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MASTER EQUATIONS WITH CANONICAL INVARIANCE

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It is shown that there exists a large variety of master equations with the property that the canonical equilibrium distribution becomes a solution on replacing the temperature with a suitable function of time. The additional requirement of detailed balance, however, strongly restricts the possibilities and uniquely leads to only three special forms of the master equation, given by equations (17), (22), and (23). These equations are the same as found by Andersen, Oppenheim, Shuler, and Weiss. Other requirements than detailed balance are also briefly discussed.

1. Introduction

Consider a physical system whose possible energy values ε form a set E of real numbers, either continuous, as in classical systems, or discrete, as in quantum systems. Let $G(\varepsilon)$ be the "structure function", i.e., the volume of phase space, or the number of states, with energy less than ε. An ensemble of such systems is described by a probability $p(\varepsilon)$ *per state*, such that

$$p(\varepsilon)\geqslant 0, \qquad \int_{-\infty}^{\infty} p(\varepsilon)\, dG(\varepsilon)=1 . \tag{1}$$

The possible distributions p span a linear space L, in which they themselves define a convex cone C.

When the system interacts with a heat bath with fixed temperature T, transitions between states with different energies are possible and p varies with time t. In many cases the time evolution of p is governed by a master equation

$$\dot{p}=Wp, \tag{2}$$

where W is a linear operator in L, independent of t. Of course, this equation can only be true when separate states having the same energy need not be distinguished, either because they have the same transition probabilities, or because equilibrium among them is rapidly established.

The master equation (2) has a time-independent equilibrium solution

$$p_\beta(\varepsilon) = e^{\psi - \beta\varepsilon}, \tag{3}$$

with $\beta = 1/kT$, and a normalizing constant ψ determined by

$$e^{-\psi(\beta)} = \int e^{-\beta\varepsilon}\, dG(\varepsilon). \tag{4}$$

We assume that p_β is the only equilibrium solution; this assumption involves no loss of generality, because when there are more the operator W is reducible and the system decomposes into subsystems, each having a single equilibrium. Moreover, we only consider systems in which every solution of (2) tends to p_β for $t \to \infty$.

On working out special applications, a number of authors ([2], [3], [8], [9]) have noticed that the master equation (2) often admits of solutions of the form

$$p_\vartheta(\varepsilon) = e^{\psi(\vartheta) - \vartheta\varepsilon}, \tag{5}$$

when for ϑ a suitable function of t is taken. Physically this means that the system evolves towards equilibrium through a sequence of equilibrium states having a well-defined temperature, which varies with time. It is of interest to know how general a feature this is.

Andersen, Oppenheim, Shuler, and Weiss ([1]) found that the existence of "canonically invariant" solutions (5) is confined to a very limited class of operators W. However, they assumed a number of additional properties of W, in particular detailed balance, and it is difficult to discern which of these properties is mainly responsible for their result. In the present note we shall first keep W as general as possible, and subsequently investigate the consequences of additional properties. Our conclusion is that there is a large class of operators W having canonical invariant solutions (5), but that the additional requirement of detailed balance leads to the limitations found in AOSW.

2. The canonical invariance condition

Suppose one has a linear equation (2) and one knows that it admits of solutions of the form (5) with a specified $\vartheta(t)$ for $t > t_1$. This implies that $\vartheta(t)$ is differentiable and

$$\lim_{t\to\infty} \vartheta(t) = \beta. \tag{6}$$

Furthermore $\vartheta(t)$ must be monotonic; for, if at any time t_2 one had $\vartheta'(t_2) = 0$, it would mean that another time-independent solution exists. Hence one can define a function $\Theta(\vartheta)$ by setting

$$\vartheta'(t) = -\Theta(\vartheta). \tag{7}$$

The domain of Θ is $\Lambda = \{\vartheta(t);\, t > t_1\}$. Also we know $\beta \in \Lambda$, $\Theta(\beta) = 0$, and

$$\Theta(\vartheta) \gtrless 0 \quad \text{for} \quad \vartheta \gtrless \beta. \tag{8}$$

We shall con fine ourselves to the case where Θ has a positive derivative at β.

