On characterizing optimality and existence of optimal solutions in Lyapunov type optimization problems

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Necessary and sufficient conditions for optimality, in the form of a duality result of Fritz-John type, are given for an abstract optimization problem of Lyapunov type. The introduction of a so-called integrand constraint qualification allows the duality result to take the form of a Kuhn-Tucker type result. Special applications include necessary and sufficient conditions for the existence of optimal controls for certain optimal control problems.

1 Introduction

In [7] P. Kaiser studied a one-dimensional problem in the calculus of variations, which, rewritten in its equivalent optimal control form, runs as follows:

$$(P_K) \quad \inf_{u \in \mathcal{U}} \{ \int_0^1 \phi(t)\sqrt{1+u^2(t)}dt : \int_0^1 u(t)dt = d \}.$$

Here $\mathcal{U} := \mathcal{L}^1_{\mathbb{R}}[0,1]$ is the set of all Lebesgue-integrable functions on $[0,1]$, $d \in \mathbb{R}$ is some constant, and $\phi \in \mathcal{L}^1_{\mathbb{R}}[0,1]$ is a strictly positive function. Let η be the essential infimum of ϕ. The main result in [7] is the following characterization of existence of an optimal solution for (P_K).

Theorem 1.1 ([7]) *An optimal solution for the problem (P_K) exists if and only if*

$$|d| \leq \int_0^1 \frac{\eta}{\sqrt{\phi^2(t)-\eta^2}}dt.$$

In the above result the integral may have the value $+\infty$ when it is improper. Observe also that, in comparison to [7], the conditions used here for ϕ are somewhat less demanding (in [7] ϕ is also supposed to be smooth).

Subsequently, P. Brandi [4] and C. Marcelli [8, 9] gave generalizations of Theorem 1.1, by replacing the integrand $\phi(t)\sqrt{1+u^2}$ with much more general expressions (including nonsmooth ones).

This work presents a new approach to study existence problems of this variety. Namely, it exploits the role played by the duality aspects of optimization problems of Lyapunov type. For such problems, which include (P_K) and the other ones mentioned above, we present Theorem 2.2, a duality result à la Fritz John; this result is of some independent interest, because its quite general form combines and extends similar results in [1, §4.3.3,§4.3.4]. Under an *integrand constraint qualification* of an apparently novel type, this duality result is applied to obtain in Theorem 3.3 a characterization of optimality for the Lyapunov type problem. Not surprisingly, this leads immediately to Corollary 3.4, which gives a necessary and sufficient condition for the *existence* of an optimal solution.

2 Duality for Lyapunov type optimization problems

Let $(T, \mathcal{T}, \mu)$ be a finite measure space and let S be a Suslin space, e.g., a Polish space. Let $\mathcal{M}_S$ be the set of all $(\mathcal{T}, \mathcal{B}(S))$-measurable functions u from T into S such that $u(T)$ is a relatively compact subset of S; here $\mathcal{B}(S)$ stands for the Borel σ-algebra on S. Let $\mathcal{U}$ be a set of $(\mathcal{T}, \mathcal{B}(S))$-measurable functions from T into S that is *decomposable* in the sense of [5, VII]. That is to say, $\mathcal{U}$ contains $\mathcal{M}_S$ and is closed for concatenations: for every pair $u, u' \in \mathcal{U}$ and every $A \in \mathcal{T}$ the concatenation $v : T \to S$, defined by $v := u$ on A and $v := u'$ on $T \backslash A$, belongs to $\mathcal{U}$.

Readers who are only interested in applications to the calculus of variations can just concentrate on the situation considered in the next example:

Example 2.1 *In case $S = \mathbb{R}^d$ the set $\mathcal{M}_S$ is obviously the set $\mathcal{L}^\infty_{\mathbb{R}^d} := \mathcal{L}^\infty(T, \mathcal{T}, \mu; \mathbb{R}^d)$ of all bounded measurable functions from T into $\mathbb{R}^d$. Moreover, $\mathcal{L}^p_{\mathbb{R}^d}$ is clearly decomposable for any $p \in \mathbb{N} \cup \{\infty\}$.*

