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SOLUBLE BOLTZMANN EQUATIONS FOR INTERNAL STATE AND MAXWELL MODELS

II. SIMILARITY SOLUTIONS

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We continue our study of the nonlinear Boltzmann equation for "diffuse" binary scattering between subsystems in a microcanonical ensemble. Exact similarity solutions of Bobylev–Krook–Wu type are found for systems of arbitrary dimensionality and in both continuous and discrete state variables. Additional similarity solutions are also derivable as expansions in Laguerre or Meixner polynomials, the required Fourier coefficients following from algebraic recursion relations.

1. Introduction

In a previous paper, referred to as I[1]), we constructed the general solution to a class of both continuous and discrete model Boltzmann equations, applicable equally to the relaxation of translational motion or internal (e.g. vibrational) degrees of freedom. The continuous models for translational relaxation describe diffuse scattering of Maxwell molecules; their discrete counterparts refer most naturally to the vibrational relaxation of degenerate oscillator modes.

The reason why these models are soluble may be sought in Truesdell's earlier work on the non-linear Boltzmann-equation for Maxwell-molecules[2]), in which he pointed out that the system of coupled moment equations can be solved sequentially, in increasing order. In fact, as we were able to show in I, it is possible to construct orthogonal polynomial moments which satisfy the identical set of coupled equations while having the distinct advantage that they form a Fourier series and thus lead directly to the desired solution by a straight-forward inversion.

The identity of the two sets of moment equations is not, of course, accidental and can be shown to arise from a fundamental symmetry in the underlying equation. (See refs. 3–5).

In this paper we will show that exact similarity solutions of the type found by Bobylev[5,6]), Krook and Wu[7]) (BKW-mode) – see also refs. 8–10 – also hold for both discrete and continuous variable versions of our models; the former tending to the latter in the appropriate classical limit. Interest in similarity solutions derives in part from the conjecture made by Krook and Wu that at least a significant class of initial distributions may relax rapidly to a similarity mode which then evolves essentially unchanged to final equilibrium. The original suggestion by Krook and Wu that this mode may be of BKW-type seems not to be supported by numerical calculations[10])*.

2. General kinetic equations and moment solutions

The BKW-similarity solutions have recently been extended to a large class of Maxwell models by Ernst[3,4]) and our first objective is to extend his results to the internal state models described in I.

We consider the energy distribution functions $P(x, t)$ with a continuous variable $x \in (0, \infty)$ or $P(i, t)$ with a discrete variable $i = 0, 1, 2, \ldots$. For the models considered in I the Boltzmann equation reads in the continuous case

$$\partial_t P(x, t) = \int_x^\infty \mathrm{d}u \int_0^u \mathrm{d}y [W_{p,p}(x; u) P(y, t) P(u - y, t) - W_{p,p}(y; u) P(x, t) P(u - x, t)], \tag{2.1a}$$

or

$$(\partial_t + 1) P(x, t) = \int_x^\infty \mathrm{d}u \int_0^u \mathrm{d}y\, W_{p,p}(x; u) P(y, t) P(u - y, t), \tag{2.1b}$$

where $W_{p,p}(x; u)$ is the symmetric Beta distribution.

$$W_{p,p}(x; u) = \frac{x^{p-1}(u - x)^{p-1}}{B(p, p) u^{2p-1}}, \tag{2.2}$$

and $B(p, q) = \Gamma(p)\Gamma(q)/\Gamma(p + q)$ is the Beta function. The discrete analogue of (2.1) is

* Observations against the validity of the conjecture have been made by Tjon[10]), Weinert et al.[11]), Barnsley and Cornille[12]), Alexanian[13]) and Hauge[14]).

$$\partial_t P(i, t) = \sum_{k=i}^{\infty} \sum_{j=0}^{k} [W_{p,p}(i; k)P(j, t)P(k-j, t) - W_{p,p}(j; k)P(i, t)P(k-i, t)]. \tag{2.3a}$$

or

$$(\partial_t + 1)P(i, t) = \sum_{k=i}^{\infty} \sum_{j=0}^{k} W_{p,p}(i;k)P(j, t)P(k-j, t), \tag{2.3b}$$

and $W_{p,p}(i;k)$ is the symmetrical negative hypergeometric (NHG) distribution,

$$W_{p,p}(i; k) = \frac{(i+1)_{p-1}(k-i+1)_{p-1}}{B(p, p)(k+1)_{2p-1}}. \tag{2.4}$$