The assertion that (2) has solutions (5) amounts to

$$\Theta(\vartheta)\{\varepsilon-\psi'(\vartheta)\}\,e^{-\vartheta\varepsilon}=We^{-\vartheta\varepsilon}\qquad(\vartheta\in\Lambda)\,. \tag{9}$$

This specifies the action of W on the set of functions $\{e^{-\lambda\varepsilon};\ \lambda\in\Lambda\}$, which form a curve $C^*\subset C$; and hence also on their linear combinations

$$p(\varepsilon)=\int_\Lambda \sigma(\lambda)\,e^{-\lambda\varepsilon}d\lambda\,, \tag{10}$$

which form a linear space $L^*\subset L$. Thus this assertion uniquely determines the operator W in L^*, but cannot tell anything about the effect of W on functions, if any, that are in L but not in L^*.

The problem can now be translated in terms of the Laplace transforms $\sigma(\lambda)$ (see Appendix I). The master equation (2) translates into a linear equation

$$\dot\sigma(\lambda)=\int_\Lambda A(\lambda\,|\,\mu)\,\sigma(\mu)\,d\mu\,. \tag{11}$$

The canonical distributions (5) translate into

$$\sigma(\lambda\,,t)=e^{\psi(\vartheta)}\delta(\lambda-\vartheta)\,.$$

The assertion that these satisfy (2) is

$$\Theta(\vartheta)\{\delta'(\lambda-\vartheta)-\psi'(\vartheta)\,\delta(\lambda-\vartheta)\}=A(\lambda\,|\,\vartheta)\,.$$

Since this must be true for all λ and ϑ in Λ, it determines A. The equation (11) is therefore

$$\dot\sigma(\lambda)=\int_\Lambda \{\delta'(\lambda-\mu)-\psi'(\mu)\,\delta(\lambda-\mu)\}\;\Theta(\mu)\,\sigma(\mu)\,d\mu\,. \tag{12}$$

Thus *we have constructed for any structure function $G(\varepsilon)$ and any time dependence $\vartheta(t)$ a master equation with the canonically invariant solution* (5).

3. Detailed balance

Define a scalar product in L by

$$(p_1\,,p_2)=\int_E e^{\beta\varepsilon}p_1(\varepsilon)\,p_2(\varepsilon)\,dG(\varepsilon)\,.$$

Detailed balance is the property of W to be symmetric in terms of this scalar product (see [5], [7])

$$(p_1\,,Wp_2)=(p_2\,,Wp_1)\,.$$

On expressing p_1, p_2 in σ_1, σ_2 with the aid of (10) one finds for the scalar product of any two functions in L

$$\begin{aligned}(p_1\,,p_2)&=\int_\Lambda\int_\Lambda \sigma_1(\lambda)\,\sigma_2(\mu)\,d\lambda d\mu\int_E e^{(\beta-\lambda-\mu)\varepsilon}dG(\varepsilon)\\&=\int_\Lambda\int_\Lambda e^{-\psi(\lambda+\mu-\beta)}\sigma_1(\lambda)\,\sigma_2(\mu)\,d\lambda d\mu\,.\end{aligned}$$

