Master's Thesis

Combinatory algebras of functions and their modest sets

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Abstract

We consider a general notion of computation on arbitrary sets, where every element of the set acts as "program", but also as "input". We have a partial function, "the application", that sends a pair (x, y) to an element $x \cdot y$. Think of this element as the result of applying program x to input y. The axioms this application has to satisfy, define the notion of partial combinatory algebra (PCA).
In this thesis we consider the set of all functions from A to A, for some infinite set A. Define an application on this set by using the idea of interrogation: a function asks questions of the form "what is your value at this element?" to a second function. Use the so called sequential functions to investigate this application. When A is the set of natural numbers, we can use topological properties. We also consider the notion of modest sets on our partial combinatory algebra, with "computable functions" between them. This notion can be related to the category of equilogical spaces.

Contents

Introduction

In parts of mathematics and theoretical computer science one considers “models of computation”. When thinking about computation, one often thinks about Turing Machines. A Turing Machine T, also called a program, is given by a finite number of instructions. Say that program T is defined on input b if and only if the program computes in a finite number of steps an output c, write $T(b) = c$. In any other case the program is not defined on that input.

In recursion theory one considers programs that have as both input and output a natural number. An important concept is to code those programs by numbers themselves. First define a coding of finite sequences of natural numbers. Assuming such a coding, say that $e \in \mathbb{N}$ is a code for a program if and only if e is a code of a finite sequence of instructions. This defines a partial function $\mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ denoted by $(e, n) \mapsto e \cdot n$ as follows: $e \cdot n$ is defined if and only if the program with code e computes a value on input n, in that case $e \cdot n$ is that value.

In this thesis we focus on the set A^A, the set of all functions from A to A, for some infinite set A. We would like to introduce a “model of computation” on this set.

Start by formalizing abstract notions of computability by considering *partial combinatory algebras* (or PCAs). The concept of a PCA was defined by S. Feferman [5] in 1975, it generalizes the notion of a total combinatory algebra introduced around 1920 by M. Schönkfinkel [9]. In this thesis we think of a particular PCA as a particular model of computation. The general idea is to consider a set D and define for every pair of elements $a, b \in D$ an application $a \cdot b$, satisfying certain properties. The application is allowed to be partial. A good intuition is to think of a as a “program” and b as the “input”. Then $a \cdot b$ is the result of applying program a to input b. We give a proper definition and find important elements in Chapter 1. We also take a look at some examples. Kleene’s second model (example 1.13) is of particular interest.

As mentioned above, our goal is to define and investigate a PCA on the set A^A. This PCA was introduced by J. van Oosten [14]. For functions $\alpha, \beta : A \to A$ we would like to find an application $\alpha \cdot \beta$. We do this by introducing the notion of interrogation (section 2.1). In such an interrogation the function α asks questions of the form “what is your value at $a \in A$?” to the function β. After α has gathered enough information about β, it gives a result (output). We stick to the idea that α may ask only a finite number of questions, otherwise there is no result. The definition of the application is strongly related to these interrogations. We introduce the notion of bisequential trees and partial bisequential functions to learn more about the application. Using these notions, it will follow that the application does indeed define a PCA on A^A, denote this PCA by $\mathcal{K}_2(A)$. In the special case that $A = \mathbb{N}$, we can describe the application using topological properties. Of course, we should start by defining a topology on $\mathbb{N}^\mathbb{N}$. In section 2.4 we discuss this in some detail.

We consider a different point of view. Given the PCA $\mathcal{K}_2(A)$ on set A^A, we can define "computable functions" between other sets X, Y. This is done by defining for every value $x \in X$ a subset $E(x) \subseteq A^A$, say that the elements of $E(x)$ "represent" x. This idea is made precise by introducing the notion of *modest sets* on a PCA. We call a function $f : X \to Y$ computable if and only if there is a program $\alpha \in A^A$ that sends each element representing x to an elements representing $f(x)$. The modest sets together with these functions form a category. In Chapter 3 we investigate this category, and prove some properties. In this context it is interesting to consider the category **Equ** of equilogical spaces. This category was first defined by D.S. Scott, take a look at [3] for some historical background. It turns out that the category of modest sets on $\mathcal{K}_2(\mathbb{N})$ (so for the special case $A = \mathbb{N}$) is equivalent to a full subcategory of **Equ**. In section 3.3 we try to generalize the result to learn more about the modest sets on $\mathcal{K}_2(A)$, for general infinite sets A. Some results remain true, however we discover that for uncountable sets A, a key result fails.

Prerequisites and Notation

This thesis is aimed primarily at master students mathematics. Below a short overview of preliminaries.

1. Recursion Theory: recommended (but not mandatory). Some concepts, like sequence coding, are similar to concepts in recursion theory. If the reader is not familiar with this field, the second part of section 1.2 and example 1.12 may be difficult to understand. For an introduction consider the first chapter of [11].

2. Topology: experience with topology is assumed. Especially in section 2.4 and moving forward. Some notes and a reference can be found at the start of that section.

3. Category Theory: in Chapter 3 we use basic notions. The first two chapters, skipping section 1.4, can be understood without any knowledge of category theory. A text like [6] offers a good introduction.

About notation: Write $\mathbb{N}$ to indicate the set of natural numbers including zero, so $\mathbb{N} = \{0, 1, \dots\}$. In this thesis we often work with partial functions. Write $f : A \rightharpoonup B$ to indicate that f is a partial function, meaning that $\mathrm{dom}(f) \subseteq A$ (equality is allowed). When writing $f : A \to B$ we implicitly assume that f is total, so $\mathrm{dom}(f) = A$. More notation will be established in the thesis itself.

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1 Partial Combinatory Algebras

This chapter is an introduction to the notion of a partial combinatory algebra (PCA). We will show basic properties and find important elements such as the fixed point operator. The last section defines assemblies and modest sets which will be used later on.

1.1 Basic definitions

To give a definition of partial combinatory algebras, we consider partial maps.

Definition 1.1 A *partial applicative structure* is a pair $(D, \cdot)$ consisting of a nonempty set D and a partial map $D \times D \rightharpoonup D$ denoted by $(a, b) \mapsto a \cdot b$. This map is called *the application* on D.

Often we just write $a \cdot b$ or ab to indicate the application of a to b. Note that the application does not have to be associative. To reduce brackets, we adopt the convention of 'association to the left': we write $a \cdot b \cdot c \cdot d$, or shorter $abcd$, instead of $((a \cdot b) \cdot c) \cdot d$.

For partial applicative structure $(D, \cdot)$ define the collection $\mathcal{C}(D, \cdot)$ of *closed terms over* $(D, \cdot)$ as the least collection satisfying:

- $D \subseteq \mathcal{C}(D, \cdot)$
- if $t \in \mathcal{C}(D, \cdot)$ and $r \in \mathcal{C}(D, \cdot)$ then $(tr) \in \mathcal{C}(D, \cdot)$

So a closed term is just a string of elements with some brackets in between. Again we reduce brackets by the convention of association to the left. For example write tru to indicate the term $((tr)u)$. Define a relation $\downarrow$ between closed terms and elements of D as the least relation satisfying:

- $a \downarrow a$ for every $a \in D$
- $tr \downarrow a$ if and only if there are $b, c \in D$ such that $t \downarrow b$, $r \downarrow c$ and $b \cdot c = a$

We write $t \downarrow a$ instead of the more formal notation $(t, a) \in \downarrow$. Say that a closed term t *denotes*, write $t\downarrow$, if and only if there is $a \in D$ such that $t \downarrow a$. In the case that a closed term denotes we make no distinction between this term and the unique element it denotes. Given closed terms t, r, define $t = r$ if and only if there is $a \in D$ such that $t \downarrow a$ and $r \downarrow a$. Define $t \simeq r$ if and only if either $t = r$ or both t and r do not denote. Using this notation we are ready to define the notion of a PCA.

Definition 1.2 Let $(D, \cdot)$ be a partial applicative structure. Call $(D, \cdot)$ a *partial combinatory algebra* (or PCA) if there are $k, s \in D$ such that for all $a, b, c \in D$:

(*i*) $kab = a$

(*ii*) $sab\downarrow$

(*iii*) $sabc \simeq ac(bc)$

Note that the elements k, s do not have to be unique. We sometimes write $(D, \cdot, k, s)$ after we have chosen certain $s, k \in D$ satisfying the definition.

To work with PCA's we require some more tools. We use a similar approach as in [13]. First generalize the definition of closed terms. Let $(D, \cdot)$ be a PCA and consider some infinite set V of variables disjoint from D. Define the collection $\mathcal{T}(D, \cdot)$ of *terms over* $(D, \cdot)$ as the least collection satisfying:

- $D \cup V \subseteq \mathcal{T}(D, \cdot)$
- if $t \in \mathcal{T}(D, \cdot)$ and $r \in \mathcal{T}(D, \cdot)$ then $(tr) \in \mathcal{T}(D, \cdot)$

Again, we reduce brackets by the same convention as before. Let t be a term and $x_1, \ldots, x_n$ be variables. When writing $t(x_1, \ldots, x_n)$ we assume that all variables occurring in t are among $x_1, \ldots, x_n$. For $a_1, \ldots, a_n \in D$, the closed term $t(a_1, \ldots, a_n)$ is defined as the term where all variables x_i are substituted by elements a_i (use induction on the construction of t to define this properly).

It is worth mentioning that the relation $\downarrow$ and $\simeq$ can be extended to all terms. Say that $t\downarrow$ and $t \simeq s$ are true iff they are true for all possible substitutions of the variables in t. Using terms and variables we can construct new terms with useful properties:

Definition 1.3 For each variable x and term t define a term $\langle x \rangle t$, the *pre-substitution of x in t*, by induction on t as follows:

$$\begin{aligned} &\langle x \rangle t := kt \quad \text{ if } t \text{ is an element } b \in D \text{ or a variable } y \text{ different from } x \\ &\langle x \rangle x := skk \\ &\langle x \rangle t_1 t_2 := s(\langle x \rangle t_1)(\langle x \rangle t_2) \quad \text{ if } t \text{ is of the form } t_1 t_2 \text{ for some terms } t_1,\, t_2 \end{aligned}$$

Using induction on t one can proof that the variables in $\langle x \rangle t$ are exactly those of t minus x, and that every substitution of the variables in $\langle x \rangle t$ does denote. We will write $\langle x_1, \ldots, x_n \rangle t$ to indicate $\langle x_1 \rangle(\langle x_2 \rangle(\ldots(\langle x_n \rangle t)\ldots))$. On first sight terms of this form are quite unwieldy, for example:

$$\begin{aligned} \langle x, y \rangle xy = \langle x \rangle(\langle y \rangle xy) = \langle x \rangle(s(\langle y \rangle x)(\langle y \rangle y)) = \langle x \rangle s(kx)(skk) \\ = s(\langle x \rangle s(kx))(\langle x \rangle skk) = s(s(\langle x \rangle s)(\langle x \rangle kx))(s(\langle x \rangle sk)(\langle x \rangle k)) = \ldots \\ \cdots = s(s(ks)(s(kk)(skk))(s(s(ks)(kk))(kk)) \end{aligned}$$

However the following results justifies the definition of these terms.

Proposition 1.4 *Let $(D, \cdot)$ be a PCA and $t = t(x, x_1, \ldots, x_n)$ be a term with $x, x_1, \ldots, x_n$ variables. Then for all $a, a_1, \ldots, a_n \in D$ we have:*

$$(\langle x \rangle t)a(a_1, \ldots, a_n) \simeq t(a, a_1, \ldots, a_n)$$

Proof: To clarify left-hand side notation, $\langle x\rangle t$ is a term and so is $(\langle x\rangle t)a$, in this last term we substitute variables $x_1, \ldots, x_n$ by elements $a_1, \ldots, a_n$. The proof is by induction on t. We have that $(\langle x\rangle b)a = kba = b$ and $(\langle x\rangle x_i)a(a_1, \ldots, a_n) = ka_ia = a_i$. Also $(\langle x\rangle x)a = skka = ka(ka) = a$. In the last case we obtain $(\langle x\rangle t_1t_2)a(a_1 \ldots, a_n) \simeq s(\langle x\rangle t_1)(\langle x\rangle t_2)a(a_1, \ldots, a_n) \simeq (\langle x\rangle t_1a(a_1, \ldots, a_n))(\langle x\rangle t_2a(a_1, \ldots, a_n))$ on which to apply the induction hypothesis. $\square$

So we can think of $(\langle x\rangle t)a$ as the term t where every occurrence of variable x is replaced by the element a. If x is the only variable in t, we really have $(\langle x\rangle t)a \simeq t(a)$.

The next theorem gives an alternative definition for the notion of PCA. It also explains the word 'combinatory', referring to combinatory completeness, used in the name. This means that every definable term can be represented by an element, made more precise below.

Theorem 1.5 *Let $(D, \cdot)$ be a partial applicative structure. Then $(D, \cdot)$ is a PCA if and only if for every $n \in \mathbb{N}$ and term $t(x_1, \ldots, x_{n+1})$ there is an element $a \in D$ such that for all $a_1, \ldots, a_{n+1} \in D$ the following holds:*

(i) $aa_1 \cdots a_n \downarrow$

(ii) $aa_1 \cdots a_{n+1} \simeq t(a_1, \ldots, a_n)$

Proof: "$\Rightarrow$" Assume k and s satisfy definition 1.2. The term $\langle x_1, \ldots, x_{n+1}\rangle t$ is closed and denotes, find the unique element $a \in D$ such that $a = \langle x_1, \ldots, x_{n+1}\rangle t$. Using proposition 1.4 it follows that $(\langle x_j, \ldots, x_{n+1}\rangle t)(a_1, \ldots, a_{j-1})a_j \simeq (\langle x_{j+1}, \ldots, x_{n+1}\rangle t)(a_1, \ldots, a_j)$ for all $1 \le j \le n + 1$. This shows that $aa_1, \ldots, a_n \simeq \langle x_{n+1}\rangle t(a_1, \ldots, a_n)$ denotes and that $aa_1, \ldots, a_{n+1} \simeq t(a_1, \ldots, a_{n+1})$.

"$\Leftarrow$" Consider term $t(x_1, x_2) := x_1$, find element k satisfying (i) and (ii) of the theorem above, then $kab = a$ for all $a, b \in D$. Also define term $r(x_1, x_2, x_3) := x_1x_3(x_2x_3)$ and find element s satisfying (i) and (ii) above, check that s satisfies definition 1.2 $\square$

Before continuing, let us make some general remarks.

Remarks. (a) In abuse of notation, when there is no confusion about the application, we often write D instead of $(D, \cdot)$ to indicate the PCA.

(b) The PCA with just one element is denoted by $\mathbb{1}$. If D is non-trivial (contains more than one element), then $k \neq s$. For suppose $k = s$, then $skk = skk(skk) = kkk(skk) = k(skk)$, the first equality follows since $skka = a$ for all $a \in D$, the second equality uses the assumption. It follows that $a = skka = k(skk)a = skk$ for all $a \in D$, so all elements are equal, contradiction.

(c) One can investigate structures not satisfying (i) of theorem 1.5. Define a *conditional partial combinatory algebra* (c-PCA) as a partial applicative structure where, in theorem 1.5, we only require (ii) to be true. We can show this is equivalent with a partial applicative structure satisfying (i) and (iii) of definition 1.2. Given a c-PCA we can find a "corresponding" PCA that has properties is common with the c-PCA. Take a look at [13] to learn more about this.