Here $(p)_n = p(p+1)\dots(p+n-1)$ is the Pochammer (rising factorial) function. The moments of the symmetric Beta distribution can be written as:

$$m^{(n)}(u) = \int_0^u x^n\, W_{p,p}(x; u)\, \mathrm{d}x = \frac{(p)_n}{(2p)_n} u^n \tag{2.5}$$

and the factorial moments of the negative hypergeometric distribution are

$$\bar{m}^{(n)}(k) = \sum_{i=n}^{k} i^{(n)} W_{p,p}(i; k) = \frac{(p)_n}{(2p)_n} k^{(n)}, \tag{2.6}$$

where $k^{(n)} = k(k-1)\dots(k-n+1)$ is the falling factorial function.

The total number of particles and total energy are conserved in both classes of models, i.e.:

$$\mu_0(t) = 1, \qquad \mu_1(t) = p\varepsilon,$$

and (2.7)

$$\bar{\mu}_0(t) = 1 \qquad \bar{\mu}_1(t) = p\varepsilon/\varepsilon_0,$$

with moments defined

$$\mu_n(t) = \frac{1}{n!} \int_0^{\infty} x^n P(x, t)\, \mathrm{d}x \tag{2.8}$$

and

$$\bar{\mu}_n(t) = \frac{1}{n!} \sum_{i=n}^{\infty} i^{(n)} P(i, t) \tag{2.9}$$

in the continuous and discrete cases, respectively.

The stationary distributions are the Gamma distribution with mean energy $p\varepsilon$, i.e.

$$P_0(x) = \Gamma(p)^{-1} x^{p-1} \varepsilon^{-p}\, \mathrm{e}^{-x/\varepsilon}, \tag{2.10}$$

and the Negative Binomial with mean energy $p\varepsilon/\varepsilon_0$

$$P_0(i) = \Gamma(p)^{-1}(i+1)_{p-1}(1-c)^p c^i \tag{2.11}$$

with $c = \varepsilon/(\varepsilon + \varepsilon_0)$. The limiting process, "continuous limit", whereby $P_0(i)$ is converted into $P_0(x)\,\mathrm{d}x$ is crucial to the interconnection of discrete and continuous models throughout this paper. This limit is defined as:

$$\begin{aligned} &i \to \infty,\ \varepsilon_0 \to 0 \quad \text{with } i\varepsilon_0 = x \text{ finite} \\ &\text{or} \\ &i \to \infty,\ c \to 1 \quad \text{with } i \log c = -x/\varepsilon \text{ finite} \end{aligned} \tag{2.12}$$

with $\mathrm{d}x \equiv \varepsilon_0 \Delta i$ whereupon (2.11) yields (2.10).

The Fourier expansion of the distribution function reads

$$P(x,t) = P_0(x) \sum_{n=0}^{\infty} \gamma_n(t) L_n^{(p-1)}(x/\varepsilon) n!/(p)_n \tag{2.13}$$

with Laguerre moments $\gamma_n(t)$ defined by the integral

$$\gamma_n(t) = \int_0^{\infty} L_n^{(p-1)}(x/\varepsilon) P(x,t)\,\mathrm{d}x. \tag{2.14}$$

The Laguerre polynomials are traditionally called Sonine polynomials in the kinetic theory of the Boltzmann equation and have the standard definition

$$L_n^{(p-1)}(x) = {}_1F_1(-k, p, x)(p)_n/n! = \sum_{l=0}^{k} (-)^l \binom{k+p-1}{k-l} \frac{x^l}{l!}. \tag{2.15}$$

The Laguerre moments satisfy the equations

$$(\partial_t + 1)\gamma_n(t) = \frac{(p)_n}{(2p)_n} \sum_{k=0}^{n} \gamma_k(t)\gamma_{n-k}(t). \tag{2.16}$$

The ordinary moments $\mu_n(t)$, defined in (2.8), satisfy an identical set. Note also that $\gamma_0(t) = 1$ and $\gamma_1(t) = 0$, as follows from the conservation laws (2.7) and from $L_1^{(p-1)}(x) = p - x$.