Hence detailed balance in the σ-language is the requirement that

$$\int_A \int_A e^{-\psi(\lambda+\mu-\beta)} \sigma_1(\lambda)\, d\lambda d\mu \int_A \{\delta'(\mu-\nu)-\psi'(\nu)\,\delta(\mu-\nu)\}\,\Theta(\nu)\,\sigma_2(\nu)\,d\nu$$

be symmetric in σ_1, σ_2. This expression simplifies to

$$\int_A \int_A e^{-\psi(\lambda+\mu-\beta)} \sigma_1(\lambda)\, d\lambda d\mu\, \{\psi'(\lambda+\mu-\beta)-\psi'(\mu)\}\,\Theta(\mu)\,\sigma_2(\mu),$$

so that the symmetry requirement is

$$\{\psi'(\lambda+\mu-\beta)-\psi'(\mu)\}\,\Theta(\mu)=\{\psi'(\lambda+\mu-\beta)-\psi'(\lambda)\}\,\Theta(\lambda). \tag{13}$$

This functional equation for ψ' and Θ is solved in Appendix II. There are two classes of solutions; the first class is given by (28),

$$\psi'(\lambda)=A+\frac{B}{\lambda-b}, \qquad \Theta(\lambda)=C(\lambda-\beta)(\lambda-b), \tag{14}$$

with constants A, B, C, b. In order to obey the condition (8) for approach to equilibrium one must have

(i) $C>0$, $b<\beta$ and $\vartheta>b$; or

(ii) $C<0$, $b>\beta$ and $\vartheta<b$.

We first consider the former alternative.

From (14) follows

$$e^{-\psi(\lambda)}=\text{const}\,.\,e^{-A\lambda}(\lambda-b)^{-B},$$

and, on inverting the Laplace transform (4):

$$G(\varepsilon)=\text{constant}=0 \qquad \text{for} \quad \varepsilon\leqslant A;$$

$$\frac{dG(\varepsilon)}{d\varepsilon}=\text{const}.(\varepsilon-A)^{B-1}e^{b\varepsilon} \text{ for} \quad \varepsilon\geqslant A.$$

By a physically immaterial shift in energy one may put $A=0$. Hence this solution describes a system with the continuous energy range $E=(0,\infty)$ and with a density of states

$$g(\varepsilon)=\varepsilon^{B-1}e^{b\varepsilon}. \tag{15}$$

Thus *we are uniquely led to the form of* $g(\varepsilon)$ *that was introduced as an Ansatz by AOSW* (see, however, their footnote 11). For a physically realizable system one must have $B>0$. Moreover, for $b>0$ the system has a temperature ceiling, which is unrealistic except perhaps for hadron fire balls [6]. It is also hard to find a system with $b<0$, because that makes the total volume of phase space finite. Yet we keep b to compare our result with AOSW.

The time dependence of the temperature is found from (7) and (14),

$$\vartheta'(t)=-C(\vartheta-\beta)(\vartheta-b),$$

which integrates to

$$\frac{\vartheta(t)-\beta}{\vartheta(t)-b}=\frac{\vartheta(t_1)-\beta}{\vartheta(t_1)-b}e^{-C(\beta-b)(t-t_1)}. \tag{16}$$

The explicit form of the master equation is found from (10) and (12)

$$\begin{aligned}\dot{p}(\varepsilon)&=\iint e^{-\lambda\varepsilon}\{\delta'(\lambda-\mu)-\psi'(\mu)\,\delta(\lambda-\mu)\}\,\Theta(\mu)\,\sigma(\mu)\,d\lambda d\mu\\&=C\int\left\{\varepsilon-\frac{B}{\lambda-b}\right\}(\lambda-\beta)(\lambda-b)\,\sigma(\lambda)\,e^{-\lambda\varepsilon}d\lambda\\&=C\left\{\varepsilon\left(\frac{\partial}{\partial\varepsilon}+b\right)+B\right\}\left(\frac{\partial}{\partial\varepsilon}+\beta\right)p(\varepsilon).\end{aligned}$$

This determines the operator W in our master equation (2). The equation is usually written in terms of the probability density *in the energy scale*, i.e., $P(\varepsilon)=g(\varepsilon)p(\varepsilon)$:

$$\dot{P}(\varepsilon)=C\left[\frac{\partial}{\partial\varepsilon}\{(\beta-b)\,\varepsilon-B\}+\frac{\partial^2}{\partial\varepsilon^2}\varepsilon\right]P(\varepsilon), \tag{17}$$

which is identical with the master equation found by AOSW for the case of continuous E.