Let $f_0, \cdots, f_m : T \times S \to (-\infty, +\infty]$ be a finite collection of $\mathcal{T} \times \mathcal{B}(S)$-measurable functions, which are such that for every $u \in \mathcal{U}$ the functions

$$\min(f_0(\cdot, u(\cdot)), 0), \cdots, \min(f_{m'}(\cdot, u(\cdot)), 0) \text{ and } |f_{m'+1}(\cdot, u(\cdot))|, \cdots, |f_m(\cdot, u(\cdot))| \tag{2.1}$$

are μ-integrable; here m', $0 \le m' \le m$, is given. Consequently, integral functionals $I_{f_0}, \cdots, I_{f_{m'}} : \mathcal{U} \to (-\infty, +\infty]$ and $I_{f_{m'+1}}, \cdots, I_{f_m} : \mathcal{U} \to \mathbb{R}$ are defined by

$$I_{f_i}(u) := \int_T f_i(t, u(t)) \mu(dt),$$

where the first $m' + 1$ integrals are interpreted in the usual way as quasi-integrals [10]. Also, let X be a subset of some vector space. Let $g_0, \cdots, g_{m'} : X \to (-\infty, +\infty]$ and $g_{m'+1}, \cdots, g_m : X \to \mathbb{R}$ be given functions. The following *Lyapunov-type* optimization problem

$$(P_L) \quad \inf_{u \in \mathcal{U}, x \in X} \{I_{f_0}(u) + g_0(x) : I_{f_1}(u) + g_1(x) \bowtie 0, \cdots, I_{f_m}(u) + g_m(x) \bowtie 0\},$$

will be studied, where $I_{f_i}(u) + g_i(x) \bowtie 0$ means $I_{f_i}(u) + g_i(x) \le 0$ for indices $i \le m'$ and $I_{f_i}(u) + g_i(x) = 0$ for indices i with $m' < i \le m$. To prevent having to consider trivialities, we suppose

$$\inf(P_L) < +\infty. \tag{2.2}$$

The following theorem characterizes the optimal solutions of (P_L) and extend the corresponding theorem in [1, §4.3.3].

Theorem 2.2 (Fritz John type duality) *(i) If $(\hat{u}, \hat{x})$ is a feasible solution of (P_L) for which there exists $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \in \{1\} \times \mathbb{R}^{m'}_+ \times \mathbb{R}^{m-m'}$ such that the following three conditions hold:*

$$\begin{aligned} \hat{u}(t) &\in \operatorname{argmin}_{s \in S} \sum_{i=0}^m \hat{\lambda}_i f_i(t, s) \text{ for a.e. } t \text{ (}s\text{-minimum principle)}, \\ \hat{x} &\in \operatorname{argmin}_{x \in X} \sum_{i=0}^m \hat{\lambda}_i g_i(x) \text{ (}x\text{-minimum principle)}, \\ 0 &= \hat{\lambda}_i (I_{f_i}(\hat{u}) + g_i(\hat{x})) \text{ for } i = 1, \cdots, m' \text{ (complementarity relations)}, \end{aligned}$$

then $(\hat{u}, \hat{x})$ is an optimal solution of (P_L).

(ii) Suppose that the measure space $(T, \mathcal{T}, \mu)$ is nonatomic, that the set X is convex, that $g_0, \cdots, g_{m'} : X \to (-\infty, +\infty]$ are convex functions and that $g_{m'+1}, \cdots, g_m : X \to \mathbb{R}$ are affine functions. If $(\hat{u}, \hat{x})$ is an optimal solution of (P_L), then there exists $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \in \{0, 1\} \times \mathbb{R}^{m'}_+ \times \mathbb{R}^{m-m'}$, $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \ne (0, \cdots, 0)$, such that the s- and x-minimum principles and the complementarity relations of part (i) all hold.

But for the assertion about the value of the Fritz John multiplier $\hat{\lambda}_0$, the statement in part (ii) of the above theorem is the converse of the statement in part (i). Observe that Theorem 2.2 places no convexity conditions whatsoever upon the integrands $f_0, \cdots, f_m$.

Before giving the proof, we briefly illustrate the usefulness of this theorem by a simple application that cannot be addressed by the results in [1] (observe that the integral functional $I_{f_0} : u \mapsto \int_0^1 u^2(t)\, dt$ of this problem is not everywhere finite on $\mathcal{L}^1_{\mathbb{R}}$, as requested in [1].)