1.2 Useful elements and recursion

In this section consider a PCA D with a certain choice for k and s. Let us write i to indicate skk and $\bar{k}$ for ki. Then we have $ia = a$ and $\bar{k}ab = b$ for all $a, b \in D$. Using definition 1.3 one can find other closed terms with nice properties.

Proposition 1.6 *Let D be a PCA, then there are closed terms p, p_0, p_1 such that for all $a, b \in D$:*

$$pab\downarrow \qquad\qquad p_0(pab) = a \qquad\qquad p_1(pab) = b$$

Also, there is a closed term C, such that for all closed terms u, v:

$$Cuvk \simeq u \qquad\qquad Cuv\bar{k} \simeq v$$

Proof: Let p be the term $\langle xyz\rangle zxy$, define p_0 as $\langle x\rangle xk$ and let p_1 be $\langle x\rangle x\bar{k}$. Then for all $a, b \in D$ it is true that $pab = \langle z\rangle zab$ denotes, and

$$p_0(pab) = (pab)k = (\langle z\rangle zab)k = kab = a$$
$$p_1(pab) = (pab)\bar{k} = (\langle z\rangle zab)\bar{k} = \bar{k}ab = b$$

We have written equality since the right hand side always denotes. For the second part let u, v be closed terms and consider $C := \langle x_1x_2y\rangle y(\langle z\rangle x_1)(\langle z\rangle x_2)k$. Note that $\langle z\rangle u$ and $\langle z\rangle v$ denote, even when u or v do not. This gives:

$$Cuvk \simeq k(\langle z\rangle u)(\langle z\rangle v)k \simeq (\langle z\rangle u)k \simeq u$$

In similar way we obtain $Cuv\bar{k} \simeq v$, the desired result. □

The first terms in the proposition are used for pairing. In the second part k and $\bar{k}$ are used as the 'Booleans' $true$ and $false$ respectively. If u, v, w are closed terms, we often write: "**if** w **then** u, **else** v" to indicate the closed term $Cuvw$.

Remark. In general one can consider other Booleans $\top, \bot \in D$, given that there is a term E such that $E\top ab = a$ and $E\bot ab = b$ for all $a, b \in D$. Using these elements it is possible to find an element C such that $Cuv\top \simeq u$ and $Cuv\bot \simeq v$ as in proposition 1.6. However we stick with the standard choice: $\top = k$, $\bot = \bar{k}$ and $E = i$.

Our next goal is to define a 'primitive recursion operator' in a PCA. First we show that it is possible to represent the natural numbers inside a PCA.

Definition 1.7 The *Curry numerals* are defined in the following way, for each $n \in \mathbb{N}$ define $\overline{n} \in D$ as follows:

$$\overline{0} = i$$
$$\overline{n+1} = p\bar{k}\overline{n}$$

Note that $p_0\overline{0} = k$ and $p_0\overline{n} = \bar{k}$ for all $n \in \mathbb{N}$. Let D be non-trivial. Suppose $k = \bar{k}$, then $k = kks = \bar{k}ks = s$, in contradiction to remark (b) from the previous section. This shows that $\overline{0} \neq \overline{n}$ for all $n \in \mathbb{N}$. Using induction on n we can show $\overline{n} \neq \overline{m}$ for all $m < n$ (apply p_1 on both terms). So in a non-trivial PCA all Curry numerals are distinct.

The following elements allow basic operations on the numerals.

Proposition 1.8 *There are elements $S, Z, P \in D$ with the property that for each $n \in \mathbb{N}$*

$$S\overline{n} = \overline{n+1}, \quad Z\overline{0} = k, \quad Z\overline{n+1} = \bar{k}, \quad P\overline{0} = \overline{0}, \quad P\overline{n+1} = \overline{n}$$

Proof: Define S as the closed term $\langle x \rangle p\bar{k}x$, let Z be the element p_0 and let P be $\langle x \rangle p_0 x\overline{0}(p_1 x)$. Check for yourself these elements have stated properties. □

Also consider the so called *fixed point operator*.

Proposition 1.9 *There is an element $z \in D$ such that for every $f \in A$:*

$$zf\downarrow \quad \text{and} \quad \text{for all } x \in D,\ zfx \simeq f(zf)x$$

Proof: Let $u := \langle vwx \rangle w(vvw)x$ and define $z := uu$. Then $zf \simeq uuf \simeq \langle x \rangle f(uuf)x$, so zf denotes, and $zfx \simeq f(uuf)x \simeq f(zf)x$. □

Using this operator and the Curry numerals we find the *primitive recursion operator*.

Theorem 1.10 *There is an element $\mathcal{R} \in D$ such that for all $g, h \in D$ and all $n \in N$:*

$$\mathcal{R}gh\overline{0} \simeq g$$
$$\mathcal{R}gh\overline{n+1} \simeq h\overline{n}(\mathcal{R}gh\overline{n})$$

Proof: Define element

$$R := \langle rghm \rangle Zm(kg)(\langle x \rangle h(Pm)(rgh(Pm)i))$$

Note that $Rrghm \simeq$ **if** Zm **then** kg **else** $\langle x \rangle h(Pm)(rgh(Pm)i)$. Now define $\mathcal{R}$ as the element:

$$\mathcal{R} := \langle ghm \rangle zRghmi$$

Applying $\mathcal{R}$ on $\overline{n+1}$ shows that:

$$\begin{aligned}
\mathcal{R}gh\overline{n+1} &\simeq zRgh\overline{n+1}i \\
&\simeq R(zR)gh\overline{n+1}i \\
&\simeq Z\overline{n+1}(kg)(\langle x \rangle h(P\overline{n+1})(zRgh(P\overline{n+1})i))i \\
&\simeq (\langle x \rangle h(P\overline{n+1})(zRgh(P\overline{n+1})i))i \simeq h\overline{n}(zRgh\overline{n}i) \simeq h\overline{n}(\mathcal{R}gh\overline{n})
\end{aligned}$$

as desired. It is a good exercise to check that the first property stated in the theorem is also true. □

The following proof uses the primitive recursion operator and gives some motivation for the name. Some knowledge about recursion theory is required to understand the proposition and the proof.

Proposition 1.11 *Let D be a non-trivial PCA and $F : \mathbb{N}^k \rightharpoonup \mathbb{N}$ be a partial recursive function. There is an element $a_F \in D$ such that for all $n_1, \ldots, n_k \in \mathbb{N}$ the following is true: if $F(n_1, \ldots, n_k)$ is defined, then $a_F \overline{n_1} \cdots \overline{n_k}$ denotes and is equal to $\overline{F(n_1, \ldots, n_k)}$. Say that a_F (weakly) represents F.*

Proof sketch: Use induction on the recursive definition of F. We only mention the two more difficult cases. Assume F is defined by primitive recursion from the functions $G : \mathbb{N}^k \to \mathbb{N}$ and $H : \mathbb{N}^{k+2} \to \mathbb{N}$, so:

$$F(0, n_1, \ldots, n_k) = G(n_1, \ldots, n_k)$$
$$F(n+1, n_1, \ldots, n_k) = H(n, F(n, n_1, \ldots, n_k), n_1, \ldots, n_k)$$

By induction hypothesis we find elements a_G and a_H representing G and H. Define $h := \langle m, r, x_1, \ldots, x_k \rangle a_H m(rx_1 \cdots x_k)x_1 \cdots x_k$. Using theorem 1.10 we find an element $a_F := \mathcal{R} a_G h$, that represents F.

For the minimization case one can find elements M and $\mathcal{M}$, like R and $\mathcal{R}$ from previous theorem, that allow minimization. We will not define those here. In this case it is possible that $a_F \overline{n_1} \cdots \overline{n_k}$ denotes while $F(n_1, \ldots, n_k)$ is not defined. □

A complete proof of the proposition can be found in [4], however the approach of this book is somewhat different from ours.

We end this section by defining a *coding of finite sequence* in D together with basic operations on these sequences. This is also mentioned in [13]. Start by defining, inductively, maps $J^n : D^n \to D$ for $n > 0$ by:

$$J^1(a) := pak$$
$$J^{n+1}(a_1, \ldots, a_{n+1}) := pa_1 J^n(a_2, \ldots, a_{n+1})$$

Now for a finite sequence $(a_0, \ldots, a_{n-1})$ of elements in D, define its code $[(a_0, \ldots, a_{n-1})]_D$ as follows:

$$[(\,)]_D := p\overline{0}\overline{0} \qquad (n = 0)$$
$$[(a_0, \ldots, a_{n-1})]_D := p\overline{n} J^n(a_0, \ldots, a_{n-1}) \qquad (n > 0)$$

We often write $[a_0, \ldots, a_{n-1}]_D$ instead. With the help of theorem 1.10 we can find, for example, elements $\mathrm{lh}, c, d \in D$ such that:

$$\begin{aligned} \mathrm{lh}[a_0, \ldots, a_{n-1}]_D &= \overline{n} \\ c\overline{m}[a_0, \ldots, a_{n-1}]_D &= a_m \ (m < n) \\ d[a_0, \ldots, a_{n-1}]_D[b_0, \ldots, b_{m-1}]_D &= [a_0, \ldots, a_{n-1}, b_0, \ldots, b_{m-1}]_D \end{aligned}$$

These elements represent basic operations on codes of sequences in D. Note that we can take lh:$= p_0$. For c first define $g := \langle x \rangle p_0(p_1 x)$ and $h := \langle mru \rangle r(p_1 u)$, check for yourself that $c := \mathcal{R}gh$ satisfies the equation. Finding element d is left to the reader.

1.3 Some examples of PCAs

To familiarize oneself with the notion of PCA, it can be helpful to consider examples. Below we take a look at three well-known examples. Our goal is to give a short introduction of the PCA, we will not go in detail.

Example 1.12 Kleene's first model
Let $\mathcal{PR}^{(1)}$ be the set of partial recursive functions $\mathbb{N} \rightharpoonup \mathbb{N}$ and define $\varphi : \mathbb{N} \to \mathcal{PR}^{(1)}$ as the standard enumeration of those functions. So $\varphi_n := \varphi(n)$ is the partial recursive function of one variable with code n. Define application on $\mathbb{N}$ as $n \cdot m := \varphi_n(m)$.

To show that there are elements satisfying definition 1.2 we make use the S-m-n theorem. Consider function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $(a, b) \mapsto a$, let f as the code of this recursive function. Let k be the code of $a \mapsto S^1_1(f, a)$. Then $k \cdot a \cdot b \simeq S^1_1(f, a) \cdot b \simeq a$, this is always defined. For element s considering a code of the partial recursive function $(a, b, c) \mapsto a \cdot c \cdot (b \cdot c)$ and apply the S-m-n theorem multiple times.

So $\mathbb{N}$ together with the application defines a PCA called Kleene's first model, often denoted by $\mathcal{K}_1$.

Example 1.13 Kleene's second model
Let $\mathbb{N}^{\mathbb{N}}$ denote the set of all functions form $\mathbb{N}$ to $\mathbb{N}$. We will define an application on this set. First consider $\mathbb{N}^*$, the set of all finite sequences of natural numbers. For $u = (n_0, \dots, n_{k-1}) \in \mathbb{N}^*$ a finite sequence, define the code of u as follows:

$$[(n_0, \dots, n_{k-1})] := \prod_{i=0}^{k-1} p_i^{1+n_i}$$

where p_i is the $(i + 1)$-th prime number. Use the convention that the empty product equals 1. Some other injective coding of finite sequences is also allowed. For $\alpha \in \mathbb{N}^{\mathbb{N}}$ let $\alpha \restriction n := (\alpha(0), \dots, \alpha(n - 1))$. In particular $\alpha \restriction 0 = (\ )$. Every $\alpha \in \mathbb{N}^{\mathbb{N}}$ defines a partial function $I_\alpha : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ in the following way:

$$\begin{aligned} I_\alpha(\beta) = n \iff \quad & \text{there is an m} \in \mathbb{N} \text{ such that } \alpha([\beta \restriction m]) = n + 1 \\ & \qquad\qquad \text{and for all } j < m \text{ it is true that } \alpha([\beta \restriction j]) = 0 \end{aligned}$$

where $\beta \in \mathbb{N}^{\mathbb{N}}$ and $n, m \in \mathbb{N}$. If such m, n do not exist, then $I_\alpha(\beta)$ is undefined.

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$ write $(n) * \alpha$ to indicate the function $f \in \mathbb{N}^{\mathbb{N}}$ defined as $f(0) = n$ and $f(n+1) = \alpha(n)$. Thinking of α as an infinite sequence, then $(n) * \alpha$ is the concatenation of sequences (n) and α. Now define the application on $\mathbb{N}^{\mathbb{N}}$ as follows: for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ define $\alpha \cdot \beta$ as the function

$$\alpha \cdot \beta(n) := I_\alpha((n) * \beta)$$

given that $I_\alpha((n) * \beta)$ is defined for all $n \in \mathbb{N}$. Otherwise $\alpha \cdot \beta$ is not defined.

The space $\mathbb{N}^{\mathbb{N}}$ together with this application is a PCA, denoted by $\mathcal{K}_2$. This can be proven by finding elements k and s satisfying definition 1.2. We will not do this for the moment.

In Chapter 2 we define an application on A^A, the set of functions from A to A, for an arbitrary infinite set A. In section 2.3 we show this application defines an PCA (by finding elements k and s), which we denote by $\mathcal{K}_2(A)$.

The PCA $\mathcal{K}_2(\mathbb{N})$ for the case $A = \mathbb{N}$ will be very similar to $\mathcal{K}_2$. In fact they are isomorphic in some sense, made more precise in the remark ending section 2.4. In that section we also define a topology on $\mathbb{N}^\mathbb{N}$ and relate it to the application. It is for example true that a function $F : \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}$ is of the form I_α for some α if and only if it is continuous with open domain (see proposition 2.12 for a similar result in $\mathcal{K}_2(A)$).

Example 1.14 Scott's Graph Model
Scott's graph model is constructed on $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. We define the application, but will not study it any further. The properties mentioned below, written out in detail, can be found in [10]. To understand the details, knowledge about topology is required. We just mention the (topological) properties without proof.

Start with an enumeration $\{e_n \,|\, n \in \mathbb{N}\}$ of all finite subsets of $\mathbb{N}$. Assuming that $m_0 < \cdots < m_{k-1}$, take for example:

$$e_n = \{m_0, \ldots, m_{k-1}\} \iff n = \sum_{i<k} 2^{m_i}$$

So we have $e_0 = \emptyset$, $e_1 = \{0\}$, $e_2 = \{1\}$, $e_3 = \{0, 1\}$ etc. Define a topology on $\mathcal{P}(\mathbb{N})$ by taking as basic opens the sets:

$$U_n := \{x \in \mathcal{P}(\mathbb{N}) \,|\, e_n \subseteq x\}$$

for $n \in \mathbb{N}$. One should check that these sets indeed define a basis. Given this topology, a function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is continuous if and only if $f(x) = \bigcup\{f(e_n) \,|\, e_n \subseteq x\}$ for all $x \in \mathcal{P}(\mathbb{N})$. Write $\mathrm{cont}(\mathcal{P}(\mathbb{N}))$ to indicate the set of all continuous functions. Also define a coding $[.\,,.]$ of pairs of natural numbers, for example:

$$[n, m] := \frac{1}{2}(n + m)(n + m + 1) + m$$

Consider the functions $\mathrm{graph}\colon \mathrm{cont}(\mathcal{P}(\mathbb{N})) \to \mathcal{P}(\mathbb{N})$ and $\mathrm{fun}\colon \mathcal{P}(\mathbb{N}) \to \mathrm{cont}(\mathcal{P}(\mathbb{N}))$ defined by

$$\begin{aligned} \mathrm{graph}(f) &:= \{[n, m] \mid m \in f(e_n)\} \\ \mathrm{fun}(u)(x) &:= \{m \,|\, \exists n(e_n \subseteq x \;\wedge\; [n, m] \in u)\} \end{aligned}$$

Using properties of continuity it follows that $\mathrm{fun}(\mathrm{graph}(f)) = f$ for every $f \in \mathrm{cont}(\mathcal{P}(\mathbb{N}))$. Furthermore the function $x \mapsto \mathrm{fun}(u)(x)$ is continuous for every $u \in \mathcal{P}(\mathbb{N})$ and it has the property that $u \subseteq \mathrm{graph}(\mathrm{fun}(u))$.