The discrete analogue of (2.13) reads:

$$P(i,t) = P_0(i) \sum_{n=0}^{\infty} \bar{\gamma}_n(t) M_n(i,p,c), \tag{2.17}$$

where the Meixner moments are defined as

$$\bar{\gamma}_n(t) = c^n \frac{(p)_n}{n!} \sum_{i=0}^{\infty} M_n(i,p,c) P(i,t), \tag{2.18}$$

and the Meixner polynomials are

$$M_k(i, p, c) = {}_2F_1(-k, -i, p, 1 - c^{-1})$$
$$= \sum_{l=0}^{\min(i,k)} \binom{i}{l}\binom{k}{l} \frac{l!(l - c^{-1})^l}{(p)_l}. \tag{2.19}$$

The Meixner moments $\bar{\gamma}_n(t)$ satisfy again the same equation as the factorial moments $\bar{\mu}_n(t)$, defined in (2.9), but this set of equations is also identical to the set (2.16) for the Laguerre moments of the corresponding continuous problem.

3. BKW-modes

The method of similarity solutions, though having interesting group-theoretic aspects (see e.g. Bluman and Cole[15]) is for our present purposes essentially a heuristic tool for finding at least a class of viable solutions to the kinetic equations (2.1) and (2.3). In short we guess functional forms for $P(x, t)$ and $P(i, t)$ which remain structurally invariant in time while tending to the known equilibrium solutions. Experience shows it to be somewhat easier to guess the solution than to determine ab initio the invariant initial distribution functions as the solutions to certain integro differential equations as required in the formal method.

In this section we present similarity solutions for closed system relaxation under the continuous Beta and discrete NHG scattering kernels. The former case has been treated using the formal similarity method by one of us[3,4]) as part of an investigation of a whole class of models reducible to the Bobylev-Krook-Wu formalism. We shall present the heuristic method here since it serves as a prototype for the less familiar discrete solution.

3.1. *The continuous Beta-model*

The Krook-Wu solution for Maxwell molecules strongly suggests that a suitable functional form to try for the solution to (2.1) is that of a low-order polynomial in x with time-dependent coefficients, multiplying a time-dependent modification of the equilibrium distribution. Taking the simplest conceivable structure of this type we therefore postulate a form

$$P^s(x, t) = (\alpha + \beta x)x^{p-1} e^{-\psi x}, \tag{3.1}$$

with $\alpha(t)$, $\beta(t)$ and $\psi(t)$ time dependent functions to be determined, if so possible, by applying (1) number conservation, (2) conservation of energy and (3) a suitably restricted initial condition $P(x, 0)$. We shall apply conditions (1) and (2) successively to show that a consistent solution of the required form is indeed possible, first reducing α and β to known functions of ψ and then solving for the latter as an explicit function of time. Thus as a first step we find from number and energy conservation that

$$\alpha = \psi^p(1 + p - p\varepsilon\psi)/\Gamma(p),$$
$$\beta = \psi^{p+1}(\varepsilon\psi - 1)/\Gamma(p). \tag{3.2}$$

The time derivatives of these are readily found for substitution in the kinetic equation, it being most convenient for our purpose to use (2.1a). Turning attention to the right hand side of the equation we see that this, after use of the detailed balance condition $W(x, u)P_0(y)P_0(u-y) = W(y, u)P_0(x)P_0(u-x)$ takes the form

$$\text{R.H.S.} = \beta^2 \int_x^\infty \mathrm{d}x \int_0^u \mathrm{d}y\, \mathrm{e}^{-\psi u} W_{p,p}(y; u) x^{p-1}(u-x)^{p-1}[y(u-y) - x(u-x)]. \tag{3.3}$$

It is clear that the inner integral comes out in terms of the moments (2.5) of the Beta distribution while the outer one involves only Gamma functions. Thus, using the values $m^{(1)}(u) = \frac{1}{2}u$, $m^{(2)}(u) = \frac{1}{2}u^2(p+1)/(2p+1)$. We can equate left and right hand sides of the substituted equation to get

$$[-x(\partial\psi/\partial t) + \partial/\partial t](\alpha + \beta x) = \frac{\beta^2\Gamma(p)}{\psi^{p+2}}\{\lambda x^2\psi^2 + (2\lambda - 1)px\psi + \lambda p(p+1)\} \tag{3.4}$$

with $\lambda = \frac{1}{2}p/(2p+1)$. After reducing everything to a ψ-dependence by use of (3.2) we may test for the existence of a solution by seeing whether equations for the coefficients of x^0, x^1 and x^2 on either side of the above are consistent. On performing the necessary algebra in each of the three cases – this proves indeed to be the case, whichever of the three conditions we take–a single first-order equation for ψ is found. This is

