb{Z}_{/2}$-graded algebra. Approaching the topic via category theory leads to straightforward generalizations of many concepts from ordinary (linear) algebra to the supercase, and the use of the appropriate braiding isomorphism

$$c_{V,W} \colon V \otimes W \longrightarrow W \otimes V \ , \qquad v \otimes w \longmapsto (-1)^{\bar{v}\,\bar{w}}\, w \otimes v \ ,$$

automatically takes into account all the minus signs that distinguish supercommutative algebra from commutative algebra. In this way we show that the symmetric and exterior algebra of a super vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ are given by

$$\mathrm{Sym}^\bullet\, V = \mathrm{Sym}^\bullet\, V_{\bar{0}} \underset{\mathbb{K}}{\otimes} \Lambda^\bullet\, V_{\bar{1}} \ , \quad \Lambda^\bullet\, V = \Lambda^\bullet\, V_{\bar{0}} \underset{\mathbb{K}}{\otimes} \mathrm{Sym}^\bullet\, V_{\bar{1}} \ ,$$

and operations on maps of supermodules over a supercommutative algebra, such as taking the supertranspose or the supertrace, involve minus signs.

However, the categorical approach does not offer a straightforward generalization of the determinant to super linear algebra. Nevertheless, there are several arguments motivating the definition

$$\mathrm{Ber}\, T \coloneqq \det(K - L\, N^{-1}\, M) \cdot \det N^{-1} \ . \tag{5.3}$$

for the superdeterminant, or Berezinian, of a map $T \colon M \longrightarrow M$ of supermodules over a superalgebra with matrix

$$\mathrm{mat}\, T = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$$

with respect to some homogeneous basis for $M$. In contrast with the determinant, the Berezinian is only defined for *even*, *invertible* endomorphisms.

We show that formula (5.3) is well defined, and is uniquely characterized by the following three axioms:

i) For maps $T$ with a block-diagonal matrix decomposition, we have

$$\mathrm{Ber} \begin{pmatrix} K & 0 \\ 0 & N \end{pmatrix} = \det K \cdot \det N^{-1} \ ;$$

ii) Multiplicativity;

iii) Compatibility with direct sums: if $M = M' \oplus M''$ is a supermodule, and $T = T' \oplus T''$ a direct sum of $T' \colon M' \longrightarrow M'$ and $T'' \colon M'' \longrightarrow M'''$, then

$$\mathrm{Ber}\, T = \mathrm{Ber}\, T' \cdot \mathrm{Ber}\, T'' \ .$$

Axioms (i) and (iii) directly follow from (5.3), but it is more work to prove that (5.3) is multiplicative.

A more satisfying and *invariant* description of the Berezinian can be given in terms of homological algebra. Suppose that $M$ is a free supermodule of finite rank over a supercommutative algebra $A$. Define the symmetric algebra $R \coloneqq \mathrm{Sym}^{\bullet}(M^{\vee})$ of the dual $M^{\vee}$ of $M$. We can view $M$ as an $R$-module via augmentation, and resolve $M$ via a minimal resolution of free $R$-modules. By applying the inner hom functor $\mathbf{Hom}_A(-,R)$ to this resolution and taking cohomology of the resulting cochain complex we compute that

$$\mathbf{Ext}^n_R(A,R) \cong \begin{cases} A^{1|0} & \text{if } n = p \text{ and } q \text{ is even ;} \\ A^{0|1} & \text{if } n = p \text{ and } q \text{ is odd ;} \\ 0 & \text{if } n \neq p \text{ .} \end{cases}$$

In the case of ordinary linear algebra, where $q = 0$, the cochain complex is the Koszul complex $K_{\bullet}(t_1, \cdots, t_p)$, which is intimately related to the exterior algebra of $M$, and the nonvanishing $\mathbf{Ext}^p_R(A,R)$ gives the top exterior power $\Lambda^p M$ of $M$. Thus, the definition

$$\mathrm{Ber}\, M \coloneqq \mathbf{Ext}^p_R(A,R)$$

generalizes the determinant of a module.

As in ordinary linear algebra, an even automorphism $T\colon M \longrightarrow M$ induces a linear map on the Berezinian $\mathrm{Ber}\, M$. The induced map is given by multiplication by an element $a_T \in A$. By checking that the map $T \longmapsto a_T$ satisfies axioms (i)–(iii) characterizing the Berezinian, we show that this prescription provides an invariant description of the Berezinian.

# Outlook

There are several aspects of the Berezinian that are worth further investigation; we point out two of them. Firstly, there is yet another way to describe the Berezinian, via odd symplectic geometry; see e.g. §5 of [7].

Secondly, an important application of super linear algebra lies in supergeometry. It would be interesting to get a better understanding of integration over Grassmann algebras and supermanifolds, and to see how the Berezinian generalizes to this case.

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