EIGENVALUES OF A DIFFUSION PROCESS WITH A CRITICAL POINT

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The eigenvalues of a Fokker–Planck equation involving a critical point have been computed by means of a simple discretization technique. The results smoothly connect the monostable case above the critical point with the bistable case below it.

1. Introduction. It is well known how monostable systems far from critical points should be treated stochastically [1]. Much attention has been paid recently also to diffusion problems in bistable potentials [2–5], again far removed from the critical point. However, of particular interest is the transition from monostability to bistability through a critical point, where the standard treatment breaks down.

An analytical analysis of the intermediate, critical region turns out to be rather intricate. For example, the potential at a critical point deviates so much from a quadratic form, that a WKB analysis becomes inaccurate especially for the dynamically important low lying eigenvalues. Also, the critical problem can not be reduced to known functions, except for a few very special models [3,6–8]. Therefore, we here present the results of a numerical analysis of the eigenvalues of a diffusion process in the quartic potential (see also ref. [4])

$$U(x) = \tfrac{1}{2}\kappa x^2 + \tfrac{1}{4}x^4 , \qquad (1)$$

for all values of the pump parameter κ. Above the critical point ($\kappa > 0$) $U(x)$ has one minimum at $x = 0$; below it ($\kappa < 0$) the potential develops two minima at $x = \pm(-\kappa)^{1/2}$, and a maximum at $x = 0$. The Fokker–Planck equation to be solved reads

$$\partial P(x,t)/\partial t = (\partial/\partial x)(\kappa + x^2)xP + \partial^2 P/\partial x^2 . \qquad (2)$$

Notably, this equation has also been studied in connection with the laser (see e.g. refs. [9–11]).

2. Lattice matrix analysis. The analysis of eq. (2) proceeds most elegantly by casting it into its equivalent self-adjoint Schrödinger form [2,3,10]. To this end one puts $P = P_0^{1/2}\mathcal{S}$, where P_0 represents the stationary solution of eq. (2), and subsequently sets $\mathcal{S}(x, t) = \mathcal{S}(x)\exp(-\mu t)$. This transforms eq. (2) into

$$\mathrm{d}^2\mathcal{S}/\mathrm{d}x^2 + [\mu - V(x)]\mathcal{S}(x) = 0, \qquad (3)$$

with

$$V(x) = -\tfrac{1}{2}\kappa + \tfrac{1}{2}(\tfrac{1}{2}\kappa^2 - 3)x^2 + \tfrac{1}{2}\kappa x^4 + \tfrac{1}{4}x^6 . \qquad (4)$$

For large positive κ this potential $V(x)$ has one minimum at $x = 0$; for large negative κ its develops one inner minimum at $x = 0$ and two outer minima (see e.g. refs [2,4,5]). For the purpose of computation eq. (3) has been put on a lattice simply as (also see refs. [12–14]) [‡1]

[‡1] We have considered direct discretization of the Fokker–Planck equation as well; and further studied both versions also including the probability conservation condition. The differences were not significant for the presented results. The Schrödinger matrices, however, behaved numerically more stable in particular for higher eigenvalues.

$$\delta_{n+1} - 2\delta_n + \delta_{n-1} + a^2[\mu - V_n]\delta_n = 0 , \qquad (5)$$

where $x_n = na; a = x_N/N; n = 0, \pm 1, \pm 2, \ldots, \pm N; \delta_n = \delta(x_n); V_n = V(x_n)$. It is useful to employ the symmetry of the potential to distinguish between even and odd solutions. Solving eq. (5) then is equivalent to finding the eigensolutions of the tridiagonal lattice matrix A, with:

for the odd solutions: $n \in [0,N]$, all elements zero except:

$$A_{n,n} = 2 + a^2 V_n ;$$

$$A_{n,n+1} = A_{n,n-1} = -1, \quad n \neq 0 ; \quad A_{0,1} = -2 ; \qquad (6)$$

for the odd solutions: $n \in [1,N]$, all elements zero except:

$$A_{n,n} = 2 + a^2 V_n ; \quad A_{n,n+1} = A_{n+1,n} = -1 . \qquad (7)$$

These matrix problems have been solved using the Fortran subroutine EIGRF from the standard IMSL computer library (EISPACK; also see ref. [15]). Convergence of the results with increasing N was good. Final runs were taken with a 100 × 100 matrix, and with only a crudely optimized range x_N (the prime criterion used being the result for the stationary eigenfunction). The obtained eigenvalues are estimated to be accurate within roughly 1% for μ_1 to 5% for μ_5. Improvements can be made easily by increasing N [‡2]

[‡2] In the table: the results for $\kappa = -15$ have been obtained from a special run with $N = 160$.