$$\partial\psi/\partial t = -\lambda\psi(\varepsilon\psi - 1). \tag{3.5}$$

A straightforward integration leads to the time dependence

$$\psi(t) = \varepsilon^{-1}[1 - \eta\mathrm{e}^{-\lambda t}]^{-1} \tag{3.6}$$

the remaining unspecified quantity η parametrizing the range of possible initial conditions. Thus, to summarize, the solution may be composed as

$$P^{\mathrm{s}}(x, t) = \Gamma(p)^{-1}x^{p-1}\psi^p\, \mathrm{e}^{-\psi x}(1 + p - p\varepsilon\psi + x\psi(\varepsilon\psi - 1)\}. \tag{3.7}$$

The general characteristics of this solution are plain enough. We may note how the term { } collapses neatly to unity since $\psi(t) \to \varepsilon^{-1}$ as $t \to \infty$, giving the equilibrium Gamma distribution $P^s(x,\infty)=P_0(x)$ of Eq. (2.10). So far as the transient $\exp(-\lambda t)$ is concerned, this varies rather weakly with p, changing from $\exp(-t/6)$ for $p=1$ to $\exp(-t/4)$ for $p \gg 1$. Although the exponential terms vary only slightly with p, there is nevertheless a powerful effect of number of degrees of freedom through the factor $[\psi(t)]^p$.

Thus we have demonstrated that the BKW-similarity solution is a special solution of eq. (2.1) for arbitrary (integer or noninteger) values of p. The $p=1$ result, for which the kernel (2.2), $W(x,u)=u^{-1}$ is, of course, identical with that derived by Tjon and Wu[9]).

For completeness we shall also calculate the simple and polynomial moments, $\mu_n^s(t)$ and $\gamma_n^s(t)$, corresponding to the similarity solution (3.7). When necessary integrations are carried out and the factors $\psi(t)$ given explicitly we have for the former

$$\mu_n^s(t)=\{1+(n-1)\eta\,e^{-\lambda t}\}(1-\eta\,e^{-\lambda t})^{n-1}\varepsilon^n(p)_n/n!, \tag{3.8}$$

and the latter

$$\gamma_n^s(t)=(1-n)\eta^n\,e^{-n\lambda t}(p)_n/n!. \tag{3.9}$$

The Fourier representation

$$P^s(x,t)=P_0(x)\sum_{n=0}^{\infty}(1-n)\eta^n\,e^{-n\lambda t}L_n^{(p-1)}(x/\varepsilon) \tag{3.10}$$

is then equivalent to (3.7)

3.2. *Discussion of the solutions*

We cannot leave tha above solutions without a brief comment on their range of physical validity. The usefulness of $P^s(x,t)$ in the above form is severly restricted by the requirement that, in addition to conserving probability and energy, it must remain positive for all $x\in(0,\infty)$ and all time $t>0$. Clearly enough, these conditions can always be violated by suitable choice of the parameter η in eq. (3.6).

It suffices to examine the original form of the similarity solution (3.1) to see that positivity for all x is equivalent to the requirements $\alpha>0$ and $\beta>0$ at a given time. For unconditional positivity, this may be translated into the pair of inequalities

$$0<\eta\,e^{-\lambda t}<\frac{1}{p+1}\quad\text{for all } t\geqq 0. \tag{3.11}$$

Now, we can make the following assertions:
(1) *Any* initial condition $P(x,0)$ which is positive at $t=0$ will necessarily remain so for all $t>0$. (2) The subclass of similarity initial conditions $P^s(x,0)$

corresponding to $\eta > 0$ has the property that, even if they are non-positive at $t = 0$, there exists a time $t = t_0 = \lambda^{-1} \ln[\eta(p+1)]$ when positivity is established for all $t > t_0$. Condition (1) is a textbook result for all Boltzmann-type equations[16]) and is self-evident from eq. (3.11). Proposition (2) follows in similar manner.

When suitable cases are computed, it becomes clear that valid initial distributions $P(x, 0)$ may only differ significantly from equilibrium values $P_0(x)$ for x appreciably above the thermal energy region. Consider the ratio

$$R(x, t) = P(x, t)/P_0(x). \tag{3.12}$$

The largest possible deviation is found by choosing $\eta = 1/(1+p)$. In this case $R(x, t)$ remains between 0.6 and 1.5 for $0.2\varepsilon < x < 5\varepsilon$, i.e. it is relatively close to 1. However, in the high energy tail $R(x, 0)$ becomes very small. Even when time becomes large, i.e., $\eta\, e^{-\lambda t} = \delta \to 0$, the ratio $R(x, t)$ still vanishes for $x \gg 1/\delta$. Of course for $x \leqq 1/\delta$ the ratio $R(x, t) \approx 1$.