The second alternative, denoted above by (ii), leads to the same time dependence (16) and the same density of states (15). However, the condition $\vartheta<b$ cannot be met by any normalizable solution, so that this alternative is spurious. Hence, the first class of solutions of the functional equation (13) for detailed balance leads uniquely to a continuous energy range $E=(0, \infty)$ with the master equation (17).

4. The discrete case

According to Appendix II the functional equation (13) has a second class of solutions

$$\psi'(\lambda)=A+\frac{B}{c-e^{-\gamma(\lambda-\beta)}}, \tag{18}$$

$$\Theta(\lambda)=C\left\{e^{\gamma(\lambda-\beta)}-1\right\}\left\{c-e^{-\gamma(\lambda-\beta)}\right\}, \tag{19}$$

with constants A, B, C, c, γ. By construction $\gamma>0$, and the condition (8) requires

(i) $c>1$, and $C>0$; or

(ii) $c<0$, and $C<0$.

We first study alternative (i).

The time dependence of the temperature is found from (19) and (7) to be

$$\frac{1-e^{-\gamma[\vartheta(t)-\beta]}}{c-e^{-\gamma[\vartheta(t)-\beta]}}=\frac{1-e^{-\gamma[\vartheta(t_1)-\beta]}}{c-e^{\gamma[\vartheta(t_1)-\beta]}}e^{-\gamma(c-1)C(t-t_1)}. \tag{20}$$

In order to find $G(\varepsilon)$ we first integrate (18) putting $B/\gamma c=\alpha$,

$$e^{-\psi(\lambda)}=\text{const.}\, e^{-(A+\gamma\alpha)\lambda}[c-e^{-\gamma(\lambda-\beta)}]^{-\alpha}.$$

We omit the constant factor, take $A+\gamma\alpha=0$ and put $ce^{-\gamma\beta}=e^{-\gamma\tau}$. Then

$$G(\varepsilon)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\varepsilon\lambda}-1}{\lambda}[1-e^{-\gamma(\lambda-\tau)}]^{-\alpha}d\lambda \tag{21}$$

vanishes for $\varepsilon<0$. For $\varepsilon>0$ the integral is evaluated in Appendix III with the following result. $G(\varepsilon)$ is constant apart from jumps at the energy levels $\varepsilon_n=n\gamma$. The magnitude of the jump, which is the weight (or degree of degeneracy) of the level, is

$$g_n=g_0\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}e^{n\gamma\tau}\qquad(n=0,1,2,\ldots).$$

In order that these weights are positive one must take $\alpha>0$.

The explicit master equation is obtained by substituting (18) and (19) into (12); a calculation similar to that in the previous section yields for the occupation probability p_n of a state belonging to level n

$$\dot{p}_n=\gamma Cc[e^{\gamma\tau}(n+\alpha)p_{n+1}+e^{-\gamma\beta}np_{n-1}-\{e^{\gamma\tau-\gamma\beta}(n+\alpha)+n\}p_n].$$

The master equation for the occupation probability $P_n=g_np_n$ *per level* is

$$\dot{P}_n=\gamma C[c(n+1)P_{n+1}+(n+\alpha-1)P_{n-1}-\{cn+(n+\alpha)\}]P_n. \tag{22}$$

In case of the alternative (ii) one may write

$$\psi'(\lambda)=A-\frac{B}{c'+e^{-\gamma(\lambda-\beta)}},$$

$$\Theta(\lambda)=C'[e^{\gamma(\lambda-\beta)}-1]\{c'+e^{-\gamma(\lambda-\beta)}\},$$

with $c'=-c>0$, $C'=-C$. The time dependence is again given by (20). In order to find $G(\varepsilon)$ we put $B/\gamma c'=\alpha'$ and obtain

$$e^{-\psi(\lambda)}=\text{const}\, e^{-(A-\gamma\alpha')\lambda}[c'+e^{-\gamma(\lambda-\beta)}]^{-\alpha'}.$$