Example 2.3 *The optimal control problem*

$$\inf_{u \in \mathcal{L}^1[0,1], x \in \mathbb{R}} \{ \int_0^1 (u^2(t) - y_{u,x}(t)) dt : x \leq 0, y_{u,x}(1) = 1 \},$$

where $y_{u,x}(t) := x + \int_0^t u(\tau)\, d\tau$, *can also be rewritten as*

$$\inf_{u \in \mathcal{L}^1[0,1], x \leq 0} \{ \int_0^1 (u^2(t) - (1-t)u(t))\, dt - x : \int_0^1 u(t)\, dt + x - 1 = 0 \}.$$

This shows that it is of the same type as (P_L), *with* $\mathcal{U} := \mathcal{L}^1_{\mathbb{R}}$, $X := \mathbb{R}_-$, $f_0(t,s) := s^2 - (1-t)s$, $g_0(x) := -x$, $m' = 0$, $m = 1$, $f_1(t,s) := s$ *and* $g_1(x) := x - 1$ *for instance. Suppose for the moment that the above problem has an optimal solution* $(\hat{u}, \hat{x})$. *Let* $(\hat{\lambda}_0, \hat{\lambda}_1) \neq (0,0)$ *be as guaranteed by Theorem 2.2(ii). Then validity of the s-minimum principle implies* $\hat{\lambda}_0 = 1$, *so* $\hat{u}(t) = (1 - t - \hat{\lambda}_1)/2$. *Also, validity of the x-minimum principle implies* $\hat{\lambda}_1 \leq 1$. *The case* $\lambda_1 = 1$ *cannot occur, for it would lead to* $\hat{u}(t) = -t/2$, *whence* $\hat{x} = 5/4 \notin X$. *So* $\lambda_1 < 1$, *which implies* $\hat{x} = 0$ *by the x-minimum principle. Solving the equality constraint for* $\hat{\lambda}_1$, *we find* $\hat{\lambda}_1 = -3/2$ *for the only remaining parameter, and this uniquely determines* $\hat{u}(t) := 5/4 - t/2$ *(and* $\hat{x} = 0$*). Next, for* $\hat{\lambda} := (1, -3/2)$ *we invoke Theorem 2.2(i) to verify optimality of the above pair* $(\hat{u}, \hat{x})$. *This amounts to retracing the preceding argument and is left to the reader. We conclude that* $\hat{y}(t) := 5t/4 - t^2/4$ *(corresponding to* $\hat{u}(t) := 5/4 - t/2$ *and* $\hat{x} = 0$*) is the unique optimal solution of the original variational problem.*

Remark 2.4 *In Theorem 2.2(i)* $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m)$ *is easily seen to be the optimal solution of the following* dual *optimization problem:*

$$(Q_L) \qquad \sup_{(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^{m'}_+ \times \mathbb{R}^{m-m'}} J(\lambda_1, \cdots, \lambda_m)$$

where $J(\lambda_1, \cdots, \lambda_m) := \int_T [\inf_{s \in S}(f_0(t,s) + \sum_{i=1}^m \lambda_i f_i(t,s))] \mu(dt) + \inf_{x \in X}\{g_0(x) + \sum_{i=1}^m \lambda_i g_i(x)\}$. *The same holds for* $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m)$ *in Theorem 2.2(ii), provided that* $\hat{\lambda}_0 = 1$. *Moreover, under the same provision* $\hat{\lambda}_0 = 1$ *Theorem 2.2(ii) can be extended as follows: irrespective of whether* (P_L) *has an optimal solution or not, there exists* $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) \in \mathbb{R}^{m'}_+ \times \mathbb{R}^{m-m'}$ *such that*

$$J(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) = \sup(Q_L) = \inf(P_L).$$

This can be derived immediately from the proof of Theorem 2.2(ii) given below.

The proof of Theorem 2.2, to which the remainder of this section is devoted, is a modification of the corresponding proof in [1, p. 354]. Observe, however, that much more general conditions are imposed here: in [1] T is an interval, and while its S is a general topological space, its integrand functions f_i are supposed to be continuous, and no allowance is made for its f_i's to take the value $+\infty$. Not surprisingly, the proof of the weak duality part (i) of Theorem 2.2 is elementary:

Proof of Theorem 2.2(i). Let $\hat{\lambda}$ be as stated. For any feasible pair (u, x) for (P_L) we obviously have $\sum_{i=0}^m \hat{\lambda}_i f_i(t, \hat{u}(t)) \leq \sum_{i=0}^m \hat{\lambda}_i f_i(t, u(t))$ a.e., by the s-minimum principle, and also $\sum_{i=0}^m \hat{\lambda}_i g_i(\hat{x}) \leq \sum_{i=0}^m \hat{\lambda}_i g_i(x)$ by the x-minimum principle. The former implies $\sum_{i=0}^m \hat{\lambda}_i I_{f_i}(\hat{u}) \leq \sum_{i=0}^m \hat{\lambda}_i I_{f_i}(u)$, so combined with the latter we find

$$I_{f_0}(\hat{u}) + g_0(\hat{x}) = \sum_{i=0}^m \hat{\lambda}_i (I_{f_i}(\hat{u}) + g_i(\hat{x})) \leq \sum_{i=0}^m \hat{\lambda}_i (I_{f_i}(u) + g_i(x)) \leq I_{f_0}(u) + g_0(x),$$

where the identity holds by the given complementarity relations, and the last ineqality by feasibility of (u, x) and the nature of the components of the vector $\hat{\lambda}$. This proves the optimality of $(\hat{u}, \hat{x})$ for (P_L). Q.E.D.

Next, we prepare the proof of part (ii) of Theorem 2.2. To begin with, let us observe that the objective function $(u, x) \mapsto I_{f_0}(u) + g_0(x)$ cannot attain the value $-\infty$, so the fact that Theorem 2.2(ii) supposes the existence of an optimal element in implies that $\iota := \inf(P_L)$ is not equal to $-\infty$; in view of (2.2), this means $\iota \in \mathbb{R}$. Let C be the set of all $r := (r_0, \cdots, r_m) \in \mathbb{R}^{m+1}$ for which there exist $u \in \mathcal{U}$ and $x \in X$ such that $I_{f_0}(u) + g_0(x) < r_0$ and $I_{f_i}(u) + g_i(x) \bowtie r_i$ for $i = 1, \cdots, m$.

Lemma 2.5 *C is a nonempty convex subset of $\mathbb{R}^{m+1}$,*

PROOF. Nonemptiness follows immediately from (2.2). To prove the convexity of C, let $r, r' \in C$ and $\alpha \in (0, 1)$ be arbitrary. By definition of C there exist (u, x) and (u', x') in $\mathcal{U} \times X$ such that for $\psi_i := f_i(\cdot, u(\cdot))$ and $\psi'_i := f_i(\cdot, u'(\cdot))$ we have $\int \psi_0 + g_0(x) < r_0$, $\int \psi'_0 + g_0(x') < r'_0$; $\int \psi_i + g_i(x) \leq r_i$, $\int \psi'_i + g_i(x') \leq r'_i$ for $1 \leq i \leq m'$ and $\int \psi_i + g_i(x) = r_i$, $\int \psi'_i + g_i(x') = r'_i$ for $i \geq m' + 1$. By (2.1) all the component functions ψ_i and ψ'_i are integrable. By an application of Lyapunov's theorem to the vector-valued measure $\nu : A \mapsto \int_A (\psi_0, \psi'_0, \cdots, \psi_m, \psi'_m)$, there exists $A \in \mathcal{T}$ such that $\nu(A) = \alpha\nu(T)$ (here we use the nonatomicity hypothesis). Let $v \in \mathcal{U}$ be the concatenation given by $v := u$ on A and $v := u'$ on $T \backslash A$. Then it is easy to see that $I_{f_i}(v) = \alpha I_{f_i}(u) + (1-\alpha) I_{f_i}(u')$ for all i, $0 \leq i \leq m$. By the given convexity/affinity of the functions g_i, it follows that $(v, \alpha x + (1-\alpha)x') \in \mathcal{U} \times X$ is such that $I_{f_0}(v) + g_0(\alpha x + (1-\alpha)x') < \alpha r_0 + (1-\alpha) r'_0$ and $I_{f_i}(v) + g_i(\alpha x + (1-\alpha)x') \bowtie \alpha r_i + (1-\alpha) r'_i$ for all $1 \leq i \leq m$. This shows that $\alpha r + (1-\alpha) r'$ belongs to C. Q.E.D.

Lemma 2.6 *The set C does not contain the vector $(\iota, 0, \cdots, 0)$.*

PROOF. An immediate consequence of the definition of C and ι.