We now define the application map $\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ as follows:

$$u \cdot x := \mathrm{fun}(u)(x)$$

Note that the application is always defined.

A function $f : (\mathcal{P}(\mathbb{N}))^k \to \mathcal{P}(\mathbb{N})$ of k-variables is continuous if and only if it is continuous in each of its variables separately. It is possible to show that $(u, x) \mapsto u \cdot x$ is continuous in both of its variables. This shows that any closed term is continuous. With some work it is possible to show that for any continuous function $f : (\mathcal{P}(\mathbb{N}))^k \to \mathcal{P}(\mathbb{N})$ there is an $u \in \mathcal{P}(\mathbb{N})$ such that:

$$ux_1 \cdots x_k = f(x_1, \dots, x_k)$$

for all $x_1, \dots, x_k \in \mathcal{P}(\mathbb{N})$. Combining the two results shows that the set $\mathcal{P}(\mathbb{N})$ together with the application satisfies (ii) of theorem 1.5, so it defines a PCA. Of course we did not prove the (non-trivial) properties mentioned above, it is just a sketch.

1.4 Assemblies and modest sets

In this section we introduce the notion of assemblies and modest sets on a PCA. We will use those again (in particular the modest sets) in Chapter 3. Use notation $\mathcal{P}^*(D) := \mathcal{P}(D) \setminus \{\emptyset\}$, the power set of D minus the empty set.

Definition 1.15 Let D be a PCA. An *assembly* X on D consists of a set $|X|$ and a map $E_X : |X| \to \mathcal{P}^*(D)$, write $X = (|X|, E_X)$. Call $|X|$ the underlying set of X. If $x \in |X|$ and $b \in E_X(x)$ say that b *represents* x.
Let X, Y be two assemblies. A *morphism* $f : X \to Y$ is a function $f : |X| \to |Y|$ with the property that there is an element $r \in D$ such that for all $x \in |X|$ and $a \in E_X(x)$, ra denotes and is an element of $E_Y(f(x))$. Say that the element r *tracks* the function f.

To give some intuition about assemblies think of the set $|X|$ as "values". Every value $x \in |X|$ is represented by a set $E_X(x)$ of "machine level representations" of that value. A certain PCA D can be seen as a model of computation. For $b \in E_X(x)$ and $a \in D$ we understand $a \cdot b$ as applying program a on some representation of x. So a morphism f tracked by r is a function that can be computed in the PCA by program r. It sends representations of x to representations of $f(x)$.

Note that the identity function on an assembly X is tracked by i. When r tracks $f : X \to Y$ and t tracks $g : Y \to Z$, then the composition $g \circ f : X \to Z$ is tracked by the element $\langle x \rangle t(rx)$. So the assemblies on D together with their morphisms form a category, denoted by $\mathrm{Ass}(D)$.

Define the subcategory of modest sets by requiring an extra property.

Definition 1.16 Let D be a PCA, an assembly X on D is called a *modest set* if for all $x, x' \in |X|$ the following is true:

$$\text{if } \; x \neq x' \; \text{ then } \; E_X(x) \cap E_X(x') = \emptyset$$

Write $\mathrm{Mod}(D)$ for the category of modest sets on D

In a modest set every element of D represents at most one value. When thinking of the PCA as a model of computation, this is a nice property to have.

It is interesting to investigate the categorical properties of $\mathrm{Ass}(D)$ and $\mathrm{Mod}(D)$. For example, it can be shown that both categories are catersian-closed, contain all finite limits and are regular. These properties can be found in the thesis by J.R. Longley [7]. We will prove some of those properties in section 3.1 and we find a category that is equivalent to $\mathrm{Mod}(D)$. In section 3.2 and 3.3 we investigate the category of modest sets on the PCA of functions defined in Chapter 2.

Remark. In his thesis, Longley introduced a definition of morphisms between PCAs. We define this for future reference, however it plays only a minor role in this thesis.

Definition 1.17 Let $(D, \cdot)$ and $(E, \star)$ be PCAs. An *applicative morphism* from D to E is a function $\gamma : D \to \mathcal{P}^*(E)$ with the property that there is an element $r \in E$ such that for all $a, a' \in D$ and $b \in \gamma(a)$, $b' \in \gamma(a')$:

$$\text{if } a \cdot a' \text{ denotes, then } r \star b \star b' \text{ denotes and } r \star b \star b' \in \gamma(a \cdot a')$$

Say that r *realizes* the applicative morphism γ.

In terms of computation: an applicative morphism γ realized by r means that any program $a \in D$ gives rise to programs $r \star b \in E$, for $b \in \gamma(a)$. These programs send representations of a' to representations of $a \cdot a'$ (if this is defined in D). We can think of γ as an "implementation" of D on E.

It can be shown that the PCAs together with the applicative morphisms define a category. We will not prove this.

2 The PCA of functions

The first sections define a PCA on the set A^A (the set of function from A to A) for some infinite set A. This PCA is thanks to J. van Oosten [14]. The application of this PCA is studied in some detail. After that, we explore the special properties of the case $A = \mathbb{N}$.

2.1 Interrogation

Let A be an infinite set. The first goal is to define an application $(\alpha, \beta) \mapsto \alpha \cdot \beta$ on the set A^A. To do this, let A^* be the set of all finite sequences of elements of A. Choose an injective function $f : A^* \to A$.[1] Use notation:

$$[a_0, \ldots, a_{n-1}] := f(a_0, \ldots, a_{n-1}) \qquad \text{for } \ a_0, \ldots, a_{n-1} \in A$$

We call $[a_0, \ldots, a_{n-1}]$ the code of finite sequence $(a_0, \ldots, a_{n-1})$. In some cases we write $[(a_0, \ldots, a_{n-1})]$ to indicate the code. Let $\hat{q}$ and $\hat{r}$ be fixed and distinct elements of A, called the 'query' and 'result'. Define the following interaction between elements of A^A.

Definition 2.1 For $\alpha, \beta \in A^A$, $a \in A$ and $u = (a_0, \ldots, a_{n-1})$ a finite sequence, call u an *a-interrogation* of β by α, if for each $j \leq n - 1$ there is an element $b \in A$ such that $\alpha([a, a_0, \ldots, a_{j-1}]) = [\hat{q}, b]$ and $\beta(b) = a_j$.

For fixed $a \in A$, any pair of functions $\alpha, \beta \in A^A$ determine an unique sequence of a-interrogations of β by α (also called the *interrogation process*), which can be finite or infinite. Note that any such sequence starts with $u = (\ )$ for the case $n = 0$.

Let $\alpha, \beta \in A^A$ and $a \in A$, we say that that $\varphi^a(\alpha, \beta)$ *is defined with value c* if and only if there is a a-interrogation $u = (a_0, ..., a_{n-1})$ of β by α such that $\alpha([a, a_0, ..., a_{n-1}]) = [\hat{r}, c]$. We call c the result of the interrogation u.

Using this we can define a partial function $A^A \times A^A \rightharpoonup A^A$ denoted by $(\alpha, \beta) \mapsto \alpha \cdot \beta$ in the following way: $\alpha \cdot \beta$ is defined if and only if for every $a \in A$, $\varphi^a(\alpha, \beta)$ is defined. In that case, $\alpha \cdot \beta$ is the function $a \mapsto \varphi^a(\alpha, \beta)$.

To understand the definitions it is helpful to consider an example.

Example 2.2 Let A be an infinite set and assume $\mathbb{N} \subseteq A$ (otherwise find injective function $g : \mathbb{N} \to A$). We define a function $\alpha \in A^A$ as follows: for any sequence $(a, a_0, ..., a_{n-1}) \in A^*$ let:

$$\begin{aligned}
&\alpha([a]) := [\hat{q}, a] && (n = 0)\\
&\alpha([a, a_0, ..., a_{n-1}]) := [\hat{q}, a_{n-1}] && \text{if } n \geq 1 \text{ and } a \neq a_{n-1}\\
&\alpha([a, a_0, ..., a_{n-1}]) := [\hat{r}, n] && \text{if } n \geq 1 \text{ and } a = a_{n-1}
\end{aligned}$$

[1] Such a function exists since A^* and A are of equal cardinality.

To make α total define $\alpha(b) := [\hat{q}, \hat{q}]$ for all elements $b \in A$ not yet in the domain. The sequence coding is injective, so α is well-defined. Then α gives rise to a partial function $A^A \rightharpoonup A^A$ defined by $\beta \mapsto \alpha \cdot \beta$. Check for yourself that $\varphi^a(\alpha, \beta)$ is defined if and only if there is $n \in \mathbb{N}$ such that $\beta^n(a) = a$, (where $\beta^n := \beta \circ ... \circ \beta$, composition n-times). So $\alpha \cdot \beta$ is defined if and only if for every $a \in A$ there is $n \in \mathbb{N}$ such that $\beta^n(a) = a$. In that case $\alpha \cdot \beta(a) = n$, where n is the least number satisfying $\beta^n(a) = a$.

Closely related to interrogations is the notion of partial sequential functions. Those functions are defined using sequential trees. We call a function finite iff it has a finite domain. For functions $p, q : A \rightharpoonup A$ we write $p \subseteq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $p(a) = q(a)$ for all $a \in \mathrm{dom}(p)$.

Definition 2.3 Let A be an infinite set. Consider a set T of finite functions $p : A \rightharpoonup A$ that contains the empty function and is ordered by inclusion. Call T a *tree* if for every element $p \in T$ the set $\{q \in T \mid q \subseteq p\}$ is linearly ordered. The empty function is the *root* of the tree. Element q is called an *immediate successor* of p, if p is the greatest element below q. A *path* through the tree is a maximal linearly ordered subset and a *leaf* is a maximal element.

A tree T is called a *sequential tree* if it has the property that for every $p \in T$ which is not a leaf, there is an element $a \notin \mathrm{dom}(p)$ such that if q is an immediate successors of p in T, then $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{a\}$. Call a sequential tree T *total* if for each non-leaf p, with corresponding element a, the set of immediate successors is the set of *all* functions satisfying $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{a\}$.

If T is a total sequential tree, then any $\alpha \in A^A$ determines an unique path through T: the set $\{p \in T \mid p \subseteq \alpha\}$. We use this in the following definition.

Definition 2.4 Call a partial function $F : A^A \rightharpoonup A$ *partial sequential* if there is a total sequential tree T and a function K from the set of leaves of T to A such that:

- For any $\alpha \in A^A$, $F(\alpha)$ is defined if and only if the path through T determined by α ends in a leaf q.
- In that case $F(\alpha) = K(q)$.

Sometimes we write $\Phi_{T,K}$ to indicate the function F.

In a moment we show the relation between interrogations and partial sequential functions. We first return to our example.

Example 2.2' Let A be an infinite set such that $\mathbb{N} \subseteq A$ and let $a \in A$. We are going to define a sequential function and compare it with $\beta \mapsto \alpha \cdot \beta(a)$ from example 2.2. The sequential function is defined by constructing a sequence $T_0 \subseteq T_1 \subseteq ...$ of sequential trees and taking the union over this sequence.

Start by defining tree T_0 consisting of the empty function, together with all finite functions p satisfying $\mathrm{dom}(p) = \{a\}$, and all p satisfying $\mathrm{dom}(p) = \{a, a_0\}$ s.t. $p(a) = a_0$. Suppose T_k is defined, we construct T_{k+1}. Denote the set of leaves of T_k by $L(T_k)$. For

$p \in L(T_k)$, the tree induces a linear order on $\mathrm{dom}(p)$, write $\mathrm{dom}(p) = \{a, a_0, ..., a_{n-1}\}$. Define sets E_p as follows: If $a_{n-1} = a$, then $E_p := \emptyset$. Otherwise define:

$$E_p := \{q : A \rightharpoonup A \mid p \subset q \ \wedge \ \mathrm{dom}(q) = \mathrm{dom}(p) \cup \{p(a_{n-1})\}\}$$

This allows us to construct the sequential tree:

$$T_{k+1} := T_k \cup (\bigcup_{p \in L(T_k)} E_p)$$

Note that $T_k \subseteq T_{k+1}$, let T be the union of the sequence $T_0 \subseteq T_1 \subseteq ...$. For any leaf $p \in T$ we can again write $\mathrm{dom}(p) = \{a, a_0, ..., a_{n-1}\}$, define $K(p) := n$. This defines a partial sequential function $\Phi_{T,K}$. Consider the element $\alpha \in A^A$ defined in the previous example. Check that $\Phi_{T,K}(\beta)$ is defined if and only if there is $n \in \mathbb{N}$ such that $\beta^n(a) = a$. In that case we have $\Phi_{T,K}(\beta) = \varphi^a(\alpha, \beta)$. If for *every* $b \in A$ there is $n \in \mathbb{N}$ such that $\beta^n(b) = b$, then $\Phi_{T,K}(\beta) = \alpha \cdot \beta(a)$. Otherwise the right-hand side is not defined.

The example makes us believe there is indeed a relation between interrogations and partial sequential functions. This is made precise in the following results.

Proposition 2.5 *Let $F_a : A^A \rightharpoonup A$, $a \in A$, be a collection of partial sequential functions. Then there is an $\alpha \in A^A$ such that for all $\beta \in A^A$ and all $a \in A$, $\varphi^a(\alpha, \beta)$ is defined if and only if $F_a(\beta)$ is defined. In that case $\varphi^a(\alpha, \beta) = F_a(\beta)$.*

Proof: Assume the functions F_a are sequential. For each $a \in A$ find sequential tree T_a and function K_a such that F_a is the function Φ_{T_a,K_a}. If $p \in T_a$, then any path through p in T_a induces a linear order on $\mathrm{dom}(p)$, write $\mathrm{dom}(p) = \{b_0, ..., b_{n-1}\}$. Also write $a_i := p(b_i)$ for $i \leq n-1$. Now define function $\alpha \in A^A$ as follows: for any $p \in T_a$, with notation as above, define:

$$\begin{aligned} \alpha([a, a_0, ..., a_{n-1}]) &:= [\hat{r}, K_a(p)] && \text{if } p \text{ is a leaf of } T_a \\ \alpha([a, a_0, ..., a_{n-1}]) &:= [\hat{q}, b_n] && \text{otherwise} \end{aligned}$$

Where b_n is the unique element of $\mathrm{dom}(q) - \mathrm{dom}(p)$ for each immediate successor q of p in the tree. Do this for all partial sequential functions F_a. Define $\alpha(c) := [\hat{q}, \hat{q}]$ for all $c \in A$ not yet in the domain of α, to make sure α is total. The reader should check that α is well-defined and that it satisfies the proposition. □

Corollary 2.6 *Let $F_a : A^A \rightharpoonup A$, $a \in A$, be a collection of partial sequential functions. Then there is an $\alpha \in A^A$ such that for all $\beta \in A^A$:*

$$\alpha \cdot \beta \textit{ is defined if and only if } F_a(\beta) \textit{ is defined for all } a \in A$$

And if $\alpha \cdot \beta$ is defined, then $\alpha \cdot \beta(a) = F_a(\beta)$ for all $a \in A$.