Putting $c'=e^{\gamma(\beta-\tau)}$ and omitting unimportant factors,

$$G(\varepsilon)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{\varepsilon\lambda}[1+e^{-\gamma(\lambda-\tau)}]^{\alpha'}d\lambda.$$

A calculation similar to that in Appendix III now leads again to energy levels with weight

$$g_n=g_0\frac{(-1)^n\Gamma(n-\alpha')}{\Gamma(-\alpha')\Gamma(n+1)}e^{n\gamma\tau}.$$

The only way to avoid negative weights is by taking for α' a non-negative integer N, so that

$$g_n = g_0 \binom{N}{n} e^{n\gamma\tau} \qquad (n=0,1,2,\dots,N).$$

The corresponding master equation for the P_n is

$$\dot{P}_n = \gamma C' [c'(n+1) P_{n+1} + (N-n+1) P_{n-1} - \{c'n + N - n\} P_n]. \tag{23}$$

Thus *the second class of solutions of* (13) *leads uniquely to discrete, equidistant energy levels, either a finite number with master equation* (23), *or an infinite number with* (22), in agreement with AOSW.

5. Other conditions

Although detailed balance applies to many physical systems it is by no means universal and, in particular, it is usually invalidated by external forces. *Are there more fundamental conditions that also restrict the possible forms of W?* Of course, the master equation must conserve total probability, but this is not an additional restriction, as shown by the following argument. The canonically invariant solutions (5), which served to construct our W, are normalized; consequently the general master equation constructed in Section 2 automatically conserves the integral (1) for these functions, and hence also for their linear combinations.

Another fundamental condition is the *positivity requirement*: when $p(\varepsilon) \geqslant 0$ at one time, it must remain non-negative at later times. We have not been able to find the full consequences of this condition, but we shall demonstrate on two examples that, on the one hand it does restrict the possibilities for W, but, on the other hand, the restriction is less severe than detailed balance. Note that all master equations we found to be consistent with detailed balance obey the positivity requirement.

The Laplace transformed master equation (12) is not suitable for discussing positivity. However, in the continuous case, $E = (0, \infty)$, it can be translated back in terms of $p(\varepsilon)$,

$$\dot{p}(\varepsilon) = \{\varepsilon - \psi'(-\partial)\}\, \Theta(-\partial)\, p(\varepsilon),$$

where $\partial = \partial/\partial\varepsilon$. For both examples we take $g(\varepsilon) = \varepsilon^{B-1}$, so that $\psi'(\lambda) = B/\lambda$. Then the master equation is

$$\begin{aligned}\dot{p}(\varepsilon) &= \{\varepsilon\partial + B\}\, \Theta(-\partial)\, \partial^{-1} p(\varepsilon)\\ &= -\varepsilon^{-B+1} \partial \varepsilon^{B} \Theta(-\partial) \int_{\varepsilon}^{\infty} p(x)\, dx.\end{aligned}$$

For the first example take $\Theta(\vartheta) = \vartheta(\vartheta - \beta)\, \Phi(\vartheta)$ with some polynominal Φ, and write the equation in terms of the more familiar $P(\varepsilon) = \varepsilon^{B-1} p(\varepsilon)$:

$$\dot{P}(\varepsilon) = \partial \varepsilon^{B} (\partial + \beta)\, \Phi(-\partial)\, \varepsilon^{-B+1} P(\varepsilon).$$

For $\Phi = C$ this reduces to (17), but for other Φ higher derivatives enter, which upset the positivity condition. Suppose, in particular, $\Phi(\vartheta) = \vartheta + a$, and take at some $\varepsilon_0 > 0$

$$P(\varepsilon_0) = 0, \quad \partial P(\varepsilon_0) = 0, \quad \partial^2 P(\varepsilon_0) > 0.$$