Lemma 2.7 *There exist $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \in \{0, 1\} \times \mathbb{R}^{m'}_+ \times \mathbb{R}^{m-m'}$, $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \neq (0, \cdots, 0)$, such that*

$$\inf_{u \in \mathcal{U}, x \in X} \sum_{i=0}^{m} \hat{\lambda}_i (I_{f_i}(u) + g_i(x)) = \hat{\lambda}_0 \inf(P_L).$$

PROOF. By Lemmas 2.5 and 2.6 the origin of $\mathbb{R}^{m+1}$ does not belong to the convex set $C - (\iota, 0, \cdots, 0)$. By a well-known separation theorem in finite dimensions [1, §1.3.3], there exists $\hat{\lambda} := (\hat{\lambda}_0, \cdots, \hat{\lambda}_m)$ in $\mathbb{R}^{m+1}$, $\hat{\lambda} \neq 0$, such that $\sum_{i=0}^m \hat{\lambda}_i r_i \geq \hat{\lambda}_0 \iota$ for all $r \in C$. It follows that $\hat{\lambda}_0, \cdots, \hat{\lambda}_{m'} \geq 0$, because $C + (\mathbb{R}^{m'+1}_+ \times \{(0, \cdots, 0)\}) = C$. Normalizing in case $\hat{\lambda}_0 > 0$ (divide all components of $\hat{\lambda}$ by $\hat{\lambda}_0$), we ensure $\hat{\lambda}_0 \in \{0, 1\}$ without loss of generality. By definition of the set C the inequality

$$\inf_{u \in \mathcal{U}, x \in X} \sum_{i=0}^{m} \hat{\lambda}_i (I_{f_i}(u) + g_i(x)) \geq \hat{\lambda}_0 \iota$$

follows easily from the above separation inequality. The converse inequality follows by considering any minimizing sequence (u_k, x_k) of (P_L) (observe that $\hat{\lambda}_1 (I_{f_1}(u_k) + g_1(u_k)), \cdots, \hat{\lambda}_1 (I_{f_{m'}}(u_k) + g_{m'}(u_k)) \leq 0$). Q.E.D.

To prove Theorem 2.2(ii), we employ a *reduction theorem* that originated in the work of Ioffe-Tichomirov [6] and Rockafellar; results of this type are essentially sophisticated measurable selection results. The present version, which comes from [2], was inspired by [5, VII]. It is stated with the following integration convention in force: for any $\mathcal{T}$-measurable function $\phi : T \to \mathbb{R}$ the integral $\int_T \psi$ is defined by $\int_T \psi := \int_T \max(\psi, 0) - \int_T \max(-\psi, 0)$, with the understanding that $(+\infty) - (+\infty)$ means here $+\infty$.

Theorem 2.8 ([2, Theorem B.1]) *For every $\mathcal{T} \times \mathcal{B}(S)$-measurable function $f : T \times S \to [-\infty, +\infty]$ and every decomposable set $\mathcal{V}$ of $(\mathcal{T}, \mathcal{B}(S))$-measurable functions from TintoS the identity*

$$\inf_{v \in \mathcal{V}} \int_T f(t, v(t)) \mu(dt) = \int_T \inf_{s \in S} f(t, s) \mu(dt)$$

holds, provided that the left hand infimum does not equal $+\infty$. Here the function $t \mapsto \inf_{s\in S} f(t,s)$ is $\mathcal{T}$-measurable.

Here we should note that the measure space $(T, \mathcal{T}, \mu)$ in [2] is complete. However, by a rather standard argument this can be lifted (e.g., see [5, III.22] and the proof of Theorem 3 in [3]).

PROOF OF THEOREM 2.2(ii). Let $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \in \{0,1\} \times \mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ be as guaranteed by Lemma 2.7. Then by the given optimality of $(\hat{u}, \hat{x})$

$$\hat{\lambda}_0(I_{f_0}(\hat{u})+g_0(\hat{x})) = \inf_{u\in\mathcal{U}, x\in X} \sum_{i=0}^{m} \hat{\lambda}_i(I_{f_i}(u)+g_i(x)) \leq \sum_{i=0}^{m} \hat{\lambda}_i(I_{f_i}(\hat{u})+g_i(\hat{x})) = \sum_{i=0}^{m'} \hat{\lambda}_i(I_{f_i}(\hat{u})+g_i(\hat{x})).$$

Since the terms $\hat{\lambda}_1(I_{f_1}(\hat{u})+g_1(\hat{x})), \cdots, \hat{\lambda}_{m'}(I_{f_{m'}}(\hat{u})+g_{m'}(\hat{x}))$ are all nonnegative, the complementarity relations follow immediately. Next, by additive separation the above yields

$$\hat{\lambda}_0(I_{f_0}(\hat{u})+g_0(\hat{x}) = \inf_{u\in\mathcal{U}, x\in X} \sum_{i=0}^{m} \hat{\lambda}_i(I_{f_i}(u)+g_i(x)) = \inf_{u\in\mathcal{U}} \sum_{i=0}^{m} \hat{\lambda}_i I_{f_i}(u) + \inf_{x\in X} \sum_{i=0}^{m} \hat{\lambda}_i g_i(x).$$