Proof: Apply proposition 2.5 to the functions F_a, the result follows by the definition of the application $(\alpha, \beta) \mapsto \alpha \cdot \beta$. □

The following result is a converse of proposition 2.5.

Proposition 2.7 *Let $\alpha \in A^A$ and $a \in A$. Then the function $F_a : A^A \rightharpoonup A$ defined by $\beta \mapsto \varphi^a(\alpha, \beta)$ is partial sequential.*

Proof: Note that the definition of interrogation still makes sense when considering finite functions $p : A \rightharpoonup A$. Let $(a_0, \dots a_{n-1})$ be an a-interrogation of p by α. We find corresponding sequence $(b_0, \dots, b_{n-1})$ such that $\alpha([a, a_0, \dots, a_{i-1}]) = [\hat{q}, b_i]$ for all $i \leq n-1$. Define T_0 as the set of all finite functions p such that there is an a-interrogation $(a_0, \dots a_{n-1})$ of p by α with the property that $\mathrm{dom}(p) = \{b_0, \dots, b_{n-1}\}$. In other words: all elements of $\mathrm{dom}(p)$ are used in the interrogation.

Claim: T_0 is a total sequential tree. To show that for $p \in T_0$ the set $\{q \in T_0 \mid q \subseteq p\}$ is linearly ordered, let $q, q' \in T_0$ with $q, q' \subseteq p$. Find a corresponding a-interrogation $(a_0, \dots, a_{n-1})$ of p by α in which all elements of $\mathrm{dom}(p)$ are used. The a-interrogations of q and q' by α, using all domain elements, are initial sequences of $(a_0, \dots, a_{n-1})$. It follows that $q \subseteq q'$ or $q' \subseteq q$. The other properties are left to the reader to check.

Let p be a leaf of T_0. Consider the following two cases: 1) the a-interrogation process of p by α is infinite, so α keeps asking for information it already knows. Or 2) there is some a-interrogation u of p by α such that $\alpha([u])$ is neither a query nor a result. In both cases find a set $\{e_0, e_1, \dots\} \subseteq A - \mathrm{dom}(p)$. Extend p by adding all finite functions of the form $p \cup r$, where $\mathrm{dom}(r) = \{e_0, \dots, e_i\}$ for some $i \in \mathbb{N}$. In any other case, do not extend p. Doing this for all $p \in T_0$ defines a sequential tree T.

For any leaf p of T there is an a-interrogation u of p by α such that $\alpha([u]) = [\hat{r}, b]$ for some b. Other cases are excluded by the previous step. Define $K(p) := b$. Check that $\Phi_{T,K}$ is the function F_a, so this function is partial sequential. □

Note that for $\alpha \in A^A$ and $a \in A$ the partial function $\beta \mapsto \alpha \cdot \beta(a)$ is in general not partial sequential. The tree is (in general) not able to check if $\alpha \cdot \beta$ is defined, we can use example 2.2 to find a counter example.

2.2 Bisequential trees

It is still not proven that the application $\alpha \cdot \beta$ defined in the previous section satisfies the definition of a PCA. However, before showing that, we need to learn more about the application. This is done by considering the notion of bisequential trees.

Definition 2.8 Let T be a set of pairs of finite functions (p, q), with $p, q : A \rightharpoonup A$, ordered by pairwise inclusion. Assume that T contains $(\emptyset, \emptyset)$, the pair of empty functions. Call T a *tree* if for every pair (p, q) the set $\{(p', q') \mid p' \subseteq p,\ q' \subseteq q)\}$ is linearly ordered. Further definitions involving this type of tree, like *immediate successors*, are similar to definition 2.3.

Call tree T a *bisequential tree* if for any non-leaf (p, q) *either* there is an element $a \notin \mathrm{dom}(p)$ such that the set of immediate successors of (p, q) is the set of all pairs (p', q) satisfying $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{a\}$, *or* there is $b \notin \mathrm{dom}(q)$ such that the set of immediate successors is the set of all pairs (p, q') satisfying $\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{b\}$. In the first case call (p, q) a $(0, a)$-point. In the second case call (p, q) a $(1, b)$-point.

If T is a bisequential tree, then any pair $\alpha, \beta \in A^A$ determines a unique path through T, the set $\{(p, q) \in T \mid p \subseteq \alpha,\ q \subseteq \beta\}$. Similar to definition 2.4 a function $G : A^A \times A^A \rightharpoonup A$ is called *partial bisequential* if there is a bisequential tree T and function K, such that $G(\alpha, \beta)$ is defined if and only of the path determined by α, β ends in a leaf (p, q). And in that case $K(p, q) = G(\alpha, \beta)$. We will again write $\Phi_{T,K}$ to indicate this function.

Bisequential trees are used primarily for finding elements $\gamma \in A^A$ with certain properties, made more precise in the following lemma. We use this lemma in the next section to show that $(A^A, \cdot)$ is indeed a PCA.

Lemma 2.9 *Let $G_a : A^A \times A^A \to A$, $a \in A$, be a collection of total biseqential functions. Then there is an element $\delta \in A^A$ such that for all $\alpha, \beta \in A^A$ and $a \in A$, $\delta \cdot \alpha$ is defined, $(\delta \cdot \alpha) \cdot \beta$ is defined and $((\delta \cdot \alpha) \cdot \beta)(a) = G_a(\alpha, \beta)$.*

Proof: For every $a \in A$ we find a bisequential tree T_a and function K_a such that G_a equals Φ_{T_a,K_a}. Since G_a is total, every path in T_a is finite.

Let $(p, q) \in T_a$. Define $P_{(p,q)} := \{(r, s) \in T_a : r \subseteq p,\ s \subseteq q\}$ to indicate to path through T_a up to (p, q). This path induces a linear ordering on the domains, write $\mathrm{dom}(p) = \{b_0, ..., b_{n-1}\}$ and $\mathrm{dom}(q) = \{d_0, ..., d_{m-1}\}$. Also define elements $a_i := p(b_i)$ and $c_j := q(d_j)$ for all $i \leq n - 1$ and $j \leq m - 1$.

Find the least number $l \leq n-1$ with the property that there is an $(0, b_l)$-point $(r, s) \in P_{(p,q)} \setminus \{(p, q)\}$ such that every $(r', s') \in P_{(p,q)} \setminus \{(p, q)\}$ with $r \subseteq r'$ and $s \subseteq s'$ is also a $(0, x)$-point, for some x. If such an l exists, consider $u := ([a, c_0, \ldots, c_{m-1}], a_l, \ldots, a_{n-1})$. If there is no such number, let $u := ([a, c_0, \ldots, c_{m-1}])$. We define the value of $\delta \in A^A$ on the element $[u]$. There are two cases, if (p, q) is not a leaf define:

$$\begin{aligned} &\delta([u]) := [\hat{q}, b] && \text{if } (p, q) \text{ is a } (0, b)\text{-point} \\ &\delta([u]) := [\hat{r}, [\hat{q}, d]] && \text{if } (p, q) \text{ is a } (1, d)\text{-point} \end{aligned}$$

for the unique $b \in A$ or $d \in A$. In the case that (p, q) is a leaf, define:

$$\delta([u]) := [\hat{r}, [\hat{r}, K_a(p, q)]]$$

Do this for all $a \in A$ and $(p, q) \in T_a$. Check this is well-defined. Complete the definition of δ by defining $\delta(c) := [\hat{r}, \hat{r}]$ for all $c \in A$ not yet in the domain.

Now consider $\alpha, \beta \in A^A$ and $a \in A$. By definition of δ and since every path through T_a is finite, it follows that $\delta \cdot \alpha$ is defined. For $(p, q) \in T_a$ on the path determined by α, β we have that $(\delta \cdot \alpha)([a, c_0, \ldots, c_{j-1}]) = [\hat{q}, b_j]$ for all $j \leq m - 1$, using notation from above. If (p, q) is a leaf, then also $(\delta \cdot \alpha)([a, c_0, \ldots, c_{m-1}]) = [\hat{r}, G_a(\alpha, \beta)]$. This shows that $((\delta \cdot \alpha) \cdot \beta)(a) = G_a(\alpha, \beta)$, convince yourself this is true. □

We can think of the following result as a 'strong converse' of lemma 2.9. The proof uses similar ideas as the proof of proposition 2.7.

Proposition 2.10 *Let $\delta \in A^A$ such that $\delta \cdot \alpha$ is defined for all $\alpha \in A^A$, let $a \in A$. Then the partial function $(\alpha, \beta) \mapsto \varphi^a((\delta \cdot \alpha), \beta)$ is partial bisequential.*

Proof: Let $\delta \in A^A$ as above and let $a \in A$. Note that any two (finite) functions $p, q : A \rightharpoonup A$ determine an unique, possible infinite, sequence:

$$(a_0^0, \dots, a_0^{n_0-1}, c_0, \dots, a_j^0, \dots, a_j^{n_j-1}, c_j, \dots) \tag{2.1}$$

Such that for every j we find an element d and for each $i \leq n_j - 1$ we find b satisfying:

$$\begin{aligned} &\delta([[a, c_0, \dots, c_{j-1}], a_j^0, \dots, a_j^{i-1}]) = [\hat{q}, b] && \text{and } p(b) = a_j^i \\ &\delta([[a, c_0, \dots, c_{j-1}], a_j^0, \dots, a_j^{n_j-1}]) = [\hat{r}, [\hat{q}, d]] && \text{and } q(d) = c_j \end{aligned}$$

So we have that $(a_j^0, \dots, a_j^{n_j-1})$ is an $[a, c_0, \dots, c_{j-1}]$-interrogation of p by δ.[2] Note that the sequence stops if δ does not give a query or result, or if the output is $[\hat{r}[\hat{r}, y]]$ for some y. It also stops if the information asked is not in the domain of q or p.

Build a sequence of bisequential trees $T_0 \subseteq T_1 \subseteq \dots$. Define T_0 as the set of pairs $(p, \emptyset)$ such that there is an $[a]$-interrogation of p by δ in which all elements of $\mathrm{dom}(p)$ are used, similar as in proposition 2.7. Suppose T_n is defined. Let $(p, q) \in T_n$ be a leaf, if the corresponding sequence defined in (2.1) is finite, consider two cases:

1. The sequence ends in $(\dots, a_j^{i-1})$ and there is a d such that

   $$\delta([[a, c_0, \dots, c_{j-1}], a_j^0, \dots, a_j^{i-1}]) = [\hat{r}, [\hat{q}, d]]$$

   then add all pairs (p, q') such that $\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{d\}$.

2. The sequence ends in $(\dots, c_{j-1})$ and there is a b such that

   $$\delta([[a, c_0, \dots, c_{j-1}]]) = [\hat{q}, b]$$

   then add all pairs (p', q) extending (p, q) such that there is an $[a, c_0, .., c_{j-1}]$-interrogation of p' by δ in which all elements of $\mathrm{dom}(p') - \mathrm{dom}(p)$ are used.

In all other cases do not extend the leaf. Doing this for all leaves of T_n defines a extended tree denoted by T_{n+1}. Define $\tilde{T}$ as the union of the sequence $T_0 \subseteq T_1 \subseteq \dots$. Let (p, q) be a leaf of $\tilde{T}$. Consider the cases where the sequence defined in (2.1) is infinite or the next value asked by δ is neither a query nor a result. In those cases let $\{e_0, e_1, \dots\} \subseteq A - \mathrm{dom}(p)$ and add all pairs (p', q) where $p' = p \cup r$ with $\mathrm{dom}(r) = \{e_0, \dots, e_i\}$ for some $i \geq 0$. This defines a tree T.

For any leaf $(p, q) \in T$ the sequence defined in (2.1) ends in $(\dots, a_j^{i-1})$ for some $i \geq 0$ and $j \geq 0$ and $\delta([[a, c_0, \dots, c_{j-1}], a_j^0, \dots, a_j^{i-1}]) = [\hat{r}, [\hat{r}, y]]$ for some y. Define $K(p, q) := y$, then $\Phi_{T,K}$ is the function $(\alpha, \beta) \mapsto \varphi^a((\delta \cdot \alpha), \beta)$. $\square$

[2] Also, one can think of $(c_0, \dots, c_m)$ as an a-interrogation of q by $\delta \cdot p$. However $\delta \cdot p$ is in general not defined, so this is just for intuition.

2.3 The elements k and s

We are almost ready to show that A^A with application $(\alpha, \beta) \mapsto \alpha \cdot \beta$ defined in section 2.1, is indeed a PCA. This is done by showing the existence of elements k and s satisfying definition 1.2. After we have established this fact, we write $\mathcal{K}_2(A) := (A^A, \cdot)$ to indicate the PCA. The theorem below is also proven in [14]. Our proof has a similar outline.

We introduce some more terminology: Let T be a bisequential tree. For $(p, q) \in T$, the *length* of the path up to (p, q) is the number of elements in $\{(p', q') \mid p' \subseteq p,\ q' \subseteq q\}$. Say that T *is of height* n if every path is of length $\leq n$ and there is a path of exactly length n. Recall that A^* is the set of all finite sequences of elements of A. For $u = (a_0, \dots, a_{n-1})$, $v = (b_0, \dots, b_{m-1})$ finite sequences, define $u * v := (a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1})$.

Theorem 2.11 *Let A be an infinite set. The set A^A together with the application $(\alpha, \beta) \mapsto \alpha \cdot \beta$ defines a partial combinatory algebra.*

Proof: Define element $k \in A^A$ in the following way: $k([[a]]) = [\hat{q}, a]$ for all $a \in A$ and $k([[a], a_0]) = [\hat{r}, [\hat{r}, a_0]]$ for all coded sequences of length two. Let k be the identity on all other elements. It is straightforward to show that $k\alpha\beta = \alpha$ for all $\alpha, \beta \in A^A$ (note that $k\alpha\beta$ is just short notation for $k \cdot \alpha \cdot \beta$).

For s we want to find total bisequential functions $G_a : A^A \times A^A \rightharpoonup A$ such that if δ is as in lemma 2.9, then $\delta\alpha\beta\gamma \simeq (\alpha\gamma)(\beta\gamma)$. Note that in this case $\delta\alpha\beta$ receives as input coded sequences of the form $[(a) * u]$, so we actually need to define functions $G_{[(a)*u]}$ for $a \in A$ and $u \in A^*$.

First consider fixed $\alpha, \beta \in A^A$, let $a \in A$, and take a look at $(\alpha\gamma)(\beta\gamma)(a)$ for some $\gamma \in A^A$. Any finite function t (think $t \subseteq \gamma$) determines a, possible infinite, sequence:

$$(a_0^0, \dots, a_0^{n_0-1}, e_0, c_0^0, \dots, c_0^{m_0-1}, f_0 \dots, a_j^0, \dots, a_j^{n_j-1}, e_j, c_j^0, \dots, c_j^{m_j-1}, f_j, \dots) \tag{2.2}$$

such that for each j and $i \leq n_j - 1$ there is a b_j^i satisfying:

$$\alpha([[a, f_0, \dots, f_{j-1}], a_j^0, \dots, a_j^{i-1}]) = [\hat{q}, b_j^i] \qquad \text{and } t(b_j^i) = a_j^i$$
$$\alpha([[a, f_0, \dots, f_{j-1}], a_j^0, \dots, a_j^{n_j-1}]) = [\hat{r}, [\hat{q}, e_j]]$$

and for each j and $i \leq m_j - 1$ there is a d_j^i satisfying:

$$\beta([e_j, c_j^0, \dots, c_j^{i-1}]) = [\hat{q}, d_j^i] \qquad \text{and } t(d_j^i) = c_j^i$$
$$\beta([e_j, c_j^0, \dots, c_j^{m_j-1}]) = [\hat{r}, f_j]$$

With this in mind we are ready to define bisequential trees $T_{(a)*u}$ together with function $K_{(a)*u}$, for $a \in A$ and u finite sequences. This is done by induction on the length of sequence u.