The practical importance of this nonlinear solution is virtually restricted to its high energy tail, the approach to the Maxwellian being clearly non-uniform. Whereas for energies in the thermal range the distribution function relaxes to the Maxwellian in a single mean collision time – and can be described by the linearized kinetic equation – for energies x much larger than thermal, $R(x, t) \approx 1$ only if $x \leqq \delta^{-1}$ or $t > T_0(x) = \lambda^{-1} \ln(x)$. The relaxation of the distribution function in its high energy tail cannot be described by the linearized kinetic equation.

3.3. *The discrete NHG model*

In searching for a similarity solution in the discrete variable case we are nothing if not dependent upon intuition – not even the most rudimentary form of theory for nonlinear differential-difference and differential-sum equations of the present kind would appear to exist. Nevertheless, on considering the previous result in the light of experience of *linear* relaxation problems, it is possible to make a judicious guess at the form which our time-dependent distribution function might take. We adopt the following

$$P^s(i, t) = (A + Bi)(i+1)_{p-1}(1 - \Psi)^i. \tag{3.13}$$

Here $A(t)$, $B(t)$ and $\Psi(t)$ are undetermined functions reminiscent of α, β and ψ, but not to be directly identified with them. In fact $\Psi(t)$ in particular is here dimensionless, in contrast to $\psi(t)$ which had dimensions energy^{-1}. The binomial $(1 - \Psi)$is designed to give unity for $\Psi \to 0$ in keeping with the exponential in ψ.

We shall now proceed as before, emphasizing the similarities between discrete and continuous treatments where possible. The first step is to write number and energy conservation conditions as

$$\sum_{i=0}^{\infty} (A + Bi)(i + 1)_{p-1}(1 - \Psi)^i = 1 \tag{3.14}$$

and

$$\varepsilon_0 \sum_{i=0}^{\infty} [(A + B)i + Bi(i - 1)](i + 1)_{p-1}(1 - \Psi)^i = p\varepsilon \tag{3.15}$$

respectively. Notice that we have rearranged the second summand so as to use the binomial theorem, as given in I. In fact, on using the moment formulae for the negative binomial we immediately obtain the connecting formulae

$$\begin{aligned} A &= \Gamma(p)^{-1}\Psi^p[1 + p - p\Psi(1 - c)^{-1}], \\ B &= \Gamma(p)^{-1}\Psi^{p+1}(1 - \Psi)^{-1}[\Psi(1 - c)^{-1} - 1], \end{aligned} \tag{3.16}$$

with $c = \varepsilon/(\varepsilon_0 + \varepsilon)$. The similarity with eqs. (3.2) indicates that we may be on the right track. We now continue in a manner parallel to our solution in section 3.1, substituting (3.13) into the kinetic equation (2.3a) using detailed-balance, the NHG moment formulae (2.6) and finally reducing both sides to functions of Ψ along by the above formulae. The result of this operation is the following

$$\begin{aligned} &\left[\left(\frac{-i}{1-\Psi}\right)\left(\frac{\partial \Psi}{\partial t}\right) + \frac{\partial}{\partial t}\right](A + Bi) \\ &= \frac{B^2\Gamma(p)}{\Psi^{p+2}}\{\lambda p(p + 1)(1 - \Psi)^2 + (2\lambda - 1)ip(1 - \Psi)\Psi + \lambda i(i - 1)\Psi^2\}, \end{aligned} \tag{3.17}$$

in which once again $\lambda = \frac{1}{2}p/(2p + 1)$ and the whole bears an obvious resemblance to eq. (3.4). The next step, of equating coefficients of i^0, i, and i^2 is again similar and leads to the following first-order equation for $\Psi(t)$:

$$\partial\Psi/\partial t = -\lambda\Psi[\Psi(1 - c)^{-1} - 1]. \tag{3.18}$$

This being of the same form as (3.5), we can immediately write the solution as

$$\Psi(t) = (1 - c)[1 - \eta\, e^{-\lambda t}]^{-1}, \tag{3.19}$$

with η again giving a parametrization of possible initial conditions. The solution for $P^s(i, t)$ is now to hand and may be composed in the form

$$P^s(i, t) = \Gamma(p)^{-1}(i + 1)_{p-1}\Psi^p(1 - \Psi)^i\left\{1 + \left(-p + \frac{i\Psi}{1 - \Psi}\right)\left(\frac{\Psi + c - 1}{1 - c}\right)\right\}. \tag{3.20}$$