Since $\sum_{i=0}^{m} \hat{\lambda}_i I_{f_i}(u) = \int_T \sum_{i=0}^{m} \hat{\lambda}_i f_i(t,u(t))\mu(dt)$ by (2.1), we have for the first infimum in the above right hand side

$$\inf_{u\in\mathcal{U}} \sum_{i=0}^{m} \hat{\lambda}_i I_{f_i}(\hat{u}) = \int_T \inf_{s\in S} \sum_{i=0}^{m} \hat{\lambda}_i f_i(t,s)\mu(dt),$$

by an application of Theorem 2.8. So if we combine the preceding results, we find

$$\hat{\lambda}_0(I_{f_0}(\hat{u})+g_0(\hat{x})) = \sum_{i=0}^{m} \hat{\lambda}_i I_{f_i}(\hat{u}) + \sum_{i=0}^{m} \hat{\lambda}_i g_i(\hat{x}) = \int_T \inf_{s\in S} \sum_{i=0}^{m} \hat{\lambda}_i f_i(t,s)\mu(dt) + \inf_{x\in X} \sum_{i=0}^{m} \hat{\lambda}_i g_i(x).$$

This immediately leads to the x-minimum principle for $\hat{x}$ and to

$$\int_T [\sum_{i=0}^{m} \hat{\lambda}_i f_i(t,\hat{u}(t)) - \inf_{s\in S} \sum_{i=0}^{m} \hat{\lambda}_i f_i(t,s)]\mu(dt) = 0.$$

In the above integral the integrand is nonnegative, which means that the integrand must be zero a.e. This proves the s-minimum principle for $\hat{u}$. Q.E.D.

3 Optimality characterization for Lyapunov type problems

Let $f_0, \cdots, f_m : T \times S \to [-\infty, +\infty]$ be $\mathcal{T} \times \mathcal{B}(S)$-measurable functions, precisely as in the previous section, satisfying (2.1). Let (P_L) be as in section 2, but, for reasons of convenience, we set all functions $g_0, \cdots, g_m$ equal to constants $-\gamma_0, \cdots, -\gamma_m$ in this section. Thus, we consider

$$(P_L) \quad \inf_{u\in\mathcal{U}} \{I_{f_0}(u) : I_{f_1}(u) \bowtie \gamma_1, \cdots, I_{f_m}(u) \bowtie \gamma_m\}.$$

Recall that $I_{f_i}(u) \bowtie \gamma_i$ means $I_{f_i}(u) \leq \gamma_i$ for $i \leq m'$ and $I_{f_i}(u) = \gamma_i$ for $m' < i \leq m$. To prevent trivialities, we again suppose (2.2).

From Theorem 2.2 we can immediately derive necessary and sufficient conditions for optimality for (P_L), by means of an *integrand constraint qualification* (ICQ) for the integrands $f_1, \cdots, f_m$. Its purpose is the same as the usual but quite different constraint qualifications for problems of the usual convex programming type (which arise from (P_L) by setting the integrands $f_0, \cdots, f_m$ identically equal to zero): that is, to guarantee that the Fritz John multiplier $\hat{\lambda}_0$ in Theorem 2.2 is nonzero.

Definition 3.1 (integrand constraint qualification) *The functions $f_1, \cdots, f_m$ are said to satisfy the ICQ if for every $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ with $(\lambda_1, \cdots, \lambda_m) \neq (0, \cdots, 0)$ there is no $u \in \mathcal{U}$ such that*

$$u(t) \in \mathrm{argmin}_{s \in S} \sum_{i=1}^{m} \lambda_i f_i(t, s) \text{ for a.e. } t,$$

that is to say, no element in $\mathcal{U}$ satisfies the s-minimum principle for a nontrivial multiplier vector $(\lambda_0, \cdots, \lambda_m)$ with $\lambda_0 = 0$.