Start with $a \in A$ and $u = (\ )$. Define $T_{(a)}$ as $\{(\emptyset, \emptyset)\}$ together with the set of all pairs $(p, \emptyset)$ satisfying $\mathrm{dom}(p) = \{[[a]]\}$. Let $(p, \emptyset)$ be a leaf, if there is $b \in A$ such that $p([[a]]) = [\hat{q}, b]$, then define $K_{(a)}(p, \emptyset) = [\hat{q}, b]$. If there is $y \in A$ such that $p([[a]]) = [\hat{r}, [\hat{r}, y]]$, define $K_{(a)}(p, \emptyset) = [\hat{r}, y]$. Otherwise let $K_{(a)}(p, \emptyset) = [\hat{q}, \hat{q}]$.

Now consider sequence $u = (u_0, \dots, u_n)$ of length $n + 1$. Let $\tilde{u} = (u_0, \dots u_{n-1})$ and assume tree $T_{(a)*\tilde{u}}$ of height at most $n + 2$ has been defined. Let (p, q) be a leaf of this tree. The unique path to (p,q) in $T_{(a)*\tilde{u}}$ determines a finite function t: Consider $(p',q') \subsetneq (p,q)$ and let i be the length of the path up to (p',q'). If (p',q') is a $(0,x)$-point and $p(x) = [\hat{q}, b]$, define $t(b) = u_{i-1}$. If (p',q') is a $(1,x)$-point and $q(x) = [\hat{q}, d]$, define $t(d) = u_{i-1}$. Do nothing in any other case. The fact that t is well-defined follows from the inductive construction of the tree (done below).

So for leaf (p,q) we have found a finite t. For this t we can find a maximal sequence of length $\leq n+1$ satisfying (2.2), where p and q have the role of α and β respectively. Extend tree $T_{(a)*\tilde{u}}$ in the following cases:

1. The sequence ends in $(a_j^0, \ldots, a_j^{i-1})$ for some $i \in \mathbb{N}$
   (a) and $p([[a, f_0, \ldots, f_{j-1}], a_j^0, \ldots, a_j^{i-1}]) = [\hat{q}, b]$, then add all pairs (p', q) with:
   $$\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{[[a, f_0, \ldots, f_{j-1}], a_j^0, \ldots, a_j^{i-1}, u_n]\}$$
   (b) and $p([[a, f_0, \ldots, f_{j-1}], a_j^0, \ldots, a_j^{i-1}]) = [\hat{r}, [\hat{q}, e_j]]$, add pairs (p, q') satisfying:
   $$\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{[e_j]\}$$

2. The sequence ends in $(c_j^0, \ldots, c_j^{i-1})$ for some $i \in \mathbb{N}$
   (a) and $q([e_j, c_j^0, \ldots, c_j^{i-1}]) = [\hat{q}, d]$, then add all pairs (p, q') such that:
   $$\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{[e_j, c_j^0, \ldots, c_j^{i-1}, u_n]\}$$
   (b) and $q([e_j, c_j^0, \ldots, c_j^{i-1}]) = [\hat{r}, f_j]$, add pairs (p', q) satisfying:
   $$\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{[[a, f_0, \ldots, f_j]]\}$$

In all other cases, do not extend the leaf. This defines a tree $T_{(a)*u}$. Let (p,q) be a leaf of this tree. If (p,q) is defined by cases 1(a) or 2(a), define $K_{(a)*u}(p,q)$ equal to $[\hat{q}, b]$ or $[\hat{q}, d]$ respectively. If (p,q) is defined by case 1(b) or 2(b) define $K_{(a)*u}(p,q) = [\hat{q}, \hat{q}]$. Otherwise (p,q) is already a leaf of $T_{(a)*\tilde{u}}$. In that case just define $K_{(a)*u}(p,q) = K_{(a)*\tilde{u}}(p,q)$ with one exception: if the maximal corresponding sequence (2.2) is of length n, ends in $(a_j^0, \ldots, a_j^{i-1})$ and $p([[a, f_0, \ldots, f_{j-1}], a_j^0, \ldots, a_j^{i-1}]) = [\hat{r}, [\hat{r}, y]]$, then define $K_{(a)*u}(p,q) = [\hat{r}, y]$. This concludes the definition.

Tree $T_{(a)*u}$ with function $K_{(a)*u}$ define a total bisequential function, write $G_{[(a)*u]}$ to indicate this function. By adding some dummy functions we obtain functions G_b for every $b \in A$. Applying lemma 2.9 we find an element $\delta \in A^A$ such that $\delta\alpha\beta([(a)*u]) = G_{[(a)*u]}(\alpha, \beta)$ for all $a \in A$, $u \in A^*$ and $\alpha, \beta, \in A^A$. For $\gamma \in A^A$, check that the following is true: if $\alpha\gamma$ and $\beta\gamma$ are defined, then $\delta\alpha\beta\gamma \simeq (\alpha\gamma)(\beta\gamma)$. To offer some help: think of u as a sequence of the form $(a_0^0, \ldots, a_0^{n_0-1}, \hat{q}, c_0^0, \ldots, c_0^{m_0-1}, \hat{q}, \ldots)$.

The most important work is done. However it is still possible that $\delta\alpha\beta\gamma$ is defined, but $\alpha\gamma$ or $\beta\gamma$ is not. To remedy this we define trees $R_{(a)*u}$ extending $T_{(a)*u}$ that also checks if $\alpha\gamma$ (and later also if $\beta\gamma$) is defined.

Let $a \in A$. Any $\alpha \in A^A$ and finite function t determine a, possible infinite, sequence $(v_0, v_1, \dots)$ such that any initial part is an a-interrogations of t by α. We adjust the definition of $T_{(a)*u}$ somewhat. Define $R_{(a)}$ equal to $T_{(a)}$. Assume $R_{(a)*\tilde{u}}$ is defined. Let $(p,q) \in R_{(a)*\tilde{u}}$ be a leaf. If $(p,q) \in T_{(a)*\tilde{u}}$ and $K_{(a)*\tilde{u}}(p,q) = [\hat{r}, y]$, let $l := 0$. If $(p,q) \notin T_{(a)*\tilde{u}}$ and there is an $i \geq 1$ such that $L_{(a)*\tilde{u}}(p,q) = [\hat{q}, v_{i-1}]$, let $l := i$. In those cases extend leaf (p,q) by adding pairs (p', q) with:

$$\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{[a, v_0, \dots, v_{l-1}]\}$$ [3]

Extend other leaves that are also in $T_{(a)*u}$ using the construction of $T_{(a)*u}$. This defines tree $R_{(a)*u}$. The function $L_{(a)*u}$ is equal to $K_{(a)*u}$, except for leaves (p', q) constructed in the first two cases. If $p'([a, v_0, \dots, v_{l-1}]) = [\hat{q}, z]$, then define $L_{(a)*u}(p', q) = [\hat{q}, v_l]$. If $p'([a, v_0, \dots, v_{l-1}]) = [\hat{r}, z]$, define $L_{(a)*u}(p', q) = [\hat{r}, y]$, where $[\hat{r}, y]$ is the result in tree $T_{(a)*u}$ on the path determined by (p', q). Otherwise define $L_{(a)*u}(p', q) = [\hat{q}, \hat{q}]$.

For $\alpha, \beta, \gamma \in A^A$ we now have: The path determined by α, β in $R_{(a)*u*v}$ ends in result $[\hat{r}, y]$ if and only if the path in $T_{(a)*u}$ ends in $[\hat{r}, y]$, and v is an a-interrogation of γ by α with result. Applying lemma 2.9 to the total bisequential function determined by $R_{(a)*u}$ we find an element $r \in A^A$. The intuition is that $r\alpha\beta\gamma(a)$ first determines the value that $(\alpha\gamma)(\beta\gamma)(a)$ would take and then checks if $(\alpha\gamma)(a)$ is defined.

We again extend $R_{(a)*u}$ to trees $S_{(a)*u}$ such that $S_{(a)*u*v*w}$ also checks if w is an a-interrogation of γ by α with a result. Applying lemma 2.9 to functions determined by $S_{(a)*u}$ we find an element $s \in A^A$, with the property that $s\alpha\beta\gamma \simeq (\alpha\gamma)(\beta\gamma)$. $\square$

As mentioned before, we write $\mathcal{K}_2(A)$ to indicate the PCA.

We close this section with two remarks about interrogations and the corresponding PCA application. Results that are related to both remarks can be found in [14].

Remarks. (a) The definition of $\mathcal{K}_2(A)$ depends on a coding $[\cdot, \dots, \cdot]$ of finite sequences and on a choice for the elements $\hat{q}$ and $\hat{r}$. In the case that A itself is a partial combinatory algebra $(A, \star)$, consider the coding $[\cdot, \dots, \cdot]_A$ defined in the last part of section 2.2. Say that $\mathcal{K}_2(A)$ is *based* on $(A, \star)$ iff in the definition of an interrogation we use $[\cdot, \dots, \cdot]_A$ for the coding and $\hat{q} = \bar{k}_A$, $\hat{r} = k_A$ (the elements $\bar{k}$ and k in A) for the query and result.

Assuming that $\mathcal{K}_2(A)$ is based on A, we can relate elements of A to functions in A^A. For example find an element $r \in A$ such that $r \star [[a]] = [\bar{k}_A, a]$ and $r \star [[a], a_0] = [k_A, [k_A, a_0]]$ for all $a, a_0 \in A$. This is done by using basic operators from section 2.2. Then $a \mapsto r \star a$ defines a partial function $A \rightharpoonup A$. Any element of A^A extending this function satisfies the property of k in definition 1.2.

(b) The definitions and results from previous sections can be generalized to the set $\mathrm{part}(A, A)$ of all *partial* functions $A \rightharpoonup A$. We give a short overview.

The definition of interrogation and, subsequently, of the application on $\mathrm{part}(A, A)$ is similar to that on A^A. Only now $\alpha \cdot \beta$ is defined for all $\alpha, \beta \in \mathrm{part}(A, A)$. Modify the

[3] It is possible that p is already defined on this element. In that case do not extend the tree, but do adjust the value of $L_{(a)*u}(p, q)$. From now on regard (p, q) as a leaf not occurring in $T_{(a)*u}$.

definition of partial sequential functions by also considering non-total trees or, equivalently, by considering function K from the set of leaves to A that are partial. The notion of bisequential functions is modified in a similar fashion.

After some adjustments, the results from previous sections still hold up. An advantage of $\mathrm{part}(A, A)$ with its application is that there are less restrictions to take in account. For example proposition 2.7 now states:

Let $\alpha \in \mathrm{part}(A, A)$ and $a \in A$. Then the function $F_a : \mathrm{part}(A, A) \rightharpoonup A$ defined by $\beta \mapsto \alpha \cdot \beta(a)$ is partial sequential.

Another example: after adjustment lemma 2.9 reads:

Let $G_a : \mathrm{part}(A, A) \times \mathrm{part}(A, A) \rightharpoonup A$, $a \in A$, be a collection of partial biseqential functions. Then there is an element $\delta \in \mathrm{part}(A, A)$ such that for all $\alpha, \beta \in \mathrm{part}(A, A)$ and $a \in A$, $((\delta \cdot \alpha) \cdot \beta)(a)$ is defined if and only if $G_a(\alpha, \beta)$ is defined, in that case they are equal.

The proof of the lemma needs only slight modifications: the tree T_a can be non-total and δ is allowed to be partial.

So it can be shown that $\mathrm{part}(A, A)$ together with its application is a PCA. Note that if it is based on $(A, \star)$, then every element of A really represents an element of $\mathrm{part}(A, A)$. For example the partial function $a \mapsto r \star a$ form previous remark is an element of $\mathrm{part}(A, A)$ that satisfies the property of k.

More results are in the article mentioned above. Take a look at [12] to learn more about the PCA for the special case $A = \mathbb{N}$ and its connection to domain theory. We will not study it any further.

2.4 Functions on the natural numbers

In this section we investigate the PCA $\mathcal{K}_2(A)$ for the special case $A = \mathbb{N}$. This is done by defining a topology on $\mathbb{N}^{\mathbb{N}}$ and expressing the application of $\mathcal{K}_2(\mathbb{N})$ in terms of continuous functions. Some familiarity with the basic notions of topology is assumed, those notions can be found in any introduction book on topology, for example [8]. We keep notation general, so we can compare our results with the general case.

Let $\mathcal{O}_{\mathbb{N}}$ be the discrete topology on $\mathbb{N}$. Using the fact that $\mathbb{N}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{N}$, define the product topology $\mathcal{O}_{\mathbb{N}^{\mathbb{N}}}$ on $\mathbb{N}^{\mathbb{N}}$. The basic opens of this topology are sets of the form:

$$O_p := \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathrm{dom}(p) \,.\, \beta(n) = p(n)\}$$

for finite functions p. From now on we assume $\mathbb{N}^{\mathbb{N}}$ is induced with the topology $\mathcal{O}_{\mathbb{N}^{\mathbb{N}}}$. In abuse of notation we write $\mathbb{N}^{\mathbb{N}}$ instead of $(\mathbb{N}^{\mathbb{N}}, \mathcal{O}_{\mathbb{N}^{\mathbb{N}}})$ to indicate the topological space.

We extend notation "$\simeq$" used in PCA $\mathcal{K}_2(\mathbb{N})$ to partial functions. If g, h are partial functions write $g(x) \simeq h(x)$ iff either both g and h are defined and equal in x, or both are not defined in x. Let us first take a look at partial functions $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$. We use the notation $\varphi^a(\alpha, \beta)$ defined in section 2.1.

Proposition 2.12 *Let $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ a partial function and let $n \in \mathbb{N}$ be arbitrary, but fixed. The following are equivalent:*

(i) there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $F(\beta) \simeq \varphi^n(\alpha, \beta)$, for all $\beta \in \mathbb{N}^{\mathbb{N}}$

(ii) F is a continuous function with open domain

Proof: "$(i) \Rightarrow (ii)$" Assume $\alpha \in \mathbb{N}^{\mathbb{N}}$ satisfies (i). We want to show that for every $m \in \mathbb{N}$, the set $F^{-1}(\{m\}) = \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \varphi^n(\alpha, \beta) = m\}$ is open. Let $\beta \in F^{-1}(\{m\})$, then there is an n-interrogation $(n_0, \dots, n_{k-1})$ of β by α such that $\alpha([n, n_0, \dots, n_{k-1}]) = [\hat{r}, m]$. Find the corresponding $v_i \in \mathbb{N}$ satisfying $\alpha([n, n_0, \dots, n_{i-1}]) = [\hat{q}, v_i]$ for $i \leq k-1$. Define finite function $p : \{v_0, \dots, v_{k-1}\} \to A$, by $p(v_i) := n_i$. Then $\beta \in O_p \subseteq F^{-1}(\{m\})$, so F is continuous. This also shows that $\mathrm{dom}(F) = \bigcup_{m \in \mathbb{N}} F^{-1}(\{m\})$ is open.