Then

$$\dot{P}(\varepsilon_0) = \{(a - \beta)\varepsilon_0 + 2B - 3\}\, \partial^2 P(\varepsilon_0) - \varepsilon_0\, \partial^3 P(\varepsilon_0).$$

For any values of a, β, B it is possible to choose $\partial^3 P(\varepsilon_0)$ such that this is negative. A similar argument shows that no polynomial Φ is compatible with the positivity condition.

For the second example, take $\Theta(\vartheta) = \vartheta - \beta$,

$$\dot{P}(\varepsilon) = \partial \varepsilon^B (\partial + \beta) \int_\varepsilon^\infty x^{-B+1} P(x)\, dx.$$

This master equation does not obey detailed balance, but it can be shown as follows that the positivity requirement is satisfied. For any $\varepsilon_1 < \varepsilon_2$

$$\frac{d}{dt} \int_{\varepsilon_1}^{\varepsilon_2} P(\varepsilon)\, d\varepsilon = -\varepsilon_2 P(\varepsilon_2) + \varepsilon_1 P(\varepsilon_1) + \beta \left[\varepsilon_2^B \int_{\varepsilon_2}^\infty x^{-B+1} P(x)\, dx - \varepsilon_1^B \int_{\varepsilon_1}^\infty x^{-B+1} P(x)\, dx \right].$$

Suppose $P(\varepsilon)$ is negative in some intervals. Identify $(\varepsilon_1, \varepsilon_2)$ with the interval farthest to the right; the possibilities $\varepsilon_1 = 0$ and $\varepsilon_2 = \infty$ are not excluded. Then the first two terms vanish since $P(\varepsilon_2) = 0$ and $\varepsilon_1 P(\varepsilon_1) = 0$. The remaining terms are

$$\beta(\varepsilon_2^B - \varepsilon_1^B) \int_{\varepsilon_2}^\infty x^{-B+1} P(x)\, dx - \beta \varepsilon_1^B \int_{\varepsilon_1}^{\varepsilon_2} x^{-B+1} P(x)\, dx,$$

and both are non-negative. Thus, as soon as P is negative in an interval its integral must increase, so that no such interval can originate.

Appendix I

According to (10) a single function p is associated with each σ, but it is not evident that a unique σ is associated with each p. Suppose σ_1 and σ_2 give the same p; then $\sigma_1 - \sigma_2 = \tau$ has the property that

$$\int_\Lambda \tau(\lambda)\, e^{-\lambda\varepsilon} d\lambda \tag{24}$$

vanishes for each $\varepsilon \in E$. If E has a continuous interval, or at least one finite accumulation point, then the analytic function (24) must vanish for all ε, and hence $\tau(\lambda) = 0$. If E consists of discrete values ε_n with no other accumulation point than infinity, the uniqueness is still guaranteed provided that (see [4])

$$\sum_n \varepsilon_n^{-1} = \infty.$$

No uniqueness, however, can be asserted when E consists of a finite point set, or of an infinite point set for which this sum converges (for instance, a Schrödinger particle in a one-dimensional box). In such cases we can only state that (12) is *sufficient* for the existence of canonically invariant solutions. As a consequence, the condition of detailed balance does not strictly exclude the possibility of systems of this type.