Example 3.2 *Let $T := (0,1)$ be equipped with Lebesgue measure μ, let $S := \mathbb{R}$, $m := 1$, and let $\mathcal{U} := \mathcal{L}_{\mathbb{R}}^p$ for $p \geq 1$.*

(a) Suppose that $f_1(t, s) := (s - \frac{1}{\sqrt{t}})^2$. Obviously, for every $\lambda_1 \neq 0$

$$\mathrm{argmin}_{s \in S} \lambda_1 f_1(t, s) = \begin{cases} \{\frac{1}{\sqrt{t}}\} & \text{if } \lambda_1 > 0, \\ \emptyset & \text{if } \lambda_1 < 0. \end{cases}$$

Hence, if $m' = 0$ then taking $\lambda_1 = -1$ shows that the ICQ does not hold for any p. Next, if $m' = 1$ then the ICQ holds whenever $p < 2$ (for then $t \mapsto t^{p/2}$ is integrable), and the ICQ does not hold when $p \geq 2$.

(b) Suppose that $f_1(t, s) := \alpha s + \beta$, where $\alpha, \beta \in \mathbb{R}$. For every $\lambda_1 \neq 0$

$$\mathrm{argmin}_{s \in S} \lambda_1 f_1(t, s) = \begin{cases} \mathbb{R} & \text{if } \alpha = 0, \\ \emptyset & \text{if } \alpha \neq 0. \end{cases}$$

Hence, the ICQ holds when $\alpha \neq 0$. It does not hold when $\alpha = 0$ (regardless of the values of p and β).

Theorem 3.3 (Kuhn-Tucker type duality) *Suppose that $(T, \mathcal{T}, \mu)$ is nonatomic and that the ICQ holds. Let $\tilde{\Lambda}$ be any subset of $\mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ which contains the set of all $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ for which there exists $u \in \mathcal{U}$ with*

$$u(t) \in \mathrm{argmin}_{s \in S} f_0(t, s) + \sum_{i=1}^{m} \lambda_i f_i(t, s) \text{ for a.e. } t.$$

For every $\hat{u} \in \mathcal{U}$ the following are equivalent:

(a) $\hat{u}$ is an optimal solution of (P_L).

(b) There exist $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) \in \tilde{\Lambda}$ such that

$$I_{f_1}(\hat{u}) \bowtie \gamma_1, \cdots, I_{f_m}(\hat{u}) \bowtie \gamma_m \text{ (feasibility)}, \tag{3.3}$$

$$\hat{u}(t) \in \mathrm{argmin}_{s \in S} f_0(t, s) + \sum_{i=1}^{m} \hat{\lambda}_i f_i(t, s) \text{ (s-minimum principle)}, \tag{3.4}$$

$$\hat{\lambda}_1 (I_{f_1}(\hat{u}) - \gamma_1) = \cdots = \hat{\lambda}_{m'} (I_{f_{m'}}(\hat{u}) - \gamma_{m'}) = 0 \text{ (complementarity)}. \tag{3.5}$$

PROOF. $(a) \Rightarrow (b)$: Let $\hat{\lambda} := (\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \in \{0,1\} \times \mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ be as guaranteed by Theorem 2.2(ii). Suppose we had $\hat{\lambda}_0 = 0$. Then the s-minimum principle of Theorem 2.2(ii) gives $\hat{u}(t) \in \mathrm{argmin}_{s \in S} \sum_{i=1}^{m} \hat{\lambda}_i f_i(t, s)$, which implies $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) = (0, \cdots, 0)$ by the ICQ. But the latter contradicts the outcome $(\hat{\lambda}_0, \cdots, \hat{\lambda}_m) \neq (0, \cdots, 0)$ of Theorem 2.2(ii). So we conclude that $\hat{\lambda}_0 = 1$. Since $\hat{u}$ satisfies the minimum principle, this means that $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) \in \tilde{\Lambda}$, by the properties of $\tilde{\Lambda}$. The feasibility of $\hat{u}$ is obvious, and the desired complementarity is another consequence of Theorem 2.2(ii).

$(b) \Rightarrow (a)$: If $(\hat{\lambda}_1, \cdots, \hat{\lambda}_m) \in \tilde{\Lambda}$ is as stated, then $\hat{u}$ and $(1, \hat{\lambda}_1, \cdots, \hat{\lambda}_m)$ obviously meet the sufficient conditions for optimality, given in Theorem 2.2(i). Q.E.D.

Corollary 3.4 *Suppose that $(T, \mathcal{T}, \mu)$ is nonatomic and that the ICQ holds. Let $\tilde{\Lambda}$ be any subset of $\mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ which contains the set of all $(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}_+^{m'} \times \mathbb{R}^{m-m'}$ for which there exists $u \in \mathcal{U}$ with*

$$u(t) \in \operatorname{argmin}_{s \in S} f_0(t,s) + \sum_{i=1}^{m} \lambda_i f_i(t,s) \text{ for a.e. } t.$$

The following are equivalent:

(a) There exists an optimal solution of (P_L).