"$(ii) \Rightarrow (i)$" Assume F is continuous with open domain. So for every $\beta \in \mathrm{dom}(F)$ there is $m \in \mathbb{N}$ and finite function p such that:

$$\beta \in O_p \subseteq \mathrm{dom}(F) \wedge \forall \gamma \in O_p \, . \, F(\gamma) = m \tag{2.3}$$

Define $\alpha([n]) := [\hat{q}, 0]$. Let $u = (n_0, \dots, n_{k-1}) \in \mathbb{N}^*$. If there is $m \in \mathbb{N}$ and finite p satisfying (2.3) such that $\mathrm{dom}(p) \subseteq \{0, \dots, k-1\}$ and $p(i) = n_i$ for all $i \in \mathrm{dom}(p)$, define $\alpha([n, n_0, \dots, n_{k-1}]) := [\hat{r}, m]$. If not let $\alpha([n, n_0, \dots, n_{k-1}]) := [\hat{q}, k]$. For $c \in \mathbb{N}$ not yet in the domain, define $\alpha(c) := [\hat{q}, \hat{q}]$. Check that $\varphi^n(\alpha, \beta) \simeq F(\beta)$ for all $\beta \in \mathbb{N}^{\mathbb{N}}$. □

Note that the direction $(ii) \Rightarrow (i)$ uses the countability of $\mathbb{N}$, but the other direction does not. Another remark is that using proposition 2.7 and proposition 2.5, with $F_a = F$ for all $a \in \mathbb{N}$, the conditions are equivalent with F being partial sequential.

We turn our attention to partial functions of the form $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$. The results below, proven for the PCA of example 1.13, can be found in the PhD thesis of A. Bauer [2]. We have modified the proofs to fit our (more general) approach. See also the remark at ending this section.

For partial functions $f, g : A \rightharpoonup B$ call g an *extension* of f iff $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(a) = g(a)$ for all $a \in \mathrm{dom}(f)$. Some extra notation: For $u = (n_0, \dots, n_{k-1}) \in \mathbb{N}^*$, define the finite function $\bar{u} : \{0, \dots, k-1\} \to \mathbb{N}$, by $\bar{u}(i) := n_i$ for all $i \leq k-1$. Also define $\mathrm{lh}(u) := k$, in particular $\mathrm{lh}((\,)) = 0$.

Theorem 2.13 (**Extension Theorem**) *Every partial continuous function $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ can be extended to a partial function of the form $\beta \mapsto \alpha \cdot \beta$.*

Proof: Let $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ be partial continuous. Consider for each $n \in \mathbb{N}$ the set

$$B_n := \{(u, m) \in \mathbb{N}^* \times \mathbb{N} \mid O_{\bar{u}} \cap \mathrm{dom}(F) \neq \emptyset \wedge \forall \beta \in O_{\bar{u}} \cap \mathrm{dom}(F) \, . \, F(\beta)(n) = m\}$$

Define $\alpha \in \mathbb{N}^{\mathbb{N}}$ in the following way:

$$\begin{array}{ll} \alpha([(n) * u]) = [\hat{q}, \mathrm{lh}(u)] & \text{if } n \in \mathbb{N},\ u \in \mathbb{N}^* \text{ and there is no } m \in \mathbb{N} \text{ s.t. } (u, m) \in B_n \\ \alpha([(n) * u]) = [\hat{r}, m] & \text{if } n \in \mathbb{N},\ u \in \mathbb{N}^*,\ m \in \mathbb{N} \text{ and } (u, m) \in B_n \\ \alpha(c) = [\hat{q}, \hat{q}] & \text{in all other cases} \end{array}$$

Note that any n-interrogation of some β by α is of the form $(\beta(0), \ldots, \beta(k-1))$. Now let $\beta \in \mathrm{dom}(F)$, $n \in \mathbb{N}$ and define $m := F(\beta)(n)$. Since F is continuous, the set $\{\gamma \in \mathbb{N}^{\mathbb{N}} \mid F(\gamma)(n) = m\}$ is open in $\mathrm{dom}(F)$, so there is a finite function $p \subset \beta$ such that $O_p \cap \mathrm{dom}(F)$ is contained in that set. Let $v := (\beta(0), \ldots, \beta(l))$ where l is the largest element of $\mathrm{dom}(p)$, we have that $(v, m) \in B_n$. This shows that $\varphi^n(\alpha, \beta)$ is defined and equal to m. This is true for all $n \in \mathbb{N}$, so $\alpha\beta$ is defined and $\alpha\beta(n) = F(\beta)(n)$. It follows that $\alpha\beta = F(\beta)$ for all $\beta \in \mathrm{dom}(F)$. $\square$

A set U is called G_δ if it is a countable intersection of open sets. The following lemma will be of help:

Lemma 2.14 *Let $U \subseteq \mathbb{N}^{\mathbb{N}}$ be a G_δ set, then there is $\nu \in \mathbb{N}^{\mathbb{N}}$ with the property:*

$$\nu \cdot \alpha \textit{ is defined if and only if } \alpha \in U$$

Proof: Let U be G_δ. Find open sets C_i such that $U = \bigcap_{i \in \mathbb{N}} C_i$. Using the fact that the base of $\mathbb{N}^{\mathbb{N}}$ is countable and that C_i is open, we find finite functions $p_{i,j}$ such that $C_i = \bigcup_{j \in \mathbb{N}} O_{p_{i,j}}$, this shows:

$$U = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} O_{p_{i,j}}$$

Define element $\nu \in \mathbb{N}^{\mathbb{N}}$ as follows:

$$\begin{array}{ll} \nu([(i) * u]) = [\hat{q}, \mathrm{lh}(u)] & \text{if } i \in \mathbb{N}, u \in \mathbb{N}^* \text{ and there is no } j \in \mathbb{N} \text{ s.t. } p_{i,j} \subset \bar{u} \\ \nu([(i) * u]) = [\hat{r}, \hat{r}] & \text{if } i \in \mathbb{N}, u \in \mathbb{N}^*, \text{ and } p_{i,j} \subset \bar{u} \text{ for some } j \in \mathbb{N} \end{array}$$

And let $\nu(n) = [\hat{q}, \hat{q}]$ for all other elements to make it total. Now consider $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\nu\alpha$ is defined. Let $i \in \mathbb{N}$, by inspecting the definition of ν we find an $k \in \mathbb{N}$ with the property that $\nu([i, \alpha(0), \ldots, \alpha(k-1)]) = [\hat{r}, \hat{r}]$. So there is $j \in \mathbb{N}$ such that $p_{i,j} \subseteq \alpha$, which implies that $\alpha \in O_{p_{i,j}}$. Doing this for every $i \in \mathbb{N}$ shows that $\alpha \in U$.

For the converse assume $\alpha \in U$. Let $i \in \mathbb{N}$, we find $j \in \mathbb{N}$ such that $\alpha \in O_{p_{i,j}}$. This is equivalent with $p_{i,j} \subseteq \alpha$, so we find $k \in \mathbb{N}$ large enough s.t. for $u = (\alpha(0), \ldots, \alpha(k-1))$ we have $p_{i,j} \subseteq \bar{u}$. It follows that $\nu([(i, \alpha(0), \ldots, \alpha(k-1))]) = [\hat{r}, \hat{r}]$, implying that $\nu\alpha(i) = \hat{r}$. Doing this for all $i \in \mathbb{N}$ shows that $\nu\alpha$ is defined. $\square$

Using this lemma in combination with the other results we can describe the application in terms of continuity.

Theorem 2.15 *Let $F : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ be a partial function. The following are equivalent:*

(i) there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $F(\beta) \simeq \alpha \cdot \beta$, for all $\beta \in \mathbb{N}^{\mathbb{N}}$

(ii) F is a partial continuous function with G_δ−domain

Proof: "$(i) \Rightarrow (ii)$" Assume that F is the function $\beta \mapsto \alpha \cdot \beta$. Let p be a finite function, note that: $F^{-1}(O_p) = (\bigcap_{n \in \mathrm{dom}(p)} \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \varphi^n(\alpha, \beta) = p(n)\}) \cap \mathrm{dom}(F)$. By proposition

2.12 every set in the finite intersection (with index set $\mathrm{dom}(p)$) is open, so $F^{-1}(O_p)$ is open in $\mathrm{dom}(F)$, this shows that F is continuous. The domain of F is given by:

$$\begin{aligned}\mathrm{dom}(F) &= \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \alpha\beta \text{ is defined}\} \\ &= \bigcap_{n\in\mathbb{N}} \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \varphi^n(\alpha,\beta) \text{ is defined}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \{\beta \in \mathbb{N}^{\mathbb{N}} \mid \varphi^n(\alpha,\beta) = m\}\end{aligned}$$

The right-hand side set is open. So $\mathrm{dom}(F)$ is a countable intersection of open sets.

"$(ii) \Rightarrow (i)$" Let F be a partial continuous function with G_δ-domain. By theorem 2.13 we find an $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\gamma\beta = F(\beta)$ for all $\beta \in \mathrm{dom}(F)$. By lemma 2.14 we find element $\nu \in \mathbb{N}^{\mathbb{N}}$ such that $\nu\beta$ is defined if and only if $\beta \in \mathrm{dom}(F)$. Now consider element $\alpha := \langle x\rangle \bar{k}(\nu x)(\gamma x)$ of $\mathbb{N}^{\mathbb{N}}$, with $\bar{k}$ as in section 1.2. Then $\alpha\beta \simeq \bar{k}(\nu\beta)(\gamma\beta)$ is defined if and only if $\beta \in \mathrm{dom}(F)$. In that case $\alpha\beta = F(\beta)$. $\square$

We promised to investigate the relation between Kleene's second model (example 1.13) and our PCA. We do this in the following remark.

Remark. As mentioned before, the last three results in this section are based on [2]. However in that thesis, as in most literature, the application on $\mathbb{N}^{\mathbb{N}}$ is defined as we have done in example 1.13. Let us write $\star$ for the application and $\lfloor\cdot,\dots,\cdot\rfloor$ for sequences coding defined in that example. Just write $\cdot$ and $[\cdot,\dots,\cdot]$ for the application and coding of $\mathcal{K}_2(\mathbb{N})$.

The PCAs defined by these applications are isomorphic in the following sense: There are elements $\gamma, \delta \in \mathbb{N}^{\mathbb{N}}$ such that for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ we have $\gamma \cdot \alpha \cdot \beta \simeq \alpha \star \beta$ and $\delta \star \alpha \star \beta \simeq \alpha \cdot \beta$. In terms of definition 1.17: the function $\alpha \mapsto \{\alpha\}$ is an applicative morphism from $(\mathbb{N}^{\mathbb{N}}, \star)$ to $(\mathbb{N}^{\mathbb{N}}, \cdot)$ and is an applicative in the other direction, realized by elements γ and δ respectively. We define elements γ and δ below. The definitions are quite technical, my advice is to focus on the general idea.

We would like γ to "simulate" the application $\alpha \star \beta$ for every $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. Start by defining $\gamma([[n]]) := [\hat{q}, \lfloor\ \rfloor]$ and $\gamma([[n, n_0, \dots, n_{k-1}]]) := [\hat{q}, \lfloor n, n_0, \dots, n_{k-2}\rfloor]$. Consider sequences of the form $u = ([n, n_0, \dots, n_{k-1}], a)$, we define γ on the code of these sequences as follows: If $a = 0$, then $\gamma([u]) := [\hat{r}, [\hat{q}, k]]$, if $a > 0$, then $\gamma([u]) := [\hat{r}, [\hat{r}, a-1]]$. Also define $\gamma([a_0, a_1, a_2]) := [\hat{r}, \hat{q}]$ and let $\gamma(m) := [\hat{q}, \hat{q}]$ on all other elements. Check that $\gamma \cdot \alpha$ always denotes and that γ has the desired property. Hint: think of n_i as the values $\beta(i)$ and a as the values $\alpha(\lfloor((n) * \beta) \upharpoonright k\rfloor)$.

Finding element δ is less straightforward. The idea is that δ gathers information about α and β, until it knows enough. Start with $\delta(\lfloor\ \rfloor) := 0$ and $\delta(\lfloor\lfloor\ \rfloor\rfloor) := 1$. We define δ on sequences $v = (\lfloor n, n_0, \dots, n_{k-1}\rfloor, m_0, \dots, m_{l-1})$. Consider finite functions $p : \{0, \dots, l-1\} \to \mathbb{N}$ and $q : \{0, \dots, k-1\} \to \mathbb{N}$ defined by $p(i) := m_i$ and $q(i) := n_i$. Find a maximal n-interrogation $u = (a_0, \dots, a_{j-1})$ of q by p. If the interrogation is finite, define δ as follows:

$$\begin{aligned}&\delta(\lfloor v\rfloor) := 0 && \text{if } [(n) * u] \notin \mathrm{dom}(p) \\ &\delta(\lfloor v\rfloor) := 1 && \text{if } p[(n) * u] = [\hat{q}, b] \text{ for some } b \\ &\delta(\lfloor v\rfloor) := y + 2 && \text{if } p[(n) * u] = [\hat{r}, y]\end{aligned}$$

In the second case it is true that $b \notin \mathrm{dom}(q)$, otherwise the interrogation u is not maximal. In any other case define $\delta(\lfloor v \rfloor) := 1$. Think of p and q as initial parts of α and β respectively. Then δ is defined on sequences of the form $\lfloor((\lfloor((n) * \beta) \restriction k \rfloor) * \alpha) \restriction l \rfloor = \lfloor \lfloor n, \beta(0), \ldots, \beta(k-2) \rfloor, \alpha(0), \ldots, \alpha(l-2) \rfloor$. Define $\delta(m) := 1$ on all the elements not yet in the domain. On should check that δ is well-defined and that $\delta \star \alpha$ denotes for every $\alpha \in \mathbb{N}^{\mathbb{N}}$. Inspecting the definition shows that $\delta \star \alpha \star \beta \simeq \alpha \cdot \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$.

3 Modest sets

In the first section the category of modest sets, as defined in section 1.4, is studied. After that we introduce a subcategory of Top, the category of topological spaces. This allows us to investigate the modest sets on $\mathcal{K}_2(A)$, for both $A = \mathbb{N}$ and A arbitrary.

3.1 The category of modest sets

To learn more about the category of modest sets on a PCA, see definition 1.16, we consider the category of partial equivalence relations on that PCA. We show these categories are equivalent.

Definition 3.1 Consider a PCA $(D, \cdot)$. A *partial equivalence relation*, denoted by $\equiv_X$, is a symmetric and transitive relation on D. We sometimes write X instead of $\equiv_X$ to indicate the relation.

Let X and Y be two partial equivalence relations on D. Call $r \in D$ *equivalence preserving* when for all $a, b \in D$ the following is true: if $a \equiv_X b$, then $r \cdot a \downarrow$, $r \cdot b \downarrow$ and $r \cdot a \equiv_Y r \cdot b$. Two equivalence preserving elements $r, t \in D$ are considered equivalent when for all $a, b \in D$: if $a \equiv_X b$, then $r \cdot a \equiv_Y t \cdot b$. A *morphism* $[\![r]\!] : X \to Y$ is an equivalence class of equivalence preserving elements of D.

The partial equivalence relations, regarded as objects, together with their morphisms form a category which is denoted by $\mathrm{Per}(D)$. The equivalence class of element i fulfils the role of identity morphism. The composition of $[\![r]\!] : X \to Y$ and $[\![t]\!] : Y \to Z$ is the morphism $[\![\langle x \rangle t(rx)]\!] : X \to Z$.

We use notation $[\![.]\!]$ for equivalence classes, instead of the usual notation $[.]$, to avoid confusion with the sequence coding of Chapter 2.