Table 1

κ	μ_1	μ_2	μ_3	μ_4	μ_5
15	15.19	30.76	46.68	62.96	79.57
10	10.28	21.10	32.41	44.18	56.37
8	8.34	17.33	26.88	36.95	47.48
6	6.44	13.66	21.54	30.00	38.97
4	4.59	10.17	16.51	23.48	31.00
2	2.85	7.01	11.98	17.62	23.85
0	1.37	4.45	8.26	12.75	17.84
− 1	0.79	3.55	6.84	10.82	15.40
− 2	0.37	3.00	5.83	9.32	13.42
− 3	0.13	2.90	5.33	8.31	11.97
− 4	0.03	3.31	5.51	7.91	11.14
− 5	0.00	4.19	6.52	8.24	10.97
− 6	0.00	5.34	8.41	9.50	11.51
− 8	0.00	7.58	13.83	14.12	14.47
− 10	0.00	9.68	18.62	18.62	18.67
− 15	0.00	14.79	29.11	29.11	29.15

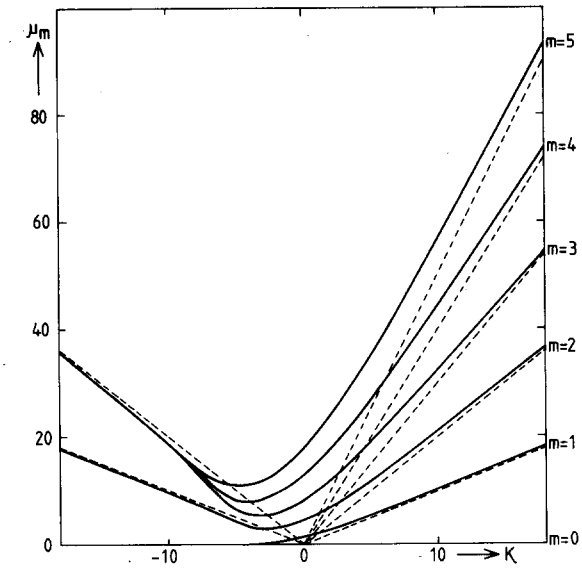

Fig. 1. Eigenvalues μ_m for $m = 0$ through 5 versus pump parameter κ near critical point.

or, probably more efficiently, by extrapolation techniques to $N = \infty$.

3. Results. In table 1 and in fig. 1 we present results for μ_1 through μ_5 for $-15 \leqslant \kappa \leqslant 15$. Clearly, as $\kappa \to +\infty$ the eigenvalues approach their monostable asymptotic values $\mu_m = m\kappa$ ($m = 0,1 \ldots$). In this limit the eigenfunctions tend to the harmonic oscillator eigenfunctions at $x = 0$. On the other hand, as $\kappa \to -\infty$ the eigenvalues group into one doublet ($\mu_0 = 0$ and μ_1), and successive singlets ($\mu_2 \approx |\kappa|$, $\mu_6 \approx 3|\kappa|$, etc.) and triplets (e.g. $\mu_3 \approx \mu_4 \approx \mu_5 \approx 2|\kappa|$). See also ref. [4]. The corresponding eigenfunctions tend to the harmonic oscillator solutions of either the (even and odd) combined outer wells ($m = 0, 1; 3, 4; 7, 8$; etc.) or the inner well ($m = 2; 5, 6; 9, 10$; etc.) of the potential $V(x)$. In the doublet $\mu_1 \sim \exp(-\frac{1}{4}\kappa^2)$ corresponds to Kramers' diffusion rate between the metastable states at $x \pm(-\kappa)^{1/2}$.

The precise eigenvalues near $\kappa = 0$ are new. For example, in contrast with the result $\mu_1 \approx \pi^{3/4}/\sqrt{2} \approx 1.66$ from a WKB analysis, and $\mu_1 \approx 2\Gamma(\frac{5}{4})/\Gamma(\frac{3}{4}) \approx 1.48$ from a variational principle calculation, the exact value in $\kappa = 0$ is seen to be $\mu_1 = 1.37$. Finally, one observes the accumulation of low lying eigenvalues in the region where $|\kappa|$ is of order unity, reflecting the slowing down

in critical dynamics. Notice, that the slowing down continues somewhat below the point $\kappa = 0$.

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