The exact solution of the set (2.16), corresponding to moments of $P^s(i, t)$ is also interesting. On forming moments of (3.20) we find after some algebra that these are simply

$$\bar{\mu}_n^s(t) = \{c + (n-1)\eta\, e^{-\lambda t}\}(c - \eta\, e^{-\lambda t})^{n-1} \frac{(p)_n}{(1-c)^n n!}. \tag{3.21}$$

The same can be done for the Meixner moments of the similarity solution. This involves more difficult algebra. The easiest way to calculate it is from the generating function for Meixner polynomials (see I). The final result is

$$\bar{\gamma}_n^s(t) = (1-n)\eta^n\, e^{-\lambda t}(p)_n/n!. \tag{3.22}$$

This is identical to (3.9) in the continuous case. This is no surprise, since the corresponding moments obey the same set of equations with the same restrictions $\gamma_0 = 1$ and $\gamma_1 = 0$. The Fourier representation of the BKW-similarity solution is in this case

$$P^s(i,t) = P_0(i) \sum_{n=0}^{\infty} (1-n)\eta^n\, e^{-n\lambda t} M_n(i,p,c)(p)_n/n! \tag{3.23}$$

and this is equivalent to (3.20).

3.4. *Discussions of solutions*

Both the structure of the equation and subsequent behaviour of the above solution parallel those of the continuous analogues (2.1) and (3.7); the limit $\Psi(t) \to 1-c$ as $t \to \infty$ guarantees evolution to the Negative Binomial distribution (2.11); the dependence upon p likewise follows the pattern previously discussed. Remarkable perhaps is the fact that the crucial "relaxation-time" λ^{-1} is unchanged on passage from the continuous to the discrete model. Similar behaviour has, however, been noticed in the spectra of linear random-scattering-models discussed in ref. 17. Finally, if we wish to recover the continuous solution in the form (3.7) we have only to apply the limit $i \to \infty$, $\varepsilon_0 \to 0$, $i\varepsilon_0 = x$ just as insection 2, remembering to identify $\Psi = \varepsilon_0 \psi$ in the limit.

A discussion of the range of validity of the discrete solution $P^s(i,t)$ can be given in much the same terms as that for the continuous case. Here one encounters the inequalities

$$0 < \eta\, e^{-\lambda t} < \min\left[\frac{1}{1+p}, c\right]. \tag{3.24}$$

Before concluding this section we might refer briefly to another discrete-variable model, that of Tjon and Wu[9]), i.e.:

$$(\partial_t + 1)P(i,t) = \sum_{k=i}^{\infty} \frac{\varepsilon_{k-i}}{k} \sum_{j=0}^{k} \varepsilon_j \varepsilon_{k-j} P(j,t)P(k-j,t), \tag{3.25}$$

where $\varepsilon_j = \frac{1}{2}$ for $j = 0$ and $\varepsilon_j = 1$ otherwise. This resembles closely our model (2.3) with $p = 1$ in (2.4), i.e.

$$(\partial_t + 1)P(i, t) = \sum_{k=i}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} P(j, t)P(k-j, t). \tag{3.26}$$

For the latter model the BKW-mode (3.20) is an exact solution, whereas for the former no exact solution is known and the BKW-mode is only an approximate solution. The model equation (3.25) was of course only constructed for numerical calculations as a discretization of the continuous model (2.1) using the trapezoidal rule.

4. Additional similarity modes

4.1. *A two-parameter class of similarity solutions*

In order to search for more similarity solutions we notice that the set of moment equations (2.16) is invariant under the group of transformations[18])

$$t^a = t + \ln a, \quad \gamma_n^a = a^{-n\lambda}\gamma_n, \tag{4.1}$$

with $a > 0$ and λ undetermined. Hence from every solution $\gamma_n(t)$ one can construct a one parameter class of solutions $\gamma_n^a(t^a) = a^{-n\lambda}\gamma_n(t)$. We may now look for similarity solutions, which are themselves invariant under this group, i.e.

$$\gamma_n^s(t) = a^{n\lambda}\gamma_n^s(t + \ln a) = e^{-n\lambda t}\gamma_n^s(0). \tag{4.2}$$

The stationary solution $\gamma_n(\infty) = \delta_{n,0}$, as well as $\gamma_0(t) = 1$ and $\gamma_0(t) = 0$ are consistent with (4.2).