Appendix II

To solve the functional equation (13) we first simplify it by setting $\lambda=\beta x$, $\mu=\beta y$ and

$$\frac{\Theta(\beta x)}{\beta\Theta'(\beta)}=u(x), \qquad \frac{\psi'(\beta x)-\psi'(\beta)}{\beta\psi''(\beta)}=v(x).$$

Then $u(1)=v(1)=0$, $u'(1)=v'(1)=1$, and

$$v(x)u(x)-v(y)u(y)=v(x+y-1)\{u(x)-u(y)\}. \tag{25}$$

Differentiate with respect to y and take $y=1$:

$$0=v'(x)u(x)-v(x).$$

Substitute this expression for u in (25)

$$\frac{v(x)^2}{v'(x)}-\frac{v(y)^2}{v'(y)}=v(x+y-1)\left\{\frac{v(x)}{v'(x)}-\frac{v(y)}{v'(y)}\right\};$$

differentiate twice with respect to y and take $y=1$:

$$-2=v''(x)\frac{v(x)}{v'(x)}-2v'(x)-v(x)h,$$

where $h=-v''(1)$.

This differential equation for v is simplified by inserting $v(x)=1/w(x)$,

$$w''-hw'+2ww'=0.$$

A first integration yields

$$w'=-(w-\tfrac{1}{2}h-\tfrac{1}{2}q)(w-\tfrac{1}{2}h+\tfrac{1}{2}q),$$

where q is an integration constant. In the case $q>0$ one obtains by a second integration

$$v(x)=2\frac{1-e^{-q(x-1)}}{q+h+(q-h)e^{-q(x-1)}}, \tag{26}$$

and accordingly

$$u(x)=\frac{1}{2q^2}\{e^{q(x-1)}-1\}\{q+h+(q-h)e^{-q(x-1)}\}. \tag{27}$$

Another class of solutions one obtains for $q=0$; the easiest way to find these is by taking the limit in (26) and (27):

$$v(x)=\frac{x-1}{1+\frac{1}{2}h(x-1)}, \quad u(x)=(x-1)\{1+\tfrac{1}{2}h(x-1)\}. \tag{28}$$

Direct substitution shows that (27) and (28) are, indeed, solutions of (25). Finally, q might be purely imaginary, but that appears not to lead to an admissible $G(\varepsilon)$.

Appendix III

The integrand in (21) has a sequence of singularities at

$$\lambda_m=\tau+2m\pi i/\gamma \quad (m=\ldots,-2,\,-1,0,1,2,\ldots).$$

When the integration path is shifted to the left each singularity gives rise to a loop, which starts at $-\infty$, encircles the singularity counter-clockwise, and returns to $-\infty$. Changing the integration variable on each loop by setting $\lambda=\lambda_m+z$ one obtains

$$G(\varepsilon)=\frac{1}{2\pi i}\sum_m\int\frac{e^{\varepsilon(\lambda_m+z)}-1}{\lambda_m+z}(1-e^{-\gamma z})^{-\alpha}\,dz, \tag{29}$$

the integral being taken over a loop encircling the origin. The summation over m can be performed

$$\sum_m\frac{e^{\varepsilon(\lambda_m+z)}-1}{\lambda_m+z}=\int_0^\varepsilon dx\,e^{x(\tau+z)}\sum_m e^{2m\pi i(x/\gamma)}=\gamma\int_0^\varepsilon dx\,e^{x(\tau+z)}\sum_{-\infty}^{\infty}\delta(x-n\gamma)=\gamma\sum_{n\gamma<\varepsilon}e^{n\gamma(\tau+z)}.$$

Inserting this identity into (29)

$$G(\varepsilon)=\sum_{n\gamma<\varepsilon}e^{n\gamma\tau}\frac{\gamma}{2\pi i}\int e^{n\gamma z}(1-e^{-\gamma z})^{-\alpha}dz.$$

For $\alpha<1$, $n\geqslant 0$ the integral is found to be

$$\frac{\gamma}{2\pi i}\int_0^\infty e^{-n\gamma s}(e^{\gamma s}-1)^{-\alpha}ds(-e^{-\pi i\alpha}+e^{\pi i\alpha})=\frac{\sin\pi\alpha}{n}\int_0^1 t^{n-1+\alpha}(1-t)^{-\alpha}\,dt=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

By analytic continuation this result extends to all α.

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