(b) There exists $(\hat{u}, (\hat{\lambda}_1, \cdots, \hat{\lambda}_m)) \in \mathcal{U} \times \tilde{\Lambda}$ for which (3.3)-(3.5) hold.

Example 3.5 ([7]) *The optimization problem (P_K), introduced in section 1, is of the same form as (P_L) with $\mathcal{U} := \mathcal{L}^1_{\mathbb{R}}$, $m' = 0$, $m = 1$, $f_0(t,s) := \phi(t)\sqrt{1+s^2}$, $f_1(t,s) := s$, $\gamma_0 := 0$ and $\gamma_1 := d$. These substitutions give*

$$\operatorname{argmin}_{s \in S} f_0(t,s) + \lambda_1 f_1(t,s) = \begin{cases} \{-\frac{\lambda_1}{\sqrt{\phi^2(t)-\lambda_1^2}}\} & \text{if } |\lambda_1| < \phi(t), \\ \emptyset & \text{otherwise,} \end{cases}$$

It follows that the set $\tilde{\Lambda}$, defined by

$$\tilde{\Lambda} := \{\lambda_1 \in \mathbb{R} : |\lambda_1| \leq \phi(t) \text{ for a.e. } t\} = [-\eta, +\eta],$$

where $\eta > 0$ stands for the essential infimum of ϕ, meets the conditions of Corollary 3.4. Also, we have

$$\operatorname{argmin}_{s \in S} \lambda_1 f_1(t,s) = \begin{cases} \mathbb{R} & \text{if } \lambda_1 = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

which shows that the ICQ holds trivially. So application of Corollary 3.4 gives the following: there exists an optimal solution of (P_K) if and only if there exists $\lambda_1 \in [-\eta, +\eta]$ with

$$G(\lambda_1) := \int_0^1 -\frac{\lambda_1}{\sqrt{\phi^2(t) - \lambda_1^2}} dt = d \tag{3.6}$$

(observe that complementarity holds automatically by $m' = 0$). Since G is obviously monotone and continuous on $[-\eta, +\eta]$, it follows that a necessary and sufficient condition for the above is

$$G(-\eta) \leq d \leq G(+\eta),$$

which, since the function G is odd, is equivalent to the condition stated in Theorem 1.1. This is regardless of whether the integrals $G(+\eta)$ and $G(-\eta)$ take values $+\infty$ and $-\infty$ (i.e., are improper) or not, because our conventions regarding integration automatically enforce integrability of $t \mapsto -\frac{\lambda_1}{\sqrt{\phi^2(t)-\lambda_1^2}}$ when (3.6) is satisfied.

See [8, 9] for more involved applications of this type; all of these have an integrand $f_0(t,s)$ that is convex in s. In contrast, the following application of Corollary 3.4 involves an integrand $f_0(t,s)$ that is both nonconvex and nonsmooth in s; therefore it is completely beyond the reach of [4, 7, 8, 9].

Example 3.6 *Let $T := (0,1)$ be equipped with Lebesgue measure μ, let $S := \mathbb{R}$, $m' = 0$, $m := 1$, and let $\mathcal{U} := \mathcal{L}^1_{\mathbb{R}}$. Further, let $f_0(t,s) := [\max(s^2-1,0)]^{\frac{1}{4}}$, $f_1(t,s) := s$, $\gamma_0 := 0$ and $\gamma_1 := d$. With these substitutions (P_L) becomes*

$$\inf_{u \in \mathcal{U}} \{ \int_0^1 [\max(u^2(t) - 1, 0)]^{\frac{1}{4}} dt : \int_0^1 u(t) dt = d \}.$$

In this simple example the optimal solutions and a fortiori *their existence/nonexistence follow by elementary considerations: If $d \leq 1$ then $\hat{u} \equiv d$ is optimal, and if $d > 1$ there is no optimal solution (consider $u_n(t) := dn\, 1_{[0,1/n]}(t)$). More formally, it follows from Corollary 3.4 that the problem has a solution if and only if $d \leq 1$: observe that the ICQ holds, just as in Example 3.5 and that we can take $\tilde{\Lambda} = \{0\}$, since*

$$\operatorname{argmin}_{s \in S} f_0(t,s) + \lambda_1 f_1(t,s) = \begin{cases} \emptyset & \text{if } \lambda_1 \neq 0, \\ (-\infty, 1] & \text{if } \lambda_1 = 0. \end{cases}$$

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