With the substitution (4.2) the moment equations reduce to the following set of algebraic recursion relations,

$$n(\lambda_n - \lambda)\gamma_n^s(0) = \frac{(p)_n}{(2p)_n} \sum_{k=2}^{n-2} \gamma_k^s(0)\gamma_{n-k}^s(0), \tag{4.3}$$

with $n \geqq 2$ and λ_n given by

$$\lambda_n = \frac{1}{n} \Lambda_n = \frac{1}{n}\left\{1 - \frac{2(p)_n}{(2p)_n}\right\}. \tag{4.4}$$

The quantity Λ_n is the eigenvalue of the linearized Boltzmann collision operator, as obtained in I.

The search for similarity solutions is the search for special initial conditions to be obtained here by solving (4.3). An analagous set of recursion relations to determine similarity solutions has been discussed by Bobylev[5]) for the case of ordinary Maxwell molecules. The recursion relation (4.3) are only soluble for certain values of the parameter λ (hence the term non linear eigenvalue). If $\gamma_2^s(0) \neq 0$, solubility of the system (4.3) requires a vanishing coefficient in front of $\gamma_2^s(0)$ on the left hand side of eq. (4.3), i.e. $\lambda = \lambda_2 = \frac{1}{2}p/(2p+1)$ and $\gamma_2^s(0)$ can be chosen arbitrarily. However, due to accidental degeneracy $\lambda_2 = \lambda_3$, the coefficient of $\gamma_3^s(0)$ on the left hand side of (4.3) also vanishes, and a second parameter can be chosen arbitrarily. This degeneracy occurs in all continuous Maxwell models[5,14]). All remaining coefficients $\gamma_n^s(0)$ with $n \geqq 4$ can then be expressed in terms of $\gamma_2^s(0)$ and $\gamma_3^s(0)$, and we list a few

$$\gamma_4^s(0) = -\frac{1}{2}\frac{(p+2)(p+3)}{p(p+1)}(\gamma_2^s(0))^2,$$
$$\gamma_5^s(0) = -\frac{1}{5}\frac{(p+3)(p+4)}{p(p+1)}\gamma_2^s(0)\gamma_3^s(0). \tag{4.5}$$

For the special case $p = 1$ the recursion relations can be solved explicitly for the complete λ_2-mode, i.e. the generating function for the coefficients $\gamma_k^s(0)$ is the Weierstrass elliptic function[19]). It also contains two arbitrary constants. The BKW-mode (3.7) or (3.20) contains only one arbitrary instant η, and is a special case of the general λ_2-mode. This can be seen by relating the two parameters in (4.5) as

$$\gamma_3^s(0) = (2/3)(p+2)\eta\gamma_2^s(0), \tag{4.6}$$

where η is defined through $\gamma_2^s(0) = \frac{1}{2}p(p+1)\eta^2$. It then follows that

$$\gamma_n^{\mathrm{BKW}}(0) = (1-n)\eta^n(p)_n/n!. \tag{4.7}$$

4.2. λ_m-*similarity modes*

There also exist similarity modes with nonlinear eigenvalues $\lambda_m < \lambda_2$ for $m > 3$, resulting from the initial conditions

$$\begin{aligned} &\gamma_l^s(0) = 0, \quad \text{for } 0 < l < m,\\ &\gamma_m^s(0) \neq 0. \end{aligned} \tag{4.8}$$

Since there is no further degeneracy of λ_m for integer values $m > 3$, all other λ_m-modes contain only *one* arbitrary constant $\gamma_m^s(0)$. Then, only coefficients $\gamma_l^s(0)$ with a label $l = nm$ ($n = 1, 2, \ldots$) are non-vanishing, and can be found from the recursion relation

$$\gamma_{nm}^s(0) = -\frac{1}{nm(\lambda_m - \lambda_{nm})}\frac{(p)_{nm}}{(2p)_{nm}}\sum_{k=1}^{n-1}\gamma_{km}^s(0)\gamma_{nm-km}^s(0). \tag{4.9}$$

For continuous models with initial conditions $R(x, 0)$, defined in (3.12) and not belonging to the Hilbert space $L_2(0, \infty)$, one can find similarity solutions where $m = q > 1$ is non integer[14]). The spectrum of nonlinear eigenvalues λ_q is continuous and doubly degenerate with $0 < \lambda_q = \lambda_{max}$, where λ_{max} is some constant depending on the model considered. The eigenvalue λ_q in (4.4) for non-integer q is defined through Gamma functions.