Remember that two categories $\mathcal{C}$, $\mathcal{D}$ are equivalent iff there is a full and faithful functor $G : \mathcal{C} \to \mathcal{D}$ that is essentially surjective on objects. This last property means that for every object Y of $\mathcal{D}$ we can find an object X of $\mathcal{C}$ such that Y is isomorphic to $G(X)$ in $\mathcal{D}$.

Proposition 3.2 *Let D be a PCA. The categories* $\mathrm{Mod}(D)$ *and* $\mathrm{Per}(D)$ *are equivalent.*

Proof: Define a functor $G : \mathrm{Mod}(D) \to \mathrm{Per}(D)$ and show this is an equivalence between categories. Let $X = (|X|, E_X)$ be a modest set on D. Define a partial equivalence relation $\equiv_{G(X)}$, also denoted by $G(X)$, on D as follows: $a \equiv_{G(X)} b$ if and only if there is an $x \in |X|$ such that $a \in E_X(x)$ and $b \in E_X(x)$. Note that the collection $\{E_X(x) : x \in |X|\}$ of non-empty pairwise disjoint subsets of D, are the equivalence classes of this relation. Let $f : X \to Y$ be a morphism in $\mathrm{Mod}(D)$, tracked by $r \in D$. Then $[\![r]\!] : G(X) \to G(Y)$ is a morphism in $\mathrm{Per}(D)$, define $G(f) = [\![r]\!]$.

This defines a full and faithful functor between the categories. To show that G is also essentially surjective, let Z be an object of $\mathrm{Per}(D)$. For $a \in D$, write $[\![a]\!]_Z := \{b \in D \,:\, a \equiv_Z b\}$ to denote the equivalence class of a. Define an object $\tilde{Z}$ of $\mathrm{Mod}(D)$ by $|\tilde{Z}| = \{[\![a]\!]_Z \,:\, a \in D \,\wedge\, a \equiv_Z a\}$ and $E_{\tilde{Z}}([\![a]\!]_Z) = [\![a]\!]_Z$. Check that $G(\tilde{Z}) = Z$, so G is (essentially) surjective on objects. $\square$

We use the category $\mathrm{Per}(D)$ instead of $\mathrm{Mod}(D)$ when that is convenient. To familiarize ourself with these categories, let us proof some properties.

Proposition 3.3 *The category* $\mathrm{Mod}(D)$ *has binary products, equalizers and is cartesian closed.*

Proof: First note that the object 1 defined by $|1| = \{*\}$ and $E_1(*) = D$ is a terminal object. If X and Y are objects of $\mathrm{Mod}(D)$, then the product $X \times Y$ is given by:

$$\begin{aligned} |X \times Y| &= |X| \times |Y| \\ E_{X \times Y}(x, y) &= \{pab \,:\, a \in E_X(x) \,\wedge\, b \in E_Y(y)\} \end{aligned}$$

for $(x, y) \in |X \times Y|$. The projections are tracked by p_0 and p_1 respectively. Consider morphisms $f, g : X \to Y$, the equalizer of this pair consist of the object Z defined by:

$$|Z| = \{x \in |X| \,:\, f(x) = g(x)\} \quad \text{and} \quad E_Z(x) = E_X(x) \text{ for } x \in |Z|$$

together with the inclusion $|Z| \to |X|$ tracked by i. Finally, to show that $\mathrm{Mod}(D)$ is cartesian closed, find for any pair of objects X, Y an exponential object. Define Y^X as follows:

$$\begin{aligned} |Y^X| &= \{f : |X| \to |Y| \,:\, f \text{ is a morphism } X \to Y\} \\ E_{Y^X}(f) &= \{r \in D \,:\, r \text{ tracks } f\} \end{aligned}$$

for $f \in |Y^X|$. The evaluation map $eval : |Y^X \times X| \to |Y|$ is tracked by the element $\langle w \rangle p_0 w(p_1 w)$. To show that Y^X together with $eval$ is indeed an exponential object, let W be an object and $g : W \times X \to Y$ be a morphism tracked by t. Then the corresponding morphism $\tilde{g} : W \to Y^X$ is tracked by $\langle wx \rangle t(pwx)$. $\square$

Note that the first three results of the proposition show that $\mathrm{Mod}(D)$ has all finite limits. The category has more properties, many of those are proven in [7]. Before moving on, we point out such a property.

Proposition 3.4 *The category* $\mathrm{Mod}(D)$ *is regular.*

One would need to show that $\mathrm{Mod}(D)$ has coequalizers of kernel-pairs and that regular epis are stable under pullback. Proofs of these properties can be found in the reference mentioned above. $\square$

3.2 Equivalence of $\mathrm{Mod}(\mathcal{K}_2(\mathbb{N}))$

Our goal is to learn more about $\mathrm{Mod}(\mathcal{K}_2(A))$ using topology. This is especially useful when considering the case $A = \mathbb{N}$. In a moment we introduce the notion of equilogical spaces, first defined by D.S. Scott. The results proven in this section can be found in [2] by A. Bauer.

Let $(T, \mathcal{O}_T)$ be a topological space, as before we just write T to indicate the space. Recall that the space is called T_0 iff for every distinct $x, y \in T$ there is an open set U such that $x \in U$ and $y \notin U$, or $y \in U$ and $x \notin U$.

Definition 3.5 An *equilogical space* X consists of a T_0 topological space $\|X\|$ together with an equivalence relation $\approx_X$ on $\|X\|$, write $X = (\|X\|, \approx_X)$.

Let X, Y be equilogical spaces. A continuous map $f : \|X\| \to \|Y\|$ is *equivariant* when for all $x, y \in \|X\|$ the following is true: if $x \approx_X y$, then $f(x) \approx_Y f(y)$. Two equivariant maps $f, g : \|X\| \to \|Y\|$ are considered equivalent when for all $x, y \in \|X\|$: if $x \approx_X y$, then $f(x) \approx_Y g(y)$. A *morphism* $[\![f]\!] : X \to Y$ is an equivalence class of equivariant maps.

The identity morphism on X is the equivalence class of the identity on $\|X\|$. If $[\![f]\!], [\![g]\!] : X \to Y$ are two morphisms, the composition $[\![g]\!] \circ [\![f]\!]$ is given by $[\![g \circ f]\!]$. So the equilogical spaces together with their morphisms form a category, denoted by **Equ**.

A topological space is called 0-dimensional iff it has a base consisting of clopen (closed and open) sets. To investigate $\mathrm{Mod}(\mathcal{K}_2(\mathbb{N}))$ we consider a full subcategory of **Equ**.

Definition 3.6 A *countable 0-equilogical space* is an equilogical space whose underlying topological space has a countable clopen base. In similar way as above this defines a category denoted by $0\text{-}\mathbf{Equ}_\omega$.

More general, let $\kappa \geq \omega$ be a cardinal. A *κ-based 0-equilogical space* is an equilogical space whose underlying topological space has a clopen base of cardinality at most κ. This gives rise to a category denoted by $0\text{-}\mathbf{Equ}_\kappa$.

We assume $\mathbb{N}^\mathbb{N}$ is induced with the topology defined in section 2.4, this topology is T_0, and has a countable clopen base. For $\alpha \in \mathbb{N}^\mathbb{N}$ write $\boldsymbol{\eta}_\alpha$ to indicate the partial function defined by $\boldsymbol{\eta}_\alpha(\beta) := \alpha \cdot \beta$ for all $\beta \in \mathbb{N}^\mathbb{N}$.

Theorem 3.7 *There s a full and faithful functor $G : \mathrm{Per}(\mathcal{K}_2(\mathbb{N})) \to 0\text{-}\mathbf{Equ}_\omega$.*

Proof: Let X be a partial equivalence relation on $\mathbb{N}^\mathbb{N}$. Define $\|G(X)\|$ as the domain of $\equiv_X$ equipped with the subspace topology inherent from $\mathbb{N}^\mathbb{N}$, note that the subspace is T_0 and has a countable clopen base. Define $\approx_{G(X)}$ as the equivalence relation $\equiv_X$. This defines a countable 0-equilogical space $G(X) = (\|G(X)\|, \approx_{G(X)})$. Let $[\![r]\!] : X \to Y$ be a morphism in $\mathrm{Per}(\mathcal{K}_2(\mathbb{N}))$. Consider the partial function $\boldsymbol{\eta}_r : \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}^\mathbb{N}$, using the forward direction of theorem 2.15 this function is continuous. Define $G([\![r]\!])$ as $\boldsymbol{\eta}_r$ restricted to $\|G(X)\|$.

Check that G preserves identity and composition, so it is a functor. Every morphism in $\mathrm{Per}(\mathcal{K}_2(\mathbb{N}))$ defines an unique equivalence class of equivariant maps, so G is faithful.

To show that G is full let $[\![f]\!] : G(X) \to G(Y)$ be a morphism in 0-$\mathbf{Equ}_\omega$. By theorem 2.13 we can find α such that $\boldsymbol{\eta}_\alpha$ is an extension of f, it follows that α is equivalence preserving between X and Y. Also $G([\![\alpha]\!]) = [\![f]\!]$, so G is full. □

Using the following lemma, the functor defined above actually shows equivalence between the categories.

Lemma 3.8 (**Embedding Lemma**) *Let $(T, \mathcal{O}_T)$ be a topological space. This space is T_0 with a countable clopen base if and only if there is an embedding from the space into* $\mathbb{N}^\mathbb{N}$

Proof: "$\Leftarrow$" Let $e : T \to \mathbb{N}^\mathbb{N}$ be an embedding. Then $e(T)$, the range of e, is a subspace of $\mathbb{N}^\mathbb{N}$. So $e(T)$ is a T_0 space with a countable clopen base. Using the embedding, these properties are also true for T.

"$\Rightarrow$" Let $\{U_n \,|\, n \in \mathbb{N}\}$ be a countable base of T consisting of clopen sets. Define function $e : T \to \mathbb{N}^\mathbb{N}$ by:

$$e(a)(n) = \begin{cases} 1 & \text{if } a \in U_n, \\ 0 & \text{otherwise.} \end{cases}$$

This function is injective since T is a T_0 space. To show that e is continuous, let $p : \mathbb{N} \rightharpoonup \mathbb{N}$ be a finite function such that $\mathrm{ran}(p) \subseteq \{0, 1\}$, note that:

$$e^{-1}(O_p) = \bigcap (\{U_n \,|\, n \in \mathrm{dom}(p) \wedge p(n) = 1\} \cup \{U_n^c \,|\, n \in \mathrm{dom}(p) \wedge p(n) = 0\})$$

If $\mathrm{ran}(p) \nsubseteq \{0, 1\}$, then $e^{-1}(O_p) = \emptyset$. For every $n \in \mathbb{N}$ the sets U_n and U_n^c are both open. So $e^{-1}(O_p)$ is a finite intersection of open sets, which is open. To show that e is an open map, note that $e(U_n) = \{\beta \in \mathbb{N}^\mathbb{N} \,|\, \beta(n) = 1\}$ is open. This proves that e is an embedding. □

Corollary 3.9 *The categories* $\mathrm{Per}(\mathcal{K}_2(\mathbb{N}))$ *and* 0-$\mathbf{Equ}_\omega$ *are equivalent.*

Proof: We show that the functor defined in theorem 3.7 is essentially surjective on objects. Let $Z = (\|Z\|, \approx_Z)$ be a countable 0-equilogical space. Using lemma 3.8 we find an embedding $e : \|Z\| \to \mathbb{N}^\mathbb{N}$. Define a partial equivalence relation $\equiv_{\tilde{Z}}$ on $\mathbb{N}^\mathbb{N}$ by:

$$e(x) \equiv_{\tilde{Z}} e(y) \Leftrightarrow x \approx_Z y$$

The domain of this relation equals $e(\|Z\|)$, the range of e. This defines an object $\tilde{Z}$ of $\mathrm{Per}(\mathcal{K}_2(\mathbb{N}))$. Consider $G(\tilde{Z})$, we have that $\|G(\tilde{Z})\| = e(\|Z\|)$, and $\approx_{G(\tilde{Z})}$ is just the relation $\equiv_{\tilde{Z}}$. Using the fact that $e : \|Z\| \to e(\|Z\|)$ is a homeomorphism, it follows that $[\![e]\!] : Z \to G(\tilde{Z})$ is an isomorphism in 0-$\mathbf{Equ}_\omega$. This shows that G is essentially surjective on objects. □

The theorem and corollary are still true if we replace $\mathrm{Per}(\mathcal{K}_2(\mathbb{N}))$ by $\mathrm{Mod}(\mathcal{K}_2(\mathbb{N}))$, using the equivalence between those categories.

Remarks. (a) One can think of **Equ** as a non-full subcategory $\mathcal{C}$ of Top, the category of topological spaces and continuous maps. For object $(\|X\|, \approx_X)$, consider the quotient topology $\|X\|/\approx_X$, with quotient map π_X. The objects of $\mathcal{C}$ are exactly those quotient topologies. For morphism $[\![f]\!] : X \to Y$, define function $\tilde{f} : \|X\|/\approx_X \to \|Y\|/\approx_Y$ by $\tilde{f}([\![x]\!]) := \pi_Y(f(x))$. Check this is well-defined and continuous. This defines the collection of morphisms in $\mathcal{C}$. Equivalence between **Equ** and $\mathcal{C}$ is clear.

(b) For a moment, we consider a different point of view. The following ideas are from the theory of "Type Two Effectivity" (TTE).

Let $(|X|, E_X)$ be a modest set on $\mathcal{K}_2(\mathbb{N})$ and $R := \bigcup\{E_X(x) : x \in |X|\}$. We consider the function $r : R \to |X|$ defined by $r(\alpha) = x \Leftrightarrow \alpha \in E_X(x)$. Using the topology on $R \subseteq \mathbb{N}^{\mathbb{N}}$, this induces a topology on $|X|$ as follows:

$$U \subseteq |X| \text{ is open} \iff r^{-1}(U) \text{ is open in } R$$

Note that $r : R \to |X|$ is a topological quotient (i.e. r is a quotient map).

Now let X be a topological space. Way say that a set $R \subseteq \mathbb{N}^{\mathbb{N}}$ together with a function $r : R \to X$ is a *representation* of X iff r is a surjective continuous function. In particular a representation r is called a *quotient representation* iff r is a topological quotient. We call a representation $r : R \to X$ *admissible* iff for every other a representation $s : S \to X$ there is a continuous function $f : S \to R$ such that $s(\alpha) = r(f(\alpha))$ for all $\alpha \in S$.

Consider the category with as objects the admissible quotient representations, and as morphisms continuous functions between the co-domains of the representations. It can be shown this is equivalent to $\mathrm{Mod}(\mathcal{K}_2(\mathbb{N}))$.

The questions one would like to answer is what topological spaces have an admissible (quotient) representation. This is done by considering sequential space. Take a look in [1] and also in [2] for more information about this and the theory of TTE.

3.3 Investigate Mod($\mathcal{K}_2(A)$)

During this section we assume A is some infinite set. Similar to the case of natural numbers we define a topology on A^A. Start with the discrete topology on A and consider the product topology on A^A. The basic open sets are again of the form:

$$O_p := \{\beta \in A^A \mid \forall a \in \mathrm{dom}(p) \,.\, \beta(a) = p(a)\}$$

for finite functions $p : A \rightharpoonup A$. We would like to use a similar approach as in previous section. However most of the proofs in section 2.4 use the countability of $\mathbb{N}$, so can be false for the general case. Let's first find out what results are still true.

Proposition 3.10 *Let $\alpha \in A^A$, the partial function $F : A^A \rightharpoonup A^A$ defined by $F(\beta) := \alpha \cdot \beta$ is continuous.*

Proof: Inspect the proofs of direction $(i) \Rightarrow (ii)$ from proposition 2.12 and $(i) \Rightarrow (ii)$ from theorem 2.15. Using the exact same proofs (module some notation) shows that

$\beta \mapsto \varphi^a(\alpha, \beta)$ is continuous for all $a \in A$ and subsequently that F is continuous. $\square$

This allows us to define a similar functor as in the case $A = \mathbb{N}$.

Proposition 3.11 *Let A be an infinite set of cardinality κ. There is a faithful functor $G : \mathrm{Per}(\mathcal{K}_2(A)) \to 0\text{-}\mathbf{Equ}_\kappa$.*

Proof: One can show that the set of all finite functions $p : A \rightharpoonup A$ has cardinality κ (there are κ many functions with domain $\{a_0, \ldots, a_{n-1}\}$ and κ many of those domains). It follows that A^A is a T_0 topological space with a clopen base of cardinality κ. The definition of G is analogous to the definition of G in proposition 3.7. Any subspace of A^A is T_0 with clopen base of cardinality at most κ. The proof proceeds in similar fashion as before, using proposition 3.10 instead of theorem 2.15. $\square$

We mention another result that does still hold:

Lemma 3.12 (**Embedding Lemma**) *Let $(T, \mathcal{O}_T)$ be a topological space and A an infinite set of cardinality κ. Then T is T_0 with a clopen base of cardinality at most κ if and only if there is an embedding from T into A^A*

Proof: For the "$\Rightarrow$" direction find a clopen base $\{U_a \mid a \in A\}$ of T, the proof is similar as the proof of lemma 3.8. $\square$

If the extension theorem (thm. 2.13) is true for a particular set A, then the functor defined in proposition 3.11 is full. In that case equivalence between categories would follow. However the theorem is not true for uncountable sets A. Consider the following counter example:

Proposition 3.13 *Let A be an uncountable set, there is a partial continuous function $F : A^A \rightharpoonup A^A$ that cannot be extended to a function of the form $\beta \mapsto \alpha \cdot \beta$*

Proof: Let $c \in A$ be fixed. Define a partial function $F : A^A \rightharpoonup A^A$ with domain:

$$\mathrm{dom}(F) := \{\beta \in A^A \mid \exists!\, d \in A \ \text{ s.t. } \ \beta(d) = c\}$$

Introduce notation $V_d := \{\beta \in A^A \mid \beta(d) = c\} \cap \mathrm{dom}(F)$, and write const_d to indicate the constant function in A^A with value d. Note that for every $\beta \in \mathrm{dom}(F)$ there is an unique element $d \in A$ such that $\beta \in V_d$, define $F(\beta) := \mathrm{const}_d$. Check that F is continuous on its domain.

Suppose, towards contradiction, that there is an $\alpha \in A^A$ such that $\alpha \cdot \beta = F(\beta)$ for all $\beta \in \mathrm{dom}(F)$. Let $a \in A$, define for each $n \in \mathbb{N}$ sequences $u_n, v_n \in A^*$ and a function $\beta_n \in A^A$ by induction. Start by defining $u_0 = (\,)$ and $v_0 = (b_0)$, where b_0 is such that $\alpha([a]) = [\hat{q}, b_0]$ (note that F is not constant on its domain, so by assumptions on α, this has to be a query). Let $\beta_0 \in \mathrm{dom}(F)$ be arbitrary but fixed.

Now assume $u_n = (a_0, \dots, a_{k_n-1})$, $v_n = (b_0, \dots, b_{k_n})$ and β_n are defined and satisfy the following three properties:

- u_n is an a-interrogation of β_n by α
- $\alpha([a, a_0, \dots a_{i-1}]) = [\hat{q}, b_i]$ for all $i \leq k_n$
- $a_i \neq c$ for all $i \leq k_n - 1$

We define sequences u_{n+1}, v_{n+1} and find a function β_{n+1}. To do this, let $d \in A$ such that $d \neq b_i$ for all $i \leq k_n$. Define $\beta_{n+1} \in V_d$ as a function satisfying $\beta_{n+1}(b_i) = a_i$ for all $i \leq k_n - 1$. Check that such a functions exists. Use the fact that $\beta_{n+1} \in \mathrm{dom}(F)$ to find an a-interrogation $(a_0, \dots, a_{l-1})$ of β_{n+1} by α with the property that $\alpha([a, a_0, \dots, a_{l-1}]) = [\hat{r}, d]$. Find sequence $(b_0, \dots, b_{l-1})$ such that $\alpha([a, a_0, \dots a_{i-1}]) = [\hat{q}, b_i]$ for all $i \leq l - 1$. As the notation suggests, these sequences extend u_n and v_n respectively.

We claim that $d \in \{b_0, \dots, b_{l-1}\}$. Proof of our claim: suppose this is not the case, then find a function γ that agrees on $\{b_0, \dots, b_{l-1}\}$ with β_{n+1}, but such that $\gamma \in V_e$ for some $e \neq d$. Then $e = \mathrm{const}_e(a) = \alpha\gamma(a) = \alpha\beta_{n+1}(a) = d$, contradiction.

Define k_{n+1} as the smallest number such that $b_{k_{n+1}} = d$. This defines sequences $u_{n+1} := (a_0, \dots, a_{k_{n+1}-1})$ and $v_{n+1} := (b_0, \dots, b_{k_{n+1}})$. These sequences, together with function β_{n+1}, satisfy the three properties listed above. Also note that $u_n \subsetneq u_{n+1}$ and $v_n \subsetneq v_{n+1}$, they are proper extensions. This concludes the induction.

Consider the sequences $u = (a_0, a_1, \dots)$ and $v = (b_0, b_1, \dots)$ extending all the finite sequences u_n and v_n. Since A is uncountable there is an element $e \in A$ not in the sequence v and there is $\beta \in A^A$ such that $\beta \in V_e$ and $\beta(b_i) = a_i$ for all $i \in \mathbb{N}$. It follows that $\alpha\beta$ is not defined, however $F(\beta) = \mathrm{const}_e$ $\lightning$. $\square$

Corollary 3.14 *Let A be uncountable. The functor $G : \mathrm{Per}(K_2(A)) \to 0\text{-}\mathbf{Equ}_\kappa$ defined in proposition 3.11 is not full.*

Proof: Let partial function $F : A^A \rightharpoonup A^A$, sets V_d and functions const_d be defined as in the proof of proposition 3.13. Assume, towards contradiction, that G is full. Define objects X and Y in $\mathrm{Per}(\mathcal{K}_2(A))$ by: $\alpha \equiv_X \beta$ if and only if there is an $d \in A$ such that $\alpha, \beta \in V_d$. And define $\alpha \equiv_Y \beta$ if and only if $\alpha = \beta = \mathrm{const}_d$ for some $d \in A$. Then $[\![F]\!] : G(X) \to G(Y)$ is a morphism in $0\text{-}\mathbf{Equ}_\kappa$. Since G is full we find $\delta \in A^A$ such that $G([\![\delta]\!]) = [\![F]\!]$, this shows that $\beta \mapsto \delta\beta$ extends F $\lightning$. $\square$

This result leads to the following question I am not (yet) able to answer.

Unanswered question. Is there a full subcategory of $0\text{-}\mathbf{Equ}_\kappa$, or a full subcategory of Top, that is equivalent to $\mathrm{Mod}(\mathcal{K}_2(A))$? In other words, is it possible to describe $\mathrm{Mod}(\mathcal{K}_2(A))$ in a “topological way”?

I'm inclined to think the answer is “no”. To show this we would like to find a categorical property of Top that is not true in $\mathrm{Mod}(\mathcal{K}_2(A))$, or vice versa. One could think of a property of continuous functions that is not true for functions of the form $\beta \mapsto \alpha\beta$.

The requirement "full" in the question is important. We can find a non-full subcategory $\mathcal{C}$ of Top that is equivalent to $\mathrm{Per}(\mathcal{K}_2(A))$. This is done in a similar way as in remark (a) from previous section. Only in this case consider quotient spaces constructed on subspaces of A^A, instead of general T_0 spaces. And equivalence preserving continuous maps between those subspaces that have an extension of the form $\beta \mapsto \alpha\beta$, instead of the more general equivariant maps.

The final result of this thesis is a weaker version of the extension theorem. I don't think it can be used to answer the question above. However the proof is non-trivial and can give some insight. The result generalizes the theorem: *every total continuous function* $A^A \to A$ *is sequential*, found in [14]. The proof is similar to the one in that article. Call finite functions $p, q : A \rightharpoonup A$ *compatible* iff $p(a) = q(a)$ for all $a \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$.

Proposition 3.15 *Every partial continuous function* $F : A^A \rightharpoonup A^A$ *with open domain can be extended to a partial function of the form* $\beta \mapsto \alpha \cdot \beta$*.*

Proof: Let $F : A^A \rightharpoonup A^A$ be a partial continuous function with open domain. Define partial functions $F_a : A^A \rightharpoonup A$, by $F_a(\beta) := F(\beta)(a)$. Let $a \in A$, notice that $\mathrm{dom}(F_a) = \mathrm{dom}(F)$ and $F_a^{-1}(\{b\})$ is open in $\mathrm{dom}(F)$ for all $b \in A$. So for $\beta \in \mathrm{dom}(F)$ there is finite function p such that $\beta \in O_p$ and F_a constant on $O_p \cap \mathrm{dom}(F)$. Since $\mathrm{dom}(F)$ is open we also find q such that $\beta \in O_q \subseteq \mathrm{dom}(F)$. Define finite r by $\mathrm{dom}(r) := \mathrm{dom}(p) \cup \mathrm{dom}(q)$, compatible with both p and q, then F_a is constant on O_r.

Doing this for all $\beta \in \mathrm{dom}(F)$ we find a set $\mathcal{B}$ of finite functions, such that for all $r \in \mathcal{B}$, F_a is constant on O_r and for every $\beta \in \mathrm{dom}(F)$ there is $r \in \mathcal{B}$ such that $\beta \in O_r$. Let p be some finite function. Write $\mathcal{B}_p$ to indicate the set of those $q \in \mathcal{B}$ that are compatible with p.

Consider finite p such that F_a is not constant on $O_p \cap \mathrm{dom}(F)$. So there are $\beta, \gamma \in O_p \cap \mathrm{dom}(F)$ with $F_a(\beta) \neq F_a(\gamma)$. Find $q, r \in \mathcal{B}_p$ satisfying $q \subset \beta$ and $r \subset \gamma$. Suppose that there is $t \in \mathcal{B}_p$ such that $(\mathrm{dom}(q) \cup \mathrm{dom}(r)) - \mathrm{dom}(p)$ is disjunct from $\mathrm{dom}(t) - \mathrm{dom}(p)$, we want to find contradiction. Then t is compatible with both q and r, so both $O_q \cap O_t$ and $O_r \cap O_t$ are not empty. Since F_a is constant on the sets O_q, O_r and O_t, it follows that F_a is constant on $O_q \cup O_r \cup O_t$, this implies $F_a(\beta) = F_a(\gamma)$, contradiction. So for all $t \in \mathcal{B}_p$ the intersection of the set $C_p := ((\mathrm{dom}(q) \cup \mathrm{dom}(r)) - \mathrm{dom}(p))$ with $\mathrm{dom}(t)$ is not empty.

Define trees $T_a^0 \subseteq T_a^1 \subseteq \dots$ as follows. Let $T_a^0 := \{\emptyset\}$, the tree with only the empty function. Suppose T_a^n has been defined, we construct tree T_a^{n+1}. Let p be a leaf of T_a^n, extend this leaf by considering cases. If F_a is constant on $O_p \cap \mathrm{dom}(F)$, or in the case this set is empty, do not extend p. In any other case F_a is not constant on $O_p \cap \mathrm{dom}(F)$, find a set C_p as defined above and write $C_p = \{c_0, \dots, c_m\}$. Add for each $0 \leq k \leq m$ all functions p' with $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{c_0, \dots, c_k\}$. Doing this for all leaves p defines tree T_a^{n+1}. Define $T_a := \bigcup_{n \in \mathbb{N}} T_a^n$ as the union of those trees.

Now let $\beta \in \mathrm{dom}(F)$ and find $q \in \mathcal{B}$ such that $\beta \in O_q$. Let p be a leaf of T_a^n that is compatible with q. By induction it is true that F_a is constant on O_p (so it is also a leaf of T_a) or the cardinality of $\mathrm{dom}(q) \cap \mathrm{dom}(p)$ is at least n. This shows there is a leaf $r \in T_a$ such that $r \subset \beta$, define $K_a(r) := F_a(\beta)$. Do this for all $\beta \in \mathrm{dom}(F)$. If there are still leaves $r \in T$ for which $K_a(r)$ is not defined, give it an arbitrary value.

This construction defines a partial sequential function Φ_{T_a,K_a}. It follows that if $\beta \in \mathrm{dom}(F)$, then $\Phi_{T_a,K_a}(\beta) = F_a(\beta)$. Of course we can find such partial sequential functions for every $a \in A$. Applying corollary 2.6 on those functions gives an element $\alpha \in A^A$ with the property that if $\beta \in \mathrm{dom}(F)$, then $\alpha\beta$ is defined and in that case $\alpha\beta(a) = F_a(\beta)$ for all $a \in A$. So the function $\beta \mapsto \alpha\beta$ is an extension of F. $\square$

Bibliography

[1] I. Battenfield, M. Schröder, and A. Simpson. A convenient category of domains. *Electronic Notes in Theoretical Computer Science*, 172:69–99, 2007.

[2] A. Bauer. *The Realizability Approach to Computable Analysis and Topology*. PhD thesis, Carnegie Mellon University, 2000.

[3] A. Bauer, L. Birkedal, and D.S. Scott. Equilogical spaces. *Theoretical Computer Science*, 315(1):35–59, 2004.

[4] M.J. Beeson. *Foundations of Constructive Mathematics: metamathematical studies*, chapter VI, pages 97–107. Springer-Verlag, 1985.

[5] S. Feferman. A language and axioms for explicit mathematics. In J.N. Crossley, editor, *Algebra and Logic*, pages 87–139. Springer-Verlag, 1975.

[6] S.M. Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 1998.

[7] J.R. Longley. *Realizability Toposes and Language Semantics*. PhD thesis, Edinburgh University, 1995.

[8] J.R. Munkres. *Topology (2nd edition)*. Prentice Hall, 2000.

[9] M. Schönkfinkel. Über die bausteine der mathematischen logik. *Mathematische Annalen*, 92:305–316, 1924. Translated in J. van Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic*, 1879-1931, pages 355-366. Harvard University Press, 1967.

[10] D.S. Scott. Lambda calculus and recursion theory. In S. Kanger, editor, *Proceedings of the 3rd Scandinavian Symposium*, volume 82 of *Studies in Logic*, pages 154–193. North Holland, 1975.

[11] R.I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic. Springer, 1987.

[12] J. van Oosten. A combinatory algebra for sequential functionals of finite type. In S.B. Cooper and J.K. Truss, editors, *Models and Computability*, pages 389–406. Cambridge University Press, 1999.

[13] J. van Oosten. *Realizabilty: an Introduction to its Categorical Side*, volume 152 of *Studies in Logic*. Elsevier, 2008.

[14] J. van Oosten. Partial combinatory algebras of functions. *Notre Dame Journal of Formal Logic*, 52(4):431–448, 2011.