4.3. *Similarity solutions versus general solutions: the Krook–Wu conjecture*

The Fourier series (3.10) and (3.23) though *not* a *finite* expression, allows us to draw some interesting conclusions by making a term by term comparison with the Fourier series for the general solution (2.13) or (2.17). The structure of the equations (2.16) for the Laguerre and Meixner moments show that the general solutions of our kinetic equations can be seen as a sum of transients of the form

$$\gamma_n(t) = \sum_{\{m_k\}}{}' \Gamma^{(n)}(\{m_k\}) \exp\left\{-t \sum_{k=2}^{n} m_k \Lambda_k\right\}, \tag{4.10}$$

where Λ_k is given in (4.4), $m_k = 0, 1, 2, \ldots$, and the prime indicates that we sum over all partitions $\{m_k\}$ under the restriction,

$$\sum_{k=2}^{n} k m_k = n. \tag{4.11}$$

For $\Gamma^{(2)}(\{m_k\}) \neq 0$ the *fastest* transients in each $\gamma_n(t)$ are those with only $m_2 \neq 0$ and/or $m_3 \neq 0$. Due to the degeneracy $\lambda_2 = \lambda_3$ we have for every allowed partition $\{m_2, m_3\}$ that the exponent in (4.10) is $m_2\Lambda_2 + m_3\Lambda_3 = n\lambda_2$. The sum of the fastest transients, proportional to $\exp(-n\lambda_2 t)$, over all Fourier components n in the series (2.13) or (2.17) represents the λ_2-similarity mode. It characterizes the time development of a system, provided only that the $\gamma_n(0)$ satisfy the recursion relations (4.3) with $\lambda = \lambda_2$, i.e. provided the system starts from a λ_2-similarity initial condition. Decay through the BKW-mode applies only under a further restriction that $\gamma_2(0)$ and $\gamma_3(0)$ are related as in eq. (4.6).

For general initial conditions slower transients will always be present in $\gamma_n(t)$. The *slowest* transients of all are those with all $m_k = 0$ except m_n (which must therefore be equal to 1), and they decay as $\exp(-t\Lambda_n)$. These are the relaxational modes of the linearized kinetic equation, discussed in I.

We want to conclude with some further discussion on the Krook–Wu conjecture, ano the evidence against its validity. By numerical solution of the Boltzmann equation for the $p = 1$ model, Tjon[10]) has found that there exist solutions $R(x, t)$ – defined in (3.12) – in which $R(x, t)$ approaches *one* (the Maxwellian) from above, whereas in the BKW-mode the approach is always from below. Furthermore, as Alexanian[13]) and Hauge[14]) have pointed out, the final approach to the Maxwellian is always from above for strictly positive $\gamma_2(0)$.

Hauge further remarks that for initial conditions outside the Hilbert space $L_2(0,\infty)$ the existence of similarity modes with a basic relaxation rate λ_q, arbitrary close to zero, would also invalidate the Krook–Wu conjecture. However, at the same time such initial conditions would invalidate the notions about the linearized Boltzmann collision operator describing exponential relaxation[5]). The choice of the correct function space, however, is an open problem in the kinetic theory of the Boltzmann equation[5,19]).

For the class of initial conditions with $\gamma_2(0) < 0$ the conjecture is valid in the rather trivial sense that the *final* approach to equilibrium in a *fixed* energy range is given by the BKW-mode plus terms of order $\exp(-3\lambda_2 t)$. In the same sense the complete λ_2-similarity mode gives the final relaxation correct up to terms of order $\exp(-\Lambda_4 t)$ (where $\Lambda_4 > 3\lambda_2 = \Lambda_3$), provided only that $\gamma_2(0)$ and $\gamma_3(0)$ do not vanish simultaneously. However, to the orders indicated both similarity modes coincide with the solution of the linearized Boltzmann equation.

From the discussion following (4.10) it is clear that the Fourier series (2.13) or (2.17) of the general solution does not approach *term by term* the Fourier series of the λ_2-similarity mode.

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